Reality as a Consensus Protocol: The Fixed-Point Computation That Implements Physics

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Abstract

Observer Patch Holography describes a world through records that local observers keep and repair where they disagree with a neighbor. If every accepted repair lowers a measure of disagreement that cannot fall forever, two repairs from one state can always reach a common state, and repair stops exactly at the consistent states, then every order of repairs ends at the same public result, up to hidden relabelings. Termination or atomic commits alone do not suffice. Fair scheduling settles eventually; a bound on wasted attempts also bounds the attempts to settle. For constraints that fix the difference across each overlap, a solution exists exactly when these differences sum to zero around every cycle, and otherwise disagreement survives every repair. Agreed records carry Born weights and rereads that return the same answer, and an operation on one observer's registers leaves every other observer's record statistics unchanged. The result passes to finer and coarser descriptions, approximate recovery and distributed implementations; noisy repair stays near it under a certificate; records of who read whose write order the events. Fields, clocks and gravity need further premises. In the executable model of the theory, agreed records and local events meet the record and local-operation premises. Under its main candidate law, Interval Echo, overlapping observers agree on every shared reading, and a prepared local priority settles a fork between two competing repairs.

1 Local repair and observable agreement

A finite graph of observer patches represents local records and overlap constraints. Accepted local repairs compute a quotient normal form when they decrease a well-founded measure, satisfy a local diamond and are complete for the declared consistent states. The result is a rewriting theorem and identifies its endpoint with no physical universe. The closure hypothesis proposes that observers recover and realize the specification of their world; selecting a particular source, state and physical interpretation requires additional premises [14].

The executable model of the programme [13] places loads 0, 1 or 2 on the ports of federations of twelve-port icosahedral cells and a private memory on every seam. One cell is one native observer and its twelve ports are its interfaces; larger observers are finite, possibly overlapping patches of cells. The reference candidate law, A-contrast-memory-spherical-v1 or the contrast-memory law, is the exchange of loads across seams plus an agreement energy that lowers the energy of a seam whose two ports both hold a load; a seam record is admitted only when the copies at both ends agree. On its agreed records the model reproduces particle stability, by a theorem that holds for

every number of cells. The main candidate law, Interval Echo, uses this seam interaction at $\kappa = 8$ and lets a finite observer patch carry its own clock. Under the second candidate law on the same carrier, the scalar channel, an object on four seams is kept for every interval and carries a prepared clock that an observer reads in units of its own reference. The laws are candidates, and their couplings are supplied. The state of that simulation is no spacetime, and its carrier is hardware of the computation. A test of the model compares distributions of admissible observer records with what observers in our world record. Observers who inhabit the computation can describe their experience as a relativistic continuum spacetime with three dimensions of space and one of time, by the proposition on smooth effective field descriptions of the flagship paper and the reconstruction of the spacetime paper, under their stated premises [14, 32]; those results consume readouts and limits that the model does not produce in the required form. Three results of this paper apply to the model [13]. Its agreed seam records are orthogonal projectors with complete outcome sets, so Theorem 7.2 gives them Born weights, Lüders conditioning and certain rereads. A seam event or a seam reading is a complete operation on the registers of one seam, so by Proposition 3.31 it leaves the record law of every observer on other registers unchanged. The free evolution of the contrast-memory seam, which Interval Echo uses, and that of the scalar channel are continuous one-parameter unitary flows on finite registers, so the Hamiltonian rigidity of Section 8 fixes each by its flow up to an energy offset. Applying this paper's quotient theorem requires a specified mismatch, protected boundary, strict descent and complete consistent-state presentation. The free evolutions are unitary and conserve their loads and objects, so for them the settling and normal-form theorems apply to the classical seam rule, whose write the contrast-memory law performs with the coin in superposition, follows with a phase and then reverses; they make no statement about the free evolutions themselves.

Overlapping observers of Interval Echo agree on what they share [13]: on the federation of 320 cells, 320 single-cell observers and 960 overlapping three-cell observers read signed repair records and track conserved mobile doubles through 2,399 transfers, and every shared checkpoint reading agrees. In a repair experiment of Interval Echo on one cell, an attempt applies the echo and accepts the write when a sharp reading finds both ends of the seam holding one load, and each accepted write lowers the sum of the squared loads. Two such repairs competing for one doubled load leave distinguishable records, so without arbitration one fork has two distinct terminal states, an outcome that the local-diamond premise of Theorem 3.9 excludes. A prepared local priority, read without being moved, makes both request orders end in one common terminal state in all six combinations of sharp preparation and priority. No confluence of all accepted repairs on the federation is proved.

The construction uses Newman's confluence lemma [19]. Termination differs from a continuing legitimate regime in self-stabilization [17], and finite linear sheaf diffusion has its own state and operator hypotheses [18].

The main results are the quotient normal-form theorem, adaptive scheduling and capacity bounds, the cycle-holonomy obstruction, gauge descent, refinement naturality and complete record stability. The derived informational order does not supply a physical metric or clock. Likewise, fixed nonzero holonomy forces residual mismatch but supplies no localized particle or maintenance-to-energy law. These distinctions determine the interfaces to the gauge, particle and spacetime constructions.

On the collar branch, repair uses exact Markov splice or a specified Petz/Fawzi–Renner recovery map. Support conditions, repair completeness and the local diamond are explicit hypotheses. The rooted-tree packet domain below realizes them jointly. Host execution time, repair attempts, record precedence and physical duration are distinct; a physical interpretation must state their maps. The consensus fixed point, a complete-history closure point and a stationary probability law are also distinct objects.

2 Patch Nets, Overlaps, and Global Consistency

Definition 2.1 (Patch net)**.** *Let $G = (V, E)$ be a finite connected graph. Each vertex $i \in V$ is an* ***observer patch*** *with finite local state space S_i. The global state space is*

$$\Sigma := \prod_{i \in V} S_i.$$

For each edge $e = \{i, j\} \in E$, let I_e be an interface alphabet and let

$$\pi_{i,e} : S_i \to I_e, \qquad \pi_{j,e} : S_j \to I_e$$

be the interface projection maps. A global state $s = (s_i)_{i \in V} \in \Sigma$ is ***consistent on edge*** *$e = \{i, j\}$ iff*

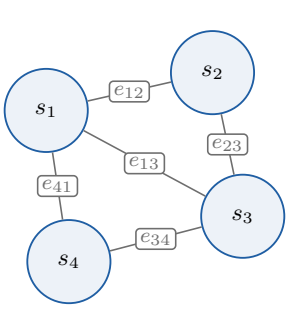
$$\pi_{i,e}(s_i) = \pi_{j,e}(s_j).$$

The global consistency set is

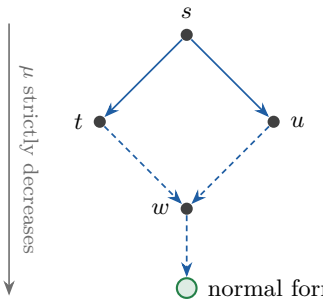
$$C := \{s \in \Sigma : \forall\, e = \{i, j\} \in E,\ \pi_{i,e}(s_i) = \pi_{j,e}(s_j)\}.$$

For exposition we use a finite pairwise-overlap graph. The hypergraph version is straightforward: replace edges by hyperedges and pairwise equality by a common interface label on each hyperedge. Nothing in the proofs depends on the pairwise restriction.

In words, each observer holds a local state, and neighboring observers share an interface through which they can compare notes. A global state is consistent exactly when all neighbors agree on their shared data. Consistency is therefore a constraint satisfaction problem, and the consistent states are its codewords.



(a) Patches i with local states s_i; on each overlap $e = \{i, j\}$ the interface projections must agree, $\pi_{i,e}(s_i) = \pi_{j,e}(s_j)$.



(b) Every one-step peak $t \leftarrow s \rightarrow u$ closes at a common w (local diamond) and every accepted repair lowers the measure μ; Newman's lemma then gives one normal form, whatever the order of repairs.

Figure 1: (a) A patch net: a global state is consistent when the interface projections agree on every overlap (Definition 2.1). (b) The finite consensus mechanism: accepted repairs strictly decrease the exact measure, every one-step peak closes in a local diamond, and Newman's lemma gives one normal form whatever the order of repairs (Theorem 3.9).

Definition 2.2 (Inconsistency potential)**.** *For each edge e, choose a weight $w_e > 0$ and a function $d_e : I_e \times I_e \to \mathbb{R}_{\geq 0}$ with $d_e(a, b) = 0 \iff a = b$. On the declared fixed-cutoff branch, d_e is the overlap score used by the local acceptance contract on that interface. Define*

$$\Phi(s) := \sum_{e=\{i,j\} \in E} w_e\, d_e(\pi_{i,e}(s_i), \pi_{j,e}(s_j)).$$

Then $s \in C \iff \Phi(s) = 0$.

Proposition 2.3 (Bare overlap nets are finite constraint codes)**.** *For every finite patch net of Definition 2.1, the consistency set* $C \subseteq \Sigma$ *is a finite constraint code whose codewords are exactly the globally overlap-consistent states. With the mismatch potential of Definition 2.2,*

$$C = \Phi^{-1}(0).$$

This proposition is the theorem-grade content of the unqualified phrase "the overlap network is a code." It supplies a finite constraint code, not a quantum error-correcting code, not a topological code, not a graph-theoretic formula for code distance, and not a Lorentzian or Einstein-geometry theorem.

Proof. Finiteness follows from $\Sigma = \prod_i S_i$ with finite S_i. The definition of C is a finite family of interface-equality constraints, so C is the set of satisfying assignments. Since each $w_e > 0$ and $d_e(a, b) = 0$ exactly when $a = b$, every term in Φ is nonnegative and vanishes exactly on a satisfied edge constraint. Therefore all edge constraints hold if and only if $\Phi(s) = 0$. □

Definition 2.4 (Bare finite consensus reduct)**.** *A bare finite consensus reduct at regulator* r *is*

$$\mathsf{Cons}_r = (\Sigma_r, \Gamma_r, Q_r, \Phi_r, \to_r, n_r, C_r, B_r),$$

where Σ_r *is the finite presentation space,* Γ_r *is the presentation-redundancy groupoid,* $Q_r = \Sigma_r/\Gamma_r$ *is the physical quotient,* Φ_r *is the mismatch functional,* $\to_r$ *is the accepted repair relation,* n_r *is the quotient normal-form map, and*

$$C_r = \Phi_r^{-1}(0)$$

is the globally overlap-consistent set, and B_r *is its boundary-data readout.*

Theorem 2.5 (Bare finite consensus is not Einstein-complete)**.** *The bare finite consensus reduct* Cons_r *does not determine a Lorentzian metric* g_{ab}*, stress tensor* T_{ab}*, Newton coupling* G*, area/edge operator* L_C*, generalized entropy* S_{gen}*, modular geometric flow, or the equation*

$$G_{ab} + \Lambda g_{ab} = 8\pi G\, T_{ab}.$$

Consequently, Einstein geometry is not a theorem of the bare finite consensus language.

Proof. The symbols of Cons_r describe finite states, quotienting, mismatch, accepted repair, normal forms, and consistency. They do not include a metric, curvature tensor, stress tensor, cap modular automorphism, area operator, or entropy-area normalization. Hence two model extensions can share the same Cons_r while assigning different geometry/stress data. One extension may attach Minkowski metric, $T_{ab} = 0$, and $\Lambda = 0$, so Einstein holds. Another may attach a Lorentzian metric g'_{ab} and stress tensor T'_{ab} for which

$$G'_{ab} + \Lambda' g'_{ab} \neq 8\pi G' T'_{ab}$$

somewhere. All bare consensus statements have the same truth value in the two extensions, while the Einstein equation has different truth values. Therefore the Einstein equation is not entailed by the bare reduct. □

The theorem excludes definability of the Einstein equation from free decorations of Cons_r: the two extensions attach geometry with no arrow from the quotient. Its machine-checked form covers the integer scalar Einstein identity on a constant demonstration tower [12]; the metric, stress, coupling, area, entropy and modular-flow clauses rest on the model extension above. Extensions whose geometry is a computable function of the quotient normal forms are the subject of the producer and composition theorems and lie outside this exclusion. Bare consensus supplies quotient normal forms, boundary data and obstruction classes. It supplies no future null cone, round-cap support chart, conformal transport, time orientation or observer-frame homogeneous space; the spacetime layer adds these with its own geometric-readout, modular, stress, entropy and calibration premises [32]. The functional Φ measures mismatch and is no energy: repair weights and counts acquire an energy meaning only through a derived maintenance-to-energy map in a calibrated clock.

3 Asynchronous Reconciliation and the Main Fixed-Point Theorem

Definition 3.1 (Recovery-derived local repair law)**.** *Fix for each patch i a finite collar chart $A_i{-}B_i{-}D_i$ around the overlaps touched by i, together with a fixed local decoder from repaired collar data back to the finite patch label at i. A **law** λ is a family of local repair maps*

$$T_i^{\lambda} : \Sigma \to \Sigma \qquad (i \in V)$$

such that T_i^{λ} changes only the state of patch i (or, more generally, only a bounded neighborhood of i), and the local update is induced by one of the declared OPH recovery moves on that collar, where $\omega_{A_iB_i}(s)$ denotes the $A_i \cup B_i$ marginal of the input collar state encoded by s:

(i) exact Markov splice on the collar, using Theorem A.1 when $I(A_i : D_i \mid B_i) = 0$; or

(ii) a declared recoverability channel

$$(\mathrm{id}_{A_i} \otimes \mathcal{R}_i)(\omega_{A_iB_i}(s)), \qquad \mathcal{R}_i = \mathcal{R}_{\sigma_i,\mathcal{N}_i},$$

with $\mathcal{R}_i$ in the Petz/Fawzi–Renner class of Definition C.3 and Theorem A.1.

The decoder back to S_i is bookkeeping for the finite patch presentation; the physical content is the repaired collar state on the declared fixed-cutoff branch. Write $s \rightsquigarrow_i t$ iff $t = T_i^{\lambda}(s) \neq s$. These are primitive proposals, not accepted physical rewrite steps. A proposal is committed only through the transactional acceptance layer below.

Definition 3.2 (Touched-overlap potential and accepted local-fit contract)**.** *For each repair site i, let*

$$E_i^{\mathrm{touch}} := \{e \in E : \text{the interface data on } e \text{ may change under } T_i^{\lambda}\}.$$

Define the touched-overlap potential

$$\Phi_i(s) := \sum_{e=\{u,v\} \in E_i^{\mathrm{touch}}} w_e \, d_e\big(\pi_{u,e}(s_u), \pi_{v,e}(s_v)\big).$$

On the declared fixed-cutoff branch, a recovery-derived primitive candidate is eligible for transactional commitment only if it strictly lowers this touched-overlap score:

$$s \rightsquigarrow_i t \implies \Phi_i(t) < \Phi_i(s).$$

This is the patch-net form of the regulator-side monotone local-fit contract carried by the declared repair package.

Definition 3.3 (Overlap-associative union-collar gluing)**.** *Fix $s \in \Sigma$ and two enabled primitive proposals $s \rightsquigarrow_i t$, $s \rightsquigarrow_j u$. Write*

$$E_{ij}^{\mathrm{touch}} := E_i^{\mathrm{touch}} \cup E_j^{\mathrm{touch}}.$$

*The declared repair branch is **overlap-associative** if the following hold.*

(i) If $E_i^{\mathrm{touch}} \cap E_j^{\mathrm{touch}} = \varnothing$, the two local proposals have disjoint support on the declared branch and therefore commute if they are committed as separate transactions.

(ii) If $E_i^{\mathrm{touch}} \cap E_j^{\mathrm{touch}} \neq \varnothing$, there is a finite union collar U_{ij} covering the interfaces in E_{ij}^{touch} such that the physical glued state on U_{ij} is parenthesization-independent on the quotient, in the sense of Proposition B.7, and the local decoders/lifts of Definition 3.1 are restriction-compatible on nested collars.

This is the concrete compatibility package used below to build canonical aggregate payloads for conflicting repair components.

Definition 3.4 (Transactional acceptance layer)**.** *Let X denote the finite physical presentation on which a repair step is being checked: before quotienting one may take $X = \Sigma$, and on the physical branch X is the quotient by hidden representatives. Registers are the finite patch, interface, sector, and record coordinates. Fix a boundary/sector/holonomy record map*

$$B : X \to \mathcal{B}$$

and a well-founded exact measure

$$\mu : X \to (W, \prec),$$

for example a lexicographic integer vector $(N_{\mathrm{hard}}, \Phi, N_{\mathrm{unresolved}})$.

A prepared transaction is a tuple

$$\tau = (R_\tau, W_\tau, \sigma_\tau, p_\tau)$$

with read set, write set, read snapshot, and payload. It commits at $x \in X$ only if all of the following hold:

(i) the snapshot is current: $x|_{R_\tau} = \sigma_\tau$;

(ii) the payload changes no register outside W_τ;

(iii) boundary, sector, holonomy, and protected record data are preserved: $B(\mathrm{Apply}_\tau(x)) = B(x)$;

(iv) exact descent holds:

$$\mu(\mathrm{Apply}_\tau(x)) \prec \mu(x).$$

*A stale, aborted, ambiguous, or obstructed transaction is not a rewrite step. The read set is **semantic-dependency-complete** as follows. Let $\mathcal{F}$ contain every finite-support functional whose value enters acceptance: the supported terms of μ, the protected boundary/sector/holonomy functions, enablement predicates, semantic-history and event-parent functions, observer-registry updates, and checkpoint-continuation functions. For a write set W, define*

$$D_{\mathcal{F}}(W) := \bigcup_{\substack{f \in \mathcal{F} \\ \mathrm{supp}(f) \cap W \neq \varnothing}} \mathrm{supp}(f).$$

Every prepared transaction satisfies $R_\tau \supseteq D_{\mathcal{F}}(W_\tau)$; its enablement and payload are functions of the read snapshot, and every acceptance functional affected by its write is revalidated at commit. For seam potentials this reads both endpoints of every seam whose score can change. Protected-support completeness and protected-conflict completeness are the restrictions of this condition to protected functionals; they are not substitutes for the full semantic closure.

At a state x, form the conflict graph of enabled primitive repair proposals, with

$$\tau \# \sigma \quad \Longleftrightarrow \quad W_\tau \cap (R_\sigma \cup W_\sigma) \neq \varnothing \text{ or } W_\sigma \cap (R_\tau \cup W_\tau) \neq \varnothing.$$

Each connected conflict component K is replaced by exactly one canonical aggregate transaction τ_K, computed on the union-collar or conflict-component support, with final declared supports R_K, W_K satisfying $R_K \supseteq D_{\mathcal{F}}(W_K)$. If aggregation or collar completion expands either support, conflicts are recomputed on the final aggregate supports and components are merged again until a fixed point is reached; distinct fixed-point aggregates are pairwise nonconflicting. These fixed-point components form a prepared source batch. After one component commits, another source-batch aggregate whose snapshot remains current and whose acceptance functionals revalidate remains admissible before later-enabled proposals are batched. Equivalently, an implementation may recompute immediately only if it certifies this surviving-component property. Primitive members of K never commit separately. Write $x \to y$ for a successful aggregate transaction commit, and let $\to^*$ be its reflexive-transitive closure. A state is a ***normal form*** *when no aggregate transaction is enabled.*

3.1 The quotient repair operator

The transactional layer above defines the physically accepted one-step relation. The object used as law is the induced finite normal-form map on the physical quotient, not a hidden-representative rewrite.

Definition 3.5 (Finite quotient repair presentation)**.** *A* ***finite quotient repair presentation*** *is a tuple*

$$\mathcal{P} = (\Sigma, \Gamma, q, Q, C_Q, B, \mu, \mathsf{A}, \prec_{\mathsf{A}})$$

where

$$Q = \Sigma/\Gamma, \qquad q : \Sigma \to Q$$

is the physical quotient by hidden representative data, $C_Q \subseteq Q$ is the quotient-level consistency set,

$$B : Q \to \mathcal{B}$$

is the protected boundary, sector, root-packet, charge, or holonomy-sector map, and

$$\mu : Q \to (W, \prec)$$

is a well-founded exact descent measure whose image on Q is finite. The finite set A consists of accepted aggregate repair transactions, equipped with a fixed total order $\prec_{\mathsf{A}}$. Each $a \in \mathsf{A}$ has a domain $D_a \subseteq Q$ and a map

$$a : D_a \to Q.$$

The accepted one-step relation is

$$x \to_{\mathcal{P}} y \quad \Longleftrightarrow \quad \exists a \in \mathsf{A},\ x \in D_a,\ y = a(x).$$

The presentation is ***OPH-admissible*** *when:*

$$(H_B) \qquad x \to_{\mathcal{P}} y \implies B(y) = B(x),$$

$$(H_{\downarrow}) \qquad x \to_{\mathcal{P}} y \implies \mu(y) \prec \mu(x),$$
$$(H_{\diamond}) \qquad \to_{\mathcal{P}} \text{ is locally confluent on } Q,$$

and

$$(H_{\text{comp}}) \qquad x \in C_Q \iff \text{no accepted aggregate transaction is enabled at } x.$$

Definition 3.6 (Local quotient repair operator)**.** *For $x \in Q$, let*

$$\mathsf{A}(x) := \{\, a \in \mathsf{A} : x \in D_a \,\}$$

be the enabled aggregate transaction set. Define

$$\mathrm{locRep}_\lambda(x) := \begin{cases} a_{\min}(x), & \mathsf{A}(x) \neq \varnothing, \\ x, & \mathsf{A}(x) = \varnothing, \end{cases}$$

where $a_{\min}$ is the $\prec_{\mathsf{A}}$-least enabled aggregate transaction.

Proposition 3.7 (Local quotient repair is total, protected, and descending)**.** *For every OPH-admissible finite quotient repair presentation,*

$$\mathrm{locRep}_\lambda : Q \to Q$$

is a total map. For every $x \in Q$,

$$B(\mathrm{locRep}_\lambda(x)) = B(x),$$

and either $\mathrm{locRep}_\lambda(x) = x$ or

$$\mu(\mathrm{locRep}_\lambda(x)) \prec \mu(x).$$

Finally,

$$\mathrm{locRep}_\lambda(x) = x \iff x \in C_Q.$$

Proof. The set A is finite and totally ordered, so a least enabled transaction exists whenever $\mathsf{A}(x) \neq \varnothing$. If no transaction is enabled, the definition returns x. Hence locRep_λ is total.

If no transaction is enabled, boundary preservation is immediate. If $a_{\min}$ is enabled, (H_B) gives $B(a_{\min}(x)) = B(x)$. The descent statement is immediate from $(H_{\downarrow})$. Since strict descent excludes $a_{\min}(x) = x$, the local operator fixes exactly those states with no enabled aggregate transaction. By (H_{comp}), these are exactly the elements of C_Q. □

Definition 3.8 (Global quotient repair operator)**.** *For $x \in Q$, define the canonical local-repair iterates by*

$$x_0 = x, \qquad x_{n+1} = \mathrm{locRep}_\lambda(x_n).$$

By Proposition 3.7, the sequence either stops or strictly descends inside the finite value set $\mu(Q)$. Hence there is a least $N(x)$ such that

$$x_{N(x)+1} = x_{N(x)}.$$

*The **global quotient repair operator** is*

$$\mathrm{Rep}_\lambda(x) := x_{N(x)}.$$

Theorem 3.9 (Global Repair is the finite quotient normal-form map)**.** *For every OPH-admissible finite quotient repair presentation,*

$$\mathrm{Rep}_\lambda : Q \to Q$$

is total and satisfies

$$\mathrm{Rep}_\lambda(x) \in C_Q, \qquad B(\mathrm{Rep}_\lambda(x)) = B(x),$$
$$\mathrm{Rep}_\lambda(\mathrm{Rep}_\lambda(x)) = \mathrm{Rep}_\lambda(x),$$

and

$$\mathrm{Rep}_\lambda(x) = x \quad \Longleftrightarrow \quad x \in C_Q.$$

If $x \to_{\mathcal{P}}^{*} y$ and y is terminal, then

$$y = \mathrm{Rep}_\lambda(x).$$

Thus

$$\mathrm{Rep}_\lambda = \overline{\mathrm{nf}}_\lambda : Q \to Q$$

is independent of the accepted asynchronous repair schedule.

Proof. Totality follows from finite descent. If $x_{n+1} \neq x_n$, then Proposition 3.7 gives

$$\mu(x_{n+1}) \prec \mu(x_n).$$

Since $\mu(Q)$ is finite and well founded, no infinite strictly descending sequence exists. Thus the canonical iteration reaches a fixed point x_N. By Proposition 3.7,

$$x_N = \mathrm{locRep}_\lambda(x_N) \quad \Longleftrightarrow \quad x_N \in C_Q,$$

so $\mathrm{Rep}_\lambda(x) \in C_Q$. Boundary preservation follows by induction from (H_B), hence $B(\mathrm{Rep}_\lambda(x)) = B(x)$. Idempotence follows because every point of C_Q is fixed by Proposition 3.7.

For schedule independence, $(H_\downarrow)$ gives termination of $\to_{\mathcal{P}}$, and $(H_\diamond)$ gives local confluence. Newman's lemma gives confluence. A terminating confluent rewrite system has a unique terminal normal form reachable from each initial state. The canonical iteration defining Rep_λ is one accepted repair execution, so it reaches that unique terminal normal form. Therefore every other maximal accepted repair execution from x reaches the same value. □

Proposition 3.10 (Validation support for local mismatch measures)**.** *Suppose the exact repair measure has finite local supports,*

$$\mu(x) = \sum_{a \in A} \mu_a(x|_{S_a}).$$

If a transaction writes W, then only terms with $S_a \cap W \neq \varnothing$ can change. Consequently, the descent validation is snapshot-local once the read set contains

$$R^\mu(W) := \bigcup_{a : S_a \cap W \neq \varnothing} S_a.$$

For the OPH overlap potential

$$\Phi(x) = \sum_{e=\{i,j\}} w_e \, d_e(\pi_{i,e}(x_i), \pi_{j,e}(x_j)),$$

this means reading both endpoints of every overlap whose score may change when a written register changes.

Proof. If $S_a \cap W = \varnothing$, the transaction leaves every register read by μ_a unchanged, so that term cancels between the pre- and post-state. Every remaining term is determined by the payload together with the restriction to $R^\mu(W)$. The displayed edge-potential formula has one two-endpoint support for each overlap term, giving the stated seam rule. □

Remark 3.11 (Inputs and branch conditions). The repair step is therefore not an abstract rewrite primitive. Its declared inputs are the fixed-cutoff collar chart, either exact Markov splice or a chosen Petz/Fawzi–Renner recovery channel, a local decoder/lift back to the finite patch presentation, and the touched-overlap local-fit contract of Definition 3.2, together with the support-local disjoint-commutation clause and the restriction-compatible union-collar package of Definition 3.3. The parenthesization-independent quotient-local glue used there is supplied by Proposition B.7 from the fixed-cutoff center-sector / higher-gauge gluing package. The actual accepted step is the validated aggregate transaction of Definition 3.4. The theorem package takes repair completeness as an explicit branch condition. On the Petz branch, full CPTP action on all inputs also requires the support clause recorded in Proposition C.4. On broader branches one must also prove that the declared union-collar compatibility is preserved under refinement or branch change.

The theorem package separates the imported repair-law data from the theorem-local inputs cleanly:

Assumption 3.12 (Repair completeness)**.** *For the accepted transactional relation $\to$, $s \in C$ if and only if no aggregate repair transaction is enabled at s.*

Normal forms are exactly the globally consistent states. The dynamics is neither too weak (missing some inconsistencies) nor too strong (repairing things that were fine).

Definition 3.13 (Verified rooted-tree packet-net domain)**.** *Fix a finite rooted tree $T = (V, E, r)$. For each non-root vertex i, write $p(i)$ for its parent and write $w_i > 0$ for the weight of the edge $\{p(i), i\}$. Let A be a finite packet alphabet with $|A| \geq 2$, and let K_i be a finite hidden-label set. The patch state space is*

$$S_i = A \times K_i, \qquad s_i = (x_i, k_i).$$

For every edge $e = \{i, j\}$, the interface alphabet is $I_e = A$, and both endpoint projections read the packet component:

$$\pi_{i,e}(x_i, k_i) = x_i, \qquad \pi_{j,e}(x_j, k_j) = x_j.$$

Thus e is consistent exactly when $x_i = x_j$. Choose the weights so that

$$w_i > \sum_{j : p(j) = i} w_j \qquad \text{for every non-root } i,$$

with an empty sum equal to 0. Define

$$\Phi(s) = \sum_{\{p(i), i\} \in E} w_i \, \mathbf{1}[x_i \neq x_{p(i)}].$$

The repair map at a non-root vertex i is

$$T_i(s)_i = (x_{p(i)}, k_i),$$

with all other vertices unchanged; if $x_i = x_{p(i)}$, the map is a no-op. The root repair map is the identity. The hidden labels k_i are acted on by arbitrary finite gauge relabelings and are not read by any interface projection.

Theorem 3.14 (Rooted-tree packet repair completeness and quotient closure)**.** *On the domain of Definition 3.13, Assumption 3.12 is a theorem. Every enabled repair strictly decreases* Φ*. Every maximal asynchronous repair run terminates at the unique state*

$$x_i = x_r \quad \text{for all } i \in V,$$

with all hidden labels k_i *unchanged. The normal-form map descends to the quotient by hidden gauge relabeling. If several tree repairs lie in one conflict component, the canonical aggregate transaction applies them in increasing tree depth and is the same as the corresponding serial tree repair path. The four-vertex instance with edges* $r-a$*,* $a-b$*,* $a-c$*, alphabet* $A = \mathbb{Z}_3$*, hidden labels* $K_i = \mathbb{Z}_2$*, and weights* $5, 1, 1$ *is exported as a verified domain record with the cited consensus code.*

Proof. First, repair completeness is immediate from the rooted tree. If $s \in C$, every edge has $x_i = x_{p(i)}$, so every non-root repair is a no-op. Conversely, if every repair is a no-op, then $x_i = x_{p(i)}$ for every non-root vertex, hence every edge is consistent and $s \in C$.

Let $i \neq r$ be enabled. Only the parent edge $\{p(i), i\}$ and child edges $\{i, j\}$ with $p(j) = i$ can change their contribution to Φ. The parent edge changes from inconsistent to consistent, contributing $-w_i$. Each child edge can increase by at most its weight. Therefore

$$\Phi(T_i(s)) - \Phi(s) \leq -w_i + \sum_{j:p(j)=i} w_j < 0.$$

So every enabled repair is accepted by the Lyapunov contract.

Since Φ takes finitely many values, every maximal repair run terminates. At a terminal state no non-root vertex differs from its parent, so the terminal packet label is x_r on every vertex. The root label and every hidden label are invariant under all repairs, so the terminal state is unique and independent of update order. Because interface projections, enabledness, Φ, and the repair maps depend only on x_i, arbitrary relabelings of the hidden K_i commute with quotienting:

$$q \circ T_i = \overline{T}_i \circ q.$$

Thus the normal-form map descends to the hidden-label quotient. □

Corollary 3.15 (Physical-law map on the verified packet domain)**.** *On the rooted-tree packet domain, every gauge-invariant observable*

$$M : \prod_i (A \times K_i) \to Y$$

has a schedule-independent repaired value

$$M(\mathrm{nf}(s)),$$

and this value depends only on the quotient class of s*. Hence the normal-form map is usable as physical law on this verified packet branch without adding a representative-level gauge-covariance assumption.*

Proof. Theorem 3.14 gives a unique terminal state for every asynchronous repair schedule and shows that the normal-form map descends to the quotient by hidden-label relabeling. A gauge-invariant observable factors through that quotient, so its value on the terminal state is independent of both the repair schedule and the hidden representative. □

Proposition 3.16 (Classical full-support Petz packet domain)**.** *Let B and D be finite packet alphabets, let the collar algebras be diagonal, and let $\mathcal{N} : \mathbb{C}^{B\times D} \to \mathbb{C}^{B}$ be the marginal channel $(\mathcal{N}p)(b) = \sum_d p(b,d)$. Fix a reference state σ_{BD} with $\sigma_B(b) > 0$ for every b. Let*

$$\gamma_\sigma := \min_{b\in B} \sigma_B(b) > 0.$$

Define the Petz recovery channel

$$\mathcal{R}_{\sigma,\mathcal{N}}(\mu)(b,d) = \mu(b)\,\sigma(d \mid b), \qquad \sigma(d \mid b) := \frac{\sigma_{BD}(b,d)}{\sigma_B(b)}.$$

Then $\mathcal{R}_{\sigma,\mathcal{N}}$ is stochastic, completely positive and trace preserving on the diagonal algebra, and ℓ^1-contractive:

$$\|\mathcal{R}_{\sigma,\mathcal{N}}(\mu) - \mathcal{R}_{\sigma,\mathcal{N}}(\nu)\|_1 \le \|\mu - \nu\|_1.$$

The support inverse is uniformly bounded on this domain by

$$\|\sigma_B^{-1/2}\| \le \gamma_\sigma^{-1/2}.$$

If a collar state has the exact classical Markov form

$$\omega_{ABD}(a,b,d) = \omega_{AB}(a,b)\sigma(d \mid b),$$

then $(\mathrm{id}_A \otimes \mathcal{R}_{\sigma,\mathcal{N}})(\omega_{AB}) = \omega_{ABD}$. The support obstruction is exact: if $\sigma_B(b) = 0$ and an input assigns mass to b, the Petz inverse on that sector is undefined unless the channel domain is restricted or a separate trace-preserving completion is declared.

Proof. The displayed formula is the finite diagonal specialization of the Petz map

$$\sigma_{BD}^{1/2}\left(\sigma_B^{-1/2}\mu\,\sigma_B^{-1/2} \otimes \mathbf{1}_D\right)\sigma_{BD}^{1/2}.$$

Full support of σ_B makes the inverse well defined, with support gap $\gamma_\sigma > 0$, hence $\|\sigma_B^{-1/2}\| \le \gamma_\sigma^{-1/2}$. Nonnegativity is immediate, and

$$\sum_{b,d} \mathcal{R}_{\sigma,\mathcal{N}}(\mu)(b,d) = \sum_b \mu(b) \sum_d \sigma(d \mid b) = \sum_b \mu(b),$$

so the map is trace preserving. Diagonal positive trace-preserving maps are completely positive on the diagonal algebra. For signed diagonal inputs,

$$\|\mathcal{R}_{\sigma,\mathcal{N}}(\mu) - \mathcal{R}_{\sigma,\mathcal{N}}(\nu)\|_1 = \sum_{b,d} |\mu(b) - \nu(b)|\,\sigma(d \mid b) = \sum_b |\mu(b) - \nu(b)|,$$

which gives the stated contraction. The exact Markov recovery identity follows by substitution. If $\sigma_B(b) = 0$, the factor $\sigma_B^{-1/2}$ has no value on that sector; mass placed there by an input is outside the Petz support domain. That is the claimed obstruction. □

Proposition 3.17 (Accepted repair moves are decreasing)**.** *For the accepted transactional repair law of Definitions 3.1–3.4, every enabled repair strictly decreases the declared exact measure:*

$$s \to t \implies \mu(t) \prec \mu(s).$$

On the scalar Φ-branch, this specializes to $\Phi(t) < \Phi(s)$.

Proof. This is part of the commit validation in Definition 3.4. For a single-site scalar-Φ transaction it reduces to the touched-overlap contract of Definition 3.2; for an aggregate conflict component it is checked on the aggregate payload before the component can commit. □

Proposition 3.18 (Termination from the OPH Lyapunov functional)**.** *Under Proposition 3.17, every repair sequence is finite; equivalently, the repair relation $\to$ is terminating.*

Proof. Along any nontrivial repair step $s \to t$, Proposition 3.17 gives $\mu(t) \prec \mu(s)$. The measure order is well founded, so an infinite strictly descending chain is impossible. On the scalar-Φ branch this is the finite-value argument on $\Phi(\Sigma)$. □

Proposition 3.19 (Finite repair step bound)**.** *Let $s_0 \in \Sigma$. Under Proposition 3.17, every accepted repair run starting at s_0 has length at most the number of distinct μ-values reachable below $\mu(s_0)$ minus one. On the scalar-Φ branch this gives*

$$\left|\{\, \Phi(s) : s \in \Sigma,\ \Phi(s) \le \Phi(s_0) \,\}\right| - 1 \le |\Phi(\Sigma)| - 1.$$

If, additionally, Φ is integer-valued, or more generally every accepted move lowers Φ by at least a fixed $\eta > 0$, then the run length satisfies

$$T(s_0) \le \left\lceil \frac{\Phi(s_0)}{\eta} \right\rceil.$$

This is a finite descent bound in repair steps only. It is not a wall-clock bound, not a probability-one scheduling statement, and not spectral or exponential convergence.

Proof. The values of μ strictly decrease along the run, so no μ-value can occur twice. This gives the finite reachable-value bound. On the scalar-Φ branch, the values of Φ strictly decrease along the run, so no value in $\{\, \Phi(s) : s \in \Sigma,\ \Phi(s) \le \Phi(s_0) \,\}$ can occur twice. This gives the finite value-set bound. If each move lowers Φ by at least η, after T moves the value has dropped by at least $T\eta$. Since $\Phi \ge 0$, $T\eta \le \Phi(s_0)$, giving the displayed ceiling bound. □

Theorem 3.20 (Adaptive repair attempts: stabilization, fairness, and work conservation)**.** *Let C be a finite OPH carrier, let x_0 be an initial record, and let σ choose one site from the current record at each attempt. Write $x_n = \mathrm{adaptiveRun}_C(n, \sigma, x_0)$. Then:*

(i) for every σ, the sequence x_n is eventually constant;

(ii) if σ is pathwise weakly fair, so that a site which remains enabled along a tail is eventually selected on that tail, the stable record is a normal form;

(iii) if σ is work conserving, so that it selects an effective repair whenever one exists, there is a normal and stable index $N \le \mathrm{mismatchCount}(x_0)$.

Proof. Every attempt either stutters or is an accepted repair step. The mismatch count never increases and strictly decreases at every nonstuttering attempt, so only finitely many genuine changes are possible. If a pathwise weak-fair run were eventually constant at a reducible record, an enabled site would remain enabled along that tail and would eventually be selected, contradicting constancy. Work conservation removes stuttering before normality, so the initial natural-number rank bounds the normal horizon. □

Theorem 3.21 (Cumulative attempt-capacity classification)**.** *Charge one unit for every canonical adaptive scheduler invocation, including an equality stutter. Then:*

(i) no function of the initial mismatch alone bounds the first normal attempt over all normalizing adaptive schedulers;

(ii) if, from every scheduler index and every reducible record, one genuine change occurs among the next $q+1$ attempts, there is a stable normal index

$$N \le (q+1)\,\mathrm{mismatchCount}(x_0);$$

(iii) the coefficient is sharp at $q = 0$: for every $m \ge 1$, a canonical independent-defect carrier has initial mismatch m and a work-conserving run whose first normal index is exactly m. The committed TwoCell source is a second sharp instance with a different patch cardinality.

Proof. For the first clause, delay the enabled TwoCell probe by selecting its edge-free site for any prescribed finite number of attempts, then select the probe. The run normalizes one attempt later while the initial mismatch stays one. For the second clause, every block of $q+1$ attempts starting from a reducible record contains a genuine change, and each such change strictly lowers the natural mismatch rank. Strong induction gives the product bound. For the sharp family, each self-loop mismatch belongs to one site and one canonical local repair removes exactly that mismatch. A state-dependent scheduler always chooses a broken site, so its mismatch count is $m - n$ through attempt m. The threshold is therefore both necessary and sufficient. The TwoCell calculation repeats the threshold-one conclusion on a nonisomorphic source. □

The premise in (ii) is a bounded stutter allowance. It is stronger than eventual fairness and weaker than work conservation when $q > 0$. The cumulative unit is only a count of chosen-site attempts; it is not a physical clock, rate, energy, bandwidth allocation, hardware quota, or fee.

Remark 3.22 (Termination is not confluence). Proposition 3.18 proves only that accepted repair runs stop. It does not prove that two update orders stop at the same physical state. Schedule-independence enters only after the local-diamond property of Proposition 3.23 is combined with termination via Newman's lemma and with repair completeness in Assumption 3.12. If two accepted repair schedules from the same initial state reach different observer-facing quotient normal forms, with no declared holonomy or higher-gauge obstruction and no mere gauge-representative difference, then the proposed repair law is not OPH-admissible as a consensus mechanism.

Proposition 3.23 (Semantic-complete transactional local diamond)**.** *For the accepted repair law of Definitions 3.1–3.4, assume semantic-dependency-complete read sets and revalidation, payload determination from the read snapshot, and a coherent canonical aggregate union-collar payload from Proposition B.7. Assume additionally the aggregate-support fixed-point closure and surviving prepared-component rule of Definition 3.4. Then every one-step quotient peak*

$$t \longleftarrow s \longrightarrow u$$

admits a quotient state v with $t \to v \leftarrow u$. Hence the transactional repair relation is locally confluent. The finite implementation receipt records the functional supports, dependency closures, read/write sets, support-closed conflict components, aggregate payload hashes, finite state transitions, and recomputed protected-record and descent checks needed to check that its concrete reference engine realizes these premises.

Proof. A one-step peak cannot choose two different payloads from one conflict component because that component has one canonical aggregate transaction. Its two steps therefore come from distinct support-closed components, say τ and σ. By the fixed-point conflict definition,

$$W_\tau \cap (R_\sigma \cup W_\sigma) = \varnothing, \qquad W_\sigma \cap (R_\tau \cup W_\tau) = \varnothing.$$

The writes are disjoint and neither changes the other's read snapshot. If an acceptance functional can change under τ, semantic dependency closure puts its support in R_τ; hence σ leaves that functional's input unchanged. Thus τ's enablement, descent comparison, protected-data check, semantic parents, checkpoint continuation, and payload remain valid after σ, and conversely. The surviving prepared-component rule prevents a later-enabled proposal from absorbing either old component before its second commit. Both second commits are legal. Since their writes are disjoint and their payloads are determined by unchanged snapshots,

$$\mathrm{Apply}_\sigma \mathrm{Apply}_\tau(s) = \mathrm{Apply}_\tau \mathrm{Apply}_\sigma(s) = v.$$

This is the local diamond on the physical quotient. □

Finite premise receipt. The executable check enumerates all states and one-step peaks of a finite conflict-component reference engine and recomputes separate countermodels for semantic closure, atomic component commits, canonical aggregate coherence, aggregate-support reclosure, surviving prepared components, snapshot-determined payloads, strict descent, repair completeness, and protected-record preservation. It verifies a concrete realization of the proposition's premises; it does not infer those premises for an arbitrary engine.

Remark 3.24 (Atomic commits do not prove the local diamond). Let $Q = \{0,1\}^2$, $s = (0,0)$, and protect $B(x_1, x_2) = x_1 x_2$. The transactions

$$\tau_1 : (0, x_2) \mapsto (1, x_2), \qquad \tau_2 : (x_1, 0) \mapsto (x_1, 1)$$

have disjoint write sets and each preserves B at s. Both lower $\mu(x_1, x_2) = 2 - x_1 - x_2$. Their peak has endpoints $(1,0)$ and $(0,1)$, while either second write would reach $(1,1)$ and change B. Neither continuation is admissible. The missing dependency is the nonlinear protected observable shared by the two writes. Protected-support and conflict completeness expose that dependency; an implementation must check the resulting quotient peak.

Theorem 3.25 (Asynchronous confluence / fixed-point law)**.** *For the accepted repair law of Definitions 3.1–3.4, assume that the induced physical relation on $Q = \Sigma/\Gamma$ is the OPH-admissible finite quotient presentation of Definition 3.5; in particular use Proposition 3.23 and Assumption 3.12 on that quotient. Each fixed initial state $s \in \Sigma$ then has a unique* quotient *normal form*

$$\overline{\mathrm{nf}}_\lambda(q(s)) = \mathrm{Rep}_\lambda(q(s)) \in q(C),$$

and every maximal asynchronous repair execution from s terminates at a representative $t \in C$ with $q(t) = \overline{\mathrm{nf}}_\lambda(q(s))$. Microscopic terminal representatives need not be equal. When one such representative must be named below, write $\mathrm{nf}_\lambda(s)$ for a choice; only its quotient is canonical. The quotient endpoint is independent of update order because descent, local confluence on the physical quotient, and repair completeness all hold; termination alone is not enough.

Proof. Proposition 3.18 gives termination, while Proposition 3.23 gives local confluence on Q. Newman's lemma [19] therefore makes the quotient relation confluent. By Assumption 3.12, its normal forms are exactly $q(C)$. Theorem 3.9 identifies the unique quotient endpoint with $\mathrm{Rep}_\lambda(q(s))$. A lifted maximal execution terminates at some microscopic representative of that orbit; no representative equality follows from quotient confluence. □

Corollary 3.26 (Objective law is schedule-independent)**.** *Let $\overline{M} : Q \to Y$ be a quotient observable, equivalently let $M : \Sigma \to Y$ factor as $M = \overline{M} \circ q$. Under the hypotheses of Theorem 3.25, $\overline{M}(\overline{\mathrm{nf}}_\lambda(q(s)))$ is independent of the asynchronous update schedule. If physical law is identified with this quotient-observable map, then physical law is objective.*

Proof. All schedules from the same initial s terminate in the same quotient normal form, so a quotient observable has the same value on every terminal representative. □

Here objectivity is identified with schedule-independent convergence on the physical quotient; Theorem 5.16 lifts this to the physical observable algebra while explicitly allowing distinct microscopic terminal representatives.

Corollary 3.27 (Canonical endpoint for weak-fair adaptive repair)**.** *Assume the completeness and confluence hypotheses of Theorem 3.25. Every pathwise weak-fair canonical adaptive run from one initial record reaches a stable consistent record equal to the canonical repair endpoint. Its image is therefore the same public fixed-point object for every such scheduler.*

Proof. Theorem 3.20 gives a reachable stable normal form. Completeness makes that normal form consistent. Confluence identifies it with the canonical repair normal form from the same start, and the public projection preserves the equality. □

Corollary 3.28 (Capacity-bounded canonical endpoint)**.** *Assume the completeness and confluence hypotheses of Theorem 3.25. Let the scheduler have waste bound q, and let the cumulative attempt budget satisfy*

$$B \geq (q+1)\,\mathrm{mismatchCount}(x_0).$$

Then the run reaches a stable consistent record at some attempt $N \leq B$. That record equals the canonical repair endpoint and maps to the same existing public fixed-point object for every scheduler satisfying the same contract.

Proof. Theorem 3.21 gives the normal index within budget. Completeness makes the record consistent. Its accepted-step reachability and confluence identify it with canonical repair, and the public projection preserves the equality. □

Remark 3.29 (Quantifier on normal-form uniqueness). Theorem 3.25 proves quotient-normal-form uniqueness from a fixed initial state; Theorem 5.2 below records representative-lift independence for a fixed physical quotient state. It does not say that all possible initial data settle to one universal quotient state. Different boundary conditions, conserved charges, root packets, holonomy sectors, or external records can legitimately determine different normal forms.

3.2 Repair as computation and its complexity

On a fixed finite patch net the accepted reconciliation dynamics is a finite-state asynchronous rewrite system. Every accepted run has at most the number of distinct reachable μ-values below its start value minus one (Proposition 3.19), so exact normal-form computation is decidable by direct iteration of accepted aggregate transactions. This bound counts accepted nontrivial repairs; scheduler attempts may stutter and are classified by Theorems 3.20 and 3.21. No uniform polynomial-time bound, complexity-class placement for growing families of nets or hardness lower bound is proved here.

Two finite computation results have distinct scopes. An input-specialized federation with one ordered sweep produces the value of any Boolean formula and a unique consensus output; since that federation is compiled from the input and the weak `RepairStep` relation permits a stuttering move that leaves a failing observer unfixed, this is straight-line satisfiability, and that relation admits a one-member fair-stuttering execution that never repairs.

The fixed computation federation is the stronger device. Its generated formula nodes are independent of the input, and it reads immutable input ports from the initial state. Its canonical

accepted step performs the selected failing node's declared write. A superincreasing dependency rank strictly descends on every genuine step; arbitrary attempts stabilize, and pathwise weak fairness promotes the stable state to consensus. Same-input normal endpoints have one output. Tail-relative pathwise weak fairness is equivalent to eventual stable consensus on a fixed run; the reverse implication is vacuous after stable consensus and supplies no recurrence. Member and register-site recurrence coincide at the generated-node granularity because output registers are unique, and recurrence is strictly stronger than tail fairness. A state-blind round-robin function is recurrent, pathwise weak-fair and bounded-waste. Its first n attempts are exactly the emission-order compiler sweep, so it reaches stable consensus and the correct output within n attempts for n emitted nodes.

The compiler gives every generated register at most one downstream consumer. Linear defect weights therefore bound every canonical accepted path by $n(n+1)/2$ for n emitted nodes. A recursively nested NAND family from that compiler has a consensus-reaching path with $8m + 1 = n^2 + 8n$, making the same-metric classification order-sharp $\Theta(n^2)$, not constant-sharp. Well-formedness alone admits fewer than 2^n accepted steps, and an explicit fanout chain attains $2^n - 1$, so the single-consumer structure is load-bearing for the quadratic bound. Arbitrarily long finite stuttering prefixes remain pathwise weak-fair, so weak fairness gives no uniform attempt horizon. A separately supplied fixed-node bounded-waste value q gives a stable horizon of $(q+1)n(n+1)/2$, and round robin supplies $q = n-1$; the one-cycle theorem is the sharper linear bound for emission-order round robin. The bounded-waste counting proof is shared with the adaptive route through a neutral ranked-attempt theorem.

Separately, a width-five permutation word program is fixed independently of the input and evaluates the formula under the stated encoding. Its formal carrier is the ambient permutation group on five points; membership of every instruction in the alternating subgroup is not a typed theorem. These results give a fixed finite repair device with accepted nontrivial repairs and fair asynchronous convergence, and a mathematical round-robin scheduler with conditional attempt counts. They supply no physical scheduler source, wall-clock horizon, rate, energy, bandwidth, fee, hardware resource, noise stability, encoding-overhead bound or physical device realization, and no identity with the width-five engine [12].

3.3 Finite support propagation and algebraic no-signalling

The concrete single-site record repair has a finite support theorem that is independent of termination and confluence. For a patch i, let $\mathcal{N}[i]$ be i together with every endpoint of an edge incident to i, and define

$$\mathrm{Grow}(S) = S \cup \bigcup_{i \in S} \mathcal{N}[i], \qquad B_0(S) = S, \qquad B_{n+1}(S) = \mathrm{Grow}(B_n(S)).$$

Theorem 3.30 (Finite repair support bound)**.** *Let L_i be the concrete* `localRepair` *map: it reads only $\mathcal{N}[i]$ and writes only the record at i. For any schedule word $w = (i_1, \dots, i_n)$, region S, and records x, y,*

$$x|_{B_n(S)} = y|_{B_n(S)} \quad \Longrightarrow \quad L_w(x)|_S = L_w(y)|_S,$$

where $L_w = L_{i_n} \circ \cdots \circ L_{i_1}$. Consequently, changing one initial record at a site outside $B_n(S)$ cannot change the final readout on S after any fixed exogenous schedule word of length n.

Proof. One repair move preserves agreement on T whenever its two inputs agree on $\mathrm{Grow}(T)$: away from the firing site it writes nothing, and at the firing site its trigger, solvability predicate, and selected repair value depend only on the closed incident-edge neighborhood. Induction on the schedule word gives the displayed implication. A one-site update outside $B_n(S)$ leaves the two

initial records equal throughout that set. The statement is machine checked for the committed repair map [12]. □

This fixed-word bound has an exact conditional adaptive extension. If the next-site scheduler is supplied together with a region R on which its choice depends, then agreement on $B_n(S \cup R)$ forces the two n-step runs to agree on $S \cup R$; equivalently, a change outside both $B_n(S)$ and $B_n(R)$ cannot alter the probe readout. Fixed words embed as record-blind schedulers, while a two-cell countermodel proves that the consultation-region term cannot be dropped. Naturality also follows under the laws of a separately declared refinement structure. These results do not produce the scheduler, consultation region, channel, or refinement maps from the source. Neither theorem supplies a converse, minimal cone, physical graph metric, propagation speed, or claim that influence reaches the cone boundary.

Proposition 3.31 (Bipartite marginal invariance)**.** *Let A, B be finite sets.*

(i) For an arbitrary real array $p : A \times B \to \mathbb{R}$ and real matrix $K : A \times A \to \mathbb{R}$ satisfying $\sum_{a'} K(a, a') = 1$, define

$$p'(a', b) = \sum_a p(a, b) K(a, a').$$

Then $\sum_{a'} p'(a', b) = \sum_a p(a, b)$ for every b.

(ii) For an arbitrary matrix M on $\mathbb{C}^A \otimes \mathbb{C}^B$ and a finite family K_r on $\mathbb{C}^A$ satisfying $\sum_r K_r^\dagger K_r = I_A$, define

$$\Phi(M) = \sum_r (K_r \otimes I_B) M (K_r^\dagger \otimes I_B).$$

Then $\mathrm{Tr}_A \Phi(M) = \mathrm{Tr}_A M$.

Proof. For the classical identity, interchange the two finite sums and use the row normalization of K. For the matrix identity, expand the partial trace in indices and contract the acted-factor indices with $\sum_r K_r^\dagger K_r = I_A$. The one-point choice $K = 2$ is a negative control: without row normalization the remote marginal is doubled. Both identities are machine checked [12]. □

These marginal identities are standard finite algebra. Their bipartite split and local lift are supplied inputs. Positivity and unit mass are additional premises when the classical arrays are read as probabilities. No physical spacelike separation, observer-region factorization, density-state realization, or laboratory operation follows. A local observable-net construction must attach disjoint observer regions to the declared factors.

4 Why Local Agreement Is Not Enough: Cycle Holonomy and Frustration

Pairwise neighbor agreement does not by itself imply global consistency.

Theorem 4.1 (Cycle-obstruction / holonomy criterion)**.** *Let A be an abelian group, and let $G = (V, E)$ be a connected graph with an arbitrary orientation on each edge. For each oriented edge $e : u \to v$, assign a label $b_e \in A$. Consider the affine consistency equations*

$$x_v - x_u = b_e \qquad \text{for every oriented edge } e : u \to v,$$

where $x_v \in A$ are unknown patch labels. A global solution $x : V \to A$ exists if and only if for every cycle $C \subseteq G$,

$$\sum_{e \in C} \varepsilon_C(e)\, b_e = 0,$$

where $\varepsilon_C(e) = +1$ if the cycle traverses e in the chosen orientation and -1 otherwise.

Proof. **Necessity.** Suppose x is a solution. Summing the edge equations around any cycle C,

$$\sum_{e:u \to v \in C} \varepsilon_C(e)\,(x_v - x_u) = \sum_{e \in C} \varepsilon_C(e)\, b_e.$$

The left side telescopes to 0 because every vertex appears once with + sign and once with − sign.

Sufficiency. Fix a root $r \in V$. For any vertex v, choose a path $P_{r \to v}$ and define

$$x_v := \sum_{e \in P_{r \to v}} \varepsilon_{P_{r \to v}}(e)\, b_e, \qquad x_r := 0.$$

If P and P' are two paths from r to v, traversing P followed by the reverse of P' yields a cycle. By the vanishing-holonomy assumption, the total signed sum is zero, so x_v is well-defined. For any edge $e : u \to v$, extending a path to u by that edge gives $x_v = x_u + b_e$. □

Corollary 4.2 (Parity triangle: pairwise consistency is not enough)**.** *Take $A = \mathbb{Z}_2$ on the triangle A-B-C-A with edge labels $b_{AB} = 0$, $b_{BC} = 0$, $b_{CA} = 1$. Each individual edge equation is satisfiable. But the global system is not: the cycle sum is $0 \oplus 0 \oplus 1 = 1 \neq 0$.*

This example shows that pairwise consistency does not imply a global solution. The obstruction is carried by the cycle.

Corollary 4.3 (Persistent frustration at fixed holonomy)**.** *On a finite connected graph with fixed edge labels b_e and strictly positive weights $w_e > 0$, define the mismatch cost*

$$\Phi_b(x) := \sum_{e:u \to v \in E} w_e\, \mathbf{1}[x_v - x_u \neq b_e]\,.$$

If the cycle-holonomy condition fails, then $\min_{x:V \to A} \Phi_b(x) > 0$. Every minimizer contains irreducible residual inconsistency.

Proof. The cost takes values in the finite set of subset sums of the positive weights, so its minimum is attained. If that minimum were zero, an assignment would satisfy every edge equation, contradicting Theorem 4.1. □

No sequence of local changes of x, holding b fixed, removes this residual; Section 4.1 records what it supplies to the matter layer.

Theorem 4.4 (Higher-gauge defect hierarchy)**.** *Let a finite overlap nerve carry crossed-module defect data (g_{ij}, h_{ijk}) for a compact crossed module*

$$H \xrightarrow{\partial} G.$$

Under local rechartings by

$$C^1(N, H) \rtimes C^0(N, G),$$

the nonabelian Čech class

$$q = [(g, h)] \in \check{H}^2(N, H \to G)$$

is invariant and classifies the full crossed-module orbit. Its neutral element is equivalent to full gauge trivialization of both levels. The higher associator alone is removable precisely when q lies in the image of the natural strict-locus map

$$\check{H}^1(N,G) \longrightarrow \check{H}^2(N, H \to G), \qquad [g] \longmapsto [(g,1)].$$

After such a strictification, endpoint-only ordinary reconciliation additionally requires an allowed strict representative with trivial represented loop holonomy. The strict-locus map need not be injective, so q need not determine one ordinary H^1 class.

Proof. The allowed rechartings are exactly the crossed-module coboundaries, so they preserve the full orbit. The neutral orbit admits the completely trivial representative. More generally, the higher associator is removed when the orbit meets the strict locus $(g,1)$; different points of that locus can be related by H-valued edge changes. Ordinary loop coherence is therefore the separate existential holonomy test stated above. □

4.1 Holonomy and the particle interpretation

With fixed offsets b, every assignment violates an edge on each cycle of nonzero holonomy. Repairs of vertex labels cannot remove this obstruction; updates of b that change its class can. The invariant therefore constrains an admitted dynamics only if that dynamics preserves the class. The residual $\min_x \Phi_b(x)$ is a cost of the whole constraint problem, not an excitation energy above its own ground state. Localization, reference subtraction, transport and a common physical energy and clock map are additional requirements.

The contrasts of the contrast-memory seam, the differences of the two loads of a seam that the reference candidate and Interval Echo both write [13], are exact port gradients and have zero cycle holonomy, so the obstruction leaves the model no residual mismatch. The seam graph has a cycle space, but the filled icosahedral face complex has vanishing first homology: face-curl-free offsets have zero holonomy on every cycle.

The localized-defect interpretation requires actual transition data and an invariant preserved by all admitted operations, together with finite excitation energy and particle-compatible dynamics. The finite holonomy theorem establishes the algebraic obstruction under its stated premises; it does not by itself identify physical matter.

5 Gauge Symmetry as Implementation Hiding

Definition 5.1 (Gauge action)**.** *Let $\Gamma = \prod_{i \in V} \Gamma_i$ act on Σ componentwise. The action is a **gauge action** if for every $e = \{i,j\} \in E$,*

$$\pi_{i,e}(\gamma_i \cdot x) = \pi_{i,e}(x) \qquad \forall\, x \in S_i,\ \forall\, \gamma_i \in \Gamma_i.$$

Gauge changes alter hidden local representations and leave overlap data fixed.

Write

$$q : \Sigma \to \Sigma/\Gamma, \qquad q(s) = [s],$$

for the gauge-orbit map. A **physical repair law** is the family of quotient-local maps

$$\overline{T}_i^{\lambda} : \Sigma/\Gamma \to \Sigma/\Gamma$$

induced by the recovery-derived collar updates of Definition 3.1. A **representative repair family** is any choice of lifts

$$T_i^\lambda : \Sigma \to \Sigma$$

such that

$$q \circ T_i^\lambda = \overline{T}_i^\lambda \circ q.$$

This is the finite patch-net form of defining the repair step first on overlap-invariant physical data and only then lifting it to hidden representatives. Once repair is formulated on the quotient, no separate gauge-covariance premise is needed.

Theorem 5.2 (Gauge quotient theorem)**.** *Under the gauge action of Definition 5.1, any representative lift of a physical repair law as just defined, and the hypotheses of Theorem 3.25,*

$$q(\mathrm{nf}_\lambda(\gamma \cdot s)) = q(\mathrm{nf}_\lambda(s)) \qquad \forall\, \gamma \in \Gamma,\ \forall\, s \in \Sigma.$$

Equivalently, the canonical normal-form map is the quotient operator constructed above:

$$\overline{\mathrm{nf}}_\lambda : \Sigma/\Gamma \to \Sigma/\Gamma, \qquad [s] \mapsto \mathrm{Rep}_\lambda([s]) = q(\mathrm{nf}_\lambda(s)).$$

Proof. Let $s \to t$ be an accepted aggregate transaction, and let $\overline{\tau}$ be its quotient payload. If $s' = \gamma \cdot s$, the representative-lift hypothesis gives an accepted lift $s' \to t'$ of the same quotient transaction, and

$$q(t') = \overline{\tau}(q(s')) = \overline{\tau}(q(s)) = q(t).$$

Thus gauge-equivalent inputs induce the same repaired orbit, and by induction the orbit reached after any repair sequence depends only on the initial orbit. Theorem 3.25 says that every maximal repair sequence from s ends at some terminal representative whose orbit is $\mathrm{Rep}_\lambda(q(s))$. Hence $q(\mathrm{nf}_\lambda(s))$ depends only on $q(s) = [s]$. This makes

$$[s] \longmapsto \mathrm{Rep}_\lambda([s])$$

well-defined on Σ/Γ. □

Corollary 5.3 (Repair respects gauge)**.** *Let $Q = \Sigma/\Gamma$, let $q : \Sigma \to Q$ be the quotient map, and let*

$$\mathrm{Rep}_\lambda^\Sigma := \mathrm{Rep}_\lambda \circ q : \Sigma \to Q$$

be the quotient-valued physical repair of a representative. Then for every $\gamma \in \Gamma$ and $s \in \Sigma$,

$$\mathrm{Rep}_\lambda^\Sigma(\gamma \cdot s) = \mathrm{Rep}_\lambda^\Sigma(s).$$

Equivalently,

$$\mathrm{Rep}_\lambda(q(\gamma \cdot s)) = \mathrm{Rep}_\lambda(q(s)).$$

If $M : Q \to Y$ is any physical observable, then

$$M(\mathrm{Rep}_\lambda^\Sigma(\gamma \cdot s)) = M(\mathrm{Rep}_\lambda^\Sigma(s)).$$

Proof. Since q is the quotient map, $q(\gamma \cdot s) = q(s)$. Therefore

$$\mathrm{Rep}_\lambda^\Sigma(\gamma \cdot s) = \mathrm{Rep}_\lambda(q(\gamma \cdot s)) = \mathrm{Rep}_\lambda(q(s)) = \mathrm{Rep}_\lambda^\Sigma(s).$$

Applying M gives the observable statement. □

Theorem 5.4 (Boundary-conditioned quotient uniqueness)**.** *Let*

$$B : \Sigma/\Gamma \to \mathcal{B}$$

record fixed external boundary data, conserved charge, root packet, holonomy sector, or task input. Assume accepted quotient repairs preserve B:

$$[s] \to [t] \implies B([s]) = B([t]).$$

For $b \in \mathcal{B}$, define the consistent quotient fiber

$$C_b := \{\, x \in q(C) : B(x) = b \,\}.$$

If every C_b has at most one element, then all initial states with the same boundary value settle to the same observer-facing quotient normal form:

$$B([s]) = B([s']) \implies \overline{\mathrm{nf}}_\lambda([s]) = \overline{\mathrm{nf}}_\lambda([s']).$$

Proof. Theorem 5.2 gives unique quotient normal forms $\overline{\mathrm{nf}}_\lambda([s])$ and $\overline{\mathrm{nf}}_\lambda([s'])$ in $q(C)$. Boundary preservation along accepted repair sequences gives

$$B(\overline{\mathrm{nf}}_\lambda([s])) = B([s]), \qquad B(\overline{\mathrm{nf}}_\lambda([s'])) = B([s']).$$

If $B([s]) = B([s']) = b$, both quotient normal forms lie in C_b. By the unique consistent extension assumption, C_b has at most one element, so the quotient normal forms are equal. □

Remark 5.5 (Same-source and cross-source quantifiers). Theorem 3.25 compares repair schedules from one fixed initial source. The stronger comparison of endpoints reached from different sources with the same protected boundary is controlled by the consistent boundary fibers. Under boundary preservation and NF $= q(C)$, Ref. [10] proves that cross-source endpoint agreement, modulo any declared silent equivalence, is equivalent to injectivity of the induced boundary map on the consistent quotient. Weak normalization and all-schedule liveness are separate premises. The layered and functional carriers below establish the injectivity premise on their declared domains. On the verified rooted-tree packet-net domain of Definition 3.13, the declared boundary map is the root-packet readback, and its injectivity modulo hidden-label gauge on the consistent set is established class-wide, together with the resulting cross-source endpoint form for the tree repair. Same-source confluence alone does not establish the injectivity premise for an arbitrary physical boundary map, and outside the declared domains it is a named per-net hypothesis with explicit failure witnesses.

5.1 A multi-edge finite carrier for boundary reconstruction

Definition 5.6 (Layered functional boundary carrier)**.** *A **layered functional boundary carrier** is a finite directed layered graph*

$$G = (V, E), \qquad V = L_0 \sqcup L_1 \sqcup \cdots \sqcup L_D,$$

with $D \geq 2$. The boundary layer is L_0. For each $v \in L_d$, $d \geq 1$, choose a nonempty parent set

$$P(v) \subseteq L_0 \sqcup \cdots \sqcup L_{d-1}$$

and a finite alphabet A_v. Each interior vertex has a deterministic local rule

$$F_v : \prod_{u \in P(v)} A_u \to A_v.$$

The carrier is genuinely multi-edge when at least two dependency edges occur and at least one layer contains more than one dependency or more than one repaired vertex.

The quotient state space is

$$Q = \prod_{v \in V} A_v,$$

and the boundary map is

$$B : Q \to \prod_{v \in L_0} A_v, \qquad B(a) = a|_{L_0}.$$

Given boundary data $b \in \prod_{v \in L_0} A_v$, define its functional extension $E(b) \in Q$ recursively by

$$E(b)_v = b_v \quad (v \in L_0),$$

and, for $v \in L_d$, $d \geq 1$,

$$E(b)_v = F_v\big((E(b)_u)_{u \in P(v)}\big).$$

Optional cross-check edges may be included by choosing finite predicates

$$\chi_e(a_u, a_v) = 0.$$

A state $a \in Q$ is consistent when every interior functional equation

$$a_v = F_v\big((a_u)_{u \in P(v)}\big)$$

holds and all cross-check predicates pass. Let C_Q be the set of consistent states. A boundary b is admissible when $E(b) \in C_Q$; equivalently,

$$C_Q = \{\, E(b) : b \text{ admissible} \,\}.$$

For each layer $d = 1, \ldots, D$, define a layer repair map $R_d : Q \to Q$ by

$$(R_d(a))_v = \begin{cases} F_v((a_u)_{u \in P(v)}), & v \in L_d, \\ a_v, & v \notin L_d. \end{cases}$$

The full staged repair sweep is

$$R_{\text{sweep}} = R_D \circ R_{D-1} \circ \cdots \circ R_1.$$

Gauge representatives may be added by replacing A_v with $A_v \times H_v$, letting Γ_v act only on H_v, and defining all boundary and consistency maps through the A_v-coordinate. The quotient is then the Q above.

Theorem 5.7 (Layered carrier proves $H_B \wedge H_{\text{fib}}$)**.** *Let G be a layered functional boundary carrier. For every initial quotient state $a \in Q$, set*

$$a^{(0)} = a, \qquad a^{(d)} = R_d(a^{(d-1)}), \quad d = 1, \ldots, D.$$

Let $b = B(a)$. Then

$$B(a^{(d)}) = b \qquad \text{for all } d = 0, \ldots, D.$$

If b is admissible, then

$$a^{(D)} = E(b) \in C_Q.$$

The consistent boundary fiber is a singleton:

$$C_Q \cap B^{-1}(b) = \{E(b)\}.$$

Thus preserved boundary data plus consistency reconstruct the observer-facing quotient bulk on this carrier.

Proof. Boundary preservation is immediate because each R_d writes only layer L_d, with $d \geq 1$. No R_d writes L_0, so $B(a^{(d)}) = B(a) = b$ for every stage.

We prove by induction on d that after stage d,

$$a_v^{(d)} = E(b)_v \qquad \text{for every } v \in L_0 \sqcup \cdots \sqcup L_d.$$

For $d = 0$, this is the definition of b. Assume the claim holds through layer $d-1$, and let $v \in L_d$. Every parent $u \in P(v)$ lies in an earlier layer, so $a_u^{(d-1)} = E(b)_u$. The layer repair therefore gives

$$a_v^{(d)} = F_v((a_u^{(d-1)})_{u \in P(v)}) = F_v((E(b)_u)_{u \in P(v)}) = E(b)_v.$$

Earlier layers are not modified by R_d, so the induction closes. At $d = D$, all layers agree with $E(b)$. If b is admissible, $E(b) \in C_Q$.

For boundary-fiber uniqueness, let $c \in C_Q \cap B^{-1}(b)$. Since c has boundary b, it agrees with $E(b)$ on L_0. Since $c \in C_Q$, every interior vertex satisfies

$$c_v = F_v((c_u)_{u \in P(v)}).$$

The same induction on layers gives $c_v = E(b)_v$ for every vertex. Hence $c = E(b)$. □

Corollary 5.8 (Boundary reconstruction by global Repair)**.** *Assume the layered carrier's layer updates are included among the accepted aggregate transactions of an OPH-admissible finite quotient repair presentation, and let $b = B(x)$ be admissible. Then*

$$\mathrm{Rep}_\lambda(x) = E(b).$$

Consequently, for every physical readout

$$\mathrm{Read} : Q \to \mathcal{Y}$$

that factors through the quotient normal form,

$$\mathrm{Read}(\mathrm{Rep}_\lambda(x)) = \mathrm{Read}(E(B(x))).$$

Proof. By Theorem 3.9,

$$\mathrm{Rep}_\lambda(x) \in C_Q \qquad \text{and} \qquad B(\mathrm{Rep}_\lambda(x)) = B(x) = b.$$

Therefore $\mathrm{Rep}_\lambda(x) \in C_Q \cap B^{-1}(b)$. Theorem 5.7 identifies this fiber with $\{E(b)\}$, so $\mathrm{Rep}_\lambda(x) = E(b)$. Applying Read gives the readout statement. □

Remark 5.9 (A sharp linear reconstruction fixture). The layered carrier is feed-forward. A linear binary update on a finite cylinder tests the same logic without that structure. A row is $x_t : \mathbb{Z}_n \to \mathbb{F}_2$ with $x_{t+1}(j) = x_t(j-1) + x_t(j+1) \pmod 2$; this is a finite test object, not a proposed physics law. On the slab $0, \ldots, t$, the two-column tube $\{j_0, j_0 + g\} \times \{0, \ldots, t\}$ determines the whole finite record exactly when

$$\gcd(g, n) = 1 \quad \text{and} \quad n \leq 2(t+1).$$

Adjacent columns reach the capacity threshold, and a non-coprime stride loses information at every horizon. A local transactional decoder that writes one non-tube cell per transaction through a bounded edge-local window, under the declared rank schedule, terminates in one pass and preserves the tube reading; at the threshold any two records with the same tube reading settle to the same normal form, which is consistent exactly when the tube fiber is realizable. On the same cylinder no single-patch frustration-free local repair satisfies the strongest $H_1 \wedge H_2 \wedge H_3$ binder form, and the canonical single-patch repair stalls at $(n, t) = (3, 2)$ on a record whose tube fiber is unrealizable. Boundary reconstruction is therefore a theorem only after the preserved boundary, the realizable-fiber condition and the repair roster are specified.

Theorem 5.10 (Functional selected-fiber uniqueness)**.** *Let $Q_b := B^{-1}(b) \subseteq \Sigma/\Gamma$ be a same-boundary quotient fiber presented on a rooted finite overlap graph with spanning tree T, root r, and patch state sets X_v. Assume the boundary value fixes the root value*

$$u_r = \beta(b),$$

and every non-root vertex has a deterministic extension map

$$f_v : X_{p(v)} \times \mathcal{B} \to X_v.$$

Define u_b recursively by

$$u_v = f_v(u_{p(v)}, b).$$

Evaluate all non-tree overlap equations, sector constraints, and holonomy checks on this candidate. If they all pass, then $C_b = \{u_b\}$. If any check fails, then $C_b = \varnothing$.

Proof. Every consistent state in Q_b must have root value $\beta(b)$. Along each tree edge, the deterministic extension equation forces the child value to be $f_v(u_{p(v)}, b)$. Induction over tree depth therefore forces every consistent state in the fiber to equal the single recursively constructed candidate u_b. The remaining constraints are exactly the non-tree overlaps, sector equations, and holonomy checks. If they pass, u_b is a consistent state and no other state can be. If one fails, the only tree-compatible candidate is inconsistent, so no element of C_b exists. □

The layered functional carrier of Theorem 5.7 is the finite multi-edge, multi-step witness that the boundary-fiber hypothesis is non-vacuous beyond one-dimensional propagation intuition.

Proposition 5.11 (Tree repair realizes the selected extension)**.** *Under the hypotheses of Theorem 5.10, choose positive weights*

$$w_v > \sum_{c:p(c)=v} w_c$$

and set

$$\Phi_T(x) = \sum_{v \neq r} w_v \, \mathbf{1}[x_v \neq f_v(x_{p(v)}, b)].$$

The repair that sets a non-root vertex to $f_v(x_{p(v)}, b)$ strictly decreases Φ_T, and the aggregate transaction for a connected tree-repair conflict component computes the same result in increasing tree depth.

Proof. Repairing v removes the v-term of weight w_v. Only children of v can become unmatched, contributing at most $\sum_{c:p(c)=v} w_c$, which is strictly smaller than w_v. Thus the tree potential decreases. For a connected conflict component, applying parent repairs before child repairs is forced by the same extension equations, and any different parenthesization has the same quotient result because each child value is a deterministic function of the repaired parent and b. □

Corollary 5.12 (Nontrivial branch elimination)**.** *Assume transactional confluence, boundary preservation, repair completeness, and Theorem 5.10. If a selected boundary fiber has $|Q_b| \geq 2$ and $C_b = \{u_b\}$, then*

$$\overline{\mathrm{nf}}_\lambda(Q_b) = \{u_b\},$$

while $Q_b \setminus C_b \neq \varnothing$. Thus multiple candidate interiors exist, inconsistent candidates are eliminated by repair, and all surviving candidates share one quotient normal form. If $C_b = \varnothing$, the branch is obstructed. If a union solver returns two physically distinct consistent quotient endpoints for the same selected data, the result is an ambiguous union repair and requires an explicit continuation gate, not selection by hash.

Remark 5.13 (Receipts do not replace the theorem). Implementation receipts for seam descent, atomic commit, the distributed local diamond, repair completeness, same-boundary multistart confluence and the quotient normal-form hash are evidence contracts for the finite theorem hypotheses above, never proof substitutes. A normal-form hash certifies that two emitted quotient normal forms agree after the declared quotienting; it may not choose between two physically distinct minimizing states. Declared booleans, labels, hashes or plots cannot override failed residuals, missing objects or undeclared quotient maps, and a receipt must expose the raw objects needed to recompute it.

Corollary 5.14 (Gauge-invariant law). *If $M : \Sigma \to Y$ is gauge-invariant ($M(\gamma \cdot s) = M(s)$ for all γ), then $M(\mathrm{nf}_\lambda(s))$ depends only on the gauge orbit $[s]$, not the representative.*

Proof. Because M is gauge-invariant, it factors through the orbit map: $M = \overline{M} \circ q$ for some $\overline{M} : \Sigma/\Gamma \to Y$. Theorem 5.2 gives

$$q\big(\mathrm{nf}_\lambda(\gamma \cdot s)\big) = q\big(\mathrm{nf}_\lambda(s)\big),$$

hence

$$M\big(\mathrm{nf}_\lambda(\gamma \cdot s)\big) = \overline{M}\big(q(\mathrm{nf}_\lambda(\gamma \cdot s))\big) = \overline{M}\big(q(\mathrm{nf}_\lambda(s))\big) = M\big(\mathrm{nf}_\lambda(s)\big).$$

□

Definition 5.15 (Physical observable algebra on the quantum lift). *Fix a finite patch region or union collar R on the declared fixed-cutoff quantum lift of Appendix B. Its **physical observable algebra** $\mathcal{A}_{\mathrm{phys}}(R)$ is the fixed-point collar algebra under the compact boundary redundancy action on the ordinary or central-defect branch, or the corresponding quotient-local algebra on the genuinely noncentral branch. The central sector projectors carried by the collar decomposition belong to $Z(\mathcal{A}_{\mathrm{phys}}(R))$. A **physical observable** is any $X \in \mathcal{A}_{\mathrm{phys}}(R)$. For a microscopic representative $s \in \Sigma$, let ω_R^s denote the induced state on $\mathcal{A}_{\mathrm{phys}}(R)$.*

Theorem 5.16 (Observable-level confluence on the quantum lift). *Under the hypotheses of Theorems 3.25 and 5.2, every initial orbit $[s] \in \Sigma/\Gamma$ has a unique quotient normal form*

$$\overline{\mathrm{nf}}_\lambda([s]) \in q(C).$$

Fix a declared region R. Let $t, u \in \Sigma$ be terminal microscopic representatives reached by representative lifts of maximal repair sequences from initial states in the same orbit $[s]$. If the induced R-collar states of t and u are representatives of the same quotient-local glued state in the sense of Proposition B.7, then for every physical observable $X \in \mathcal{A}_{\mathrm{phys}}(R)$,

$$\omega_R^t(X) = \omega_R^u(X).$$

Hence all physical observables converge to the same overlap-consistent values even when the microscopic terminal representatives differ by gauge relabelings globally or by sector/higher-gauge relabelings inside one declared quotient-local glued state.

Proof. Theorems 3.25 and 5.2 give the unique quotient normal form $\overline{\mathrm{nf}}_\lambda([s]) = [\mathrm{nf}_\lambda(s)] \in q(C)$. Let t and u be as stated. By Corollary B.9, representatives of the same quotient-local glued state induce the same state on the physical observable algebra $\mathcal{A}_{\mathrm{phys}}(R)$, including the central sector projectors carried by that algebra. Therefore $\omega_R^t(X) = \omega_R^u(X)$ for every $X \in \mathcal{A}_{\mathrm{phys}}(R)$. □

Remark 5.17 (Inputs and boundary for observable-level confluence). Theorem 5.16 does not add a new repair hypothesis beyond the confluence and fixed-cutoff gluing package. Its inputs are exactly the quotient-level confluence theorem, quotient descent of the repair law, Definition 5.15, and Corollary B.9 from the fixed-cutoff union-collar gluing package. The boundary item is extension of the same statement to broader refinement-stable branches where the declared union-collar compatibility is only approximate or where the chosen physical observable algebra itself changes under refinement.

Corollary 5.18 (Inert ancillary refinement does not change physical law). *Let $K = \prod_i K_i$ be a finite ancillary state space and define $\Sigma^\eta := \Sigma \times K$. Lift the repair maps by*

$$T_i^{\lambda,\eta}(s,k) := (T_i^\lambda(s), k).$$

If $M^\eta : \Sigma^\eta \to Y$ ignores the ancillary factor,

$$M^\eta(s,k) = M(s),$$

with M gauge-invariant on Σ, then

$$M^\eta(\mathrm{nf}_\lambda^\eta(s,k)) = M(\mathrm{nf}_\lambda(s))$$

for all $(s,k) \in \Sigma^\eta$.

Proof. Because the ancillary factor is inert,

$$\mathrm{nf}_\lambda^\eta(s,k) = (\mathrm{nf}_\lambda(s), k).$$

Therefore $M^\eta(\mathrm{nf}_\lambda^\eta(s,k)) = M(\mathrm{nf}_\lambda(s))$, and Corollary 5.14 supplies gauge-orbit independence. □

Physical uniqueness therefore holds on the quotient by gauge or implementation hiding. The same statement is unchanged under inert ancillary stabilization.

5.2 Implementation hiding and the gauge layer

Gauge here means implementation hiding: a change of hidden local representation that leaves every overlap readout fixed (Definition 5.1). This subsection supplies the gauge layer with three exact inputs. Physical law factors through the quotient Σ/Γ, and repair defined first on overlap data and then lifted needs no separate gauge-covariance premise (Theorem 5.2, Corollaries 5.3 and 5.14). Holonomy classes, abelian or crossed-module, are invariants of that quotient (Theorems 4.1 and 4.4); they are the part of the overlap transport that implementation hiding cannot remove. At fixed cutoff, overlap rechartings generate a compact boundary redundancy group whose invariant collar algebra has the central sector decomposition of Appendix B; these central sectors are the edge-sector seed that the gauge layer transports.

The gauge layer adds what the quotient does not determine [15]. On a branch carrying an explicit compact-gauge refinement receipt, coherent pullback functors reconstruct a compact group from the tensor-generated zero-obstruction sectors; zero obstruction there means central or higher-associator strictifiability together with an allowed strict representative of trivial represented holonomy. Separately, complete reversible response and endogenous overlap transport force the local Lie algebra $\mathfrak{u}(1) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(3)$ under that paper's response premises. The consensus theorems fix no gauge group, Lie algebra, connection dynamics, coupling or matter representation: they hold verbatim for every redundancy group that preserves overlap data. A transformation that changes overlap data changes the compatibility problem; it hides no implementation.

Gauge and matter meet at the holonomy. A defect in the sense of Section 4.1 is a transport class that implementation hiding cannot remove, and the matter hypothesis requires that class to be carried and protected by native transport.

5.3 Finite packet closure and the simulation criterion

Definition 5.19 (Finite packet closure simplex)**.** *Assume the finite fixed-cutoff consensus branch of Theorems 3.25 and 5.2, so the physical quotient $Q := \Sigma/\Gamma$ is finite and carries the schedule-independent quotient normal-form map*

$$\overline{\mathrm{nf}}_\lambda : Q \to Q.$$

Let

$$N_\lambda := \overline{\mathrm{nf}}_\lambda(Q) = q(C)$$

be the quotient normal-form set. The ***finite packet simplex*** *on Q is*

$$\Delta(Q) := \left\{ \mu : Q \to \mathbb{R}_{\geq 0} \;\middle|\; \sum_{x \in Q} \mu(x) = 1 \right\},$$

and the ***finite packet closure map*** *is the pushforward*

$$\mathcal{C}_\lambda : \Delta(Q) \to \Delta(Q), \qquad \mathcal{C}_\lambda(\mu) := \left(\overline{\mathrm{nf}}_\lambda\right)_* \mu,$$

equivalently

$$\mathcal{C}_\lambda(\mu)(y) = \sum_{\substack{x \in Q \\ \overline{\mathrm{nf}}_\lambda(x) = y}} \mu(x).$$

Theorem 5.20 (Finite packet-quotient closure map)**.** *On the finite fixed-cutoff consensus branch of Definition 5.19, the map $\mathcal{C}_\lambda$ is an affine continuous idempotent self-map of $\Delta(Q)$. Its image is exactly the normal-form simplex*

$$\Delta(N_\lambda) = \{\mu \in \Delta(Q) \mid \mathrm{supp}(\mu) \subseteq N_\lambda\}.$$

Hence the fixed points of $\mathcal{C}_\lambda$ are exactly the packets supported on quotient normal forms. For every initial quotient state $[s] \in Q$,

$$\mathcal{C}_\lambda(\delta_{[s]}) = \delta_{\overline{\mathrm{nf}}_\lambda([s])}.$$

Proof. Because $\overline{\mathrm{nf}}_\lambda : Q \to Q$ is a set map on a finite set, its pushforward on probability packets is affine and continuous in the finite-dimensional simplex topology.

By Theorems 3.25 and 5.2, the quotient normal form is schedule-independent and terminal, so

$$\overline{\mathrm{nf}}_\lambda \circ \overline{\mathrm{nf}}_\lambda = \overline{\mathrm{nf}}_\lambda.$$

Pushforward therefore gives

$$\mathcal{C}_\lambda^2 = \left(\overline{\mathrm{nf}}_\lambda\right)_* \left(\overline{\mathrm{nf}}_\lambda\right)_* = \left(\overline{\mathrm{nf}}_\lambda \circ \overline{\mathrm{nf}}_\lambda\right)_* = \mathcal{C}_\lambda,$$

so $\mathcal{C}_\lambda$ is idempotent.

If $\nu = \mathcal{C}_\lambda(\mu)$, then every point in $\mathrm{supp}(\nu)$ lies in the image of $\overline{\mathrm{nf}}_\lambda$, namely N_λ. Thus $\mathrm{im}(\mathcal{C}_\lambda) \subseteq \Delta(N_\lambda)$. Conversely, if $\nu \in \Delta(N_\lambda)$, then $\overline{\mathrm{nf}}_\lambda(y) = y$ for every $y \in N_\lambda$, so

$$\mathcal{C}_\lambda(\nu) = \nu.$$

Hence $\Delta(N_\lambda) \subseteq \mathrm{im}(\mathcal{C}_\lambda)$, proving $\mathrm{im}(\mathcal{C}_\lambda) = \Delta(N_\lambda)$. The same identity shows that ν is a fixed point if and only if it is supported on N_λ. The Dirac-mass formula is the pushforward of a point mass under $\overline{\mathrm{nf}}_\lambda$. □

Theorem 5.20 concerns one fixed quotient carrier whose repair law is terminating, confluent and quotient-descended. It gives no closure map for arbitrary state-and-law data and no uniqueness or stability estimate beyond this finite branch.

Fixed points are not simulations. A fixed-point equation holds for a selected finite OPH packet, but it does not define one. The content lies in the further clauses below.

Definition 5.21 (OPH simulation certificate)**.** *On the branch of Definition 5.19 write* $n := \overline{\mathrm{nf}}_\lambda$*, let* $B : Q \to \mathcal{B}$ *be a boundary/sector map preserved by accepted quotient repairs, fix* $b \in \mathcal{B}$*, and set* $Q_b := B^{-1}(b)$*,* $C_b := N_\lambda \cap Q_b$*. A selected pair* (b, u) *with* $u \in C_b$*, equivalently the packet* $\mathfrak{U}_u := \delta_u$*, has an* ***OPH simulation certificate*** *when:*

(S1) ***Endogenous update.*** *The local maps are the recovery-derived collar maps of Definition 3.1, descend to the quotient and read only declared patch, overlap, recovery, decoder and acceptance data.*

(S2) ***Observer-readable records.*** *A declared observer-accessible region carries an exact record presentation (Definition 7.1) with at least two nonzero orthogonal event projectors, collected in a map* $\mathrm{Read} : Q \to \mathcal{Y}_{\mathrm{rec}}$ *whose terminal value depends only on the quotient normal form.*

(S3) ***Overlap repair.*** *Accepted moves strictly lower the declared touched-overlap score, normal forms are exactly the consistent states, and* n *is terminating and schedule independent.*

(S4) ***Nontrivial branch elimination.*** $C_b = \{u\}$*,* $Q_b \setminus C_b \neq \varnothing$ *and* $n(Q_b) = \{u\}$*; a candidate with nonzero holonomy or higher-gauge obstruction counts as obstructed, never as an alternative world.*

(S5) ***Implementation and clock closure.*** *For auxiliary implementation and clock sets with physical projection* $\widetilde{p} : Q \times \mathcal{E}_{\mathrm{ext}} \times \Theta_{\mathrm{ext}} \to Q$*, every admissible lifted update and readout satisfy* $p \circ \widetilde{n} = n \circ p$ *and* $\widetilde{\mathrm{Read}} = \mathrm{Read} \circ p$*. Schedules and repair counters are proof data, and any history is read from records in the terminal quotient state.*

Proposition 5.22 (Selected packet and fixed-point firewall)**.** *Assume Definitions 3.1, 3.2 and 3.3, Assumption 3.12, and the hypotheses of Theorems 5.2, 5.16 and 7.2. If a preserved value* b *has* $C_b = \{u\}$ *and* $Q_b \setminus C_b \neq \varnothing$*, and the selected observer surface has a nontrivial exact record presentation whose order data are quotient-observable, then* δ_u *has an OPH simulation certificate, and* $n(u) = u$*,* $\mathcal{C}_\lambda(\delta_u) = \delta_u$*. The converse fails: for any set* X *with* $|X| > 1$*, the identity map fixes every point and every point minimizes the constant functional, while no proper candidate basin collapses to any point, so clause (S4) fails.*

Proof. (S1) is Definition 3.1 with quotient descent from Theorem 5.2; (S2) follows from Theorems 7.2 and 5.16; (S3) from Definition 3.2 and Theorem 3.25. For $x \in Q_b$, boundary preservation and completeness give $n(x) \in C_b = \{u\}$, which is (S4); the rooted-tree packet domain of Theorem 3.14 shows the clause is nonvacuous. The update and readout are defined on the quotient alone, schedule independence removes the scheduler, and Corollary 5.18 removes inert carrier enlargement, giving (S5). Theorem 5.20 gives $\mathcal{C}_\lambda(\delta_u) = \delta_{n(u)} = \delta_u$. The identity-map example proves the non-implication. □

A certificate fails if an accepted update reads undeclared oracle, carrier or clock data; if the record algebra is trivial or not quotient-readable; if a move fails strict descent; if terminal states differ from consistent states; if two schedules or two candidates in the selected fiber give different physical normal forms; if the fiber has nothing to eliminate; or if a carrier or clock lift changes the projected update or readout.

6 Refinement-Limit Consensus Classes

The finite consensus theorem is the operational object used by the continuum branches. To pass from one finite patch net to a refining family, the paper uses an explicit inverse-limit package with named compatibility clauses.

Definition 6.1 (Separated cofinal refinement consensus system)**.** *Let $(R, \preceq)$ be a directed refinement set. For each $r \in R$, let*

$$Q_r := \Sigma_r/\Gamma_r$$

be the finite physical quotient state space of a patch presentation, let

$$n_r : Q_r \to Q_r$$

be the finite-stage quotient normal-form map supplied by Theorems 3.25 and 5.2, and let

$$h_r : Q_r \to \mathcal{H}_r$$

be the finite-stage holonomy or higher-gauge obstruction map supplied by Theorems 4.1 and 4.4. For $r \preceq s$, assume restriction maps

$$\rho_{sr} : Q_s \to Q_r, \qquad \chi_{sr} : \mathcal{H}_s \to \mathcal{H}_r,$$

with $\rho_{rr} = \mathrm{id}$, $\chi_{rr} = \mathrm{id}$, and the cocycle identities

$$\rho_{tr} = \rho_{sr} \circ \rho_{ts}, \qquad \chi_{tr} = \chi_{sr} \circ \chi_{ts} \qquad (r \preceq s \preceq t).$$

The system is a separated cofinal refinement consensus system when:

1. ***Normal-form naturality:***
$$\rho_{sr} \circ n_s = n_r \circ \rho_{sr} \qquad (r \preceq s).$$
2. ***Holonomy naturality:***
$$\chi_{sr} \circ h_s = h_r \circ \rho_{sr} \qquad (r \preceq s).$$
3. ***Visible separation:*** *two compatible families in $\varprojlim Q_r$, or in $\varprojlim \mathcal{H}_r$, are equal whenever their projections agree on a cofinal subset of stages.*

Theorem 6.2 (Refinement-limit consensus and holonomy classes)**.** *For any separated cofinal refinement consensus system, the formulas*

$$n_\infty((x_r)_r) := (n_r(x_r))_r, \qquad h_\infty((x_r)_r) := (h_r(x_r))_r$$

define maps

$$n_\infty : \varprojlim Q_r \to \varprojlim Q_r, \qquad h_\infty : \varprojlim Q_r \to \varprojlim \mathcal{H}_r.$$

The class $n_\infty(x)$ is the unique schedule-independent refinement-limit normal form of x. The class $h_\infty(x)$ is the refinement-limit holonomy obstruction. The pair $(n_\infty(x), h_\infty(x))$ is the refinement-limit consensus class. In the inverse-limit topology, finite-stage normal forms and holonomy classes converge to these two classes: for every stage r, all refinements $s \succeq r$ restrict to the fixed values

$$\rho_{sr}(n_s(x_s)) = n_r(x_r), \qquad \chi_{sr}(h_s(x_s)) = h_r(x_r).$$

The relation $h_\infty(x) = 0$ holds if and only if every finite projection has zero finite-stage obstruction. If $h_\infty(x) \neq 0$, visible separation gives a finite stage that witnesses the nonzero obstruction. If two refinement-limit candidates have the same normal-form and holonomy projections on a cofinal tail, then they determine the same refinement-limit consensus class.

Proof. Let $x = (x_r)_r \in \varprojlim Q_r$, so $\rho_{sr}(x_s) = x_r$ for $r \preceq s$. Normal-form naturality gives

$$\rho_{sr}(n_s(x_s)) = n_r(\rho_{sr}(x_s)) = n_r(x_r),$$

so $(n_r(x_r))_r$ is a compatible family and n_∞ is well defined. The same argument with holonomy naturality gives

$$\chi_{sr}(h_s(x_s)) = h_r(\rho_{sr}(x_s)) = h_r(x_r),$$

so h_∞ is well defined.

Each finite $n_r(x_r)$ is independent of update schedule by Theorems 3.25 and 5.2. Therefore every finite projection of $n_\infty(x)$ is schedule-independent, and visible separation makes the inverse-limit class unique. The displayed restriction identities are exactly the cylinder convergence statement in the inverse-limit topology.

The zero-obstruction statement follows from the definition of the inverse-limit zero class. The identity $h_\infty(x) = 0$ holds exactly when every projection $h_r(x_r)$ is zero. If $h_\infty(x) \neq 0$, at least one finite projection is nonzero, and any cofinal tail containing a refinement of that stage carries the same nonzero projected obstruction by holonomy naturality. The last claim is visible separation applied to the compatible normal-form and holonomy families. □

Remark 6.3 (Scope of the refinement theorem). Theorem 6.2 is a controlled inverse-limit bridge. It proves persistence, exhaustion, and collapse of the consensus data once the OPH refinement system supplies the restriction maps and the two naturality clauses in Definition 6.1. It leaves uniform complexity bounds, automatic fair-block noisy-consensus certificates, and automatic normal-form naturality for arbitrary changing repair laws as separate branch conditions. It does not identify this inverse limit with a physical continuum or spacetime.

Definition 6.4 (Repair morphism). *Let $(Q_s, \to_s, C_s, n_s)$ and $(Q_r, \to_r, C_r, n_r)$ be two finite quotient repair systems covered by Theorems 3.25 and 5.2. A map*

$$\rho_{sr} : Q_s \to Q_r$$

*is a **repair morphism** when:*

(i) every fine repair step $x \to_s y$ maps to a coarse repair path or a stutter,

$$\rho_{sr}(x) \to_r^* \rho_{sr}(y);$$

(ii) fine consistency maps into coarse consistency,

$$\rho_{sr}(C_s) \subseteq C_r.$$

For a concrete local repair law this is checked by giving, for every fine repair generator, a coarse witness word in the accepted coarse repair generators or the empty word, and by verifying that the restriction maps carry every fine consistency equation to a coarse consistency equation.

Theorem 6.5 (Normal-form naturality from repair morphisms). *If $\rho_{sr} : Q_s \to Q_r$ is a repair morphism, then*

$$\rho_{sr} \circ n_s = n_r \circ \rho_{sr}.$$

Proof. Fix $x \in Q_s$. Since $x \to_s^* n_s(x)$, repeated use of the step-simulation clause gives

$$\rho_{sr}(x) \to_r^* \rho_{sr}(n_s(x)).$$

The consistency-preservation clause gives $\rho_{sr}(n_s(x)) \in C_r$, because $n_s(x) \in C_s$. Thus $\rho_{sr}(n_s(x))$ is a coarse normal form reachable from $\rho_{sr}(x)$. The coarse system has a unique normal form reachable from each initial quotient state, so $\rho_{sr}(n_s(x)) = n_r(\rho_{sr}(x))$. □

Definition 6.6 (Holonomy cochain morphism)**.** *For obstruction maps $h_s : Q_s \to \mathcal{H}_s$ and $h_r : Q_r \to \mathcal{H}_r$, a map $\chi_{sr} : \mathcal{H}_s \to \mathcal{H}_r$ is a **holonomy cochain morphism** when the edge, cycle, or crossed-module cochains that compute h_s restrict to the cochains that compute h_r, modulo the declared quotient identifications. Equivalently,*

$$\chi_{sr} \circ h_s = h_r \circ \rho_{sr}.$$

Remark 6.7 (How exact refinement naturality is discharged). Definition 6.1 may be used as an interface contract, but on an implemented exact branch the normal-form part is proved by Theorem 6.5 from explicit coarse witness words for fine repairs. The holonomy part is proved by the cochain morphism check of Definition 6.6. Approximate RG branches use the controlled defects of Definition 6.8 instead.

Definition 6.8 (Controlled coarse-graining / reconciliation square)**.** *Fix refinement stages $r \preceq s$. Let Q_r, Q_s be the physical quotient state spaces, n_r, n_s their finite-stage normal-form maps, and h_r, h_s their holonomy or higher-gauge obstruction maps. Let*

$$\rho_{sr} : Q_s \to Q_r, \qquad \chi_{sr} : \mathcal{H}_s \to \mathcal{H}_r$$

be the coarse-graining maps on quotient states and obstruction data. Equip Q_r and $\mathcal{H}_r$ with pseudometrics d_r^Q and $d_r^{\mathcal{H}}$ that define the macroscopic readout scale at stage r.

The square is $(\varepsilon_{sr}^n, \varepsilon_{sr}^h)$-controlled when, for every $x \in Q_s$,

$$d_r^Q(\rho_{sr}(n_s(x)),\, n_r(\rho_{sr}(x))) \le \varepsilon_{sr}^n,$$

and

$$d_r^{\mathcal{H}}(\chi_{sr}(h_s(x)),\, h_r(\rho_{sr}(x))) \le \varepsilon_{sr}^h.$$

*The errors are **cofinally vanishing** when, for every fixed macroscopic stage r and every $\delta > 0$, there is a refinement stage $s_0 \succeq r$ such that $\varepsilon_{sr}^n, \varepsilon_{sr}^h < \delta$ for all $s \succeq s_0$.*

Theorem 6.9 (Coarse-graining commutes with reconciliation up to controlled error)**.** *Suppose the refinement square of Definition 6.8 is $(\varepsilon_{sr}^n, \varepsilon_{sr}^h)$-controlled. Define the coarse-stage consensus readout*

$$\mathcal{C}_r(y) := (n_r(y), h_r(y))$$

and the fine-then-coarse readout

$$\mathcal{C}_{s\to r}^{\mathrm{fine}}(x) := (\rho_{sr}(n_s(x)), \chi_{sr}(h_s(x))).$$

Then, for every fine state $x \in Q_s$, the product readout distance between $\mathcal{C}_{s\to r}^{\mathrm{fine}}(x)$ and $\mathcal{C}_r(\rho_{sr}(x))$ is bounded by

$$\max\{\varepsilon_{sr}^n, \varepsilon_{sr}^h\}.$$

Thus reconciling at the fine stage and then coarse-graining gives the same macroscopic law data as coarse-graining first and reconciling at the coarse stage, up to the declared control errors. If the exact naturality clauses of Definition 6.1 hold, then $\varepsilon_{sr}^{n} = \varepsilon_{sr}^{h} = 0$. If the errors are cofinally vanishing, then at every fixed macroscopic stage the supremum product-pseudometric distance between the two readouts tends to zero along the refinement tail. Equality of inverse-limit cylinder values additionally requires a compatible limiting family and a separated quotient or other completeness/separation hypotheses; it does not follow from vanishing pseudometric defects alone.

Proof. The normal-form component of the product readout is bounded by the first inequality in Definition 6.8, and the obstruction component is bounded by the second. Taking the maximum gives the stated product bound. Exact naturality is precisely the special case

$$\rho_{sr} \circ n_s = n_r \circ \rho_{sr}, \qquad \chi_{sr} \circ h_s = h_r \circ \rho_{sr},$$

so both errors vanish there. Cofinal vanishing means that, for any fixed coarse stage and any desired readout tolerance, all sufficiently fine presentations put the two readouts within that tolerance uniformly over the fine state. This proves asymptotic zero distance and nothing stronger without the additional limit compatibility and separation clauses stated in the theorem. □

Remark 6.10 (Scope of the coarse-graining theorem). Theorem 6.9 is not a claim that arbitrary renormalization maps commute with arbitrary repair laws. Its conclusion requires control of the two square defects ε_{sr}^{n} and ε_{sr}^{h}, derived or estimated from the chosen coarse-graining channel, recovery map, decoder, and obstruction readout. The exact refinement theorem is the zero-defect case; approximate RG matching is theorem-grade only when these defects are explicitly controlled.

7 Record Algebras and the Operator Observation Layer

Inputs used here. From the fixed-cutoff collar package we use only the observer-accessible finite-dimensional algebra on one completed compare/write/verify slice, the declared pointer and overlap-sector projectors read on that same slice, and the trace-distance control on restored accessible states when restoration is invoked. No continuum lift and no broader observer-metaphysical assumption is used here.

Definition 7.1 (Exact record algebras and approximate record presentations)**.** *Fix one completed observer-accessible slice at cycle t, and let $\mathcal{A}_t^{\mathrm{acc}}$ be the corresponding finite-dimensional accessible algebra. An* exact record presentation *is a family of orthogonal projectors $\{\widehat{P}_a(t)\}_a \subset Z(\mathcal{A}_t^{\mathrm{acc}})$ whose generated algebra*

$$\mathcal{Z}_{\mathrm{rec}}(t) := \mathrm{Alg}\big(\{\widehat{P}_a(t)\}_a\big)$$

is finite and commutative.

An approximate record presentation *on the same declared readout slots is a family of projectors $\{P_a(t)\}_a \subset \mathcal{A}_t^{\mathrm{acc}}$ together with an exact record presentation $\{\widehat{P}_a(t)\}_a$ such that*

$$\delta_{\mathrm{rec}}(t) := \max_a \|P_a(t) - \widehat{P}_a(t)\|$$

is finite.

Theorem 7.2 (Record algebra, declared Born–Lüders instrument, and quantitative stability)**.** *Fix one completed observer-accessible slice at cycle t, supply a density state ρ_t, use its trace valuation for event probabilities, and declare the selective outcome instrument to be the Lüders map for the displayed record projectors.*

1. *The exact record projectors generate a finite commutative central algebra* $\mathcal{Z}_{\mathrm{rec}}(t) \subset Z(\mathcal{A}_t^{\mathrm{acc}})$.

2. *For every event* E *in the finite event algebra generated by* $\mathcal{Z}_{\mathrm{rec}}(t)$,

$$\mathbb{P}_t(E) = \mathrm{Tr}(\rho_t \widehat{P}_E(t)), \qquad \rho_t|_E = \frac{\widehat{P}_E(t)\rho_t\widehat{P}_E(t)}{\mathrm{Tr}(\rho_t\widehat{P}_E(t))}.$$

   The conditional state is defined only when $\mathbb{P}_t(E) > 0$*; the probability formula also applies to zero-weight events.*

3. *Conditional on observing* E *with nonzero weight and applying the declared Lüders instrument, an immediate reread with the same projector has probability* 1*. The same conclusion after an intervening accepted operation requires the explicit persistence hypothesis that the resulting state remains supported in* $\widehat{P}_E(t)$*.*

4. *If* $\{P_a(t)\}_a$ *is an approximate record presentation with modulus* $\delta_{\mathrm{rec}}(t)$*, then*

$$\|[P_a(t), P_b(t)]\| \le 4\,\delta_{\mathrm{rec}}(t)$$

   for all declared record projectors $P_a(t), P_b(t)$*. For every declared elementary record event* a *and every restored accessible state* $\widetilde{\rho}_t$ *satisfying*

$$\|\widetilde{\rho}_t - \rho_t\|_1 \le \varepsilon,$$

   one has

$$\left|\mathrm{Tr}(\widetilde{\rho}_t P_a(t)) - \mathrm{Tr}(\rho_t \widehat{P}_a(t))\right| \le \varepsilon + \delta_{\mathrm{rec}}(t).$$

Proof. Because the exact record projectors lie in the center of $\mathcal{A}_t^{\mathrm{acc}}$, they generate a finite commutative central algebra. The supplied trace valuation and declared Lüders instrument give the two formulas in (2); the projector algebra does not by itself select that instrument. This proves (1) and (2) under the displayed premises. The finite-matrix steps behind (1) and (2) are machine-checked in the companion Lean 4 event-algebra development [11]: the commutative span of a projective partition and its containment in the center of the partition commutant, the trace-preserving expectation onto that span, preservation of the partition Born statistics, the Lüders fixed-point law, and compatibility of partition averaging with conditioning are theorem-level exports with declaration-level axiom audits. It is the *partition-averaged* conditional state that equals the normalized block projector. Conditioning alone retains the state inside a higher-rank block; for example, a pure state supported on a rank-two block remains pure, whereas the normalized block projector has rank two.

After the declared conditioning on a nonzero-weight event E, the state lies in the range of $\widehat{P}_E(t)$, so an immediate reread of the same projector has probability 1. More generally, the same calculation applies after an intervening operation when its output state is explicitly assumed to remain in that range. This gives (3); absence of a touched-support marker by itself is not used as a mathematical substitute for support preservation.

For (4), write

$$[P_a(t), P_b(t)] = [P_a(t) - \widehat{P}_a(t), P_b(t)] + [\widehat{P}_a(t), P_b(t) - \widehat{P}_b(t)],$$

because $[\widehat{P}_a(t), \widehat{P}_b(t)] = 0$. Since every projector has operator norm at most 1,

$$\|[P_a(t), P_b(t)]\| \le 2\|P_a(t) - \widehat{P}_a(t)\| + 2\|P_b(t) - \widehat{P}_b(t)\| \le 4\,\delta_{\mathrm{rec}}(t).$$

For the probability bound,

$$\left|\mathrm{Tr}(\widetilde{\rho}_t P_a(t)) - \mathrm{Tr}(\rho_t \widehat{P}_a(t))\right| \le \left|\mathrm{Tr}((\widetilde{\rho}_t - \rho_t) P_a(t))\right| + \left|\mathrm{Tr}(\rho_t (P_a(t) - \widehat{P}_a(t)))\right|.$$

The first term is bounded by $\|\widetilde{\rho}_t - \rho_t\|_1 \|P_a(t)\| \le \varepsilon$, and the second by $\|\rho_t\|_1 \|P_a(t) - \widehat{P}_a(t)\| \le \delta_{\mathrm{rec}}(t)$. Hence the total error is at most $\varepsilon + \delta_{\mathrm{rec}}(t)$. □

Two companion results address the declared valuation and instrument of the theorem. Complete native record histories with a supplied initial law and fair coins have additive finite-sum weights, and on a commutative record algebra those weights are the trace pairing of a unique diagonal density, with no trace rule supplied and no selection of off-diagonal entries of an ambient matrix state [12]. Within complete finite Kraus instruments, unit joint probability of the assigned label and a trusted return, on a trusted basis and on one uniform coherent state per block, is equivalent to the Lüders map on every block, and certified approximate probabilities give a uniform error for arbitrary untouched references [11].

The theorem gives a fixed-cutoff observation surface. It does not show that richer branches keep physically relevant pointer surfaces close to one central record algebra, or that the same control survives refinement and continuation.

8 Finite Observable Nets on the Consensus Tower

The record algebra extends, in the accompanying Lean development [12], to a machine-checked interface family in which the observer is a typed object. The base object is a consensus tower: a regulator-indexed family of finite matrix models carrying observer and record fibres, observer-indexed record orders, commutative public record subalgebras of the private matrix algebra, selected states and generators, and functorial refinement maps with compatibility laws. Every statement in this section is a finite structure over declared data; physical regions, instruments, clocks, spacetime and continuum objects require the higher layers.

The operational observerhood receipt. An operational observer over a consensus tower carries six complete clause groups and the own-observer half of a seventh: (i) a bounded accessible star subalgebra containing the observer's public record algebra; (ii) self-readback, a map into the public record algebra fixing it; (iii) durable records annihilated by the observer's generator; (iv) record-conditioned interventions preserving the accessible interface; (v) no-regress prediction in the observer's record order; (vi) continuation, commuting with the refinement maps; and (vii) readable evidence, a readout equal to the trace pairing of each record element with the observer's state and fixed by every intervention. The cross-observer half of (vii) is a separate fixed-regulator receipt over two observers and one access cut: distinct labels, accessible algebras agreeing with the cut, every committed record restricting to the same element at the owner-region meet, agreeing own readouts and one nonzero common restriction. A proper-meet witness on diagonal algebras with corner-evaluation restrictions has record representatives that differ before restriction and agree nontrivially after it, and mutating one shared corner breaks the receipt. Refinement-natural cross-observer evidence and higher-overlap coherence are not supplied. The base witness is a one-regulator tower with private algebra $M_2(\mathbb{C})$, diagonal public algebra, three records in strict chain order, Hamiltonian generator $\mathrm{diag}(1, 2)$ and intervention by conjugation with $\mathrm{diag}(1, -1)$; the generator is nonzero and moves the raising matrix unit, so durability is a computed fixed-point fact. Four negative controls each break one field: an erasing readback, the identity generator, a resetting prediction and a

constant readout. The receipt selects no unique observer, attaches no instrument and supplies no source realization of the tower data.

Access cuts and the common-origin theorem. A finite regional observer net enriches one consensus tower with a finite region order, per-region local star subalgebras, declared contravariant restrictions, elementwise commutation on declared-disjoint regions, regional expectations, and idempotent local repair. An access cut over such a net assigns each observer one declared support region and one accessible star subalgebra squeezed between the observer's public record algebra and the regional algebra of the declared region, with the declared restrictions below that region preserving accessibility; stability under the local repair idempotents is a separate strengthening receipt. The observer-local algebra at a region is the meet of the regional local algebra with the accessible algebra, and the regional receipts (isotony, elementwise commutation on declared-disjoint regions, restriction stability, selected-state expectation compatibility, regional-repair fixed points, and remote nondisturbance) transport to this meet. Unique descent into the observer-local algebra is conditional on a separate accessible-glue receipt; ambient unique descent alone does not place a glued section in the accessible algebra, and the universal singleton-family packet keeps the receipt type inhabited without a nontrivial gluing claim.

A finite event world binds one packet to each event, with fixed finite event, chart, and packet carriers, declared active masks, chart ownership tying chart supports to observer regions, packet-derived support, record, and coordinate readouts, and the visibility law that an active visible event has support inside the seeing chart's region. The common-origin theorem returns, for every active visible chart/event pair (i, e), three conjuncts from one packet p_e:

$$\operatorname{supp}(\mathsf{p}_e) \preceq R_{O(i)}, \qquad \operatorname{rec}(\mathsf{p}_e) \in \mathcal{L}_{O(i)}(\operatorname{supp}(\mathsf{p}_e)) \cap \mathcal{P}_{O(i)}, \qquad x_i(e) = \operatorname{geo}_i(\mathsf{p}_e),$$

with $O(i)$ the declared chart owner, $R_{O(i)}$ its observer region, $\mathcal{L}_{O(i)}$ its observer-local algebra, and $\mathcal{P}_{O(i)}$ its public record algebra: event support below the owner's region, record observable in the meet of the observer-local algebra at that support with the owner's public algebra, and the soldering chart coordinate as a readout of the same packet. An event precedence adapter carries strict order data on event slots, tied to the owner's record order on labeled events and identified with no repair schedule, modular parameter, clock reading, proper time, or global time function; where the events carry certified commit provenance, the source-derived construction below generates that order, and an exact adapter is its verifier rather than a choice of it. A selected coarse-to-fine refinement (slot maps for events, charts, and packets, with naturality for masks, ownership, regions, supports, packets, records, and visibility, plus a declared recharting map) transports the full common-origin conclusion, and raw event candidates with a declared semantic-coincidence setoid descend to the event quotient; a reflexive symmetric relation without transitivity admits no such setoid and is closed to the quotient constructor. The witness world reuses the dimension-two cut, whose interposition chain is strict (scalar public algebra inside the diagonal accessible algebra inside the private matrix algebra), and its record readout separates the two packets.

Source-derived event precedence from mismatch provenance. Event precedence need not be declared where the source supplies provenance. In the companion machine-checked package, every quotient-visible register version carries the identifier of its last semantic writer, and each commit certifies the register versions on which its acceptance genuinely depends, with frame and stamp laws fixing unwritten registers and recording written ones. The `causalSupp` field is supplied semantic data: Lean checks that it lies inside the read set, while producer and independent-verifier receipts must establish that it is sound and complete for genuine dependence. It is not inferred by

a kernel-checked dependence analysis. One commit is a direct semantic parent of another exactly when the child certifies a register that the parent wrote, the child's pre-commit snapshot names the parent as that register's writer, and the child's pre-commit value equals the parent's post-commit value. A writer label without the write and value witnesses does not create an edge. The generated informational order is the transitive closure of these edges. Under one abstract ancestry-rank witness that increases across every edge, the generated order is irreflexive, transitive, and asymmetric, and it inhabits the same strict-order interface the event precedence adapter consumes. It is the least strict transitive relation containing the direct parent edges, so an adapter that contains every edge and asserts no comparability beyond the generated order carries exactly the generated order. Geometry does not use the numerical values of this acyclicity witness; it recomputes canonical source height from authenticated parenthood. On the committed chain witness world, a two-commit log generates the declared adapter order in full, so the declaration there is a verifier rather than a choice.

An overlap's mismatch score reads only the committed values on its support, so the score is writer-blind: two snapshots with identical values carry identical scores on every overlap, and a two-register countermodel realizes both parent attributions on value-identical snapshots. A static residual therefore determines no causal arrow; provenance is load-bearing for direction. A commit whose write set misses the support leaves the score unchanged, and a repair whose certified support does not cite the injected version acquires no injection-to-response edge. Under dependency-complete read sets, any two proposals able to move one seam score overlap read-against-write in both directions and land in one aggregate conflict component. Any aggregation map that is constant on conflict components assigns them one aggregate label; an empty-read countermodel shows the completeness clause is load-bearing. A four-commit log realizes the minimal branching interval: one injection, two responses each certifying exactly the injected register, and one answer certifying both outputs generate the four-element Boolean diamond with incomparable responses, while a parent-child pair alone spans the two-element interval. The serialization of the two responses is invisible to this order, because the second response's snapshot contains the first response's output without certifying it. Executed histories carry three further machine-checked controls. One adjacent independent swap preserves the authenticated direct-parent relation and its transitive closure when the two executions both satisfy fresh-ID and duplicate-free hypotheses and the commits' write sets avoid each other and each other's certified supports. The general equivalence-generated swap chain preserves only raw writer citation, not arbitrary labeled payloads. Executor stutters are removed before commits are formed. A dependent pair marks the boundary: swapping a writer past a reader that certifies the written register changes the edge set. On such histories the append-only acyclicity rank is derived from the execution position rather than declared. Independently of that witness, the geometry layer computes canonical source height from authenticated parenthood alone, zero at roots and one plus the maximum direct-parent height otherwise, proves that it is the attained longest authenticated-parent-chain length, and proves strict increase on every parent edge and generated ancestry. Under fresh identifiers, duplicate-free execution, and visible-support hypotheses, a commit writing only outside the visible register carrier changes no visible mismatch and supplies no outgoing direct raw or authenticated parent edge to a visibly supported child; incoming edges into the hidden commit are not excluded. A freshly injected mismatch persists at full strength through a continuation whose commits never write its overlap support. A refinement receipt assumes that its map carries every direct parent edge into the coarse reflexive precedence; under that assumption it transports the generated order and its causal intervals through coincidence merges. The reversed assignment on the committed diamond-to-chain witness inhabits no such receipt. The generated order is informational. No theorem here supplies invariance over the full admissible implementation class, event separation, physical causal faithfulness, or a Lorentzian continuum.

The generated structure meets two established programs. In causal set theory, a locally finite partial order represents proto-causality [1, 2]. The OPH order is finite and therefore locally finite, but its proved interpretation is semantic read-from dependence. It is thus causal-set-like rather than a physically attached causal set. The exact-embedding continuum route requires an order embedding whose image has the approximately Poisson count density needed for a causal-set faithful embedding, together with manifoldlikeness and dimension [3, 4, 5]. In a past- and future-distinguishing continuum, causal order fixes conformal geometry [7, 8]; the volume datum fixes the conformal factor. The standard Lorentz-invariant continuum approximation uses Poisson sprinkling [6]; the OPH commit log is not a Poisson sprinkling, and its cardinality is not a spacetime volume. Neither continuum existence nor the causal-set Hauptvermutung follows from a finite order. In distributed computing the generated relation is an authenticated read-from dependence suborder and analogue of happened-before [9]; record or program order contributes only when a later commit genuinely consumes the earlier record. Its OPH-specific content is the binding to quotient-visible mismatch, certified read supports, writer-blind static scores, and the strict-order interface of the finite event world.

Every finite strict partial order compiles exactly into an abstract authenticated log, with rank derived from predecessor cardinality. In this generic compiler the authenticated-parent relation is the whole supplied order, including pairs outside its Hasse reduction. This is an expressivity theorem for the grammar, not a dynamics or execution-history selection theorem. The exact Boolean-diamond inhabitant shows that the interface is not restricted to chains. Placement of such logs on the rank-three source carrier with a real time axis, one-way and two-way cone compatibility, the source-direction Einstein implication and a conservative source-record family with a flat causal-order limit are developed in the spacetime layer under its own placement, speed, count–volume and calibration premises [32]. None of them follows from the generated order alone.

Character-block coexistence of regional algebras. For every $n \geq 2$ the full matrix algebra $M_n(\mathbb{C})$ admits no unital complex-algebra homomorphism into the scalars: the matrix ring is simple, so such a homomorphism is injective, while the off-diagonal matrix unit is a nonzero element whose square is zero. No unital complex-algebra homomorphism into the scalars accepts an anticommuting pair of units. On this interface, at any regulator of nonzero dimension, a region above a commutative-algebra region carries no anticommuting pair, and every full matrix factor of dimension at least two above a scalar overlap region is excluded: the tensor-factor local-algebra picture with trivial overlap is impossible. The inhabitant locating the boundary is a dimension-four net with three regions: a bottom overlap region carrying the scalar algebra and two declared-disjoint maximal regions carrying the block algebras $\mathrm{diag}(a, a, A)$ and $\mathrm{diag}(B, b, b)$ with arbitrary two-by-two blocks. Both regional algebras are noncommutative, neither contains the other, their elements commute pairwise, and each admits a character onto the scalars through its one-dimensional corner block, which supplies the star-homomorphic restriction retractions the interface demands. The block shape is forced: a noncommutative regional algebra compatible with a scalar drop must carry a one-dimensional block, and the witness realizes that minimal shape. The declared overlap algebra is a proper subalgebra of the set intersection of the two block algebras; the regional repair maps are identities. The construction provides no genuine coverage semantics for a source-produced net.

Hamiltonian rigidity of the private block. The reversible private dynamics behind these interfaces is forced by continuity [12]. On one full finite matrix block, every complex-linear Leibniz endomorphism is an inner derivation, and every star-compatible one is the commutator action $x \mapsto (-i)(Hx - xH)$ of a self-adjoint generator, unique up to an additive real multiple of the

identity. A real-parameter group of star automorphisms with pointwise continuous orbit maps is conjugation by $\exp(t(-i)H)$ for one time-independent self-adjoint H: continuity upgrades to differentiability at zero because the short-interval average of the norm-continuous matrix family is invertible. On a general finite private algebra, the center of any unital star subalgebra is spanned by a minimal orthogonal central resolution of the identity, unique up to permutation; the subalgebra is a finite direct sum of full matrix blocks; and a pointwise-continuous automorphism group fixes every block and acts by one Hamiltonian per block, with block fixing derived and the block swap realized on the square two-block algebra. The parameter is a real number, and no physical clock or source-selected Hamiltonian is claimed. The free evolutions of the executable model lie inside the reversibility hypothesis [13]: the contrast-memory seam, which the reference candidate and the main candidate law Interval Echo share, and the scalar channel each give a continuous one-parameter unitary flow on finite registers, so each is fixed by its flow up to an energy offset, and its parameter is no observer's clock; the clock of Interval Echo is a register that an observer patch carries. On each seam the local term of the contrast-memory law is a self-adjoint generator conjugated by a Hermitian unitary contrast write, acting on the two loads and the private seam memory.

Finite covariant-net consequences and limitations. The finite observer net determines a category of event regions and causal embeddings together with a covariant observable functor [12]. Its causal embeddings are proof-only order data, so the category is thin and carries no Lorentzian geometry. In particular, these carriers provide no region reflection, antiunitary conjugation, distinguished vacuum, spin structure, symmetry-group action, Lorentzian Cauchy-embedding class, translation action, or spectrum condition. Consequently they lack the data needed to formulate CPT, spin–statistics, relative-Cauchy stress response, or Haag–Ruelle scattering. These are limitations of the mathematical carrier, not negative theorems about QFTs built from richer data.

The same module encodes seven structural questions and kernel-checks their partition by mathematical conclusion. Four of them (CPT, spin–statistics, relative-Cauchy stress response, and particle scattering) lack defining geometric or spectral data on this carrier. Full field/action reconstruction likewise cannot be inferred from the finite static identities alone. Of the two remaining questions, the restricted endomorphism surrogate collapses as described next, while the finite KMS identity coexists with the net- and public-algebra counterexamples below. Thus the seven-way classification is a statement about what follows from the specified carrier.

The finite sector theorem has a narrower exact content. Every bijective localized endomorphism of the witness net is inner and hence equivalent to the identity under the declared finite equivalence relation. This does not constitute DHR reconstruction: the carrier supplies no source-selected vacuum representation, quasi-local net, transportability structure, or reconstructed gauge group. The selected tower states nevertheless induce mathematical GNS representations of the completed colimit, as described below. Innerness of this restricted class supplies no charge or flavor quantum number.

Two further finite results delimit thermality and field reconstruction. The oriented-face curvature has kernel exactly the port gradients, and its five-neighbor seam action is gauge invariant exactly for conserved seam currents. This is a static finite action identity; it supplies no temporal dynamics, charge–current continuity map, source-selected physical current, physical matter or Spin action, spacetime carrier, or continuum limit. On a private finite block, the normalized Gibbs state satisfies the algebraic KMS identity. On the constructed example net, however, every functional is KMS for every grading-preserving flow at every complex time, while every pointwise-continuous group of star automorphisms of the public record algebra is the identity, a statement that leaves continuous row-stochastic public semigroups untouched. The finite Gibbs identity therefore does

not establish thermality of the observer net.

Natural powers give one exact constraint on a fixed-cutoff scattering construction: in every Hausdorff topological group, $g^n \to a$ forces $g = 1$. Hence a nontrivial exact unitary update at fixed finite cutoff cannot itself settle to an ordinary matrix-topology large-time limit. The same module proves $(g^n)^{-1}g^n = 1$, so relative evolution can converge even when the individual powers do not. Any fixed-cutoff direct-power construction that uses U^n itself as its asymptotic carrier cannot use full convergence of that sequence; it must change the asymptotic carrier or dynamics, or use a projected or weaker convergence notion. At fixed finite dimension, full weak-operator convergence does not evade the obstruction because the standard finite-dimensional operator topologies coincide. Comparison dynamics, selected projected scalar or observable readouts, an infinite-dimensional weak limit after a continuum or infinite-volume passage, open-system evolution, and finite-time or recurrence-aware readouts lie outside this no-go. The theorem constructs no S-matrix, asymptotic state, optical theorem, cross section, or interacting continuum limit.

Restriction naturality on authenticated event regions. Let a finite regional net have local algebras $\mathcal{A}_r(U)$, contravariant restrictions $R^r_{V,U}$ for $V \subseteq U$, algebra refinements j_{rs}, and regional repair maps P^r_V. Restriction is additional algebraic data; isotony alone does not supply it. Suppose repair is natural,

$$j_{rs}P^r_V = P^s_{V_s}j_{rs},$$

where V_s is the refined region. The following compatibility is a separate hypothesis on the restricted domain:

$$R^r_{V,U}X = P^r_V X \qquad (X \in \mathcal{A}_r(U)). \tag{1}$$

Proposition 8.1 (Finite restriction and refinement square)**.** *Under these hypotheses,*

$$j_{rs}R^r_{V,U}X = R^s_{V_s,U_s}j_{rs}X. \tag{2}$$

Let each event e of an authenticated semantic log carry a declared region U_e, with $U_e \subseteq U_f$ on every authenticated direct-parent edge. Generated reflexive precedence then defines restrictions from the algebra at f to that at e, with identity and contravariant composition. If a certified event map preserves those direct edges in target generated precedence and satisfies $U'_{F(e)} = (U_e)_s$, the same square holds for these event restrictions.

Proof. For the algebra square, substitute (1), apply repair naturality, and substitute the target compatibility. Inclusion extends from direct edges to generated precedence by transitivity; the restriction identity and tower law give identity and composition. The event map preserves generated precedence by induction on an authenticated path, so the target restriction is typed. Its region compatibility identifies its algebra square with (2). □

An inhabited example decorates the committed four-event Boolean diamond and three-event chain by a two-dimensional diagonal algebra. The injection event has the scalar algebra, and every response and answer has the full diagonal algebra. Restriction to the root sends $\mathrm{diag}(x_0, x_1)$ to $x_0 I$; in particular, $\mathrm{diag}(1, 0)$ is changed. The authenticated coarsening maps the diamond labels $(0, 1, 2, 3)$ to $(0, 1, 1, 2)$ and merges two distinct responses. Its region decorations agree, and the restriction square is inhabited. The reversed assignment $(2, 1, 1, 0)$ violates authenticated precedence. The finite identity, composition, compatibility, and coarsening statements are machine checked [12].

The algebra regulator in this example is constant. Region decoration and edge preservation are explicit data; neither physical regions nor a source-selected refinement family are inferred.

These are restrictions of commutative finite algebras, without an assertion of homomorphic scalar restriction for matrix factors. No normed limit restriction, physical locality, spacetime identification, or physical time-slice property follows.

Joint carriers of retained observer pairs. Two regional diamonds carry source-counted correlation data of retained observer pairs from one bounded companion run: a two-slot conditional-expectation net for the pair 86/88 on a joint carrier of dimension 182, and one for the support-disjoint pair 86/247 on a joint carrier of dimension 169. Both transport onto one constant consensus-tower stage by the same specified pattern, with observer 86's checkpoint family as commutative public layer, the uniform selected state and the template nontriviality receipts. An exact shared-observer theorem relates them: the two source-derived joint paths have equal left label components at every step, equal left occupation counts and equal induced diagonal hinge states, each equal to the source marginal state of observer 86. At the anchored-net level the carriers are not identified, and no map between the two anchored nets is constructed [12].

That carrier-level obstruction is removed below the net level by the common finite carrier $(\mathrm{obs86}, \mathrm{obs88}, \mathrm{obs247}) \in \mathrm{Fin}(13) \times \mathrm{Fin}(14) \times \mathrm{Fin}(13)$. Its coordinate projections recover both complete 32-row source paths and their occupation tables. The diagonal triple counted state is positive with trace one, and explicit linear positive trace-preserving marginals recover both counted pair states. Four triple checkpoint projectors have normalized partial traces recovering both anchored checkpoint partition members, and on each of the 31 actual adjacent source transitions a triple reindexing transports the source projector and intertwines with both pair marginals. All three final-to-initial source counts are exactly zero, so the endpoint is not wrapped to the first row. This is a common state, path and checkpoint coupling, not a canonical gluing theorem: the pair of marginals is noninjective on triple matrices, so the row alignment selects the coupling.

The one-step time-slice interface. `TimeIndexedNetEvolution` is a finite one-step interface: a finite index with successor, a region family, per-step ambient star automorphisms mapping each regional algebra to its successor, and admitted data in the current regional algebra whose evolved images generate the successor algebra. A full inhabitant exists over the 86/247 net on cyclic Fin(32): it follows the 31 actual adjacent source-row pairs and adds one terminal-to-initial bookkeeping automorphism that is not a source transition, with observer 86's lifted source generators as data and the left region at every step. The natural single-projector walk candidate is a negative control: its evolved singleton cannot generate the top regional algebra. On this net, an evolution family that maps each regional algebra onto itself forces a step-invariant region family, every net-compatible slice family over the walk-step evolution is constant, and a one-step region change of any inhabitant exchanges the two slot regions. A slot-alternating inhabitant composes the walk step with the internal slot exchange and alternates observer 86 and observer 247 generators; its region family is not step invariant, and the exchange factor is a constructed symmetry, not a source transition. The interface has no multi-step cocycle, Cauchy geometry, clock or source-selected dynamics, and implies no time-slice theorem.

Joins of the two carriers. At the diagonal layer, which carries every source-counted state, the two carriers glue along the 13-dimensional observer-86 hinge. Fiber-uniform sections of slot marginalization embed the hinge into the 182- and 169-dimensional layers, and the join is the pushout over the hinge. Both embeddings are injective and agree on the hinge, the join has dimension $338 = 182 + 169 - 13$, strictly below the direct-sum dimension 351, and the intersection of the two images is exactly the embedded hinge. A hinge restriction retracts the join, under

which the two transported counted states restrict to the same observer-86 occupation law; the ambient anchorings and the 31 walk-step transports descend to one join evolution intertwined by both embeddings. Region by region, the record layer of each diamond is exactly the set of diagonal restrictions of the specified regional algebra, with dimensions $1, 13, 14, 182$ and $1, 13, 13, 169$; the induced net on the join is isotone, both embeddings are net morphisms, the conditional expectations restrict to the record layer and the join evolution preserves every induced region. Two negatives delimit this layer: the gluing submodule is not multiplicatively closed, so the pushout carries no algebra structure, and the record layer is commutative while the left regional algebra is not.

At the operator layer, adjoining the unused observer's identity gives injective unital star homomorphisms of the full M_{182} and M_{169} pair algebras into M_{2366}, whose images intersect in exactly $M_{13} \otimes I_{14} \otimes I_{13}$ and generate the triple algebra; the shared hinge is noncommutative [16]. Both embeddings intertwine all 31 adjacent path-read permutations, and ordered products give finite interval composition with no invented terminal-to-initial edge. Partial traces recover the counted pair states and are not multiplicative. For the 86/247 carrier the boundary between the layers is an exact asymmetry: the record carrier embeds into the joint matrix algebra as an injective unital star homomorphism compatible with the regional algebras and expectations, while the left-slot conditional expectation is positive, unital and trace preserving but not multiplicative. No map from the joint algebra to the record carrier is multiplicative on the left regional algebra and restricts to the record-layer morphism, and no unital algebra homomorphism carries that regional algebra into the record carrier at all. The tensor split and path permutations are supplied adapters; physical regions, preparations, a clock, Cauchy geometry and a state-preserving tower morphism are not derived, and observer 86/88 source supports overlap, so tensor slot locality does not establish physical support disjointness.

The normed completion of the colimit. The consensus tower supplies a directed family of finite matrix algebras with star homomorphisms as connecting maps. Its filtered colimit carries an inductive-limit seminorm, the infimum over declared refinements of one representative of the refined stage norm, independent of the representative by contractivity. The stage norms of one representative lie in a finite set fixed by the spectrum of a^*a, so the infimum is attained, and a class has seminorm zero exactly when some declared refinement annihilates its representative. The seminorm is therefore a norm satisfying the C*-identity, and the uniform completion is a C*-algebra. The closures of the limit local algebras are isotone, independent of the presenting regulator, elementwise commuting on declared-disjoint regions, and equal along the order-theoretic Cauchy embeddings, each instantiated on a finite witness net with a norm-one element. Two controls fix the boundary: a connecting map annihilating the unit gives a nonzero stage element of limit seminorm zero, and a diagonal scaling that is not a star map increases a matrix-unit norm. The construction is algebraic and order-theoretic and supplies no time-indexed physical evolution or physical attachment.

A selected-state Hilbert-space representation of the completed net. Equip the completed colimit with the canonical C*-spectral order. For every positive complex-linear functional ω normalized by $\omega(1) = 1$, the GNS construction gives a complex Hilbert space $\mathcal{H}_\omega$ and a unital star representation π_ω of the completed colimit by bounded operators. The completion image Ω_ω of the algebra unit is explicit and obeys

$$\|\Omega_\omega\| = 1, \qquad \langle \Omega_\omega, \pi_\omega(a)\Omega_\omega \rangle = \omega(a), \qquad \overline{\{\pi_\omega(a)\Omega_\omega : a\}} = \mathcal{H}_\omega.$$

Sending every completed regional algebra to its image under the same representation preserves isotony, refinement independence, elementwise locality for the declared-disjoint regions, and the

order-theoretic Cauchy-embedding equality. Images are used rather than identified with their sources because normalization does not imply that the representation is faithful.

For the specified tower interface, the functional need not be added as an unrelated input. Let ξ_r be an explicitly supplied observer family compatible with every regulator refinement, and let ρ_{r,ξ_r} be the selected density state carried by the tower. State naturality makes

$$\omega_\xi(\iota_r(X)) := \mathrm{Tr}(\rho_{r,\xi_r} X)$$

well defined on the algebraic colimit. The finite state-expectation bound extends it continuously to the completed colimit, where it is positive for the spectral order and normalized. Hence ω_ξ supplies the GNS Hilbert space and representation above. The one-regulator witness instantiates this construction, and its GNS unit class is nonzero with norm one. This yields a mathematical Hilbert-space representation of the selected-state completed net.

Taken alone, this construction's coherent observer family and finite density states are declared structural data, not a state selected by an experimental source. Neither they nor Ω_{ω_ξ} are identified as a physical vacuum. No Hamiltonian or physical dynamics, energy–momentum, field content, Lorentzian attachment, continuum or infinite-volume limit, interacting theory, or physical time-slice theorem follows [12].

An exact finite two-site Hamiltonian benchmark. As a separate finite-dimensional test of Hamiltonian structures outside the completed-net attachment, take

$$\mathcal{H}_{\mathrm{I}} = \mathbb{C}^2 \otimes \mathbb{C}^2 \cong \mathbb{C}^4, \qquad H_{\mathrm{I}} = \frac{1}{2}(1 - Z \otimes Z).$$

The full matrix algebra acts faithfully by bounded operators on $\mathcal{H}_{\mathrm{I}}$. Exact matrix proofs give $H_{\mathrm{I}}^* = H_{\mathrm{I}}$, $H_{\mathrm{I}} \geq 0$, and $H_{\mathrm{I}}^2 = H_{\mathrm{I}}$. The explicit unit vector $|00\rangle$ and density $\rho_{00} = |00\rangle\langle 00|$ obey

$$\||00\rangle\| = 1, \qquad H_{\mathrm{I}}|00\rangle = 0, \qquad H_{\mathrm{I}}\rho_{00} = \rho_{00}H_{\mathrm{I}} = 0, \qquad \mathrm{Tr}(\rho_{00}) = 1.$$

Thus ρ_{00} is a normalized stationary ground-state density; the orthogonal unit state $|11\rangle$ also has zero energy, so the ground space is degenerate and no unique vacuum is selected. The propagator $U(t) = \exp(-itH_{\mathrm{I}})$ is unitary and obeys $U(s+t) = U(s)U(t)$; its conjugation flow $\alpha_t(X) = U(t)XU(-t)$ is multiplicative, fixes ρ_{00}, and preserves commutation between jointly evolved left- and right-slot observables. The bond is genuinely interacting in the exact finite sense that H_{I} is not any sum $H_L \otimes 1 + 1 \otimes H_R$ of one-site Hamiltonians; in particular it belongs to neither one-site matrix-algebra image. The evolution is nontrivial: for $X_L = X \otimes 1$,

$$\frac{d}{dt}\bigg|_{t=0} \alpha_t(X_L) = -i[H_{\mathrm{I}}, X_L] \neq 0,$$

with the displayed matrix entry equal to i. Deleting only the $Z \otimes Z$ interaction leaves the scalar Hamiltonian $H_{\mathrm{del}} = \frac{1}{2}1$, makes the full conjugation flow static, and makes the same infinitesimal generator zero. These are zero-residual identities on the declared four-dimensional carrier.

Taken alone, this is an exact coupled two-spin lattice benchmark, not an interacting quantum field theory and not a Hamiltonian attachment to $(\mathcal{H}_{\omega_\xi}, \pi_{\omega_\xi})$ above. Its tensor factors are declared sites rather than Lorentzian regions; H_{I}, its normalization, and the state are supplied rather than selected by an OPH source; and t has no physical clock or energy calibration. Equal-time commutation here preserves one finite tensor-product commutator and is not a microcausality theorem. No continuum or infinite-volume limit, field algebra, Lorentz covariance, spectrum condition, renormalization flow, particle interpretation, scattering construction, detector readout, or physical time-slice theorem follows.

The finite Hamiltonian, regional diamond, and selected GNS on one carrier. The two preceding constructions compose in one typed finite model. Use the canonical basis equivalence $\mathrm{Fin}(2) \times \mathrm{Fin}(2) \simeq \mathrm{Fin}(4)$ to place the two-spin algebra on a one-regulator consensus tower T_{I}. Its selected density is exactly ρ_{00}, and its stored generator is not the constant tower adaptor's zero map but

$$\delta_{\mathrm{I}}(X) = -i[H_{\mathrm{I}}, X].$$

On the same carrier, the scalar/left/right/top conditional-expectation diamond has commuting declared-disjoint slot algebras, and its left and right algebras jointly generate $M_4(\mathbb{C})$. Let ω_{I} be the tower-selected functional, let $(\mathcal{H}_{\mathrm{I}}^{\mathrm{GNS}}, \pi_{\mathrm{I}}, \Omega_{\mathrm{I}})$ be its GNS triple, and define

$$h_{\mathrm{I}} = \iota(H_{\mathrm{I}}), \qquad \widehat{H}_{\mathrm{I}} = \pi_{\mathrm{I}}(h_{\mathrm{I}}), \qquad \widehat{U}_{\mathrm{I}}(t) = \pi_{\mathrm{I}}(\iota(U(t)))\,.$$

The exact composite representation from $M_4(\mathbb{C})$ through the colimit completion to bounded GNS operators is injective. Consequently the ground-vector equations do not arise from a collapsed quotient:

$$\|\Omega_{\mathrm{I}}\| = 1, \qquad \widehat{H}_{\mathrm{I}} \neq 0, \qquad \widehat{H}_{\mathrm{I}}^{*} = \widehat{H}_{\mathrm{I}}, \qquad \widehat{H}_{\mathrm{I}}\Omega_{\mathrm{I}} = 0.$$

The Hamiltonian belongs to the represented top region, while the represented left and right Pauli observables retain their regional memberships and commute. The mapped propagators are unitary, form an additive one-parameter group, and obey the exact intertwining identity

$$\pi_{\mathrm{I}}(\iota(\alpha_t(X))) = \widehat{U}_{\mathrm{I}}(t)\pi_{\mathrm{I}}(\iota(X))\,\widehat{U}_{\mathrm{I}}(-t).$$

Interaction remains load bearing after representation: $\pi_{\mathrm{I}}(\iota(-i[H_{\mathrm{I}}, X_L])) \neq 0$, and this first-order response is outside the represented left algebra, whereas the same represented response is zero after deleting the coupling.

The left conditional expectation E_L is trace preserving but does not preserve the selected pure state:

$$E_L(1 \otimes Z) = 0, \qquad \mathrm{Tr}(\rho_{00}(1 \otimes Z)) = 1.$$

Thus the construction is a finite regional diamond together with co-located interacting C*-dynamics on its ambient algebra, represented on the tower-selected GNS Hilbert space. The dynamics need not preserve each regional algebra: indeed, the nonzero first-order response above leaves the represented left algebra. The failure of state preservation is a property of that particular trace-preserving expectation, not a failure of a global state to restrict to the regional algebras. It therefore supplies a useful incompatibility diagnostic but is not itself the criterion separating a finite net from a physical QFT. The regulator, factorization, density, Hamiltonian, and flow parameter are declared rather than source produced or calibrated; the ground sector is degenerate; and Lorentzian localization, a spectrum condition, continuum or infinite-volume control, renormalization flow, field and particle content, scattering, detector readback, and a physical time-slice theorem are absent.

A source-counted finite history Hamiltonian on the selected GNS space. The retained eight-history packet supplies a separate finite attachment in which the state and diagonal action entries are source counted rather than freely chosen. For each binary path $g = (s_0, s_1, s_2)$, let $n_g > 0$ be its exact window count among the 1754 retained windows and let

$$S_g = \mathbf{1}_{s_0 \neq s_1} + \mathbf{1}_{s_1 \neq s_2}.$$

On $M_8(\mathbb{C})$, define

$$\rho_{\mathrm{hist}} = \mathrm{diag}\left(\frac{n_g}{1754}\right), \qquad H_{\mathrm{hist}} = \mathrm{diag}(S_g).$$

The source receipts prove every n_g positive and prove the displayed state-change formula for S_g. Exact matrix proofs then give

$$\rho_{\mathrm{hist}} \geq 0, \quad \operatorname{Tr} \rho_{\mathrm{hist}} = 1, \quad H_{\mathrm{hist}} \geq 0, \quad H_{\mathrm{hist}}^{*} = H_{\mathrm{hist}}, \quad [\rho_{\mathrm{hist}}, H_{\mathrm{hist}}] = 0,$$

with H_{hist} nonzero and non-scalar. The selected energy retains the counted source value

$$\operatorname{Tr}(\rho_{\mathrm{hist}} H_{\mathrm{hist}}) = \frac{197}{1754}.$$

Place this pair in one consensus-tower stage with stored generator $\delta_{\mathrm{hist}}(X) = -i[H_{\mathrm{hist}}, X]$. The selected density constructs a normalized GNS triple $(\mathcal{H}_{\mathrm{hist}}, \pi_{\mathrm{hist}}, \Omega_{\mathrm{hist}})$, and the composite representation of $M_8(\mathbb{C})$ is injective. Thus the represented Hamiltonian is nonzero and self-adjoint. Its cyclic matrix coefficient is exactly 197/1754, which also proves

$$\pi_{\mathrm{hist}}(H_{\mathrm{hist}})\Omega_{\mathrm{hist}} \neq 0.$$

The matrix unit E_{01} supplies a nonclassical control:

$$\big(-i[H_{\mathrm{hist}}, E_{01}]\big)_{01} = i,$$

so the represented commutator response is nonzero. The exponential propagators are unitary, obey the additive group law, intertwine exactly with the represented Heisenberg flow, and leave ρ_{hist} stationary.

The same carrier has an exact tensor-local history decomposition. The source encoding identifies its basis with $((s_0, s_1), s_2) \in (\mathbb{Z}/2)^3$. Tensor-factor embeddings define the one-slot algebras $\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2$, the generated adjacent interval algebras $\mathcal{A}_{01} = \mathcal{A}_0 \vee \mathcal{A}_1$ and $\mathcal{A}_{12} = \mathcal{A}_1 \vee \mathcal{A}_2$, and $\mathcal{A}_{012} = M_8(\mathbb{C})$. The inclusions are isotone and the separated endpoint algebras obey exact locality,

$$[\mathcal{A}_0, \mathcal{A}_2] = 0.$$

Writing the two adjacent domain-wall projectors as

$$H_{01} = \operatorname{diag}(\mathbf{1}_{s_0 \neq s_1}), \qquad H_{12} = \operatorname{diag}(\mathbf{1}_{s_1 \neq s_2}),$$

gives the residual-free local decomposition

$$H_{\mathrm{hist}} = H_{01} + H_{12}, \qquad H_{01} \in \mathcal{A}_{01}, \qquad H_{12} \in \mathcal{A}_{12}.$$

Both bonds are positive, and their separately counted source energies are

$$\operatorname{Tr}(\rho_{\mathrm{hist}} H_{01}) = \frac{94}{1754}, \qquad \operatorname{Tr}(\rho_{\mathrm{hist}} H_{12}) = \frac{103}{1754}.$$

For $\delta_{\mathrm{hist}}(X) = -i[H_{\mathrm{hist}}, X]$, locality removes the remote bond exactly, so

$$\delta_{\mathrm{hist}}(\mathcal{A}_0) \subseteq \mathcal{A}_{01}, \qquad \delta_{\mathrm{hist}}(\mathcal{A}_2) \subseteq \mathcal{A}_{12}.$$

These are first-commutator finite-support statements, not a continuum finite-speed theorem. The single source functional $\omega(X) = \operatorname{Tr}(\rho_{\mathrm{hist}} X)$ restricts to compatible states on every displayed algebra: for each inclusion $\mathcal{A}_U \subseteq \mathcal{A}_V$, $\omega_V|_{\mathcal{A}_U} = \omega_U$. The injective selected-GNS representation transports the endpoint locality, the two bond memberships, the Hamiltonian decomposition, and the generator-support statements without collapse. This compatible restriction is the state compatibility required

of the finite net; it does not assert a source-state-preserving conditional expectation from the full algebra onto every region.

This is a source-counted finite three-history-slot local dynamical system on its selected GNS Hilbert space, not a physical QFT. The retained run was hash-pinned, but the empirical law and repair-count postprocessing were not jointly preregistered. The packet fixes the diagonal density and the two repair-change contributions; the full $M_8(\mathbb{C})$ quantum extension, tensor-factor locality reading, off-diagonal coherence probe, one-regulator tower, and real flow parameter are mathematical adapters. The nonzero energy of Ω_{hist} prevents its promotion to a vacuum or ground vector. The three successive history coordinates are not calibrated physical times or Lorentzian regions, and there is no physical clock or energy calibration, spectrum condition, continuum or infinite-volume control, renormalization flow, field/particle interpretation, scattering, detector readback, or physical time-slice theorem.

The rational record layer carries a finite slot-indexed conditioning family. For every slot region $S \subseteq \{0,1,2\}$, conditioning the committed window law on the slots in S defines a map E_S on $\mathbb{Q}$-valued record observables that is rational-linear, unital, pointwise positive, localized, idempotent, and tower-compatible, $E_S \circ E_T = E_S$ for $S \subseteq T$. The full-region map is the identity because the three slots separate the eight histories, the empty-region map is the constant empirical mean, and every E_S preserves that mean. The domain walls are fixed points of their own interval maps, every regional map preserves the bond means 94/1754 and 103/1754 and the action mean 197/1754, and the late-slot conditional weights are 383/415 in the $(s_0, s_1) = (0,0)$ cell and 2/89 in the $(0,1)$ cell.

The matrix obstruction has a fixed target and hypotheses. No complex-linear map from $M_8(\mathbb{C})$ into this particular $\mathcal{A}_{01}$ can both satisfy the $\mathcal{A}_{01}$-bimodule law and preserve the committed state ω: such a map would assign the slot-two projector a single conditional weight over $\mathcal{A}_{01}$, whereas the two cells demand 383/415 and 2/89. This does not rule out ordinary AQFT nets of inclusions with state restriction, other or enlarged target algebras, ancillary boundary memory, generalized expectations, or non-bimodule positive or completely positive channels. The finite-probability conditioning identities and wall coefficients transport to represented diagonal elements on the selected GNS carrier (Lean module `SourceHistoryExpectationNet` [12]).

Boundary-fibre circularity control. Over the bare consensus tower (finite presentation and quotient states, a mismatch functional, a strictly descending step relation, an idempotent normal form, a protected boundary map and coarse maps), a signature family that factors through the protected boundary on consistent states and is complete on consistent states forces the boundary fibre: equal boundary values imply equal consistent states. Instantiating the signature as the protected boundary itself makes the factoring hypothesis trivial and makes the completeness hypothesis definitionally identical to the conclusion; the degeneration is recorded as a definitional equivalence, so the circularity is itself a theorem, and the result is substantive only for a signature constructed independently of the boundary and proved complete on its own grounds. A selected settled branch (one normal-form-fixed, coherent quotient point per regulator) is an explicit premise with a concrete inhabitant and an explicit failing candidate, and an event and geometry readout fragment carries normal-form-invariant, coarsely natural packet readouts that separate two consistent states on the same concrete tower [12].

9 Distributed Presentations of One Finite Universe

The implementation closure clause in Definition 5.21 has a distributed form. A worker partition is allowed to accelerate or package the computation, but it is not a new physical ingredient. The

physical question is whether the worker run presents the same finite quotient repair system as the monolithic carrier.

Definition 9.1 (Finite global OPH carrier)**.** *At refinement stage r, a finite global OPH carrier consists of a finite patch graph*

$$G_r = (V_r, E_r),$$

finite patch state sets S_i, interface alphabets I_e, endpoint maps $\pi_{i,e} : S_i \to I_e$, an implementation-hiding group Γ_r, a boundary/sector map B_r, a mismatch potential Φ_r, accepted quotient repairs $\to_r$, and an observer-readable map

$$\mathrm{Read}_r : Q_r \to \mathcal{Y}_r, \qquad Q_r := \left(\prod_{i \in V_r} S_i \right) / \Gamma_r.$$

Let $C_r \subseteq Q_r$ be the quotient consistency set. On the finite branch covered by Theorems 3.25 and 5.2, the accepted quotient repair relation has a unique normal-form map

$$n_r : Q_r \to C_r.$$

A one-universe input is the pair $\mathbf{U}_r = (\mathfrak{U}_r, q_0)$, where $\mathfrak{U}_r$ is the carrier data above and $q_0 \in Q_r$. A worker partition is not part of $\mathfrak{U}_r$.

Definition 9.2 (Worker presentation and physical projection)**.** *Let $\kappa : V_r \to W$ assign every patch to one authoritative worker. The cut set is*

$$E_\kappa^{\mathrm{cut}} := \{\{i, j\} \in E_r : \kappa(i) \neq \kappa(j)\}.$$

A distributed presentation for κ has worker-owned states, ghost or halo copies, message queues, transaction records, retries, worker identifiers, checkpoints, event logs, and scheduler state. Let $\widetilde{Q}_{r,\kappa}$ be its quotient by purely local worker bookkeeping that leaves authoritative physical states unchanged. The physical projection

$$p_{r,\kappa} : \widetilde{Q}_{r,\kappa} \to Q_r$$

keeps only the authoritative patch states, quotients by Γ_r, and discards queues, retry metadata, worker labels, and external clocks. Let

$$D_r := \bigsqcup_\kappa \{\kappa\} \times \widetilde{Q}_{r,\kappa}, \qquad P_r(\kappa, \tilde{q}) := p_{r,\kappa}(\tilde{q}).$$

A distributed readout is physical when it factors as

$$\widetilde{\mathrm{Read}}_r = \mathrm{Read}_r \circ P_r.$$

Definition 9.3 (Admissible distributed event)**.** *For $d, d' \in D_r$, a distributed event $d \Rightarrow d'$ is **admissible** when one of the following holds.*

(a) ***Linearizable physical commit.*** *There is a nonempty monolithic accepted repair path*

$$P_r(d) \to_r^* P_r(d')$$

whose repair word is recorded as the event's linearization witness.

(b) ***Physical stutter.*** *$P_r(d') = P_r(d)$. Message delivery, halo refresh, prepare, abort, checkpoint, worker start or stop, idempotent replay, and repartition metadata are allowed only in this class unless they also carry a physical commit witness.*

(c) ***Certified rollback.*** *There is an earlier committed state d_j in the same run history such that $P_r(d') = P_r(d_j)$, and the rollback certificate names that committed projection root. Transparent restart from a committed frontier is a stutter.*

Proposition 9.4 (Normal form is constant on an accepted reachability cone)**.** *If $q \to_r^* q'$, then*

$$n_r(q') = n_r(q).$$

Proof. The state q' is reachable from q. The state $n_r(q')$ is a normal form reachable from q', hence also reachable from q. Theorem 3.25 and Theorem 5.2 give the unique quotient normal form reachable from q, so $n_r(q') = n_r(q)$. □

Theorem 9.5 (Distributed realization of one finite OPH universe)**.** *Fix a finite global carrier $\mathbf{U}_r = (\mathfrak{U}_r, q_0)$ satisfying Theorems 3.25 and 5.2. Let*

$$d_0 \Rightarrow d_1 \Rightarrow \cdots \Rightarrow d_m$$

be a finite distributed execution in D_r with $P_r(d_0) = q_0$, and assume every event is admissible in the sense of Definition 9.3. Then, for every t,

$$q_0 \to_r^* P_r(d_t), \qquad n_r(P_r(d_t)) = n_r(q_0).$$

If the final projection is a monolithic normal form, then

$$P_r(d_m) = n_r(q_0).$$

For every physical observable or observer readout $M : Q_r \to Y$,

$$M(P_r(d_m)) = M(n_r(q_0))$$

whenever $P_r(d_m)$ is normal. In particular, $\widetilde{\mathrm{Read}}_r(d_m) = \mathrm{Read}_r(n_r(q_0))$ for every distributed readout that factors through P_r.

Proof. Induct on t. The claim is true at $t = 0$. Suppose it holds at t. For a linearizable physical commit, admissibility gives

$$P_r(d_t) \to_r^* P_r(d_{t+1}),$$

so $q_0 \to_r^* P_r(d_{t+1})$, and Lemma 9.4 gives

$$n_r(P_r(d_{t+1})) = n_r(P_r(d_t)) = n_r(q_0).$$

For a physical stutter, $P_r(d_{t+1}) = P_r(d_t)$, so both statements are unchanged. For a certified rollback, $P_r(d_{t+1})$ is the projection of an earlier committed state; the induction hypothesis gives reachability from q_0 and equality of normal forms for that projection. This proves the two displayed identities for every t.

If $P_r(d_m)$ is a monolithic normal form, then it is the unique quotient normal form reachable from q_0, hence $P_r(d_m) = n_r(q_0)$. The observable and readout statements follow by applying M or Read_r to this equality and using $\widetilde{\mathrm{Read}}_r = \mathrm{Read}_r \circ P_r$. □

Corollary 9.6 (Partition, schedule, and restart invariance)**.** *Any two admissible distributed executions of the same finite carrier* $\mathbf{U}_r$*, with the same initial quotient state* q_0*, have the same terminal quotient observables and observer-readable outputs once their final projections are monolithic normal forms. The statement is independent of partition* κ*, worker count, scheduler choices, restart history, and repartition metadata.*

Proof. Apply Theorem 9.5 to each execution. Both final projections equal $n_r(q_0)$, so every quotient observable and every readout factoring through the projection has the same value. □

Proposition 9.7 (Restart stabilization separates safety from liveness)**.** *Assume* Q_r *is finite, every physical commit strictly descends the accepted potential until normality, every rollback returns to an earlier committed projection, only finitely many progress-erasing rollbacks occur, and after the last such rollback the scheduler is fair enough to commit an enabled repair whenever the projected state is not normal. Then the distributed run reaches a monolithic normal form after finitely many physical commits.*

Proof. Safety is Theorem 9.5. For liveness, ignore stutters and the finitely many progress-erasing rollbacks before the last one. After that time, each nonnormal projected state takes a strict descending accepted repair after finitely many scheduler steps. The set of possible potential values below the post-rollback value is finite, so strict descent can occur only finitely many times before a normal projected state is reached. □

Theorem 9.8 (Distributed refinement cube)**.** *Let* $r \preceq s$*. Suppose* $\rho_{sr} : Q_s \to Q_r$ *is a repair morphism, the corresponding holonomy maps form a cochain morphism, and distributed presentations at both stages are exact in the sense of Theorem 9.5. If the lifted restriction* $\widetilde{\rho}_{sr} : D_s \to D_r$ *commutes with physical projection,*

$$P_r \circ \widetilde{\rho}_{sr} = \rho_{sr} \circ P_s,$$

then distributed execution and coarse restriction commute on normal forms and readouts:

$$P_r(\widetilde{\rho}_{sr}(d_m^s)) = n_r(\rho_{sr}(q_0^s)) = \rho_{sr}(n_s(q_0^s))$$

whenever the fine final projection is normal. The same statement holds for holonomy readouts via χ_{sr}*.*

Proof. The fine distributed theorem gives $P_s(d_m^s) = n_s(q_0^s)$. Projection compatibility gives

$$P_r(\widetilde{\rho}_{sr}(d_m^s)) = \rho_{sr}(n_s(q_0^s)).$$

Theorem 6.5 identifies this value with $n_r(\rho_{sr}(q_0^s))$. The holonomy statement is exactly the cochain-morphism identity of Definition 6.6. □

Certificate contract. A run that claims Theorem 9.5 must let a verifier reconstruct its premises without trusting worker narration: the global carrier and its hash, the monolithic graph, the initial quotient state, the partition map, cut-interface records with restriction maps, a global observer registry, a committed event log with a linearization witness for each physical commit, stutter and rollback records naming committed projection roots, and a final monolithic normal form with its recomputed readout. Missing, stale, shard-local or synthetic replacements for these fields fail closed.

9.1 Quotient chart transport and a neutral readout

The distributed certificate above proves that a worker presentation is one finite quotient system. A separate step is needed before observer rows from different shards may be read as a common neutral geometry.

Definition 9.9 (Common quotient chart atlas)**.** *For a shard s, let Σ_s be raw records, let Γ_s be the groupoid of declared presentation-only moves such as gauge representative changes and local port relabelings, and set*

$$\overline{Q}_s := \Sigma_s/\Gamma_s, \qquad X_s := n_s(\overline{Q}_s),$$

where n_s is the schedule-independent normal-form map. An interface atlas is a family of domains $U_{st} \subseteq X_s$, $U_{ts} \subseteq X_t$, and bijections

$$\tau_{ts} : U_{st} \to U_{ts}$$

satisfying identity, inverse, and cocycle laws, with zero closed-path holonomy on graph-shaped systems. A channel registry assigns each channel c a metric space (Z_c, d_c), weight $w_c > 0$, and local features $F_{s,c} : D_{s,c} \subseteq X_s \to Z_c$. The channel descends through the atlas when both domain membership and values are transported:

$$x \in D_{s,c} \iff \tau_{ts}x \in D_{t,c}, \qquad F_{t,c}(\tau_{ts}x) = F_{s,c}(x).$$

Theorem 9.10 (Quotient-visible neutral readout)**.** *Under Definition 9.9, the relation generated by interface transport on $\bigsqcup_s X_s$ is an equivalence relation and defines*

$$Q_{\mathrm{vis}} := \left(\bigsqcup_s X_s\right) / \sim_\tau .$$

The canonical maps $X_s \to Q_{\mathrm{vis}}$ agree with transport, and zero closed-path holonomy makes them injective. Every compatible family of local channel maps descends uniquely to partial maps $F_c : Q_{\mathrm{vis}} \dashrightarrow Z_c$. Therefore, on the complete channel domain

$$Q_\star = \{x : F_c(x) \text{ exists for every } c \in \mathcal{C}_\star\},$$

fix $p \geq 1$ and assume that $\mathcal{C}_\star$ is finite. More generally, an infinite declared channel set is admitted only if

$$\sum_{c \in \mathcal{C}_\star} w_c \, d_c(F_c(x), F_c(y))^p < \infty \qquad \text{for every } x, y \in Q_\star.$$

The product formula

$$d_{\mathrm{neu}}(x, y) = \left(\sum_{c \in \mathcal{C}_\star} w_c \, d_c(F_c(x), F_c(y))^p\right)^{1/p}$$

is then a finite-valued pseudometric. It is a metric only after quotienting by zero-distance feature collisions, or after proving that the declared channel family separates points.

Proof. Identity, inverse, and cocycle laws give reflexivity, symmetry, and transitivity. The transport compatibility of local features is exactly the universal-property condition for descent through the quotient. The weighted product distance is independent of the representative and finite by the finite-channel or pairwise-summability premise. Its triangle inequality is Minkowski's inequality applied to the channel metrics. The only possible failure of identity of indiscernibles is equality of all declared feature values, which is removed by quotienting feature-collision classes or by a joint-separation proof. □

Pairwise available-channel comparison does not give a metric: different pairs can share different channels and violate the triangle inequality. Presentation invariance requires a bijection of Q_{vis} together with channel isometries, and Euclidean claims require the double-centered Gram test $B = -\frac{1}{2}H(D^{\circ 2})H \succeq 0$. The section certifies a common quotient metric or pseudometric only; a Riemannian or Lorentzian identification belongs to the spacetime layer.

10 Noisy Fair-Block Approximate Consensus

The exact consensus theorem supplies the quotient normal-form target. A noisy implementation needs one more quantitative certificate: complete fair blocks must contract expected distance to that target. This section records the conditional bridge from local noisy repair to long-run observer-facing approximate consensus.

Theorem 10.1 (Global noisy approximate consensus under fair-block contraction)**.** *Let (Q, d_Q) be the observer-facing quotient state space of a fixed exported OPH patch federation, and let $\mathcal{N} \subset Q$ be the exact quotient normal-form set supplied by the finite repair theorem on that quotient. Define*

$$D(q) := d_Q(q, \mathcal{N}) = \inf_{n \in \mathcal{N}} d_Q(q, n).$$

Let $q_{t+1} = \widetilde{T}_{i_t,t}(q_t)$ be a noisy asynchronous repair process adapted to a filtration $(\mathcal{F}_t)_t$. Assume:

(G1) ***Exact quotient target.*** *The ideal repair relation on Q has the exact OPH normal-form package: finite exact descent, the semantic-complete transactional local-diamond theorem with its concrete premise receipt, and repair completeness, so every fixed initial quotient state has a schedule-independent exact normal form in $\mathcal{N}$.*

(G2) ***Fair asynchronous blocks.*** *There are stopping times*

$$0 = \tau_0 < \tau_1 < \tau_2 < \cdots$$

such that each interval $[\tau_m, \tau_{m+1})$ contains enough local repairs to expose and act on every active mismatch class required by the exact transaction/confluence and repair-completeness certificate, with uniformly bounded length $\tau_{m+1} - \tau_m \le L$. Write the noisy block map as

$$\widetilde{B}_m := \widetilde{T}_{i_{\tau_{m+1}-1},\, \tau_{m+1}-1} \circ \cdots \circ \widetilde{T}_{i_{\tau_m},\, \tau_m}.$$

(G3) ***Uniform block contraction toward normal form.*** *There are constants $0 < \lambda < 1$ and $\varepsilon \ge 0$ such that, for every block m and every reachable quotient state q at time τ_m,*

$$\mathbb{E}\Big[D\Big(\widetilde{B}_m(q)\Big) \mid \mathcal{F}_{\tau_m}\Big] \le \lambda D(q) + \varepsilon.$$

(G4) ***Controlled within-block excursions.*** *There are constants $A \ge 1$ and $\beta \ge 0$ such that, for every $t \in [\tau_m, \tau_{m+1})$,*

$$\mathbb{E}[D(q_t) \mid \mathcal{F}_{\tau_m}] \le A\, D(q_{\tau_m}) + \beta.$$

Then, at fair-block times,

$$\mathbb{E}[D(q_{\tau_m})] \le \lambda^m D(q_0) + \frac{1 - \lambda^m}{1 - \lambda}\, \varepsilon,$$

and hence

$$\limsup_{m \to \infty} \mathbb{E}[D(q_{\tau_m})] \le \frac{\varepsilon}{1 - \lambda}.$$

If the block endpoints τ_m are deterministic, then at all intermediate asynchronous times,

$$\limsup_{t\to\infty} \mathbb{E}[D(q_t)] \le A\,\frac{\varepsilon}{1-\lambda} + \beta.$$

Thus the process has a controlled block-indexed expected tube, and the same certificate gives the displayed calendar-time tube on the deterministic-block branch. On that branch, if $\varepsilon = \beta = 0$, then $D(q_t) \to 0$ in L^1, hence also in probability, along the full asynchronous run.

Proof. Let $D_m := D(q_{\tau_m})$. Assumption (G3), applied to the realized state q_{τ_m}, gives

$$\mathbb{E}[D_{m+1} \mid \mathcal{F}_{\tau_m}] \le \lambda D_m + \varepsilon.$$

Taking expectations,

$$\mathbb{E}[D_{m+1}] \le \lambda\,\mathbb{E}[D_m] + \varepsilon.$$

Iterating the scalar recursion gives

$$\mathbb{E}[D_m] \le \lambda^m D(q_0) + \varepsilon \sum_{r=0}^{m-1} \lambda^r = \lambda^m D(q_0) + \frac{1-\lambda^m}{1-\lambda}\varepsilon.$$

Taking $m \to \infty$ yields the fair-block limsup bound, including when the endpoints are stopping times. For the remaining statements assume that the endpoints are deterministic. Then the index $m = m(t)$ containing a fixed calendar time t is deterministic, and Assumption (G4) gives

$$\mathbb{E}[D(q_t)] \le A\,\mathbb{E}[D(q_{\tau_m})] + \beta.$$

Substitution of the fair-block estimate and then taking the limsup over all intermediate times gives

$$\limsup_{t\to\infty} \mathbb{E}[D(q_t)] \le A\,\frac{\varepsilon}{1-\lambda} + \beta.$$

If $\varepsilon = \beta = 0$, then $\mathbb{E}[D(q_{\tau_m})] \le \lambda^m D(q_0) \to 0$. Since $D \ge 0$, convergence in expectation to zero implies convergence in probability at block times. Assumption (G4) then gives $\mathbb{E}[D(q_t)] \le A\,\mathbb{E}[D(q_{\tau_m})]$ inside each block, so the same L^1 and probability convergence holds along the full asynchronous run. □

Remark 10.2 (Stopping-time blocks can bias a calendar-time readout). The deterministic-endpoint condition cannot be omitted from the displayed calendar-time constant. For an exact finite counterexample, let the block-endpoint distance be a two-state chain $D_m \in \{0, 1\}$, with

$$\Pr(D_{m+1} = 1 \mid D_m = 0) = \tfrac{1}{4}, \qquad \Pr(D_{m+1} = 1 \mid D_m = 1) = \tfrac{3}{4}.$$

It obeys (G3) with $\lambda = 1/2$, $\varepsilon = 1/4$, and its stationary mean is $1/2$. Keep the distance constant within each block, assigning length one to a zero block and length two to a unit block. Thus $L = 2$, $A = 1$, and $\beta = 0$, even pathwise inside each block. The calendar-time chain has three states $0, 1_{\text{first}}, 1_{\text{second}}$, with transition matrix

$$\begin{pmatrix} 3/4 & 1/4 & 0 \\ 0 & 0 & 1 \\ 1/4 & 3/4 & 0 \end{pmatrix}.$$

Its unique stationary law is uniform, so its limiting expected distance is $2/3 > \varepsilon/(1-\lambda) = 1/2$. Sampling the block containing a fixed time favors longer blocks. A random containing-block index therefore cannot be substituted into the estimate for a fixed block index. Random block endpoints need their own occupation-time or conditional sampling bound to obtain a calendar-time constant.

Corollary 10.3 (Approximate observer-facing schedule independence)**.** *Let $M : Q \to Y$ be an observer-facing readout into a metric space (Y, d_Y), and suppose M is L_M-Lipschitz. Fix an exact sector ζ whose exact normal-form set is the singleton $\mathcal{N}_\zeta = \{n_\zeta\}$, and apply Theorem 10.1 on that sector, so that $D(q) = d_Q(q, n_\zeta)$, with deterministic block endpoints. Then*

$$\limsup_{t\to\infty} \mathbb{E}[d_Y(M(q_t), M(n_\zeta))] \le L_M \left(A \frac{\varepsilon}{1-\lambda} + \beta \right).$$

For two noisy asynchronous schedules q_t and q'_t started in the same exact singleton sector and satisfying the same certificate,

$$\limsup_{t\to\infty} \mathbb{E}[d_Y(M(q_t), M(q'_t))] \le 2L_M \left(A \frac{\varepsilon}{1-\lambda} + \beta \right).$$

Proof. The Lipschitz condition gives

$$d_Y(M(q_t), M(n_\zeta)) \le L_M d_Q(q_t, n_\zeta) = L_M D(q_t).$$

The first bound follows from Theorem 10.1. The two-schedule bound follows from the triangle inequality through $M(n_\zeta)$ and applying the first estimate to both schedules. □

Corollary 10.4 (High-probability noisy consensus tube)**.** *Assume the block-distance process also admits the pathwise decomposition*

$$D_{m+1} \le \lambda D_m + \varepsilon + \xi_{m+1}, \qquad \mathbb{E}[\xi_{m+1} \mid \mathcal{F}_{\tau_m}] = 0, \qquad |\xi_{m+1}| \le b$$

almost surely. Then, for every $a > 0$,

$$\Pr\left[D_m > \lambda^m D_0 + \frac{1-\lambda^m}{1-\lambda}\varepsilon + a \right] \le \exp\left(-\frac{a^2(1-\lambda^2)}{2b^2} \right).$$

Proof. Unrolling the recursion gives

$$D_m \le \lambda^m D_0 + \frac{1-\lambda^m}{1-\lambda}\varepsilon + \sum_{r=1}^{m} \lambda^{m-r}\xi_r.$$

If $b = 0$, the noise vanishes almost surely and the displayed exceedance probability is zero; the exponential bound is understood in that limiting sense. Suppose henceforth that $b > 0$. The final term is a weighted martingale sum with increments bounded by $|\lambda^{m-r}\xi_r| \le \lambda^{m-r} b$. Azuma–Hoeffding gives

$$\Pr\left[\sum_{r=1}^{m} \lambda^{m-r}\xi_r > a \right] \le \exp\left(-\frac{a^2}{2\sum_{r=1}^{m} \lambda^{2(m-r)} b^2} \right).$$

Since $\sum_{r=1}^{m} \lambda^{2(m-r)} \le (1-\lambda^2)^{-1}$, substitution yields the claimed bound. □

Proposition 10.5 (Finite fair-block contraction certificate)**.** *Let Q be finite, let $\mathfrak{B}_{\mathrm{fair}}$ be a finite list of noisy fair-block types, and let $K_B(q, q')$ be the Markov kernel induced by block type B. Require that the type B_m is chosen at block start and is $\mathcal{F}_{\tau_m}$-measurable, and that the implemented block has the certified conditional law*

$$\Pr(q_{\tau_{m+1}} = q' \mid \mathcal{F}_{\tau_m}) = K_{B_m}(q_{\tau_m}, q') \quad \textit{almost surely.}$$

Randomized choices may be included in the block-start filtration, provided this conditional-law identity remains valid. If there are constants $0 < \lambda < 1$ *and* $\varepsilon \geq 0$ *such that*

$$\sum_{q' \in Q} K_B(q, q') D(q') \leq \lambda D(q) + \varepsilon \qquad \forall q \in Q, \quad \forall B \in \mathfrak{B}_{\mathrm{fair}},$$

then Assumption (G3) of Theorem 10.1 holds for any run whose fair blocks satisfy this implementation contract.

Proof. By the certified conditional-law identity, conditioning on the entire block-start filtration (including the chosen type B), the conditional expectation of D after the block is exactly $\sum_{q'} K_B(q, q') D(q')$. The displayed inequality is therefore Assumption (G3), uniformly over all reachable states and fair-block types. □

Selecting a block type using its as-yet-unobserved output does not meet this contract. For example, for one fair bit U, the two maps with outputs U and $1 - U$ each have expected distance $1/2$. Choosing the map whose output is one gives expected distance one. Their marginal kernels therefore do not certify that selected implementation. Adaptation inside a block is permitted only when its complete adapted law is the certified kernel.

Finite certificate route and constants. For a finite exported packet net, Proposition 10.5 gives the route from the finite quotient Q and its exact normal forms $\mathcal{N}$ to a distance table $D(q)$, noisy fair-block kernels K_B and a (λ, ε) certificate. Here λ is the net contraction of one completed fair block, ε the per-block irreducible recovery, record, readout, calibration and environmental noise, A and β bound within-block expansion and noise, and L is the fairness horizon. In collar implementations ε may absorb Petz-domain truncation, Fawzi–Renner recovery error, approximate central-record error, detector noise and coarse-graining defect. The theorem does not follow from the exact finite repair theorem alone, which supplies only the target $\mathcal{N}$.

11 Interfaces to the higher layers

The layer supplies five typed outputs, each consumed under the receiving layer's own premises.

1. **Public normal form and records.** The quotient normal form, its boundary data and the central record algebra of Theorem 7.2 are the public world that the observer layer reads, retains and acts on.

2. **Holonomy residue.** Cycle and crossed-module holonomy classes, with their refinement-limit classes, are the invariants on which the matter hypothesis builds defects (Section 4.1).

3. **Implementation hiding.** The quotient by changes that preserve overlap data, with its fixed-cutoff edge sectors, is the input of the gauge layer (Section 5.2).

4. **Informational order.** Authenticated provenance generates a finite order and its source height. The spacetime layer attaches causal, metric and clock meaning to it only through placement, count–volume, dimension and calibration premises [32].

5. **Refinement and implementation control.** Natural restriction systems, controlled coarse-graining squares, distributed presentations and fair-block certificates relate finite stages and implementations to one quotient object.

Approximate control is collar-local: Theorem A.1 gives the exact-Markov modulus and the one-shot recovery bound, and Theorem 7.2 the record-stability bound. Long-run noisy control requires a fair-block certificate; Ref. [10] adds a receipt-only endpoint estimate from residual and inverse-observation moduli, which supplies no fair-block contraction. The protocol and code extensions of Appendix C are separate from the core theorems: a Byzantine-agreement reading needs its certificate, quorum and timing contract, and a quantum-code reading needs a code subspace, error family, recovery map and, for distance claims, a topological-code certificate.

A Quantum/Algebraic Lift: Markov-Collar Splice Theorem

This appendix records the algebraic splice statement relating the finite patch-net model to the OPH collar formalism.

For this collar lemma, write the support-local algebra-state-record reduct of an observer patch as

$$O_{\mathrm{red}} = (P, \mathcal{A}(P), \rho, R),$$

where P is the support-screen patch, $\mathcal{A}(P)$ the local von Neumann algebra, ρ the local state, and R the record algebra. The full operational observer also carries overlap interface algebras and restriction maps, allowed update and repair instruments, and checkpoint data used for continuation.

Theorem A.1 (Markov-collar splice theorem, exact and controlled)**.** *Suppose a collar tripartition A-B-D has the EC-aligned exact Markov decomposition*

$$\rho_{ABD} = \bigoplus_{\alpha} p_\alpha \, \rho^{(\alpha)}_{Ab^\alpha_L} \otimes \rho^{(\alpha)}_{b^\alpha_R D}$$

over the preselected edge factors. This displayed form is the hypothesis: by the spacetime and Einstein paper's Markov-split alignment analysis it is strictly stronger than $I(A : D \mid B)_\rho = 0$, which by HJPW yields such a factorization only over a state-dependent split of B. Let $\sigma^{(\alpha)}_{b^\alpha_R D'}$ be any family of normalized environment states compatible with the same right-boundary sectors. Define

$$\rho'_{ABD'} = \bigoplus_{\alpha} p_\alpha \, \rho^{(\alpha)}_{Ab^\alpha_L} \otimes \sigma^{(\alpha)}_{b^\alpha_R D'}.$$

Then for every observable X supported on $A \cup b_L$,

$$\mathrm{Tr}(X \rho'_{ABD'}) = \mathrm{Tr}(X \rho_{ABD}).$$

Fix one finite-dimensional collar model and let

$$\mathfrak{M}_{A:B:D} := \{\tau_{ABD} : I(A : D \mid B)_\tau = 0\},$$

with exact-Markov distance modulus

$$\delta^{\mathrm{M}}_{A:B:D}(\varepsilon) := \sup \left\{ \inf_{\tau \in \mathfrak{M}_{A:B:D}} \|\omega - \tau\|_1 : I(A : D \mid B)_\omega \le \varepsilon \right\}.$$

Then

$$\delta^{\mathrm{M}}_{A:B:D}(\varepsilon) \to 0 \qquad (\varepsilon \downarrow 0).$$

On a fixed faithful collar class with lower floor $\lambda_* > 0$, this qualitative modulus can be sharpened to a collar-local rate

$$\delta^{\mathrm{M},\lambda_*}_{A:B:D}(\varepsilon) \le C_{A:B:D,\lambda_*} \, \varepsilon^{\theta_{A:B:D,\lambda_*}},$$

by the compact real-analytic Lojasiewicz inequality applied to $I(A:D\mid B)$. The constants depend on the fixed collar model and floor; they are not a dimension-free stability theorem for arbitrary tripartite systems. Let $\mathfrak{M}^{\mathrm{EC}}_{A:B:D}\subset\mathfrak{M}_{A:B:D}$ be the states that also satisfy the Markov-split alignment hypothesis on the preselected edge factors. For a declared controlled family $\mathcal{F}$, define

$$\delta^{\mathrm{M,EC}}_{\mathcal{F};A:B:D}(\varepsilon):=\sup\left\{\inf_{\tau\in\mathfrak{M}^{\mathrm{EC}}_{A:B:D}}\|\omega-\tau\|_1:\omega\in\mathcal{F},\ I(A:D\mid B)_\omega\le\varepsilon\right\}.$$

Convergence of the generic exact-Markov modulus does not imply convergence of this aligned modulus. Hence, if $\omega\in\mathcal{F}$, $I(A:D\mid B)_\omega\le\varepsilon$, and the controlled family separately satisfies $\delta^{\mathrm{M,EC}}_{\mathcal{F};A:B:D}(\varepsilon)\to 0$, choose $\widetilde{\omega}_\varepsilon\in\mathfrak{M}^{\mathrm{EC}}_{A:B:D}$ so that

$$\|\omega-\widetilde{\omega}_\varepsilon\|_1\le\delta^{\mathrm{M,EC}}_{\mathcal{F};A:B:D}(\varepsilon),$$

the corresponding exact splice $\widetilde{\omega}'_\varepsilon$ satisfies

$$\left|\mathrm{Tr}(X\omega)-\mathrm{Tr}(X\widetilde{\omega}'_\varepsilon)\right|\le\|X\|_\infty\,\delta^{\mathrm{M,EC}}_{\mathcal{F};A:B:D}(\varepsilon)$$

for every observable X supported on $A\cup b_L$.

Independently, if $I(A:D\mid B)_\omega\le\varepsilon$, then there exists a recovery map $\mathcal{R}_{B\to BD}$ such that

$$\|\omega_{ABD}-(\mathrm{id}_A\otimes\mathcal{R}_{B\to BD})(\omega_{AB})\|_1\le 2\sqrt{1-e^{-\varepsilon}}\le 2\sqrt{\varepsilon}.$$

Proof. The exact splice statement is the usual blockwise factorization argument:

$$\mathrm{Tr}(X\rho'_{ABD'})=\sum_\alpha p_\alpha\ \mathrm{Tr}\Big(X\,\rho^{(\alpha)}_{Ab^\alpha_L}\Big)\,\mathrm{Tr}\Big(\sigma^{(\alpha)}_{b^\alpha_R D'}\Big).$$

Each right factor is normalized, so the value agrees with the same computation for ρ_{ABD}.

For the controlled statement, compactness of the fixed finite-dimensional state space and continuity of conditional mutual information imply $\delta^{\mathrm{M}}_{A:B:D}(\varepsilon)\to 0$: otherwise one could find a sequence with $I(A:D\mid B)\to 0$ staying a fixed trace distance away from every exact Markov state, contradicting convergence of a subsequence to an exact Markov limit point. This generic fact does not prove the separately assumed aligned-modulus convergence: the splice identity requires $\widetilde{\omega}_\varepsilon$ to be EC-aligned over the preselected edge factors, while small conditional mutual information supplies closeness only to the full exact Markov set. Once such an EC-aligned $\widetilde{\omega}_\varepsilon$ is chosen with the declared family-specific bound, the exact splice identity gives

$$\left|\mathrm{Tr}(X\omega)-\mathrm{Tr}(X\widetilde{\omega}'_\varepsilon)\right|=\left|\mathrm{Tr}[X(\omega-\widetilde{\omega}_\varepsilon)]\right|\le\|X\|_\infty\,\delta^{\mathrm{M,EC}}_{\mathcal{F};A:B:D}(\varepsilon).$$

The final inequality is trace/operator-norm duality applied to the separately supplied EC-aligned modulus. It is not the Fawzi–Renner recovery bound [31], which guarantees a recovered comparison state rather than this EC-aligned exact Markov state. □

Exact splice identities are used in two regimes only: literal EC-aligned exact Markovity, or a controlled collar family on one fixed finite-dimensional model with $\delta^{\mathrm{M,EC}}_{\mathcal{F};A:B:D}(\varepsilon)\to 0$. Fawzi–Renner recovery supplies a recovered comparison state, and the generic fixed-collar modulus supplies an exact-Markov comparison on a possibly different split.

For a faithful finite state ρ_{ABD} with $K_X=-\log\rho_X$, the exact finite identity

$$\mathrm{Tr}\,\rho_{ABD}(K_{AB}+K_{BD}-K_B-K_{ABD})=I_\rho(A:D\mid B)$$

is an expectation identity only; it does not make conditional mutual information a state-independent local stress source. Near a full-rank exact Markov state σ, a perturbation $\rho(t)=\sigma+tX+O(t^2)$ has $\frac{\mathrm{d}}{\mathrm{d}t}I_{\rho(t)}(A:D\mid B)|_{t=0}=0$, while a fixed-reference modular-energy variation is generically linear in t. Recovery bounds therefore prove no local stress-source theorem.

B Fixed-Cutoff Realization, Quotient Repair, and Edge Centers

This appendix makes the fixed-cutoff realization and the collar boundary data explicit at the finite patch-net level. On the collar branch the local repair step is read from exact Markov splice or a declared Petz/Fawzi–Renner recovery move; the recovery move gives a recovered comparison state, while exact splice requires exact Markovity or a controlled fixed-collar replacement modulus. Representative repair maps are lifts of the resulting quotient-local update.

B.1 Quotient Repair and UV Underdetermination

At fixed cutoff, each regulator cell x carries a finite-dimensional factor $\mathfrak{h}_x$, patch algebras are finite type-I algebras, and gauge-as-gluing is realized as a compact boundary redundancy action on cut data. The physical repair law therefore belongs on the overlap-invariant quotient rather than on hidden representatives. If $q : \Sigma \to \Sigma/\Gamma$ is the quotient by boundary redundancy and $\overline{T}_i$ is the physical quotient update, a representative-level map T_i is only required to be a lift satisfying

$$q \circ T_i = \overline{T}_i \circ q.$$

Hence $q(T_i(\gamma \cdot s)) = q(T_i(s))$ for gauge-equivalent inputs: quotient descent is structural, and representative-level covariance is bookkeeping. The accepted recovery-derived moves must satisfy the repair-completeness, support-local disjoint-commutation, nested-collar restriction-compatibility and Petz-domain clauses on the declared branch.

Proposition B.1 (Ancilla-stable UV underdetermination)**.** *Let a fixed-cutoff OPH realization be stabilized by finite ancillary factors K_P in a fixed product state, with observable patch algebras embedded as $\mathcal{A}(P) \otimes \mathbf{1}_{K_P}$ and repair dynamics acting trivially on the ancillas. Then observable expectations on the physical subalgebras, overlap data, the local-Gibbs branch, the collar conditional mutual information $I(A : D \mid B)$, the Fawzi–Renner remainder, the collar Markov modulus, and the quotient normal form are unchanged. Thus the fixed-cutoff theorem package determines the UV branch only modulo such ancillary stabilization together with gauge or implementation hiding, not a unique microscopic presentation.*

Proof. Product ancillas leave physical observables unchanged, cancel additively inside conditional mutual information, and are inert under the repair maps. Hence every invariant listed above is unchanged. □

B.2 Derived Boundary Data and Ordinary EC

Proposition B.2 (Derived boundary gluing datum)**.** *Choose a finite regulator chart for the patches meeting along a connected cut Σ. Because the local overlap algebras are finite-dimensional matrix algebras, any overlap-consistent recharting is an inner automorphism and is implemented by a unitary on the cut Hilbert space. The compact closure of the subgroup generated by these recharting unitaries is a compact boundary redundancy group K_Σ. If triple-overlap defects are central, the projective composition law lifts to a compact central extension $\widehat{K_\Sigma}$; on the ordinary branch one simply sets $\widehat{K_\Sigma} = K_\Sigma$. A genuinely noncentral 2-group defect is the only obstruction to reducing the overlap transition system to an ordinary compact group action.*

Theorem B.3 (Derived EC decomposition)**.** *Under the fixed-cutoff regulator realization above, and on the ordinary or central-defect branch, the collar Hilbert space is*

$$\mathcal{H}_{B_\delta} = (\tilde{\mathcal{H}}_{B_L} \otimes \tilde{\mathcal{H}}_{B_R})^{\widehat{K}_\Sigma} \cong \bigoplus_\alpha \left(\mathcal{H}_{b_L^\alpha} \otimes \mathcal{H}_{b_R^\alpha} \right),$$

and the center of the collar algebra is generated by the block projectors:

$$Z(\mathcal{A}(B_\delta)) = \bigoplus_\alpha \mathbb{C} \cdot \mathbf{1}_\alpha.$$

The right half-collar carries the contragredient representation because it sees inverse transport across the same cut.

This is the finite patch-net origin of the collar center used by the Markov, record and observer constructions. EC provides the kinematic block structure; exact Markovity is an additional state hypothesis.

B.3 Higher-Gauge Replacement on the Genuinely Noncentral Branch

Proposition B.4 (Derived higher-gauge cut datum)**.** *On the genuinely noncentral branch, weak overlap gluing on a connected cut Σ is encoded by a compact crossed module*

$$\mathbb{K}_\Sigma = (H_\Sigma \xrightarrow{\partial_\Sigma} G_\Sigma, \triangleright)$$

with defect class

$$q_\Sigma \in \check{H}^2(N_\Sigma, H_\Sigma \to G_\Sigma),$$

and compact higher-gauge change system

$$\mathcal{T}_\Sigma = C^1(N_\Sigma, H_\Sigma) \rtimes C^0(N_\Sigma, G_\Sigma).$$

Theorem B.5 (Higher-gauge EC decomposition and defect transport)**.** *On the genuinely noncentral branch,*

$$\mathcal{H}^{2g}_{B_\delta} = (\tilde{\mathcal{H}}_{B_L} \otimes \tilde{\mathcal{H}}_{B_R})^{\mathcal{T}_\Sigma} \cong \bigoplus_\lambda (\mathcal{H}_{b_L^\lambda} \otimes \mathcal{H}_{b_R^\lambda}),$$

and

$$Z(\mathcal{A}_{2g}(B_\delta)) = \bigoplus_\lambda \mathbb{C} \cdot \mathbf{1}_\lambda.$$

The full crossed-module orbit q_Σ is also invariant under local rechartings and classifies fixed-cutoff genuinely noncentral gluing data. The higher associator is removable iff q_Σ lies in the image of

$$\check{H}^1(N_\Sigma, G_\Sigma) \longrightarrow \check{H}^2(N_\Sigma, H_\Sigma \to G_\Sigma), \qquad [g] \longmapsto [(g, 1)].$$

That map need not be injective. Strict endpoint-only ordinary transport additionally requires at least one allowed strict representative with trivial represented loop holonomy.

Corollary B.6 (Exact Markov plus split alignment adds the state factorization)**.** *On either the ordinary/central branch described in the main consensus theorem or the genuinely noncentral higher-gauge branch, if in addition*

$$I_\omega(A_\delta : D_\delta \mid B_\delta) = 0$$

and *the state satisfies the Markov-split alignment hypothesis (its HJPW decomposition of $\mathcal{H}_{B_\delta}$ can be chosen to be the EC decomposition itself; see the spacetime and Einstein paper's Markov-split alignment definition), or one passes to the explicitly stated idealized recoverability limit that supplies both conditions, then*

$$\rho_{A_\delta B_\delta D_\delta} = \bigoplus_\alpha p_\alpha \left(\rho_{A_\delta b_L^\alpha} \otimes \rho_{b_R^\alpha D_\delta} \right).$$

EC therefore gives the kinematic block decomposition, while exact Markovity plus split alignment is the extra state input that gives the EC-aligned HJPW normal form. Exact Markovity alone yields the normal form only over a state-dependent HJPW split, which the spacetime and Einstein paper's Bell-pair example shows can be transposed relative to the EC factors.

Proposition B.7 (Certified parenthesization-invariant union-collar gluing)**.** *Fix a finite union collar U built from two overlapping local repair collars on the declared fixed-cutoff branch. Assume one of the following payload receipts:*

(i) *an exact aligned Markov construction, including the HJPW split-alignment condition, whose nested collar recovery maps satisfy the commuting-square and pentagon identities;*

(ii) *a specified canonical recovery construction, such as a fixed Petz/Markov splice, together with those commuting-square and pentagon identities; or*

(iii) *a primitive aggregate state ω_U whose restrictions to every constituent and nested collar equal the declared local payloads.*

On the ordinary or central-defect branch, assume in addition that changes of representative act through the declared boundary-redundancy action. On the genuinely noncentral branch, assume the corresponding coherence identities hold in the crossed-module change system $\mathcal{T}_\Sigma$. Then the physical quotient-local glued state on U is independent of parenthesization.

Proof. In cases (i) and (ii), the commuting-square identities identify every two-step restriction and the pentagon identity identifies all iterated parenthesizations. In case (iii), every parenthesization is a restriction of the same primitive state ω_U. The additional quotient clause removes only the declared boundary representative action or its crossed-module analogue, so all parenthesizations define one quotient-local state. □

Remark B.8 (Marginals and sectors do not determine the aggregate state)*.* Central sector labels and compatible proper marginals do not imply any of the three payload receipts. The uniform even-parity and odd-parity distributions on three bits have identical one- and two-bit marginals and different tripartite states. The states

$$|\mathrm{GHZ}_\pm\rangle = \frac{|000\rangle \pm |111\rangle}{\sqrt{2}}$$

give the quantum counterpart: all two-party reductions agree while the global states differ. Approximate recoverability therefore carries a coherence defect in addition to its local recovery error.

Corollary B.9 (Physical observables are invariant on one quotient-local glued state)**.** *Let U be a finite union collar on the declared fixed-cutoff branch, and let ω_U, ω'_U be two microscopic representatives of the same quotient-local glued state from Proposition B.7. Then every physical observable X on the collar fixed-point / quotient-local algebra has the same expectation in both representatives:*

$$\mathrm{Tr}(X\omega_U) = \mathrm{Tr}(X\omega'_U).$$

In particular the same holds for the central sector projectors and for any observer-accessible record observable generated from them on that same declared surface.

Proof. On the ordinary or central-defect branch, Proposition B.7 says the two representatives differ only by the boundary-redundancy action inside one fixed sector block. The fixed-point collar algebra and its central block projectors are invariant under that action, so their expectation values agree. On the genuinely noncentral branch, the same proposition says the two representatives differ only inside one $\mathcal{T}_\Sigma$-orbit, and the quotient-local physical algebra $\mathcal{A}_{\mathrm{phys}}(U)$ of Definition 5.15 is defined precisely on that orbit space. Therefore the induced physical state and all expectations of physical observables agree there as well. □

C Protocol and Code Extensions

The core theorems are finite rewriting statements. Stronger readings of the same formalism need their own certificates; the table lists them.

Desired statement	**Required certificate**
Overlap network is a code	finite constraint-code data of Proposition 2.3
Global repair is a physical normal-form map	$\mathrm{Rep}_\lambda = \overline{\mathrm{nf}}_\lambda : Q \to Q$, total, idempotent, schedule-independent, landing in C_Q
Repair converges	finite exact descent; confluence additionally needs semantic-complete transactions, coherent canonical aggregate gluing and repair completeness
Distance equals min-cut	topological-code certificate with homological logicals and matching systole/min-cut geometry
Corrects t corrupted carriers	code projector, error family, Knill–Laflamme condition and certified $t < d/2$ distance bound
Exponential convergence	declared transfer or channel operator with stationary projection and spectral gap
Long-run noisy approximate consensus	fair-block contraction certificate $(\lambda, \varepsilon, A, \beta, L)$; deterministic block endpoints for the calendar-time constant
Cross-view BFT safety and liveness	the certificate, lock, quorum, authentication and timing contract of Definition C.1 and (A1)–(A6)

Byzantine agreement under a prepared-lock contract

The Byzantine fault model is that of Lamport, Shostak and Pease [21].

Definition C.1 (QBFT-style protocol)**.** *A consensus protocol is* QBFT-style *in this analysis if it satisfies the following five structural properties. Same-view safety uses (P1)–(P2), cross-view safety additionally uses the lock and monotone-view parts of (P1) and (P4), and the liveness clause uses (P3)–(P5) together with the timing and connectivity assumptions below.*

(P1) ***One-vote-per-view.*** *Each nonfaulty node supports at most one value per view number. If the protocol has prepare, prepared-certificate acceptance, and commit phases, its messages in all three phases refer to that same value. A node that has supported a value in view v ignores any later conflicting request in view v. View participation is monotone: after entering view w, a nonfaulty node never signs a prepare, prepared-certificate acceptance, or commit message for $v < w$, and never adopts a durable lower-view lock.*

(P2) ***Certificate semantics.*** *A raw prepared certificate* $\mathrm{PC}(v, x)$ *contains q distinct, unforgeable, authenticated prepare votes for value x in view v. A lock certificate* $\mathrm{LC}(v, x)$ *contains that* $\mathrm{PC}(v, x)$ *together with q distinct authenticated acceptance acknowledgements. These acknowledgements are provisional: they do not change the acceptor's voting lock. A raw prepared*

certificate without q acceptance acknowledgements cannot justify a new view or a commit vote. A nonfaulty validator that receives the assembled $\mathrm{LC}(v, x)$ verifies it and records the durable lock (v, x) before signing a commit vote. A decision certificate contains q distinct authenticated commit votes, each referring to the same valid $\mathrm{LC}(v, x)$. In the classical exact-size case $q = 2f + 1$. For larger validator sets at fixed fault budget, the threshold must scale so that (A6) holds. A nonfaulty observer finalises a value only after accepting such a valid decision certificate.

(P3) ***DLS-style view-change.*** *If no certificate is produced within a timeout, every nonfaulty node increments the view number by one and a new leader is selected by a fixed fair deterministic rule. Once the post-GST timeout bound of (A1) is active, the pacemaker brings all nonfaulty nodes through successive views and reaches a nonfaulty leader within at most $f + 1$ consecutive views.*

(P4) ***Prepared-certificate dissemination, commit lock, and justified new view.*** *The assembler disseminates a valid $\mathrm{PC}(v, x)$ and obtains the q authenticated acceptance acknowledgements of $\mathrm{LC}(v, x)$, then disseminates the assembled lock certificate before collecting commits. Each validator's authenticated view-change message carries its highest valid lock certificate, if any. A new leader collects q such messages. If it issues a proposal, that proposal carries the value of their highest-view lock certificate; it may carry a fresh value only if none is present. The proposal carries the complete new-view justification. Every nonfaulty validator verifies the signatures, quorum size, and highest-certificate selection. An unlocked validator may support a fresh proposal only when the verified justification contains no lock certificate. Otherwise it votes only for its current lock value, except that the value selected by a valid q-message new-view justification supersedes its local assembled-lock-certificate lock. This authorizes the prepare vote without adopting a lower-view durable lock; receipt of the assembled current-view lock certificate records the replacement before commit. This rule applies without a local test for whether an earlier commit set reached q: it reconciles every predecision lock, while the proof below shows that quorum intersection makes the selected value compatible whenever a decision certificate exists.*

(P5) ***Post-GST nonfaulty progress.*** *In a sufficiently long post-GST view led by a nonfaulty node, that leader collects the required new-view messages and emits the proposal selected by (P4). Every nonfaulty validator that receives a valid proposal promptly sends its prepare vote. The nonfaulty leader disseminates the resulting valid prepared certificate; every nonfaulty validator verifies it and returns a provisional acceptance acknowledgement. After assembling the lock certificate, the leader disseminates it; every nonfaulty recipient verifies it, records or advances its durable lock, and sends its commit vote. After accepting a valid decision certificate it finalises and relays that certificate. Each of these fixed protocol phases completes within $O(\Theta)$, with Θ defined in (A1).*

The Istanbul BFT / QBFT protocol family [23, 24] motivates this pattern; (P1)–(P5), including their exact certificate, pacemaker, lock, and progress semantics, must be checked against an implementation rather than inferred from the family name.

Assumptions A1–A6.

(A1) **Partial synchrony [22].** Fixed but initially unknown bounds Δ (per-edge message delay) and Φ (local processing time) hold after GST. Together with the certified routes of (A4), they

supply an effective end-to-end protocol-phase bound Θ that includes route length, processing, and the fixed number of message steps in one phase. Pacemaker timeouts eventually exceed the corresponding per-view bound. *Safety* holds without extra timing assumptions; *liveness* holds after the Global Stabilisation Time (GST).

(B2) **Byzantine fault model.** At most f observers behave arbitrarily; the remaining $n - f$ are nonfaulty.

(C3) **Classical exact sizing for this fixed-quorum theorem.** $n = 3f + 1$, so that $q = 2f + 1$ certificates have a nonfaulty overlap witness. The usual resilience condition $n \geq 3f + 1$ is necessary for the fault model, but when $n > 3f + 1$ a fixed $q = 2f + 1$ quorum is insufficient for the overlap used in the safety proof; one must either impose (A6) directly or choose a threshold q with $2q \geq n + f + 1$.

(D4) **Strong quorum connectivity.** Every quorum Q with $|Q| = q$ is strongly connected within G: for any $u, v \in Q$ there is a directed path in G contained entirely in Q. This is strictly stronger than requiring the overlap graph of quorums to be connected, and is needed to propagate signed votes within a quorum.

(E5) **Message authentication.** All messages carry unforgeable digital signatures.

(F6) **OPH quorum overlap.** Any two quorums Q_a, Q_b of size q satisfy $|Q_a \cap Q_b| \geq f + 1$. For $q = 2f + 1$ this is guaranteed by the exact sizing $n = 3f + 1$ in (A3); for general n it is the separate threshold condition $2q \geq n + f + 1$, not a consequence of $n \geq 3f + 1$ alone. The liveness clause additionally requires $q \leq n - f$, so a certificate can be formed without Byzantine participation.

Theorem C.2 (QBFT safety and liveness)**.** *Under (A1), (A2), and (A4)–(A6), any consensus protocol satisfying (P1)–(P5) of Definition C.1 and run over the OPH observer graph satisfies:*

(i) ***Cross-view safety.*** *No two nonfaulty observers finalise conflicting patch states, whether in the same view or in different views.*

(ii) ***Liveness.*** *From the first post-GST view in which the pacemaker timeout bound is active, every nonfaulty observer finalises within $O((f+1)\Theta)$ wall-clock time, provided $q \leq n - f$. No bound from GST to timeout activation is claimed.*

(iii) ***Certificate-contract threshold.*** *Joint feasibility of the safety threshold $2q \geq n + f + 1$ and Byzantine-independent liveness threshold $q \leq n - f$ forces $n \geq 3f + 1$; the exact-size choice $n = 3f + 1, q = 2f + 1$ attains both.*

Sizing. Assumption (A3) is the classical exact-size way to discharge (A6): when $n = 3f + 1$ and $q = 2f + 1$ the required overlap is automatic, and for general n (A6) must be checked directly. The prepared-lock rule proves safety only for the protocol just defined, never from the name "QBFT." Under these assumptions no two nonfaulty observers finalise conflicting patch states.

Proof. Same-view safety. Suppose O_a and O_b finalise $s_a \neq s_b$ in one view. By (P2), each required a certificate of q votes: sets Q_a, Q_b. By (A6), $|Q_a \cap Q_b| \geq f + 1$. In the classical exact-size case this is the inclusion-exclusion calculation $|Q_a \cap Q_b| \geq (2f + 1) + (2f + 1) - (3f + 1) = f + 1$. By (A2), at most f are Byzantine, so $Q_a \cap Q_b$ contains a nonfaulty O^*. By (P1), O^* voted for at most

one value. Contradiction. The identical overlap argument applies to two prepared certificates in the same view, using their prepare-vote quorums.

Cross-view safety. Suppose $\mathrm{DC}(v, x)$ and $\mathrm{DC}(w, y)$ are conflicting decision certificates with $v < w$. The first certificate refers to a valid $\mathrm{LC}(v, x)$. Let C_x be its q committers. The nonfaulty subset $L_x \subseteq C_x$ has size at least $q - f$, and every member of L_x received the assembled lock certificate and recorded (v, x) before committing. Every size-q prepare or view-change quorum P intersects that actual lock set because

$$|L_x \cap P| \geq (q - f) + q - n = 2q - n - f \geq 1$$

by (A6).

Assume for contradiction that a raw prepared certificate for a value different from x exists above view v, and choose one with least view $r > v$. Its proposal carries a valid q-message new-view justification J_r. Since $|L_x| \geq q - f$ and $|J_r| = q$, choose a nonfaulty $O^* \in L_x \cap J_r$. Validator O^* signed a commit for $\mathrm{LC}(v, x)$, and later sent its authenticated view-r change. Monotone view participation in (P1) orders its view-v lock and commit before that view-r message: it cannot send a view-r change and subsequently return to commit in view v. Its view-change message therefore reports $\mathrm{LC}(v, x)$, or a later valid lock certificate. By minimality of r, every raw prepared certificate underlying such a later reported lock certificate below view r carries x. Hence O^*'s report carries x.

The justification J_r therefore cannot be certificate-free. It also cannot contain a conflicting highest lock certificate, because that certificate contains a conflicting raw prepared certificate in a view below r, contradicting the minimality of r. Rule (P4) consequently selects x, so nonfaulty validators do not prepare the alleged conflicting value in view r. Thus no higher conflicting prepared certificate exists, and (P2) excludes the alleged $\mathrm{DC}(w, y)$.

This induction is also the lock-transfer invariant for view change. Every valid q-message new-view justification after a decision contains a nonfaulty report from the decision's $q - f$ commit-lock set, and its highest valid report carries the decided value.

Liveness and certificate threshold. From the first post-GST view whose pacemaker timeout is active, (A1) makes every failed-view timeout and every fixed protocol phase $O(\Theta)$. Rule (P3) reaches a nonfaulty leader within at most $f + 1$ consecutive views. Since $q \leq n - f$, nonfaulty validators can supply all q new-view, prepare, prepared-certificate acceptance, and commit messages without Byzantine participation.

Consider the valid q-message justification used by the successful new view. If it contains no lock certificate, (P4) selects a fresh value. Otherwise it selects its unique highest-view lock certificate; same-view uniqueness follows from the prepare-quorum overlap. Every nonfaulty validator may replace a local assembled-lock-certificate predecision lock with that valid selection, including a lock whose local commit set never reached a decision certificate. If a decision certificate exists, the $q - f$ nonfaulty commit-lock set proved above intersects the q-message justification. Its report forces the selected lock certificate to carry the decided value, since the safety induction excludes a conflicting certificate. Thus at least q nonfaulty validators can support the selected proposal in either case, even when some of their orphan locks were omitted from the new-view messages.

Rule (P5) executes the proposal, prepare, provisional acceptance, lock-certificate dissemination, commit, and decision-relay phases, so every nonfaulty observer finalises within $O((f + 1)\Theta)$ from that activation point. Eventual timeout activation yields eventual liveness, but the theorem does not bound its delay after GST.

Finally, $q \leq n - f$ and $2q \geq n + f + 1$ imply

$$2(n - f) \geq n + f + 1,$$

hence $n \geq 3f+1$ (with the integer convention specified in (A6)). At $n = 3f+1$, $q = 2f+1$ attains both inequalities. This is a threshold statement for the declared certificate contract, not a universal resilience claim for every authenticated protocol model.

Note on FLP. Fischer, Lynch, Paterson [20] is an impossibility result for fully asynchronous systems; it does not bear on achievability under partial synchrony (A1). □

Executable receipt and negative controls. A finite executable check exhausts all certificate and new-view quorum pairs for $(n, f, q) = (1, 0, 1), (4, 1, 3), (6, 1, 4), (7, 2, 5)$; the third checks the general threshold with $q < n - f$. It checks the certificate-overlap witness, independent prepare-signer, provisional-acceptor and committer quorums, the $q - f$ lower bound on nonfaulty commit locks, their transfer into every new-view quorum, and the next-view lock transition against every candidate prepare quorum, including an omitted higher orphan lock, together with post-GST traces for every nonfaulty reference leader. Separate finite traces remove each load-bearing safety or liveness premise. The lock-free $n = 4, f = 1, q = 3$ trace decides x from $\{B, N_1, N_2\}$ in view zero and $y \neq x$ from $\{B, N_2, N_3\}$ in view one while respecting one-value-per-validator in each view. The receipt is not a general protocol-model checker; the arbitrary-view induction is the proof above.

The Petz repair map

Definition C.3 (OPH Repair Map: Petz form)**.** *Let $\sigma \in \mathcal{D}(\mathcal{H})$ be a full-rank reference state and $\mathcal{N} : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ a quantum channel. The* OPH repair map *is*

$$\mathcal{R}_{\sigma,\mathcal{N}}(\rho) := \sigma^{1/2}\,\mathcal{N}^{\dagger}(\mathcal{N}(\sigma)^{-1/2}\,\rho\,\mathcal{N}(\sigma)^{-1/2})\,\sigma^{1/2},$$

where $\mathcal{N}^{\dagger}$ is the adjoint channel and inverses are taken on $\operatorname{supp}(\mathcal{N}(\sigma))$.

The closest-point trace-distance projection $\arg\min_{\tau\in\mathcal{S}} \frac{1}{2}\|\rho - \tau\|_1$ is a different object: it is variational and not CPTP in general. All properties below refer to Definition C.3.

Proposition C.4 (Petz map CPTP on its domain)**.** *Let σ have full support on $\mathcal{H}$.*

(a) $\mathcal{R}_{\sigma,\mathcal{N}}$ is completely positive.

(b) $\mathcal{R}_{\sigma,\mathcal{N}}$ is trace-preserving on $\operatorname{supp}(\mathcal{N}(\sigma))$*, i.e., on inputs ρ for which $\mathcal{N}(\sigma)^{-1/2}\rho\mathcal{N}(\sigma)^{-1/2}$ is well-defined.*

(c) If additionally $\mathcal{N}(\sigma)$ has full rank on $\mathcal{K}$, then $\mathcal{R}_{\sigma,\mathcal{N}}$ is CPTP on all of $\mathcal{B}(\mathcal{K})$.

If $\mathcal{N}(\sigma)$ is not *full rank on $\mathcal{K}$, then either (i) the domain must be restricted to* $\operatorname{supp}(\mathcal{N}(\sigma))$*, or (ii) pseudoinverses must replace the inverses (generalised Petz map; cf. [27]), or (iii) a regularisation $\mathcal{N}(\sigma) \mapsto \mathcal{N}(\sigma) + \varepsilon\mathbf{1}$ must be introduced. Note that full-rank σ does* not *prevent $\mathcal{N}(\sigma)$ from being rank-deficient: the channel may map the support of σ into a strict subspace of $\mathcal{K}$. In the OPH setting, whether $\mathcal{N}(\sigma)$ is full rank depends on the specific overlap channel and must be verified for the chosen analytic channel model. The finite OPH repair theorem instead uses the declared Lyapunov descent law on a finite patch net.*

Proof. Complete positivity follows from composing three CP operations:

(i) sandwiching by $\mathcal{N}(\sigma)^{-1/2}(\cdot)\mathcal{N}(\sigma)^{-1/2}$ on $\operatorname{supp}(\mathcal{N}(\sigma))$;

(ii) $\mathcal{N}^{\dagger}$;

(iii) sandwiching by $\sigma^{1/2}(\cdot)\sigma^{1/2}$.

Trace preservation in the full-rank case: Petz [25]; Fagnola–Umanità [26]. □

Analytic refinements of the repair map are separate conditions. A strictly contractive quotient repair map T with coefficient $\lambda < 1$ has at most one fixed point; when one exists, the iterates of an implemented map within ε of T approach it to within $\varepsilon/(1-\lambda)$. Exponential convergence of a declared transfer or channel operator requires a spectral gap $\|\mathcal{T}^t - \Pi_\star\| \le Ce^{-\delta t}$ on the nonstationary subspace. Neither condition is used by the finite normal-form theorem, where termination follows from strict descent of accepted repairs. Strong fairness imposes no probability space on schedules, no spectral gap and no message-delay bound, so it yields no probability-one, exponential or wall-clock statement [20].

Codes

Theorem C.5 (No free min-cut for bare overlap graphs)**.** *Let $G = (V, E)$ be any connected graph with $|V| \ge 2$. The graph G alone does not determine the Hamming distance of the consistency set C of a finite overlap net on G. In particular, the same graph can realize a binary constraint code of distance 1 or a binary repetition code of distance $|V|$.*

Proof. Set $S_i = \{0, 1\}$ for every vertex. For the repetition realization, choose $I_e = \{0, 1\}$ and let both endpoint readouts be the identity. Then every edge imposes $x_i = x_j$, so the only global codewords are $00\cdots0$ and $11\cdots1$, whose Hamming distance is $|V|$.

For the trivial-overlap realization, keep the same vertex state spaces but let every endpoint readout be the constant map to 0. Then every binary assignment is globally consistent, so the minimum Hamming distance among distinct codewords is 1. The graph is unchanged. Therefore distance is a property of the code realization (state spaces, readout maps, logical dictionary, metric, and error model), not of the bare overlap graph. □

Definition C.6 (Topological-code realization certificate)**.** *An OPH overlap network may be treated as a QECC/topological code only after supplying a tuple*

$$\mathsf{TCert} = (K, \mathcal{H}_{\mathrm{phys}}, \mathcal{H}_{\mathrm{code}}, \partial_2, \partial_1, S_X, S_Z, \mathcal{L}_X, \mathcal{L}_Z, \mathcal{E}, \mathcal{R}),$$

where $C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0$ is a chain complex over $\mathbb{F}_2$, the physical carriers live in $\mathcal{H}_{\mathrm{phys}}$, the protected subspace is $\mathcal{H}_{\mathrm{code}}$, S_X, S_Z are stabilizer or gauge checks, $\mathcal{L}_X, \mathcal{L}_Z$ are logical-operator classes, $\mathcal{E}$ is a declared error family, and $\mathcal{R}$ is a recovery map or recovery family.

Theorem C.7 (Certified topological-code distance and min-cut)**.** *Suppose a certificate* TCert *of Definition C.6 is supplied, with logical classes identified as*

$$\mathcal{L}_X \simeq H_1(K; \mathbb{F}_2), \qquad \mathcal{L}_Z \simeq H^1(K; \mathbb{F}_2),$$

and with boundary conditions excluding lower-weight trivial representatives. Then the certified distance is

$$d = \min\left\{ \min_{\ell \in \mathcal{L}_X \setminus 0} |\ell|, \min_{\ell^\star \in \mathcal{L}_Z \setminus 0} |\ell^\star| \right\}.$$

Only in geometries where this homological systole equals the relevant graph min-cut may one write $d = \mathrm{mincut}(G_{\mathrm{OPH}})$ [29, 30].

Proof. This is the standard stabilizer/topological-code distance statement once the chain complex, checks, logical representatives, and boundary conditions are declared. The min-cut equality is an additional geometric identification of that homological minimum with a graph cut. By Theorem C.5, it cannot be inferred from the bare graph. □

Theorem C.8 (QECC resilience under a supplied code certificate)**.** *Assume a genuine code subspace $\mathcal{H}_{\text{code}} \subseteq \mathcal{H}_{\text{phys}}$ with projector Π, and let $\mathcal{E}_t$ be the declared set of errors supported on fewer than t corrupted patches or physical carriers. If*

$$\Pi E_a^\dagger E_b \Pi = \alpha_{ab} \Pi \qquad \forall E_a, E_b \in \mathcal{E}_t,$$

then there exists a recovery channel correcting all errors in $\mathcal{E}_t$ [28]. If the supplied certificate also gives distance d, then all errors of weight $t < d/2$ are correctable. No such resilience statement follows from the bare overlap graph.

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