The Cycle Polynomial of Permutation Groups

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Context

These are notes created to develop an understanding of the cycle polynomial, conceptualised by Peter J. Cameron and Jason Semeraro.

Contents

1 Introduction

The cycle polynomial of a finite permutation group is an interesting construction.

Definition 1 (Cycle polynomial)**.** The *cycle polynomial* of a permutation group G acting on a set Ω of size n is defined as:

$$F_G(x) = \sum_{g \in G} x^{c(g)}$$

where $c(g)$ is the number of cycles of g on Ω, including fixed points.

Example. Consider the permutation group $\{e, (1\,2), (3\,4), (1\,2)(3\,4)\} \cong V_4$ acting on the set $\{1, 2, 3, 4\}$. Its cycle polynomial is:

$$F_{V_4}(x) = x^4 + 2x^3 + x^2 \tag{1}$$

Remark. The cycle polynomial will never have a constant term, since the minimum number of cycles of a permutation group can have is 1. It is obviously a polynomial of n^{th} order because of the identity permutation.

Proposition 1. If a is an integer, then $F_G(a)$ is a multiple of $|G|$.

Proof. $\square$