

Solutions to  
Gravitation: Foundations and Frontiers  
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# Chapter 1

## Special relativity

1. *Light clocks*. Perpendicular:

$$ct' = \sqrt{c^2 - v^2}t$$
$$t = \frac{t'}{\sqrt{1 - v^2/c^2}} = \gamma t'$$

Parallel: Using proper time -

$$dt = \gamma d\tau$$
$$\frac{L'}{c+v} + \frac{L'}{c-v} = \gamma \frac{2L}{c}$$
$$L' \frac{2c}{c^2 - v^2} = \gamma \frac{2L}{c}$$
$$L' = L \frac{1 - v^2/c^2}{\sqrt{1 - v^2/c^2}} = \frac{L}{\gamma}$$

2. *Superluminal motion*.

$$\Delta t' = t'_2 - t'_1$$
$$t'_1 - t_1 \approx \frac{v}{c} \Delta t \cos \theta$$
$$t'_2 - t_1 = L/c$$
$$\Delta t' = \Delta t (1 - (v/c) \cos \theta)$$
$$v_{app} = v \Delta t \sin \theta = \frac{v \sin \theta}{1 - (v/c) \cos \theta}$$

Rewriting this expression and plotting it for  $v = 0.99c$ :

$$v_{app} = \frac{v \sqrt{1 - \cos^2 \theta}}{1 - v \cos \theta}, \quad c = 1$$

### 3. The strange world of four-vectors.

(a) This is evident from taking the inner product, since the magnitudes add up.

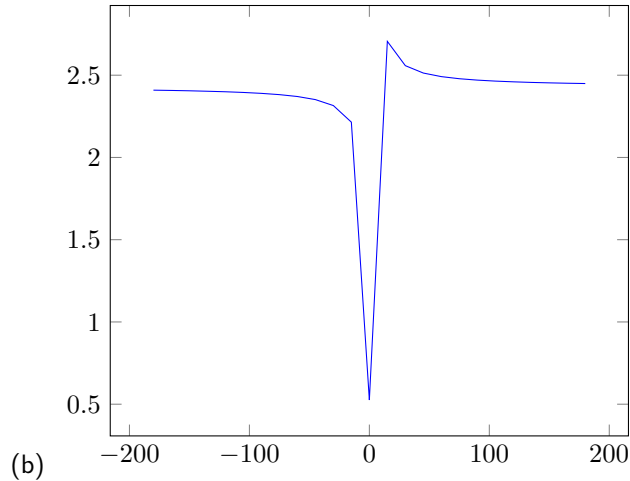
$$(a^i + b^i)(a_i + b_i) = a^i a_i + b^i b_i + 2(a^i b_i)$$

(b) Let  $k^i$  be a non-zero null vector. If non-zero  $a^i$  is a vector orthogonal to  $k^i$ , then  $a_i k^i = 0$ .

### 4. Focused to the front.

(a) Using a Lorentz transformation on the 'time' and 'space' components of the four-vector  $k^a = (\omega, \omega \mathbf{n}/c)$ :

$$\begin{aligned}\omega_R &= \gamma \left[ \omega_L - \frac{\omega_L v \cos \theta_L}{c} \right] = \gamma \omega_L (1 - (v/c) \cos \theta_L) \\ \omega_R \cos \theta_R &= \gamma \left[ \omega_L \cos \theta_L - \frac{\omega_L v}{c} \right] = \gamma \omega_L [\cos \theta_L - (v/c)] \\ \frac{\omega_R \cos \theta_R}{\omega_R} &= \frac{\gamma \omega_L [\cos \theta_L - (v/c)]}{\gamma \omega_L [1 - (v/c) \cos \theta_L]} \\ \mu_R &= \frac{\mu_L - (v/c)}{1 - (v \mu_L/c)}\end{aligned}$$



(c) The solid angle is found by taking the differential of  $\mu_R$ :

$$\begin{aligned}d\Omega &= \int \sin \theta \, d\theta \, d\phi = - \int d(\cos \theta) \, d\phi \\ d(\cos \theta_R) &= \left[ \frac{(v/c)[\cos \theta_L - (v/c)]}{[1 - (v/c) \cos \theta_L]^2} + \frac{1}{[1 - (v/c) \cos \theta_L]} \right] d(\cos \theta_L) \\ d\Omega' &= \frac{1}{\gamma^2} \frac{d\Omega}{[1 - (v/c) \cos \theta]^2}\end{aligned}$$

The energy is:

$$d\mathcal{E}' = \hbar \omega = \gamma d\mathcal{E} [1 - (v/c) \cos \theta]$$

The time is:

$$dt' = \gamma dt$$

Therefore, the energy emitted per unit time into a given solid angle in the rest frame is ( $c = 1$ ):

$$\begin{aligned} \left( \frac{d\mathcal{E}'}{dt' d\Omega'} \right)_{\text{rest}} &= \gamma^2 (1 - v \cos \theta)^3 \left( \frac{d\mathcal{E}}{dt d\Omega} \right)_{\text{lab}} \\ \left( \frac{d\mathcal{E}}{dt d\Omega} \right)_{\text{lab}} &= \frac{(1 - v^2)^2}{(1 - v \cos \theta)^3} \left( \frac{d\mathcal{E}'}{dt' d\Omega'} \right)_{\text{rest}} \end{aligned}$$

If the emission is isotropic,  $d\Omega = d\Omega' = 4\pi$ :

$$\left( \frac{d\mathcal{E}}{dt} \right)_{\text{lab}} = d\Omega \left( \frac{d\mathcal{E}'}{dt' d\Omega'} \right)_{\text{rest}} = \left( \frac{d\mathcal{E}'}{dt'} \right)_{\text{rest}}$$

### 5. Transformation of antisymmetric tensors.

$$A^{i'k'} = L_k^{k'} L_i^{i'} A^{ik} = \frac{\partial x^{k'}}{\partial x^k} \frac{\partial x^{i'}}{\partial x^i} A^{ik}$$

If  $A^{ik} = -A^{ki}$  then:

$$A^{k'i'} = -A^{i'k'}?$$

### 6. Practice with completely antisymmetric tensors.

- (a)
- (b) Multiplying each index by the metric tensor:

$$\epsilon^{abcd} = g^{ai} g^{bj} g^{ck} g^{dl} \epsilon_{ijkl} =$$

- (c)
- (d)

### 7. A null curve in flat spacetime.

$$\eta_{ij} x^i x^j = -t^2 + x^2 + y^2 + z^2 = 0$$

### 8. Shadows are Lorentz invariant.

### 9. Hamiltonian form of action - Newtonian mechanics.

$$\begin{aligned} \mathcal{A} &= \int_{t_2}^{t_1} dt [p\dot{q} - H(p, q)] \\ \delta\mathcal{A} &= \int_{t_2}^{t_1} dt \left[ \dot{q} \delta p + p \delta \dot{q} - \frac{\partial H}{\partial p} \delta p - \frac{\partial H}{\partial q} \delta q \right] = 0 \end{aligned}$$

Using  $\delta\dot{q} = \dot{d}(\delta q)$  and integrating by parts:

$$\int_{t_2}^{t_1} dt p \delta\dot{q} = \cancel{p \delta q \Big|_{t_2}^{t_1}} - \int_{t_2}^{t_1} dt \dot{p} \delta q$$

$$\therefore \int_{t_2}^{t_1} dt \left[ \left( \dot{q} - \frac{\partial H}{\partial p} \right) \delta p - \left( \dot{p} + \frac{\partial H}{\partial q} \right) \delta q \right] = 0$$

Since  $\delta q = 0$  at the endpoints, the second term is zero; therefore, the first term in parentheses should be zero, leading to the equations of motion:

$$\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q}$$

### 10. Hamiltonian form of action - special relativity.

$$\mathcal{A} = \int_{\lambda_1}^{\lambda_2} d\lambda \left[ p_a \dot{x}^a - \frac{1}{2} C \left( \frac{H}{mc^2} + mc^2 \right) \right], \quad H = \eta_{ab} p^a p^b$$

$$\delta\mathcal{A} = \int_{\lambda_1}^{\lambda_2} d\lambda \left[ \delta p_a \dot{x}^a + p_a \delta \dot{x}^a - \frac{1}{2} \left[ \delta C \left( \frac{H}{mc^2} + mc^2 \right) + C \left( \frac{2p^a \delta p_a}{mc^2} \right) \right] \right] = 0$$

$$= \int_{\lambda_1}^{\lambda_2} d\lambda \left[ \dot{x}^a - \frac{C}{mc^2} p^a \right] \delta p_a - \dot{p}_a \delta x^a - \frac{1}{2} \left[ \frac{H + m^2 c^4}{mc^2} \right] \delta C = 0$$

Each term must individually be zero, giving the equations of motion:

$$\frac{p^a}{c^2} = \frac{m \dot{x}^a}{C} \implies \dot{p}_a = 0, \quad H = -m^2 c^4 = \eta_{ab} p^a p^b, \quad C = 1$$

### 11. Hitting a mirror. Using $E = -p^i u_i$ in the rest frame of the mirror:

$$\vec{p}_1 = (h\nu_1/c, h\nu_1 \cos \theta/c, h\nu_1 \sin \theta/c, 0), \quad \vec{p}_2 = (h\nu_2/c, -h\nu_2 \cos \phi/c, h\nu_2 \sin \phi/c, 0)$$

$$E = -\frac{\gamma h\nu_1}{c} [1 + (v/c) \cos \theta](c) = -\frac{\gamma h\nu_2}{c} [1 - (v/c) \cos \phi](c)$$

$$\frac{\nu_2}{\nu_1} = \frac{c + v \cos \theta}{c - v \cos \phi}$$

By momentum conservation in the perpendicular direction,  $\nu_2/\nu_1 = \sin \theta / \sin \phi$ :

$$\frac{\sin \theta/2 \cos \theta/2}{\sin \phi/2 \cos \phi/2} = \frac{c + v(\cos^2 \theta/2 - \sin^2 \theta/2)}{c - v(\cos^2 \phi/2 - \sin^2 \phi/2)}$$

$$\left( \frac{\tan \theta/2}{\tan \phi/2} \right) = \frac{\sec^2 \theta/2 c + v}{\sec^2 \phi/2 c - v}$$

### 12. Photon-electron scattering.

(a) *Compton scattering.* The four-momentum components of the photon and electron before the collision are:

$$\vec{p}_1 = (h\nu_i/c, h\nu_i/c, 0, 0), \quad \vec{p}_2 = (mc, 0, 0, 0)$$

The four-momentum components of the scattered photon and electron after the collision are:

$$\vec{p}_3 = (h\nu_f/c, h\nu_f \cos \theta/c, h\nu_f \sin \theta/c, 0), \quad \vec{p}_4 = (p^0, p^1, p^2, p^3)$$

Let  $c = 1$ . Using the conservation of four-momentum, the four-momentum of the scattered electron is:

$$\vec{p}_4 = \vec{p}_1 + \vec{p}_2 - \vec{p}_3 = \begin{bmatrix} h(\nu_i - \nu_f) + m \\ h\nu_i - h\nu_f \cos \theta \\ -h\nu_f \sin \theta \\ 0 \end{bmatrix}$$

The four-momentum of a particle is  $\vec{p} \cdot \vec{p} = -m^2$ . Finding the magnitude of the scattered electron's four-momentum:

$$\begin{aligned} \vec{p}_4 \cdot \vec{p}_4 &= \begin{bmatrix} h(\nu_i - \nu_f) + m \\ h\nu_i - h\nu_f \cos \theta \\ -h\nu_f \sin \theta \\ 0 \end{bmatrix} \cdot \begin{bmatrix} h(\nu_i - \nu_f) + m \\ h\nu_i - h\nu_f \cos \theta \\ -h\nu_f \sin \theta \\ 0 \end{bmatrix} \\ &= -(h(\nu_i - \nu_f) + m)^2 + (h\nu_i - h\nu_f \cos \theta)^2 + (-h\nu_f \sin \theta)^2 = -m^2 \end{aligned}$$

Expanding the expression and cancelling terms:

$$-2h^2\nu_i\nu_f(1 - \cos \theta) + 2hm(\nu_i - \nu_f) = 0$$

Rearranging the terms, the final expression for the wavelength shift in Compton scattering is:

$$\lambda' - \lambda = \frac{h}{m}(1 - \cos \theta), \quad \lambda' = \frac{1}{\nu_f}, \quad \lambda = \frac{1}{\nu_i}$$

(b) *Inverse Compton scattering.*

13. *More practice with collisions.*

14. *Relativistic rocket.*

15. *Practice with equilibrium distribution functions.*

(a)

$$S^a(x^i) = c \int \frac{d^3\mathbf{p}}{E_{\mathbf{p}}} p^a \frac{2j+1}{h^3} [\exp(-\theta + \beta p^0) - \epsilon]^{-1}$$

$n = u_i S^i$ . Moving to the comoving frame of the centre of mass of the system with velocity  $u^i$ :

$$n = -S^0 = c \int \frac{d^3\mathbf{p}}{E_{\mathbf{p}}} p^0 \frac{2j+1}{h^3} [\exp(-\theta + \beta p^0) - \epsilon]^{-1}$$

Let  $p = \sinh \chi$ , then mapping to polar coordinates:

$$n = \frac{4\pi c(2j+1)}{h^3} \int_0^\infty \frac{\sinh^2 \chi \cosh \chi d\chi}{\exp(\beta \cosh \chi - \theta) - \epsilon}$$

16. *Projection effects.*

17. *Relativistic virial theorem.* The conservation law implies the following:

$$\begin{aligned}
 \partial_0 T^{00} x^\alpha x^\beta &= -\partial_\mu T^{0\mu} x^\alpha x^\beta \\
 &= -\partial_\mu [T^{0\mu} x^\alpha x^\beta] + T^{0\mu} \partial_\mu [x^\alpha x^\beta] \\
 &= -\partial_\mu [T^{0\mu} x^\alpha x^\beta] + T^{0\mu} [\partial_\mu x^\alpha x^\beta + x^\alpha \partial_\mu x^\beta] \\
 &= -\partial_\mu [T^{0\mu} x^\alpha x^\beta] + T^{0\alpha} x^\beta + T^{0\beta} x^\alpha
 \end{aligned}$$

Taking the time derivative and using the fact that partial derivatives commute:

$$\begin{aligned}
 \partial_0 (\partial_0 T^{00}) x^\alpha x^\beta &= -\partial_\mu [\partial_0 T^{0\mu}] x^\alpha x^\beta \\
 \partial_0^2 T^{00} x^\alpha x^\beta &= -\partial_\mu [\partial_0 T^{0\mu} x^\alpha x^\beta] + \partial_0 T^{0\alpha} x^\beta + \partial_0 T^{0\beta} x^\alpha \\
 &= -\partial_\mu [\partial_0 T^{0\mu} x^\alpha x^\beta] - \partial_\nu T^{\nu\alpha} x^\beta - \partial_\nu T^{\nu\beta} x^\alpha, \quad (\partial_i T^{i\alpha} = 0) \\
 &= -\partial_\mu [\partial_0 T^{0\mu} x^\alpha x^\beta + T^{\mu\alpha} x^\beta + T^{\mu\beta} x^\alpha] + 2T^{\alpha\beta}, \quad (T^{\mu\alpha} \partial_\mu x^\beta = T^{\alpha\beta})
 \end{aligned}$$

Integrating both sides of the equation and using the divergence theorem for the condition that  $T^{ij} = 0$  outside a compact region in space:

$$\begin{aligned}
 \int d^3x \partial_0^2 T^{00} x^\alpha x^\beta &= \int d^3x [-\partial_\mu (\partial_0 T^{0\mu} x^\alpha x^\beta + T^{\mu\alpha} x^\beta + T^{\mu\beta} x^\alpha) + 2T^{\alpha\beta}] \\
 \therefore \frac{d^2}{dt^2} \int d^3x T^{00} x^\alpha x^\beta &= 2 \int d^3x T^{\alpha\beta}
 \end{aligned}$$

18. *Explicit computation of spin precession.* The four-velocity and four-acceleration are:

$$\begin{aligned}
 u^i &= (\gamma, \gamma \mathbf{v}) = (\gamma, -\gamma r \omega \sin \omega t, \gamma r \omega \cos \omega t, 0) \\
 a^i &= \gamma (\dot{\gamma}, \dot{\gamma} \mathbf{v} + \dot{\mathbf{v}} \gamma) = -\gamma^2 \omega^2 (0, x, y, 0)
 \end{aligned}$$

Using the equation of motion for a moving particle with spin and separating the space and time components:

$$\begin{aligned}
 \frac{dS^j}{d\tau} &= u^j (S^k a_k) \\
 S^k a_k &= -\gamma^2 \omega^2 (xS^x + yS^y) \\
 \frac{d}{dt} [(yS^x - xS^y)] &= -\omega (xS^x + yS^y)
 \end{aligned}$$

19. *Little group of the Lorentz group.*

## Chapter 2

# Scalar and electromagnetic fields in special relativity

1. *Measuring the  $F^{ab}$ .*

2. *Schrödinger equation and gauge transformation.* The transformed Schrödinger equation is:

3. *Four-vectors leading to electric and magnetic fields.*

(a) Taking dot products:

$$E^i u_i = F^{ij} u_j u_i = 0$$
$$B^a u_a = \frac{1}{2} \epsilon^{abcd} F_{cd} u_a u_b = \frac{1}{2} (*F)^{ab} u_a u_b = 0$$

since  $u_i u_j$  is symmetric and  $F^{ij}$  and its dual are antisymmetric.

(b)

$$u^a E^b - u^b E^a = (F^{bi} u^a - F^{ai} u^b) u_i$$
$$\epsilon^{ab}_{cd} u^c B^d = \frac{1}{2} \epsilon^{ab}_{cd} \epsilon^{di}_{jk} u^c u_i F^{jk}$$

4. *Hamiltonian form of action - charged particle.*

$$\mathcal{A} = \int_{\lambda_1}^{\lambda_2} \left[ P_a \dot{x}^a - \frac{1}{2} C \left( \frac{H}{mc^2} + mc^2 \right) \right], \quad H = \eta_{ij} (P^i - qA^i) (P^j - qA^j)$$
$$\delta \mathcal{A} = \int_{\lambda_1}^{\lambda_2} \left[ \delta P_a \dot{x}^a + P_a \delta \dot{x}^a - \frac{1}{2} \left[ \delta C \left( \frac{H}{mc^2} + mc^2 \right) + C \left( \frac{2(P^i - qA^i)(\delta P_i - q\delta A_i)}{mc^2} \right) \right] \right] d\lambda = 0$$



5. *Three-dimensional form of the Lorentz force.* Using the electromagnetic tensor equation of motion  $u^i = (\gamma c, \gamma \mathbf{v})$ :

$$\begin{aligned}
 m \frac{du^i}{d\tau} &= q F^{ik} u_k \implies mc\gamma \frac{du^0}{dt} = q F^{0\alpha} u_\alpha, & \frac{du^0}{d\tau} &= \gamma \frac{du^0}{dt} \\
 mc\gamma \frac{d\gamma}{dt} &= q\gamma[(\mathbf{E}/c) \cdot \mathbf{v}] \implies \frac{1}{\gamma^3} \left( m\mathbf{v} \cdot \frac{d\mathbf{v}}{dt} \right) = q(\mathbf{E} \cdot \mathbf{v}), & \frac{d\gamma}{dt} &= \frac{1}{\gamma^3} \left( \frac{\mathbf{v}}{c^2} \cdot \frac{d\mathbf{v}}{dt} \right) \\
 m \frac{du^\alpha}{dt} &= q F^{\alpha\beta} u_\beta = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \\
 \left[ \frac{1}{\gamma^3} \left( \frac{m\mathbf{v}}{c^2} \cdot \frac{d\mathbf{v}}{dt} \right) \mathbf{v} + m\gamma \frac{d\mathbf{v}}{dt} \right] &= q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \\
 \implies \frac{d\mathbf{v}}{dt} &= \frac{q}{m\gamma} \left[ \mathbf{E} + \mathbf{v} \times \mathbf{B} - \frac{1}{c^2} (\mathbf{E} \cdot \mathbf{v}) \mathbf{v} \right]
 \end{aligned}$$

6. *Pure gauge impostors.* This can be transformed into polar coordinates ( $x = \cos \theta, y = \sin \theta$ ) and evaluated ( $A_r = 0$ , obviously):

$$\begin{aligned}
 A_\theta &= \frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial \theta} = 1, \\
 \oint_{C=x^2+y^2} \mathbf{A} \cdot d\mathbf{s} &= \int_0^{2\pi} d\theta = 2\pi
 \end{aligned}$$

The reason why  $\mathbf{A}$  is not a pure gauge mode is because  $f$  is singular at the origin, making it non-differentiable at that point. It is also a non-removable singularity, so no analytic continuation can be performed.

7. *Pure electric or magnetic fields.*  
 8. *Elegant solution to non-relativistic Coulomb motion.*

(a) Since the angular momentum is conserved,  $d\mathbf{J}/dt = 0$ :

$$\begin{aligned}
 \frac{d}{dt}(\mathbf{p} \times \mathbf{J}) &= \frac{d\mathbf{p}}{dt} \times \mathbf{J} + \mathbf{p} \times \frac{d\mathbf{J}}{dt} \\
 f(r)\hat{\mathbf{r}} \times \mathbf{r} \times m\mathbf{v} &= mf(r)[\mathbf{r}(\hat{\mathbf{r}} \cdot \mathbf{v}) - \mathbf{v}(\hat{\mathbf{r}} \cdot \mathbf{r})] \\
 &= -mf(r)r \frac{d\mathbf{r}}{dt} = -mf(r)r^2 \frac{d\hat{\mathbf{r}}}{dt}
 \end{aligned}$$

Therefore, if  $f(r)r^2 = -|\alpha|$ , then:

$$\begin{aligned}
 \int \frac{1}{m|\alpha|} \frac{d}{dt}(\mathbf{p} \times \mathbf{J}) dt &= \int \frac{d\hat{\mathbf{r}}}{dt} dt \\
 \implies \frac{\mathbf{p} \times \mathbf{J}}{m|\alpha|} - \hat{\mathbf{r}} &= \mathbf{e}
 \end{aligned}$$

where  $\mathbf{e}$  is a conserved vector independent of  $t$ , arising as a constant of the integration.

(b) Note that  $\mathbf{p}$  and  $\mathbf{J}$  are perpendicular:

$$\begin{aligned}
 \mathbf{e} \cdot \mathbf{e} &= \left[ \frac{\mathbf{p} \times \mathbf{J}}{m|\alpha|} - \hat{\mathbf{r}} \right] \cdot \left[ \frac{\mathbf{p} \times \mathbf{J}}{m|\alpha|} - \hat{\mathbf{r}} \right] \\
 \Rightarrow |\mathbf{e}|^2 &= \frac{|\mathbf{p} \times \mathbf{J}|^2}{m^2|\alpha|^2} - 2 \left( \frac{\hat{\mathbf{r}} \cdot \mathbf{p} \times \mathbf{J}}{m|\alpha|} \right) + 1 \\
 &= \frac{p^2 J^2}{m^2|\alpha|^2} - 2 \left( \frac{J^2}{m|\alpha|r} \right) + 1, \quad J^2 = (\mathbf{r} \times \mathbf{p}) \cdot \mathbf{J} \\
 &= 1 + \frac{2EJ^2}{m|\alpha|^2}, \quad E = \frac{p^2}{2m} - \frac{|\alpha|}{r}
 \end{aligned}$$

(c)

$$\begin{aligned}
 \mathbf{e} \cdot \mathbf{r} &= \frac{\mathbf{r} \cdot \mathbf{p} \times \mathbf{J}}{m|\alpha|} - \mathbf{r} \cdot \hat{\mathbf{r}} \\
 er \cos \theta &= \frac{J^2}{m|\alpha|} - r \\
 \Rightarrow r(\theta) &= \frac{J^2/m|\alpha|}{1 + e \cos \theta}
 \end{aligned}$$

(d)

$$E = \frac{p^2}{2m} - \frac{|\alpha|}{r} - \frac{|\beta|}{r^2}$$

9. More on uniformly accelerated motion.

## **Chapter 3**

# **Gravity and spacetime geometry: the inescapable connection**

## Chapter 4

# Metric tensor, geodesics and covariant derivative

(a) *Practice with metrics.*

(b)  $x = 2 \tan(\theta/2) \cos \phi$ ,  $y = 2 \tan(\theta/2) \sin \phi$ . Since the coordinates are spherical:

$$\begin{aligned}g_{ab} &= \eta_{ij} \frac{\partial X^i}{\partial x^a} \frac{\partial X^j}{\partial x^b}, \quad X^i = \theta, \phi, \quad x^a = x, y, \quad \eta_{ij} = \begin{bmatrix} 1 & 0 \\ 0 & \sin^2 \theta \end{bmatrix} \\ \frac{\partial \theta}{\partial x^a} &= \frac{\partial}{\partial x^a} \left[ 2 \tan^{-1} \left( \frac{\sqrt{x^2 + y^2}}{2} \right) \right], \quad \frac{\partial \phi}{\partial x^a} = \frac{\partial}{\partial x^a} [\tan^{-1}(y/x)] \\ \frac{\partial \theta}{\partial x^a} &= \frac{4x^a}{\sqrt{x^2 + y^2}(x^2 + y^2 + 4)} = \cos^2(\theta/2) \cos \phi, \quad \cos^2(\theta/2) \sin \phi \\ \frac{\partial \phi}{\partial x} &= -\frac{y}{x^2 + y^2} = -\frac{\sin \phi}{2 \tan(\theta/2)}, \quad \frac{\partial \phi}{\partial y} = \frac{x}{x^2 + y^2} = \frac{\cos \phi}{2 \tan(\theta/2)} \\ g_{xx} &= \left( \frac{\partial \theta}{\partial x} \right)^2 + \sin^2 \theta \left( \frac{\partial \phi}{\partial x} \right)^2 \\ &= \left[ \cos^2(\theta/2) \cos^2 \phi + \frac{\sin^2 \theta \sin^2 \phi}{4 \sin^2(\theta/2)} \right] \cos^2(\theta/2) = \cos^4(\theta/2) \\ g_{yy} &= \left( \frac{\partial \theta}{\partial y} \right)^2 + \sin^2 \theta \left( \frac{\partial \phi}{\partial y} \right)^2 \\ &= \left[ \cos^2(\theta/2) \sin^2 \phi + \frac{\sin^2 \theta \cos^2 \phi}{4 \sin^2 \theta/2} \right] \cos^2(\theta/2) = \cos^4(\theta/2) \\ \therefore ds^2 &= \cos^4 \left( \frac{\theta}{2} \right) [dx^2 + dy^2]\end{aligned}$$

(c)  $\phi = x$ ,  $\theta = 2 \tan^{-1} e^{-y}$ .

$$g_{ab} = \eta_{ij} \frac{\partial X^i}{\partial x^a} \frac{\partial X^j}{\partial x^b}, \quad X^i = \theta, \phi, x^a = x, y$$
$$\frac{\partial \theta}{\partial x} = 0, \quad \frac{\partial \phi}{\partial x} = 1, \quad \frac{\partial \theta}{\partial y} = -\frac{2}{e^y + e^{-y}} = -\operatorname{sech} y, \quad \frac{\partial \phi}{\partial y} = 0$$