Heaps and Applications

Arjit Seth
Manipal Institute of Technology, Manipal University

Context

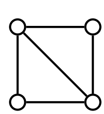
These are notes created based on a talk presented by Dr. K. N. Raghavan from The Institute of Mathematical Sciences (IMSc), Chennai at the *Computer Applications based on Modern Algebra* conference at Manipal Institute of Technology. The talk was mostly for publicity of Xavier Viennot's lectures at IMSc, uploaded on the `matscience` YouTube channel. The talk covers an introduction to the chromatic polynomial and its relation to acyclic orientations, Stanley's theorem and an informal introduction to heaps.

Contents

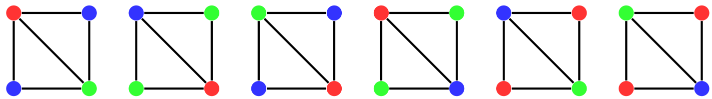
1 Chromatic Polynomial

George David Birkhoff attacked the four color problem by constructing a polynomial which characterises the number of vertex colourings for a variable number of colours, called the *chromatic polynomial*. The motivation is provided by example:

Example. Let G be the following graph:



The minimum number of colours required to colour this graph is 3 because of the two diagonally adjacent vertices. There are 6 ways of colouring this graph with 3 colours:



An exact colouring requires use of all λ colours in the graph, called a λ-colouring. There are obviously 4! exact 4-colourings, since each vertex is coloured differently. To obtain a general formula for λ-colourings, consider the following argument: There are λ colours one can use for the top left vertex and $\lambda - 1$ colours for the bottom right vertex; these conditions immediately determine that the other two vertices can only be coloured with $\lambda - 2$ colours. Therefore, the general number of λ-colourings for this graph is given by the polynomial:

$$\lambda(\lambda - 1)(\lambda - 2)^2$$

This is the chromatic polynomial of the graph, which will be defined shortly. Using this formula, the total number of 4-colourings (including ways of colouring the graph using 3 colours) is 48.

Definition 1 (Chromatic Polynomial). The *chromatic polynomial* $\gamma_G(\lambda)$ of a graph G counts the number of its proper vertex colourings with λ colours. Its general formula is given by:

$$\gamma_G(\lambda) = \sum_{k \geq 1} \gamma'_G(k) \binom{\lambda}{k}$$

where $\gamma'_G(k)$ denotes the number of exact k-colourings of G.

Remark. Note that the number of vertices n of the graph need not be mentioned since $\gamma'_G(k) = 0$ if $k > n$, obviously.

As is customary in any investigation, inserting 'forbidden' values into the polynomial, such as negative integers, is good experimentation. Let's try this for the previous graph:

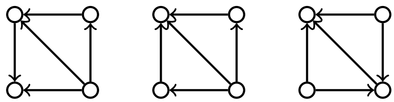
$$\gamma_G(-1) = 6 \times \binom{-1}{3} + 24 \times \binom{-1}{4} = 18$$

Surprisingly enough, this the number of different acyclic orientations of the graph!

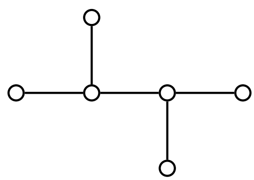
2 Acyclic Orientations

Definition 2 (Acyclic Orientation)**.** An orientation of each edge of a graph G such that no cycle in the graph is a cycle in the resulting directed graph.

Example. Three different acyclic orientations of the previous graph are:



Example. Let T be the following *tree*:



λ colours can be used to colour the first vertex, and $\lambda - 1$ colours can be used for the rest of them. Therefore, the chromatic polynomial of T and the number of its acyclic orientations are:

$$\gamma_T(\lambda) = \lambda(\lambda - 1)^{n-1}$$
$$\gamma_T(-1) = (-1)^n(2)^{n-1}$$

Example. Let K_n denote the *complete graph* with n vertices. Since every vertex is adjacent to all other vertices, the number of colours one can use for the rest of the vertices reduces by 1 after colouring each vertex. Therefore, its chromatic polynomial and number of acyclic orientations are:

$$\gamma_{K_n}(\lambda) = \lambda(\lambda - 1)\ldots(\lambda - n + 1)$$
$$\gamma_{K_n}(-1) = (-1)^n n!$$

Theorem 1 (Stanley's Theorem)**.** *The chromatic polynomial of a graph G with n vertices has the following property:*

$$\gamma_G(-1) = \sum_{k\geq 1}^{n} \gamma'_G(k)(-1)^k = (-1)^n[\text{\# of acyclic orientations of } G]$$