Calculus I: Chain Rule

Manhattan College Interview Presentation

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Presentation Outline

Assumed/Preliminary Knowledge

The course is taught from James Stewart's Calculus: Early Transcendentals 9th ed

Students have covered Chapter 2 (Limits and Derivatives) and Chapter 3.1-3.3.

The knowledge from chapter 2 that will be applied today will be

- Chapter 2
  1. 2.7 Derivatives and Rates of Change
  2. 2.8 Derivatives as a Function
- Chapter 3
  1. 3.1 Derivatives of Polynomials and Exponential Functions
  2. 3.2 The Product and Quotient Rules
  3. 3.3 Derivatives of Trigonometric Functions

What is The Chain Rule?

Let $u = g(x)$ be a differentiable function of x.
Let $f(u)$ be a differentiable function of u.
Then the composite function $F = f \circ g$ defined by $F(x) = f(g(x))$ is differentiable at x and F' is given by the product

$$F'(x) = f'(g(x)) \cdot g'(x). \tag{1}$$

Example 1: Find $F'(x)$ if $F(x) = \sqrt{x^2+1}$

Solution: We can break up the outer function, $f(u) = \sqrt{u}$ and the inner part, $g(x) = x^2 + 1$. The corresponding derivatives are

$$f'(u) = \frac{1}{2}u^{-\frac{1}{2}}, g'(x) = 2x.$$

Substituting into Equation 1,

$$F'(x) = \frac{1}{2}(x^2+1)^{-\frac{1}{2}} \cdot 2x \tag{2}$$

which simplifies to

$$\frac{x}{\sqrt{x^2+1}} \tag{3}$$

Intuition behind the Chain Rule

The chain rule can be intuited by relating the derivatives of related objects. Suppose driving speed, dy/du, is twice as fast as my biking speed, du/dx, and biking speed is 5x faster than my walking speed dy/dx, then my driving speed is $5 \cdot 2 = (dy/du) \cdot (du/dx)$ times faster than my walking speed[1].

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \tag{4}$$

Power Rule; Find $F'(x)$, if $F(x) = (x^3 - 1)^{100}$

Solution: We can break up the outer function, $f(u) = u^{100}$ and the inner function, $g(x) = x^3 - 1$. The corresponding derivatives are

$$f'(u) = 100x^{99}, g'(x) = 3x^2.$$

Substituting into Equation 1,

$$F'(x) = 100(x^3 - 1)^{99} \cdot (3x^2) \tag{5}$$

[1] Note that this notation is NOT a quotient but shows how the derivative can be broken up using a intermediate variable u

Examples: Daisy-chaining The Chain Rule

Find $F'(x)$, if $F(x) = \sin(\cos(\tan(x)))$
We can break up the chain rule into 2 steps working from outer functions towards the inner functions.

1. Let $f(u) = \sin(u), g(x) = \cos(tan(x))$
2. $f'(u) = \cos(u), g'(x) = \frac{d}{dx}\cos(\tan(x))$
3. The first execution of the Chain Rule is

$$F'(x) = \cos(\cos(\tan(x))) \cdot \frac{d}{dx}\cos(\tan(x)). \tag{6}$$

4. Applying the chain rule to $g'(x)$, we let $h(v) = \cos(v)$ and $k(x) = \tan(x)$
5. $h'(v) = -\sin(v), k'(x) = -\sec^2(x)$
6. The second execution of the Chain Rule is

$$g'(x) = -\sin(\tan(x)) \cdot (-\sec^2(x)) \tag{7}$$

7. with that, we can finally express

$$F'(x) = \cos(\cos(\tan(x))) \cdot (-\sin(\tan(x))) \cdot (-\sec^2(x)). \tag{8}$$

$$F'(x) = \cos(\cos(\tan(x)))\sin(\tan(x))(\sec^2(x)). \tag{9}$$

Examples: Is The Product Rule or Quotient Rule the Chain Rule?

Suppose we are tasked with finding $F'(x)$ of $F(x) = (x^3 - 2x)^2$.

1. By the chain rule:

$$F'(x) = 2(x^3 - 2x) \cdot (3x^2 - 2), \tag{10}$$

   where,

$$f(u) = u^2, g(x) = x^3 - 2x \tag{11}$$

2. By the product rule: by setting $u = v = (x^3 - 2x)$

$$\begin{aligned} F'(x) \quad &= u'v + v'u \quad &= (3x-2)(x^3-2x) + (3x-2)(x^3-2x) \quad (12) \\ &&= 2(x^3-2x)(3x-2) \quad (13) \end{aligned}$$

Would the Quotient Rule Show Similar behavior?
Try finding $F'(x)$ for

$$F(x) = \frac{1}{\sqrt[3]{x^2+1}} \tag{14}$$

Examples: Using the Chain Rule with Trig. Identities

Trigonometric Identities in Chapter 3 Section 3, are difficult to memorize and intuit. Restating the Pythagorean identities can be readily memorized:

$$\sin^2(x) = 1 - \cos^2(x) \tag{15}$$
$$\sec^2(x) = 1 + \tan^2(x) \tag{16}$$
$$\csc^2(x) = 1 + \cot^2(x) \tag{17}$$

While Chapter 2 gives enough information to work differentiate these functions without using the Chain Rule, we can simply take $f(u) = u^2$ and $g(x) = \sin(x), \sec(x), \csc(x)$ respectively.

Find $F'(x)$ if $F(x) = sin^2(x)$

By the chain rule:

$$F'(x) = 2(\sin(x)) \cdot \cos(x) \text{ or} \tag{18}$$
$$F'(x) = -2(\cos(x)) \cdot (-\sin(x)) \tag{19}$$

where,

$$f(u) = u^2, g(x) = \sin(x) \text{ or } \cos(x) \tag{20}$$

Try Equations (16) and (17) at home!

Class Challenges Worked Out at Home

Find $F'(x)$, if $F(x) = (x^2+1)^{-1/3}$

Let $f(u) = u^{-1/3}, g(x) = x^2+1$ such that $f'(u) = \frac{1}{3}u^{-4/3}, g'(x) = 2x$
By the Chain Rule:

$$F'(x) = -\frac{1}{3}(x^2+1)^{-4/3} \cdot 2x \tag{21}$$

Find $F'(x)$ if $F(x) = sec^2(x)$

$$F'(x) = \quad 2(\sec(x)) \cdot \sec(x)\tan(x) \text{ or} \tag{22}$$
$$F'(x) = \quad 2(\tan(x)) \cdot (\sec^2(x)) \tag{23}$$
$$f(u) = u^2, g(x) = \sec(x) \text{ or } \tan(x) \tag{24}$$

Find $F'(x)$ if $F(x) = csc^2(x)$

$$F'(x) = \quad 2(\csc(x)) \cdot (-\csc(x)\cot(x)) \text{ or} \tag{25}$$
$$F'(x) = \quad 2(\cot(x)) \cdot (-\csc^2(x)) \tag{26}$$
$$f(u) = u^2, g(x) = \csc(x) \text{ or } \cot(x) \tag{27}$$