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ELEMENTS OF MATHEMATICS, VOL.I

[For Higher Secondary (+2) Students]

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PREFACE

Today it is hardly necessary to talk about the importance of Mathematics in shaping all the successive human civilisations culminating in the present modern world. Whatever has been said or will be said about it are too little. Famous Indian Mathematician Bhaskaracharya, born in A.D.1114 conceived Mathematics as one embodied in the following verse:

यथा शिखा मयूराणां
नागानाम् मणयौर्यथा ।
तद्वत् वेदाङ्ग शास्त्राणां
गणितं मूर्द्धनिस्थितम् ।।

- Vedanga Jyotish by Lagda (About 1100 B.C)

(As crest in a peacock's feather, jewel in a Cobra's hood, Mathematics is the crest-jewel of all scientific knowledges.)

Mathematics, as a continuous human endeavour, seeks to capture the natural laws in the form of supreme abstract formulations and as such it has to depend upon infallible logic yielding the conclusions as eternal and absolute truth. It is a sublime discipline where falsehood or any inaccuracy is not entertained.

Since the study of Mathematics has become inescapable for the acquisition of any scientific knowledge, be it the farfetched subject like music or language, it is necessary to make the study of Mathematics more absorbing and interesting. The only way to do this is to encourage the students to pick up pen and paper and start solving the problems themselves. Just as one learns swimming only after entering inside the water, one enjoys the taste of the sweets only after putting it inside the mouth, Mathematics is learnt only through problem solving and this is the shortest route. No amount of lecturing on 'swimming' can equip one to swim.

The authors of the book, working under diverse constraints, are not fully certain if they have lived upto the expectations and aspirations of the members of the Orissa Mathematical Society in particular and teachers, students and the public in general.

Any suggestions for the improvement of the book shall be greatefully acknowledged in bringing out the successive editions.

The authors are greateful to the authorities of the Council of Higher Secondary Education and The Text Book Bureau for the patience and care in bringing out the present volumes in its present form.

Prof. G. Das

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Real Number System

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Mathematical Reasoning

How can it be that mathematics, being after all a product of human thought independent of experience, is so admirably adapted to the objects of reality ?

- Einstein

1.0 Historical Introduction

During the early phase of civilization, mathematics was primarily computational and intuitive in nature. Certain "rules of thumb" were devised by ingenuous minds to study problems of day-to-day life. The same practice prevails more or less even now at our elementary school level. This does not mean that whatever is taught at that level is incorrect. Only one is not accustomed to stating explicitly all assumptions upon which the conclusions are based. Consequently, though valid conclusions might have been drawn on many occasions, the exact canons of logic leading to these are not in view. This could be quite hazardous, if not rectified at a later stage, by forcing one to make erroneous judgements.

With the advent of Greeek civilzation towards the sixth century B.C. the concept of "proof" took root in mathematics. After the first treatise on logic "The Organon", was compiled by the great Greek philosopher Artistotle (fourth century B.C). Euclid of Alexandria (300 B.C.) wrote thirteen volumes of the book "Elements" on geometry which made ample use of the Aristotelian Logic. Until the later half of the nineteenth century, "Elements" had been deemed as perfection par excellence, but with the passage of time, various omissions, commissions and inconsistencies were detected in it by David Hilbert (1862 - 1943) and others, though that did not shelve the Euclidean geometry altogether since its basic propositions were largely in conformity with experience. With suitable rectifications in reasoning, while Euclidean geometry was restored once more, new geometries sprang up too. This was a miraculous victory of logical reasoning.

In the history of mathematics, similar happenings have taken place time and again. It happened once when persons like Bishop Berkley were critical about the very basis of the reasoning used in Calculus that was developed by Newton and Leibnitz (who were unaware of the already developed Indian treatment of the subject), to study motion and properties of geometric figures. Answers to these criticisms were, of course, promptly forwarded by other mathematicians and the method of Calculus became a great success in science. It was through many meandering routes around the turn of the century, that a necessity was felt to recast the entire edifice of mathematics in the language of sets. A lead was provided by the German mathematician Georg Cantor (1845 - 1918) who was drawn to it while investigating questions pertaining to trigonometric series and series of real numbers. It was, however, soon evident that certain logical contradictions were inherent in the system itself. The intrinsic beauty of

Cantor's reasoning which incidentally provided a proof of the abundant existence of transcendentals prompted mathematicians to find remedies by sharpening this reasoning. The theory that evolved in this way is termed as Mathematical Logic.

Subsequently, Turing and Godel made deeper studies of the system. This too gave rise to another interesting development. The logic that was deemed as the purest of pure mathematics now became a tool for the building of modern digital computers. It is interesting to note that the hypothesis of Turing and Post was ultimately realised by the Hungarian mathematician John Von Neumann (1903 - 1957).

In view of the fact that mathematics is essentially deductive in nature, it is inseparable from logic. Its propositions are deduced from the basic assumptions and given premises, in accordance with the laws of logic. The baisc assumptions of a mathematical system must be clearly stated at the start. Their careful choice gives immense freedom to the system by combining several mathematical structures together. Some of these structures might be relevant for applications to problems of life. The basic assumptions may be so formulated as to make the physical state amenable to mathematical handling, also including some formal structures at the same time. In this task too, logic has a role to play. A good grounding in logic can, therefore, serve a twofold purpose. Students can use it to set their mathematics right or else see how, for example, the computer hardware is designed.

It is worth mention that the scope of logic has widened vastly to cater to the present day requirements in the form of multivalent logics. Few examples along this line are the three valued logic established independently by J. Lukasiewicz (1920) and E. Post (1921). They also introduced many-valued logic. In fact, Buddha (563 B.C. to 483 B.C) is reported to have introduced multivalent logic along with his philosophy of Madhyam Marg. This logic has made a come back success through the pioneering work of L.A.Zadeh (1963) in the form of fuzzy logic.

Our scope here is, however, quite limited. So we confine ourselves only to the most basic concepts, principles and methods of logic, in order that valid conclusions can be drawn from given premises. An informal use of real numbers, rationals, integers, sets and function will be made in the sequel which will be treated systematically in due course. We use the logic which is mainly Aristotelian, furthered by the mathematician and logicist George Boole (1815-1864).

1.1 Mathematically acceptable Statements :

While presenting an argument or during a conversation, we express our ideas through sentences. Sentences are of various types, such as a question, an exclamation, an order or a wish, statement of a fact and so on. All types of sentences are not suitable for logical investigations. The following definition specifies, in an informal manner, the type of sentence that is admissible for common logical reasoning. Such sentences are called **statements** or **propositions.**

Definition : A proposition (or mathematically acceptable statement) is a declarative sentence which is either true or false, but not both.

This means that a proposition is a sentence which makes a definite assertion (of some fact) and it can be said in unequivocal terms if that assertion is true or false. It also appeals to common sense that a statement cannot be both true and false at the same time. For instance, it is possible to say definitely if Delhi is capital of India or Delhi is not capital of India and certainly it cannot be both. Hence 'Delhi is capital of India' and 'Delhi is not capital of India' are both propositions as the first is true and the second is false.

Truth value of a proposition : When a proposition is true, we say that its truth value is T and, if false, its truth value is F.

Consider a few examples of propositions along with their truth values mentioned against each.

"The earth moves round the sun", (T)

"A triangle has three angles", (T)

"Two is greater than five", (F)

"There is no even prime number", (F)

"The sun rises in the east every morning", (T)

Interrogative or exclamatory sentences, wishes, orders are examples of sentences that are not statements. Consider the following sentences, for instance :

"Make hay while the sun shines"	(an advice)
"Where are you going ?"	(a question)
"May God grant you long life." "Wish you a happy birthday."	(wishes)
"How beautiful is that flower !"	(exclamation)
"Bring a glass of water."	(order or command)

These are not statements as it is meaningless to talk of their truth values.

Sentences like

x is less than 5,

u is the father of v

are also not statements since they involve variables, though they become statements when the variables x, u, v are specified.

Look at the following sentences.

"Socrates was a wiseman."

"Ramesh is rich."

"Pintu is young."

"Raju is a good teacher".

These are also not statements since they contain words like "wise", "rich", "young" "good" whose truth or otherwise cannot be asserted in the absence of any measuring yardstick. Such sentences are called **fuzzy propositions** whose discussion is outside the scope of this book.

Note : Remember that a sentence cannot be called a statement if

i) It is an exclamation.

ii) It is an order or request.

iii) It is a question.

iv) It involves variable time such as 'yesterday', 'to day', 'tomorrow' and etc.

v) It inovolves variable places such as 'here', 'there', 'everywhere' and etc.

vi) It involves pronouns such as 'she', 'he', 'they' and etc.

vii) It involves adjectives/undefined terms or words such as 'good', 'beautiful', 'wise' and etc.

Example - 1:

Check whether the following sentences are statements. Give reason for your answer.

i) 7 is less than 5.

ii) Please open the door.

iii) The sun is a star.

iv) Ram is an intelligent student.

v) If $x = 1$, $x^2+2x+5 = 7$.

vi) There is no even prime number.

vii) x is less than 5.

viii) Moon is a star.

ix) Mind your own business.

x) Be punctual.

Solution :

i) "7 is less than 5" is false, so it is a statements.

ii) This is not a statement as it is meaningless to talk of its truth value. It is a request.

iii) This is a statement as it is true.

iv) This is not a statement since it contains the undefined term intelligent.

v) This is a statement since x is defined and truth value this statement is F.

vi) This is a statement since its truth value is F. (2 is the only even prime)

vii) This is not a statement since x is unknown.

viii) This is a statement as its truth value is F.

ix) This is not a statement as it is meaningless to talk of its truth value.

x) This is not a statement as it is an order.

Connectives, Compound Proposition :

A proposition is normally denoted by small Roman letters like p, q, r, s etc. to facilitate their abstract study. However, two or more propositions can be combined to form new propositions. There are four such key words and phrases, called **connectives,** playing a major role in combining propositions, which are :

or, and, only if, if and only if.

A proposition can be modified by the word 'not' which is also taken as a unary connective. **We shall elaborate upon these connectives including the modifier "not" in due course.** A proposition in which one or more of these connectives appear, is called a **composite** or **compound** proposition while its individual constituents are called its **prime components**. Remember that a proposition is prime by choice and not by itself. Consider the following statement.

"If it is Sunday or Thursday, then the orderly is on leave."

This is a composition of the following propositions :

p : It is Sunday

q : It is Thursday

r : It is Sunday or Thursday.

s : The orderly is on leave.

We may choose either p, q, s or r, s as the prime components of the given proposition.

Example-2

Find the connectives and component statements of the following compound statements and check whether they are true or false.

i) 2 is even and prime.

ii) 30 is divisible by 2, 3 and 5.

iii) All rational or irrational numbers are real numbers.

iv) Insat and moon are satellites of earth.

v) The constituents of water are oxygen and nitrogen.

vi) 12 is multiple of 2, 3 and 6.

vii) Arjun was son of Kunti and Pandu.

viii) A student qualifying either KVPY (Krishore Vaigyanik Protsahan Yojana) or IITJEE Advance can get admission in I.I.Sc. (Indian Institute of Science)

ix) Sun and eletric bulbs are luminous objects.

Solution :

i) The component statements are

p : 2 is even

q : 2 is prime

Here both the component statements are true and connecting word is 'and'.

ii) The component statement are

p : 30 is divisible by 2

q : 30 is divisible by 3

r : 30 is divisible by 5

Here above three prime statements are true and the connecting word is ‘and’.

iii) The component statement are

p : All rational numbers are real numbers

q : All irrational numbers are real numbers

Here both statements are true and connecting word is ‘and’.

iv) The component statement are

p : Insat is satellite of earth

q : Moon is satellite of earth

Both the statements are true and connecting word is ‘and’.

v) The component statements are

p : The constituent of water is oxygen

q : The constituent of water is nitrogen

The first statement is true and the second is false. The connecting word is ‘and’.

vi) The component statements are

p : 12 is multiple of 2

q : 12 is multiple of 3

r : 12 is multiple of 6

All the three statements are true and connecting word is ‘and’.

vii) The component statements are

p : Arjun was son of Kunti

q : Arjun was son of Pandu

Here both the statements are true and connecting word is ‘and’.

viii) The component statements are

p : A student qualifying KVPY can get admission in IISc.

q : A student qualifying IITJEE Advance can get admission in IISc.

Here both the statements are true and connecting word is ‘or’.

ix) The component statements are

p : Sun is a luminous object

q : Electric bulbs are luminous objects

Here both the statements are true and connecting word is ‘and’.

1.2 Negation

If proposition p is modified by the word "not", a new sentence results. We call it the **negation** of p and denote it by the symbol $\sim p$.

Consider the following propositions :

p : $2 + 3 = 5$

q : Six is not a prime number.

r : Three is less than one.

s : Nitin went to school today and yesterday.

The respective negations are :

$\sim p : 2 + 3 \neq 5$

(one can also write $\sim p$ as "2 +3 does not equal 5.")

$\sim q$: It is not true that six is not a prime number.

(It is also correct to write $\sim q$ as "Six is a prime number.")

$\sim r$: Three is not less than one

(One can also write $\sim r$ as "it is not true that three is less than one." It is not correct to write ~ r as "One is greater than three".)

$\sim s$: It is not true that Nitin went to school today and yesterday.

(It is not correct to write ~ s as Nitin did not go to school today and yesterday.)

It is often convenient to use the phrase "it is not true that" while writing negations of composite statements, when the use of the word, "not" is ambiguous, as in case of ~ s above.

We assume that **the negation of any proposition is a proposition.** Also, in case of the examples p, q, r, sited above, a proposition and its negation are seen to have opposite truth values. This fact is stated below as an axiom.

Axiom of negation :

For any proposition p, if p is true, then $\sim p$ is false and if p is false, then $\sim p$ is true.

The axiom of negation can be represented by the following table.

p	$\sim p$
T	F
F	T

The negation Table

1.3 Conjunction

Consider the following two propositions.

Raju went to the market.

Raju bought a note book.

These two propositions can be combined by using the connective "and" to form the following new sentence : Raju went to the market and bought a note book. Similarly, if p and q are two propositions, they can be combined by using the connective "and". The new sentence is called their **conjunction** and is written as $p \wedge q$, the symbol $\wedge$ standing for the word "and". The conjunction $p \wedge q$ becomes a proposition when a definite truth value is assigned to it. This is done by means of the following axiom.

Axiom of conjunction :

A conjunction $p \wedge q$ is true if both p and q are true and false if at least one of p, q is false.

The axiom of conjunction can be represented by the following table :

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

The conjunction table

In English language, a conjunction is sometimes expressed without using the word "and" explicitly as in the following examples.

I stepped into the bus, but got down near the college.

Ram came in, while Hari went out.

The night is dark, though stars are shining.

It is clear from the context of each that it is a conjunction.

There are also cases when 'and' is used, but it is not a connective; for example, 'the concert is a combination of vocal and instrumental music'. It is not a compound statement and 'and' is not a connective.

Example - 3

Write the truth values of the conjunctions :

i) $\sqrt{4} = 2$ and 2 is a prime

ii) $\sqrt{4}$ = -2 and -2 is an integer

iii) $\sqrt{3}$ is irrational and 3 is a complete square

iv) $\sqrt{3}$ is rational and 3 is a complete square

Solution :

Taking the first statement as p and the second as q in each case, the truth value of the conjuction $p \wedge q$ is demonstrated below, using the axiom of conjunction :

	p	q	$p \wedge q$	
i)	T	T	T	(Both p and q are true)
ii)	F	T	F	($\sqrt{4}$ only means positive square root of 4, so p is false; q is true)
iii)	T	F	F	(p is true, q is false)
iv)	F	F	F	(Both p and q are false)

Thus, statement- i is true, all others are false.

1.4 Disjunction

When two statements *p, q* are combined by the word "or" to form a new sentence, the latter is called their **disjunction** and is written as $p \vee q$. Consider the following propositions :

p : Sunday is a holiday

q : Thursday is a holiday

The disjunction $p \vee q$ then represents the sentence

$p \vee q$: Either Sunday is a holiday or Thursday is a holiday.

A few more examples of disjunction are :

Three is greater than five or it is not.

Nine is a prime or divisible by two.

The number $\sqrt{2}$ is either rational or irrational.

You shall come or I will go.

The disjunction of two propositions is taken to be a proposition and the rule to determine its truth value is as follows :

Axiom of disjunction :

A disjunction $p \vee q$ is true if at least one of p, q is true and false if both p and q are false.

The axiom of disjunction can be represented by the following table :

p	*q*	*p* *q*
T	*T*	*T*
T	*F*	*T*
F	*T*	*T*
F	*F*	*F*

The disjunction table

Note :

The connective 'or' is used in both exclusive and inclusive sense.

If $p \vee q$ is exclusive, then either of p or q can be true, not both. But in the inclusive case both the statements can be true in addition to either being true.

Take a few examples :

Example-4

i) A student can have Odia or Sanskrit as MIL in a higher secondary class.

ii) An employee either goes on leave or attends to his duty.

iii) The product of two real numbers is positive if both are positive or both are negative.

iv) If x equals 2 or -2, then x^2 equals 4.

Observe that in all these compound statements 'or' is used in exclusive sense.

Now take another set of example and observe that use of 'or' is in inclusive sense.

v) In a Zoo you can see reptiles or birds as you like.

vi) In a restaurant you can take veg. or non-veg. items.

vii) If p is a prime or a. counting number, then necessarily it is positive.

viii) Our skin becomes wet if exposed to rain or we swim in a pool.

1.5 Conditional

A proposition of the type "if p then q" is called a **conditional.** It can also be written as "p is sufficient for q", "q is necessary for p", "p only if q", "q provided that p" and so on. In symbols, we write $p \to q$. Here p is called the **antecedent** (or hypothesis) and q, the **consequent** (or conclusion).

"If in Δ ABC, $\angle$C is right-angled, then $AB^2 = BC^2 + AC^2$" is an example of a conditional in which the antecedent is 'in Δ ABC, $\angle$C is right angled' and the consequent is '$AB^2 = BC^2 + AC^2$.' A conditional is thus formed by combining two statements by the connective "if... then".

Axiom of conditional :

A conditional $p \to q$ is false only when p is true and q is false. In all other cases it is true.

Let p : Nita has ten rupees.

q : She will buy a bottle of Jam.

Then $p \to q$: If Nita has ten rupees, then she will buy a bottle of Jam.

The tabular representation of the axiom of conditional is as follows.

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

(The table for conditional)

We may notice that in the above axiom of conditional the falsity of '$p \rightarrow q$' occurs only when p is true and q is false.

The conditional $p \rightarrow q$ is frequently read as "if p then q" or "p only if q". We do not read $p \rightarrow q$ as "p implies q". The word "implies', for the time being, is reserved to be indicated by some other symbol which is closely associated with the connective $\rightarrow$. (See sc-1.7)

1. 6 Converse, biconditional, inverse, contrapositive and contradiction

Given a conditional $p \rightarrow q$, three other propositions related to it can be framed. These are :

(i) **Converse :** $q \rightarrow p$

(ii) **Inverse :** $\sim p \rightarrow \sim q$

(iii) **Contrapositive :** $\sim q \rightarrow \sim p$

The conjunction of a conditional $p \rightarrow q$ and its converse $q \rightarrow p$ is called a **biconditional** and is written as $p \leftrightarrow q$. Thus $p \leftrightarrow q$ is the same as $(p \rightarrow q) \wedge (q \rightarrow p)$. Equivalent ways of expressing a biconditional $p \leftrightarrow q$ are :

p if and only if q

p iff q

q if and only if p

p is necessary and sufficient for q

q is necessary and sufficient for p

The statement : "10 is a prime iff it has no proper divisor" is an example of a biconditional. Since a biconditional $p \leftrightarrow q$, is the conjunction of the conditional $p \rightarrow q$ and its converse $q \rightarrow p$, it follows from the axiom of conjunction that $p \leftrightarrow q$ is true when and only when $p \rightarrow q$ and $q \rightarrow p$ are both true. The table given below shows that this happens when p, q are both true or both false.

p	q	$p \rightarrow q$	$q \rightarrow p$	$\sim p$	$\sim q$	$\sim p \rightarrow \sim q$	$\sim q \rightarrow \sim p$	$\sim p \vee q$	$p \leftrightarrow q$
T	T	T	T	F	F	T	T	T	T
T	F	F	T	F	T	T	F	F	F
F	T	T	F	T	F	F	T	T	F
F	F	T	T	T	T	T	T	T	T

Table for conditional, converse, inverse, contrapositive, biconditional

The truth table of biconditional is given below for sake of emphasis.

P	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

When two composite propositions formed out of the same prime components have the same truth value for each possible combination of truth values of their prime components, we call them equivalent and write

$$P_1 \equiv P_2 \text{ (or } P_1 \Leftrightarrow P_2\text{)}$$

The above table shows that

(i) $p \rightarrow q$ and $\sim p \vee q$ are either both true or both false and hence

$$p \rightarrow q \equiv \sim p \vee q$$

(ii) The converse $q \rightarrow p$ and inverse $\sim p \rightarrow \sim q$ are either both true or both false, so that

$$q \rightarrow p \equiv \sim p \rightarrow \sim q.$$

The equivalence or otherwise, of composite propositions can be easily verified by constructing a **truth table** by which we mean a table in which all possible truth values are assigned to the prime components and corresponding truth values of the composite propositions are then computed by using the axioms described earlier. We have already constructed such tables. Note that if the number of prime components is 2, then the table contains $2^2 = 4$ horizontal rows. (What happens for n prime components ?)

Some further examples of equivalent propositions are the following :

(i) $p \equiv \sim(\sim p)$

(ii) $p \vee q \equiv \sim(\sim p \wedge \sim q)$

(iii) $\sim(p \wedge q) \equiv \sim p \vee \sim q$

(iv) $\sim(p \vee q) \equiv \sim p \wedge \sim q$

(iii) and (iv): De Morgan's laws

(v) $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

(vi) $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

These can be verified by constructing corresponding truth tables. We only construct a truth table for verifying (vi).

p	q	r	$q \wedge r$	$p \vee (q \wedge r)$	$p \vee q$	$p \vee r$	$(p \vee q) \wedge (p \vee r)$
T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T
F	T	F	F	F	T	F	F
F	F	T	F	F	F	T	F
F	F	F	F	F	F	F	F

1. 7 Tautology, Implication, Double implication

If a composite proposition p is always true for all possible assignment of truth values to its prime components, then it is called a **tautology.** Here also, the truth table of a given proposition determines if it is a tautology or not.

The following are some examples of tautology.

(i) $p \vee (\sim p)$ **(Law of excluded middle)**

(ii) $\sim (p \wedge \sim p)$ **(Law of contradiction)**

(iii) $p \leftrightarrow \sim (\sim p)$ **(Law of double negation)**

The above three tautologies are amongst some of the widely known laws of classical (that is, Aristotelian) logic.

(iv) $\left(p \wedge (p \to q)\right) \to q$

(v) $\left((p \to q) \wedge (p \to r)\right) \to \left(p \to (q \wedge r)\right)$

(vi) $\left((p \to q) \wedge \left(q \to r\right)\right) \to \left(p \to r\right)$ **(Principle of syllogism)**

(Principle of syllogism, is a basic principle of mathematical reasoning.)

(vii) $\left(\sim q \to \sim p\right) \leftrightarrow \left(p \to q\right)$ **(Law of contrapositive)**

Law of the contrapositive is a special case of the well known classical principle of **reductio *ad* absurdum** which was widely used by Euclid in his axiomatic treatment of geometry and has been in use since then in mathematical proofs. This principle simply asserts that a conditional $p \to q$ can be proved to be true if $p \wedge \sim q$ implies a **contradiction** making use of other established tautologies, if necessary. **By "contradiction" we mean a proposition which is false for all possible assignments of truth values to its prime components.** For example $p \wedge \sim p$ is a contradiction.

Consider, for instance, the following proposition from geometry :

The base angles of an isosceles triangle are congruent.

This is proved by supposing that in a given isosceles triangle, the base angles are not congruent and then arriving at a conclusion that an angle is congruent to a proper part of it which forms a contradiction in conjunction with other axioms of geometry. The conclusion, therefore logically follows from the hypothesis of the proposition.

We construct a table below to prove (vii) and leave the rest to the reader.

p	q	$\sim p$	$\sim q$	$-q \to \sim p$	$p \to q$	$(\sim q \to \sim p) \leftarrow \to (p \to q)$
T	T	F	F	T	T	T
T	F	F	T	F	F	T
F	T	T	F	T	T	T
F	F	T	T	T	T	T

If a conditional $p \to q$ is a tautology, then we say that p implies q and we write $p \Rightarrow q$ to indicate this implication.

Note : In order to avoid a point of confusion regarding the use of notations $\rightarrow$ and $\Rightarrow$ some explanation is given below.

First note that the symbol $\rightarrow$ is a connective and for two statements p and q (true or false), $p\rightarrow q$ is another statement which may be true or false as per the axioms of conditional.

On the other hand **the symbol $\Rightarrow$ is not a connective**. For example we cannot write p $\vee (q \Rightarrow r) \wedge (r \Rightarrow p)$.

The statement $p\Rightarrow q$ means that $p\rightarrow q$ is always a true statement i.e. from the truth of p, truth of q follows. In other words it means that q is true whenever p is true. It is trivial that $p\rightarrow q$ is true when p is false. Thus $p \Rightarrow q$ means that $p \rightarrow q$ is a tautology. Thus in any logical argument once we assume that the hypothesis p is true if we have to automatically accept that the conclusion q is true, then we use the symbol $p\Rightarrow q$. Here there is no possibility of q being false when p is true.

In simple words $p \Rightarrow q$ means : if the statement p is true then the statement q is true. (see1.8)

Similary if a biconditional $p \leftrightarrow q$ is a tautology, we say that p implies and is implied by q. We write $p \Leftrightarrow q$ to indicate this double implication.

$p \Rightarrow q$ is read as "p implies q".

$p\Leftrightarrow q$ is read as "p implies and is implied by q".

It should be noted that neither $\Rightarrow$ is a connective nor $p \Rightarrow q$ is a statement formula. Similar is the case for $\Leftrightarrow$ and $p \Leftrightarrow q$. We construct below truth tables for $p\rightarrow p$, $p\Lambda (p\rightarrow q) \rightarrow q$.

p	$p \rightarrow p$
T	T
F	T

p	q	$p\rightarrow q$	$p \Lambda (p\rightarrow q)$	$p \Lambda (p\rightarrow q) \rightarrow q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

From the above tables we observe that the conditionals $p\rightarrow p$ and $p\Lambda(p\rightarrow q) \rightarrow q$ are tautologies.

Hence $p \Rightarrow p$ and $p\Lambda (p\rightarrow q) \Rightarrow q$.

The table for $(\sim q \rightarrow \sim p) \leftrightarrow (p \rightarrow q)$ given in this section illustrates that $(\sim q \rightarrow \sim p) \leftrightarrow (p \rightarrow q)$ is a tautology.

So it is a double implication.

Hence $(\sim q \rightarrow \sim p) \Leftrightarrow (p \rightarrow q)$,

i.e. $\sim q \rightarrow \sim$ p *and* $p\rightarrow q$ are equivalent statements.

1.8 Quantifiers

Quantifiers are used to describe variables in relation to a predicate. We mention two types of quantifiers, 'existential' and 'universal'.

The existential quantifier is symbolized as '$\exists$', read as 'there exists' or 'there is' and the symbol for universal quantifier is '$\forall$', read as 'for all' or 'for every'.

Few examples are in order.

Example-5

i) 'There is a prime which is even' can be written, using existential quantifier, as :

$(\exists p)$ (p is an even prime)

'is an even prime' is the predicate which has been quantified by 'p'.

We know, there is only one such prime.

This is symbolically expressed as :

$(\exists! p)$ (p is an even prime) (In fact, 2 is the only even prime)

'$\exists!$' is read as 'there uniquely exists' or 'there exists only one'.

ii) We know that square of every odd number is also odd and vice versa. This is expressed, using universal quantifier, as

$\forall\, n \in N$, n is odd $\Leftrightarrow n^2$ is odd.

iii) $\forall\, x, y \in R$, $xy = yx$ and $x + y = y + x$

iv) $\forall\, x \in R,\ x^2 \geq O$

Later you will come across variety of cases of use of quantifiers.

Negation of a quantified statement :

This will be clear through the examples :

Example-6

i) $\sim (\forall\, x \in R,\ x^2 \geq 0) \equiv \exists x \in R,\ x^2 < O$

or $\exists x \in R,\ \sim (x^2 \geq 0)$

ii) $\sim (\exists x,\ \text{Sin}\, x = 0) \equiv \forall\, x \in R,\ \mathit{Sin} x > 0$ or $\text{Sin}\, x < 0$

iii) $(\forall\, n \in N)$ ($n^2 - 79n + 1601$ is a prime)

Negation of this statement is

$(\exists n \in N)$ ($n^2 - 79n + 1601$ is not a prime)

Remark :

We have both the statement and its negotion at hand. Which one, you think, is true ?

You can veryfy that 'n^2-79n+1601 is a prime for as many upto 79 verifications, starting from n=1.

As in case of the so called 'experimental proof' of elementary geometry, we are likely to jump to the conculusion (?) that the proposition is true ! (We, infact, verify only three figures in elementary geometry!)

But once you put n=80, $n^2-79n+1601 = 1681 = 41^2$ which is not a prime.

This proves that the second statement, i.e. negation of the first, is true.

Thus, only one example is sufficient to disprove the first statement. Such type of examples are known as **counter examples** and are of great importance in mathematics. If we fail to prove something, we cannot say that it is false. But if we give an example where it is false, then and we disprove the proposition.

We shall come back to similar such examples when we discuss 'proof by induction' in a later chapter.

Exercises- 1(a)

1. Choose the correct answer from the given choices :

Negation of 'Paris is in France and London is in England' is

A) Paris is in England and London is in France.

B) Paris is not in France or Landon is not in England.

C) Paris is in England or London is in France.

D) Paris is not in France and London is not in England.

ii) The conditional $(p\wedge q) \rightarrow p$ is

A) a tautology B) a contradiction

C) neither a tautology nor a contradiction

D) non of these

iii) Which of the following is a contradiction ?

A) $(p\wedge q) \wedge \sim (p\wedge q)$ B) $p\vee (\sim p\wedge q)$

C) $(p\rightarrow q) \rightarrow p$ D) None of these

iv) Which of the following is logically equivalent to $\sim(\sim p \rightarrow q)$

A) $p \wedge q$, B) $p\wedge\sim q$, C) $\sim p\wedge q$ D) $\sim p\wedge\sim q$

v) $(\sim(\sim p)) \wedge q$ is equivalent to

A) $\sim p \wedge q$ B) $p \wedge q$ C) $p \wedge \sim q$ D) $\sim p \wedge \sim q$

vi) If p:It rains today, q: I go to school, r: I shall meet any friend and s: I shall go for a movie, then which of the following is the proposition :

If it does not rain or if I do not go to school, then I shall meet any friend and go for a movie ?

A) $\sim(p \wedge q) \rightarrow (r \wedge s)$ B) $\sim(p \wedge \sim q) \rightarrow (r \wedge s)$

C) $\sim(p \wedge q) \rightarrow (r \vee s)$ D) non of these

vii) Which of the following is true ?

A) $p \rightarrow q \equiv \sim p \rightarrow \sim q)$ B) $\sim(p \rightarrow \sim q) \equiv \sim p \wedge q)$

C) $\sim(\sim p \rightarrow \sim q) \equiv \sim p \wedge q)$ D) $\sim(p \leftrightarrow q) \equiv [\sim(p \rightarrow q) \wedge \sim(q \rightarrow p)]$

viii) The Inverse of the proposition $(p \wedge \sim q) \rightarrow r$ is

A) $\sim r \rightarrow (\sim p \vee q)$ B) $\sim p \vee q \rightarrow \sim r$

C) $r \rightarrow (p \wedge \sim q)$ D) non of these

ix) The contrapositive of $(p \vee q) \rightarrow r$ is

A) $r \rightarrow (p \vee q)$ B) $\sim r \rightarrow (p \vee q)$

C) $\sim r \rightarrow (\sim p \wedge \sim q)$ D) $p \rightarrow (q \vee r)$

x) Which of the following is inverse of the proposition : “If a number is a prime, then it is odd.

A) if a number is not prime, then it is odd.

B) if a number is not a prime, then it is not odd.

C) if a number is not odd then it is not a prime.

D) if a number is not odd, then it is a prime.

2. Give examples, **five** in each case, of sentences that are

(i) propositions

(ii) not propositions

3. Illustrate the use of all connectives and the modified ‘not’ in five separate examples of propositions.

4. Try to construct an example of a proposition involving all connectives and also the modifier ‘not’.

5. Which of the following sentences are propositions and which are not ? Write with reasons:

(i) $2 < 5$

(ii) Is $9 < 3$?

(iii) x is greater than 100.

(iv) Why are you crying ?

(v) May God grant you long life.

(vi) Cuttack is a big city.

(vii) It is possible that there is life in Mars.

(viii) Ram is a friend of Hari.

(ix) $x^2 - x + 1 = 0$.

(x) Oh ! What a scenery ?

(xi) You must go to school everyday.

(xii) It was raining yesterday.

6. Write down negations of each of the following :

(i) If you read, you will pass.

(ii) John is a friend of Thomas.

(iii) Fifteen is greater than five.

(iv) Either Pramod is clever or he is laborious.

(v) Money is necessary for happiness.

(vi) It is raining and Mahanadi is flooded.

(vii) Pen is mightier than sword.

(viii) $|x|$ is equal to either x or $-x$

(ix) It is raining and it is cool.

(x) $3+6>8$ and $2+3 < 6$

7. Translate the following propositions into symbolic form, stating the prime components in each case.

(i) If you do not work hard, you will repent.

(ii) Jamini wil be rewarded if and only if he is punctual.

(iii) If there is a will, there is a way.

(iv) Time and tide waits for none.

(v) 29 is a prime number which is a sum of two squares.

(vi) Life is short, but virtue is lasting.

(vii) If the boy is poor, then he will be hungry and if he is hungry, then he cannot be honest.

(viii) A year consists of twelve months while a month does not consist of more than thrityone days.

(ix) If the government cannot solve the unemployment problem, then public opinion will rise against it which will lead to a strengthening of opposition.

(x) Chinu and Minu went to Calcutta, but Minu came back earlier since she lost all her money.

8. Let $p, q\ r$ denote respectively the statements : "you are honest", "you are laborious" and "you will receive a promotion". Translate the following statements into English language :

   (i) $(p \vee q) \to r$

   (ii) $\sim r \to \sim p$

   (iii) $\sim (p \vee q) \to \sim r$

   (iv) $[r \vee (\sim p)] \to \sim q$

   (v) $p \wedge q \wedge r$.

9. Construct truth tables for the following and indicate which of these are tautologies :

   (i) $p \wedge q \to p \vee q$

   (ii) $p \wedge q \to p$

   (iii) $p \wedge (p \to q) \to q$

   (iv) $p \to p \wedge q$

   (v) $p \to (\sim p \to q)$

   (vi) $\sim p \wedge (p \wedge q) \to q$

   (vii) $(p \vee \sim q) \wedge (q \vee \sim p)$

   (viii) $p \to (\sim q \wedge r)$

   (ix) $(p \to q) \to [(q \to r) \to (p \to r)]$

   (x) $p \vee q \to \sim (p \wedge q)$

   (xi) $(p \to \sim p) \to \sim p$

   (xii) $(\sim p \vee p) \to (\sim q \vee q)$

   (xiii) $((p \wedge q) \to p) \to q$

   (xiv) $(p \leftrightarrow q) \wedge (q \leftrightarrow r) \to (p \leftrightarrow r)$

   (xv) $[p \to (p \vee q)] \to [(q \to (p \wedge q)]$

10. If p has truth value T, what can be said about the truth values of

    (i) $\sim p \wedge q \to p \vee q$

    (ii) $p \vee q \to \sim p \wedge q$

11. Determine the truth values of $p \leftrightarrow \sim q$ when the biconditional $p \leftrightarrow q$ has truth value (i) F, (ii) T

12. State the converse, inverse and contrapositive of each of the following propositions, stating it as a conditional, wherever necessary.

    (i) If ABC is equilateral, then its three angles are congruent.

    (ii) If Gopal is clever, then he is rich.

    (iii) $p \to \sim q$

    (iv) Sum of two odd integers is even.

    (v) The square of an integer is a natural number.

    (vi) A paralellogram which is inscribed in a circle is a rectangle.

    (vii) The ground being wet, there has been rainfall at night.

13. Are the following pair of statements negation of each ather ?

    i) The number π is not a rational number.

       The number π is not an irrational number.

    ii) The number π is a rational number.

       The number π is an irrational number.

14. Write the component statement of the following compound statements and check whether the compound statement is true or false.

i) 24 is multiple of 4 and 6

ii) The school is closed if there is a holiday or a sunday.

iii) 7 is an rational number or an irrational number.

iv) 57 is divisible by 2 or 3.

v) All things have two eyes and two legs.

vi) 2 is an even number and a prime number.

vii) Every parallelogram is a trapezium or a rhombus.

15. Identity the Quantifiers of the following statements.

i) There exists a number which is equal to its square.

ii) For every real number x, x is less than x+1

iii) There exists a capital for every state of India.

iv) For all $x \in R$, $\sin^2 x + \cos^2 x = 1$

v) There exists an even prime number other than 2.

vi) For every negative interger x, x^3 is also a negative integer.

vii) For every real number x, $x^2 \neq x$.

16. Write the negation of following statement :

i) Every living person is not 150 year old.

ii) There exists $x \in N$, $x+3=10$

iii) All the students completed their homework.

iv) There exists a number which is equal to its square.

v) For every real number x, $x+4$ is greater than x.

vi) Everyone who lives in India is an Indian.

1.9 Validity of Statements

A Statement is said to be valid or invalid according as it is true or false.

So validating a statement is the process of showing it to be true.

This depends upon which of the connectives, modifiers or quantifiers are used in the statement.

i) Validity of statement with 'AND'

Steps To validate $p \wedge q$,

Step-1 Show statement 'p' is true

Step-2 Show statement 'q' is true

ii) Validity of statement with ‘or’

Casc-1 : Assuming that is falsc, show that q must bc truc

Case-2 : Assuming that q is false, show that p must be true.

iii) Validity of statement with “If ....then”

If p and q are two mathematical statements, then to show ‘r: if p then q’ is true, we can adopt any of the following methods :

a) Direct Method : Assume p is true and show q is true ie $p \Rightarrow q$.

b) Contrapositive Method : Assume ~q is true and show ~p is true ie $\sim q \Rightarrow \sim p$.

c) Contradiction Method : Assume that p is true and q is false and obtain a contradiction from assumption.

d) By giving a counter example : In mathematics, counter examples are used to disprove the statement : ie to prove the given statement r is false we give a counter example. Consider the following statement. “r: All prime numbers are odd”. Now the statement ‘r’ is false as 2 is both prime and even number.

iv) Validity of the statement with “If and only if”.

If p and q are two statements

then to show the statement “r : p if and only if q” is true, we proceed as follows :

Step-1 : Show if p is true then q is true.

Step-2 : Show if q is true then p is true.

Example-7

Given below are two statements

p : 25 is a multiple of 5

q : 25 is a multiple of 8

Write the compound statements connecting these two statements with ‘and’ and ‘or’. In both cases check the validity of the compound statement.

Solution :

i) By using connective “and”, the compound statement is “25 is multiple of 5 and 8”. We know that 25 is a multiple of 5. So, the statement p is true but “25 is a multiple of 8” is false i.e the statement q is false.

Hence the compound statement $p \wedge q$ is false.

ii) By using connective 'or' the compound statement is "25 is multiple of 5 or 8".

Since the statement 'p' is true and the statement 'q' is false, the compound statement $p \vee q$ is true.

Example-8

Show that the statement

p : "If x is a real number such that $x^3 + x = 0$" then $x = 0$ is true, by (i) Direct method (ii) method of contradiction and (iii) method of contrapositive.

Solution :

Let q are r be the statements given by

q : x is a real number such that $x^3+x = 0$

r : x is 0

Then p : If q, then r.

i) Direct Method : Let q be true

$\Rightarrow x$ is a real number such that $x^3+x = 0$

$\Rightarrow x$ is a real number such that $x(x^2+1) = 0$

$\Rightarrow x = 0$ ($\because x \in R$, so $x^2+1 \neq 0$)

$\Rightarrow$ r is true.

Thus q is true $\Rightarrow$ r is true. Hence p is true.

ii) Method of contrapositive: Let 'r' be not true. then,

r is not true

$\Rightarrow x \neq 0, x \in R$

$\Rightarrow x\ (x^2+1) \neq 0, x \in R$

$\Rightarrow$ q is not true

Example-9

By giving counter example, show that the following statement is false.

If n is a odd integer, then n is prime.

Solution :

It is in the form "if p then q". We have to show that this is false. To show this we look for an odd integer n which is not a prime number. 9 is one such number. So n = 9 is a counter example. Thus we conclude that the given statement is false.

Example-10

Check the validity of the statement

"The integer n is odd iff n^2 is odd"

Solution :

Let p and q be the statements given by

p : the integer n is odd.

q : n^2 is odd.

The given statement is "p if and any if q".

In order to check its validity, we have to show the validity of the following statements.

Step-1 : "If p, then q"

Step-2 : "If q, then p"

Step-1 : "If p then q" is given by

If n is an odd integer then n^2 is odd"

Let us assume that n is odd. Then,

$n = 2m + 1, m \in Z$

$\Rightarrow n^2 = (2m + 1)^2$

$\Rightarrow n^2 = 4m(m + 1) + 1$

$\Rightarrow n^2$ is odd. $(\because 4m(m+1)$ is even)

Thus, n is odd then n^2 is odd.

$\therefore$ "If p then q" is true.

Step-2 : 'If q then p" is given by

If n^2 is odd then n is odd.

To check the validity of this statement, we will use contrapositive method. So, let 'n' be an even integer.

$\Rightarrow n = 2k$, for $k \in Z$

$\Rightarrow n^2 = 4k^2$

$\Rightarrow n^2$ is an even integer

$\Rightarrow n^2$ is not an odd integer.

Thus, n is not odd $\Rightarrow n^2$ is not odd :

$\therefore$ "If q then p" is true.

Hence, from step-1 and step-2, we conclude that "p if and only if q" is true.

Exercises- 1 (b)

1. Check the validity of the following statements.

   i) p : 100 is a multiple of 5 and 4.

   ii) q : 125 is a multiple if 5 and 7.

   iii) r : 60 is a multiple of 3 or 5.

2. Check the validity of the statements given below by the method given against each.

   i) "The sum of an irrational number and a rational number is irrational" (by contradiction method).

   ii) "If n is a real number with $n > 3$, then $n^2 > 9$ (by method of contradiction).

   iii) "If x, y are integers such that xy is odd then both x and y are odd" (by method of contrapositive)

   iv) "If x is an integer and x^2 is even then x is also even" (method of contrapositive)

3. By giving counter examples, show that the following statements are not true :

   i) If measures of all the angles of a triangle are equal, then the triangle is an obtuse angled triangle.

   ii) For every real number x and y, $x^2=y^2 \Rightarrow x=y$.

   iii) The equation $x^2-1=0$ does not have any root lying between 0 and 2.

4. Check the validity of "If I do not work, I sleep.

   If I am worried, I will not sleep. There fore, if I am worried, I will work".

5. Let a and b be integers. By the law of contrapositive prove that if ab is even then either a is even or b is even :

●

CHAPTER 2

Sets

No one can expel us from the paradise which Cantor has created for us.

- Hilbert

2.1 Sets :

Mathematics is understood to be the science of numbers, magnitude and forms, but its scope has so vastly expanded that this sort of definition is hardly comprehensive enough to characterise mathematics. Its scope has been expanded to such an extent that two branches of mathematics do not appear to be having anything in common. This has grown to the absurd limit of resembling the assertions of the fabled seven blind men who were set on describing what an elephant is like. Could mathematics be looked at more organically ? Indeed, it has been demonstrated that such a scheme of putting mathematics on a common footing is possible. As mentioned in section 1.0, earliest attempts in this direction were made by G. Cantor. Cantor's theory of sets was subject to severe criticism because of certain contradictions that were observed to have crept into his theory, by Russel, Burali- forti and many others. However, logicians and mathematicians of the early twentieth century were luckily able to sort out the anomalies in Cantor's methods and salvage the central theme of his ideas that were about to be consigned to oblivion. As it is beyond our present scope and objective to embark upon a formal theory of sets, we do not make any such attempt here and start with a rather informal approach.

The word 'set' and other similar words are frequently used in our day-to-day vocabulary, such as a **set** of cards, a **pack** of wolves, a **swarm** of bees, a **bunch** of keys, a **shoal** of fish, a **herd** of cattle, a **pride** of lions, a **flock** of sheep, a **group** of children and so on. Such phrases express the common notion of 'set', but experience shows that any attempt to produce technical definitions of each and every word of our vocabulary is bound to be futile. Indeed, if we look up at the meaning of a word in the dictionary, we find it composed of certain words which are, in turn, explained in terms of certain other words. The process, when continued indefinitely tends to be ultimately circular, since our vocabulary is **finite.** It follows that in every scheme of logical construction, there ought to be certain **primitive** or **undefined** words in terms of which every other concept of the system can be explained. In our present treatment of sets, we take 'set' as an **undefined** term. Mathematicians like John Von Neumann attempted to define the word 'set' in terms of an undefined term 'class' and a number of whole books have been written on the subject, but we do not wish to dwel upon those here. We shall be primarily concerned with sets of specific and well defined objects.

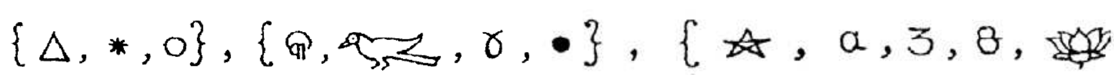
Thus, the undefined word 'set' is informally understood here as a **definite collection** of **well defined** objects which are called its **elements** (or members). By this, we mean that given an object x and a set A, it is possible to state exclusively if x is **an element of A** or **it is not.** In addition, if x, y are given elements of a certain given set, it is possible to state exclusively if $x = y$ or if $x \neq y$. As an example, let A denote the collection of all small letters in the English alphabet. If x denotes the natural number 1, then x is **not** an element of **A** and secondly, any two letters of the alphabet are seen to be distinct. Thus, A is a set.

It is natural to call two sets A and B **equal** if they have the same elements and we then write A = B.

We may describe a set by specifying its elements. This is done in two ways. One way is to **enlist** or **enumerate** all its elements. In that case we usually write all the elements, separated by commas and enclosed within braces. For example, the set representing the English (that is, Roman) alphabet can be written as

$$\{a, b, c, d, e, f, g, h, i, j, k, l, m, n, o, p, q, r, s, t, u, v, w, x, y, z\}$$

Similarly, $\{1, 2, 3, 4, 5\}$ represents the set consisting of the natural numbers 1, 2, 3, 4, 5 (and only these). The following are also examples of sets.



Observe that there is **apparently** no restriction as to which objects can be put together in a list to form a set, but if there are too many elements in a set, this method may not be convenient or even practicable. If we want to express the set of Indian citizen in a **tabular form**, the task would be stupendous. Even when we are able to write names of 500 Indian citizens on one page of a standard crown size book, one would need about 1800,000 pages (amounting to 3600 books, each consisting of 500 pages). The reader can easily construct examples of sets, describing each by a list which is impractical. To circumvent this, we have another way of describing a set, that is by devising a criterion to decide if a given object is an element of the set in question, or not. Thus, we may write the set of all Indian citizens simply as

$$\{ x : x \text{ is an Indian national} \}$$

(we are not concerned in this context as to how Indian citizenship is to be determined). Similarly, the set of natural numbers and the set of integers can be expressed respectivelyas $\{n : n \text{ is a natural number}\}$.

$$\{n : n \text{ is an integer} \}.$$

We can, however, express the set of natural numbers and integers also in a list by writing respectively $\{0, 1, 2, 3, ...\}$,

$$\{......, -3, -2, -1, 0, 1, 2, 3, ...\}$$

Remark As usual

$$N = \{1, 2, 3, ...\}$$

denotes the set of **positive** natural numbers. Since 0 is a natural number (see Chapter 4), we may write N* to denote the set of **all** natural numbers; that is,

$$N^* = \{0, 1, 2, ...\}$$

Note that in the above case, we have mentioned only some of the elements explicity and have used dots in place of others. This is because we can never exhaust all the elements by explicit writing, however hard we may try. In our previous example of the set of Indian citizens, one could at least, in principle, write down the complete list of elements, ignoring the time constraint and other factors. In case of natural numbers and integers, this is not possible since to any list of natural numbers or integers, a new natural number or integer could be added, that is not there already. This is beacuse of the presence of an **'infinity'** of elements in these, a concept to which we shall return later.

Nevertheless, we have now seen that an 'infinite' can at times be expressed by a list, provides that the elements that are explicitly mentioned, must clearly indicate which other elements are there in the list.

Expressing a set by a list of all its elements is called **defining** (or **describing) a set by extension** (or by **tabular method**) whereas expressing a set by a criterion (or rule) which decides its membership, is called **defining the set by intention** (or by **set builder method** or by **method of specification**). To define a set by intention, a proposition $p(x)$ involving x (since x is a variable, it is customary to call $p(x)$ a **propositional variable**) is to be found such that $p(x)$ has truth value **T** for and only for elements x of A. We call $p(x)$, the **defining property** of A and write $A = \{x : p(x)\}$.

Under this scheme, the set representing the English alphabet can be written as

$\{x : x$ is a letter of the English alphabet$\}$

Some further examples are :

$\{n : n$ is an integer$\}$, $\{n : n$ is a natural number$\}$,

$\{x : x$ is a student studying in Ravenshaw college during 1997$\}$ and so on.

The reader should construct an example of a set that can be expressed through different defining properties.. A set may also be expressed by a list as well as by a defining property as in case of the set $\{3, 6, 9\}$ which can also written as $\{3k : k = 1, 2, 3\}$ or $\{x : x = 3$ or $x = 6$ or $x = 9\}$

Since a **set is specified by its elements the order in which the elements are listed in a tabular form is immaterial.** For instance, $\{a, b, c,\} = \{b, c, a,\} = \{c, a, b\}$ etc

If A is given set and x is a given object, we write

$x \in A$ (read as x belong to A) if x **is** an element of A, and

$x \notin A$ (read as x does not belongs to A) if x **is not** an element of A.

For technical reasons, we talk of a set having **no** elements. Indeed, if $p(x)$ denotes a proposition which is false for every x to which it applies, such as "$x \neq x$", then $p(x)$ defines the set $\{x : p(x)\}$ which has no element. We call this set the **empty** (or **null**) set and write it as ϕ . Note that the empty set is **unique** since the membership relation is well defined in this case.

Warning : Before embarking upon the study of sets, a word of 'caution' is due. We must not suppose that every property about objects defines a set. Unrestricted use of defining properties may lead to contradictions. For instance, consider the property

$p(X) : X \notin X.$

Nothing looks unusual about such a property. Many sets have this property; for instance, a book contains pages, but is not a page by itself; a library consists of books, but it is not a book and so on. Now suppose that this property defines the set $R = \{X : X \notin X\}$. It must be possible to determine if $R \in R$ or $R \notin R$. However, if $R \in R$, then the defining property of R implies that

R $\notin$ R which contradicts the supposition that R $\in$ R. Similarly, the supposition 'R $\notin$ R' confers on R the right to be an element of R, again leading to a contradiction. The only plausible conclusion is that the property "X $\notin$ X" cannot define a set. This contradiction is the essence of the famous **Russell's paradox.**

In order to avoid this paradox, we may have to always ensure that any set that we talk about, is not a member of itself. It is also convenient to choose a 'largest' set in any given context, called the **universal set** and confine our studies to elements of the universal set only. This set may vary in different contexts, but in a given setup, the universal set should be so specified that no occasion arises, ever to digress from it. Otherwise, there is every danger of colliding with paradoxes such as the Russell's paradox. We shall return to this topic later.

2.2 Subset and Power set

We now come to some basic concepts about set.

Definition :

Let A, B be sets. We say that A is a **subset** of B, or B is a **superset** of A and write

$A \subset B$ (or $B \supset A$)

if every element of A is an element of B; that is, $x \in A \Rightarrow x \in B$.

Example 1 :

The following are some examples of subsets.

(i) The set N of positive natural numbers is a subset of the set Z of integers.

(ii) Let A denote the set of all animals and M stands for the set of all mammals. Then $M \subset A$.

(iii) Let A be the English alphabet and V denote the set of vowels in it. Then $V \subset A$.

(iv) Let C be the set of articles in Indian constitution and F denote the set of articles in it about fundamental rights. Then $F \subset C$.

Examples can be multiplied indefinitely. Definition of subsets implies that **every set A is a subset of itself.** If $A \subset B$ and B contains at least one element that is not in A (so that $A \neq B$), we call A a **proper subset** of B. Thus, B is not a proper subset of itself. The set {1, 2, 3) is a proper subset of {1, 2, 3, 4).

Definition of subset implies that the **empty set ϕ is a subset of every set.** Indeed, the subset criterion is **vacuously** fulfilled by ϕ in the sense that there is no element in ϕ to contradict it.

We also note that two sets A and B are equal if and only if $A \subset B$ and $B \subset A$. This property of sets is called the **property of extension.** This important property is often used to prove the equality of sets. For example, consider the sets :

$A = \{n : n \in Z \text{ and } n \geq 0\}$,

$B = \{n : n \text{ is a natural number}\}$

Since every nonnegative integer is a natural number, $A \subset B$ and since every natural number is nonnegative integer, $B \subset A$. Hence $A = B$.

The following properties are easily deducible, where X, Y, Z are sets.

(i) $X = Y \Leftrightarrow [x \in X \Leftrightarrow x \in Y]$

(ii) $X = X$

(iii) $X = Y \Rightarrow Y = X$

(iv) If $X = Y$ and $Y = Z$, then $X = Z$

(v) If $X \subset Y$ and $Y \subset Z$, then $X \subset Z$

If A , B are sets and A is not a subset of B, then A must have atleast one element which is not in B. We write $A \not\subset B$ to express this fact.

If A is any set, the collection of all its subsets is another set which we call the **power set** of A and denote it by P (A). Some examples are cited below.

Example 2 :

(a) $A = \phi$, $P(A) = \{\phi\}$

(b) $A = \{a\}$, $P(A) = \{\phi, A\}$

(c) $A = \{a, b\}$, $P(A) = \{\phi, \{a\}, \{b\}, A\}$

(d) A is any set, $P(A) = \{B : B \subset A\}$

EXERCISES 2 (a)

1. Construct five different examples of sets. Describe each with the help of a proposition.

2. Give an example of a set which has exactly 10 elements and express it through a defining property.

3. Is it possible to express every set through a defining property ? Justify your answer.

4. If $\{x: p_1(x)\} = \{x : p_2(x)\}$, show for each x, $p_1(x)$ and $p_2(x)$ have the same truth value.

5. For each of the following words, write down the set of letters forming that word :

   (i) Administration, (ii) Misrepresentation, (iii) Mathematics,

   (iv) Concurrence, (v) Demonstration.

6. State with reason, which of the following are sets and which are not :

   (i) All big rivers of India.

   (ii) All natural numbers having at least one prime factor.

   (iii) All sincere students of Ravenshaw College during the academic year 1998-99.

   (iv) All real numbers with negative square.

   (v) All citizens of India earning more than Rs. 10,000/- per month.

   (vi) All college teachers who are citizens of India.

   (vii) All finite subsets of the set Z of integers.

   (viii) Collection of all sets.

   (ix) Collection of all winged horses.

(x) Collection of all residents of Orissa who will live for more than 100 years.

7. Write the following sets in the form of lists :

(i) $\{x : x$ is a prime number and $1 < x < 100)$

(ii) $\{x : x$ is an odd integer$\}$

(iii) $\{x : x = 1$ or $x = 2$ or $x = 3\}$

(iv) $\{x : x$ can be written as a sum of two odd integers$\}$

(v) set of all natural numbers that are divisible by 5.

8. Write the following sets in the intention (or specification) form :

(i) $\{a\}$ (ii) ϕ (iii) $\{1, 2\}$

(iv) $\{1, 2, 3, 4, 5\}$ (v) $P(\phi)$ (vi) $\{1, 3, 9, 27\}$

9. Determine if the set A is a proper subset of the set B where A and B are as given below.

(i) $A = \{1, 2, 3, ...\}$

$B = \{x : x$ is a rational number$\}$

(ii) $A = \{x : x$ is a prime number$\}$

$B = \{2n - 1 : n = 1, 2, 3, ...\}$

(iii) $A = \{-1, 1, 3\}$

$B = \{x : x \in R$ and $x^3 - 2x^2 - x + 2 = 0\}$

(iv) $A = \{1, 2, 3, 4\}$

$B = \{n \in N : n$ is a diviser of $60\}$

10. For each of the following pairs of sets A, B, determine if $A \subset B$ or $A \not\subset B$:

(i) $A = \phi$, $B = \{\phi\}$

(ii) $A = \{x : x$ is an integer$\}$,

$B = \{3x : x$ is an integer$\}$

(iii) $A = \{x : x$ is an odd integer$\}$,

$B = \{x : x$ is real and not an even integer$\}$

(iv) $A = \{x : x$ is an integer which is both even and odd$\}$,

$B = \{x : x$ is an integer and $x \neq x\}$

(v) $A = \{a, b, c\}$. $B = \{\{a\}, \{b\}, \{c\}\}$

11. Determine the truth of falsity of the following propositions with reasons :

(a) $\{1, 2\} \not\in \{1, 2, 3\}$

(b) $A \subset A$ for any set A

(c) Every set has a proper subset.

(d) Every set is a proper subset of the same set.

(e) For any object x, there is a set A such that $x \in$ A.

(f) For every object x, there is a set A such that, $x \notin$ A.

(g) If A, B, C are sets, then either A = B or A $\subset$ B or B $\subset$ A.

(h) $a \in \{\{a\}\}$

(i) $a \in \{\{a, b\}, b\}, a \neq b.$

(j) If A is a proper subset of B and B is a subset of C, then A a is proper subset of C.

(k) A $\subset \phi$ if and only if A = ϕ.

12. Write down the power set of :

(i) $\{a, b, c\}$ (ii) $\{a, \{a\}\}$ (iii) ϕ (iv) $\{\phi\}$ (v) $\{a, \{a\}, \{a, b\}\}$, (vi) $\{\{\phi\}\}$

13. Prove that P(A) $\subset$ P(B) if and only if A $\subset$ B. When is the inclusion P(A) $\subset$ P(B) proper ?

14. A set can be **finite** or **infinite** (as understood in an informal way). For instance, {1,2,3,4} is a finite set whereas Z is an infinite set. The number of elements of a set A, denoted by |A|, is called its **cardinal number.** Without going into the necessary technicalities, we may just observe that

$$|\phi| = 0, |\{x_1, x_2, \text{............} x_n\}| = n.$$

Two sets A and B are called **similar** if they have the same cardinal number. Thus, the sets {1, 2, 3} and {2, 4, 6} are similar. We write A ~ B to express the fact that A and B are similar. Now, answer the following questions.

(i) What are the cardinal numbers of the following sets ?

$\{\phi\}$, $\{a, \{a, b\}\}$, {Z}, {0,5}, {0,{5}}, $\{a, b, \{a, b\}$, $\{\{\phi\}\}$, $\{\phi, \{\phi\}\}$

(ii) For any natural number n, give an example of a set A such that |A| = n.

(iii) Determine the cardinal number of the set $\{x : x$ is real and $x^3 - x^2 + x - 1 = 0\}$

15. Which of the following sets are finite and which are infinite ?

(i) The set N of positive natural numbers

(ii) The set Z of integers

(iii) The set Q of rational numbers

(iv) The set R of real numbers

(v) The set of prime numbers

(vi) The set of even integers

(vii) The set of human beings

(viii) The set of integers less than 10

16. Verify that

$|P(\phi)| = 2^0$

$|P(\{a\})| = 2^1$

$|P(\{a, b\})| = 2^2$

$|P(\{a, b, c\})| = 2^3$

17. Find the number of elements of :

(i) $P(P(\phi))$ (ii) $P(P(P(\phi)))$ (iii) $P(P(P(P(\phi))))$

18. Prove by method of induction that if A has n elements, then

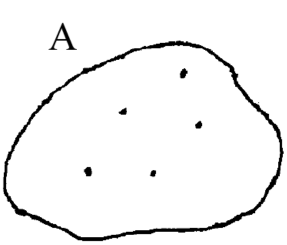
$|P(A)| = 2^n$

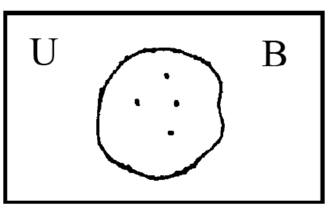
19. Can you say how many elements P(P(A)) has if A has n elements ?

2.3 Operations with sets, Venn diagrams

Out of given sets, new sets can be formed in various ways. Subsets and power set are some examples. More examples are provided by set operations to which we turn now.

To understand the complex properties of sets and their mutual relations, diagramatical representations of sets were introduced by Venn in 1880, popularly called **Venn diagrams.** Venn's method of presenting sets through diagrams is indeed an extension of an earlier one introduced by Euler in 1770. Euler used circular regions to denote domains of terms in a given proposition. In a Venn diagram, a set is represented by the points of a region bounded by a simple closed curve. Circles and rectangles are often employed. A subset is represented by a subregion (see figure 9) and an outer rectangular boundary may be used to denote the universal set (also called the **universe of discourse)** as in figure 10.





Venn diagrams provide beautiful visual aids to understand set operations and associated properties. These diagrams possess no **power of proof** and are sometimes inadequate to express a situation such as that of the empty set, but their role as a tool to understand complex situation must be recognized.

Union :

Suppose that a broadcasting organization seeks to announce in English and in Hindi. So they look for announcers who can speak English and those who can speak Hindi. If E denotes the set of candidates speaking English and H denotes the set of candidates speaking Hindi, then any one from the set E and similarly everyone from the set H can be a candidate for the job of announcer for the broadcasting organization. This new set of candidates is called the **union** of the sets E and H and is written as $E \cup H$ whose elements are those candidates who can speak either English or Hindi or both. We are now led to the following definition.

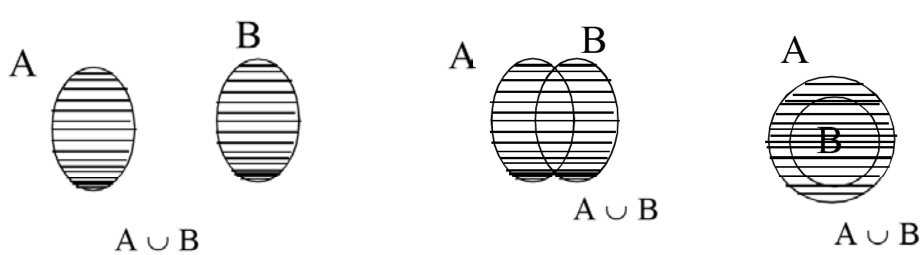
Definition. If A, B are sets, then their **union,** denoted by $A \cup B$, is the set

$$A \cup B = \{x : x \in A \text{ or } x \in B\}$$

Since the sets A, B can have common elelments, the word 'or' within the bracket above is used in the 'inclusive' sense which means that $A \cup B$ consists of **all** those elements x that belong to **at least** one of the sets A, B and it can very well belong to both A and B. This interpretation of the word 'or' will be throughout adhered to unless otherwise specified.

The concept of union can be illustrated through Venn diagram (see figure 11). The shaded region in each case represents $A \cup B$.

(i) (ii) (iii)



Example 3

$\{1, 2\} \cup \{2, 3, 4\} = \{1, 2, 3, 4\}$

$\{1, 2, 3,\} \cup \{\{1, 2\}, \{2\}, 3\} = \{1, 2, 3, \{1, 2\}, \{2\}\}$

$Z = \{x : x \text{ is an even integer}\} \cup \{x : x \text{ is an odd integer}\}$

Facts : We note the following easily provable properties, where A, B, C are any sets :

(a) $A \cup A = A$ **(Law of idempotence of union)**
(b) $A \cup \phi = A$ **(Law of identity)**
(c) $A \cup B = B \cup A$ **(Commutative property)**

(d) $A \subset B \Rightarrow A \cup C \subset B \cup C$

In particular $A \subset A \cup C$ for any C,

(e) $A \cup B = B$ if and only if $A \subset B$

(f) $A \cup (B \cup C) = (A \cup B) \cup C$ **(Associative property).**

A proof of property (*f*) will be given in a later section. Properties (c) and (*f*) enable us to write $A \cup B \cup C$ unambiguously for the union of three sets (and that of more sets similarly or by induction). One way of proving (c) (also (*f*)) is by using properties of logical disjunction. We shall, however, give a different proof of (*f*).

Intersection

Consider a situation in which an examination is conducted on two subjects, say, mathematics and physics. A candidate has to qualify in both the subjects in order to pass the examination. If $P = \{x : x$ has passed in physics$\}$ and $M = \{x : x$ has passed in mathematics$\}$, then the set of persons who have qualified in the examination is $(x : x$ has passed in both mathematics and physics$\} = \{x : x \in M$ and $x \in P\}$ This last set is called the **intersection** of the sets M and P and we write it as $M \cap P$. This prompts us to make the following definition.

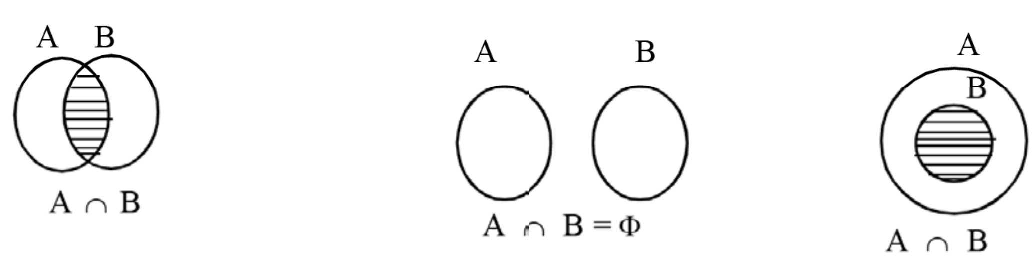
Definition : If A, B are sets, then their **intersection**, denoted by $A \cap B$, is the set

$$A \cap B = \{x : x \in A \text{ and } x \in B\}$$

Thus, $A \cap B$ is the set of all those elements which are common to both the sets A and B. It may happen that two sets A and B have no common elements that is, $A \cap B = \phi$. We then say that A and B are **disjoint** (or **non overlapping**).

In figure 12 below, intersection is expressed through Venn diagrams by shaded regions.

(i) (ii) (iii)



In figure12(i) and figure 12(iii) the shaded regions represent $A \cap B$, But figure 12(ii) corresponds to the case in which, A, B are disjoint, so that $A \cap B = \phi$.

Example 4

(i) $\{0, 1, 2, 3, 4\} \cap \{1, 5, 6\} = \{1\}$

(ii) $\{1, 3\} \cap \{2, 4\} = \phi$

(iii) $\{1\} \cap \{(1)\} = \phi$

Facts : The following properties are easily proved :

(i) $A \cap A = A$ (ii) $A \cap \phi = \phi$ (iii) $A \cap B \subset A$, $A \cap B \subset B$,

(iv) $A \cap B =$ A if and only it $A \subset B$, (v) $A \cap B = B \cap A$

In addition, the associative law and the distributive laws also hold. We shall prove the last two laws in section 1.13 .

As in case of union, intersection can be extended to more than two sets.

Difference, Symmetric difference

If A, B are sets, it is sometimes required to consider elements of A that are **not** in B. A new set is formed thereby which we denote by $A \setminus B$ (or $A - B$) and indeed

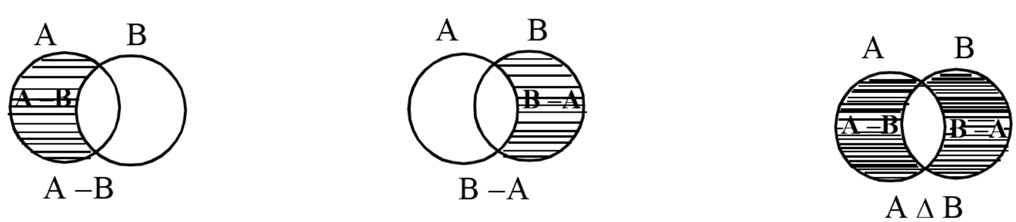
$A \setminus B = \{x : x \in A \text{ and } x \notin B)$

The set $B \setminus A$ can be similarly defined. Such sets are called **difference sets.** The **symmetric difference** of two sets. A, B, denoted by $A \Delta B$, is defined to be the set

$A \Delta B = (A \setminus B) \cup (B \setminus A)$

A Venn diagramatic representation of difference and symmetric difference is shown in figure 13.

(i) (ii) (iii)



In the example of selection of announcers for a broadcasting organization if E and H denote the sets of candidates speaking English and Hindi respectively, then $E \setminus H$ is the set of candidates speaking English but not Hindi and $E \Delta H$ denotes the set of candidates who can speak either English or Hindi, but not both.

As yet another example $Z - N$ is the set of non positive integers. The following properties are easily verified.

(i) $A \cup B = (A \Delta B) \cup (A \cap B)$

(ii) $A \Delta A = \phi$, $A \Delta B = B \Delta A$

Earlier, while talking about paradoxes in set theory, we talked of the necessity of a **universal set** in a given context. This is often necessary to avoid trivialities in arguments. For instance, when we are talking about prime numbers we obviously confine ourselves to the set of integers. No one makes a statement like "$\sqrt{2}$ is not a prime number". Similarly, when one talks about a set of non- Oriyas, one is certainly talking in the context of human beings at least. Or, when someone makes a statement like "none but Gandhi could do such a feat", he or she is obviously talking in the context of human beings. Here the set of human beings can be taken as the **universe of discourse.** In the earlier example about prime numbers, the set of integers, or even the set of natural numbers can be taken to be the universal set.

If U is our universal set in a particular context and $A \subset U$, the difference $U \setminus A$ is called the **complement** of A (with respect to U). We denote it by A' or A^c. In general, $A \setminus B$ is called the (relative) **complement of** B **with respect to** A. However, the symbol A' is used only for the complement of A with respect to the universal set. The following facts are obvious :

(i) $\phi' = U$, $U' = \phi$ **(ii)** $A \cup A' = U, A \cap A' = \phi$

(iii) $(A')' = A$ **(iv)** $A \subset B \Leftrightarrow B' \subset A'$ **(v)** $A \setminus B = A \cap B'$

2.4 Further results about sets

It was pointed out in section 1.12 that union and intersection are commutative set operations, in the sense that

$A \cup B = B \cup A,\ A \cap B = B \cap A$

for all sets A, B. The proofs are easy consequences of definitions and the commutative properties of disjunction and conjunction of statements. We employ the commutative properties without mention. Next we take up proofs of associative and distributive properties of union and intersection in a direct manner. The following facts which are easy to prove, turn out to be helpful in those proofs. For all sets A, B, we have

$A \subset A \cup B$, $A \cap B \subset A$ **(1)**

$A \subset B \Rightarrow A \cup B = B,\ A \cap B = A$ **(2)**

$B \subset A$ and $C \subset A \Rightarrow B \cup C \subset A$ and

similarly $A \subset B$ and $A \subset C \Rightarrow A \subset B \cap C$ **(3)**

$B \subset C \Rightarrow A \cup B \subset A \cup C,\ A \cap B \subset A \cap C$ **(4)**

THEOREM 1 : If A, B, C are sets, then

(i) $(A \cup B) \cup C = A \cup (B \cup C)$ **(Associativity of union)**

(ii) $(A \cap B) \cap C = A \cap (B \cap C)$ **(Associativity of intersection)**

(iii) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ **(Distributive property)**

(iv) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ **(Distributive property)**

Proof : The above identities are results of the form U = V, where U, V are sets. It is convenient to show that $U \subset V$ and $V \subset U$ in each case, whence the conclusion follows.

(i) We have

$(A \cup B) \cup C \subset (A \cup B) \cup (B \cup C)$ (by (1) and (2)

$\subset [A \cup (B \cup C)] \cup (B \cup C)$ (by 4)

$\subset A \cup (B \cup C)$ (by 2)

$\Rightarrow (A \cup B) \cup C \subset A \cup (B \cup C)$

and conversely

$A \cup (B \cup C) \subset (A \cup B) \cup (B \cup C)$ (by 4)

$\subset [(A \cup B) \cup C] \cup (B \cup C)$ (by 4)

$= (A \cup B) \cup C$ (by 4 and 2)

This proves (i)

(ii) $(A \cap B) \cap C \supset (A \cap B) \cap (B \cap C)$ (by 4)

$\supset [A \cap (B \cap C)] \cap (B \cap C)$ (by 4)

$= A \cap (B \cap C)$ (by 2)

$\Rightarrow (A \cap B) \cap C \supset A \cap (B \cap C)$

The proof of $A \cap (B \cap C) \supset (A \cap B) \cap C$ is similar.

(iii) $A \cup (B \cap C) \subset A \cup B$ (by 4) and similarly $A \cup (B \cap C) \subset A \cup C$

$\Rightarrow A \cup (B \cap C) \subset (A \cup B) \cap (A \cup C)$ (by 3)

On the other hand, suppose that

$x \in (A \cup B) \cap (A \cup C)$ **(5)**

If $x \in A$, then $x \in A \cup (B \cup C)$. if $x \notin A$, then (5) implies that $x \in B \cap C$

$\Rightarrow x \in A \cup (B \cap C)$. Hence (5) implies that $x \in A \cup (B \cap C)$, that is,

$(A \cup B) \cap (A \cup C) \subset A \cup (B \cap C)$

the proof of (iii) is complete

Proof of (iv) is similar and left to the reader.

An interesting feature of the above theorem is that (ii) and (iv) are obtained from (i) and (iii) respectively by interchanging $\cup$ and $\cap$. We can call (i) and (ii) **dual theorems** of each other and similarly (iii) and (iv) are duals of each other. The results contained in theorem 2 below, called **De-Morgan's laws,** provide another instance of duality.

THEOREM 2 :

If A, B are sets, then

(i) $(A \cup B)' = A' \cap B'$

(ii) $(A \cap B)' = A' \cup B'$

Proof (i) we have

$A \subset A \cup B \Rightarrow (A \cup B)' \subset A'$

and similarly, $(A \cup B)' \subset B'$, Hence $(A \cup B)' \subset A' \cap B'$

Conversely, $x \in A' \cap B' \Rightarrow x \notin A$ and $x \notin B$

$\Rightarrow x \notin A \cup B \Rightarrow x \in (A \cup B)'$

$\Rightarrow A' \cap B' \subset (A \cup B)'$

This completes the proof of (i)

Taking $A = A'$ and $B = B'$ in (i), we have

$(A')' \cap (B')' = (A' \cup B')'$

$\Rightarrow A \cap B = (A' \cup B')'$

$\Rightarrow (A \cap B)' = ((A' \cup B')')' = A' \cup B'$,

which proves (ii) and the proof of the theorem is complete.

Alternative Proof of De Morgan's Laws

Proof. (i) $x \in (A \cup B)'$

$\Leftrightarrow x \notin (A \cup B)$

$\Leftrightarrow x \notin A$ and $x \notin B$ $(\because \sim(p \vee q) = \sim p \wedge \sim q)$

$\Leftrightarrow x \in A'$ and $x \in B'$

$\Leftrightarrow x \in A' \cap B'$

(ii) $x \in (A \cap B)'$

$\Leftrightarrow x \notin (A \cap B)$

$\Leftrightarrow x \notin A$ or $x \notin B$ $(\because \sim(p \wedge q) = \sim p \vee \sim q)$

$\Leftrightarrow x \in A'$ or $x \in B'$

$\Leftrightarrow x \in A' \cup B'$

De Morgan's laws can be extended to any finite number of sets, by induction (it is also true for and infinite collection of sets which will not concern us here). These results are also true when complements are replaced by relative complements. For instance, the following is true when, X, A, B are any sets.

$X - (A \cup B) = (X - A) \cap (X - B)$

$X - (A \cap B) = (X - A) \cup (X - B)$

To prove the first one, for instance, we have

$X - (A \cup B) = X \cap (A \cup B)'$

$= X \cap (A' \cap B')$ (by Thereom 2)

$= (X \cap X) \cap (A' \cap B')$

$= (X \cap A') \cap (X \cap B')$ (by associativity and commutativity of $\cap$)

$= (X - A) \cap (X - B)$

De Morgan's laws can be illustrated as follows. Let A and B denote respectively the sets of people who play cricket and tennis. Then $(A \cup B)'$ denotes the set of people who do not play either game and that is precisely the set of people common to the set A' of people not playing cricket and the set B' of people not playing tennis, so that $(A \cup B)' = A' \cap B'$. Similarly, $(A \cap B)'$ can be illustrated.

We can now prove :

Example 5 :

(i) $A \cap (B \Delta C) = (A \cap B) \Delta (A \cap C)$

(ii) $A \Delta B = (A \cup B) - (A \cap B)$

Proof :

(i) $x \in A \cap (B \Delta C)$

$\Leftrightarrow x \in A$ and $x \in B \Delta C$

$\Leftrightarrow x \in A$ and ($x \in B$ and $x \notin C$ or $x \in C$ and $x \notin B$)

$\Leftrightarrow$ ($x \in A$ and $x \in B$ and $x \notin C$) or ($x \in A$ and $x \in C$ and $x \notin B$)

$\Leftrightarrow$ ($x \in A \cap B$ and $x \notin (A \cap C)$) or ($x \in A \cap C$ and $x \notin (A \cap B)$)

$\Leftrightarrow$ ($x \in (A \cap B) - (A \cap C)$) or ($x \in (A \cap C) - (A \cap B)$)

$\Leftrightarrow x \in (A \cap B) \Delta (A \cap C)$

Note that in the third step of the proof we used the logical formula

$$p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$$

when p, q, r are statements. Also in the fifth step of the proof we have used

$$x \notin (A \cap B) \Leftrightarrow x \notin A \text{ or } x \notin B .$$

and since already it is given that $x \in A$, we have in this case

$$x \notin (A \cap B) \Leftrightarrow x \notin B.$$

(ii) $x \in (A \cup B) - (A \cap B)$

$\Leftrightarrow x \in (A \cup B)$ and $x \notin (A \cap B)$

$\Leftrightarrow x \in (A \cup B)$ and ($x \notin A$ or $x \notin B$)

$\Leftrightarrow$ ($x \in A$ or $x \in B$) and ($x \notin A$ or $x \notin B$)

$\Leftrightarrow$ [$x \in A$ and ($x \notin A$ or $x \notin B$)]

or [$x \in B$ and ($x \notin A$ or $x \notin B$)]

$\Leftrightarrow$ ($x \in A$ and $x \notin B$) or ($x \in B$ and $x \notin A$)

$\Leftrightarrow x \in A - B$ or $x \in B - A$

$\Leftrightarrow x \in (A - B) \cup (B - A)$

$\Leftrightarrow x \in A \Delta B$

Note that in the fifth step of the proof we have used the logical formula

$(p \vee q) \wedge (r \vee s) = [p \wedge (r \vee s)] \vee [q \wedge (r \vee s)]$

This completes the proof.

Example 6 : Show that

(i) $A - (B \cup C) = (A - B) - C$

(ii) $(A \cap B) - C = A \cap (B - C)$

Proof :

(i) $x \in A - (B \cup C)$

$\Leftrightarrow x \in A$ and ($x \notin B \cup C$)

$\Leftrightarrow x \in A$ and ($x \notin B$ and $x \notin C$)

$\Leftrightarrow$ ($x \in A$ and $x \notin B$) and $x \notin C$

$\Leftrightarrow x \in (A - B)$ and $x \notin C$

$\Leftrightarrow x \in (A - B) - C$

Hence

$A - (B \cup C) = (A - B) - C$

(ii) $x \in (A \cap B) - C$

$\Leftrightarrow (x \in A$ and $x \in B)$ and $x \notin C$

$\Leftrightarrow x \in A$ and $(x \in B$ and $x \notin C)$

$\Leftrightarrow x \in A \cap (B - C)$

Example 7 : Show that

$$\mathbf{B - \bigcup_{i=1}^{n} A_i = \bigcap_{i=1}^{n} (B - A_i)}$$

Proof :

$x \in B - \bigcup_{i=1}^{n} A_i$

$\Leftrightarrow x \in B$ and $x \notin \bigcup_{i=1}^{n} A_i$

$\Leftrightarrow x \in B$ and $x \notin A_i$ for each i

$\Leftrightarrow x \in B - A_i$ for each i

$\Leftrightarrow x \in \bigcap_{i=1}^{n} (B - A_i)$

2.5 Cardinality of finite sets

We now consider another significant fact about finite sets. It can be informally seen that if A and B are two finite sets that are disjoint, then the number of elements in $A \cup B$ is precisely the sum of the number of elements of A and that of B, that is,

$$|A \cup B| = |A| + |B| \text{ since } A \cap B = \phi \qquad \textbf{(6)}$$

What happens if A, B are not disjoint ?

The answer is simple since

$A \cup B = (A \setminus B) \cup B.$

Since the sets $A \setminus B$ and B are disjoint,

$|A \cup B| = |A \setminus B| + |B|$, by (6)

and since A is the disjoint union of $A \setminus B$ and $A \cap B$,

$|A| = |A \setminus B| + |A \cap B|$

so that finally

$|A \cup B| = |A \setminus B| + |B|$

$= |A| - |A \cap B| + |B|$

$= |A| + |B| - |A \cap B|$ **(7)**

This result can be extended to three or more finite sets by repeated application of (7).

Example 8 :

Suppose that a class consists of a set S of 100 students, 70 of which pass in geometry and 60 pass in algebra. If no one failed in both the subjects, can we determine the number of students who passed in both algebra and geometry ?

We proceed like this. Let A and B denote respectively the sets of students who passed in algebra and in geometry. As per given information

$|S| = 100$, $|A| = 60$, $|B| = 70$, $S = A \cup B$.

Here $A \cap B$ denotes the set of students who passed in both algebra and geometry. By (7) we have

$|S| = |A| + |B| - |A \cap B|$

that is,

$100 = 60 + 70 - |A \cap B|$

$\Rightarrow |A \cap B| = 130 - 100 = 30.$

EXERCISES 2 (b)

1. An examination was conducted in Physics, Chemistry and Mathematics. If P, C, M denote respectively the sets of students who passed in Physics, in Chemistry and in Mathematics, express the following sets using union, intersection and difference symbols.
   (a) Set of candidates who passed in mathematics and chemistry, but not in physics;
   (b) Set of candidates who passed in all the three subjects;
   (c) Set of candidates who passed in mathematics only;
   (d) Set of candidates who failed in mathematics, but passed in at least one subject.
   (e) Set of candidates who passed in at least two subjects;
   (f) Set of candidates who failed in one subject only.
2. What can you say about the sets A, B if (i) $A \cup B = \phi$ (ii) $A \Delta B = \phi$ (iii) $A \setminus B = \phi$, (iv) $A \setminus B = A$, (v) $A \cap B = U$, where U is universal set, (vi) $A \setminus B = U$?
3. Are difference and symmetric difference commutative ? Give reasons.
4. If $B \subset C$, is it true that $A \setminus B = A \setminus C$? Is this result true when difference is replaced by symmetric difference ? Give reason.
5. Prove the following :
   (i) $(A \setminus B) \setminus C = (A \setminus C) \setminus B = A \setminus (B \cup C)$
   (ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
   (iii) $A \Delta (B \Delta C) = (A \Delta B) \Delta C$

(iv) $A \subset B \Leftrightarrow B' \subset A' \Leftrightarrow A' \cup B = U$
$\Leftrightarrow B' \cap A = \phi$, where U is the universal set.

(v) $A \cup B = U$ and $A \cap B = \phi \Rightarrow B = A'$

(vi) $A \cup B = A$ for all $A \Rightarrow B = \phi$

6. Prove all the results of the sections 1.13 and 1.14, that are stated without proof.

7. Prove that

$$A - \bigcup_{i=1}^{n} B_i = \bigcap_{i=1}^{n} (A - B_i)$$

$$A - \bigcap_{i=1}^{n} B_i = \bigcup_{i=1}^{n} (A - B_i)$$

Hint : Prove for $n = 2$ and apply induction.

8. Prove that

$|A \cup B \cup C| = |A| + |B| + |C| + |A \cap B \cap C| - |A \cap B| - |B \cap C| - |C \cap A|$

9. If X and Y are two sets such that $X \cup Y$ has 20 objects, X has 10 objects and Y has 15 objects; how many objects does $X \cap Y$ have ?

10. In a group of 450 people, 300 can speak Hindi and 250 can speak English. How many people can speak both Hindi and English ?

11. In a group of people, 37 like coffee, 52 like tea and each person in the group likes at least one of the two drinks. 19 people like both tea and coffee, how many people are in the group ?

12. In a class of 35 students, each student likes to play either cricket or hockey. 24 students like to play cricket and 5 students like to play both the games; how many students play hockey ?

13. In a class of 400 students, 100 were listed as taking apple juice, 150 as taking orange juice and 75 were listed as taking both apple as well as orange juice. Find how many students were taking neither apple juice nor orange juice ?

14. In a group of 65 people, 40 like cricket, 10 like both cricket and tennis. How many like tennis only and not cricket ? How many like tennis ?

15. In a survey it was found that 21 people liked product A, 26 liked product B and 29 liked product C. If 14 people liked products A and B, 12 people liked products C and A, 14 people liked products B and C and 8 liked all the three products. Find how many liked products C only.

CHAPTER 3

Relations and Functions

***It is true that a mathematician, who is not somewhat of a poet,* will never be a perfect mathematician.**

- Weierstrass

3.1 Introduction

Among various kinds of relations we are familiar with, father, son, brother, sister, husband, wife are a few common examples. Let us try to see how we use the terms. We say Dasaratha is father of Rama. Similarly Babur is father of Humayun and Janaka is father of Sita etc. Here the statement "Dasaratha is father of Rama" is but an example of fatherhood. But what then constitutes fatherhood ? We may say that fatherhood is collection of all the **Pairs** like Dasaratha and Rama, Babur and Humayun, Janaka and Sita etc. But they are just not pairs. We can not interchange the position of Dasaratha and Rama and say "Rama is father of Dasaratha". This would make things ridiculus. If we just say "Dasaratha is the father" it is incomplete if we do not mention of Rama at all. So what is important is a **pair of names in a definite order** such as (Dasaratha, Rama). Such relations involving pair are called **binary relations.** Thus a **certain** collection of ordered pairs stands for a definite binary relation as in this case. The ordered pairs (Dasaratha, Rama), (Babur, Humayun), (Janaka, Sita) are examples of ordered relation of fatherhood. So fatherhood would stand for a certain collection of ordered pairs of human beings (**provided** that the set of human beings is the universal set under consideration). So is the relation of brotherhood, sonhood, sisterhood etc. But in each case they are different collections of ordered pairs of human beings.

Before we go to define the term binary relation formally, we define some basic terms.

(i) Ordered Pairs :

If $\{a, b\}$ is a set consisting of the elements a, b (a, may be equal to b), it is called a **pair.** It we specify a to be the **first** component and b to be the **second** component, then we call $\{a, b\}$ an **ordered pair** and write it as (a, b).

In general, (a, b) is different from (b, a). Two ordered pairs (a, b), (c, d) are said to be equal that is, $(a, b) = (c, d)$ if and only if $a = c, b = d$. For example $(a,b)=(2\text{-}1) \Leftrightarrow a=2, b= -1$

We now define the **cartesian product** (also called **product** simply) of two sets.

(ii) Cartesian Product:

Definition :

If A, B are non-empty sets, then their **cartesian product,** denoted by $A \times B$, is defined to be the set

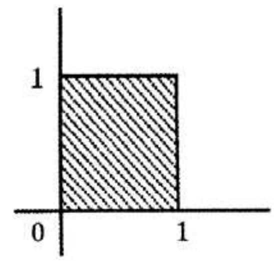
$$A \times B = \{(a, b) : a \in A, b \in B\}$$

For example $A=\{2,3\}$, $B=\{a,b,c\} \Rightarrow A \times B = \{(2,a), (2,b), (2,c), (3,a), (3,b), (3,c)\}$.

Note that $A \times B \neq B \times A$

The word 'cartesian' carries with it reminiscences of the two- dimensional plane, also called the **cartesian plane** after Rene Descartes which is just the product $R \times R$, where R denotes the set of real numbers.

If I is the closed interval [0, 1] on R, then $I \times I$ is a square, as shown in figure the figure.



The concept of an ordered pair can be analogously extended to **ordered** ***n*-** **tuples** of the form $(x_1, x_2, \ldots\ldots x_n)$ and so the product $A_1 \times A_2 \times \ldots\ldots \times A_n$ of the n nonempty sets A_1, $A_2, \ldots\ldots A_n$ can be easily defined. The product $R \times R \times R = R^3$ is our familiar three dimensional space.

It is easy to observe that if A and B are two sets such that $|A| = m$ and $|B| = n$, m and n being nonnegative integers, $|A \times B| = mn$.

By **convention, the product** $A_1 \times A_2 \times \ldots \times A_n = \phi$ **if and only if at least one of the component sets is the empty set** ϕ .

Example 1 :

(i) Let $A = \{a, b, c\}$ $B = \{1, 2\}$, then $A \times B = \{(a, 1), (a, 2), (b, 1), (b, 2), (c, 1), (c, 2)\}$

(ii) Let $A = \{x, y\}$, then $A \times A = \{(x, x), (x, y), (y, x), (y, y)\}$

(iii) $\phi \times A = \phi$ (iv) $R \times R = \{(x, y) : x \in R, y \in R\}$

In the above example (i) one may notice that $(a, 1) \in A \times B$, but $(1, a) \notin A \times B$.

Thus in general $(x, y) \neq (y, x)$ that is why (x, y) is called an ordered pair. Thus if $(x, y) = (r, s)$, then we must have $x = r$, $y = s$.

Sometimes we write $A \times A$ as A^2. It is evident that if $(x, y) \in A^2$. Then $(y, x) \in A^2$.

Now we prove some simple results regarding product.

Example 2 :

Let A, B, C be sets. Then

(i) $A \times (B \cap C) = (A \times B) \cap (A \times C)$ (ii) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

Proof :

$(x, y) \in A \times (B \cap C)$

$\Leftrightarrow x \in A, y \in B \cap C$

$\Leftrightarrow x \in A, y \in B$ and $y \in C$

$\Leftrightarrow (x \in A, y \in B)$ and $(x \in A, y \in C)$

$\Leftrightarrow (x, y) \in A \times B$ and $(x, y) \in A \times C$

$\Leftrightarrow (x, y) \in (A \times B) \cap (A \times C)$

Hence $A \times (B \cap C) = (A \times B) \cap (A \times C)$

The proof of (ii) is similar and is left to the reader.

EXERCISES 3 (a)

1. Compute the product $A \times B$ when

   (i) $A = \{0\} = B$ (ii) $A = \{a, b\}, B = \{a, b, c\}$ (iii) $A = Z, B = \phi$

2. If $|A| = m$, $|B| = n$, what can you say about :

   (i) $|A \times B|$ (ii) $|P(A) \times P(B)|$

3. Find x, y if

   (i) $(x, y) = (-3,2)$ (ii) $(x + y, 1) = (1, x-y)$ (iii) $(2x + y, 1) = (x, 2x + 3y)$

4. If $A \times B = B \times A$ then what can you say about A and B ?

5. $| A \times B | = 6$. If $(-1, y), (1, x), (0, y)$ are in $A \times B$,

   Write the other elements in $A\times B$, where $x \neq y$.

3.2 Relations

We first observe that a binary relation expresses a certain link between two objects such as 'x is father of y' or 'x is employer of y' or 'x is equal to y' etc. In certain cases x and y can not be interchanged without altering the nature of the relationship as in the case of 'x is father of y'. Thus the pair (x, y) must be considered as an ordered pair and not just pair. Of course in the case 'x is equal to y, the role of x and y can be interchanged without affecting the relationship. The totality of all such ordered pairs defines a particular relation since an ordered pair is not included in this totality if and only if it does not satisfy the relation. A relation can thus be considered as a set of ordered pairs (x, y) where x and y are taken from two specified sets A and B respectively (which may be identical at times.)

Now we have the following

Definition :

Let A and B be two arbitrary sets. **A binary relation from (or on) A to B is a subset of $A \times B$.**

If $A = B$, f is called relation **on** A. If $(x, y) \in f$, we often write this as xfy and say that x is related to y **through** f. Sometimes instead of f, we can use R for relation and write xRy if $(x, y) \in R$.

Remark :

Since $\phi \subset A \times B$, ϕ is a relation from A to B. Also, as $A \times B \subseteq A\times B$, $A \times B$ is also a relation from A to B.

If H = set of all human beings

M = set of all men

W = set of all women,

then fatherhood is a certain subset of $M \times H$, brotherhood is certain other subset of $M \times H$, where as motherhood is a subset of $W \times H$. So if $f \subset M \times H$ stands for the fatherhood relation, then $(a, b) \in f$ means that a is father of b. Similarly if $f \subset M \times H$ be the brotherhood relation, then $(a, b) \in f \Rightarrow a$ is brother of b. (Note that a is brother of b need not imply that b is brother of a . b could be sister of a). A sonhood relation is yet another subset of $\boxed{H \times M \text{ or}}$ $M \times H$.

Ramark :

We shall confine our attention to binary relation **only.** Of course there are other relations which are not binary. Just as for defining binary relations we take the subsets of cartesian product of two sets, similarly for defining other relations, involving three, four or more elements, we have to take the subsets of the cartesian products of concerned three, four or many sets. In what follows, a relation shall always mean binary relations unless otherwise stated. The sets of integers, rational numbers and real numbers will be denoted by Z, Q, R respectively.

We now take up more technical details of binary relations

Domain, Codomain and Range of a relation:

Definitions :

Let $f \subset A \times B$, that is, let f be a relation from A to B. Then the **domain** of f written as dom$f$ or D_f is defined by

dom$f$ (or D_f) = $\{x \in A : (x, y) \in f$ for some $y \in B\}$;

The set B is called the codomain of f.

the **range** of f, written as *rngf* (or R_f) = $\{y \in B : (x,y) \in f$ for some $x \in A\}$;

the **inverse** of f, written as f^{-1}, is a relation from B to A defined by

$f^{-1} = \{(y, x) : (x, y) \in f\}$

It is obvious that the domain of f is a subset of A and range of f is a subset of B. It is also clear that the f^{1} is a subset of B and range of f^{-1} is a subset of A.

Like father, mother, brother, sister we have also relations like " less than", "equal to", "similar to", "congruent to", "parallel to", "perpendicular to" etc.

We explain the above ideas in the following examples.

Example 3 :

Let A = {Dasaratha, Rama, Laxman, Sita, Janaka, Humayun, Akbar}. Take B = A. Then the relation of fatherhood on A is given by

R_1 = {(Dasaratha, Rama), (Dasaratha, Laxman), (Janaka, Sita), (Humayun, Akbar)}; the daughterhood relation is given by R_2 = {(Sita, Janaka)};

the brotherhood relation is R_3 = {(Rama, Laxman), (Laxman, Rama)},

the sonhood relation is R_4 = {(Rama, Dasaratha), (Laxman, Dasaratha), (Akbar, Humayun)},

Also dom R_1 = {Dasaratha, Janaka, Humayun},

rngR_1 = {Rama, Laxman, Sita, Akbar},

R_1^{-1} = (inverse of R_1) = {(Rama, Dasaratha), (Laxman, Dasaratha), (Sita, Janaka), (Akbar, Humayun)},

dom R_2 = {Sita},

rng R_2 = {Janaka},

R_2^{-1} = {(Janaka, Sita)}

dom R_3 = {Rama, Laxman},

rng R_3 = {Laxman, Rama} = *dom* R_3

R_3^{-1} = {(Laxman, Rama), (Rama, Laxman)} = R_3

dom R_4 = {Rama, Laxman, Akbar}

rng R_4 = {Dasaratha, Humayun}

R_4^{-1} ={(Dasaratha, Rama), (Dasaratha, Laxman), (Humayun, Akbar)}.

Example 4 :

(i) Let $A = B = \{1, 2, 3, 4, 5\}$ and let $f = \{(x, y) \in A \times B : x = y\}$

Then, $f = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5)$

$dom\, f = rng\, f = A$

$f = f^{-1}$

(ii) Let $A = \{1, 2, 3\}$, $B = \{2, 3, 4\}$ Let $f = \{(x, y) \in A \times B : x < y\}$

$g = \{(x, y) \in A\times B : x > y\}$

Then $f = \{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\}$

$dom\, f = \{1, 2, 3\} = A$

$rng\, f = \{2, 3, 4\} = B$

$f^{-1} = \{(2, 1), (3, 1), (4, 1), (3, 2)\ (4, 2), (4, 3)\}$

$g = \{(3, 2)\}$

Example 5 :

Let A = B = R and $f = \{(x, y) \in A \times B : x^2 + y^2 = 1\}$

Then $dom\, f = [-1, 1] = rng\, f, f = f^{-1}$

Example 6 :

Let A = B = R and let $f = \left\{(x, y) : y = \frac{1}{x}\right\}$

Then $dom\, f = rng\, f = R - \{0\}$ and $f = f^{-1}$

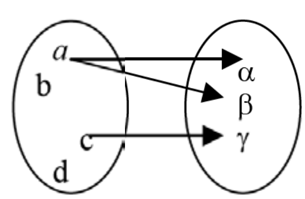
Example 7 :

Let A = B = R and let $f = \{(x, y) \in A \times B : y^2 = x\}$
The $dom\, f = \{x \in R : x \geq 0\}$
$rng\, f = R$
$f^{-1} = \{(x, y) : y = x^2\}$, dom$f^{-1}$ = R, rng$f^{-1} = \{y \mid y \geq 0\}$

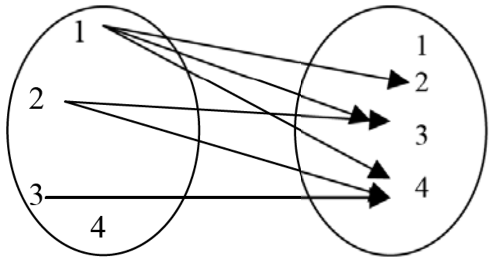
Diagramatic representation of a relation

A relation between elements of a set A and the elements of a set B can be represented by a diagram as follows :

Represent the elements of A and B as points in the plane by drawing the Venn diagrams. If $(a, b) \in f$, draw a line connecting a and b with an arrow pointing in the direction of b. The collection of all such lines with arrow head shall represent the relation f. For example if A = $\{a, b, c, d\}$, B = $\{\alpha, \beta, \gamma\}$ then the relation $\{(a, \alpha), (a, \beta)\ (c, \gamma)\}$ is pictorially represented as follows :



The following picture represents a relation from C = {1, 2, 3, 4} to D = C given by {(1, 2), (1, 3), (1,4), (2, 3), (2, 4), (3, 4)}



One-many, many -one, one- one relations

If for the relation $f \subset A \times B$ $(a,b) \in f$, $(a, b') \in f$ and $b \neq b'$ then we say that the relation f is **one-many.** In case of a one- many relation, there shall be more than one line from a single element of A to more than one element of B.

A relation $f \subset A \times B$ is called **many-one** if $(a, b) \in f$, $(a',b) \in f$ and $a \neq a'$; and a relation f is called **One-One** if $(a, b) \in f$, $(a, b') \in f \Rightarrow b = b'$

and $(a,b) \in f$, $(a', b) \in f \Rightarrow a = a'$

In a way, a one-one relation is neither many -one nor one- many.

For example, fatherhood clearly is a one- many relation whereas sonhood is a many- one relation. Brotherhood is both one-many and many-one. Husbandhood or wifehood (if there is no polygamy nor polyandry) is a one-one relation.

Less than is both one-many and many one. Equality is clearly one- one. Similarity of triangles is a one- many and many-one relation.

EXERCISES 3 (b)

1. Let A = $\{a, b, c\}$, B = $\{1, 2\}$

(a) Determine all the relations from A to B and determine the domain, range and inverse of each relation.

(b) Determine all the relations from B to A.

(c) Is there any relation which is both a relation from A to B and from B to A ? How many ?

(d) Of all the relations from A to B, identify which relations are many-one, one-many and one-one and represent these diagramatically.

2. Are the following sets relations :

(i)	ϕ from A to B	(ii)	$A \times B$ from A to B
(iii)	$A \times \phi$ from A to ϕ	(iv)	$\phi \times B$ from ϕ to B
(v)	$\phi \times \phi$ from ϕ to ϕ	(vi)	$\phi \times C$ from A to B
(vii)	$\phi \times \phi$ from A to B		

Determine the domain, range and inverse of each of the relations mentioned above.

3. Express the following relations on A to B in each case in tabular form :

   (i) $A = \{n \in N : n \leq 10\}$, $B = N$

   $f = \{(x, y) \in A \times B : y = x^2\}$

   (ii) $A = B = R$

   $f = \{(x, y) : x^2 + y^2 = 1 \text{ and } |x - y| = 1\}$

   (iii) $A = \{1, 2, 3, 4,\}$, $B = \{1, 2, 3, 4, 5\}$

   $f = \{(x, y) : 2 \text{ divides } 3x + y\}$

4. A and B are nonempty sets such that $|A| = m$, $|B| = n$. How many relations can be defined from A to B ? (Remember that the number of relations is the number of subsets of $A \times B$).

5. Give an example of a relation f such that

   (i) $dom\, f = rng\, f$ (ii) $dom\, f \subset rng\, f$

   (iii) $dom\, f \supset rng\, f$ (iv) $f \cup f^{-1} = \phi$

   (v) $f = f^{-1}$ (vi) $f \cap f^{-1} \neq \phi$

6. Let $R = \{(a, a^3) \mid a \text{ is a prime number less than } 10\}$

   Find (i) R, (ii) dom R, (iii) *rngR,* (iv) R^{-1}, (v) dom R^{-1}, (vi) *rng*R^{-1}.

7. Let A = {1,2,3,4,6} and Let R be a relation on A

   defined by

   R = {(a,b)| a divides b}

   Find (i) R, (ii) dom R, (iii) rngR, (iv) R^{-1}, (v) domR^{-1}, (vi) rngR^{-1}.

3.3 Functions

Someone defined mathematics as the study of sets and functions. However brief, the definition conveys the importance of functions in mathematics in the least number of words. Relations and functions are amongst examples of words chosen from day-to-day vocabulary that have acquired deep mathematical significance. Curiously enough, their technical meaning is not very far apart from the common usage. This point of view is evident from the examples that we have already considered for relation and that we shall consider for function in the present section.

The significance of the concept of function has been realized since antiquity, but it was the great mathematician Leibnitz (1647–1716), one of the inventors of calculus, who is credited to have coined the word **function** in 1694.

The functional notation $f(x)$ was invented by Euler in 1734. Thinking of the immense simplification that these inventions brought subsequently to mathematics it is difficult to overemphasize their ingenuity. Yet this was inevitable. The vast upsurge of analytic ideas that swept the realm of mathematics during the last two centuries, could never possibly have broken the shores, without the function concept and an appropriate symbol for it.

A relation, as defined in the previous section, does not exclude the possibility that a given element can be associated with several elements at the same time. For example, a mother can have several children, an integer can have several divisors, a polygon has several vartices and so on. On the otherhand there are relations in which a given element is associated with **exactly one element** as in case of the child-mother relationship, a real number and its square, a circle and its centre and many others. Then latter type of relations are examples of some important special types of relations called 'function' which are largely responsible for the vast application of mathematics to various branches of human knowledge.

We now formulate the definition of function in precise mathematical terms.

Definition :

A relation f from X to Y is called a **'function'** if it satisfies the following two conditions :

(i) $D_f = dom\, f = X$ (ii) $(x, y) \in f$ and $(x, z) \in f \Rightarrow y = z$.

Thus a function from (on) X to Y is a relation whose domain is the whole of X and is not one-many.

Some of the English synonyms for the word function are :

mapping, map, transformation, transform, operator, correspondence.

If $f \subset X \times Y$ is a function, we often write this as $f : X \to Y$ or $X \xrightarrow{f} Y$. and say that f is a function from X to Y, or on X to Y or f maps X into Y.

If $(x, y) \in f$, then we write $y = f(x)$

and say that y is the **value** of the function at x or the **image** of x under f. Observe that there can be no ambiguity in writing $f(x)$ because it is impossible to have more than one value for $f(x)$ for a given x in X, by the definition of function. Thus a function $f : X \to Y$ is known if we know the element $f(x)$ in Y for every $x \in X$. That is why sometimes a function $f : X \to Y$ is defined by a **rule :** $y = f(x)$ which associates to every $x \in X$, a **unique** element $f(x) \in Y$. We shall soon realise that a great simplification is achieved by writing the function in the above functional relation.

Since a function is primarily a relation the definitions of domain, codomain and range of a function are similar to those of a relation.

The set Y is called **co-domain** of f. It is evident that $R_f = rng\,(f) \subseteq Y$ and this inclusion may be proper as in the case of the function $\{(n, n^2) : n \in Z\}$

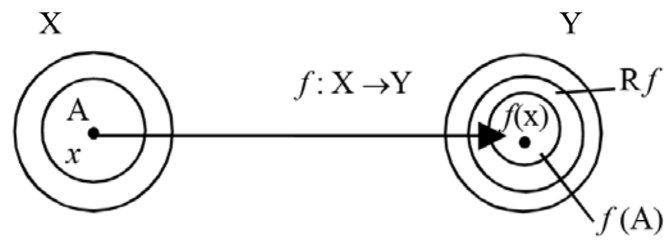
For any subset A of X, the **image** of A under f is the set $f(A) = \{f(x) : x \in A\}$ and for any B $\subseteq Y$, the **pre-image** of B is the set $f^{-1}(B) = \{x \in X : f(x) \in B\}$

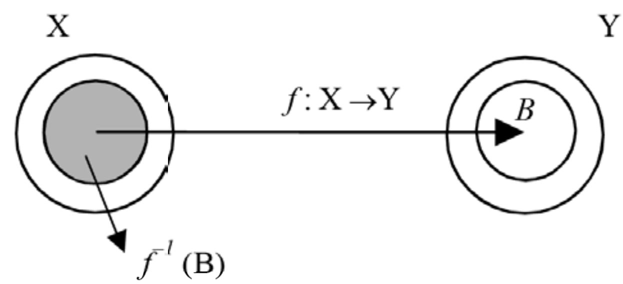
The range of f is defined by

$R_f = rng(f) = f(X)$ = the set of all images of elements of X under f.

Diagramatic Representation of a function:

The concepts of a function discussed above are diagramatically represented below:





By varying x over X, we determine $f(x)$ for each $x \in X$ and so the function $f : X \to Y$ can also be written as

$f = \{(x, f(x)) : x \in X\}$

and x is called the **variable.**

Before we go over the deeper aspects of functions, we take some examples.

Example 8 :

(i) Suppose we are observing the position of a particle moving in a straight line. We know that at every instant of time t the particle has a unique position on the line. If we agree that every point of a straightline corresponding to a real number with some convenient point on it, representing the real number 0, then the position of the particle on the line at time t is represented by the real number $x(t)$. We observe that with motion of time, for every time t, there is one and only one position represented by $x(t)$ as we do not expect the particle to be occupying two different positions at the same time (nothing prevents, though, the particle from occupying the same position at two different times). Suppose the time is represented by real numbers, 0 being a certain time deemed as the **present** time, then all time of **future** is represented by the positive real numbers and the **past** by the negative real numbers. Then the position of particle moving in a straight line is, in fact, a function

$$x : R \to R$$

which associates the position $x(t)$ to the time t.

(ii) Look at the railway time table. It records time when the train arrives at a station and departs from the station. We can look upon this as a function whose domain consists of union of these disjoint intervals of time and the range consists of certain cities where the train stops. Thus for every time in the mentioned interval we have a city where the train is.

The table does not say anything about the train's position at a time after its departure from a station and before its arrival at the next. This does not disqualify it to be a function if we take its domain to be the union of interval of time depicting the time of arrival and departure at a certion station. However if you look at the left of the column where names of the cities appear, you would find some numbers. They are the distances of the cities recorded from the station from which the train started. Thus the time table also records the distances of the train from the stations of its starting at certain times. Though it does not record the distance of the train at a time when it is between two cities, it is definitely at some distance from the starting station at every moment of time. So a railway time table may be thought of as a record of values of a function at certain times showing the distance of the train, at that time, from the station of starting. This obviously does not show the distance at every moment of time as it is quite impossible to do so since there are infinitely many points of time between departure time of the train from a station and arrival time at the next station. Thus a time table to be deemed as a **recording function**, the distance at various times really does not specify the function completely but gives it values at certain times indicating the function.

(iii) The motion of a particle is well described if we know its position $\vec{r}\,(t)$ at any given time t. So if R^3 represents the three dimensional space and R the time, then the motion of a particle is described by a function.

$\vec{r} : R \to R^3$

It is possible that a particle can be at the same point at two different times, but never at two different positions at any given time.

It is also sometimes useful to know the function by an explicit relationship like

$$s(t) = ut + \frac{1}{2}\, at^2$$

which tells the distance covered by a particle in time t. This is not to suggest that we are always lucky in having such an analytic expression of the value of the function in terms of its argument or variables. Often we do not know and it becomes a problem of theoretical physics to predict such a relationship. It was the genius of Galileo who gave the law of falling bodies :

$$s(t) = \frac{1}{2}\, gt^2$$

assuming of course that the gravity does not vary and there is no air resistance. We have in fact, a **rule** which tells us where the particle would be at which time.

This surely represents a function. However we are not so lucky to have a neat formula like this which would tell the distance of the train from the station of its starting at a given time as discussed in (ii).

(iv) The Doordarsan morning news on a particular day tells us the temperature of four metros recorded at 5.30 a.m. on that day. This is an example of a function with domain the metros and range the real numbers quantifying the temperatures (Surely a city can not have two different temperatures at the same time on the same day.)

(v) Look at your college time table. What does it record on a particular day ? It records the subjects to be taught at different times of the day. So the time table can be considered as a function whose domain is the set of periods and the range is the set of subjects of your combination.

(vi) The rate chart with the postmaster where for a definite weight, a definite postage is shown is an example of a function (which is known as **post office function**) :$p = f$ (w). (p meaning postage for the mail of weight w).

(vii) We know the solubility of a salt varies with temperature. So the relation which associates to every temperature a definite solubility is a function of temperature.

(viii) Price list in a shop indicates the list of prices for each item at a particular time. An item does not have two different prices at the same time, in the same shop. We may deem it as a function whose domain consists of different items and the range their prices.

(ix) Let A denote a set of individuals and B be the set of their respective mothers (supposed alive). If for every $x \in$ A, M (x) denote, the mother of x, then M defines a function from A to B. Thus the child-mother relationship is a function, whereas the mother-child relation cannot be a function as a mother may have more than one child.

In the above examples we considered some cases of functions which arise in nature and in life situations. Many such functions emerge from every aspect of human activities and scientific knowledge. The association of a country with its capital, a circle with its radius, a triangle with its area, the marks secured by a student in a particular paper, the score of a player in a particular match, the height of an individual, the distance of a city from the capital and billions of similar instances are examples of functions. Now we take up some typical examples from mathematics.

Example 9

Let A = {1,2,3,4,5}, B={a,b,c}

consider f_1={(1,a), (2,b), (3,b), (4,a), (5,b)}

Obviously $f_1 \subset$ A× B

Here dom $f_1 = \{1,2,3,4,5\} = A$

Codomain of $f_1 = \{a,b,c\} = B$

$rng\, f_1 = \{a,b\} \subset B$

Moreover every element of A has a unique image in B, Hence f_1 is a function from A to B.

Now for the above sets A and B consider

$$f_2 = \{(1,b), (2,a), (3,c), (5,a)\}$$

and $$f_3 = \{(1,a), (2,b), (3,a), (4,b), (5,c), (1,c)\}$$

Here f_2 is not a function from A to B since dom $f_1 = \{1,2,3,5\} \neq A$ as the element 4 of A has no image in B.

Also f_3 is not a function from A to B since the element $1 \in A$ has two different images viz. a and c, i.e. two different ordered pairs $(1,a)$ and $(1,c)$ in f have the same first component.

Example 10 :

Let $f: N \to N$ be defined by $f(n) = 2n$, $n \in N$, Then every element of N has its unique image in N. So f is a function from N to N. Clearly $f(1) = 2$, $f(2) = 4$, $f(3) = 6$ and so on.

Here dom $f = N =$ codomain of f and $rngf = \{2,4,6,...\}$ which is a subset of N.

3.4 Real valued functions and their graphs

A function $f: A \to B$ where $B \subseteq R$ is called a real valued function or simply a real function. Thus when $f: A \to B$ is a real function, for $x \in A$, there exists a unique $y \in B$, such that $y = f(x)$, the image of x under f.

In the following we shall consider real valued functions when dom $f = A \subseteq R$.

Determination of Domain and Range of a Real Function

It has already been mentioned previously that a function is also a relation and the defination of domain and range of a relation also apply to those of a function.

Let $f: A \to B$ where A and B are subsets of R.

Then

(i) dom $f = D_f = A$, (ii) Codomain of $f = B$, (iii) $rngf = f(A) \subseteq B$

A real function $f: A \to B$ is very often defined by a rule that expresses the image $y \in B$ in terms of its pre-image $x \in A$.

For example $y = 2x$, $y = x^2 + 1$, $y = \dfrac{1}{x}$ are real functions without explicitly mentioning their domains.

In such cases, to determine the domain of a function we find the **maximal set** (union of all such sets) on which $y = f(x)$ has a meaningful expression. For example $y = f(x) = 2x$ and $y = f(x) = x^2 + 1$ are defined for all $x \in R$. On the otherhand $f(x) = \frac{1}{x}$ is not defined for $x = 0$. Hence the maximal set for which $f(x) = \frac{1}{x}$, turns out to be R - $\{0\}$. The maximal set for which a function $f(x)$ is defined is called its '**natural domain**' or simply the 'domain' of f.

Determination of range of real functions in simple cases can be done with ease, but it takes some effort in general.

The knowledge of graph is useful for determining range. Before we discuss graphs, we take up few examples.

Example- 11

Find the domain of $f(x) = x + \frac{1}{\sqrt{x}}$

Solution :

Observe that $\frac{1}{\sqrt{x}}$ cannot be defined when $x \leq 0$. Hence f is defined for all $x \in R$ except $x \leq 0$.

Thus, $\text{dom} f = R - (-\infty, 0] = \{x \in R \mid x > 0\} = R^+$

Example - 12

What is the domain of the function

$$f(x) = \frac{x}{x^2 - 3x + 2}$$

Solution :

The function $\frac{x}{x^2 - 3x + 2}$ is not defined when the denominator $x^2 - 3x + 2 = 0$

Now

$x^2 - 3x + 2 = 0$

$\Rightarrow (x-1)(x-2) = 0$

$\Rightarrow x = 1$ or 2

Thus the domain of this function is R-$\{1,2\}$.

Example - 13

Find the domain and range of the following functions:

(i) $f(x) = \dfrac{1}{1-x^2}$, (ii) $f(x) = \sqrt{1-x^2}$

Solution :

(i) $f(x) = \dfrac{1}{1-x^2}$

Clearly f is not defined when $1-x^2 = 0$.

Now $1- x^2 = 0 \Rightarrow x = \pm 1$

So dom f = R - {-1, 1}

Also, $y = f(x) = \dfrac{1}{1-x^2} \Rightarrow x = \pm\sqrt{1-\dfrac{1}{y}}$

Clearly x is defined when

$1-\dfrac{1}{y} \geq 0$, i.e. when $y < 0$ or $y \geq 1$,

$\therefore$ Range of f = $\{y \in R \mid y < 0 \text{ or } y \geq 1\} = (-\infty, 0) \cup [1, \infty)$

(ii) $f(x) = \sqrt{1-x^2}$

Here f is defined if $1-x^2 \geq 0$,

or if $-1 \leq x \leq 1$

$\therefore$ dom f = [-1, 1]

Let y $= \sqrt{1-x^2}$

$\Rightarrow y^2 = 1-x^2 \Rightarrow x = \pm\sqrt{1-y^2}$

So x is defined when $1- y^2 \geq 0$, i.e. when $y^2 \leq 1$, i.e. $-1 \leq y \leq 1$

$\therefore$ *rngf* = [0, 1]

Graphs

Most of the elementary real functions we shall discuss have one distinctive feature that they can be depicted graphically and are often better understood by studying their properties as reflected in the respective graphs to be defined shortly.

We are aware of the fact that every ordered pair (a, b) of real numbers corresponds to a unique point of the Cartesian plane and vice versa. This enables us to define the graph of certain functions as follows :

Definition

Let $f: A \to R$, where $A \subseteq R$. Then the set

$\mathbf{G_f = \{(x, f(x)) \mid x \in dom f\}}$ **is called graph of *f*.** Note that the graph of f corresponds to *a* set of points on the Cartesian plane.

The relation between a function $f: A \to R$ and the graph is analogous to that between R and the corresponding number line.

The graph of a relation can be similarly defined as the set of points in the plane corresponding to its elements. Certain familiar curves in geometry such as the circles

$$C = \{(x, y) : x^2 + y^2 = r^2\}$$

arise in this way. Obviously C is not given by a function since both (x, y) and $(x, -y)$ are in it, whenever $x^2 + y^2 = r^2$.

It is sometimes, but not always possible to plot a graph geometrically. **The characteristic function X_Q of the set of rationals in R does not possess a geometrically constructible graph.** Such graphical construction, whenever possible, is an important method of representing a function.

For most of the elementary real functions which include polynomial, trigonometric functions and their inverses, exponential and logarithmic functions, their various algebraic combinations and compositions, graphical constructions are generally possible. The task may, however, be quite cumbersome at times and necessitates good knowledge of calculus, geometry and computational techniques.

Just as it is not always possible to plot a graph, it may not be possible to reconstruct a relation from a geometrical curve, but the graph conveys, almost invariably, a picture of the inverse. We shall elaborate upon this in due course.

How to plot a graph

In order to plot a graph, the following elementary devices may prove to be of use, particularly in case of elementary functions.

(a) Compute a reasonable number of functional values and look particularly for points at which the graph meets the co-ordinate axes. If $0 \in D_f$, then $(o, f(0))$, is the point at which it meets the Y– axis and if $0 \in R_f$, the solutions of $f(x) = 0$ give points at which it meets the X - axis. Note that the graph of a function meets the Y - axis at most once while it may meet the X-axis several times. (why ?)

(b) Care may be taken to detect points at which the graph shows a particular tendency such as rising upward, stooping downward and so on. Besides calculus, elementary techniques also work at times.

(c) The **line** of **symmetry**, if any, may be detected so that the complete graph can be known by a knowledge of only one side.

(d) The points on the graph, thus obtained may be plotted and then joined in free hand by a smooth curve.

Some real functions and their graphs.

Example 14

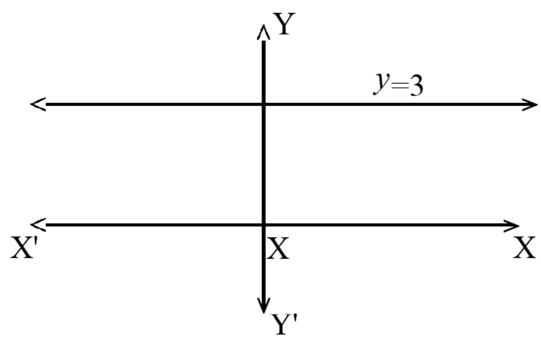
Constant function :

A function f : A $\rightarrow$ R is said to be a constant function if these is a real number k such that $f(x) = k$, for all $x \in A$.

Here dom $f = A \subseteq R$, $rngf$ = {k} which is a singleton.

The function $f = \{(0,1), (\sqrt{2}, 1), (-\frac{1}{3}, 1)\}$ is a constant function with domain $\{0, \sqrt{2}, -\frac{1}{3}\}$ and range {1}.

The graph of the constant function $f(x) = 3, x \in R$ is the line parallel to x - axis as shown:



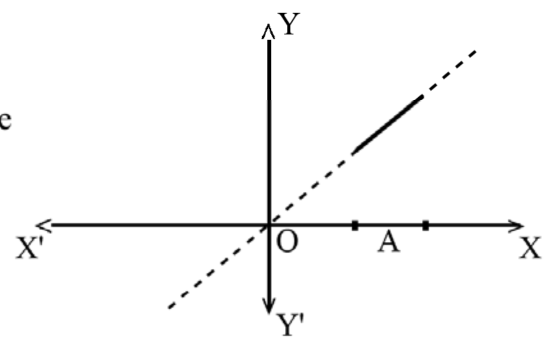
Example 15

Identity Function :

For any nonempty set $A \subseteq R$ the function f : A $\rightarrow$ A defined by $f(x) = x$ for all $x \in A$ is called the identity function on A. It is denoted by id_A.

For the identity function f, dom $f = rng\, f$.

The graph of id_A on $A \subseteq R$ is part of the straight line through the origin as shown.



(Here A is supposed to be an interval as shown in the figure.)

Example 16

Polymonial function :

A function f : A $\rightarrow$ R defined by

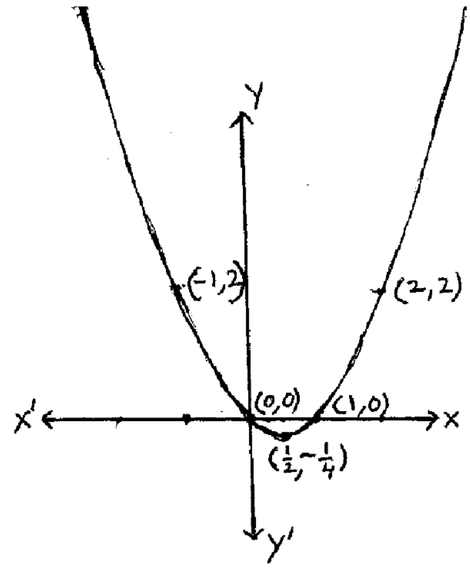
$f(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n$, where n is a nonnegative integer and $a_0, a_1, ..., a_n$ are real constants with $a_n \neq 0$, is called a polynominal function or simply *a* polynomial of degree n.

$2x + 3,\ 1+x^2,\ x^3- x^2+2,\ x^4+x$ are examples of polynominals of degree 1,2,3 and 4 respectively and they are respectively called linear, quadratic, cubic and biquadratic polynomials.

A polynomial $f(x)$ is defined for all real $x \in A \subseteq R$.

Consider the quadratic polynomial function $f(x) = y = x(x-1)$. We plot five points on it as per the following table :

x	0	$\frac{1}{2}$	1	2	-1
y	0	$-\frac{1}{4}$	0	2	2



It is noted that

(a) y is negative and decreases from $x = 0$ to $x = \frac{1}{2}$ and then increases till its meets the X - axis.

(b) It passes through (2, 2) and rises steadily.

(c) y is positive for negative x, passes through (-1, 2) and rises steadily as shown in the figure. The resulting graph is a parabola.

Example 17

Rational function :

The function $f(x) = \dfrac{x^2+1}{2x^3+x^2-1}$ is the quotient of two polynomial functions. Such a function is an example of a rational function. In general a function $f(x) = \dfrac{p(x)}{q(x)}$, where p (x) and q (x) are polynomials with q (x) $\neq 0$ for all $x \in$ dom$f$, is called a rational function.

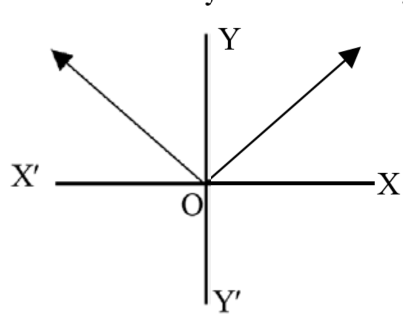
Example 18

Modulus function :

The **modulus function** f: $\mathrm{R} \to \mathrm{R}$ is defined by $f(x) = |x| = \begin{cases} x \ (x \geq 0) \\ -x \ (x < 0) \end{cases}$.

The modulus function is also known as absolute value function. Its domain is R and range is $\mathrm{R}^+ \cup \{0\} = \{x \in \mathrm{R} \mid x \geq 0\}$.

The graph of $y = |x|$ consists of two rays as shown in figure.



Example 19

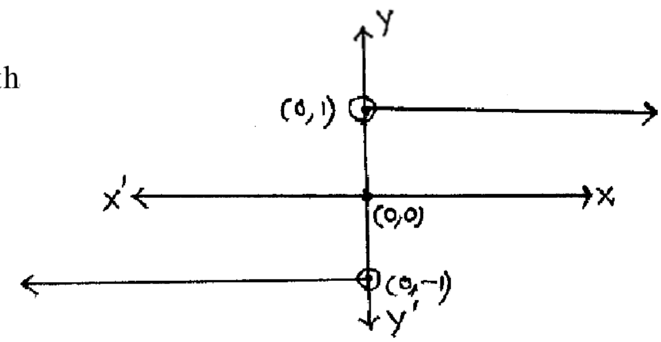
The **Signum function** on R is defined by $sgn\, x = \begin{cases} \dfrac{x}{|x|} & (x \neq 0) \\ 0 & (x = 0) \end{cases}$.

The range of *sgn x* is $\{-1, 0, 1\}$.

From the definition of sgn x if follows th

(i) $x = |x|\ sgn\ x$ when $x \neq 0$.

The graph of sgn x is as shown :



Example 20

Exponential function :

An **exponential function** is defined by $f(x) = a^x$ $(a > 0, a \neq 1), x \in R$. The fact that a^x exists for every $x \in R$ whenever $a > 0$ follows from the theory of real numbers. The following properties can also be proved.

(a) $a^{x+y} = a^x a^y, a > 0$

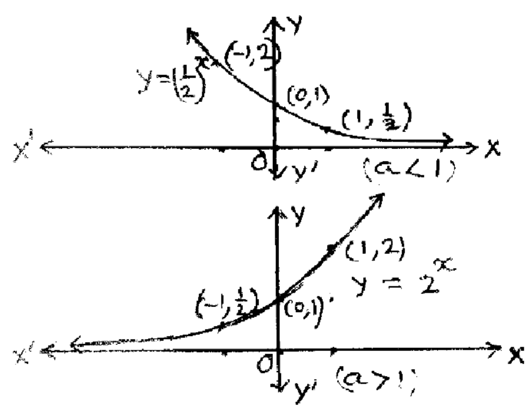
$(a^x)^y = a^{xy}, x, y \in R$

(b) $a^x = 1$ iff $x = 0$

(c) If $a > 1$, $a^x > a^y$ iff $x > y$.

(d) If $a < 1$, then $a^x > a^y$ iff $x < y$

It follows that the exponential function as defined above is one - to - one and monotonic (increasing or decreasing according as $a > 1$ or $a < 1$).



(e) a^x is closer to the X - axis as x recedes away from zero along negative values.

The graph of $f(x) = y = 2^x$ is shown in Graphs of $y = 3^x$, $y = 4^x$ etc. can be similarly plotted, but the growth is so rapid that for even for a value like $a = 4$, the graph connot be accommodated on the space provided here for $x \geq 2$.

A comparision with the graphs of $y = x^2$ or $y = x^3$ will show that 2^x grows much more rapidly than x^2 or x^3 or indeed than x^n for any n.

It is clear that dom f = R and rng $f = R^+$.

Example 21

Logarithmic function

The function f defined by $f(x) = \log_a x$, $(a > 0, a \neq 1)$ where $y = \log_a x \Leftrightarrow a^y = x$ is called the logarithmic function to the base a.

Here dom $\log_a = R^+$, rng $log_a = R$.

Properties of logarithmic function which can be easily derived from definition are given below:

(i) $\log_a (xy) = \log_a x + \log_a y$

(ii) $\log_a (x/y) = log_a x - \log_a y$

(iii) $\log_a x = 0 \Leftrightarrow x = 1$

(iv) $\log_x x = 1$

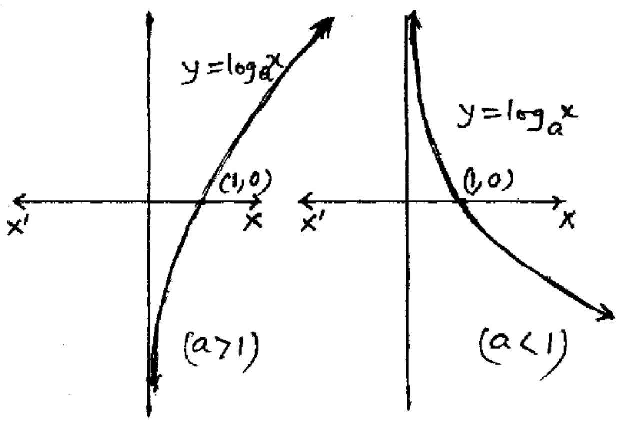
(v) $\log_a x = 1/\log_x a, \ x \neq 1$

(vi) $\log_a x = \log_b x \, . \, \log_a b$.

(vii) If $a > 1$, $\log_a x > \log_a y$ iff $x > y$

and if $a < 1$, $\log_a x < \log_a y$ iff $x > y$

(viii) $\dfrac{\log_a x}{\log_a y} = \log_y x$, $(y \neq 1)$



The accompanying figures show the graphs of $\log_a x$ *for* $a > 1$ and $a < 1$ respectively. Both the graphs meet the x- axis at (1, 0) and never meet the y- axis.

Example 22

Greatest Integer function

The function f defined by $f(x) = [\,x\,]$

Where $[\,x\,]$ is the greatest integer not greater than x (less than or equal to x) is called the **greatest integer function.**

From the difinition if follows that if n is an integer,

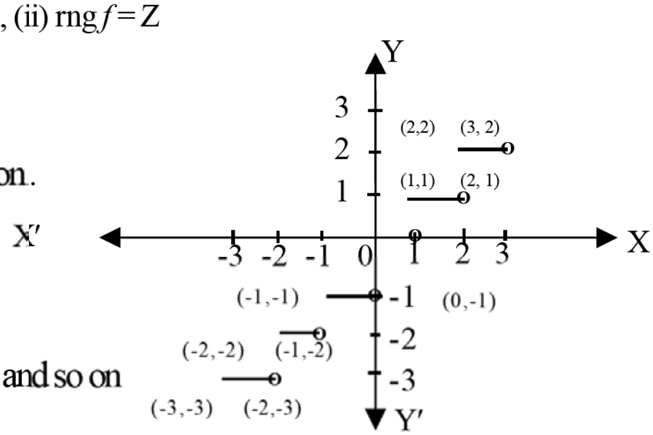
$[\,x\,] = n$ for $n \le x < n + 1$

It is clear from definition that (i) dom $f = R$, (ii) rng $f = Z$

So f(x) [x] = $\begin{cases} 0 \text{ if } 0 \le x < 1 \\ 1 \text{ if } 1 \le x < 2 \\ 2 \text{ if } 2 \le x < 3 \text{ and so on.} \end{cases}$

and

$\begin{cases} -1 \text{ if } -1 \le x < 0 \\ -2 \text{ if } -2 \le x < -1 \\ -3 \text{ if } -3 \le x < -2 \text{ and so on} \end{cases}$



The graph of $y = f(x) = [x\,]$ is plotted above

The graph consists of infinitely many closed open parallel line segments

This graph looks like steps.

Note that the **greatest integer function is a step function with jumps at integral points.**

3.5 Different categories of functions

We have discused examples of some real functions. In mathematics we often categorise numbers, sets, matrices etc. into different types (e.g. rational, irrational, finite, infinite etc.) for the sake of companison and ease of systematic study. Similarly we also have some categories of functions.

It is worth mention that these categories are not exclusive. A particular function may come in more than one categories.

(i) **Algebraic functions**

A function which can be generated by a variable by a finite number of algebraic operations such as addition, subtraction, multiplication, division, square, square root etc. is called an algebraic function : Polynomial function, the **rational** function (the function which is obtained as the ratio of two polynomial functions such as

$$\frac{x^2 + x + 1}{x^3 + 2x^2 + x + 5}$$

are algebraic functions; so also are the functions of the type

$\sqrt{x^2 + x + 1}$, $(x + 2)^{\frac{1}{5}}$ $(x^5 + 1)$, $(2x + 1)^{3,}$ etc.

(ii) **Transcendental functions**

All functions which are not algebraic are called transcendental functions. Examples of such functions are given below.

(a) **Trigonometric function :**

sine : $R \rightarrow [-1, 1]$

cosine : $R \rightarrow [-1, 1]$

tangent : $R' \rightarrow R$ Where $R' = R - \{(2n + 1)\frac{\pi}{2} : n \in Z\}$

cotangent : $R'' \rightarrow R$ where $R'' = R - \{n\pi : n \in Z\}$

secant : $R' \rightarrow R$

cosecant : $R'' \rightarrow R$

The above trigonometric functions are abbreviated as sin, cos, tan, cot, sec and cosec respectively. Note that the domains of tan, cot, sec and cosec could not be all of R but a truncated R

(b) **Inverse Trigonometric functions**

These functions shall come up in Vol-II.

(c) **Exponential functions**

Functions of the type a^x, $a > 0$ and $a \neq 1$, e^x etc are called exponential functions. More generally functions of the type $x^{\sin x}$, $(\cos x)^{\log x}$ are also transcendental.

(d) **Logarithmic functions**

Functions such as $\log_a x$, $\log_e (1+x)$. etc.

(e) **Irrational powers** of positive real numbers such as $x^{\sqrt{2}}$ etc.

Exponential and logarithmic functions will be discussed later.

We have already discused exponential and logarithimic functions while discusing graphs.

(iii) Odd and Even Functions

A function f is called odd if $f(-x) = -f(x)$ and it is called even if $f(-x) = f(x)$ for all $x \in D_f$.

Since $\sin(-\theta) = -\sin\theta$ and $\cos(-\theta) = \cos\theta$, it follows that sine is an odd and cosine is an even function. The functions

$$f(x) = x^2,$$
$$f(x) = x^4,$$
$$f(x) = x^2 + x^4 + x^6$$

are examples of even functions. The functions

$$f(x) = x,$$
$$f(x) = x^3,$$
$$f(x) = x + x^3$$

are examples of odd functions. Whether a function has to be either even or odd ? Consider

$$f(x) = x + x^2$$

which is neither odd nor even. Similarly the functions

$$f(x) = \cos x + \sin x,$$
$$f(x) = e^x$$

are neither odd nor even.

What about the constant function

$$f(x) = a\ ?$$

It is even. It is also odd if $a = 0$

If f is a real function, then the function

$$g(x) = \frac{f(x) + f(-x)}{2}$$

is an **even** function and

$$h(x) = \frac{f(x) - f(-x)}{2}$$

is an **odd** function (verify this).

Since $f(x) = \dfrac{f(x) + f(-x)}{2} + \dfrac{f(x) - f(-x)}{2} = g(x) + h(x)$

it follows that **every function can be written as the sum of an odd and an even function.**

(iv) Periodic Functions

A function f is called periodic with period k if $f(x + k) = f(x)$ for some constant $k \neq 0$. The least positive value of k for which $f(x + k) = f(x)$ holds is called the **Fundamental period** of f. If k is a period of f, then any non zero integral multiple of k is also a period of f (prove).

Periods of trigonometric functions shall be discussed in detail in chapter-4 (Art. 4.3)

It can be shown that if $f(x+k) = f(x)$ then $f(ax+b) = f(a(x+\frac{k}{a})+b)$ which shows that $f(ax+b)$ is also periodic with period k/a.

Example : $f(x) = x - [x]$ is periodic with period 1.

3.6 Algebra of real value functions

We have been familiar with examples of functions. In fact mathematics has an inexhaustive stock of a vast variety of functions. It is also possible to obtain many more functions by combining the known ones in various ways. For instance, the algebraic combinations sucn as $f+g$, fg, f/g, $|f|$ and the composition fog give rise to new functions.

Before taking up definitions of addition, subtraction etc. of functions, let us first define **equality of two functions**.

Equality : If f_1 and f_2 are two relations on X to Y, then they are called equal if they are equal as subsets of X $\times$ Y. When applied to functions, this means that two functions $f_1, f_2 : X \to Y$ are equal (written as $f_1 = f_2$) if $\{(x, f_1(x)) : x \in X\} = \{(x, f_2(x)) : x \in X\}$

that is,

$f_1(x) = f_2(x)$ for each $x \in X$, Note that $dom f_1\, dom f_2$

Though the co-domain Y does not occur explicitly in the definition of equality, it must be understood that the co-domain is the same for both the functions. If co-domains for f_1 and f_2 are different, then even if $f_1(x) = f_2(x)$ for all $x \in X$, then the functions are treated as different. For example if $f_1 : R \to R$, $f_2 : R \to R^+ \cup \{0\}$ are difined by $f_1(x) = x^2$, $f_2(x) = x^2$, then even though $f_1(x) = f_2(x)$, $x \in R$, they represent two different functions.

Also the two function $g_1 : R^+ \to R$, $g_2 : R \to R$ defined by $g_1(x) = x^2$, $g_2(x) = x^2$ are different as their domains are different even if their functional values are same.

Thus in the definition of equality of functions f_1 and f_2 it must be understood that $f_1(x) = f_2(x)$, $x \in \text{dom} f_1$, **provided** that $dom f_1 = \text{dom} f_2$ and co-domain of f_1 = co-domain of f_2.

The reason for this definition will be clear when we discuss inverse funcitons later.

Now we come to our main objective, the algebra of real valued functions.

Definition :

Let $f, g : X \to R$. Then

(i) the addition of two functions, written as $f + g$ is defined by

$(f+g)(x) = f(x) + g(x)$, $(x \in X)$

(ii) the multiplication fg is defined by $(fg)(x) = f(x)\, g(x)$, $(x \in X)$

(iii) The quotient f/g is defined by $(f/g)\,(x) = \dfrac{f(x)}{g(x)}$, $x \in X$ provided that $g(x) \neq 0$.

Warning : In order that the addition (so also the subtraction), multiplication, quotient of two functions are meaningful, it is necessary that

(a) the domains of all participating functions should be same (or at least should have non-empty intersection on which functions are defined), i.e. if f:A $\to$ R and g : B $\to$ R

then f+g : A $\cap$ B $\to$ R ; provided A $\cap$ B $\neq \phi$ and the same restrictions apply for f-g, f g and $\dfrac{f}{g}$.

(b) (b) the co-domain must be endowed with additive and multiplicative structure which should be closed with respect to these structures.

For example, for the functions

$f : [0, 1] \to R$ where $f(x) = x$ and

$g : [2, 3] \to R$ where $g(x) = x^2$

$f + g$ is not defined. Similarly for functions

$f : [0, 1] \to [0, 1]$ where $f(x) = x$ and

$g : [0, 1] \to [0, 1]$ where $g(x) = x^2$

$f + g$ is not defined as $f(x) + g(x)$ goes beyond the co-domain [0, 1] and reach out to [0, 2]. Thus, for the definition to be meaningful, we may take

$f : [0, 1] \to [0, 2]$ or R.

$g : [0, 1] \to [0, 2]$ or R.

Also for X = {Ram, Gopal}, Y= {Potato, Tomato}

$f_1 : X \to Y, f_2 : X \to Y$, defined by f_1 : {(Ram, Potato), (Gopal, Tomato)}

f_2 : {(Ram, Tomato), (Gopal, Potato)}

$f_1 + f_2$ is not defined as f_1 (Ram) + f_2 (Ram) = Potato + Tomato is meaningless as the co-domain is not endowed with additive structure.

Example 23

f : R $\to$ R and g : R $\to$ R such that f (x) = $2x + 3$, g (x) = x^2+9, x $\in$ R.

Find the sum function and its value at x= 1

Solution

$(f+g)\,(x) = f\,(x) + g\,(x) = (2x+3) + x^2 + 9 = x^2 + 2x + 12$

The value of (f+g) (x) at x =1 is $1^2+2.1+12=15$,

i.e. $(f+g)\,(1) = 15$.

Example 24

Let f and g be two real functions defined by

$f(x) = |x|, \ \forall x \in R$ and $g(x) = x, \ \forall x \in R$

find $f+g$, $f-g$, fg, $\frac{f}{g}$

Solution

The domain of both the functions f and g being R, the intersection of these domains is also R.

Therefore $f+g : R \to R$ such that

$$(f+g)(x) = f(x) + g(x) = \begin{cases} 2x \text{ if } x \geq 0 \\ 0 \quad \text{if } x < 0 \end{cases}$$

Similarly,

$$(f-g)(x) = \begin{cases} 0 \text{ if } x \geq 0 \\ -2x \text{ if } x < 0 \end{cases}$$

$$(fg)(x) = \begin{cases} x^2 \text{ if } x \geq 0 \\ -x^2 \text{ if } x < 0 \end{cases}$$

$$\left(\frac{f}{g}\right)(x) = \begin{cases} 1 \text{ if } x > 0 \\ -1 \text{ if } x < 0 \end{cases}$$

Note that since $g(0) = 0$, $dom\left(\frac{f}{g}\right) = R - \{0\}$

Hence $\left(\frac{f}{g}\right)(x)$ is not defined for $x = 0$

Example 25

Let $f : [-3, \infty) \to R$ such that $f(x) = \sqrt{x+3}$

and $g : R - (-2, 2) \to R$ such that $g(x) = \sqrt{x^2 - 4}$

Find $f+g$, $\frac{f}{g}$.

Solution

Have $dom f = D_f = [-3, \infty)$ and $dom g = D_g = R - (-2, 2)$

$\therefore D_f \cap D_g = [-3, -2] \cup [2, \infty)$ which is the domain of $f + g$.

Now $f + g : [-3, -2] \cup [2, \infty) \to R$ such that

$(f + g)(x) = f(x) + g(x) = \sqrt{x+3} + \sqrt{x^2 - 4}$.

since $g(x) = 0$ for $x = \pm 2$, we have exclude -2 and 2 from $D_f \cap D_g$

So $\frac{f}{g} : [-3, -2) \cup (2, \infty) \to R$ such that

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{\sqrt{x+3}}{\sqrt{x^2-4}},$$

EXERCISES 3 (c)

1. Give an example of a relation which is not a fuction.

2. If X and Y are sets containing m and n elements respectively then what is the total number of fuctions from X to Y ?

3. Find the domain of the following functions :

(i) $\sqrt{9-x^2}$ (ii) $\frac{x}{1+x^2}$ (iii) $1 - |x|$ (iv) $\frac{1}{x^2-1}$

(v) $\frac{\sin x}{1+\tan x}$ (vi) $\frac{x}{|x|}$ (vii) $\frac{1}{x+|x|}$ (viii) $\sqrt{\log\frac{12}{(x^2-x)}}$

(ix) $[x] - x$ (x) $\frac{1}{\sqrt{1-x^2}}$ (xi) $\log(\sin x)$

4. Find the range of the following functions:

(i) $\frac{x^2-1}{x^2+1}$ (ii) $\sqrt{x-1}$ (iii) $[x] - x$ (iv) $\frac{x}{1-x}$

(v) $\frac{x}{1+x^2}$ (vi) $\frac{1}{2-\cos 3x}$ (vii) $\log_{10}(1-x)$ (viii) $\sqrt{1+x^2}$

5. Find the domain and range of the following functions:

(i) $\frac{x^2}{1+x^2}$ (ii) $\sqrt{2x-3}$ (iii) $\log_e |x - 2|$

6. Give an example of a step function

on $[-1,3] = \{x \in R \mid -1 \le x \le 3\}$

7. Let X = $\{a,b,c\}$, Y = $\{1,2,3,4\}$

(a) Find out which of the following relations are functions and which are not and why :

(i) $\{(a,1), (a,2), (b,3), (b,4)\}$ (ii) $\{(a,2), (b,3), (c,4)\}$

(iii) $\{(a,3), (b,1), (a,4), (c,2)\}$ (iv) $\{(a,1), (b,1, (c,1)\}$

(v) $\{(a,2), (b,1), (c,1)\}$ (vi) $\{(a,a), (b,b), (c,c)\}$

(b) Find the domain and range of those relations in (*a*) which are functions.

(c) Identify the constant function, if any.

(d) Identity the Identity function, if any.

8. Find $f\left(\sqrt{2}\right)$ and $f\left(-\sqrt{3}\right)$ for the function

$$f(x) = \begin{cases} x^2 \text{ if } x < 0 \\ x \text{ if } 0 \le x \le 1 \\ \frac{1}{x} \text{ if } x > 1 \end{cases}$$

9. Find x for which the value of $f(x) = x^2 - 4x + 3$ is

(i) 0 , (ii) -1

10. Find the value/values of x for which the following functions are not defined :

(i) $\frac{x^2-4}{x-2}$ (ii) $\frac{\sin x}{x}$ (iii) $\frac{\log\cos x}{\sec x}$

11. Let $f(x) = \sqrt{1+x}$, $g(x) = \sqrt{1-x}$

find (i) f + g (ii) f - g (iii) fg (iv) $\frac{f}{g}$

Also find the domain in each case.

12. If $(x) = \log_e\left(\dfrac{1-x}{1+x}\right)$, then prove that $f(x) + f(y) = f\left(\dfrac{x+y}{1+xy}\right)$

13. Let $f = \{(-1,4), (2,7), (-2,11), (0,1), (1,2)\}$ be a quadratic polynomial from Z to Z. Find $f(x)$.

14. Sketch the graphs of the following functions :

(i) $f(x) = x^3$, (ii) $f(x) = 1+\dfrac{1}{x^2}$, (iii) $f(x) = (x-1)^2$

●

Trigonometric Functions

A line is not made up of points.

- Aristotle

4.0 Historical Introduction

The word 'trigonometry' is derived from the Greek words 'trigonon' and 'metron' which mean 'angle' and 'measure'.

This subject was originally developed to solve geometric problems involving triangles. The Hindu mathematicians Aryabhatta, Varahamihira, Brahmagupta and Bhaskara have a lot of contributions to trigonometry. It is interesting to note that the words sine, cosine have entered the latin mathematical vocabulary as mistranslations of certain words from Arabic which were originally from Sanskrit. Besides Hindu mathematicians, ancient Greek and Arabic mathematicians also contributed a lot of this subject. Applications of this subject have now widened. Currently Trigonometry is used in many areas such as science of seismology, designing electrical circuits, describing the shape of an atom, predicting the height of tides in the ocean, analysing musical tones and studying the occurence of sun spots. Attempts are also being made to use trigonometric functions in forecasting fluctuations in the stock market.

4.01 Fundamental concepts - a recapitulation : (Angles and angle-measure, arc-measure)

We recapitulate some concepts of geometry which form the basis of our presentation of trigonometry as well as co-ordinate geometry in chapter-11. The theory of trigonometric functions can be developed independent of geometry, which is beyond our scope at present. So we make the development geometry - based.

These concepts, based on set theory, have been introduced by pioneers like Moritz Pasch (1843-1930), David Hilbert (1862-1943) and G.D. Birkhoff (1884-1944). A quick review of those concepts have been done, which are essential in understanding the set-theoretic meaning of angle-measure and trigonometric arguments when they are arbitrary real numbers.

1. Points, Lines and Planes : These are undefined terms. There are several axioms which determine their mutual relationship usually studied in the geometry- curriculum of high - school level mathematics. One such axiom states that lines and planes are sets of points. These entities, being undefined, have variable meanings and interpretations. Any sort of physical or mental objects that satisfy the axioms governing these terms, can be regarded as manifestations or models of points, lines and planes.

2. Collinearity : Three distinct points are said to be collinear when they all belong to a line.

[If P is a point, L is a line and $P \in L$, we usually say : P lies on L or L passes through P.]

3. The concept of distance, The ruler postulate : We always talk about the distance of one point from the other. In our day - to - day practice, we measure distance by assigning a nonnegative real number to a pair of points by some instrumental device such as a scale.

Let us take any two points P, Q $\in \mathscr{P}$ (a plane) and call the distance between them as d (P, Q). You can easily comprehend that distance 'd' is a function from the cartesian product $P \times P$ onto the set of nonnegative real numbers. The following axiom, known as ruler postulate, highlights the mathematical assumption inherent in the act of measurement.

Ruler postulate : Let $\mathscr{P}$ be a plane. There is a mapping $d: \mathscr{P} \times \mathscr{P} \to \{x \in R : x \geq 0\}$ so that for every line L $\subseteq \mathscr{P}$ there exists a bijective map, f : L $\to$ R, such that for all P, Q $\in$ L, d (P, Q) = $|f$ (Q) $- f$ (P)$|$.

Note : 1. For assigning real numbers to the points of a line we first require a unit of measurement. This is provided by $d : \mathscr{P} \times \mathscr{P} \to \{x \in \mathrm{R} : x \geq 0\}$.

2. The bijective map $f : L \to R$ is called a co-ordinate system or scale for L. f (P) is called the co-ordinate of the point P, lying on L, in the scale 'f'. Nothing prevents us from choosing different co-ordinate system by varying f. A change in f results in a different scale of measurement.

3. The distance d (P, Q) is also denoted as PQ and is also called **'length'**. It follows from the axiom that

$\forall$ P, Q $\in \mathscr{P}$, (i) PQ ≥ 0, PQ $= 0 \Leftrightarrow$ P coincides with Q

(ii) PQ = QP.

The triangle inequality PQ + QR $\geq$ PR for P, Q, R $\in P$ follows with the additional assumption that we take the same scale of measurement for different lines in the plane.

4. Betweenness : If A, B, C are distinct and collinear, we say, B lies between A and C and write A – B – C, if and only if, AC = AB + BC.

Note that A – P – B $\Leftrightarrow$ B – P – A.

Segment and Ray :

Notation : If A and B are distinct points on the line L, we write L = $\overleftrightarrow{AB}$. Clearly if C is another point on L, then

L = $\overleftrightarrow{AB} = \overleftrightarrow{BC} = \overleftrightarrow{CA} = \overleftrightarrow{AC}$ etc.

The set $\overline{AB}$ = {A, B} $\cup$ {P : A – P – B} is called a **segment** (usually called the segment or line - segment joining the points A and B).

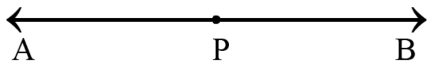
Clearly $\overline{AB} = \overline{BA}$ and $\overline{AB} \subset \overleftrightarrow{AB}$.

A and B are called the end points of $\overline{AB}$ and length of $\overline{AB}$ is defined as AB.

The set $\overrightarrow{AB} = \overline{AB} \cup$ { P : A – B – P } is called a **ray** with vertex A and passing through B.

If C is any other point satisfying either A – C – B or A – B – C, then obviously $\overrightarrow{AB} = \overrightarrow{AC}$

Two rays $\overrightarrow{PA}$ and $\overrightarrow{PB}$ are called **opposite rays** if $\overrightarrow{PA} \cup \overrightarrow{PB} = \overleftrightarrow{AB}$



(Opposite rays)

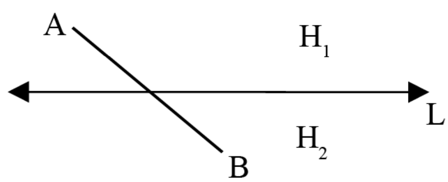
Notes : (i) $\overline{AB} \subset \overrightarrow{AB} \subset \overleftrightarrow{AB}$ (ii) $\overrightarrow{AB} \cup \overrightarrow{BA} = \overleftrightarrow{AB}$ (iii) $\overrightarrow{AB} \cap \overrightarrow{BA} = \overline{AB}$

(iv) $\overline{AB}$, $\overrightarrow{AB}$ and $\overleftrightarrow{AB}$ are sets of points, whereas AB is a real number.

5. **Convex Set :** A set of points S is called convex if $\forall$ A, B $\in$ S; $\overline{AB} \subseteq$ S. Planes and lines are convex sets.

6. Plane - Separation postulate : If L is a line in a plane *P*, then the set of points of *P*, not on L, are divided into two disjoint, nonempty and convex sets H_1 and H_2 such that

$A \in H_1, B \in H_2 \Rightarrow \overline{AB} \cap L \neq \phi$ i,e $\overline{AB}$ intersects L.

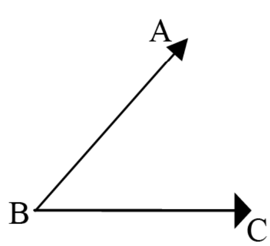


Notes : (1) H_1 and H_2 are called the **half - planes** or the **sides** of L. L is called the edge of the half planes H_1 and H_2.

(By definition of edge, a half plane and its edge are disjoint)

(2) The half planes of $\overline{AB}$ and $\overrightarrow{AB}$ are defined to be those of $\overleftrightarrow{AB}$.

7. Angle : If A, B, C are three distinct and noncollinear points, then the set $\angle ABC = \overrightarrow{BA} \cup \overrightarrow{BC}$ is called an angle (read as angle ABC)



Interior of an angle : Consider $\angle ABC$. Let us call the side of $\overline{BC}$ containing A, as the A – side of $\overline{BC}$ and the side of $\overline{AB}$ containing C as the C–side of $\overline{AB}$. **Then interior of** $\angle ABC$,denoted as Int$\angle ABC$, is defined as the intersection of the A– side of $\overline{BC}$ and the C– side of $\overline{AB}$.

Observe that an angle and its interior are disjoint sets.

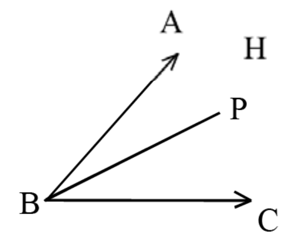
Measuring angles :

Angle– measure : With an angle $\angle ABC$ we associate a positive real number m$\angle ABC$, called measure of $\angle ABC$, by the following axiom :

Protractor postulate :

There is a function m from the set of all angles on to $\{x : x \in R, 0 < x < \alpha\}$ such that

1. If $\overrightarrow{BC}$ is on the edge $\overleftrightarrow{BC}$ of a half–plane H, then for every $r \in (0, \alpha)$, there is exactly one ray $\overrightarrow{BA}$, with $A \in H$ such that $m\angle ABC = r$



2. If $P \in$ Interior $\angle ABC$, then $m\angle ABP + m\angle PBC = m\angle ABC$.

$$\left\{ \begin{array}{l} m\angle ABC = r \\ m\angle ABC = m\angle ABP + m\angle PBC \end{array} \right\}$$

Notes : (i) m is known as the angle - measure function.

(ii) The act of assigning a positive real number to an angle, in conformity with the postulate, is called measuring the angle and the instrument making it possible is called 'a protractor'. A protractor may be a physical object or even an abstract formula.

(iii) If we take $\alpha = \pi$, 180 and 200, then we get the angle measures in radians (written as r^c or simply r), degrees (written as r^0) and grades (written as r^g) respectively. A change in α results in a new protractor.

(iv) If θ is measure of an angle, then $0 < \theta < 180$ (in degree - measure), $0 < \theta < \pi$ (in radian - measure) and $0 < \theta < 200$ (in grade - measure).

Conversion of degrees into radians and vice-versa :

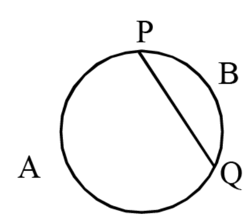
From the above definition, it is obvious that

$$1 \text{ degree} = \frac{\pi}{180} \text{ radian and}$$

$$1 \text{ radian} = \frac{180}{\pi} \text{ degrees}$$

Without mention of 'radian', 'degree' or 'grade', it is to be understood as radian measure.

(8) Arc and arc - measure : If P and Q arc two distinct points on a circle, then $\overline{PQ}$ is called a **chord** of the circle. The set of points of the circle lying on a particular side of $\overline{PQ}$, together with P and Q, is called an **arc** of the circle.



P and Q are called the **end points** of the arc and $\overline{PQ}$, the **corresponding chord** of the arc. Obviously, if A and B are two points of the circle on the opposite sides of $\overline{PQ}$, then they belong to the two different arcs of the circle on both sides of $\overline{PQ}$. These two arcs can be denoted as $\overparen{PAQ}$ and $\overparen{PBQ}$ and are called **opposite arcs** of each other whose union is the circle. $\overline{AB}$ is called the **common chord** of the opposite arcs.

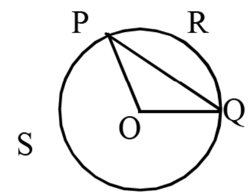
Semicircle, minor and major arcs.

Semicircle : If the end points of an arc are collinear with the centre of the circle, then the arc is called a semicircle.

There are two semicircles on the opposite sides of any diameter.

Minor and major arcs :

If any point on an arc other than its end points and the centre of the circle lie on opposite sides of the corresponding chord, then the arc is said to be a **minor arc**.



The opposite arc of a minor arc is called a **major arc**. In the accompanying figure, $\overset{\frown}{PRQ}$ is a minor arc and $\overset{\frown}{PSQ}$ is a major arc. We can also write $\overset{\frown}{PRQ}$ as the minor arc $\overset{\frown}{PQ}$ or $\overset{\frown}{PQ}$ (minor). Similarly $\overset{\frown}{PSQ}$ may be written as major arc $\overset{\frown}{PQ}$ or $\overset{\frown}{PQ}$ (major).

Arc - measure : Let P and Q be points on a circle with centre O. We define the arc - measure function m on the set of all arcs of this circle as follows :

(i) $m\ \overset{\frown}{PQ}$ (minor) $= m\angle POQ$

(ii) $m\ \overset{\frown}{PQ}$ (Semicircle) $= \pi$

(iii) $m\ \overset{\frown}{PQ}$ (major) $= 2\pi - m\angle POQ$

Notes : (1) We express $m\overset{\frown}{PQ}$ in radians, degrees or grades by expressing $m\angle POQ$ accordingly in the corresponding measures and replacing π by 180 in case of degree measure and 200 in case of grade measure.

(2) The range of the arc measure function is $(0, 2\pi)$ or $(0, 360)$ or $(0, 400)$ respectively in case of radian, degree or grade measure. In other words, if θ is measure of an arc, then $0 < \theta < 2\pi$ (radian measure) or $0 < \theta < 360$ (degree measure) or $0 < \theta < 400$ (grade measure). The inequalities are strict.

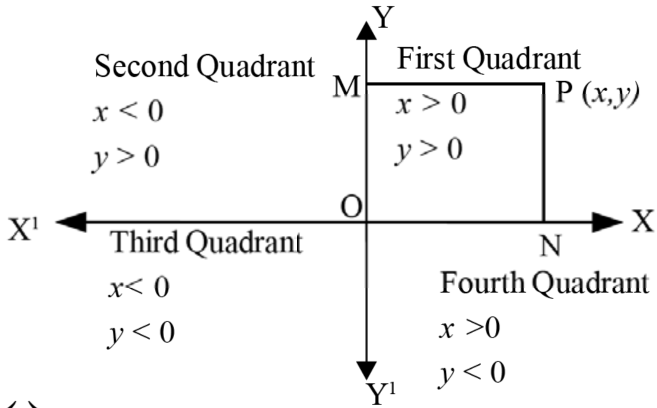
Unless otherwise specified (eg θ^0 or θ^g), θ is taken to be in the radian measure both in case of arc and angle measures.

4.1 Trigonometric Functions
(Their signs, domains and ranges)

To define trigenometric functions we first build up the fundamentals by introducing the **Rectangular Cartesian Co-ordinate system and the Polar Co-ordinate system** which are essntial to the understanding of the meaning and signs of trigonometric functions for any real value θ without involving the intuitive concept of rotation of a ray.

(Terms like 'Rotation' and 'Translation' shall be introduced in due course, later in the sequel.)

Rectangular Cartesian Co-ordinate System : We know from the ruler - postulate that there is one to one correspondence between the set of points on a line and the set of real numbers, R. We make use of this correspondence to establish a one to one correspondence between the set of points on a plane and the cartesian product R ×R.



Let $\overleftrightarrow{X'X}$ and $\overleftrightarrow{Y'Y}$ be a pair of mutually perpendicular lines on a plane, intersecting each other at the point O.

The point O is called the **Origin** and $\overleftrightarrow{X'X}$ and $\overleftrightarrow{Y'Y}$, the ***x* – axis** and the ***y* – axis** repectively. Taken together, these two lines are called **cordinate axes**.

Choosing the same scale of measurement for both the co-ordinate axes we build up a co-ordinate system as follows :

Let P be a point on the plane and let the lines through P perpendicular to x and y axes meet them respectively at N and M. We associate a pair of real numbers x and y with the point P according to the following definition.

$$x = \begin{cases} \text{ON, if N} \in \overrightarrow{\text{OX}} \\ -\text{ON, if N} \in \overrightarrow{\text{OX'}} \end{cases} \quad , \quad y = \begin{cases} \text{OM, if M} \in \overrightarrow{\text{OY}} \\ -\text{OM, if M} \in \overrightarrow{\text{OY'}} \end{cases}$$

The real numbers x and y thus associated with the point P are called its x – co-ordinate or abscissa and y – coordinate or ordinate respectively, the fact being symbolised as $P \equiv (x, y)$ or simply as P (x, y). The x and y – co-ordinates together are called the Cartesian co-ordinates of P and are denoted by the ordered pair (x, y). It is obvious from the above definition that given an ordered pair $(x, y) \in R \times R$ we can get exactly one point in the plane and conversely.

The set of co-ordinates we obtain in the above process is known as the rectangular Cartesian System of co-ordinates, named after Rene Descartes, the French mathematician and philosopher who developed co-ordinate Geometry.

Notes : (1) It follows from the definition of co-ordinates that the **origin has co-ordinates (0,0)**.

(2) The sets of points $\{(x, 0) : x > 0\}$, $\{(x, 0) : x < 0\}$, $\{(0, y) : y > 0\}$ and $\{(0, y) : y < 0\}$ are known as the **positive *x* – axis, negative *x* – axis, positive *y* – axis and negative *y* – axis** respectively.

(3) The side of x – axis that contains the positive y – axis is called the **upper half – plane** of x - axis and the side that contains the negative y - axis is called the **lower half - plane** of x - axis. By **'above' or 'below' x – axis**, we shall mean the upper and lower half – plane of x – axis respectively.

(4) The y – co-ordinate of P can also be PN or – PN according as P is in the upper half – plane or lower half- plane of x – axis.

(5) The plane of x and y - axis is known as the **Co-ordinate plane** or the **Cartesian plane**.

The Four Qudrants :

The set of points not on the co-ordinate axes is divided into four disjoint subsets known as quadrants.

First quadrant Q_1 = Interior $\angle XOY$

Second quadrant Q_2 = Interior $\angle X^1OY$

Third quadrant Q_3 = Interior $\angle X^1OY^1$

Fourth quadrant Q_4 = Interior $\angle XOY^1$.

By the definition of Cartesian coordinates,

(i) $P(x, y) \in Q_1 \Leftrightarrow x > 0, y > 0$

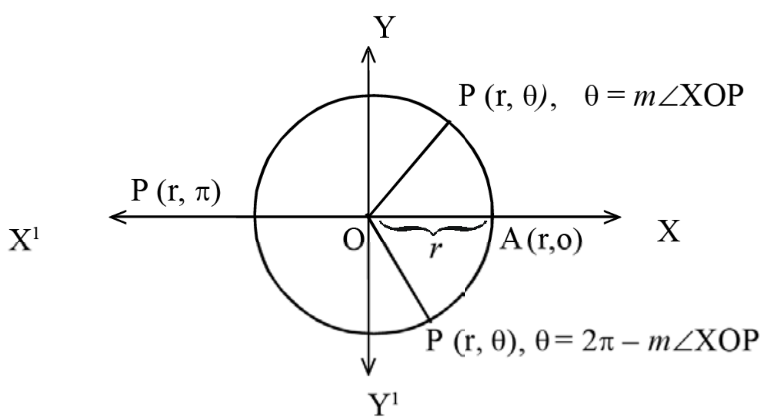
(ii) $P(x,y) \in Q_2 \Leftrightarrow x < 0, y > 0$

(iii) $P(x,y) \in Q_3 \Leftrightarrow x < 0, y < 0$

(iv) $P(x,y) \in Q_4 \Leftrightarrow x > 0, y < 0$.

Polar coordinate system :

To develop the polar coordinate system we consider the position of a point in the cartesian plane with respect to the positive x – axis $\overrightarrow{OX}$ and the origin O. In this system $\overrightarrow{OX}$ and O are known as the **initial ray** and the **pole** respectively.



Definition :

A point P in the plane has polar co-ordinates (r,θ); $r, \theta \in R$ where r = OP and θ is defined as follows :

(1) If P coincides with the pole, θ is any real number. (obviously $r = 0$ in this case)

(2) If P is different from the pole, let a circle of radius OP intersect the initial ray at A We define :

(i) If P lies on the initial ray i.e. coincides with A, then $\theta = 0$

(ii) If P lies in the upper half - plane of the initial ray, $\theta = m\overset{\frown}{AP}$ (minor) i.e $\theta = m\angle XOP$

(iii) If P lies on $\overrightarrow{OX'}$, $\theta = \pi$ (measure of the semicircle in the upper half - plane of x - axis)

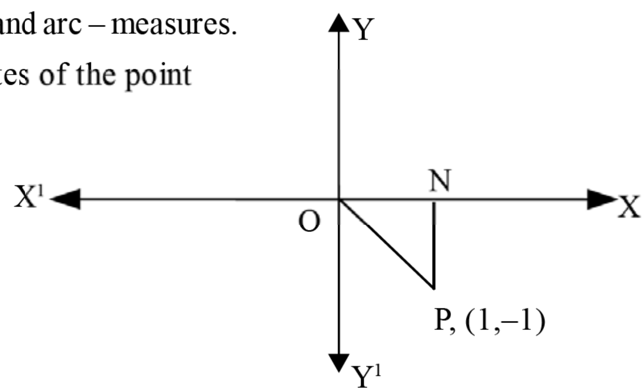
(iv) If P lies in the lower half - plane of the inital ray, $\theta = m\overset{\frown}{AP}$ (major) i.e. $\theta = 2\pi - m\angle XOP$.

N.B. (1) If a point P has polar co-ordinates (r, θ) we write $P \equiv (r, \theta)$ or simply $P(r, \theta)$.

(2) The concept of minor and major arcs or semicircle have been brought into the definition simply to give a geometric significance to θ in $P(r, \theta)$. The definition is complete even without the clauses involving a circle and arc – measures.

Illustration : Let us find the polar co-ordinates of the point having Cartesian co-ordinates $(1, -1)$

Let P be the given point,



Here ON = NP = 1

$\therefore OP = \sqrt{2}$ and $m\angle XOP = 45°$.

(As ΔPON is isosceles and right - angled)

Taking $\overrightarrow{OX}$ as the initial ray and origin O as the pole, the point P has polar co-ordinates (r, θ) where $r = OP = \sqrt{2}$ and $\theta = 2\pi - m\angle XOP = 315°$. ($\because$ $P \in$ lower half - plane of initial ray)

Thus we have $P(\sqrt{2}, 315°)$

Extended definition (Polar co-ordinates) :

The definition of polar co-ordinates given earlier restricts r and θ respectively to $[0, \infty)$ and $[0, 2\pi)$. We now liberate both of them onto R by the following extension.

Definition of (r, θ) for $r, \theta \in R$

(1) For $r \in R$

$$(r, \theta) \equiv \begin{cases} (r, \theta + 2\pi) & \text{if} \quad \theta < 0 \\ (r, \theta - 2\pi) & \text{if} \quad \theta \geq 2\pi \end{cases}.$$

(2) For $\theta \in R$, $(r, \theta) \equiv (-r, \theta + \pi)$, if $r < 0$.

(1) and (2) together give the definition.

Note : if $\theta < 0$, by definition $(r, \theta) \equiv (r, \theta + 2\pi)$.

If again $\theta + 2\pi < 0$, then

$(r, \theta) \equiv (r, \theta + 2\pi) \equiv (r, \theta + 4\pi) = ... = (r, \theta + 2n\pi)$

until $0 \le \theta + 2n\pi < 2\pi$.

Similarly for $\theta \ge 2\pi$, we have $(r, \theta) \equiv (r, \theta - 2n\pi)$

Thus in general $(r, \theta) \equiv (r, \theta \pm 2n\pi)$; $n \in N$

or $\boxed{(r, \theta) \equiv (r, \theta + 2n\pi);\ n \in Z}$

Applying the above formula to $(r, \theta) \equiv (-r, \theta + \pi)$

we get $\boxed{(r,\theta) \equiv (-r, \theta + (2n+1)\pi), n \in Z}$

Illustration : (i) $(-3, 765°) \equiv (-3, 45°) \equiv (3, 225°)$

(ii) $(r, -30°) \equiv (r, 330°)$ (Prove)

Non uniqueness of polar co-ordinates :

It is evident from the definitions of Cartesian and polar co-ordinates that given the Cartesian or polar co-ordinates, a point can be uniquely determined in a plane.

Conversely, given a point in a plane its Cartesian co-ordinates are unique, but its polar co-ordinates vary.

The following are examples as well as the reasons of nonuniqueness of polar co-ordinates of a point.

1. For the pole $r = 0$,but θ can be any real number.

(2) (i) $(r, \theta) \equiv (r, \theta + 2n\pi)$, $n \in Z$

(ii) $(r, \theta) \equiv (-r, \theta + (2n + 1)\pi)$, $n \in Z$.

Illustration :

$(2, 450^0) \equiv (2, 90^0)$

[Def (1)] $\equiv (-2, 270^0)$

[Def (2)]

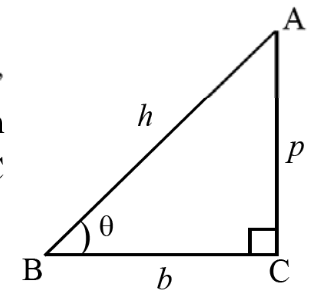
N.B. While defining polar coordinates some authors refer to r as 'radius vector' and θ as 'vectorial angle' But, as incase of cartesian coordinates, r and θ are simply real numbers.

Trigonometric ratios and trigonometric functions :

We make a review of definitions known from high school - level mathematics.

If θ is an angle measure such that $0 < \theta < \frac{\pi}{2}$, then the six trigonometric ratios : sine, cosine, tangent, cotangent, secant, cosecant for θ, abbreviated as $\sin\theta$, $\cos\theta$, $\tan\theta$, $\cot\theta$, $\sec\theta$, $\operatorname{cosec}\theta$ have been defined as follows :

Let ABC be a right angled triangle, with $m\angle ACB = \frac{\pi}{2}$. Let m



$\angle ABC = \theta$. Then $0 < \theta < \frac{\pi}{2}$. Corresponding to θ, the lengths AC, BC and AB are known as p (perpendicular), b (base) and h (hypotenuse) respectively. [observe that corresponding to $\angle BAC$ p, b and h are respectively BC, AC and AB.]

Definition : 1

$$\sin\theta = \frac{p}{h} = \frac{AC}{AB},\ \cos\theta = \frac{b}{h} = \frac{BC}{AB},\ \tan\theta = \frac{p}{b} = \frac{AC}{BC}$$

$$\cot\theta = \frac{b}{p} = \frac{BC}{AC},\ \sec\theta = \frac{h}{b} = \frac{AB}{BC},\ \operatorname{cosec}\theta = \frac{h}{p} = \frac{AB}{AC}.$$

[We can also define : $\tan\theta = \frac{\sin\theta}{\cos\theta}$, $\cot\theta = \frac{\cos\theta}{\sin\theta}$ or $\frac{1}{\tan\theta}$, $\sec\theta = \frac{1}{\cos\theta}$, $\operatorname{cosec}\theta = \frac{1}{\sin\theta}$]

Note : (1) We also write sinθ as sin $\angle ABC$ and similarly for the other trigonometric ratios. (2) It follows from the equality of ratios of lengths of corresponding sides of similar triangles that these trigonometric ratios are independent of the lengths of the particular sides $\overline{AB}$, $\overline{BC}$, $\overline{AC}$ of ΔABC. They actually depend on $m\angle ABC$ i.e. θ.

Definition – 2 : For angle - measure $\frac{\pi}{2}$ we define

$$\sin\frac{\pi}{2} = 1,\ \cos\frac{\pi}{2} = 0,\ \cot\frac{\pi}{2} = \frac{\cos\frac{\pi}{2}}{\sin\frac{\pi}{2}} = 0,\ \operatorname{cosec}\frac{\pi}{2} = \frac{1}{\sin\frac{\pi}{2}} = 1$$

$\tan\frac{\pi}{2}$ and $\sec\frac{\pi}{2}$ are not defined.

Note : $\sin\frac{\pi}{2}$, $\cos\frac{\pi}{2}$, $\cot\frac{\pi}{2}$ and $\operatorname{cosec}\frac{\pi}{2}$ have not been defined as ratios of lengths. So we do not use the term 'trigonometric ratio' for them. We shall replace this term by the more general term 'trigonometric function' in due course.

Definition – 3:

0 or 0° is not an angle - measure. However, we define $\sin 0° = 0$, $\cos 0° = 1$, $\tan 0° = \frac{\sin 0°}{\cos 0°} = 0$, $\sec 0° = \frac{1}{\cos 0°} = 1$.

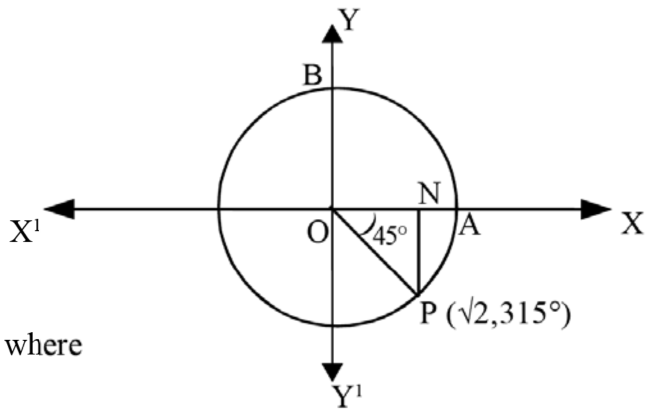
Note : cot0° and cosec0° are not defined.

For the same reason as above we do not use the term 'trigonometric ratio' for sin0°, cos0°, tan0° and sec0°.

Trigonometric functions for θ ∈ R.

Let $(r, \theta)_x$ and $(r, \theta)_y$ denote respectively the x – and y – co-ordinates of a point whose polar co-ordinates are (r, θ); $r > 0$ and $\theta \in R$.

Illustration :



In the figure r = OP = $\sqrt{2}$

$\theta = m\ \widehat{ABP} = 315°$.

$\therefore$ Polar co-ordinates of P are given by (r, θ);

where r = $\sqrt{2}$ and θ = 315°.

P has Cartesian co-ordinates (x, y), where

x = ON (= 1, in this case)

y = – PN (= – 1, in this case).

$\therefore (\sqrt{2}, 315°)_x = ON, (\sqrt{2}, 315°)_y = -PN$

i,e $(\sqrt{2}, 315°)_x = 1$ and $(\sqrt{2}, 315°)_y = -1$.

Similarly $(2, \frac{\pi}{3})_x = 1$ and $(2, \frac{\pi}{3})_y = \sqrt{3}$.

The six trigonometric functions are given by the following.

Definitions :

(i) sine : $R \to [-1, 1]$; $\sin\theta = \dfrac{(r,\theta)_y}{r}$; $r > 0, \theta \in R$

(ii) cosine : $R \to [-1, 1]$; $\cos\theta = \dfrac{(r,\theta)_x}{r}$; $r > 0, \theta \in R$.

(iii) tangent : $R - \{(2n+1)\frac{\pi}{2} : n \in Z\} \to R$;

$$\tan\theta = \frac{(r,\theta)_y}{(r,\theta)_x} \text{ or } \frac{\sin\theta}{\cos\theta};\ r > 0,\ \theta \in R - \{(2n+1)\ \frac{\pi}{2} : n \in Z\}$$

(iv) cotangent : $R - \{n\pi : n \in Z\} \to R$;

$$\cot\theta = \frac{(r,\theta)_x}{(r,\theta)_y} \text{ or } \frac{\cos\theta}{\sin\theta};\ r > 0,\ \ \theta \in R - \{n\pi : n \in Z\}$$

(v) secant : $R - \{(2n+1)\ \frac{\pi}{2} : n \in Z\} \to R - (-1, 1)$;

$$\sec\theta = \frac{r}{(r,\theta)_x} \text{ or } \frac{1}{\cos\theta};\ r > 0\ \theta \in R - \{(2n+1)\frac{\pi}{2} : n \in Z\}$$

(vi) cosecant : $R - \{n\pi : n \in Z\} \to R - (-1, 1)$

$$\text{cosec}\theta = \frac{r}{(r,\theta)_y} \text{ or } \frac{1}{\sin\theta};\ r > 0\ \ \theta \in R - \{n\pi : n \in Z\}$$

N.B. By taking r = 1 in the above definitions we get the trigonometric functions from a unit circle.

Notes :

1. From $(r, \theta) \equiv (r, \theta + 2n\pi)$, $n \in Z$, we get :

$\boxed{\begin{array}{l} \sin\theta = \sin(\theta + 2n\pi),\ \cos\theta = \cos(\theta + 2n\pi);\ n \in Z \\ \text{Similar relations also hold for } \tan\theta, \cot\theta, \sec\theta, \text{ and } \operatorname{cosec}\theta \text{ in their respective} \\ \text{domains In general if 'T' is a trigonometric function, then } T(\theta) = T(\theta + 2n\pi),\ n \in Z \end{array}}$

We shall discuss more when discussing period of a trignometric function.

2. $(r, \theta)_x = 0$ for $\theta = (2n+1)\frac{\pi}{2}$, $n \in Z$ and $(r, \theta)_y = 0$ for $\theta = n\pi$, $n \in Z$. For this reason $\{(2n+1)\frac{\pi}{2} : n \in Z\}$ and $\{n\pi : n \in Z\}$ have been excluded from the domains of tan, sec and cot, cosec functions respectively.

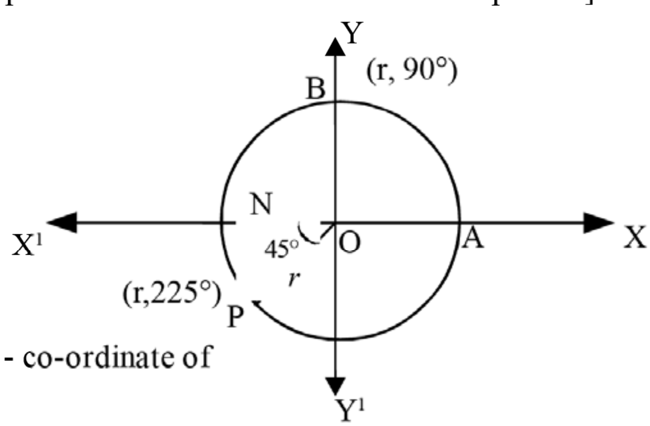
3. It follows from the equality of the ratios of lengths of corresponding sides of similar triangles that the values of the trigonometic functions for a given value of θ, do not depend on any particular value chosen for $r > 0$.

4. For $r > 0$ and $0 < \theta < \frac{\pi}{2}$, (r, θ) is in the first quadrant where $(r, \theta)_x$ and $(r, \theta)_y$ are both positive. Therefore these definitions agree with the definitions of the six trigonometric ratios given earlier.

5. **Relation Between Cartesian and Polar co-ordinates :** If $P(r,\theta) \equiv P(x, y)$, then by definition of sine and cosine functions, $\boxed{x = r\cos\theta}$ and $\boxed{y = r\sin\theta}$.

[Here (r,θ) and (x, y) are respectively polar and cartesian coordinates of the point P].

Illustrations :



1. In the figure r = OP and $m\widehat{ABP} = 225°$. So the point P has polar co-ordinates $(r, 225°)$. $(r, 225°)_x$ = x - co-ordinate of P $= -ON = -\frac{r}{\sqrt{2}}$ and $(r,225°)_y$ = y - co-ordinate of P $= -PN = -\frac{r}{\sqrt{2}}$

$$\therefore\ \sin 225° = \frac{(r,225°)_y}{r} = \frac{-\frac{r}{\sqrt{2}}}{r} = -\frac{1}{\sqrt{2}},$$

$$\cos 225° = \frac{(r,225°)_x}{r} = \frac{-\frac{r}{\sqrt{2}}}{r} = -\frac{1}{\sqrt{2}}.$$

Similarly other functional values such as tan 225°, cot225° etc. can be obtained.

2. Find $\sin\theta$ and $\cos\theta$ for $\theta = 810°$

We use note (1),

$\sin 810° = \sin(810° - 2 \times 360°)$; taking n = -1 in

$\sin\theta = \sin(\theta + 2n\pi)$; $n \in Z$

$\therefore \sin 810° = \sin 90° = 1$

and similarly $\cos 810° = \cos 90° = 0$.

Note that $\text{cosec } 810° = \dfrac{1}{\sin 810°} = 1$, but tan810° and sec810° are not defined.

Signs of Trigonometric Functions

The ASTC (All, Sin, Tan, Cos) Rule : It follows from the definition of coordinates of a point in a plane and the definitions of the trigonometric functions that

(i) $0 < \theta < \frac{\pi}{2} \Rightarrow (r, \theta) \in$ 1st quadrant, for $r > 0 \Rightarrow (r, \theta)_x, (r, \theta)_y > 0$

$\Rightarrow$ All of $\sin\theta$, $\cos\theta$, $\tan\theta$, $\cot\theta$, $\sec\theta$, and $\text{cosec}\theta$, are positive.

(ii) $\frac{\pi}{2} < \theta < \pi \Rightarrow (r, \theta) \in$ 2nd quadrant, for $r > 0$.

$\Rightarrow (r, \theta)_x < 0, (r, \theta)_y > 0$

$\Rightarrow \sin\theta > 0$. (All the rest, except $\text{cosec}\theta$, are negative)

(iii) $\pi < \theta < 3\frac{\pi}{2} \Rightarrow (r, \theta) \in$ 3rd quadrant, for $r > 0$.

$\Rightarrow (r, \theta)_x,\ (r, \theta)_y < 0$

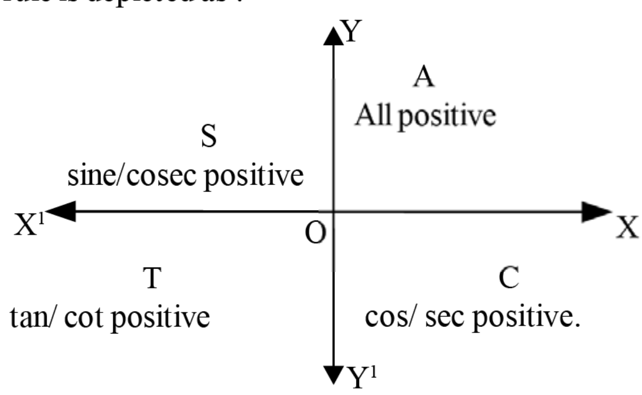
$\Rightarrow \tan\theta > 0$. (All the rest, except $\cot\theta$, are negative)

(iv) $3\frac{\pi}{2} < \theta < 2\pi \Rightarrow (r, \theta) \in$ 4th quadrant, for $r > 0$.

$\Rightarrow (r, \theta)_x > 0, (r, \theta)_y < 0$

$\Rightarrow \cos\theta > 0$. (All the rest, except $\sec\theta$, are negative)

Pictorially the rule is depicted as :



The ranges of trigonometric functions :

It follows from the definitions that

$|\sin\theta| = \dfrac{|(r,\theta)_y|}{r} = \dfrac{|y|}{r} \le 1$ ($\because$ $|y| \le$ r. We write $(r, \theta)_x = x$ and $(r, \theta)_y = y$)

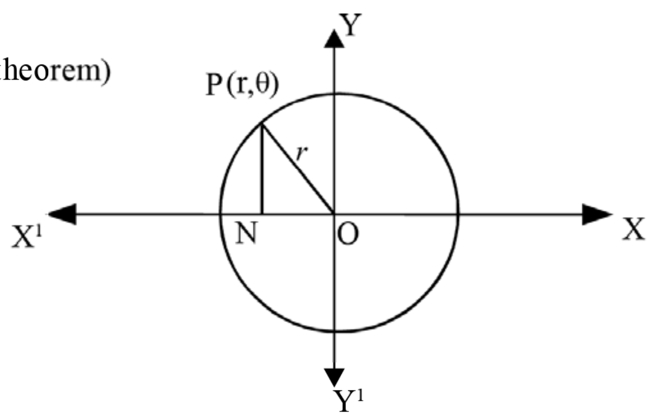
$\Rightarrow -1 \le \sin\theta \le 1$, for $\theta \in R$

similarly $-1 \le \cos\theta \le 1, \theta \in R$.

So range of sine as well as cosine function is [– 1, 1].

From this it follows that cosec $\theta \ge 1$, for $0 < \sin\theta \le 1$ and cosec $\theta \le -1$, for $-1 \le \sin\theta < 0$. Thus cosec $\theta \in R - (-1, 1)$ or $(-\infty, -1] \cup [1, \infty)$. for $\theta \in R - \{n\pi;\ n \in Z\}$. By similar considerations for secθ, it follows that the **range of secant as well as cosecant function is $(-\infty, -1] \cup [1. \infty)$.**

$$|\tan\theta| = \left|\frac{(r,\theta)_y}{(r,\theta)_x}\right| = \left|\frac{y}{x}\right| = \sqrt{\frac{y^2}{x^2}}$$



But $r^2 = OP^2 = ON^2 + PN^2$ (Pythagoras theorem)

(For $\theta \ne n\pi, n \in Z$)

$= |x^2| + |y^2| = x^2 + y^2$.

$$\therefore\ |\tan\theta| = \sqrt{\frac{r^2 - x^2}{x^2}} = \sqrt{\frac{r^2}{x^2} - 1}$$

$$\Rightarrow \tan\theta = \pm\sqrt{\frac{r^2}{x^2} - 1}.$$

Since $\dfrac{r^2}{x^2} \ge 1$ ($\dfrac{r^2}{x^2} = 1$. when $\theta = n\pi, n \in Z$), it is obvious that tanθ assumes any real number as its value, depending upon the value of x in $[-r, r] - \{0\}$, for $\theta \in R - \{(2n+1)\frac{\pi}{2} : n \in Z\}$. By similar considration for cotθ, it follows that the **range of tangent as well as cotangent function is R.**

The domains of these trigonometric functions have been given in their definitions.

A note on the variable of a trigonometric function :

It may be observed that the definitions we have given for trigonometric functions of a real variable, are primarily based on geometric considerations. But, at higher levels, when we discuss infinite series, we shall be in a position to define the trigonometric functions independent of any geometric consideration. Moreover, our definitions of trigonometric functions shall also be applicable to complex variables i.e. we shall be able to talk about sinz, cosz etc. when z is a complex number.

The variable θ in our definitions is same as an angle - measure in radians when $0 < \theta < \pi$, which is also the same as an arc - measure for a minor arc of a circle. But if $\pi \le \theta < 2\pi$, θ is no longer an angle - measure. At that time it measures a major arc of a circle. So, in this case, θ should not be called an angle - measure or angle.

When $\theta = 0$, it is not an angle⁻ measure or arc - measure. **For these reasons, it is most general and logical, at this stage, to refer to the variable of a trigonometric function not as an 'angle' but as a real number or argument.**

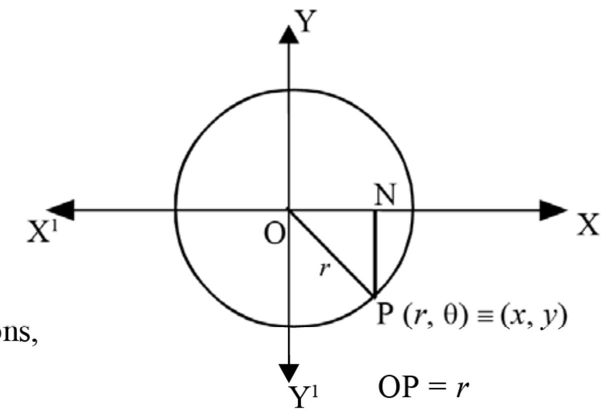
Many a time we come accross phrases like 'negative angles', 'multiple angles', 'sum of angles', and the like. Though these are not valid in in accordance with contemporary set-theoretic definition of an angle, they are still in vogue as the trigonometric arguments such as 'θ' were unitially used for $0° < \theta < 180°$ which were actually angle-measures.

4.2. The fundamental trigonometric identities :

Theorem – 1 :

$\sin^2\theta + \cos^2\theta = 1;\ \theta \in R$

Proof : Let the point P (r, θ), $(r > 0,\ \theta \in R)$, in polar co-ordinates, have Cartesian co-ordinates (x, y). Then, by definition of the sine and cosine functions,



$\sin\theta = \dfrac{y}{r}$ and $\cos\theta = \dfrac{x}{r}$.

Let $\overline{PN} \perp \overleftrightarrow{X'X}$. In the right – angled triangle PON, by Pythagoras – theorem, $OP^2 = PN^2 + ON^2$ .... (i)

From the definition of Cartesian coordinates we have : $|x| = ON$, $|y| = PN$.

$\therefore$ (i) $\Rightarrow r^2 = |y|^2 + |x|^2 = y^2 + x^2$

$$\Rightarrow \frac{y^2}{r^2} + \frac{x^2}{r^2} = 1 \ \Rightarrow \sin^2\theta + \cos^2\theta = 1;\ \text{for } \theta \in R. \square$$

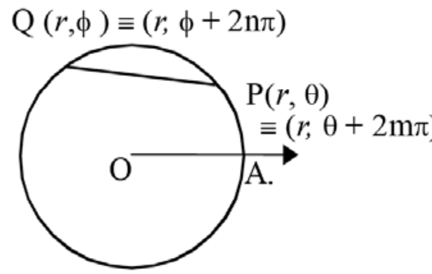
Corollary : $\sec^2\theta = 1 + \tan^2\theta$, $\operatorname{cosec}^2\theta = 1 + \cot^2\theta$; $\theta \in R$

Before proving the next two theorems, we discuss two properties of arc – measures in relation to polar co-ordinates of points on a circle. [The proofs of Lemma -1 and Lemma -2 are not for examination.]

Lemma - 1 : Let P (r, θ) and Q (r, ϕ), in polar coordinates, be two distinct points on a circle of radius r, with centre at the pole. Then measure α of an arc of the circle, with P and Q as end points, can be expressed as :

$$\alpha = \phi - \theta + 2k\pi;\ k \in Z.$$

Proof : Let the circle intersect the initial ray at A.



We suppose that neither of P and Q coincides with A.

We can find $m, n \in Z$ such that

$0 < \theta + 2m\pi < 2\pi$ and

$0 < \phi + 2n\pi < 2\pi$ and (Equality does not arise because of the assumption : A does not coincide with P or Q)

By the definition of polar co-ordinates, we have $m\overset{\frown}{AP} = \theta + 2m\pi$ and $m\overset{\frown}{AQ} = \phi + 2n\pi$. [The arcs $\overset{\frown}{AP}$ and $\overset{\frown}{AQ}$ will be minor or major arcs depending upon the positions of the points P and Q in the upper and lower half planes of $\overleftrightarrow{OA}$]

P and Q being distinct, $\theta + 2m\pi \neq \phi + 2n\pi$.

Suppose $\theta + 2m\pi < \phi + 2n\pi$.

$\Rightarrow m\overset{\frown}{AP} < m\overset{\frown}{AQ}$.

Then, by the definition of polar co-ordinates, P is a point on $\overset{\frown}{AQ}$, different from A or Q.

$\Rightarrow m\overset{\frown}{AQ} = m\overset{\frown}{AP} + m\overset{\frown}{PQ}$; where $\overset{\frown}{AP} \subset \overset{\frown}{AQ}$ and $\overset{\frown}{PQ} \subset \overset{\frown}{AQ}$.

$\Rightarrow m\overset{\frown}{PQ} = \phi + 2n\pi - (\theta + 2m\pi) = \phi - \theta + 2(n - m)\pi$ ...... (1)

Putting $k = n - m$ and $\alpha = m\overset{\frown}{PQ}$, we get

$\alpha = \phi - \theta + 2k\pi;\ k \in Z$ .... (A)

If we suppose $\phi + 2n\pi < \theta + 2m\pi$ i.e. $m\overset{\frown}{AQ} < m\overset{\frown}{AP}$, we have $m\overset{\frown}{AP} = m\overset{\frown}{AQ} + m\overset{\frown}{QP}$; where $\overset{\frown}{AQ} \subset \overset{\frown}{AP}$ and $\overset{\frown}{QP} \subset \overset{\frown}{AP}$.

$\Rightarrow m\overset{\frown}{QP} = m\overset{\frown}{PQ} = m\overset{\frown}{AP} - m\overset{\frown}{AQ} = \theta - \phi + 2(m - n)\pi$ .... (2)

There are two arcs with $\overline{PQ}$ as the common chord. The measure of one such arc is expressed by (2). The measure of the other arc is $2\pi - \{\theta - \phi + 2(m - n)\pi\}$

$= \phi - \theta + 2(n - m + 1)\pi$ .... (3)

P(r;θ) ≡ (x,y)

We choose the arc whose measure is given by (3) to satisfy the requirement of the lemma. So, taking α as the measure of this arc i.e. $\alpha = \phi - \theta + 2(n - m + 1)\pi$ and putting $k = n - m + 1$, we get

$\alpha = \phi - \theta + 2k\pi$.................. (B)

A and B complete the proof.

In case one of P and Q coincides with A, the proof is trivial and is left to the reader.

Definition (Congruent ares) : Two arcs on the same circle are congruent of they have the same measure.

(Note that we cannot have congruent arcs on different circles).

Lemma – 2 :

If $\theta + 2m\pi$ and $\theta + 2n\pi$ ($\theta \in$ R and $m, n \in$ Z) express arc – measures on the same circle then the arcs must be congruent.

Proof : Let $\alpha = \theta + 2m\pi$, $\beta = \theta + 2n\pi$ express arc measures.

Therefore $0 < \alpha < 2\pi$ and $0 < \beta < 2\pi$.

Suppose $\alpha < \beta$. Then $0 < \beta - \alpha < \beta < 2\pi$.

$\Rightarrow 0 < 2\,(n - m)\pi < 2\pi \;\Rightarrow\; 0 < n - m < 1$; which is impossible since $m, n \in$ Z. So .$\alpha < \beta$

Similarly it can be proved that $\beta < \alpha$.

Therefore $\alpha = \beta$,which means that the arcs whose measures are expressed by $\theta + 2m\pi$ and $\theta + 2n\pi$, are congruent.□

Theorem – 2 :

cos (– θ) = cosθ and sin (– θ) = – sinθ ; θ ∈ R

Proof : If $\theta = 2k\pi$, k $\in$ Z, then $(r, \theta) \equiv (r, 2k\pi) \equiv (r, 0)$ $(r > 0)$ and $(r, -\theta) \equiv (r, -2k\pi) \equiv (r, 0)$ $(r > 0)$.

$$\therefore \cos(-\theta) = \frac{(r,-\theta)_X}{r} = \frac{(r,0)_X}{r} = \frac{r}{r} = 1.$$

$$\text{Also } \cos\theta = \frac{(r,\theta)_X}{r} = \frac{(r,0)_X}{r} = \frac{r}{r} = 1; \text{ So that } \cos(-\theta) = \cos\theta.$$

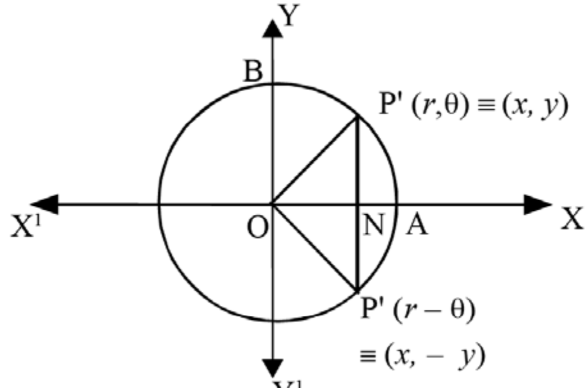
$$\text{Again } \sin(-\theta) = \frac{(r,-\theta)_Y}{r} = \frac{(r,0)_Y}{r} = 0$$

$$\text{and } \sin\theta = \frac{(r,\theta)_Y}{r} = \frac{(r,0)_Y}{r} = 0.$$

$\therefore$ sin $(-\theta) = -$ sinθ. Similar results also hold for $\theta = (2k+1)\pi$.

Now suppose $\theta \neq k\pi$; k $\in$ Z.

Let P (r, θ) and P' $(r, -\theta)$ $(r > 0)$, in polar co-ordinates, be points on a circle of radius r and centre at the pole, O. Let the circle intersect the initial ray at A. Then obviously, A has polar co-ordinates $(r, 0)$. It can be easily proved that P(r,θ) and P'$(r, -\theta)$ lie on opposite sides of $\overleftrightarrow{X'X}$ i.e. x – axis and hence $\overline{P'P}$ intersects the x– axis. Let the point of intersection be N.



By Lemma -1, measure α of an arc with end points A and P and measure β of an arc with end points A nd P' are expressible as : $\alpha = \theta + 2k_1\pi$, and $\beta = \theta + 2k_2\pi$; $k_1, k_2 \in Z$.

So, by lemma– 2, $\alpha = \beta$.

$\Rightarrow \overline{AP}$ and $\overline{AP'}$ are corresponding chords of congruent arcs of a circle.

$\Rightarrow AP = AP'$. It now follows from elementary geometry that $\overline{PP'} \perp \overleftrightarrow{OA} \Rightarrow \overline{PP'} \parallel$ y- axis $\Rightarrow$ P and P' lie on the same side of y - axis (Plane – separation postulate)

Also it follows from elementary geometry that PN = P'N. Therfore P and P' have equal x- co-ordinates, but opposite y - co-ordinates (Same in absolute value, but opposite in sign).

$\therefore$ 1f (x, y) be the Cartesian co-ordinates of P (r, θ), then cartesian co-ordinates of P' $(r, -\theta)$ are given by $(x, -y)$.

It follows from the relation between Cartesian and polar co-ordinates that

$$x = r \cos \theta , y = r \sin \theta \qquad ....(1)$$

and $x = r \cos(-\theta), -y = r \sin(-\theta)$ ....(2)

(1) and (2) imply $\cos(-\theta) = \cos\theta$, $\sin(-\theta) = -\sin\theta$; $\theta \in R$.

Corollary : $\tan(-\theta) = -\tan\theta$; $\theta \in R$.

N.B. : The identities : **$\sin(-\theta) = -\sin\theta$ and $\cos(-\theta) = \cos\theta$ show that sine is an odd function and cosine is an even function.** Similarly tangent is an odd function.

You can take similar decisions regarding the remaining trigonometric functions.

Theorem - 3 (Addition theorem)

For $\alpha. \beta, \in R$

(i) $\cos(\alpha + \beta) = \cos\alpha \cos\beta - \sin\alpha \sin\beta$

(ii) $\sin(\alpha + \beta) = \sin\alpha \cos\beta + \cos\alpha \sin\beta$

Proof : - We shall prove the first one. The second will follow as one of the corrollaries.

Let A (1, 0), B (1, α) , C (1, $\alpha + \beta$), D (1, $-\beta$), all in polar co-ordinates, lie on a circle of unit radius and centre at the pole.

We first suppose A and C are distinct points. From this, it can be easily proved that B and D are also distinct points and vice versa.

By lemma- 1, measure θ of an arc with $\overline{AC}$ as the corresponding chord is given by

$$\theta = \alpha + \beta + 2k_1 \pi; \quad \kappa_1 \in Z........ (1)$$

and measure ϕ of an arc with $\overline{DB}$ as the corresponding chrod is given by

$$\phi = \alpha + \beta + 2k_2 \pi; \quad \kappa_2 \in Z........ (2)$$

By lemma (2), $\overline{AC}$ and $\overline{DB}$ are chords of congruent arcs on a circle and, therefore,

$AC = DB$.

Next, if A and C are coincident then B and D are also coincident so that AC = 0 = DB.

Thus, in any case, we have AC = DB.

In cartesian co-ordinates we can express the points as follows: A(1, 0), B $(\cos\alpha, \sin\alpha)$, C $(\cos(\alpha+\beta), \sin(\alpha+\beta))$, D $(\cos(-\beta), \sin(-\beta))$ i.e. D $(\cos\beta, -\sin\beta)$ (by theorem - 2).

Now, AC = DB

$\Rightarrow \{\cos(\alpha+\beta)-1\}^2 + \sin^2(\alpha+\beta) = (\cos\alpha - \cos\beta)^2 + (\sin\alpha + \sin\beta)^2$

$\Rightarrow \cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$. (after simplification)

Corollary – 1 :

$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

Proof :- Replace β by $-\beta$ in $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$ and apply theorem - 2.

Corollary –2 : (i) $\cos\left(\frac{\pi}{2}-\theta\right) = \sin\theta$

(ii) $\sin\left(\frac{\pi}{2}-\theta\right) = \cos\theta$, for $\theta \in R$.

Proof :- (i) For any real number θ, the result follows by putting $\alpha = \pi/2$ and $\beta = -\theta$ in $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$.

(ii) Replace θ by $\frac{\pi}{2} - \theta$ in $\cos\left(\frac{\pi}{2}-\theta\right) = \sin\theta$, which holds for any $\theta \in R$.

Corollary -3 : $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$; for $\alpha, \beta \in R$.

Proof :- Replacing θ by $\alpha+\beta$ in $\sin\theta = \cos\left(\frac{\pi}{2}-\theta\right)$,

$$\sin(\alpha+\beta) = \cos\left(\frac{\pi}{2}-(\alpha+\beta)\right) = \cos\left(\left(\frac{\pi}{2}-\alpha\right)+(-\beta)\right)$$

$$= \cos\left(\frac{\pi}{2}-\alpha\right)\cos(-\beta) - \sin\left(\frac{\pi}{2}-\alpha\right)\sin(-\beta)$$

$$= \sin\alpha\cos\beta - \cos\alpha(-\sin\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta . \square$$

Corollary -4 : $\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$; $\alpha, \beta \in R$.

Proof :- Replace β by $-\beta$ in corollary – 3.

Corollary -5 : For $\alpha, \beta, \gamma \in R$.

(i) $\tan(\alpha+\beta) = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$, **(ii)** $\tan(\alpha-\beta) = \dfrac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$

(iii) $\tan(\alpha+\beta+\gamma) = \dfrac{\tan\alpha + \tan\beta + \tan\gamma - \tan\alpha\tan\beta\tan\gamma}{1 - \tan\alpha\tan\beta - \tan\beta\tan\gamma - \tan\gamma\tan\alpha}$

Proof is left as exercise for the reader.

Some of useful deductions from the addition theorem.

For $\theta \in R$

1. (i) $\sin(\pi - \theta) = \sin\theta$ (ii) $\cos(\pi - \theta) = -\cos\theta$

 (iii) $\tan(\pi - \theta) = -\tan\theta;\ \theta \neq (2n+1)\frac{\pi}{2},\ n \in Z$

 (iv) $\sin(2\pi - \theta) = -\sin\theta$; (v) $\cos(2\pi - \theta) = \cos\theta$

 (vi) $\tan(2\pi - \theta) = -\tan\theta,\ \theta \neq (2n+1)\frac{\pi}{2},\ n \in Z.$

Proof : Apply the addition theorem and its corollaries.

2. Replacing θ by $-\theta$ in the above we get

 (i) $\sin(\pi + \theta) = -\sin\theta$ (ii) $\cos(\pi + \theta) = -\cos\theta$ (iii) $\tan(\pi + \theta) = \tan\theta$

 (iv) $\sin(2\pi + \theta) = \sin\theta$ (v) $\cos(2\pi + \theta) = \cos\theta$ (vi) $\tan(2\pi + \theta) = \tan\theta$;

 with obvious restriction on θ for the tangent function.

3. We have already proved that $\cos(\frac{\pi}{2} - \theta) = \sin\theta$, $\sin(\frac{\pi}{2} - \theta) = \cos\theta$

 Replacing θ by $-\theta$, we get

 (i) $\cos(\frac{\pi}{2} + \theta) = -\sin\theta$ (ii) $\sin(\frac{\pi}{2} + \theta) = \cos\theta$

4. For $\theta \in R$, $\sin(\pi + \theta) = -\sin\theta \Rightarrow \sin(2\pi + \theta) = \sin(\pi + (\pi + \theta))$

 $= -\sin(\pi + \theta) = -(-\sin\theta) = (-1)^2\sin\theta.$

 Proceeding inductively we get, $\boxed{\sin(n\pi + \theta) = (-1)^n \sin\theta}$

 Similarly $\boxed{\cos(n\pi + \theta) = (-1)^n \cos\theta}$

 and $\tan(n\pi + \theta) = \tan\theta$.

 These results can be proved for $n \in Z$ applying the fact that sine is an odd function and cosine is an even function.

 By repeated application of $\sin(\frac{\pi}{2} + \theta) = \cos\theta$ using

 $\sin(\pi + \theta) = -\sin\theta$ it can be proved that,

if *n* is an odd integer.

$$\sin(n\frac{\pi}{2} + \theta) = (-1)^{\frac{n-1}{2}}\cos\theta,\ \cos(n\frac{\pi}{2} + \theta) = (-1)^{\frac{n+1}{2}}\sin\theta.,\ \tan(n\frac{\pi}{2} + \theta) = -\cot\theta$$

For even n, the previous formulae will take care.

Multiple and submultiple arguments.

For an argument (variable)θ, usually 2θ, 3θ etc. are called its mutliples and $\frac{\theta}{2}$, $\frac{\theta}{3}$ etc. are called its submultiples.

[For arguments θ and ϕ, $\theta \pm \phi$ is usually called a compound argument.]

(A) (i) $\sin 2\alpha = 2\sin\alpha \cos\alpha$ (ii) $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = 2\cos^2\alpha - 1 = 1 - 2\sin^2\alpha$. (1)

(iii) $\tan 2\alpha = \dfrac{2\tan\alpha}{1-\tan^2\alpha}$; $\alpha \neq (2n+1)\dfrac{\pi}{2}$. (2)

Proof : Take $\alpha = \beta$ in the addition theorem.

(B) (i) $\sin 3\alpha = 3\sin\alpha - 4\sin^3\alpha$ (ii) $\cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha$ (3)

(ii) $\tan 3\alpha = \dfrac{3\tan\alpha - \tan^3\alpha}{1-3\tan^2\alpha}$; $\alpha \neq (2n+1)\dfrac{\pi}{2}$. (4)

Proof : Write $3\alpha = 2\alpha + \alpha$. Apply the addition theorem and the above formula for 2α.

(C) Putting $\alpha = \dfrac{\theta}{2}$ in (A)

(i) $\sin\theta = 2\sin\dfrac{\theta}{2}\cos\dfrac{\theta}{2}$ (ii) $\cos\theta = \cos^2\dfrac{\theta}{2} - \sin^2\dfrac{\theta}{2} = 2\cos^2\dfrac{\theta}{2} - 1 = 1 - 2\sin^2\dfrac{\theta}{2}$

(iii) $\tan\theta = \dfrac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}}$. (5)

From these three we can obtain

(iv) $\sin^2\dfrac{\theta}{2} = \dfrac{1}{2}(1-\cos\theta)$ (v) $\cos^2\dfrac{\theta}{2} = \dfrac{1}{2}(1+\cos\theta)$

(vi) $\tan\dfrac{\theta}{2} = \dfrac{\sin\theta}{1+\cos\theta} = \dfrac{1-\cos\theta}{\sin\theta}$. (6)

(D) Putting $\alpha = \dfrac{\theta}{3}$ in (B)

(i) $\sin\theta = 3\sin\dfrac{\theta}{3} - 4\sin^3\dfrac{\theta}{3}$ (ii) $\cos\theta = 4\cos^3\dfrac{\theta}{3} - 3\cos\dfrac{\theta}{3}$

(iii) $\tan\theta = \dfrac{3\tan\frac{\theta}{3} - \tan^3\frac{\theta}{3}}{1-3\tan^2\frac{\theta}{3}}$ (7)

(E) $\sin C + \sin D = 2\sin \frac{C+D}{2} \cos \frac{C-D}{2}$ .... (8)

$\sin C - \sin D = 2\cos \frac{C+D}{2} \sin \frac{C-D}{2}$ .... (9)

$\cos C + \cos D = 2\cos \frac{C+D}{2} \cos \frac{C-D}{2}$ .... (10)

$\cos C - \cos D = 2\sin \frac{C+D}{2} \sin \frac{D-C}{2}$ .... (11)

Proof : We have $\sin(\alpha+\beta) + \sin(\alpha-\beta) = 2\sin\alpha\cos\beta$ .... (i)

$\sin(\alpha+\beta) - \sin(\alpha-\beta) = 2\cos\alpha\sin\beta$ .... (ii)

(using expansion of $\sin(\alpha+\beta)$ and $\sin(\alpha-\beta)$)

Now, putting $\alpha+\beta = C$ and $\alpha-\beta = D$, $\alpha = \frac{C+D}{2}$, $\beta = \frac{C-D}{2}$.

Writing (i) and (ii) in terms of C and D we get :

$$\sin C + \sin D = 2\sin \frac{C+D}{2} \cos \frac{C-D}{2}$$

$$\sin C - \sin D = 2\cos \frac{C+D}{2} \sin \frac{C-D}{2}.$$

We also have : $\cos(\alpha-\beta) + \cos(\alpha+\beta) = 2\cos\alpha\cos\beta$ .... (iii)

$\cos(\alpha-\beta) - \cos(\alpha+\beta) = 2\sin\alpha\sin\beta$ .... (iv)

Putting $\alpha-\beta = C$, $\alpha+\beta = D$, $\alpha = \frac{C+D}{2}$, $\beta = \frac{D-C}{2}$

Writing (iii) and (iv) in terms of C and D we get :

$$\cos C + \cos D = 2\cos \frac{C+D}{2} \cos \frac{D-C}{2} = 2\cos \frac{C+D}{2} \cos \frac{C-D}{2}$$

($\because$ Cosine is even function)

$$\cos C - \cos D = 2\sin \frac{C+D}{2} \sin \frac{D-C}{2} \ \square.$$

Example 1 :

Express sin 1000° as the trigonometric ratio of an acute angle.

since $1000° = 2 \times 360° + 280°$, $\sin 1000° = \sin(2 \times 360° + 280°)$

$= \sin 280° = \sin(180° + 100°)$ $[\because \sin(n\pi + 0) = (-1)^n \sin 0]$

$= -\sin 100°$ $[\sin(\pi + \theta) = -\sin\theta]$

$= -\sin(90° + 10°)$

$= -\cos 10°$ $\quad\quad\quad\quad\quad$ $[\sin(90° + \theta) = \cos\theta]$

or dividing 1000° by 90° we get $1000° = 11 \times 90° + 10°$

$$\therefore \sin 1000° = \sin(11 \times 90° + 10°) = (-1)^{\frac{11-1}{2}} \cos 10°$$

$$= -\cos 10°.$$

$$\left[\because \sin\left(n\frac{\pi}{2} + \theta\right) = (-1)^{\frac{n-1}{2}} \cos\theta;\ n \text{ odd}\right]$$

Example 2 :

Find the values of (i) tan (–900°), (ii) sin 1230°, (iii) cos (–1020)°

(i) $\tan(-900°)$ $\quad$ ($\because$ tan is an odd function)

$= -\tan 900° = -\tan(10 \times 90° + 0) = -\tan 0 = 0.$

(ii) $\sin 1230°$

$= \sin(3 \times 360° + 150°) = \sin 150°$

$= \sin(180° - 30°) = \sin 30° = \frac{1}{2}.$

(iii) $\cos(-1020°)$

$= \cos(1020°)$ $\quad$ ($\because$ cos is even)

$= \cos(2 \times 360° + 300°)$

$= \cos 300°$

$= \cos(180° + 120°) = -\cos 120°$

$= -\cos(180° - 60°) = \cos 60° = \frac{1}{2}.$

Example 3 :

Show that the equation $\cos\theta = a + \frac{1}{a}$ does not have a solution if $a \neq 0$ is real.

Proof : We have

$$\left(a + \frac{1}{a}\right)^2 = \left(a - \frac{1}{a}\right)^2 + 4a\,\frac{1}{a} = \left(a - \frac{1}{a}\right)^2 + 4.$$

Therefore $\left(a + \frac{1}{a}\right)^2 \geq 4.$

$$\Rightarrow \quad a + \frac{1}{a} \geq 2 \text{ or } a + \frac{1}{a} \leq -2$$

$\Rightarrow$ $\cos\theta \geq 2$ or $\cos\theta \leq -2$ (impossible).

EXERCISES 4 (a)

1. State which of the following are positive ?

(i) $\cos 271°$	(ii) $\sec 73°$	(iii) $\sin 302°$
(iv) $\operatorname{cosec} 159°$	(v) $\sec 199°$	(vi) $\operatorname{cosec} 126°$
(vii) $\cos 315°$	(viii) $\cot 375°$	

2. Express the following as trigonometric ratios of some acute angles.

(i) $\sin 1185°$	(ii) $\tan 235°$	(iii) $\sin(-3333)°$
(iv) $\cot(-3888°)$	(v) $\tan 458°$	(iv) $\operatorname{cosec}(-60°)$
(vii) $\cos 500°$	(viii) $\sec 380°$	

3. Find the domains of tangent and cotangent functions.
4. Determine the ranges of sine and cosine functions.
5. Find a value of A when $\cos 2A = \sin 3A$.
6. Find the value of $\cos 1° . \cos 2° ..... \cos 100°$.
7. Find the value of $\cos 24° + \cos 5° + \cos 175° + \cos 204° + \cos 300°$.
8. Evaluate $\tan\frac{\pi}{20} . \tan\frac{3\pi}{20} . \tan\frac{5\pi}{20} . \tan\frac{7\pi}{20} . \tan\frac{9\pi}{20}$.
9. Show that $\dfrac{\sin^3(180°+A).\tan(360°-A).\sec^2(180°-A)}{\cos^2(90°+A)\operatorname{cosec}^2 A.\sin(180°-A)} = \tan^3 A$.
10. If $A = \cos^2\theta + \sin^4\theta$, than prove that for all values of θ, $\frac{3}{4} \le A \le 1$.

4. 3 Periodicity of Trigonometric functions and their graphs :

We have proved that $\sin(x + 2\pi) = \sin x$, $\cos(x + 2\pi) = \cos x$, $\tan(x + \pi) = \tan x$ for all x in the corresponding domains of the functions.

Consider the sine function.. Suppose $0 < k < 2\pi$ and $\sin(x + k) = \sin x$, for all x.

Putting $x = 2\pi - k$. we get $\sin 2\pi = 0 = \sin(2\pi - k)$.

$\Rightarrow 2\pi - k = n\pi$ (By definition of sine function)

$\Rightarrow k = (2 - n)\pi$ which is a multiple of π .

Since $0 < k < 2\pi$, it follows that $n = 1$

$\therefore\ k = \pi$.

Thus $\sin(x + k) = \sin x$ implies that $\sin(\pi + x) = \sin x$ which is a contradiction.

Therefore 2π is the smallest positive number for which $\sin(x + 2\pi) = \sin x$, for all x and consequently 2π is the period of the sine function.

From cos $(x+2\pi)$ = cos$x$ and tan $(x+\pi)$ = tan x it follows similarly that periods of cosine and tangent functions are respectively 2π and π

cosec x and sec x being respectively reciprocals of sin x and cos x have period 2π. For the same reason cot x has period π as tan x. For the same reason cot x has period as tan x.

This means that **sin x, cos x, sec x and cosec x repeat their values at internals of 2π. Also tan x and cot x do the same at intervals of π.**

The periodic property of trigonometric functions is of great advantage in mathematics.

We shall make use of periodicity to sketch the graphs of trigonometric functions.

Illustration : Refering to Section 3.5 of Ch-3 on periodic functions you can prove that cos$3x$ as well as sin$3x$ has period $\frac{2\pi}{3}$.

Exercise :

Find periods of :

(i) Sin$5x$, (ii) Cos$6x$+Sin$9x$.

Also verify that Cos x^2 is not periodic.

[(i) $\frac{2\pi}{5}$, (ii) $\frac{2\pi}{3}$]

Graphs of Trigonometric Functions

Graph of y = sin x

Since the function has period 2π, it is sufficient to choose an internal of length 2π and study the behaviour of the function. We take the internal $(-\pi,\pi)$. You can choose any other interval also; simply its length has to be 2π.

We make the following observations:

(i) For $-\pi \le x \le -{}^{\pi}/_{2}$, sin x decreases from 0 to -1,

For $-\frac{\pi}{2} \le x \le 0$, increases from -1 to 0,

For $0 \le x \le \frac{\pi}{2}$, increases from 0 to 1, and

For $\frac{\pi}{2} \le x \le \pi$, decreases from 1 to 0.

(ii) $|\sin x| \le 1$, i.e. $-1 \le \sin x \le 1$, so its graph lies between the lines y = -1 and y = 1.

Later it will be proved that sin$x$ is a 'continuous function', which means that its graph has no breaks and it takes all the valus between -1 and 1

Actually for graphing a function several other informatins are necessary and at times essential, but we cannot have access to those at present and we proceed with this much of informations.

With these informations we sketch the graph of $y = \sin x$, as given in the figure.

(In all the graphs the argaments 'x' and functional values 'y' are plotted respectively along x and y-axes.

Graph of y = cos *x*

In the interval [-π, π]

(i) For $-\pi \le x \le 0$, cox x increases from -1 to 1 and

for $0 \le x \le \pi$, decreases from 1 to -1

cos x equals 0 at $x = -\frac{\pi}{2}$ and $\frac{\pi}{2}$

(ii) $1\cos x 1 \le 1$, This graph also lies between y = -1 and y = 1. As in case of sin x, cos x also takes all the value between -1 and 1.

N.B. By periodicity the graphs of sin x and cos x can be symmetrically extended beyond π and -π.

Graph of y = cosec *x*

(i) cosec x is not defined at $x = -\pi$, but nearer and nearer we take the values of x to $-\pi$, in $(-\pi, -\frac{\pi}{2})$, limitlessly larger becomes the absolute values of cosec x and its sign remains negotive, At $x = -\frac{\pi}{2}$, cosec $x = -1$

This phenomenon is stated precisely as cosec x increases from $-\infty$ to -1 in the interval $\left(-\pi, -\frac{\pi}{2}\right]$

(ii) It decreases from -1 to $-\infty$ in $\left[-\frac{\pi}{2}, 0\right)$

and is undefined at $x = 0$.

(iii) It decreases from ∞ to 1 in $\left(0, \frac{\pi}{2}\right]$ and again increases from 1 to ∞ in $\left[\frac{\pi}{2}, \pi\right)$ and is undefined at $x = \pi$.

The graph can be symmetrically extended for both right and left by periodicity.

Graph of y = sec *x*

(i) sec x decreases from -1 to $-\infty$ in $\left[-\pi, -\frac{\pi}{2}\right)$ and is undefined at x = $-\frac{\pi}{2}$

(ii) It again decreases from ∞ to 1 in $\left(-\frac{\pi}{2}, 0\right]$ and increases from 1 to ∞ in $\left[0, \frac{\pi}{2}\right)$ and is undifined at $x = \frac{\pi}{2}$.

(iii) It increases from -∞ to -1 in $\left(\frac{\pi}{2}, \pi\right]$

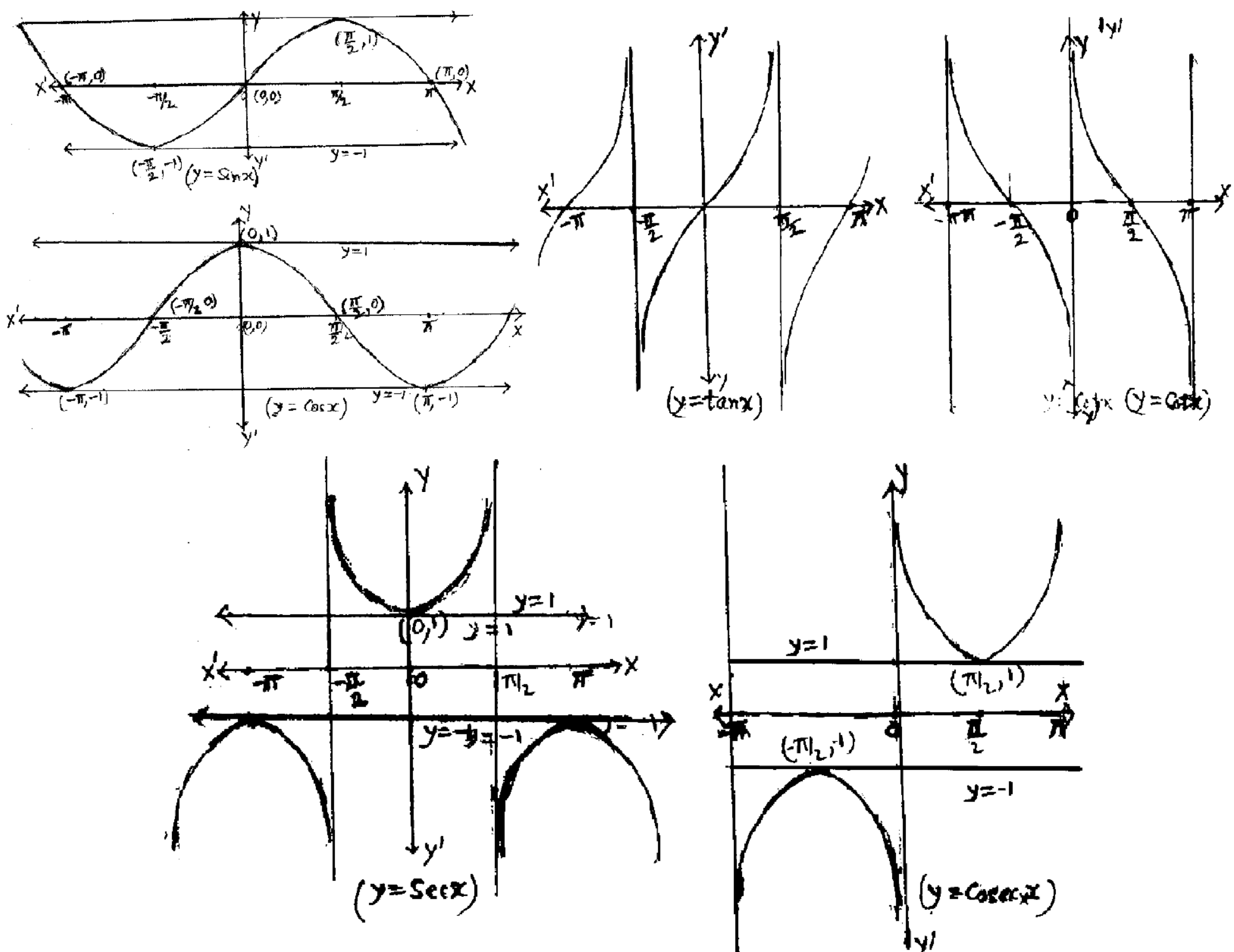
Based on these observations the graph is sketched in the figure.

Graph of y = tan *x*

tan x is not defined at $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. It increases from -∞ to 0 in $\left(-\frac{\pi}{2}, 0\right]$ and then from 0 to ∞ in $\left[0, \frac{\pi}{2}\right)$. The graph is shown in figure.

The graph of y = cot *x* can be similarly sketched.

Sketch of the Graphs.



Example 4 :

Prove that $\dfrac{\sin A.\sin 2A + \sin 3A.\sin 6A}{\sin A.\cos 2A + \sin 3A.\cos 6A} = \tan 5A.$

Proof : L.H.S. = $\dfrac{\sin A.\sin 2A + \sin 3A.\sin 6A}{\sin A.\cos 2A + \sin 3A.\cos 6A}$

$$= \frac{2\sin A.\sin 2A + 2\sin 3A.\sin 6A}{2\sin A.\cos 2A + 2\sin 3A.\cos 6A}$$

$$= \frac{\cos(2A - A) - \cos(2A + A) + \cos(6A - 3A) - \cos(6A + 3A)}{\sin(2A + A) - \sin(2A - A) + \sin(6A + 3A) - \sin(6A - 3A)}$$

$$= \frac{\cos A - \cos 3A + \cos 3A - \cos 9A}{\sin 3A - \sin A + \sin 9A - \sin 3A}$$

$$= \frac{\cos A - \cos 9A}{\sin 9A - \sin A} \qquad \text{(Apply (10) and (11))}$$

$$= \frac{2\sin 5A.\sin 4A}{2\cos 5A.\sin 4A} = \tan 5A = \text{R.H.S.} \qquad \text{(Proved)}$$

Example 5 :

Prove that

$\cos 2A + \cos 2B + \cos 2C + \cos 2(A + B + C) = 4\cos(B + C)\cos(C + A)\cos(A + B)$

Proof : L.H.S. $= \cos 2A + \cos 2B + \cos 2C + \cos 2(A + B + C)$

$= (\cos 2A + \cos 2B) + \{\cos 2C + \cos 2(A + B + C)\}$

$= 2\cos(A + B)\cos(A - B) + 2\cos(A + B + 2C)\cos(A + B)$

$= 2\cos(A + B)\{\cos(A - B) + \cos(A + B + 2C)\}$

$= 2\cos(A + B) . 2\cos(C + A)\cos(B + C)$

$= 4\cos(A + B)\cos(B + C)\cos(C + A) =$ R. H. S.

Example 6 :

Prove that

(i) $\sin 2A = \dfrac{2\tan A}{1 + \tan^2 A}$ (12)

(ii) $\cos 2A = \dfrac{1 - \tan^2 A}{1 + \tan^2 A}$. (13)

Proof :

(i) $\sin 2A = 2\sin A.\cos A$

$= 2\dfrac{\sin A}{\cos A} . \cos^2 A = 2\dfrac{\tan A}{\sec^2 A} = \dfrac{2\tan A}{1 + \tan^2 A}$.

(ii) $\cos 2A = \cos^2 A - \sin^2 A$

$= \cos^2 A - \cos^2 A . \dfrac{\sin^2 A}{\cos^2 A}$

$= \cos^2 A(1 - \tan^2 A) = (1 - \tan^2 A) / \sec^2 A = \dfrac{1 - \tan^2 A}{1 + \tan^2 A}$.

Example 7 :

Find the value of sin 18°, cos 18°, sin 36° and cos36°.

Solution :

We know that $\cos(3 \times 18°) = \cos 54° = \sin 36° = \sin(2 \times 18°)$

$\Rightarrow 4\cos^3 18° - 3\cos 18° = 2\sin 18° \,.\, \cos 18°$

$\Rightarrow \cos 18° \,(4\cos^2 18° - 3 - 2\sin 18°) = 0$

$\Rightarrow \cos 18° \,(4\sin^2 18° + 2\sin 18° - 1) = 0$

$\Rightarrow 4\sin^2 18° + 2\sin 18° - 1 = 0 \quad (\because \cos 18° \neq 0)$

$\Rightarrow \sin 18° = \dfrac{-1 \pm \sqrt{5}}{4}$

But sin 18° is positive. Thus $\sin 18° = \dfrac{\sqrt{5}-1}{4}$.

Again $\cos 18° = \sqrt{1-\sin^2 18°} = \dfrac{1}{4}\sqrt{10+2\sqrt{5}}$.

Using the equation $\sin(3 \times 36°) = \sin(2 \times 36°)$

we can deduce $\cos 36° = \dfrac{\sqrt{5}+1}{4}$, $\sin 36° = \dfrac{1}{4}\sqrt{10-2\sqrt{5}}$.

Example 8 :

Prove that $\sin 5\theta = 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$

Proof : $\sin 5\theta = \sin(3\theta + 2\theta)$

$= \sin 3\theta \,.\, \cos 2\theta + \cos 3\theta \,.\, \sin 2\theta$

$= (3\sin\theta - 4\sin^3\theta)(1 - 2\sin^2\theta) + 4(\cos^3\theta - 3\cos\theta)\, 2\sin\theta.\cos\theta$

$= (3\sin\theta - 4\sin^3\theta)(1 - 2\sin^2\theta) + 2\cos^2\theta\sin\theta\,(4\cos^2\theta - 3)$

$= (3\sin\theta - 4\sin^3\theta)(1 - 2\sin^2\theta) + 2(1 - \sin^2\theta)\sin\theta\,(1 - 4\sin^2\theta)$

$= (8\sin^5\theta - 10\sin^3\theta + 3\sin\theta) + 2\sin\theta\,(4\sin^4\theta - 5\sin^2\theta + 1)$

$= 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$.

Example 9 :

Prove that

(i) $$\sin A = \frac{2\tan\frac{A}{2}}{1+\tan^2\frac{A}{2}} \tag{14}$$

(ii) $$\cos A = \frac{1-\tan^2\frac{A}{2}}{1+\tan^2\frac{A}{2}} \tag{15}$$

Proof :

Replacing 2A by A in (12) and (13) we get (14) and (15).

Example 10 :

Given sinA, find $\sin\frac{A}{2}$, $\cos\frac{A}{2}$ and $\tan\frac{A}{2}$.

Solution :

We know that

$$2\sin\frac{A}{2}\cdot\cos\frac{A}{2} = \sin A$$

$$\sin^2\frac{A}{2} + \cos^2\frac{A}{2} = 1$$

$$\text{so that } \left(\sin\frac{A}{2}+\cos\frac{A}{2}\right)^2 = 1+\sin A$$

$$\left(\sin\frac{A}{2}-\cos\frac{A}{2}\right)^2 = 1-\sin A$$

$$\Rightarrow \sin\frac{A}{2} + \cos\frac{A}{2} = \pm\sqrt{1+\sin A}$$

$$\sin\frac{A}{2} - \cos\frac{A}{2} = \pm\sqrt{1-\sin A}$$

$$\Rightarrow \sin\frac{A}{2} = \frac{1}{2}\{\pm\sqrt{1+\sin A} \pm\sqrt{1-\sin A}\} \qquad \textbf{(16)}$$

$$\cos\frac{A}{2} = \frac{1}{2}\{\pm\sqrt{1+\sin A} \mp\sqrt{1-\sin A}\}. \qquad \textbf{(17)}$$

The appropriate sign for R.H.S of (16) and (17) is taken in the following manner.

$$\sin\frac{A}{2} + \cos\frac{A}{2} = \sqrt{2}\sin\left(\frac{A}{2}+\frac{\pi}{4}\right)$$

$$\sin\frac{A}{2} - \cos\frac{A}{2} = \sqrt{2}\sin\left(\frac{A}{2}-\frac{\pi}{4}\right).$$

If the range of A is given then one can find out the quadrant in which $\left(\frac{A}{2}+\frac{\pi}{4}\right)$ or $\left(\frac{A}{2}-\frac{\pi}{4}\right)$ lies. Thus the sign of $\sin\frac{A}{2}+\cos\frac{A}{2}$ and $\sin\frac{A}{2}-\cos\frac{A}{2}$ can be determined.

$$\text{Again } \tan\frac{A}{2} = \frac{\sin\frac{A}{2}}{\cos\frac{A}{2}} = \frac{\sin A}{1+\cos A} = \frac{\sin A}{1\pm\sqrt{1-\sin^2 A}}$$

Example 11 :

Given cosA, find the values of $\sin\frac{A}{2}$, $\cos\frac{A}{2}$ and $\tan\frac{A}{2}$.

Solution :

We know that $\sin^2\dfrac{A}{2} = \dfrac{1}{2}(1 - \cos A)$, $\cos^2\dfrac{A}{2} = \dfrac{1}{2}(1 + \cos A)$

$$\Rightarrow \sin\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{2}} \qquad \textbf{(18)}$$

$$\cos\frac{A}{2} = \pm\sqrt{\frac{1+\cos A}{2}} \qquad \textbf{(19)}$$

$$\tan\frac{A}{2} = \frac{\sin\frac{A}{2}}{\cos\frac{A}{2}} = \pm\sqrt{\frac{1-\cos A}{1+\cos A}}\,. \qquad \textbf{(20)}$$

In particular cases ± sign is replaced by a single sign applying ASTC rule.

Example 12 :

Given tanA, find $\tan\dfrac{A}{2}$.

Solution :

We know that $\tan A = \dfrac{2\tan\frac{A}{2}}{1-\tan^2\frac{A}{2}}$

$$\Rightarrow \tan^2\frac{A}{2} + 2\cot A\tan\frac{A}{2} - 1 = 0 \qquad \textbf{(21)}$$

There are two values of $\tan\dfrac{A}{2}$, when tan A is given. They are the roots of the quadratic equation (21).

Example 13 :

Find the values of $\sin 22\dfrac{1}{2}^\circ$ and $\cos 22\dfrac{1}{2}^\circ$.

Solution :

From Example 11 we have

$$\sin\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{2}}$$

$$\cos\frac{A}{2} = \pm\sqrt{\frac{1+\cos A}{2}}\,.$$

Putting A = 45° we get

$$\sin 22\frac{1}{2}^\circ = \sqrt{\frac{1}{2}(1-\cos 45^\circ)} = \frac{1}{2}\sqrt{(2-\sqrt{2})}$$

$$\cos 22\frac{1}{2}^\circ = \sqrt{\frac{1}{2}(1+\cos 45^\circ)} = \frac{1}{2}\sqrt{(2+\sqrt{2})}\,.$$

$22\frac{1}{2}^\circ$ is in first quadrant. So + ve sign is taken.

Example 14 :

Prove that $\sin x = 2^n \cos\frac{x}{2} \cos\frac{x}{2^2} . \cos\frac{x}{2^3} \dots \cos\frac{x}{2^n} \sin\frac{x}{2^n}$ where n is a positive integer.

Proof : We have

$$\sin x = 2 \sin \frac{x}{2} . \cos \frac{x}{2}$$

$$\sin \frac{x}{2} = 2 \sin \frac{x}{2^2} . \cos \frac{x}{2^2}$$

$$\sin \frac{x}{2^2} = 2 \sin \frac{x}{2^3} . \cos \frac{x}{2^3}$$

....

....

$$\sin \frac{x}{2^{n-1}} = 2 \sin \frac{x}{2^n} . \cos \frac{x}{2^n}$$

Thus $\sin x = 2^n \cos \frac{x}{2} . \cos \frac{x}{2^2} . \cos \frac{x}{2^3} \dots . \cos \frac{x}{2^n} . \sin \frac{x}{2^n}$.

Example 15 :

If $A + B + C = \pi$, prove that

(i) $\sin 2A + \sin 2B + \sin 2C = 4 \sin A. \sin B. \sin C$

(ii) $\cos A + \cos B - \cos C = 4 \cos \frac{1}{2} A \cos \frac{1}{2} B. \sin \frac{1}{2} C - 1$

(iii) $\sin^2 A + \sin^2 B + \sin^2 C - 2 \cos A. \cos B. \cos C = 2$

(iv) $\cos\frac{1}{2} A - \cos\frac{1}{2} B + \cos\frac{1}{2} C = 4 \cos \frac{\pi + A}{4} \cos \frac{\pi - B}{4} \cos \frac{\pi + C}{4}$

Proof :

(i) L.H.S. $= \sin 2A + \sin 2B + \sin 2C$

$= 2 \sin A. \cos A + 2 \sin (B + C). \cos (B - C)$ [Applying (8)]

$= 2 \sin A. \cos A + 2 \sin A. \cos (B - C)$ [$\sin (B + C) = \sin (\pi - A) = \sin A$]

$= 2 \sin A \{\cos A + \cos (B - C)\}$

$= 2 \sin A \{\cos (B - C) - \cos (B + C)\}$

$= 2 \sin A . 2 \sin B. \sin C = 4 \sin A. \sin B. \sin C =$ R.H.S.

(ii) L.H.S. $= \cos A + \cos B - \cos C$

$= 1 + \cos A + \cos B - \cos C - 1$

$= 2 \cos^2 \frac{A}{2} + 2 \sin \frac{B+C}{2} . \sin \frac{C-B}{2} - 1$

$$= 2\cos^2\frac{A}{2} - 2\cos\frac{A}{2} \,.\, \sin\frac{B-C}{2} - 1 \qquad \left[\begin{array}{l}\because A+B+C=\pi \\ \Rightarrow \dfrac{B+C}{2} = \dfrac{\pi}{2} - \dfrac{A}{2}\end{array}\right]$$

$$= 2\cos\frac{A}{2}\left(\cos\frac{A}{2} - \sin\frac{B-C}{2}\right) - 1$$

$$= 2\cos\frac{A}{2}\left(\sin\frac{B+C}{2} - \sin\frac{B-C}{2}\right) - 1$$

$$= 2\cos\frac{A}{2} \,.\, 2\cos\frac{B}{2} \,.\, \sin\frac{C}{2} - 1$$

$$= 4\cos\frac{A}{2} \,.\, \cos\frac{B}{2} \,.\, \sin\frac{C}{2} - 1 = \text{R. H. S}$$

(iii) L.H.S. $= \sin^2 A + \sin^2 B + \sin^2 C - 2\cos A \,.\, \cos B \,.\, \cos C$

$= 1 - \cos^2 A + 1 - \cos^2 B + 1 - \cos^2 C - 2\cos A.\cos B.\cos C$

$= 3 - \cos^2 A - \cos^2 B - \cos^2 C - 2\cos A.\cos B.\cos C$

$$= \frac{1}{2}\,[6 - 2\cos^2 A - 2\cos^2 B - 2\cos^2 C - 4\cos A.\cos B.\cos C]$$

$$= \frac{1}{2}\,[6 - (1 + \cos 2A) - (1 + \cos 2B) - 2\cos^2 C - 4\cos A.\cos B.\cos C]$$

$$= \frac{1}{2}\,[4 - (\cos 2A + \cos 2B) - 2\cos^2 C - 4\cos A.\cos B \,.\, \cos C]$$

$$= \frac{1}{2}\,[4 - 2\cos(A+B)\cos(A-B) - 2\cos^2 C - 4\cos A.\cos B.\cos C]$$

$$= \frac{1}{2}\,[4 + 2\cos C\cos(A-B) - 2\cos^2 C - 4\cos A.\cos B.\cos C]$$

$$= \frac{1}{2}\,[4 + 2\cos C\left\{\cos(A-B) + \cos(A+B)\right\} - 4\cos A.\cos B.\cos C]$$

$$= \frac{1}{2}\,[4 + 4\cos A.\cos B.\cos C - 4\cos A.\cos B.\cos C]$$

$= 2 =$ R. H. S.

(iv) L.H.S. $= \cos\frac{1}{2}A - \cos\frac{1}{2}B + \cos\frac{1}{2}C$

$$= -2\sin\frac{A+B}{4} \,.\, \sin\frac{A-B}{4} + \sin\frac{A+B}{2}$$

$$= -2\sin\frac{A+B}{4} \,.\, \sin\frac{A-B}{4} + 2\sin\frac{A+B}{4} \,.\, \cos\frac{A-B}{4}$$

$$= 2\sin\frac{A+B}{4}\left(\cos\frac{A+B}{4} - \sin\frac{A-B}{4}\right)$$

$$= 2\sin\frac{A+B}{4}\left\{\sin\left(\frac{\pi}{2}-\frac{A+B}{4}\right)-\sin\frac{A-B}{4}\right\}$$

$$= 2\sin\frac{A+B}{4}\times 2\cos\frac{\pi-B}{4}.\sin\frac{\pi-A}{4}$$

$$= 4\sin\frac{A+B}{4}.\cos\frac{\pi-B}{4}.\sin\frac{\pi-A}{4}$$

$$= 4\sin\frac{\pi-C}{4}\cos\frac{\pi-B}{4}.\sin\frac{\pi-A}{4}\quad\begin{bmatrix}A+B+C=\pi\\ \Rightarrow\frac{A+B}{4}=\frac{\pi-C}{4}\end{bmatrix}$$

$$= 4\cos\left(\frac{\pi}{2}-\frac{\pi-C}{4}\right)\cos\frac{\pi-B}{4}\cos\left(\frac{\pi}{2}-\frac{\pi-A}{4}\right)$$

$$= 4\cos\frac{\pi+A}{4}.\cos\frac{\pi-B}{4}\cos\frac{\pi+C}{4} = \text{R. H. S.}$$

Example 16 :

Find the maximum and minimum values of $3\sin x + 4\cos x$.

Let $3 = r\cos\alpha$, $4 = r\sin\alpha$,

so that $r = \sqrt{3^2+4^2} = 5$.

Now $3\sin x + 4\cos x = r(\sin x\cos\alpha + \cos x\sin\alpha) = 5\sin(x+\alpha)$.

We know that the maximum and minimum values of $\sin(x+\alpha)$ are 1 and -1 respectively.

Thus the maximum and minimum values of $3\sin x + 4\cos x$ are 5 and -5 respectively.

Example 17 :

If $\dfrac{\cos^4\alpha}{\cos^2\beta}+\dfrac{\sin^4\alpha}{\sin^2\beta}=1$, show that $\dfrac{\cos^4\beta}{\cos^2\alpha}+\dfrac{\sin^4\beta}{\sin^2\alpha}=1$.

Solution :

Let $\cos^2\alpha = x$ and $\cos^2\beta = y$, then

$$\frac{\cos^4\alpha}{\cos^2\beta}+\frac{\sin^4\alpha}{\sin^2\beta}=1$$

$$\Rightarrow \frac{x^2}{y}+\frac{(1-x)^2}{1-y}=1$$

$\Rightarrow x^2 - x^2y + y - 2xy + x^2y = y - y^2$

$\Rightarrow (x-y)^2 = 0 \qquad \Rightarrow x = y$

$\Rightarrow \cos^2\alpha = \cos^2\beta$ thus $\sin^2\alpha = \sin^2\beta$

Therefore, $\dfrac{\cos^4\beta}{\cos^2\alpha}+\dfrac{\sin^4\beta}{\sin^2\alpha}=\dfrac{\cos^4\alpha}{\cos^2\alpha}+\dfrac{\sin^4\alpha}{\sin^2\alpha}=1.$

Example 18 :

If $\sin\theta + \text{cosec}\,\theta = 2$, show that $\sin^n\theta + \text{cosec}^n\theta = 2$ for all positive integers n.

Solution :

$\sin\theta + \text{cosec}\,\theta = 2$

$\Rightarrow \sin^2\theta - 2\sin\theta + 1 = 0$

$\Rightarrow (1 - \sin\theta)^2 = 0$

$\Rightarrow \sin\theta = 1$ so $\text{cosec}\,\theta = 1$

$\therefore \sin^n\theta + \text{cosec}^n\theta = (1)^n + (1)^n = 2.$

EXERCISES 4 (b)

1. In the following questions, write T for true and F for false statements.

(i) If $\tan x + \tan y = 5$ and $\tan x . \tan y = \frac{1}{2}$, then $\cot(x + y) = 10$.

(ii) $\sqrt{3}\,(1 + \tan 15°) = 1 - \tan 15°$.

(iii) If θ lies in third quadrant, than $\cos\frac{\theta}{2} + \sin\frac{\theta}{2}$ is positive.

(iv) $2\sin 105° . \sin 15° = \frac{1}{2}$.

(v) If $\cos A = \frac{1}{2}$, $\cos B = 1$ then $\tan\frac{A+B}{2} . \tan\frac{A-B}{2} = 1$.

(vi) $\cos 15° \cos 7\frac{1}{2}^{\circ} . \sin 7\frac{1}{2}^{\circ} = 1$.

(vii) $\sin 20° (3 - 4\cos^2 70°) = \frac{\sqrt{3}}{2}$.

(viii) $\sqrt{3}\,(3\tan 10° - \tan^3 10°) = (1 - 3\tan^2 10°)$.

(ix) $$\frac{2\tan 7\frac{1}{2}^{\circ}\left(1-\tan^2 7\frac{1}{2}^{\circ}\right)}{\left(1+\tan^2 7\frac{1}{2}^{\circ}\right)} = 1.$$

(x) The minimum value of $\sin\theta . \cos\theta$ is $(-1)^2$.

2. In the following questions, fill in the gaps with correct answer chosen from the brackets.

(i) If α and β lie in first the second quadrants respectively, and if $\sin\alpha = \frac{1}{2}$, $\sin\beta = \frac{1}{3}$, then sin

$(\alpha+\beta) =$ ——— . $\left(\frac{1}{2\sqrt{3}}+\frac{\sqrt{2}}{3}, \frac{1}{2\sqrt{3}}-\frac{\sqrt{2}}{3}, \frac{-1}{2\sqrt{3}}+\frac{\sqrt{2}}{3}\right)$

(ii) If $\tan\alpha = \frac{1}{2}$, $\tan\beta = \frac{1}{3}$, then $\alpha+\beta =$ ————. $\left(\frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{3}\right)$

(iii) The value of $\frac{\cos 15° + \sin 15°}{\cos 15° - \sin 15°} =$ ——— $\left(\frac{\sqrt{3}}{2}, \sqrt{3}, \frac{1}{\sqrt{3}}\right)$

(iv) If $\frac{1+\sin A}{\cos A} = \sqrt{2}+1$, then the value of $\frac{1-\sin A}{\cos A}$ is —— $\left(\frac{1}{\sqrt{2}-1}, \sqrt{2}-1, \sqrt{2}+1\right)$

(v) $\sin 105°. \cos 105° =$ —— $\left(\frac{1}{2}, -\frac{1}{4}, -\frac{1}{2}\right)$

(vi) $2 \sin 67\frac{1}{2}^\circ \cos 22\frac{1}{2}^\circ =$ —— $\left(1-\frac{1}{\sqrt{3}}, 1+\frac{1}{\sqrt{2}}, -1+\frac{1}{\sqrt{2}}\right)$

(vii) $\sin 35° + \cos 5° =$ —— $(2 \cos 25°, \sqrt{3} \cos 25°, \sqrt{3} \sin 25°)$

(viii) $\sin^2 24° - \sin^2 6° =$ ——— $\left(\frac{\sqrt{5}+1}{8}, \frac{\sqrt{5}-1}{8}, \frac{\sqrt{5}-1}{4}\right)$

(ix) $\sin 70° (4 \cos^2 20° - 3) =$ ——— $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\sqrt{3}\right)$

(x) $\cos 3\theta + \sin 3\theta$ is maximum if $\theta =$ ——— $(60°, 15°, 45°)$

(xi) $\sin 15° - \cos 15°$ is ——— . [$\frac{1}{2}$, 0, Positive, Negative]

(xii) If θ lies in the third quadrant and $\tan\theta = 2$ then the value of $\sin\theta$ is ——.

$\left(\frac{1}{\sqrt{5}}, -\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}, -\frac{2}{\sqrt{5}}\right)$

(xiii) The correct expression is ———.

$(\sin 1° > \sin 1, \sin 1° < \sin 1, \sin 1° = \sin 1, \sin 1° = \frac{\pi}{180} \sin 1)$

(xiv) The correct expression is ——— .

$(\tan 1 > \tan 2, \tan 1 < \tan 2, \tan 1 = \frac{1}{2} \tan 2, \tan 1 < 0)$

3. Prove the following :

(i) $\sin A. \sin(B-C) + \sin B \sin(C-A) \sin C. \sin(A-B) = 0$

(ii) $\cos A \sin(B-C) + \sin B \sin(C-A) + \cos C \sin(A-B) = 0$

(iii) $\dfrac{\sin(B-C)}{\sin B.\sin C}+\dfrac{\sin(C-A)}{\sin C.\sin A}+\dfrac{\sin(A-B)}{\sin A.\sin B}=0$

(iv) $\tan^2 A-\tan^2 B=\dfrac{\sin(A+B).\sin(A-B)}{\cos^2 A.\cos^2 B}$.

4. Prove the following :

(i) $\tan 75° + \cot 75° = 4$

(ii) $\sin^2 18° + \cos^2 36° = \dfrac{3}{4}$

(iii) $\sin 18°.\cos 36° = \dfrac{1}{4}$

(iv) $\sin 15° = \dfrac{\sqrt{3}-1}{2\sqrt{2}}$

(v) $\cot\dfrac{\pi}{8}-\tan\dfrac{\pi}{8}=2$

(vi) $\dfrac{\cos 9°+\sin 9°}{\cos 9°-\sin 9°}=\tan 54°$

(vii) $\tan 10° + \tan 35° + \tan 10°.\tan 35° = 1$.

5. Prove the following :

(i) $\cot 2A=\dfrac{\cot^2 A-1}{2\cot A}$

(ii) $\dfrac{\sin B}{\sin A}=\dfrac{\sin(2A+B)}{\sin A}-2\cos(A+B)$

(iii) $\dfrac{\sin 2A+\sin 2B}{\sin 2A-\sin 2B}=\dfrac{\tan(A+B)}{\tan(A-B)}$

(iv) $\dfrac{\cot A-\tan A}{\cot A+\tan A}=\cos 2A$

(v) $\dfrac{\sin 2A+\sin 5A-\sin A}{\cos 2A+\cos 5A+\cos A}=\tan 2A$

(vi) $\cot A-\tan A=2\cot 2A$

(vii) $\cot A-\operatorname{cosec} 2A=\cot 2A$

(viii) $\dfrac{\cos A-\sin A}{\cos A+\sin A}=\sec 2A-\tan 2A$

(ix) $\tan\theta\,(1+\sec 2\theta)=\tan 2\theta$

(x) $\dfrac{\sin A+\sin B}{\sin A-\sin B}=\tan\dfrac{A+B}{2}.\cot\dfrac{A-B}{2}$

(xi) $\sin 50° - \sin 70° + \sin 10° = 0$

(xii) $\cos 80° + \cos 40° - \cos 20° = 0$

(xiii) $8\sin 10°.\sin 50°.\sin 70° = 1$

(xiv) $4\sin A\,\sin(60°-A)\sin(60°+A)-\sin 3A=0$

(xv) $\tan 3A-\tan 2A-\tan A=\tan 3A\tan 2A\tan A$

6. Prove the following :

(i) $\tan\dfrac{A}{2} = \left(\sqrt{\dfrac{1-\cos A}{1+\cos A}}\right)$ (ii) $\sqrt{\dfrac{1+\sin A}{1-\sin A}} = \tan\left(\dfrac{\pi}{4}+\dfrac{A}{2}\right)$

(iii) $\dfrac{1+\tan\dfrac{A}{2}}{1-\tan\dfrac{A}{2}} = \sec A + \tan A$ (iv) $\sec\theta + \tan\theta = \tan\left(\dfrac{\pi}{4}+\dfrac{\theta}{2}\right)$

(v) $\cot\dfrac{A}{2} = \dfrac{\sin A}{1-\cos A}$.

7. Find the maximum value of the following.

(i) $5\sin x + 12\cos x$

(ii) $24\sin x - 7\cos x$

(iii) $2 + 3\sin x + 4\cos x$

(iv) $8\cos x - 15\sin x - 2$.

8. Answer the following :

(i) If $\tan A = \dfrac{13}{27}$, $\tan B = \dfrac{7}{20}$ and A, B are acute, show that $A + B = 45°$.

(ii) If $\tan\theta = \dfrac{b}{a}$, find the value of $a\cos 2\theta + b\sin 2\theta$.

(iii) If $\sec A - \tan A = \dfrac{1}{2}$ and $0 < A < 90°$ then show that $\sec A = \dfrac{5}{4}$.

(iv) If $\sin\theta + \sin\phi = a$ and $\cos\theta + \cos\phi = b$, then show that $\tan\dfrac{1}{2}(\theta+\phi) = \dfrac{a}{b}$.

(v) If $\tan\theta = \dfrac{a\sin x + b\sin y}{a\cos x + b\cos y}$, then show that $a\sin(\theta - x) + b\sin(\theta - y) = 0$.

(vi) If $A + C = B$, show that $\tan A . \tan B . \tan C = \tan B - \tan A - \tan C$.

(vii) If $\tan A = \dfrac{1}{5}$, $\tan B = \dfrac{2}{3}$ show that $\cos 2A = \sin 2B$.

(viii) If $\cos 2A = \tan^2 B$, then show that $\cos 2B = \tan^2 A$.

In ABC, prove that

(ix) $\tan\dfrac{B+C}{2} = \cot\dfrac{A}{2}$.

(x) $\cos(A+B) + \sin C = \sin(A+B) - \cos C$.

If $A + B + C = \pi$ and $\cos A = \cos B . \cos C$ show that (xi and xii)

(xi) $\tan B + \tan C = \tan A$

(xii) $2\cot B . \cot C = 1$.

9. Prove the following :

(i) $\cos(A-D)\sin(B-C)+\cos(B-D)\sin(C-A)+\cos(C-D)\sin(A-B)=0$

(ii) $\sin 2A+\sin 2B+\sin 2(A-B)=4\sin A.\cos B.\cos(A-B)$

(iii) $\cos 2A+\cos 2B+\cos 2(A-B)+1=4\cos A.\cos B.\cos(A-B)$

(iv) $\sin 2A+\sin 2B+\sin 2C-\sin 2(A+B+C)=4\sin(B+C)\sin(C+A)\sin(A+B)$

(v) $\sin A+\sin 3A+\sin 5A=\sin 3A(1+2\cos 2A)$

(vi) $\sin A-\sin 3A+\sin 5A=\sin 3A(2\cos 2A-1)$

(vii) $\cos(A+B)+\sin(A-B)=2\sin(45°+A)\cos(45°+B)$

(viii) $\cos(120°+A)\cos(120°-A)+\cos(120°+A)\cos A+\cos A(\cos 120°-A)+\dfrac{3}{4}=0$

(ix) $\cos 4A-\cos 4B=8(\cos A-\cos B)(\cos A+\cos B)(\cos A-\sin B)(\cos A+\sin B)$.

10. Prove the following :

(i) $\dfrac{1-\tan^2(45°-A)}{1+\tan^2(45°-A)}=\sin 2A$

(ii) $\dfrac{\cos A+\sin A}{\cos A-\sin A}-\dfrac{\cos A-\sin A}{\cos A+\sin A}=2\tan 2A$

(iii) $\dfrac{1-\cos 2A+\sin 2A}{1+\cos 2A+\sin 2A}=\tan A$

(iv) $\dfrac{\sin(A+B)+\cos(A-B)}{\sin(A-B)+\cos(A+B)}=\sec 2B+\tan 2B$

(v) $\dfrac{\cos 7\alpha+\cos 3\alpha-\cos 5\alpha-\cos\alpha}{\sin 7\alpha-\sin 3\alpha-\sin 5\alpha+\sin\alpha}=\cot 2\alpha.$

(vi) $\dfrac{\sin\theta+\sin 3\theta+\sin 5\theta+\sin 7\theta}{\cos\theta+\cos 3\theta+\cos 5\theta+\cos 7\theta}=\tan 4\theta$

11. Prove the following :

(i) Express $4\cos A.\cos B\cos C$ as the sum of four cosines.

(ii) Express $\cos 2A+\cos 2B+\cos 2C+\cos 2(A+B+C)$ as the product of three cosines.

12. Prove the following :

(i) $\cos^6 A-\sin^6 A=\cos 2A\left(1-\dfrac{1}{4}\sin^2 2A\right)$

(ii) $\cos^6 A+\sin^6 A=\dfrac{1}{4}(1+3\cos^2 2A)$

(iii) $\cos^3 A.\cos 3A+\sin^3 A.\sin 3A=\cos^3 2A$

(iv) $\sin^4\theta=\dfrac{3}{8}-\dfrac{1}{2}\cos 2\theta+\dfrac{1}{8}\cos 4\theta$

(v) $\cot 3A=\dfrac{\cot^3 A-3\cot A}{3\cot^2 A-1}$

(vi) $\tan 4\theta = \dfrac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$

(vii) $\dfrac{1}{\tan 3A - \tan A} - \dfrac{1}{\cot 3A - \cot A} = \cot 2A$

(viii) $\dfrac{\cot A}{\cot A - \cot 3A} - \dfrac{\tan A}{\tan 3A - \tan A} = 1.$

13. Find the value of $\sin 3°$, $\cos 3°$, $2\sin\dfrac{\pi}{32}$.

14. If $\sin A + \sin B = a$ and $\cos A + \cos B = b$, show that

(i) $\tan (A + B) = \dfrac{2ab}{b^2 - a^2}$

(ii) $\sin (A + B) = \dfrac{2ab}{b^2 + a^2}$

(iii) $\cos (A + B) = \dfrac{b^2 - a^2}{b^2 + a^2}$

15. Prove the following :

(i) $\dfrac{1 + \sin A - \cos A}{1 + \sin A + \cos A} = \tan\dfrac{A}{2}$

(ii) $8\sin^4\dfrac{1}{2}\theta - 8\sin^2\dfrac{1}{2}\theta + 1 = \cos 2\theta$

(iii) $\cos^4\dfrac{\pi}{8} + \cos^4\dfrac{3\pi}{8} + \cos^4\dfrac{5\pi}{8} + \cos^4\dfrac{7\pi}{8} = \dfrac{3}{2}$

(iv) $\cos^2\dfrac{\alpha}{2}(1 - 2\cos\alpha)^2 + \sin^2\dfrac{\alpha}{2}(1 + 2\cos\alpha)^2 = 1.$

16. Prove the following :

(i) $\sin 20° . \sin 40° . \sin 60° . \sin 80° = \dfrac{3}{16}$

(ii) $\cos 36° . \cos 72° . \cos 108° . \cos 144° = \dfrac{1}{16}$

(iii) $\cos 10° . \cos 30° . \cos 50° . \cos 70° = \dfrac{3}{16}$

(iv) $\cos 20° . \cos 40° . \cos 60° . \cos 80° = \dfrac{1}{16}$

(v) $\tan 6° . \tan 42° . \tan 66° . \tan 78° = 1$

{Hints : Use the identity $\tan 3A = \tan A \tan(60° - A)\tan(60° + A)$}.

17. Prove the following :

(i) $\cot 7\frac{1}{2}^\circ = \sqrt{6}+\sqrt{3}+\sqrt{2}+2$

(ii) $\cot 22\frac{1}{2}^\circ = \sqrt{2} + 1$

(iii) $\cot 37\frac{1}{2}^\circ = \sqrt{6}-\sqrt{3}-\sqrt{2}+2$

(iv) $\tan 37\frac{1}{2}^\circ = \sqrt{6}+\sqrt{3}-\sqrt{2}-2$

(v) $\cos\frac{\pi}{16} = \frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2}}}$.

18. (i) If $\sin A = K \sin B$, prove that $\tan\frac{1}{2}(A - B) = \frac{K-1}{K+1}\tan\frac{1}{2}(A + B)$.

(ii) If $\alpha \cos(x + \alpha) = b \cos(x - \alpha)$, show that $(a + b)\tan x = (a - b)\cot\alpha$.

(iii) An angle θ is divided into two parts α, β such that $\tan\alpha : \tan\beta = x : y$. Prove that

$\sin(\alpha - \beta) = \frac{x-y}{x+y}\sin\theta$.

(iv) If $\sin\theta + \sin\phi = a$, $\cos\theta + \cos\phi = b$, show that $\frac{\sin\frac{\theta+\phi}{2}}{a} = \frac{\cos\frac{\theta+\phi}{2}}{b} = 2\frac{\cos\frac{\theta-\phi}{2}}{a^2-b^2}$

(v) If $a\cos\alpha + b\sin\alpha = c = a\cos\beta + b\sin\beta$ then prove that

$$\frac{a}{\cos\frac{1}{2}(\alpha+\beta)} = \frac{b}{\sin\frac{1}{2}(\alpha+\beta)} = \frac{c}{\cos\frac{1}{2}(\alpha-\beta)}$$

(vi) Prove that $\left(\frac{\cos A+\cos B}{\sin A-\sin B}\right)^n + \left(\frac{\sin A+\sin B}{\cos A-\cos B}\right)^n = 2\cot^n\frac{A-B}{2}$ or zero according as n is even or odd.

19. (i) If $(1-e)\tan^2\frac{\beta}{2} = (1 + e)\tan^2\frac{\alpha}{2}$, prove that $\cos\beta = \frac{\cos\alpha - e}{1-e\cos\alpha}$.

(ii) If $\cos\theta = \frac{\cos A - \cos B}{1-\cos A.\cos B}$ prove that one of the values of $\tan\frac{\theta}{2}$ is $\tan\frac{A}{2}.\cot\frac{B}{2}$.

(iii) If $\tan\theta = \frac{\sin x.\sin y}{\cos x+\cos y}$, then prove that one of the values of $\tan\frac{1}{2}\theta$ is $\tan\frac{1}{2}x.\tan\frac{1}{2}y$.

(iv) If $\sec(\phi + \alpha) + \sec(\phi - \alpha) = 2\sec\phi$, show that $\cos\phi = \pm\sqrt{2}\cos\frac{\alpha}{2}$.

(v) If $\tan A + \tan B = a$ and $\cot A + \cot B = b$, then show that $\cot(A + B) = \frac{1}{a} - \frac{1}{b}$.

(vi) If $\cot\theta = \cos(x+y)$ and $\cot\phi = \cos(x-y)$ show that $\tan(\theta-\phi) = \dfrac{2\sin x \,.\, \sin y}{\cos^2 x + \cos^2 y}$.

(vii) If $\tan\beta = \dfrac{n^2 \sin\alpha \,.\, \cos\alpha}{1-n^2\sin^2\alpha}$, then show that $\dfrac{\tan(\alpha-\beta)}{\tan\alpha} = 1 - n^2$.

(viii) If $2\tan\alpha = 3\tan\beta$, then prove that $\tan(\alpha-\beta) = \dfrac{\sin 2\beta}{5-\cos 2\beta}$.

(ix) If α, β are acute angles and $\cos 2\alpha = \dfrac{3\cos 2\beta - 1}{3-\cos 2\beta}$, then prove that $\tan\alpha = \sqrt{2}\tan\beta$.

20. If $A + B + C = \pi$, then prove the following.

(i) $\cos 2A + \cos 2B + \cos 2C + 1 + 4\cos A.\cos B.\cos C = 0$

(ii) $\sin 2A + \sin 2B - \sin 2C = 4\cos A.\cos B.\sin C$

(iii) $\cos A + \cos B + \cos C = 1 + 4\sin\frac{1}{2}A.\sin\frac{1}{2}B.\sin\frac{1}{2}C$

(iv) $\sin A + \sin B - \sin C = 4\sin\frac{1}{2}A.\sin\frac{1}{2}B.\cos\frac{1}{2}C$

(v) $\cos^2 A + \cos^2 B + 2\cos A.\cos B.\cos C = \sin^2 C$

(vi) $\sin^2\frac{A}{2} + \sin^2\frac{B}{2} + \sin^2\frac{C}{2} = 1 - 2\sin\frac{A}{2}.\sin\frac{B}{2}.\sin\frac{C}{2}$.

(vii) $\sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2} = 4\sin\frac{\pi-A}{4}\sin\frac{\pi-B}{4}.\sin\frac{\pi-C}{4} + 1$.

(viii) $\cos^2\frac{A}{2} + \cos^2\frac{B}{2} - \cos^2\frac{C}{2} = 2\cos\frac{A}{2}.\cos\frac{B}{2}.\sin\frac{C}{2}$

(ix) $\sin(B+2C) + \sin(C+2A) + \sin(A+2B) = 4\sin\frac{B-C}{2}.\sin\frac{C-A}{2}.\sin\frac{A-B}{2}$.

21. (i) Show that $(2\cos\theta - 1)(2\cos 2\theta - 1)(2\cos 2^2\theta - 1) \ldots\ldots (2\cos 2^{n-1}\theta - 1) = \dfrac{2\cos 2^n\theta + 1}{2\cos\theta + 1}$.

(ii) Show that $2^n\cos\theta.\cos 2\theta.\cos 2^2\theta \ldots\ldots \cos 2^{n-1}\theta = 1$, if $\theta = \dfrac{\pi}{2^n+1}$.

(iii) Prove that $\dfrac{\tan 2^n\theta}{\tan\theta} = (1+\sec 2\theta)(1+\sec 2^2\theta)\ldots\ldots(1+\sec 2^n\theta)$.

22. If $x + y + z = xyz$, prove that

(i) $\dfrac{x}{1-x^2} + \dfrac{y}{1-y^2} + \dfrac{z}{1-z^2} = \dfrac{4xyz}{(1-x^2)(1-y^2)(1-z^2)}$.

(ii) $\dfrac{3x-x^3}{1-3x^2}+\dfrac{3y-y^3}{1-3y^2}+\dfrac{3z-z^3}{1-3z^2}=\dfrac{3x-x^3}{1-3x^2}\cdot\dfrac{3y-y^3}{1-3y^2}\cdot\dfrac{3z-z^3}{1-3z^2}$.

(Hints : Put $x = \tan\alpha$, $y = \tan\beta$ $z = \tan\gamma$ and get $\alpha+\beta+\gamma=\pi$)

23. If $\dfrac{\sin^4\alpha}{a}+\dfrac{\cos^4\alpha}{b}=\dfrac{1}{a+b}$, show that $\dfrac{\sin^8\alpha}{a^3}+\dfrac{\cos^8\alpha}{b^3}=\dfrac{1}{(a+b)^3}$.

4.4 Trigonometric Equations

Equations involving one or more trigonometric functions of a variable are called trigonometric equations.

$\sin x = \dfrac{1}{2}$ (i)

$\cos x = \dfrac{\sqrt{3}}{2}$ (ii)

$\sec x = \dfrac{1}{2}$ (iii)

$\tan x = \tan\alpha$ (iv)

$\sin x = 5$ (v)

are examples of trigonometric equations.

The solutions of an equation are those values of x which satisfy the equation.

A trigonometric equation may or may not have a solution. The equations (iii) and (v) have no solution.

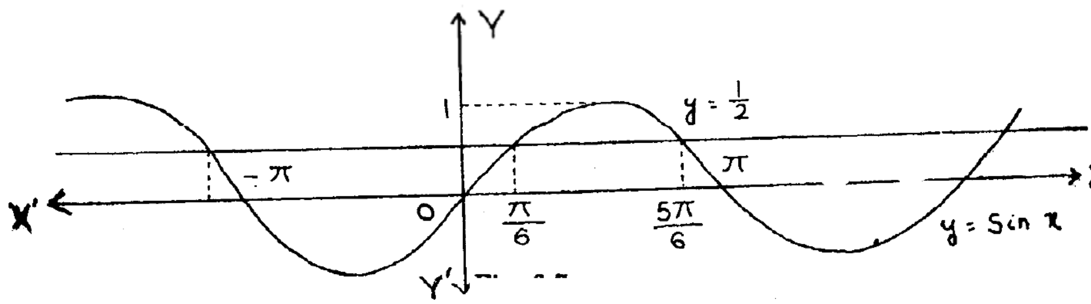
Let us discuss about the solution of the equation $\sin x = \dfrac{1}{2}$. The solution of this equation are the abscissae of points of intersection of the curve $y = \sin x$ and $y = \dfrac{1}{2}$ as shown in fig 6.7. Two cases may be considered.

Case 1 - If $x \in [0, 2\pi]$, then the solution set of the equation $\sin x = \dfrac{1}{2}$ is $\left\{\dfrac{\pi}{6}, \dfrac{5\pi}{6}\right\}$.

Case 2 - If $x \in R$, due to the periodicity of sine function the solution set of the equation is

$$\left\{\frac{\pi}{6}, \frac{\pi}{6} \pm 2\pi, \frac{\pi}{6} \pm 4\pi, \ldots\right\} \cup \left\{\frac{5\pi}{6}, \frac{5\pi}{6} \pm 2\pi, \ldots\right\}$$

$$= \left\{\frac{\pi}{6} + 2n\pi, \frac{5\pi}{6} + 2n\pi \mid n \in Z\right\}$$



Solutions of a trigonometric equation can be classified into two categories.

(i) The solutions considered over the entire set R are called **general solutions.**

(ii) The solutions restricted to the set $[0, 2\pi]$ are called **principal solutions.**

Note that the solution set of the equation $\sin x = \frac{1}{2}$ occuring in case 1 consists of principal solutions and those occuring in case 2 consist of general solutions.

The general solutions of some standard equations :

Theorem 3 :

The general solutions of :

(i) $\sin x = 0$ are $x = n\pi, n \in Z$ ... (1)

(ii) $\cos x = 0$ are $x = (2n+1)\frac{\pi}{2}, n \in Z$ ... (2)

(iii) $\sin x = \sin \alpha$ are $x = (-1)^n \alpha + n\pi, n \in Z$ ... (3)

(iv) $\cos x = \cos \alpha$ are $x = 2n\pi \pm \alpha, n \in Z$ ... (4)

(v) $\tan x = \tan \alpha$ are $x = n\pi + \alpha, n \in Z$ ... (5)

where α is a constant.

Proof :(i) and (ii) are obvious :

(iii) $\sin x = \sin \alpha$

$\Rightarrow \sin x - \sin \alpha = 0$

$\Rightarrow 2 \cos \frac{x+\alpha}{2} . \sin \frac{x-\alpha}{2} = 0$

$\Rightarrow \cos \frac{x+\alpha}{2} = 0$ or $\sin \frac{x-\alpha}{2} = 0$

$\Rightarrow \frac{x+\alpha}{2} = (2k+1)\frac{\pi}{2}$ or $\frac{x-\alpha}{2} = k\pi, k \in Z$ (with the aid of (1) and (2))

$\Rightarrow x = (2k+1)\pi - \alpha$ or $x = 2k\pi + \alpha, k \in Z$

$\Rightarrow x = (-1)^{2k+1} \alpha + (2k+1)\pi$ or $x = (-1)^{2k} \alpha + 2k\pi, k \in Z$

$\Rightarrow x = (-1)^n \alpha + n\pi, n \in Z.$

(iv) $\cos x = \cos \alpha$

$\Rightarrow \cos x - \cos \alpha = 0$

$\Rightarrow -2 \sin \frac{x+\alpha}{2} \sin \frac{x-\alpha}{2} = 0$

$\Rightarrow \sin \frac{x+\alpha}{2} = 0$ or $\sin \frac{x-\alpha}{2} = 0$

$\Rightarrow \frac{x+\alpha}{2} = n\pi$ or $\frac{x-\alpha}{2} = n\pi$, $n \in Z$ (with the aid of (1))

$\Rightarrow x = 2n\pi - \alpha$, or $x = 2n\pi + \alpha$, $n \in Z$

$\Rightarrow x = 2n\pi \pm \alpha$, $n \in Z$.

(v) $\tan x = \tan \alpha$

$\Rightarrow \frac{\sin x}{\cos x} = \frac{\sin \alpha}{\cos \alpha}$

$\Rightarrow \sin x. \alpha \cos \alpha - \cos x. \sin \alpha = 0 \Rightarrow \sin (x - \alpha) = 0$

$\Rightarrow x - \alpha = n\pi$, $n \in Z$ (with the aid of (1))

$\Rightarrow x = n\pi + \alpha$, $n \in Z$.

Example 19 : Solve

(i) $\sin \theta = 1$ (ii) $\cos \theta = 1$ (iii) $\tan \theta = 1$

Solution :

(i) $\sin \theta = 1 \Rightarrow \sin \theta = \sin \frac{\pi}{2}$

$\Rightarrow \theta = (-1)^n \frac{\pi}{2} + n\pi$, $n \in Z$. (Using formula (3))

(ii) $\cos \theta = 1 \Rightarrow \cos\theta = \cos 0$

$\Rightarrow \theta = 2n\pi \pm 0$ (Using formula (4))

$\Rightarrow \theta = 2n\pi$, $n \in Z$.

(iii) $\tan \theta = 1 \Rightarrow \tan \theta = \tan \frac{\pi}{4}$

$\Rightarrow \theta = n\pi + \frac{\pi}{4}$, $n \in Z$.

Example : 20 : Solve : $\cos \theta + \cos 2\theta + \cos 3\theta = 0$

Solution :

$\cos\theta + \cos 2\theta + \cos 3\theta = 0$

$\Rightarrow \cos 2\theta + \cos\theta + \cos 3\theta = 0$

$\Rightarrow \cos 2\theta + 2 \cos 2\theta . \cos \theta = 0$

$\Rightarrow \cos 2\theta (1 + 2 \cos\theta) = 0$

$\Rightarrow \cos 2\theta = 0$ or $1 + 2\cos\theta = 0$

$\Rightarrow \cos 2\theta = 0$ or $\cos\theta = -\frac{1}{2} = \cos\frac{2\pi}{3}$

$\Rightarrow 2\theta = \frac{(2n+1)\pi}{2}$ or $\theta = 2n\pi \pm \frac{2\pi}{3}$ [Using (2) and (4)]

$\Rightarrow \theta = (2n+1)\frac{\pi}{4}$ or $\theta = 2n\pi \pm \frac{2\pi}{3}$, $n \in Z$.

Example 21 : Solve : $2\cos^2\theta + 3\sin\theta = 0$

Solution : $2\cos^2\theta + 3\sin\theta = 0$

$\Rightarrow 2 - 2\sin^2\theta + 3\sin\theta = 0$

$\Rightarrow 2\sin^2\theta - 3\sin\theta - 2 = 0$

$\Rightarrow (\sin\theta - 2)(2\sin\theta + 1) = 0$

$\Rightarrow \sin\theta = 2$ (impossible) or $2\sin\theta + 1 = 0$

$\Rightarrow \sin\theta = -\frac{1}{2} = \sin\frac{7\pi}{6}$ $\left[\because \sin\frac{7\pi}{6} = \sin\left(\pi + \frac{\pi}{6}\right) = -\sin\frac{\pi}{6} = -\frac{1}{2}\right]$

$\Rightarrow \theta = (-1)^n\frac{7\pi}{6} + n\pi$, $n \in Z$.

Example 22 : Solve $a\cos\theta + b\sin\theta = c$, $\left(c \le \sqrt{a^2+b^2}\right)$

Solution :

Let $a = r\cos\alpha$, $b = r\sin\alpha$,

then $r = \sqrt{a^2+b^2}$, $\tan\alpha = \frac{b}{a}$.

The equation reduces to $r\cos\alpha \,.\, \cos\theta + r\sin\alpha .\, \sin\theta = c$

$\Rightarrow r\cos(\theta - \alpha) = c$

$\Rightarrow \cos(\theta - \alpha) = \frac{c}{r} = \frac{c}{\sqrt{a^2+b^2}} \le 1$ $\left(\because c \le \sqrt{a^2+b^2}\right)$

$\Rightarrow \cos(\theta - \alpha) = \cos\beta$ $\left[\text{taking } \cos\beta = \frac{c}{\sqrt{a^2+b^2}}\right]$

$\Rightarrow \theta - \alpha = 2n\pi \pm \beta$

$\Rightarrow \theta = 2n\pi + \alpha \pm \beta$, $n \in Z$.

Example 23 : Solve $\cos\theta + \sqrt{3}\sin\theta = 2$.

Solution :

Comparing this with the problem of previous example $a = 1$, $b = \sqrt{3}$, $c = 2$,

we get $r = \sqrt{a^2+b^2} = \sqrt{4} = 2$ and

$\tan\alpha = \frac{\sqrt{3}}{1} = \sqrt{3}$ which gives $\alpha = \frac{\pi}{3}$.

The equation reduces to $r\cos(\theta-\alpha)=c$

$$\Rightarrow 2\cos\left(\theta-\frac{\pi}{3}\right)=2 \Rightarrow \cos\left(\theta-\frac{\pi}{3}\right)=\cos 0$$

$$\Rightarrow \theta-\frac{\pi}{3}=2n\pi,\ n\in Z \Rightarrow \theta=2n\pi+\frac{\pi}{3},\ n\in Z.$$

Example 24 : Solve $r\sin\theta=3$ and $r=4(1+\sin\theta)$ for $r\geq 0$, $0\leq\theta\leq 2\pi$.

Solution :

Eliminating $\sin\theta$, we have $r=4\left(1+\frac{3}{r}\right)$, or $r^2-4r-12=0$

$\Rightarrow (r-6)(r+2)=0$ and

since $r\geq 0$, $r=6$.

Thus $\sin\theta=\frac{1}{2}\Rightarrow\theta=\frac{\pi}{6},\frac{5\pi}{6}$.

EXERCISES 4 (c)

1. Fill in the blanks choosing correct answer from the brackets.

(i) The number of solutions of $2\sin\theta-1=0$ is _____. (one, two, infinite)

(ii) If $\cos\alpha=\cos\beta$, then $\alpha+\beta=$ _____. $(0, \pi, 2\pi)$

(iii) The number of solution(s) of $2\sec\theta+1=0$ is _____ (zero, one, infinite)

(iv) If $\tan\theta=\tan\alpha$ and $90°<\alpha<180°$, then θ can be in _____ quadrant.(1st, 3rd, 4th)

(v) If $\tan x$, $\tan 2x$. $\tan 7x=\tan x+\tan 2x+\tan 7x$, then $x=$ _____. $\left(\frac{\pi}{4},\frac{\pi}{5},\frac{\pi}{10}\right)$

(vi) For —— value of θ, $\sin\theta+\cos\theta=\sqrt{2}$. $\left(\frac{\pi}{4},\frac{\pi}{2},\frac{\pi}{3}\right)$

(vii) The number of values of x for which $\cos^2 x=1$ and $x^2\leq 4$ is _____. (1, 2, 3)

(viii) In the first quadrant the solution of $\tan^2\theta=3$ is _____ $\left(\frac{\pi}{2},\frac{\pi}{3},\frac{\pi}{4}\right)$

(ix) The least positive value of θ for which $1+\tan\theta=0$ and $\sqrt{2}\cos\theta+1=0$ is_____.
$\left(\frac{\pi}{4},\frac{3\pi}{4},\frac{5\pi}{4}\right)$

(x) The least positive value of x for which $\tan 3x=\tan x$ is ——. $\left(\frac{\pi}{2},\frac{\pi}{3},\pi\right)$

2. Find the princpal solution of the following equations :

(i) $\sin\theta = \sin 2\theta$

(ii) $\sqrt{3}\sin\theta - \cos\theta = 2$

(iii) $\cos^2\theta + \sin\theta + 1 = 0$

(iv) $\sin 4x + \sin 2x = 0$

(v) $\sin x + \cos x = \frac{1}{\sqrt{2}}$.

3. Find the general solutions of the following equations :

(i) $\cos 2x = 0$

(ii) $\sin (x^\circ + 40^\circ) = \frac{1}{\sqrt{2}}$

(ii) $\sin 5\theta = \sin 3\theta$

(iv) $\tan ax = \cot bx$

(v) $\tan^2 3\theta = 3$.

4. Solve the following :

(i) $\tan^2 x + \sec^2 x = 3$

(ii) $4\sin^2 x + 6\cos^2 x = 5$

(iii) $3\sin x + 4\cos x = 5$

(iv) $3\tan x + \cot x = 5\operatorname{cosec} x$

(v) $\cos x + \sqrt{3}\sin x = \sqrt{2}$

(vi) $\sin 3x - 2\cos^2 x = 0$

(vii) $\sec\theta + \tan\theta = \sqrt{3}$

(viii) $\cos 2\theta - \cos\theta = \sin\theta - \sin 2\theta$

(ix) $\sin\theta + \sin 2\theta + \sin 3\theta + \sin 4\theta = 0$

(x) $\cos 2x^\circ + \cos x^\circ - 2 = 0$

(xi) $\tan\theta + \tan 2\theta = \tan 3\theta$

(xii) $\tan\theta + \tan\left(\theta + \frac{\pi}{3}\right) + \tan\left(\theta + \frac{2\pi}{3}\right) = 3$

(xiii) $\cot^2\theta - \tan^2\theta = 4\cot 2\theta$

(xiv) $\cos 2\theta = (\sqrt{2}+1)\left(\cos\theta - \frac{1}{\sqrt{2}}\right)$

(xv) $\sec\theta - 1 = (\sqrt{2} - 1)\tan\theta$

(xvi) $3\tan^2\theta - 2\sin\theta = 0$

(xvii) $4\cos x.\cos 2x.\cos 3x = 1$`

(xviii) $\cos 3x - \cos 2x = \sin 3x$

(xix) $\cos x + \sin x = \cos 2x + \sin 2x$

(xx) $\tan x + \tan 4x + \tan 7x = \tan x.\tan 4x.\tan 7x$.

(xxi) $2(\sec^2\theta + \sin^2\theta) = 5$

(xxii) $(\cos x)^{\sin^2 x - \frac{3}{2}\sin x + \frac{1}{2}} = 1$

(Hints: $\cos x \neq 0$ and $\sin^2 x - \frac{3}{2}\sin x + \frac{1}{2} = 0$)

4.5 Application of sine and cosine formula (Properties of triangles)

In any triangle ABC, there are six parts. These are three sides and three angles. The lengths of sides $\overline{BC}$, $\overline{CA}$ and $\overline{AB}$ are denoted by a, b, c and the angles $\angle CAB$, $\angle ABC$, $\angle BCA$ by A, B, C respectively. We know that sum of the lengths of two sides of a triangle is greater than the length of the third one and the sum of the measures of the angles is equal to π. These six parts are not independent of one another. Some important formulae on the relationship between these six parts are given below.

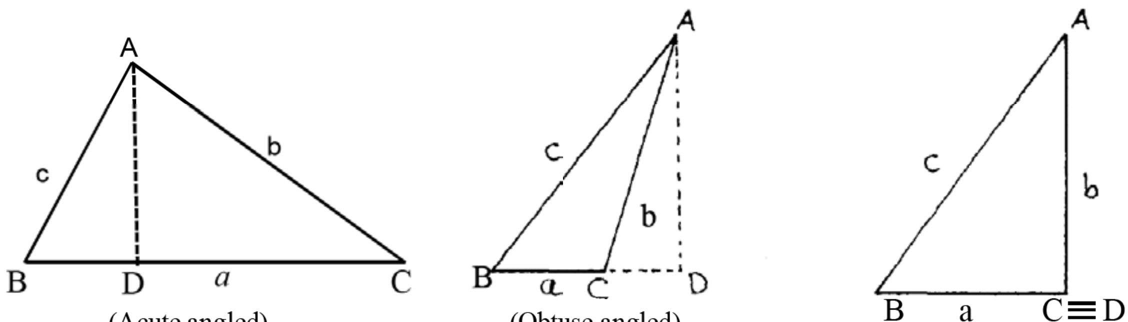
Sine formula :

In any triangle, the lengths of the sides are proportional to the sines of opposite angles. i.e.

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}. \qquad (1)$$

Proof :

Let us consider a tirangle ABC. it can be in three forms, viz. an acute angled triangle, an obtuse angled triangle and a right angled triangle.



(Acute angled) (Obtuse angled) (Right angled)

From the vertex A, draw $\overline{AD}$ perpendicular to $\overleftrightarrow{BC}$.

In Δ ABD, AD = AB sin B = c sin B (In all the Δ's)

In Δ ACD, AD = AC sin C = b sin C (In the acute angled Δ)

AD = b sin (π – C) = b sin C. (In the Obtuse angled Δ)

$\therefore$ b sin C = c sin B

$$\Rightarrow \frac{b}{\sin B} = \frac{c}{\sin C}.$$

Similarly, by drawing perpendicular from the vertex B on $\overline{CA}$ we can show that

$$\frac{a}{\sin A} = \frac{c}{\sin C}.$$

Hence $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$.

Further in the right angled Δ, C is a right angle.

$\therefore \sin A = \frac{a}{c}$, $\sin B = \frac{b}{c}$, $\sin C = 1 = \frac{c}{c}$

$$\therefore \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = c.$$

Thus in all cases, $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$.

Some Important results :

(1) In any ΔABC, $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$ whre

R is the radius of the circumscribing circle of the ΔABC.

(2) If Δ denotes the area of the Δ ABC, then $\Delta = \frac{abc}{4R}$.

The proof of these two results are left as exercises for the readers.

Cosine formula :

In any ABC

$$a^2 = b^2 + c^2 - 2bc \cos A \quad (2)$$
$$b^2 = c^2 + a^2 - 2ca \cos B \quad (3)$$
$$c^2 = a^2 + b^2 - 2ab \cos C. \quad (4)$$

Proof : Consider Δ ABC where C is an acute angle. By geometry,

$AB^2 = BC^2 + CA^2 - 2BC.\ CD$

$\Rightarrow c^2 = a^2 + b^2 - 2ab \cos C.$

Considering Δ ABC where C is an obtuse angle,

$AB^2 = BC^2 + CA^2 + 2BC.\ CD$

$\Rightarrow c^2 = a^2 + b^2 + 2ab \cos (\pi - C)$

$= a^2 + b^2 - 2ab \cos C.$

Lastly, considering Δ ABC where C is a right angle, $AB^2 = BC^2 + CA^2$

$\Rightarrow c^2 = a^2 + b^2 - 2ab \cos C \quad (\because \cos C = 0)$

Hence for all measures of C, $c^2 = a^2 + b^2 - 2ab \cos C$.

Similarly other two relations (2) and (3) can be established.

Tangent formula :

In any Δ ABC, $\tan \frac{B-C}{2} = \frac{b-c}{b+c} \cot \frac{A}{2}$ (5)

$\tan \frac{C-A}{2} = \frac{c-a}{c+a} \cot \frac{B}{2}$ (6)

$\tan \frac{A-B}{2} = \frac{a-b}{a+b} \cot \frac{C}{2}$. (7)

Proof : In any Δ ABC, we have

$$\frac{b}{c} = \frac{\sin B}{\sin C} \qquad \text{(Formula (1))}$$

$$\Rightarrow \frac{b-c}{b+c} = \frac{\sin B - \sin C}{\sin B + \sin C} = \frac{2\cos\frac{B+C}{2} . \sin\frac{B-C}{2}}{2\sin\frac{B+C}{2} . \cos\frac{B-C}{2}}$$

$$\Rightarrow \frac{b-c}{b+c} = \cot\frac{B+C}{2} . \tan\frac{B-C}{2}$$

$$\Rightarrow \frac{b-c}{b+c} = \tan\frac{A}{2} . \tan\frac{B-C}{2} \qquad (\because A + B + C = \pi)$$

$$\Rightarrow \tan\frac{B-C}{2} = \frac{b-c}{b+c} = \cot\frac{A}{2}.$$

Similarly the other two formula (6) and (7) can be derived.

The Projection Formula :

In any Δ ABC,

$a = b \cos C + c \cos B$ (8)

$b = c \cos A + a \cos C$ (9)

$c = a \cos B + b \cos A.$ (10)

Proof :

In the acute angled Δ ABC, BC = BD + CD = AB cos B + AC cos C

$\Rightarrow a = c \cos B + b \cos C.$

In the obtuse angled Δ ABC, BC = BD – CD

= AB cos B – AC cos (π – C)

$= c \cos B - b \cos (\pi - C)$

$= c \cos B + b \cos C.$

$\Rightarrow a = c. \cos B + b \cos C.$

In the right angled Δ ABC, BC = AB cos B

$\Rightarrow a = c \cos B = c \cos B + b \cos C.$ $(\because \cos C = 0)$

Thus in all cases $a = c \cos B + b \cos C.$

This proves (8).

Similarly (9) and (10) can be established.

Area of a Triangle (Heron's formula);

The area of a triangle is given by $\Delta = \sqrt{s\,(s-a)(s-b)(s-c)}$ (11)
where $2s = a + b + c$ is the perimeter of the triangle.

Proof :

We know that the area of a Δ ABC is given by $\Delta = \frac{1}{2}ab\ \sin C$

$$\Rightarrow \Delta^2 = \frac{1}{4}\ a^2b^2 \sin^2 C = \frac{1}{4}\ a^2b^2\,(1-\cos^2 C)$$

$$= \frac{1}{4}\ a^2b^2\,(1-\cos C)\,(1+\cos C) \qquad \text{(A)}$$

Using cosine formula, $1 - \cos C = 1 - \dfrac{a^2+b^2-c^2}{2ab}$

$$= \frac{c^2-(a-b)^2}{2ab} = \frac{(c+a-b)(c-a+b)}{2ab}$$

$$= \frac{(2s-2b)(2s-2a)}{2ab} = \frac{2(s-a)(s-b)}{ab}. \qquad \text{(B)}$$

Similarly, $1 + \cos C = \dfrac{2s(s-c)}{ab}$. (C)

Putting (B) and (C) in (A) we get,

$$\Delta^2 = \frac{1}{4}\ a^2b^2\,\frac{2(s-a)(s-b)}{ab}\cdot\frac{2s(s-c)}{ab}$$

$$= s\,(s-a)\,(s-b)\,(s-c) \Rightarrow \Delta = \sqrt{s\,(s-a)(s-b)(s-c)}.$$

Theorem 4 :

$$\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}},\ \sin\frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{ac}},\ \sin\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}$$

$$\cos\frac{A}{2} = \sqrt{\frac{s\,(s-a)}{bc}},\ \cos\frac{B}{2} = \sqrt{\frac{s\,(s-b)}{ca}},\ \cos\frac{C}{2} = \sqrt{\frac{s\,(s-c)}{ab}}$$

$$\tan\frac{A}{2} = \sqrt{\frac{(s-b)\,(s-c)}{s(s-a)}},\ \tan\frac{B}{2} = \sqrt{\frac{(s-c)\,(s-a)}{s(s-b)}},\ \tan\frac{C}{2} = \sqrt{\frac{(s-a)\,(s-b)}{s(s-c)}}.$$

Proof :

We have $2\sin^2\dfrac{A}{2} = 1 - \cos A = 1 - \dfrac{b^2+c^2-a^2}{2bc}$

$$= \frac{a^2-(b^2+c^2-2bc)}{2\,bc} = \frac{a^2-(b-c)^2}{2\,bc}$$

$$= \frac{(a-b+c)(a+b-c)}{2bc} = \frac{2(s-b)\,2(s-c)}{2bc}$$

$$\Rightarrow \sin^2\frac{A}{2} = \frac{(s-b)(s-c)}{bc} \quad \Rightarrow \sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}.$$

(The positive square root is taken since $\frac{A}{2} < 90°$)

Again $2\cos^2\frac{A}{2} = 1 + \cos A = 1 + \frac{b^2+c^2-a^2}{2bc}$

$$= \frac{2bc+b^2+c^2-a^2}{2bc} = \frac{(b+c)^2-a^2}{2bc}$$

$$= \frac{(b+c+a)(b+c-a)}{2bc} = \frac{2s(2s-2a)}{2bc}$$

$$\Rightarrow \cos^2\frac{A}{2} = \frac{s(s-a)}{bc} \Rightarrow \cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}.$$

Now $\tan\frac{A}{2} = \dfrac{\sin\frac{A}{2}}{\cos\frac{A}{2}} = \sqrt{\dfrac{(s-b)(s-c)}{s(s-a)}}$

Similarly the other trigonometric ratios for $\frac{B}{2}$ and $\frac{C}{2}$ can be obtined.

Corollary 1 :

$$\sin A = \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)} = \frac{2\Delta}{bc}$$

$$\sin B = \frac{2}{ca}\sqrt{s(s-a)(s-b)(s-c)} = \frac{2\Delta}{ca}$$

$$\sin C = \frac{2}{ab}\sqrt{s(s-a)(s-b)(s-c)} = \frac{2\Delta}{ab}$$

Corollary 2 :

$$\tan\frac{A}{2} = \frac{\Delta}{s(s-a)},\ \tan\frac{B}{2} = \frac{\Delta}{s(s-b)},\ \tan\frac{C}{2} = \frac{\Delta}{s(s-c)}.$$

Proof :

$$\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = \sqrt{\frac{s(s-a)(s-b)(s-c)}{s^2(s-a)^2}} = \frac{\Delta}{s(s-a)}.$$

Similarly other relations can be derived.

Example 25 :

Show that in any triangle

$a^2(\sin^2 B - \sin^2 C) + b^2(\sin^2 C - \sin^2 A) + c^2(\sin^2 A - \sin^2 B) = 0.$

Proof :

We have the formula $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = \frac{1}{k}$ where $k \neq 0$ is a constant

$\Rightarrow$ $\sin A = ka$, $\sin B = kb$, $\sin C = kc$.

Now L.H.S. $= k^2 a^2 (b^2 - c^2) + k^2 b^2 (c^2 - a^2) + k^2 c^2 (a^2 - b^2)$

$= k^2 \{a^2 (b^2 - c^2) + b^2 (c^2 - a^2) + c^2 (a^2 - b^2)\} = k^2 \times 0 = 0 =$ R.H.S.

Example 26 :

In any Δ ABC, prove that $\cos \frac{B-C}{2} = \frac{b+c}{a} \sin \frac{A}{2}$.

Proof : R.H. S. $= \frac{b+c}{a} \sin \frac{A}{2}$

$$= \frac{2R\sin B + 2R\sin C}{2R\sin A} . \sin \frac{A}{2} \left(\because \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R \right)$$

$$= \frac{\sin B + \sin C}{\sin A} . \sin \frac{A}{2} = \frac{2\sin \frac{B+C}{2} . \cos \frac{B-C}{2}}{2\sin \frac{A}{2} . \cos \frac{A}{2}} \sin \frac{A}{2}$$

$$= \frac{\cos \frac{A}{2} . \cos \frac{B-C}{2}}{\cos \frac{A}{2}} \left(\because \sin \frac{B+C}{2} = \cos \frac{A}{2} \right)$$

$$= \cos \frac{B-C}{2} = \text{L. H. S.}$$

Example 27 :

If $(a^2 + b^2) \sin (A - B) = (a^2 - b^2) \sin (A + B)$, prove that the triangle is either isosceles or right angled.

Proof : $(a^2 + b^2) \sin (A - B) = (a^2 - b^2) \sin (A + B)$

$$\Rightarrow \frac{\sin(A+B)}{\sin(A-B)} = \frac{a^2+b^2}{a^2-b^2}$$

$$\Rightarrow \frac{\sin(A+B)+\sin(A-B)}{\sin(A-B)-\sin(A-B)} = \frac{a^2}{b^2}$$

$$\Rightarrow \frac{\sin A.\cos B}{\cos A.\sin B} = \frac{\sin^2 A}{\sin^2 B} \qquad \text{(sine formula)}$$

$\Rightarrow \sin A . \cos B. \sin^2 B - \cos A. \sin B. \sin^2 A = 0$

$\Rightarrow \frac{1}{2} \sin A. \sin B (\sin 2B - \sin 2A) = 0$

$\Rightarrow \sin 2B - \sin 2A = 0$ ($\because$ $\sin A \neq 0$ and $\sin B \neq 0$)

$\Rightarrow 2\cos(A + B).\ \sin(B - A) = 0$

$\Rightarrow 2\cos C.\ \sin(A - B) = 0$

$\Rightarrow \cos C = 0$

$\Rightarrow C = \frac{\pi}{2}$ so that the Δ is right angled.

If $\sin(A - B) = 0$, then $A = B$, thus the triangle is isosceles.

Example 28 :

If in the Δ ABC, $a\sin^2\frac{C}{2} + c\sin^2\frac{A}{2} = \frac{b}{2}$, then show that a, b, c are in A.P.

Proof : Applying theorem 13, $a\sin^2\frac{C}{2} + c\sin^2\frac{A}{2} = \frac{b}{2}$

$$\Rightarrow a\,\frac{(s-a)(s-b)}{ab} + c\,\frac{(s-b)(s-c)}{bc} = \frac{b}{2}$$

$$\Rightarrow \frac{s-b}{b}\,(2s - a - c) = \frac{b}{2}$$

$\Rightarrow 2(s - b) = b$

$\Rightarrow a + c = 2b$. ($\because 2s = a + b + c$)

$\therefore$ a, b and c are in AP.

EXERCISES 4 (d)

1. Fill in the blanks choosing correct answer from the brackets.

(i) In Δ ABC, b = ———. ($b\cos B + c\cos C$, $a\cos A + c\cos C$, $c\cos A + a\cos C$)

(ii) If $a\cot A = b\cot B$, then Δ ABC is ——— . (isosceles, right angled, equilateral)

(iii) In Δ ABC if $b\sin C + c\sin B = 2$, then $b\sin C$ = ———. (0,1, 2)

(iv) In Δ ABC if $\frac{\text{coaA}}{a} = \frac{\cos B}{b} = \frac{\cos C}{c}$, then the triangle is ———.

(equilateral, isosceles, scalene)

(v) If $\sin A = \sin B$ and $b = \frac{1}{2}$, then a = ———. $\left(2, \frac{1}{2}, 1\right)$

(vi) In Δ ABC if A = 60°, B = 45°, than $a : b$ = ——. $\left(\sqrt{2}{:}\sqrt{3}, \sqrt{6}{:}2, \sqrt{3}{:}2\right)$

(vii) In Δ ABC if $b^2 + c^2 < a^2$, then —— angle is obtuse. (A, B, C)

(viii) If $a\cos B = b\cos A$, then $\cos B$ = ——— . $\left(\frac{c}{a}, \frac{a}{2c}, \frac{c}{2a}\right)$

(ix) If $a = b \cos C$, than —— angle is a right angle. (A, B, C)

(x) If $a = 12$, $b = 7$, $C = 30°$, then $\Delta =$ ——— . (42, 84, 21)

2. Prove that

(i) $a \sin A - b \sin B = c \sin (A - B)$.

(ii) $b \cos B + c \cos C = a \text{ coc } (B - C)$.

(iii) If $(a + b + c)(b + c - a) = 3\, bc$, then $A = 60°$.

(iv) If $\frac{b+c}{5} = \frac{c+a}{6} = \frac{a+b}{7}$, then $\sin A : \sin B : \sin C = 4 : 3 : 2$.

(v) If $A : B : C = 1 : 2 : 3$, then $\sin A : \sin B : \sin C = 1 : \sqrt{3} : 2$.

(vi) If $b^2 + c^2 - a^2 = bc$, then $A = 60°$.

(vii) If $A : B : C = 1 : 2 : 7$, then $c : a = \left(\sqrt{5}+1\right) : \left(\sqrt{5}-1\right)$.

3. (i) If $\cos A = \frac{12}{13}$, $\cos B = \frac{5}{13}$, then find $a : b$.

(ii) If $a = 7, b = 3$, $c = 5$, then find A.

(iii) If $a = 8$, $b = 6$, $c = 4$, find $\tan \frac{B}{2}$.

(iv) If $\frac{a}{\sec A} = \frac{b}{\sec B}$ and $a \neq b$ then find C.

(v) If $a = 48$, $b = 35$, $C = 60°$, then find c.

In Δ ABC prove that (Q4 – Q 26)

4. $a \sin (B - C) + b \sin (C - A) + c \sin (A - B) = 0$.

5. $$\frac{\sin(B - C)}{\sin(B+C)} = \frac{b \cos C - c \cos B}{b \cos C + c \cos B}.$$

6. $$\sum \frac{a^2 \sin(B - C)}{\sin(B+C)} = 0.$$

7. $a^2(\cos^2 B - \cos^2 C) + b^2(\cos^2 C - \cos^2 A) + c^2(\cos^2 A - \cos^2 B) = 0$.

8. $$\frac{b^2 - c^2}{a^2} \sin 2A + \frac{c^2 - a^2}{b^2} \sin 2B + \frac{a^2 - b^2}{c^2} \sin 2C = 0.$$

9. $$\frac{a^2(b^2 + c^2 - a^2)}{\sin 2A} = \frac{b^2(c^2 + a^2 - b^2)}{\sin 2B} = \frac{c^2(a^2 + b^2 - c^2)}{\sin 2C}.$$

10. $$\sum \frac{\cos A}{\sin B.\ \sin C} = 2.$$

11. $(a^2 - b^2 + c^2) \tan B = (a^2 + b^2 - c^2) \tan C$.

12. $(b^2 - c^2) \cot A + (c^2 - a^2) \cot B + (a^2 - b^2) \cot C = 0$.

13. $$\frac{b+c}{a} = \frac{\cos B + \cos C}{1 - \cos A}.$$

14. $\Sigma a^3 \sin(B - C) = 0$.
15. $(b + c) \cos A + (c + a) \cos B + (a + b) \cos C = a + b + c$.
16. $2 (bc \cos A + ca \cos B + ab \cos C) = a^2 + b^2 + c^2$.
17. $a (b^2 + c^2) \cos A + b (c^2 + a^2) \cos B + c (a^2 + b^2) \cos C = 3 abc$.
18. $a^3 \cos (B - C) + b^3 \cos (C - A) + c^3 \cos (A - B) = 3 abc$.
19. $a (\cos B + \cos C) = 2 (b + c) \sin^2 \frac{A}{2}$.
20. $(b + c - a) \tan \frac{A}{2} = (c + a - b) \tan \frac{B}{2} = (a + b - c) \tan \frac{C}{2}$.
21. $(b + c - a) \left(\cot \frac{B}{2} + \cot \frac{C}{2}\right) = 2a \cot \frac{A}{2}$.
22. $(a - b)^2 \cos^2 \frac{C}{2} + (a + b)^2 \sin^2 \frac{C}{2} = c^2$.
23. $1 - \tan \frac{A}{2} . \tan \frac{B}{2} = \frac{c}{s}$.
24. $(b - c) \cot \frac{A}{2} + (c - a) \cot \frac{B}{2} + (a - b) \cot \frac{C}{2} = 0$.
25. $\cot A + \cot B + \cot C = \dfrac{a^2 + b^2 + c^2}{4\Delta}$.
26. $a^2 \cot A + b^2 \cot B + c^2 \cot C = 4\Delta$.
27. If $\frac{1}{a+c} + \frac{1}{b+c} = \frac{3}{a+b+c}$, then prove $C = 60°$.
28. If $a = 2b$ and $A = 3B$, find the measures of the angles of the triangle.
29. If $a^4 + b^4 + c^4 = 2c^2 (a^2 + b^2)$. prove that $m\angle ACB = 45°$ or $135°$.
30. If $x^2 + x + 1$, $2x + 1$ and $x^2 - 1$ are lengths of sides of a triangle, then prove that the angle of highest measure, measures $120°$.
31. If $\cos B = \dfrac{\sin A}{2 \sin C}$, prove that the triangle is isosceles.
32. If $a \tan A + b \tan B = (a + b) \tan \frac{1}{2}(A + B)$ prove that the triangle is isosceles.
33. If $(\cos A + 2 \cos C) : (\cos A + 2 \cos B) = \sin B : \sin C$, prove that the triangle is either isosceles or right- angled.
34. If $\cos A = \sin B - \cos C$, prove that the triangle is right angled.
35. If a^2, b^2, c^2 be in A.P, prove that cot A, cot B, cot C are also in A.P.
36. If $\sin A : \sin C = \sin (A - B) : \sin (B - C)$, prove that a^2, b^2, c^2 are in A.P.
37. If the side-lengths a, b and c are in A.P., then prove that $\cos \frac{1}{2}(A - C) = 2 \sin \frac{1}{2} B$.
38. If the side-lengths a, b, c are in A.P., prove that $\cot \frac{1}{2} A$, $\cot \frac{1}{2} B$, $\cot \frac{1}{2} C$ are in A.P.

CHAPTER 5

Principle of Mathematical Induction

Believe nothing, merely because you have been told it, or because it is traditional, or because you have imagined it.

- Buddha

5.1 Introduction

There are several principles guiding a mathematical proof. Mention was made in section 1.7, of the following principles.

(i) Principle of syllogism.

(ii) Principle of reductio ad absurdum

(iii) Law of the contrapositive.

The principle of syllogism is one of the oldest methods of **direct proof.** By repeated application of this principle, the premises,

$$p_1 \rightarrow p_2 \text{ , } p_2 \rightarrow p_3 \text{ .... } p_{n-1} \rightarrow \text{ p}_\text{n}$$

lead to the conclusion $p_1 \rightarrow \text{p}_\text{n}$

The principle of reductio ad absurdum and the law of contradiction are instances of **indirect methods of proof** the latter being a special case of the former. Earlier, we mentioned the two laws : Law of the excluded middle and the principle of contradiction, which together make up the axiom of negation. Both of these laws are implicit in the principle of reductio and absurdum and also in thc principlc of syllogism.

Over and above these, there is another principle, called the **principle of mathematical induction**, which comes handy on many occasions while dealing with propositions about **natural numbers** 0, 1, 2, 3,...... This principle is not deducible from the rules mentioned earller. It is indeed an axion about natural numbers that was introduced by Peano (1889) and also by Dedekind (1888) independently. (Ref. Real Number System in Appendix)

5.2 The Principle of Induction

***If p_n is a proposition about a natural number n such that p_0 is true and the truth of p_n for any natural number n implies* that of p_{n+1}, *then p_n* is true for every natural number *n* .**

Remark

(i) It must be borne in mind that the initial proposition **need not always *be P_0***. It can be any p_k and if the other hypothesis holds, the conclusion is valid for each natural number $n \geq k$.

(ii) One may try to prove the induction principle by arguing that since $p_n \Rightarrow p_{n+1}$ for all *n,* the truth of p_0 implies that of p_1 which, in turn, implies that of p_2 and continuing the argument, p_n is true for each n. However, such an argument can not be sustained since our logical system admits only arguments involving **finite** sequences of premises, and conclusions. This is not the case with the principle of induction which involves, in reality an infinite sequence of premises as well as an infinite number of conclusions.

(iii) In the induction hypothesis we verify p_0 to be true and then show that $p_k \Rightarrow p_{k+1}$. without taking into account the truth of $p_1, p_2, ... p_{k-1}$. In some situation $p_k \Rightarrow p_{k+1}$ cannot be deduced without assuming the truth of $p_1, p_2 ..... p_{k-1}$.

This situation is handled by a **stronger** version of the principle of mathematical induction which we state below.

If p_n is a proposition about a natural number n such that

(i) p_0 is true

(ii) for any $k \in N$, $(p_1 \wedge p_2 \wedge .p_3 \wedge .... \wedge p_k) \Rightarrow p_{k+1}$

then p_n is true for every natural number n.

In view of this stronger version, the previous principle is also called the principle of *weak* induction. Example 3 illustrates the use of the strong induction. Example 1 and 2 use the principle of weak induction.

In view of the peano-axiom on natural numbers, the set of natural numbers is the best inductive subset of R, the set of real number. It may be mentioned that $S \subseteq R$ is called an **inductive set if 0∈S and m∈S implies m+1∈S**. Thus any inductive subset of R contains the set of natural numbers.

Example 1

For any natural number $n \geq 1$,

$$1^3 + 2^3 + 3^3 + ... + n^3 = \left[\frac{n(n+1)}{2}\right]^2$$

Proof : For any natural number $n \geq 1$, Let p_n denote the proposition

$$1^3 + 2^3 + 3^3 + ... + n^3 = \left[\frac{n(n+1)}{2}\right]^2$$

Clearly $p_1 : 1^3 = \left[\frac{1.(1+1)}{2}\right]^2$ is true since each side is 1. Assume the truth of p_n. To complete the argument by induction, we must show that p_{n+1} is true, that is,

$$1^3 + 2^3 + 3^3 + ... + (n+1)^3 = \left[\frac{1}{2}\,(n+1)\,(n+2)\right]^2$$

Now $1^3 + 2^3 + ... + n^3 + (n+1)^3$

$$= \left[\frac{n(n+1)}{2}\right]^2 + (n+1)^3 \text{ (as } p_n \text{ is true)}$$

$$= (n+1)^2\left[\frac{n^2}{4} + (n+1)\right]$$

$$= (n+1)^2\left[\frac{n^2+4(n+1)}{4}\right] \frac{(n+1)^2\ (n+2)^2}{4}$$

$$= \left[\frac{(n+1)(n+2)}{2}\right]^2$$

So that p_{n+1} is true whenever p_n is true.

So p_1 being true, it follows from this that p_2, p_3 ... and in general p_n is true for all n, thus the result follows by induction.

Example 2 :

$4^n + 15n - 1$ is divisible by 9 for all n .

Proof :

Let p_n denote the above statement. Clearly p_0 is true since $4^0 + 15.0 - 1 = 0$ is divisible by 9.

Suppose that p_n is true for any n. Then $4^n + 15n - 1 = 9m$ (say) for some natural number m. Now

$$4^{n+1} + 15(n+1) - 1$$
$$= 4.4^n + 15n + 14$$
$$= 4(4^n + 15n - 1) - 60n + 4 + 15n + 14$$
$$= 4.9m - 45n + 18$$
$$= 9(4m - 5n + 2)$$

and this is divisible by 9. Hence p_{n+1} is true. Now by the method of induction p_n is true for all n.

Example 3 :

Prove that every integer $n \geq 2$ can be factored as a product of primes.

Proof : Let p_n be the statement :

Every integer $n \geq 2$ can be factored as a product of primes.

For $n = 2$, p_2 is true.

Assume that every integer k, $2 \leq k < n$, can be factored as a product of primes, i.e. p_3, p_4, p_{n-1} are assumed to be true...................... (1)

Now if n is prime, then n itself is the only prime factor in its factorisation.

On the other hand if n is composite, let $n = ab$, whre a, b are integers such that $2 \le a < n$ and $2 \le b < n$. Since $a < n$, by induction hypothesis (1), a is a product of primes. Similarly b is a product of primes.

Hence $n = ab$ is a product of primes.

Thus p_n is true for every integer n.

Example 4

Bernoulli's inequality

Prove that for every $x \in R$, $x \ge -1$;

$(1+x)^n \ge 1+ nx$ for every positive integer n.

Proof : Let P_n denote the above statement.

Obviously P_1 is true, Suppose P_n is true for some positive integer n.

New, $P_n \Rightarrow (1+x)^n \ge 1+ nx$

$\Rightarrow (1+x)^{n+1} \ge (1+ nx)\ (1+x)$

(Multiplying both sides by $1+x$ which is nonnegative, so that the inequality is preserved, by properties of real numbers.)

$= 1+(n+1)\ x + nx^2$

$\ge 1+(n+1)x$

$\Rightarrow P_{n+1}$ is true.

The result follows by induction, since P_1 is true.

Exercise-5

Prove the following by induction :

1. $1 + 2 + \ldots\ldots + n = \dfrac{n(n+1)}{2}$

2. $1^2 + 2^2 + \ldots\ldots.. + n^2 = \dfrac{n(n+1)\ (2n+1)}{6}$

3. $1 + r + r^2 + \ldots\ldots + r^n = \dfrac{r^{n+1} - 1}{r - 1}$

4. $5^n - 1$ is divisible by 4.

5. $7^{2n} + 2^{3n-3}\, 3^{n-1}$ is divisible by 25 for any natural number $n \ge 1$.

6. $7.5^{2n-1} + 2^{3n+1}$ is divisible by 17 for every natural number $n \ge 1$.

7. $4^{n+1} + 15n + 14$ is divisible by 9 for every natural number $n \geq 0$.

8. $3^{2(n-2)} + 7$ is divisible by 8 for every natural number $n \geq 2$.

9. $5^{2n+2} - 24n - 25$ is divisible by 576 for all n.

10. $\frac{1}{1.2} + \frac{1}{2.3} + \ldots\ldots + \frac{1}{n(n+1)} = \frac{n}{n+1}$

11. $1.3 + 2.4 + 3.5 + \ldots\ldots + n(n+2) = \frac{n(n+1)(2n+7)}{6}$

12. $x^n - y^n = (x-y)(x^{n-1} + x^{n-2}y + \ldots + xy^{n-2} + y^{n-1})$; $x, y \in R$

[Hint: Write $x^{n+1} - y^{n+1} = x(x^n - y^n) + y^n(x-y)$]

13. $1+3+5+\ldots+(2n-1) = n^2$

14. $2^n > n$; n is a natural number

15. $(1,2,3 \ldots, n)^3 > 8(1^3+2^3+3^3+\ldots + n^3)$, for $n>3$.

16. $\frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{3n+1} > 1$ for every positive integer n.

●

CHAPTER 6

Complex Numbers and Quadratic Equations

The moving power of mathematical invention is not reasoning but imagination.

- De Morgan

6.1 Introduction

In real numbers one of the properties was $x^2 \geq 0$ for every real x. This means $x^2 = -1$ has no solution in real numbers. This is much the same way as $x^2 = 2$ had no solution when our numbers were rational. But we did extend our scheme of numbers to include solution of the equation $x^2 = 2$. Now the question is "can we similarly extend our scheme of numbers so that $x^2 = -1$ has a solution ? The interesting thing is that it has been possible to do so. The necessity becomes clear when we try to solve a quadratic equation

$$ax^2 + bx + c = 0$$

with $b^2 - 4ac < 0$. The problem is that in the usual formula for solution of a quadratic equation we have a square root of $b^2 - 4ac$. But in the field of real numbers negative numbers have no square roots. So the obvious thing to do is to recognise $\sqrt{-1}$ as a new kind of number and augment our number system to include this new number which we call as an **imaginary** number. If we represent $\sqrt{-1}$ by the symbol i and write $i^2 = i.i = \sqrt{-1} \cdot \sqrt{-1} = -1$. You should not confuse $\sqrt{-1} \cdot \sqrt{-1}$ with $\sqrt{a}.\sqrt{b}=\sqrt{ab}$, which is valid only for $a, b \geq 0$. Otherwise, putting $a=b=$ -1, you will get the fallacious result $\sqrt{-1} \cdot \sqrt{-1} = \sqrt{(-1).(-1)} = \sqrt{1} = 1$. Remember, $\sqrt{1}$ only means 1, not $\pm$ 1 or -1. Thus our new numbers shall consist of elements of the type $a+b\sqrt{-1} = a + ib$, where a, b are real numbers. A number of the from $z = a + ib$ will be called a complex number.

6.2 Algebra of Complex Numbers

We define addition of two complex numbers $a+ib$ and $c+id$ by the rule :

$$(a+ib) + (c+id) = (a+c) + i\,(b+d).$$

We can easily verify that all the laws for addition in the case of real numbers such as **(i) closure law, (ii) associative law, (iii) existence of additive identity, (iv) existence of additive inverse and (v) commutative law also hold for addition of complex numbers**. Here we take **0+i0** as the **additive identity** and $-a + (-b)i$ as the **additive inverse of** $a+ib$ **and write it as** $-(a+ib)$.

We define subtraction of two complex numbers $a+ib$ and $c+id$ by the rule :

$$(a+ib) - (c+id) = (a-c) + i\,(b-d)$$

Now you can verify that associative law and commutative law do not hold for subtraction of two complex numbers as in the case of real numbers.

Now we define multiplication of two complex numbers by the rule :

$(a+ib)(c+id) = (ac-bd) + i(ad+bc)$

Again we can verify the closure law, associative law and commutative law. Besides, multiplication also obeys distributive law :

For three complex numbers z_1, z_2 and z_3, we have

$z_1(z_2+z_3) = z_1 z_2 + z_1 z_3 = (z_2+z_3) z_1$

If we agree to denote **0+i0 by the symbol 0** (the same symbol as used in R) then

0 (a+ib) = 0 which is same as 0+i0. Similarly if we agree to denote **1+0i as 1**, then we would get 1.(a+ib) = (a+ib)1=a+ib. In otherwords, 1 is the **multiplicative identity** of the new set of numbers.

We call this new set of numbers **complex numbers** and denote it by $\mathbb{C}$. Observe that those elements of $\mathbb{C}$ which are of the form $a + 0i$ behave much the same as real numbers including the property $(a + io)(c + io) = ac + io$ which makes it self contained as far as multiplication and even addition and subtraction are concerned. It is as if we could identify $a + 0i$ with the real number a. In other words $\{a + io : a \in R\}$ looks exactly like R. This is what makes R a part of $\mathbb{C}$ and $\mathbb{C}$ an extension of R. Thus those numbers of the type $a + ib$ shall be called purely real if $b = 0$. What if we consider the numbers of the type $a + ib$ *with* $a = 0$? We see that such numbers are self contained as far as addition and subtraction are concerned but **not** so when it comes to mutiplication. Indeed $(0 + ib)(0 + id) = -bd + i0$ which is not of the type $0 + \beta i$, rather of the type $\alpha + 0i$, which we call **purely real number** and identify this with the real number α . The numbers of the form $\mathbf{0 + ib}$ are called **purely imaginary** numbers . A complex number has a real part and an imaginary part : $a + ib$ has real part a whereas b is its imaginary part. We usually write a complex number by a single letter. So if $\mathbf{z = a + ib}$, then

the real part of z is a and imaginary part, b.

We write this symbolically

Re z = a, Im z = b.

If Im $z = 0$ the we call z purely real. Some times product of two complex numbers becomes purely real even when they are not individually purely real number eg.

$(2 + 3i)(2 - 3i) = 13.$

Complex Conjugates

If z is the complex number $a + ib$ then we call the complex number $a - ib$ its complex conjugate and denote it by $\bar{z}$. So if $z = a + bi$, $\bar{z} = a - bi$. We observe that

(i) $\overline{(\bar{z})} = z$ **(ii)** $z\bar{z} = a^2 + b^2 = \bar{z}z$.

Thus product of a complex number with its complex conjugate is not only a real number but also a non negative number. So

(iii) $z\,\bar{z} = 0$ if and only if $z = 0$.

6.3 Geometrical representation of a complex number

We represent a complex number z=a+ib by an ordered pair (a,b) which is identified as a point (a,b) on the Cartesian plane (Refer Cartesian - coordinates in chapter-4, section 4.1). In other words, we can say that a point (x,y) on the cartesian plane represents a complex number $x+iy$.

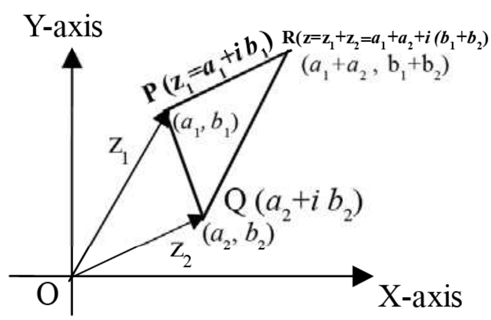
Since every ordered pair (a,b) in the Cartesian product $R \times R$ or R^2 is uniquely represented by a point on the Cartesian plane, the above representation of complex numbers by points is also unique.

The establishes a one-to-one correspondence between this set of complex numbers $\mathbb{C}$ and R^2. We already know that there is a one-to-one correspondence between the set of real numbers and the *x-axis*. Therefore, the reason for which *x-axis* is called the real line, is carried over to call the Cartesian plane as the **complex plane**.

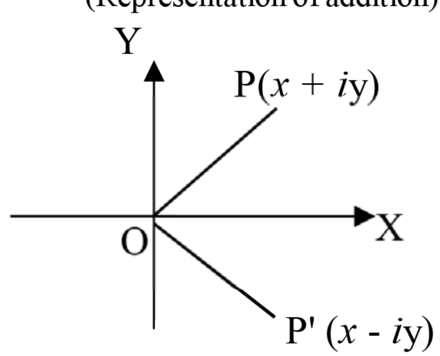
If Re z=0 then z = 0+iy is identified as the point (0,y) which is on y-axis. Similarly if Im z = 0 then z=x+i0 is identified as the point (x,0) which is on x-axis. Therefore y-axis is called imaginary axis. As usual, x-axis is called real line or real axis.

The identification of a complex number on a plane was proposed by Jean-Robert Argand (1768-1822). Hence the complex plane is also called Argand plane and the corresponding diagrams for representation of a complex number are known as Argand diagrams.

We observe that if $z_1 = a_1 + ib_1$, $z_2 = a_2 + ib_2$ are represented by the points $P(a_1, b_1)$ and $Q(a_2, b_2)$ respectively and O is the origin then the point $z_1 + z_2$ is represented by the point $R(a_1+a_2, b_1+b_2)$. You can easily prove that OQRP is a parallelogram. Later, when you come to vectors in Vol-II you will get to know that the sum resembles the parallelogram law of addition of vectors. In fact the vector $\overrightarrow{OP}$ is said to denote the complex number z_1 and similarly $\overrightarrow{OQ}$ and $\overrightarrow{OR}$ denote z_2 and $z_1 + z_2$ respectively. The vector $\overrightarrow{QP}$ is the vector representation of the complex number $z_1 - z_2$.



(Representation of addition)

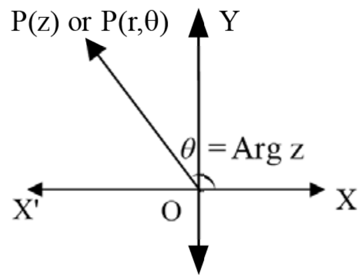


(Representation of complex conjugate)

Now if $z = x+iy$ is a point on the Argand plane then its **reflection** in the x-axis represents its complex conjugate $\bar{z} = x - iy$. We have seen before that $z\bar{z} = \bar{z}z = x^2 + y^2 \geq 0$. But $x^2 + y^2$ is the square of the distance of the point P(x, y) representing the complex number $x+iy$ from the origin. In other words OP = $\sqrt{x^2+y^2}$. $\sqrt{\mathbf{x^2+y^2}}$ **is called the modulus** of the complex number $x+iy$. We denote the modulus of a complex number z by | z |. Thus $|\mathbf{z}| = |\boldsymbol{x+iy}| = \sqrt{x^2+y^2}$. It is easy to derive that $\mathbf{z\bar{z} = \bar{z}z = |z|^2}$.

Polar Representation of a complex Number

Let the point P have polar coordinates (r,θ) (refer Section 4.1 of chapter-4).



(Polar representation of a complex number)

Since P has Catesian coordinates (x,y) we have

$x = r\cos\theta,\ y = r\sin\theta$ . ... ***(a)***

Clearly, $r = \sqrt{x^2+y^2}$ and $\tan\theta = \dfrac{y}{x},\ x \neq 0$ .... ***(b)***

It is now convenient to define $z = x+iy = r(\cos\theta + i\sin\theta)$ .... **(c)**

Any value of θ for which (a) or (b) or (c) holds is called **an argument** of z denoted by $\theta = \arg z$. When z = 0 (=0+ i0) arg z is not defined. Since sine and cosine are periodic functions of period 2π, z has an infinite number of arguments any two of which differ by a multiple of 2π. We assign a unique value to arg z by restricting θ to the interval (–π, π] i.e. $-\pi < \theta \leq \pi$. If argz is thus restricted, it is called **the principal argument** and is denoted by Arg z or Pr.arg z. Hence

$$-\pi < \text{Arg } z \leq \pi .$$

From equation (b), $\theta = \tan^{-1}(y/x)$ and while considering the principal values of $\tan^{-1}(y/x) \in (-\pi/2, \pi/2)$. Arg z is determined as follows :

$$\text{Arg } z\ (= \text{Pr.arg } z) = \begin{cases} \tan^{-1}\dfrac{y}{x} & \text{if } x>0 \\ \pi + \tan^{-1}\dfrac{y}{x} & \text{if } x<0,\ y>0 \\ -\pi + \tan^{-1}\dfrac{y}{x} & \text{if } x<0,\ y<0 \\ \dfrac{\pi}{2} & \text{if } x=0,\ y>0 \end{cases}$$

$$-\frac{\pi}{2} \qquad \text{if } x = 0,\ y < 0$$

where $-\frac{\pi}{2} < \tan^{-1}\frac{y}{x} < \frac{\pi}{2}$.

[N.B.: You will be formally introduced to functions like $\tan^{-1}\frac{\mathbf{y}}{\mathbf{x}}$ in Vol-II when inverse trigonometric functions will be discussed. For the time being, you may take $\tan^{-1}\frac{\mathbf{y}}{\mathbf{x}}$ as θ such that $\tan\theta = \frac{\mathbf{y}}{\mathbf{x}}$]

For example, Arg $(-\sqrt{3}-i) = -\pi + \tan^{-1}\left(\frac{-1}{-\sqrt{3}}\right) = -\pi + \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = -\pi + \frac{\pi}{6} = -\frac{5\pi}{6}$; Similarly Arg $(-\sqrt{3}+i) = \frac{5\pi}{6}$, Arg $(\sqrt{3}-i) = -\frac{\pi}{6}$.

Exercise : Find Arg $\left(\frac{-1+\sqrt{3}i}{2}\right)$ and Arg $\left(\frac{-1-\sqrt{3}i}{2}\right)$. $\left[\text{Ans. } \frac{2\pi}{3}, -\frac{2\pi}{3}\right]$

An interesting property of complex numbers of the type cos α + *i sin*α .

$$(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta) = \cos(\alpha+\beta) + i\sin(\alpha+\beta) \qquad \textbf{(1)}$$

$$(\cos\alpha + i\sin\alpha)(\cos\beta - i\sin\beta) = \cos(\alpha-\beta) + i\sin(\alpha-\beta) \qquad \textbf{(2)}$$

Now if z_1 and z_2 are two complex numbers expressed as in (c) by

$$z_1 = r_1 \cos\theta_1 + i\sin\theta_1 \text{ and } z_2 = r_2 \cos\theta_2 + \mathrm{i}\sin\theta_2$$

then $z_1 z_2 = r_1 r_2 [\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)]$ [Using identity **(1)** above]

and $\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)]$ provided $z_2 \neq 0$ [Using identity **(2)** above]

Observing that $|\cos\alpha + i\cos\alpha| = 1$, we get

(i) $|z_1 z_2| = r_1 r_2 = |z_1||z_2|$

(ii) $\left|\frac{z_1}{z_2}\right| = \frac{r_1}{r_2} = \frac{|z_1|}{|z_2|}$

(iii) $\arg z_1 z_2 = \theta_1 + \theta_2 = \arg z_1 + \arg z_2$

(iv) $\arg\left(\frac{z_1}{z_2}\right) = \theta_1 - \theta_2 = \arg z_1 - \arg z_2$

Note that (iii) and (iv) are not true in general if we consider the principal value of arguments. If z is a nonzero complex number, we can wirte $z = r(\cos\theta + i\sin\theta) = r\{\cos(\theta+2n\pi) + i\sin(\theta+2n\pi)\}$ where θ is the principal value of arg *z* and *n* is an integer.

So $\arg z = \theta + 2n\pi$. This gives a set of values for arguments of z where any pair of these values differ by a multiple of 2π. Inview of this (iii) and (iv) are interpreted as set equations.

Note that if $z \neq 0$, $\arg z + \arg \bar{z} = 2n\pi$, where n is an integer or zero.

6.4 Inequality involving complex numbers

The triangle inequality : For two complex numbers z_1 and z_2

$|z_1 + z_2| \leq |z_1| + z_2|$

equality holds if and only if $\arg z_1$ differs from $\arg z_2$ by a multiple of 2π.

(It may be emphasized once again that complex number system is not ordered, i.e. there is no inequality among complex numbers. However the triangle inequality is an inequality involving modulus of complex numbers.)

Proof : Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ where

$|z_1| = r_1$, $\arg z_1 = \theta_1$, $|z_2| = r_2$, $\arg z_2 = \theta_2$.

Hence $|z_1 + z_2|^2 = |(r_1\cos\theta_1 + r_2\cos\theta_2) + i(r_1\sin\theta_1 + r_2\sin\theta_2)|^2$

$= r_1^2 + r_2^2 + 2r_1r_2\cos(\theta_1 - \theta_2)$ $\quad [\because |x + iy|^2 = x^2 + y^2]$

But $\cos\theta \leq 1$ for every real θ. So we have

$|z_1 + z_2|^2 \leq r_1^2 + r_2^2 + 2r_1r_2 = (r_1 + r_2)^2 = (|z_1| + |z_2|)^2$ which proves the result.

Now equality holds if and only if $\theta_1 - \theta_2$ is a multiple of 2π. Geometrically this means that in the parallelogram OPRQ, OR < OP+PR.

Example 1 :

(i) $(3 + 7i) . (5 - 2i) = 3. 5 - 7.2i^2 + 7.5i - 3.2i = 15 + 14 + i(35-6) = 29 + i29$

(ii) The multiplicative inverse of $3 + 5i$ is given by

$$(3 + 5i)^{-1} = \frac{1}{3+5i} = \frac{3-5i}{(3+5i)(3-5i)} = \frac{3-5i}{3^2+5^2} = \frac{3}{34} - \frac{5}{34}i$$

Example 2 :

Express $\dfrac{2+3i}{5-2i}$ in the $x + iy$ form

Solution :

$$\frac{2+3i}{5-2i} = \frac{(2+3i)(5+2i)}{(5-2i)(5+2i)} = \frac{10-6+i(15+4)}{5^2+2^2} = \frac{4+19i}{29} = \frac{4}{29} + i\frac{19}{29}$$

Example 3 :

If z_1 and z_2 are two complex numbers then show that

(a) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$

(b) $\overline{z_1 z_2} = \overline{z_1} \cdot \overline{z_2}$

(c) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$ if $z_2 \neq 0$.

Solution :

(a) Let $z_1 = a + ib$, $z_2 = c + id$. Then $z_1 + z_2 = (a + c) + i(b + d)$

$\Rightarrow \overline{z_1 + z_2} = (a + c) - i(b + d)$.

$\overline{z_1} + \overline{z_2} = (a - ib) + (c - id) = (a + c) - i(b + d) = \overline{z_1 + z_2}$

(b) Left to the reader.

(c) $\frac{z_1}{z_2} = \frac{a+ib}{c+id} = \frac{a+ib}{c+id} \cdot \frac{c-id}{c-id} = \frac{ac+bd}{c^2+d^2} + i\frac{bc-ad}{c^2+d^2}$

$$\frac{\overline{z_1}}{\overline{z_2}} = \frac{a-ib}{c-id} = \frac{a-ib}{c-id} \cdot \frac{c+id}{c+id} = \frac{ac+bd}{c^2+d^2} - i\frac{bc-ad}{c^2+d^2} = \overline{\left(\frac{z_1}{z_2}\right)}.$$

Example 4 :

Solve the equation : $x^3 = 1$

and obtain **properties of the cube roots of unity**.

Solution :

Since $x^3 - 1 = (x-1)(x^2 + x + 1)$, it follows that the cube roots of one are given by $x = 1$, $x^2 + x + 1 = 0$. But the solutions of the quadratic equation $x^2 + x + 1 = 0$ are given by

$$\alpha = \frac{-1+\sqrt{3}i}{2},\ \beta = \frac{-1-\sqrt{3}i}{2}$$

where we may notice the following important facts :

(i) $\overline{\alpha} = \beta$, $\overline{\beta} = \alpha$.

In other words, **the two complex roots of the equation are conjugate of each other.**

(ii) $\alpha^2 = \beta$, $\beta^2 = \alpha$

that is, **the square of any one complex root is the other complex root.**

(iii) If we denote ω to be one complex root, then the other complex root is ω^2 and therefore the **three roots are 1, ω, ω^2**.

(iv) Since ω is a root of the equation $x^3 = 1$, we obtain $\boldsymbol{\omega^3 = 1}$

and as consequence of this we obtain : $\omega^4 = \omega$, $\omega^5 = \omega^2$, $\omega^6 = (\omega^3)^2 = 1$ $\omega^{3n} = 1$,

$\omega^{3n+1} = \omega$, $\omega^{3n+2} = \omega^2$, $n \in \mathbf{N}$.

(v) Since ω is a root of the equation $x^2 + x + 1 = 0$, we obtain $\mathbf{1 + \omega + \omega^2 = 0}$, that is, **the sum of the three cube roots of unity vanishes.**

Example 5 :

Find the cube roots of a where a is a non-zero real number.

Solution :

Suppose thet $x^3 = a$ and suppose that $a > 0$. Then as $a^{\frac{1}{3}}$ is a positive real number, we can rewrite the above equation as

$$\left(\frac{x}{a^{\frac{1}{3}}}\right)^3 = 1$$

whose solutions are $\dfrac{x}{a^{\frac{1}{3}}} = 1, \omega, \omega^2$

so that $x = a^{\frac{1}{3}}, a^{\frac{1}{3}}\omega, a^{\frac{1}{3}}\omega^2$ are the three roots of the given equation.

In the case $a < 0$, we write the given equation as $(-x)^3 = (-a)$ or as

$$\left(\frac{-x}{(-a)^{\frac{1}{3}}}\right)^3 = 1$$

where solutions are given by $\dfrac{-x}{(-a)^{\frac{1}{3}}} = 1, \omega, \omega^2$

that is, $x = -(-a)^{\frac{1}{3}}, -\omega(-a)^{\frac{1}{3}}, -\omega^2(-a)^{\frac{1}{3}}$.

Remark

By using De Moivre's theorem (given later) the cube roots of unity can also be obtained.

EXERCISES 6 (a)

1. Multiply $\left(2\sqrt{-3} + 3\sqrt{-2}\right)$ by $\left(4\sqrt{-3} - 5\sqrt{-2}\right)$

2. Multiply $\left(3\sqrt{-7} - 5\sqrt{-2}\right)$ by $\left(3\sqrt{-2} + 5\sqrt{-2}\right)$

3. Multiply $(i + i)$ by $(a - bi)$

4. Multiply $\left(x-\frac{1+\sqrt{-3}}{2}\right)$ and $\left(x-\frac{1-\sqrt{-3}}{2}\right)$

Express with rational denominator.

5. $\frac{1}{3-\sqrt{-2}}$

6. $\frac{3\sqrt{-2}+2\ (-5)}{3\sqrt{-2}-2\sqrt{-2}}$

7. $\frac{3+2i}{2-5i}+\frac{3-2i}{2+5i}$

8. $\frac{a+xi}{a-xi}+\frac{a-xi}{a+xi}$

9. $\frac{(x+i)^2}{x-i}+\frac{\left(x-\sqrt{-1}\right)^2}{(x+i)}$

10. $\frac{(a+i)^3-(a-i)^3}{(a+i)^2-(a-i)^2}$

11. Find the value of $(-i)^{4n+3}$; when n is a positive integer.

12. Find the square of $\sqrt{9+40i}+\sqrt{9-40i}$.

13. Express in the form $a+ib$:

(i) $\frac{3+5i}{2-3i}$ (ii) $\frac{\sqrt{3}-i\sqrt{2}}{2\sqrt{3}-i\sqrt{3}}$ (iii) $\frac{(1+i)^2}{3-i}$

(iv) $\frac{(a+ib)^3}{a-ib}-\frac{(a-ib)^3}{a+ib}$ (v) $\frac{1+i}{1-i}$

14. Express the following points geometrically in the Argand plane

(i) 1, (ii) $3i$ (iii) -2

(iv) $3+2i$ (v) $-3+i$ (vi) $1-i$

15. Show that the following numbers are equidistant from the origin :

$\sqrt{2}+i,\ 1+i\sqrt{2},\ i\sqrt{3}$

16. Express each of the above complex numbers in the r, θ form

If 1, ω, ω^2 are the three cube roots of unity, prove that

17. $(1+\omega^2)^4=\omega$ 18. $(1-\omega+\omega^2)(1+\omega-\omega^2)=4$

19. $(1-\omega)(1-\omega^2)(1-\omega^4)(1-\omega^5)=9$

20. $(2+5\omega+2\omega^2)^6=(2+2\omega+5\omega^2)^6=729$

21. $(1-\omega+\omega^2)(1-\omega^2+\omega^4)(1-\omega^4+\omega^2)$ ... to $2n$ factors $=2^{2n}$.

22. Prove that $x^3+y^3+z^3-3xyz=(x+y+z)(x+\omega y+\omega^2 z)(x+y\omega^2+z\omega)$

23. If $x=a+b$, $y=a\omega+b\omega^2$, $z=a\omega^2+b\omega$, show that

(1) $xyz=a^3+b^3$ (2) $x^2+y^2+z^2=6ab$ (3) $x^3+y^3+z^3=3(a^3+b^3)$

24. If $ax+by+cz=X$, $cx+by+az=Y$, $bx+ay+cz=Z$ show that

$(a^2+b^2+c^2-ab-bc-ca)(x^2+y^2+z^2-xy-yz-zx)=X^2+Y^2+Z^2-YZ-ZX-XY$.

6.5 De - Moivre's Theorem (for integral index)

Statement :

$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$, for any integer n.

Proof : Case I (When n is a positive integer)

We prove this by induction : The above statement is true for $n = 1$ and suppose that it is true for any fixed positive integer k, that is, let.

$(\cos\theta + i\sin\theta)^k = \cos k\theta + i\sin k\theta$.

So $(\cos\theta + i\sin\theta)^{K+1}$

$$= (\cos\theta + i\sin\theta)^k(\cos\theta + i\sin\theta)$$
$$= (\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta)$$
$$= \cos(k+1)\theta + i\sin(k+1)\theta$$

by using indentity (1) given in section 6.3.

Now the result follows by the principle of Mathematical induction.

Case II. (When n is a negative integer)

Let $n = -m$ where m is a positive integer. To prove our contention we need only to observe that

$$\frac{1}{\cos\theta + i\sin\theta} = \cos\theta - i\sin\theta,$$

which follows from (2). So we have

$(\cos\theta + i\sin\theta)^n = (\cos\theta + i\sin\theta)^{-m}$

$$= \frac{1}{(\cos\theta + i\sin\theta)^m} = \frac{1}{\cos m\theta + i\sin m\theta} \text{ by case I}$$

$= \cos m\theta - i\sin m\theta = \cos n\theta + i\sin n\theta$

as $\sin(-\alpha) = -\sin\alpha$.

De-Moivre's Theorem (for rational index)

Let p and q be two integers with $q > 0$. Now

$$\left(\cos\frac{p}{q}\theta + i\sin\frac{p}{q}\theta\right)^q = (\cos p\theta + i\sin p\theta) = (\cos\theta + i\sin\theta)^p \qquad (3)$$

This gives $\cos\frac{p}{q}\theta + i\sin\frac{p}{q}\theta$ as one of the values of $(\cos\theta + i\sin\theta)^{\frac{p}{q}}$.

Why we say that this is one of the values is simply because

$$\left(\cos\left(\frac{p\theta}{q} + \frac{2\pi}{q}\right) + i\sin\left(\frac{p\theta}{q} + \frac{2\pi}{q}\right)\right)^q$$

$= (\cos(p\theta + 2\pi) + i\sin(p\theta + 2\pi) = \cos p\theta + i\sin p\theta = (\cos\theta + i\sin\theta)^p$.

This gives another solution to (3). In fact (3) has exactly q solutions namely

$$\cos\left(\frac{p\theta}{q}+\frac{2\pi k}{q}\right)+i\sin\left(\frac{p\theta}{q}+\frac{2\pi k}{q}\right) \text{ for k} = 0, 1, 2, \ldots\ q-1.$$

The reader can easily see that for $k \geq q$ there will be repetition of earlier values.

6.6 Application

(i) General solution of the equation

$x^n = 1$

where n is a positive integer. (4)

Since $1 = \cos 2\pi k + i \sin 2\pi k$

by what we have done above one sees that all the solution of (4) are

$$x = \cos\frac{2\pi k}{n} + i\sin\frac{2\pi k}{n},\ k = 0, 1, 2, \ldots\ n-1.$$

So for $n = 3$ we have the solutions

$x = \cos\frac{2\pi k}{3} + i\sin\frac{2\pi k}{3}$, $k = 0, 1, 2$ where $k = 0$ corresponds to 1 and $k = 1, 2$ correspond to the imaginary cube roots of unity. (see example 4)

General solution of the equation

$$x^n = a \tag{5}$$

where a is a complex number, is given as follows :

If $a = r(\cos\alpha + i\sin\alpha)$ then it is easily seen by the above arguments that the n solutions to (5) are

$$z = r^{\frac{1}{n}}\left(\cos\left(\frac{\alpha}{n}+\frac{2k\pi}{n}\right)+i\sin\left(\frac{\alpha}{n}+\frac{2k\pi}{n}\right)\right),\ k = 0, 1, 2, \ldots, n-1.$$

(ii) Finding square roots of a complex number $p + iq$ $p, q \in \mathbf{R}$.

Let $z = p + iq = r(\cos\alpha + i\sin\alpha)$ where $r = \sqrt{p^2+q^2}$, $r\cos\alpha = p$, $r\sin a = q$.

By what we have done about solution of (5) the two square roots of z shall be

$$r^{\frac{1}{2}}\left(\cos\frac{\alpha}{2}+i\sin\frac{\alpha}{2}\right),\ r^{\frac{1}{2}}\left(\cos\left(\frac{\alpha}{2}+\pi\right)+i\sin\left(\frac{\alpha}{2}+\pi\right)\right).$$

But $\cos\left(\frac{\alpha}{2}+\pi\right) = -\cos\frac{\alpha}{2}$, $\sqrt{\frac{1-\cos\alpha}{2}} = \sin\frac{\alpha}{2}$.

So we have the two roots :

$$r^{\frac{1}{2}}\left(\sqrt{\frac{1+\cos\alpha}{2}}+i\sqrt{\frac{1-\cos\alpha}{2}}\right),\ -r^{\frac{1}{2}}\left(\sqrt{\frac{1+\cos\alpha}{2}}+i\sqrt{\frac{1-\cos\alpha}{2}}\right)$$

which are $\left(\sqrt{\frac{r+r\cos\alpha}{2}}+i\sqrt{\frac{r-r\cos\alpha}{2}}\right),\ -\left(\sqrt{\frac{r+r\cos\alpha}{2}}+i\sqrt{\frac{r-r\cos\alpha}{2}}\right)$

On substitution for $p = r\cos\alpha$ we get them to be

$$\left(\sqrt{\frac{r+p}{2}}+i\sqrt{\frac{r-p}{2}}\right) \text{ and } -\left(\sqrt{\frac{r+p}{2}}+i\sqrt{\frac{r-p}{2}}\right).$$

However if we are required to find the square root of a complex number we can also do so without involving De-Moivre's theorem. This is explained by the following example.

Example 6 :

Obtain the square roots of $3 + 4i$.

Solution :

Let $x, y \in R$ such that $(x + iy) = \sqrt{3+4i}$. Then

$$x^2 - y^2 + i\,2xy = 3 + 4\,i$$

Equating the real and imaginary parts

$$x^2 - y^2 = 3 \qquad (6)$$

$$2xy = 4. \qquad (7)$$

Now $(x^2 + y^2)^2 = (x^2 - y^2)^2 + 4x^2y^2 = 3^2 + 4^2 = 25.$

But since $x^2 + y^2$ is non - negative, we have $x^2 + y^2 = 5.$

Hence $x^2 + y^2 = 5.$

Solving equations (6) and (8) we obtain (8)

$$\left.\begin{array}{ll} x^2 = 4 & ie.\ x = \pm 2 \\ y^2 = 1 & ie.\ y = \pm 1 \end{array}\right\}$$

From these possible values, we have to choose correct values. It follows from equation (7) that the product xy is a positive number. So both x and y must have the same sign. Thus if $x = 2$, then $y = 1$ and if $x = -2$, then $y = -1$.

Hence the square roots of $3 + 4i$ are $2 + i$ and $-2 - i$.

(iii) **Other uses of complex numbers in Geometry**

One easily sees that if z_1, z_2 are two complex numbers represented by the points P and Q then $\dfrac{nz_1 + mz_2}{m+n}$ is a complex number represented by the point on the line segment PQ which divides it in the ratio $m : n$.

This fact can be used to show that **medians of a triangle are concurrent;** so also the bisectors of the angles of triangle.

Again, when we discuss circles (chapter-12) and their equations you will get to know that for any complex number α and $r > 0$,

$$\{z : |z - \alpha| = r\}$$

is a circle of radius r and centre at the point α . (By point α, we mean the point which the complex number α represents)

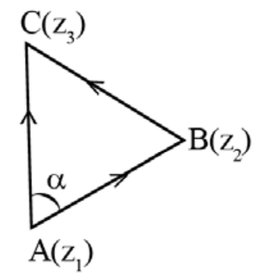
Also $\left\{ z : \left| \dfrac{z-a}{z-b} \right| = k \right\}$

too is a circle if $k \neq 1$; otherwise, it is a straight line. (That it is a line for k=1 will be evident after knowing general equation of a straight line in chapter-11)

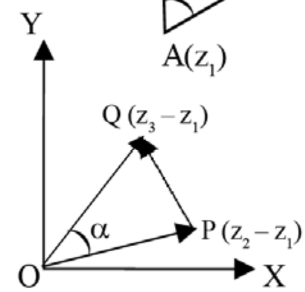
(iv) A useful result : If z_1, z_2, z_3 are three complex numbers represented by the vertices A, B, C, respectively of a triangle described in anticlockwise sense, then

$$\frac{-b-\sqrt{b^2-4ac}}{2a}(\cos\alpha + i\sin\alpha)$$

where $\alpha =$ m$\angle$CAB.



Proof : Let P and Q represent the numbers $z_2 - z_1$ and $z_3 - z_1$ respectively. Considering the fact that a complex number z can be represented by the vector joining the origin to the point z, it is easily seen that ΔOPQ and ΔABC are congruent.



(OP $=|\overrightarrow{OP}| = |z_2 - z_1| = |\overrightarrow{AB}| =$ AB. similarly for other sides)

Hence $\dfrac{AC}{AB} = \dfrac{OQ}{OP}$ and m$\angle$QOP = m$\angle$CAB

$$\therefore \left| \frac{z_3 - z_1}{z_2 - z_1} \right| = \frac{OQ}{OP} = \frac{AC}{AB}$$

and $\arg\left(\dfrac{z_3 - z_1}{z_2 - z_1}\right) =$ m$\angle$CAB $= \alpha$ (say)

$$\Rightarrow \quad \frac{z_3 - z_1}{z_2 - z_1} = \frac{AC}{AB}(\cos\alpha + i\sin\alpha)$$

It follows that the lines joining the points z_1 and z_2 and that joining z_3 and z_4 are perpendicular iff $\arg[(z_1 - z_2)/(z_3 - z_4)] = \pm\frac{\pi}{2}$ i.e. iff $(z_1 - z_2)/(z_3 - z_4)$ is purely imaginary.

6.7 Solution of Quadratic Equations

We mentioned earlier that our need for complex numbers arose while searching for a solution of as simple an equation as simple an equation as $x^2+1=0$. Now we know that this equation has two solutions $x = i, -i$.

Let us consider a quadratic polynomial p(x) $= ax^2+bx+c$; $a, b, c \in R$, $a \neq 0$. If for $\alpha \in R$, *p(α)=0 we say that α is a* root of p(x) which is same as saying α is a root of the equation p(x)=0.

We know that $\alpha_1 = \dfrac{-b+\sqrt{b^2-4ac}}{2a}$ and $\alpha_2 = \dfrac{-b-\sqrt{b^2-4ac}}{2a}$ are the real roots of the quation p(x) = 0 if b^2-4$ac \geq 0$.

But If b²-4ac < 0, we need to search for roots in $\mathbb{C}$. But does p(x) = 0 have a root at all in $\mathbb{C}$? The answer is provided by the **fndamental theorem of algebra** which we state without proof:

'Every polynomial equation of degree ≥ 1 has at least one root in $\mathbb{C}$'.

More specifically we have :

'Every polynomical equation of degree n has n roots in $\mathbb{C}$'. So now the equation $ax^2+bx+c=0$, $a,b,c \in \mathbb{C}$, $a \neq 0$ has two roots

$$\alpha_1 = \frac{-b+\sqrt{b^2-4ac}}{2a}, \alpha_2 = \frac{-b-\sqrt{b^2-4ac}}{2a}$$

However, in solution of quadratic equations, we shall take a, b, c as real numbers.

Example : 7

Find all the roots of $4x^2+8x+13=0$.

Solution

The two roots are given by

$$\alpha_1 = -1+\frac{3}{2}\boldsymbol{i} \text{ and } -1-\frac{3}{2}\boldsymbol{i},\ \alpha_2 = \frac{-8-\sqrt{8^2-4\times4\times13}}{2\times4}$$

Simplifying the discriminant $\sqrt{8^2-4\times4\times13} = \sqrt{-144} = 12\boldsymbol{i}$

Hence the roots are $-1 + \dfrac{3}{2}\boldsymbol{i}$ and $-1 - \dfrac{3}{2}\boldsymbol{i}$, after simplification.

N.B. Observe that in a quadratic equation $ax^2 + bx+c = 0$ with real coefficients $a,b,c,$ complex roots occur in conjugate pairs.

For example if a quadratic equation $ax^2 + bx+c = 0$ has 3+4i as a root then the other root must be 3-4i .

Example- 8

One of the roots of $x^3-x^2+x-1=0$ is i. So that this equation has a exactly one real root.

Solution :

Since i or 0+i is a complex root the other complex root must be its conjugate pair 0-i. By the fundamental theorem of algebra, this equation, being of degree = 3 must have exactly 3 roots. Two of the roots are 0+i and 0-i. So the other root must be real, because if it is complex, its conjugate pair must also be a root. Therefore, the third root has to be real number.

EXERCISES 6 (b)

1. If z_1 and z_2 are two complex number then show that

   $|1 - z_1\overline{z_2}|^2 - |z_1 - z_2|^2 = (1 - |z_1|^2)(1 - |z_2|^2)$.

2. If a, b, c are complex numbers satisfying $a + b + c = 0$ and $a^2 + b^2 + c^2 = 0$ then show tha $|a| = |b| = |c|$.

3. What do the following represent ?

   (i) $\{z : |z - a| + |z + a| = 2c\}$ where $|a| < c$

   (ii) $\{z : |z - a| - |z + a| = c\}$

   (iii) What happens in (i) if $|a| \geq c$?

4. Given $\cos\alpha + \cos\beta + \cos\gamma = \sin\alpha + \sin\beta + \sin\gamma = 0$,

   show that $\cos 3\alpha + \cos 3\beta + \cos 3\gamma = 3\cos(\alpha + \beta + \gamma)$.

5. **Binomial Theorem for complex Numbers**

   Show that $(a + b)^n = a^n + {}^nC_1\, a^{n-1} b + \ldots\ldots + {}^nC_r\, a^{n-r} b^r + \ldots\ldots + b^n$ where $a, b \in C$ and n a positive integer (use induction on n, rule of multiplication of complex numbers and the relation ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$).

6. Use Binomial theorem and De Moiver's theorem to show

   $$\cos 3\theta = 4\cos^3\theta - 3\cos\theta, \quad \sin 3\theta = 3\sin\theta - 4\sin^3\theta.$$

   Express $\cos n\theta$ as sum of product of powers of $\sin\theta$ and $\cos\theta$. Do the same thing for $\sin n\theta$.

7. Find square roots of

   (i) $-5 + 12\sqrt{-1}$ (ii) $-11 - 60\sqrt{-1}$

   (iii) $-47 + 8\sqrt{-1}$ (iv) $-8 + \sqrt{-1}$

   (v) $a^2 - 1 + 2a\sqrt{-1}$ (vi) $4ab - 2(a^2 - b^2)\sqrt{-1}$.

8. Find the values of cos 72°, ...

Solution : Note that $72° = \frac{2\pi}{5}$ radians. Let $\alpha = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$. Thus α is one of the roots of the equation $x^5 = 1$ whose roots are,

$$\cos\frac{2k\pi}{5} + i\sin\frac{2k\pi}{5},\ k = 0, 1, 2, 3, 4.$$

But $x^5 - 1 = (x - 1)(x^4 + x^3 + x^2 + x + 1)$. So $x^5 - 1 = 0$ has $x = 1$ as one of the roots which corresponds to $k = 0$. So all other roots must be roots of the equation $x^4 + x^3 + x^2 + x + 1 = 0$. Let us try to solve the above equation. Since $x \neq 0$ we might divide both sides by x^2 to get

$$x^2 + x + 1 + \frac{1}{x} + \frac{1}{x^2} = 0$$

or $x^2 + \frac{1}{x^2} + x + \frac{1}{x} + 1 = 0$ (1)

Now if $x + \frac{1}{x} = 2y$, then $4y^2 = x^2 + \frac{1}{x^2} + 2$

So we can rewrite equation (1) as

$$4y^2 + 2y - 1 = 0 \tag{2}$$

which is a quadratic equation admitting solutions :

$$y = \frac{-2 \pm \sqrt{20}}{8} = \frac{-1 \pm \sqrt{5}}{4}.$$

But what is y ? If we write

$$x_k = \cos\frac{2k\pi}{5} + i\sin\frac{2k\pi}{5}$$

we get all complex roots as x_1, x_2, x_3, x_4.

But $x_1 = \overline{x_4}$ and $= x_3 = \overline{x_2}$. So y can be either Re x_1 or Re x_3.

It is easily seen that Re α = Re $x_1 > 0$ and Re $x_3 < 0$. In other words we have

$\cos 72° = \frac{-1+\sqrt{5}}{4}$ and $\cos 216° = \frac{-1-\sqrt{5}}{4}$.

9. Find the value of $\cos 36°$

10. Evaluate $\cos\frac{2\pi}{17}$ using the eqation $x^{17} - 1 = 0$.

11. Solve the equations

(i) $z^7 = 1$, (ii) $z^3 = i$, (iii) $z^6 = -i$, (iv) $z^3 = 1 + i$.

12. If $\sin\alpha + \sin\beta + \sin\gamma = \cos\alpha + \cos\beta + \cos\gamma = 0$, show that

(i) $\sin 3\alpha + \sin 3\beta + \sin 3\gamma = 3\sin(\alpha + \beta + \gamma)$

(ii) $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = \cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{3}{2}$.

13. If $x + \frac{1}{x} = 2\cos\theta$, show that $x^n + \frac{1}{x^n} = 2\cos n\theta$.

14. If $x_r = \cos\alpha_r + i\sin\alpha_r$, $r = 1, 2, 3$, and $x_1 + x_2 + x_3 = 0$, show that $\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} = 0$.

15. Show that $\left(\frac{1 + \sin\theta + i\cos\theta}{1 + \sin\theta - i\,\mathrm{cis}\,\theta}\right)^n = \cos\left(\frac{n\pi}{2} - n\theta\right) + i\sin\left(\frac{n\pi}{2} - n\theta\right)$.

16. If α and β are roots of $x^2 - 2x + 4 = 0$, then show that

$\alpha^n + \beta^n = 2^{n+1}\cos\frac{n\pi}{3}$.

17. For a positive integer n show that

(i) $(1 + i)^n + (1 - i)^n = 2^{\frac{n+2}{2}}\cos\frac{n\pi}{4}$

(ii) $(1+i\sqrt{3})^n + (1-i\sqrt{3})^n = 2^{n+1}\cos\frac{n\pi}{3}$.

18. Let $x + \frac{1}{x} = 2\cos\alpha$, $y + \frac{1}{y} = 2\cos\beta$, $z + \frac{1}{z} = 2\cos\gamma$. Show that

(i) $2\cos(\alpha+\beta+\gamma) = xyz + \frac{1}{xyz}$

(ii) $2\cos(p\alpha + q\beta + r\gamma) = x^p y^q z^r + \frac{1}{x^p y^q z^r}$.

19. Solve $x^9 + x^5 - x^4 = 1$.

20. Find the general value of θ if

$(\cos\theta + i\sin\theta)(\cos 2\theta + i\sin 2\theta)\ .....\ (\cos n\theta + i\sin n\theta) = 1$.

21. If $z = x + iy$ show that $|x| + |y| \le \sqrt{2}\,|z|$.

22. Show that

$\text{Re}\,(z_1 z_2) = \text{Re}\,z_1\,\text{Re}\,z_2 - \text{Im}z_1\text{Im}z_2$

$\text{Im}\,(z_1 z_2) = \text{Re}\,z_1\,\text{Re}\,z_2 + \text{Im}z_2\text{Im}z_1$

23. What is the value of $\arg\omega + \arg\omega^2$?

24. If $|z_1| \le 1$, $|z_2| \le 1$, show that

$|1 - z_1\overline{z_2}|^2 - |z_1 - z_2|^2 = (1 - |z_1|^2)(1 - |z_2|^2)$

Hence or otherwise show that

$\left|\frac{z_1 - z_2}{1 - z_1\overline{z_2}}\right| < 1$ if $|z_1| < 1$, $|z_2| < 1$.

25. If $z_1^2 + z_2^2 + z_3^2 - z_1z_2 - z_2z_3 - z_3z_1 = 0$, show that

$|z_1 - z_2| = |z_2 - z_3| = |z_3 - z_1|$.

26. If $|a| < |c|$ show that there are complex numbers z satisfying $|z - a| + |z + a| = 2|c|$.

27. Solve $\frac{(1-i)x + 3i}{2+i} + \frac{(3+2i)y + i}{2-i} = -i$ where $x, y, \in R$.

28. If $(1 + x + x^2)^n = p_0 + p_1 x + p_2 x^2 + ... + p_{2n} x^{2n}$, then prove that $p_0 + p_3 + p_6 + ... = 3^{n-1}$.

29. Find the region on the Argand plane on which z satisfies

(i) $1 < |z - 2i| < 3$

(ii) $\arg\left(\frac{z}{z+i}\right) = \frac{\pi}{2}$ [Hint Arg $(x + iy) = \frac{\pi}{2} \Rightarrow x = 0, y > 0$]

CHAPTER 7

Linear Inequalities

In most sciences one generation tears down what another has built and what one has established another undoes. In mathematics alone each generation builds a new story to the old structure.

- Hankel

7.0 Introduction :

You apply algebra to solve apparently difficult problems of arithmatic. The unknown value or values to be determined are taken as x or x, y both and a single or a pair of equations are framed using the given conditions and the solution of the problem reduces to solving some equations.

But in some cases we do not get equations out of the given conditions, rather we get inequations or inequalities.

Take an example :

Example -1

A person's BMI (Body Mass Index) is calculated as $\frac{w}{h^2}$; where w is weight in kilograms and h is height in meter.

BMI has the following implications related to health :

(i) BMI < 20 (Some also keep this figure at 18.5) $\Rightarrow$ under weight

(ii) 20 < BMI < 25 $\Rightarrow$ Normal

(iii) 25 < BMI < 30 $\Rightarrow$ Over weight

(iv) BMI > 30 Obese (fat).

Let us now determine the range of weight of a person of height 1.6 meter so that his BMI is within the normal range, i.e. he is neither underweight nor overweight or fat.

To answer the question, we have to determine the range for the weight of the person so that his BMI is normal, which means

$$20 < \frac{w}{h^2} < 25$$

Note that here we do not get any equations in the unknown 'w'.

Putting h = 1.6 we get

$20 \times 1.6^2 < w < 25 \times 1.6^2$

or $51.2 < w < 64$

Thus we may say that the body mass index of the person will be in the normal range if he maintains his weight within 51.2 kg to 64 kg, which means that his weight should not be less than 51.2 kg and more than 64 kg.

Mark that the BMI under the conditions, normal and over weight has been expressed in terms of combination of inequalities or inequations of first degree in the variable w (the index of w here is 1).

Study of real life problems in several diverse areas as economics, finance, Optimization problems involves inequalities. Inequalities are also extremely useful in expressing and building up concepts of pure mathematics and theoretical sciences.

Definition:

Inequalities involving variable or variables in first degree are called linear inequalities.

Example - 2

(i) $2x + 3 > 0$

(ii) $-3x + 5 \leq 7$

(It is a combination of an inequality $-3x + 5 < 7$ and an equality $-3x + 5 = 7$)

(iii) $x \geq 1$

(iv) $y < -2$

(v) $3x + 4y < 7$

(vi) $x + 2 < 2x + 3 \leq x + 5$

(vii) $ax + by \leq c$

(viii) $ax + by \geq c$

These are all examples of linear inequalities in one or two variables.

Inequalities of the type $ax^2+bx+c \geq 0$ or $ax^2+bx+c \leq c$ $(a \neq 0)$ are examples of quadratic inequalities.

In this chapter we shall deal with linear inequalities with one or two variables or unknowns.

It may be noted that an inequality with '$\leq$' sign (less than or equal) or '$\geq$' sign (greater than or equal) is called a **slack inequality**. An inequality with '<' or '>' sign is called strict inequality.

7.1 Soultion of a linear inequality

Definition :

The set of values of the variables which satisfy the inequality is called solution of the inequality. (Usually a solution of an inequality is expressed in the form of a smpler inequality or a set of numbers.)

The following facts which are derivable from the properties of the real number system are helpful in solution of inequalities.

(i) Addition or subtraction of the same number does not affect an nequality in the sense that the greater or smaller side remain as before.

Example : $x < y \Leftrightarrow x + r < y + r,$ for any $r \in$ R.

(ii) Multiplication or division by a possitive number does not affect an inequality.

$a < b \Leftrightarrow b - a > 0 \Leftrightarrow \alpha\,(b\text{-}a) > 0,\ \alpha > 0$

$\Leftrightarrow \alpha b - \alpha a > 0 \Leftrightarrow \alpha\, b > \alpha\, a$

$or\ \ \alpha a < \alpha b.$

Similarly the case of division can be explained; as $\alpha > 0$ implies $\dfrac{1}{\alpha} > 0$ (by properties of R)

(iii) Multiplication or division by a negative number reverses an inequality.

$a < b \Leftrightarrow b - a > 0 \Leftrightarrow \alpha\,(b - a) < 0,\ for\ \alpha < 0$

(because product of a negative and a positive number is negative; by properties of R)

$\Leftrightarrow \alpha b - \alpha a < 0 \Leftrightarrow \alpha b < \alpha a$

or $\alpha a > \alpha b$

(The inequality is reversed)

Similarly the case of division can be explained; as $\alpha < 0$ implies $\dfrac{1}{\alpha} < 0$

(reciprocal of a negative number is also negative).

(iv) Taking reciprocals, whenever they exist, reverses an inequality, e.g.

$3 < 5 \Rightarrow \dfrac{1}{3} > \dfrac{1}{5}$

(v) Multiplication by 0, coverts an inequality into equality as both sides become 0.

Now let us come to solving inequalities.

N.B.: (i) Whenever we say that a number is positive or negative, it is necessarily a real number. A complex number cannot be positive or negative.

(ii) If $a, b \in R$ exactly one of the following is true; $a = b$, $a<b$ or $a>b$. But for complex numbers we have only $a=b$ or $a \neq b$; there is no inequality.

Example-3

Solve $20x < 500$ in

(i) positive integers

(ii) integers

(iii) real numbers

along with representation on the number line.

Solution:

(i) In positive integers, solution - set is given by

$S = \{n \mid 1 \leq n < 25\}$ which includes 24 solutions

$n = 1, 2, 3, ..., 24$

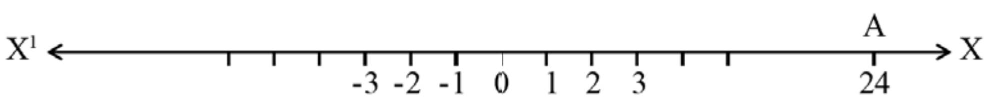
(ii) In integers, solution set $S = \{k \mid k \in Z \;\&\; k < 25\}$

or $S = \{ ..., -3, -2, -1, 0, 1, 2, ..., 24\}$ which is an infinite set of solutions:

(iii) In real numbers, solution set $S = \{ x \mid x \in R, x < 25\}$

or $(-\infty, 25)$ which is an infinite set.

Representation on the number line



(i) Solution in positive integers are shown by points marked

1, 2, 3, ..., 24

(ii) Integral solutions are represented by points marked

..., -3, -2, -1, 0, 1, 2, ..., 24

(iii) Solution in R is represented by all the points on the ray $\overrightarrow{AX'}$

Example-4

Solve

(i) $x+2 < 2x+3 \leq x+5$

(ii) $\dfrac{5x+17}{8} \geq x - \dfrac{1}{2}$

Represent the solution-sets on the number line.

Solution:

(i) $x + 2 < 2x + 3 \leq x+5$

or $2 < x + 3 \leq 5$ (Subtracting x)

or $-1 < x \leq 2$ (Subtracting 3)

$\therefore$ Solution-set $S = \{x \in R \mid -1 < x \leq 2\}$

or $(-1, 2]$

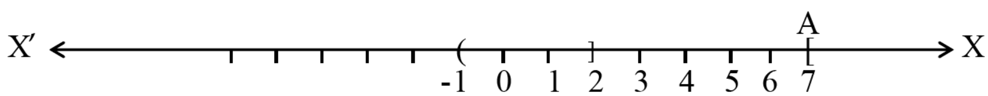
(ii) Multiplying by 8,

$5x + 17 \leq 8x - 4$ or $21 \leq 3x$ or $x \geq 7$

or $S = \{x \in R \mid x \geq 7\}$ or $S = [7, \infty)$

Any one of these gives the solution.

Representation on the number line :



The values of x, in case of (i) are all the real numbers in between -1 and 2 including 2, i.e. the semiclosed or semi open interval $(-1, 2]$ (open at -1 and closed at 2).

Similarly, in case of (ii) , the solution set is the ray $\overrightarrow{AX}$, i.e. all the real numbers on it; where A represents the number 7.

Example-5

A person purchased four items A, B, C and D for rupees 1000, 950, 800 and 750 and sold the first three items at rupees 1200, 1140 and 944. Find out, what should be the minimum selling price of the fourth item. if the person wants at least 18% profit on his total investment.

Solution:

Total purchase price in rupees

$= 1000+950+800+750=3500$

Minimum selling price in rupees at a profit of at least 18%

$$= 3500\left(1+\frac{18}{100}\right)=4130$$

Taking x to be the least selling price of item D,

$1200+1140+944+x \geq 4130$

$\Rightarrow x+3284 \geq 4130$

$\Rightarrow x \geq 846$

$\therefore$ Minimum selling price of item D is 846 rupees.

Example-6

Find all pairs of natural numbers differing by 3, both of which are greater than 2, such that their sum is less than 28.

Solution:

The solution-set is $\{(x, x+3) | x \in \mathrm{N}, x > 2, 2x+3 < 28\}$

Now,

$2x+3 < 28 \Rightarrow x < \dfrac{25}{2} \Rightarrow x \leq 12 \;(\because x \in \mathrm{N})$

$2 < x \leq 12 \Rightarrow 3 \leq x \leq 12$, so the required pairs are (3,6), (6,9), (9,12) and (12,15).

Example-7

Solve $| x-a | < \delta$ (pronounced as delta) ; $x, a \in \mathrm{R}$.

Solution:

Recall the definition of modulus function in Chapter - 3.

By definition,

(i) for $x \geq a$, $| x - a | = x - a$

$\therefore | x - a | < \delta \Leftrightarrow x - a < \delta \Leftrightarrow x < a + \delta$

Combining with $x \geq a$, we get $a \leq x < a + \delta$ ... ... (1)

(ii) for $x < a$, $| x - a | = -(x - a) = a - x$

$\therefore | x - a | < \delta \Leftrightarrow a - x < \delta \Leftrightarrow a - \delta < x$

Combining with $x < a$, we get $a - \delta < x < a$ ... ... (2)

Taking into account the solution sets in (1) and (2), we get

$a - \delta < x < a + \delta$, which includes values of x satisfying both $x \geq a$ and $x < a$.

Thus, $| x - a | < \delta \Leftrightarrow a - \delta < x < a + \delta$

This is an extremely important result which has applications in subsequent chapters and further.

Exercises - 7 (a)

1. Determine whether the solution-set is finite or infinite or empty :

 (i) $x < 1000$, $x \in N$

 (ii) $x < 1$, $x \in Z$ (set of integers)

 (iii) $x < 2$, x is a positive integer

 (iv) $x < 1$, x is *a* positive integer

2. Solve as directed :

 (i) $5x \leq 20$ in positive integers, in integers.

 (ii) $2x + 3 > 15$ in integers, in natural numbers.

 Do you mark any difference in the solution-sets ?

 (iii) $5x + 7 < 32$ in integers, in nonnegative integers.

 (iv) $-3x - 8 > 19$, in integers, in real numbers.

 (v) $|x - 3| < 11$, in N and in R.

3. Solve as directed :

 (i) $2x + 3 > x - 7$ in R (ii) $\frac{x}{2}+\frac{7}{3} < 3x-1$ in R

 (iii) $\frac{x}{2}-\frac{x}{3}+\frac{x}{5}\leq\frac{11}{3}$ for nonnegative real numbers.

 (iv) $2(3x-1) < 7x+1 < 3(2x+1)$ for real values.

 (v) $7(x-3) \leq 4(x+6)$, for nonnegative integral values.

 (vi) Convert to linear inequality and solve for natural numbers :

 $(x - 2)(x - 3) < (x + 3)(x - 1)$.

 (vii) Solve in R , $\frac{x}{2}+1\leq 2x-5 < x$.

 Also find its solution in N.

 (viii) Solve in R and also in Z :

$$\frac{3x+1}{5} \geq \frac{x+2}{3} - \frac{5-3x}{5}$$

4. Solve $|x - 1| > 1$ and represent the solution on the number line.

   [Exhaustive hints : By definition of modulus function

   For $x - 1 \geq 0$ or $x \geq 1$, $|x - 1| > 1 \Leftrightarrow x - 1 > 1 \Leftrightarrow x > 2 \Leftrightarrow x \in (2, \infty)$

   For $x - 1 < 0$ or $x < 1$, $|x - 1| > 1 \Leftrightarrow -(x - 1) > 1$

   $\Leftrightarrow x - 1 < -1$ (multiplication by - 1 reverses the inequality)

   $\Leftrightarrow x < 0 \Leftrightarrow x \in (-\infty, 0)$

   $\therefore$ Solution set is the union, $(-\infty, 0) \cup (2, \infty)$.

   Show this as two disjoint open intervals on the number line, i.e., real line.]

5. Solve in R and represent the solution on the number line.

   (i) $|x - 5| < 1$ (ii) $\frac{x}{5} < \frac{2x+1}{3} + \frac{1-3x}{6}$

   (iii) $2x + 1 \geq 0$ (iv) $\frac{x-1}{2} \leq \frac{x+1}{3} < \frac{3x-1}{6}$

6. In a triangle ABC; AB, BC and CA are x , $3x + 2$ and $x + 4$ units respectively where $x \in$ N. Find the lengths of its sides.

   (Hint : Apply triangle - inequality)

7. The length of one side of a paralleogram is 1 c.m. shorter than that of its adjacent side. If its perimeter is at least 26 c.m., find the minimum possible lengths of its sides.

8. The length of the largest side of a quadrilateral is three times that of its smallest side. Out of the other two sides, length of one is tweice that of the smallest and the other is 1 c.m. longer than the smallest. If the perimeter of the quadrilateral is at most 36 c.m., then find the maximum possible lengths of its sides.

9. Find all pairs of consecutive odd numbers each greater than 20, such that their sum is less than 60.

10. Find all pairs of even numbers each less than 35, such that their sum is at least 50.

7.2 Graphical solution of linear inequalities in two variables

As you know, linear inequalities in two variables are presented in the form $ax + by + c \leq 0$ or $ax + by + c \geq 0$, which may also be strict.

Consider, $2x + 3y < 5$

According to definition, the solutions are those values of x and y which satisfy $2x + 3y < 5$.

Taking a particular value of x, i.e. $x = 1$, we get $3y < 3$ or $y < 1$. Thus the real values of y corresponding to $x = 1$, belong to the set $\{y \mid y \in R, y < 1\} = (-\infty, 1)$.

If we consider all possible real values of x, then we have no other way than describing the solution-set as $S=\{(x,y) \mid 2x+3y < 5\}$, which is just a restatement of the inequality.

Ofcourse, we can pick up as many solutions as we like by taking particular values of x or y, as we have already done, taking $x = 1$.

Thus, we are convinced that $2x+3y < 5$ has infinitely many solutions for $x, y \in R$.

Now the question arises : how to graph these solutions or, for that matter, how to visualise these solutions on the Cartesian plane, i.e. the plane of x and y - axes ?

From highschool geometry you know that $2x+3y-5=0$ or, in general, $ax + by + c = 0$ represents a line on the plane. In other words the coordinates (x,y) of all the points on that line satisfy $ax + by + c = 0$ and conversely any point $P(x,y)$ whose coordinates x and y satisfy $ax+by+c=0$ lies on this line.

What about the points whose coordinates do not satisfy $ax+by+c= 0$? Obviously those points lie off the line, i.e. not on the line $ax + by + c = 0$. So for all such points $P(x,y)$, we have either $ax + by + c > 0$ or $ax + by + c < 0$.

Thus, the values of x and y satisfying $2x+3y < 5$ or $2x+3y -5 < 0$, are the coordinates of points $P(x,y)$ lying outside the line $2x + 3y - 5= 0$.

We shall now make use of a very fascinating property of a line on a plane which will be proved in chapter-11 (Section 11.4) while discussing 'position of a point with respect to a line', which states :

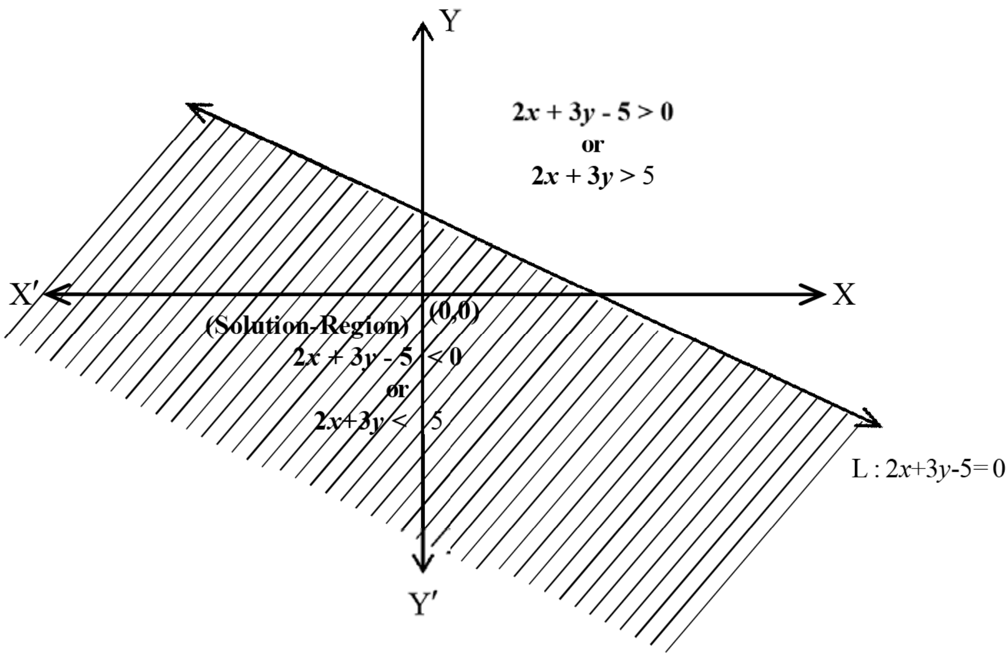
Two points $P(x_1,y_1)$ and $Q(x_2, y_2)$ lie on the same or opposite sides of a line $ax+by+c=0$ according as ax_1+by_1+c and ax_2+by_2+c have same or opposite signs.

This solves our problem !

Putting $x = 0, y = 0$ in $2x+3y-5$ we get $2x + 3y - 5 = -5 < 0$. Therefore all the points (x, y) for which $2x + 3y - 5 < 0$ (or $2x + 3y <5$) lie on the same side of the line where the origin lies.

This side of the line $2x + 3y - 5= 0$ is called the Solution-Region (SR) of $2x+3y < 5$.

This fact is diagramatically, represented below:



(Solution-Region of $2x + 3y < 5$)

N.B.

(1) If you remember the 'plane-separation postulate' discussed in chapter-4 (Act 4.01) you can easily notice that the solution-region is a half plane, with the line $2x + 3y - 5= 0$ as its edge.

(2) Since the inequality is strict, i.e. $2x + 3y < 5$, the points on the line do not comprise the solution.

(3) The graph of the solution of $2x + 3y < 5$ is the solution-region which has been shaded.

Corollary : The graph of the solution of $2x + 3y > 5$ is the other half plane of $2x + 3y - 5=0$ which does not contain the origin. This region has not been shaded.

(4) In case of a slack-inequality like $ax+by+c \leq 0$ or $ax+by+c \geq 0$ the solution-region includes the line $ax+by+c=0$.

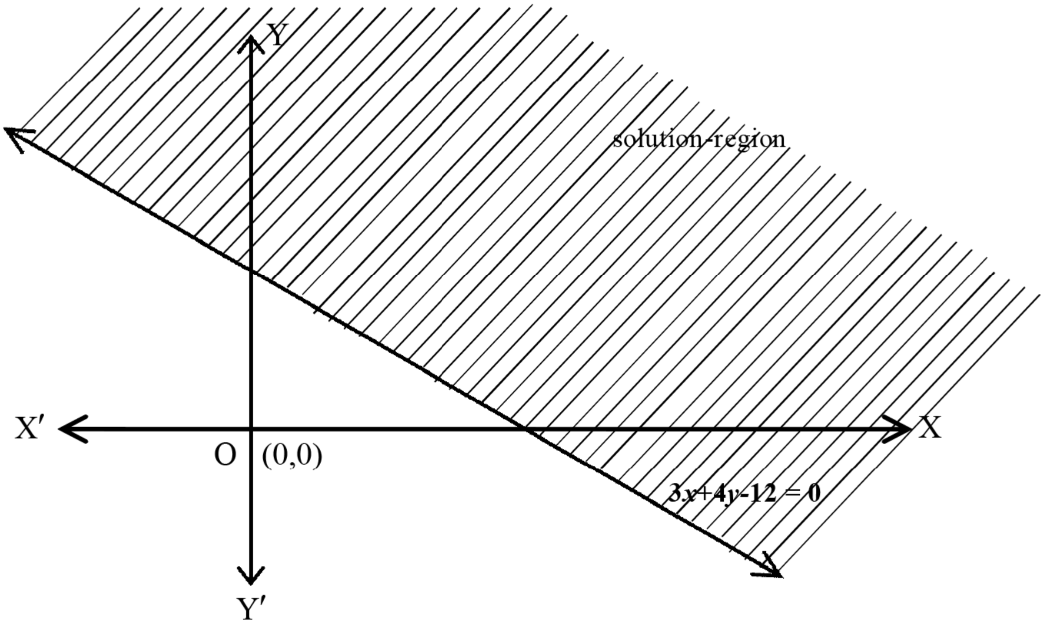
Example-8

Solve graphically $3x+4y \geq 12$.

Solution:

Putting $x = 0, y = 0$

We see that $3x + 4y - 12 < 0$, i.e. $3x+y < 12$

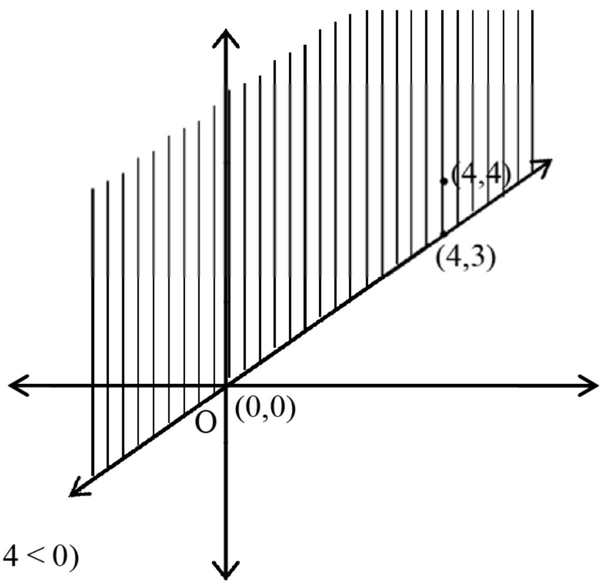


$\therefore$ Solution region of $3x + 4y \geq 12$ is the other side of the line not containing the origin. Since the inequality is slack, the paints satisfying $3x + 4y = 12$, i.e. the line itself is included in the solution region which is shaded.

Example-9

Solve graphically $3x < 4y$

Solution:



The line $3x - 4y= 0$ is drawn.

(Just by taking two points whose coordinate satisfy the equation of a line, you can draw its graph, as you do in high school)

Now, we have to locate a point whose coordinate satisfy $3x - 4y < 0$.

(4,4) is such a point. (Notice that $3 \times 4 - 4 \times 4 < 0$)

So all point satisfying $3x - 4y < 0$ or $3x < 4y$ lie on the half-plane of $3x - 4y = 0$ containing (4,4). Solution region has been shaded.

Exercises - 7(b)

Solve graphically :

1. $x < y$
2. $3x+4y \geq 12$
3. $x - y > 0$
4. $x + 2y -5 \leq 0$
5. $7x - 4y < 14$
6. $x + 8y+10 > 0$

7. $5x + 6y < 12$ 10. $x + y \geq 1$

8. $-3x + y > 0$ 11. $x \leq 0$

9. $3x + 8y > 24$ 12. $y > 5$

7.3 Graphical solution of system of linear inequalities in two variables

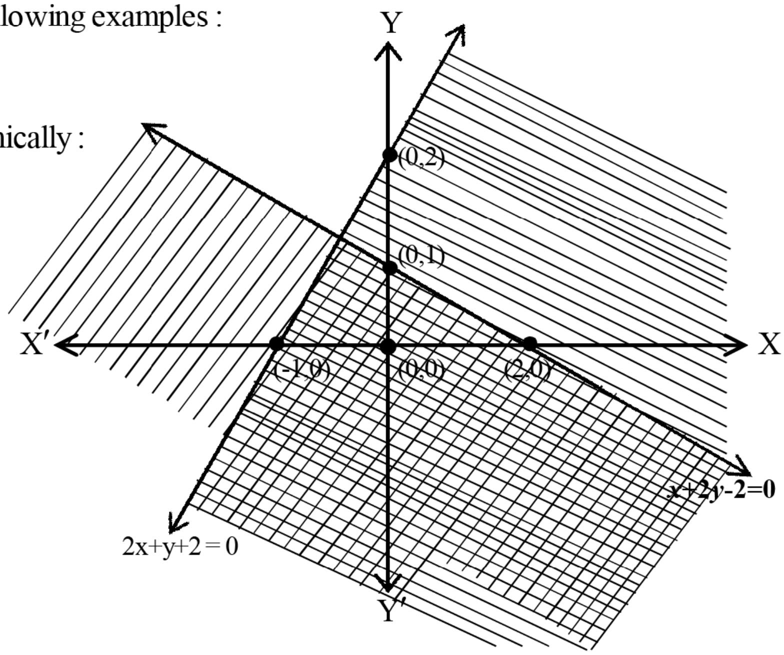
This is explained throught the following examples :

Example-10

Solve the following system graphically :

$x + 2y < 2$

$2x - y + 2 \geq 0$



Solution:

The lines

$x + 2y - 2 = 0$

and $2x - y + 2 = 0$ are drawn.

(0, 0) satisfies

$x + 2y - 2 < 0$, i.e. $x + 2y < 2$.

$\therefore$ Solution - region of $x + 2y < 2$ is the shaded half plane of $x + 2y - 2 = 0$ containing the origin.

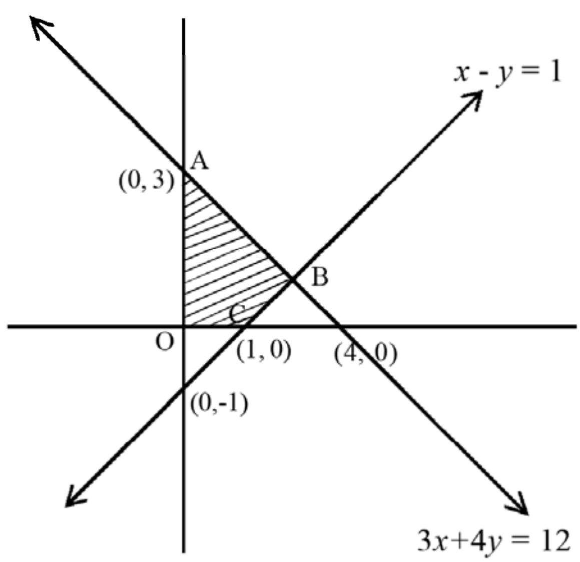
Also (0,0) satisfies $2x - y + 2 > 0$, So solution region of $2x - y + 2 > 0$ is the shaded half-plane of $2x - y + 2 = 0$, containing origin and also including the line $2x - y + 2 = 0$.

The common double shaded portion is the solution-region of the system.

Example-11

Solve graphically the system :

$3x + 4y < 12$, $x - y < 1$, $x \geq 0$, $y \geq 0$.



Solution:

The solution regions of $3x + 4y < 12$ and $x - y < 1$ are the respective sides of the lines $3x + 4y - 12 = 0$, $x - y - 1 = 0$ containing the origin.

Solution regions of $x \geq 0$ and $y \geq 0$ are the right half-plane of y-axis and the upper half-plane of x-axis.

So the solution-region of the system is the shaded portion bounded by the polygon OABC including the sides $\overline{OA}$ and $\overline{OC}$, but excluding $\overline{AB}$ and $\overline{BC}$, as shown in figure.

Exercises-7 (c)

Solve the following systems of linear inequalities graphically :

1. $2x - y \geq 0$, $x - 2y \leq 0$, $x \leq 2$, $y \leq 2$.

   (Hint : you may consider the point (2,2) to determine SR of the first two inequalities)

2. $x - y < 1$, $y - x < 1$
3. $x - 2y + 2 < 0$, $x > 0$
4. $x - y + 1 \geq 0$, $3x + 4y \leq 12$, $x \geq 0$, $y \geq 0$
5. $x+y > 1$, $3x - y < 3$, $x - 3y + 3 > 0$
6. $x > y$, $x < 1$, $y > 0$
7. $x < y$, $x > 0$, $y < 1$

●

CHAPTER 8

Permutations and Combinations

A lightening clears the air of impalpable vapours, so an incisive paradox frees the human intelligence from the lethargic influence of latent and unsuspected assumptions. Paradox is the slayer of prejudice.

- Sylvester

8.0 Historical Background

In one form or the other counting originated with the primitive man.. As long as it was within the limits of his fingers and toes he did not find any difficulty. But as soon as it crossed the limits he found out other devices and at later stage invented the numbers. Even with numbers when counting became a laborious and time taking affair, and it was practically impossible to count, he invented new concepts and ideas which gave rise to permutations and combinations. This idea was known to the people during the Vedic period. During the period of Jainism, that is around 6th century B.C., certain formulae on permutations and combinations were known to the Indians. It was also known to Chinese around 3rd century B.C. However the first book on permutation and combination appeared during the 17th century written by Jacob Bernoulli (1654 – 1704 A.D.)

8.1 How to count without counting

We begin with the following problem. Suppose that we have four vegetables : potato, tomato, brinjal and cauliflower. We are to cook a curry choosing any two of these vegetables. If the order in which the vegetables are chosen does not matter, then how many different curries can be cooked ignoring the way of cooking or spices used ? Let us count the possibilities {potato, tomato}, {potato, brinjal}, {potato, cauliflower}, {tomato, brinjal}, {tomato, cauliflower}, {brinjal, cauliflower}, these being the only possible combinations of vegetables making a total of six different curries. If the number of vegetables were more or number of vegetables allowed in a curry were different, then it would be cumbersome to enumerate all possible combinations and then count all the possibilities.

We illustrate another idea by another example of this kind. Suppose that we have four distinguishable balls and six numbered boxes. If we are to distribute the balls into the boxes, how many different ways of distribution can be there ? Suppose that the set of balls is $\{r, b, g, w\}$ and the boxes are numbered 1, 2, 3, 4, 5, 6. In one arrangement, r may be placed in 1, b in 2, g in 3, w in 4. But this is certainly different from the distribution in which r is put in 2, b is put in 1, g is put in 4 and w is put in 3. Look at the picture below.

1	2	3	4	5	6
r	b	g	w		

1	2	3	4	5	6
b	r	w	g		

The reader is encouraged to try a few more arrangements. Do not attempt to write all the possible arrangements with pictures as above. There will be as many as 1296 such arrangements ! (How we arrived at this figure shall be explained later). Another similar example is to find the number of three digit positive integers with only two digits, namely, 1 and 0. The reader sees here that 100, 101, 110 are a few of those numbers. We furnish a few more examples for the reader to think about :

Example 1 :

Suppose A is a set of n elements and B is a set with m elements. How many functions can there be from A to B ?

Example 2 :

How many of the functions in Example 1 are one - to - one, with (i) $m = n$, (ii) $m < n$, (iii) $m > n$?

Example 3 :

Suppose I have to distribute 20 sweets, all of same kind, among five children. If every child is to receive at least one sweet, in how many ways can I distribute the sweets if I don't care for justice or fairplay ?

Example 4 :

How many words can be formed using five given letters of an alphabet whose length does not exceed 6 if we are not to bother whether any juxtaposition of the letters makes a meaningful word in the vocabulary or not ?

Example 5 :

Suppose that I have to travel from Delhi to Calcutta and back in such a way that I can take a train while travelling from Delhi to Calcutta whereas I am only to take a flight by planes from Calcutta to Delhi. If there are 10 different trains leaving Delhi for Calcutta and 5 flights from Calcutta to Delhi, in how many ways can I perform a round trip ?

Example 6 :

If A is a set of n elements, how many bijections are there from A to itself ?

We see in the examples above that sometimes the order in which we choose things is important and sometimes not. For example, if we are to make a word of length 3 using the letters t, a and b then 'bat' is not the same word as 'tab' whereas in using vegetable for a curry the order in which we choose the vegetables is usually immaterial. The reader is to keep in mind the difference in these situations.

The reader must have seen that in choosing possibilities we are putting together objects or elements, a process which is called in ordinary parlance, **combination**. But we shall use the word **combination** in a specific sense to distinguish it from the case where the order in which they are chosen is important too. These would be called **permutations**. But before we formally define any of these terminologies, let us discuss Example 5. When it is the case of a trip from Delhi to Calcutta and back let t_1, t_2, t_3 ..... t_{10} be the trains that run between Delhi and Calcutta in a day and let $f_1, f_2, f_3, f_4\, f_5$ be the flights from Calcutta to Delhi. The possibilities of round trips are (t_1,f_1), (t_1,f_2) ......, (t_1,f_5),...., (t_{10},f_5) where (t_1,f_1) means that the trip consists of going from Delhi to Calcutta by t_1 and returning from Calcutta to Delhi by the flight f_1. In general (t_1, f_1) would mean the trip in which one travels from Delhi to Calcutta by the train t_1 and returns by the flight f_1. We see clearly that there are exactly 10 such possibilities for every travel from Delhi to Calcutta by train and there are axactly 5 different ways of return. So, in all, there can be $10 \times 5 = 50$ ways of round trips. We can now state the following in general.

A. Counting Principle :

If we are to choose one element from a set A with *n* elements first and then one element from a set B with *m* elements, then the total number of ways we can make a choice is exactly m.n.

This is but what one would expect, as the problem is one of finding out the number of elements in the cartesian product $A \times B$. It is clearly possible to extend this principle to more than two sets.

EXERCISES 8 (a)

1. What is the total number of functions that can be defined from the set $\{1, 2\}$ to the set $\{1,2,3\}$?
2. A die of six faces marked with the integers 1, 2, 3, 4, 5, 6, one on each face, is thrown twice in succesion, what is the total number of outcomes thus obtained ?
3. Five cities A, B, C, D, E are connected to each other by straight roads. What is the total number of such roads ?
4. What is the total number of diagonals of a given pentagon ?
5. There are two routes joining a city A to a city B and three routes joining B to another city C. In how many ways can a person perform a journey from A to C ?
6. How many different four lettered words can be formed by using the four letters *a, b, c, d* while the letters can be repeated ?
7. What is the sum of all three digit numbers formed by using the digits 1, 2, 3 ?
8. How many different words with two letters can be formed by using the letters of the word JUNGLE, each containing **one** vowel and **one** consonant ?
9. There are four doors leading to the inside of a cinema hall. In how many ways can a person enter into it and come out ?

8.2 Permutation

We can use the above observations to look at the following problem :

Suppose that there are n distinguishable objects $a_1, a_2, ...a_n$. We are to arrange r of them **in order** in a row. How many such arrangements can be there ?

Observe that we are talking of arranging the objects in an order. This means that the order in which we arrange them is important also. For example, if $r = 3$, the arrangement $a_1\, a_2\, a_3$ is different from the arrangement $a_2 a_1 a_3$. Since we are to arrange r things in a row, it is useful to think of the first position in the arrangement, the second position in the arrangement, ..., the *rth* position in the arrangement. So if we are to place one of the elements from the set $\{a_1, a_2, ..., a_n\}$ in the first position there are exactly n ways we can do it. Now when it comes to placing an object in the second position, we have only $n-1$ objects left to choose from as one has already been taken out to occupy the first position. Then for the third place, there are only $n-2$ ways to choose from. So for the *rth* position, there are $n-r+1$ ways to choose from. Hence the total number of ways of choosing r things in a row from a list of n distinguishable objects is $n(n-1)$ .... $(n-r+1)$.

It is customary to call this the number of **permutations** of r things chosen from a list of n distinguishable objects. We denote this by ${}^{n}P_{r}$ and sometimes people denote it by P(n, r). So we have, for $1 \le r \le n$,

$$\boxed{{}^{n}P_{r} = n(n-1)\,(n-2)\,.......(n-r+1)} \qquad ....\textbf{(3.1)}$$

What is then the number of ways in which n things can be arranged in a row ? What we are asking is a formula for ${}^{n}P_{n}$.

It is clear for (3.1) that when n is a positive integer ${}^{n}P_{n} = n\,(n-1)\,.....\,3.\,2.\,1.$

We denote this expression as ***n* ! (read as factorial *n*)**. As an instance, if *a, b, c* are to be arranged in a row, then there are 3.2.1 = 6 ways of doing it.

These are

a b c, a c b, b a c, b c a, c a b, c b a.

We find from (3.1) that for $1 \le r \le n-1$,

$$\boxed{{}^{n}P_{r} = \frac{n!}{(n-r)!}} \qquad ....\textbf{(3.2)}$$

We cannot take $r = n$ in (3.2) since we shall then have ${}^{n}P_{n} = \frac{n!}{0!}$ and the symbol 0! does not carry any meaning as yet. It is therefore convenient to define 0! = 1, so that formula (3.2) also holds for $r = n$. **Moreover, one can also define ${}^{n}P_{0} = 1$ for all nonnegative integers *n*** and then (3.2) is valid for all integers n, r such that $0 \le r \le n$.

Let us illustrate these ideas by a few examples.

Example 7 :

How many 3 - digit numbers can be constructed using the digits 1, 2, 3, 4 without any digit being repeated ?

Solution :

A 3 - digit number has three places assigned to digits, namely, the unit's place, the ten's place and the hundred's place. Our problem is to fill up these places out of the four given digits 1, 2, 3, 4. The answer is ${}^{4}P_{3} = \frac{4!}{1!} = 1 \times 2 \times 3 \times 4 = 24.$

Example 8 :

How many integer between 10 and 100 (both inclusive) consist of distinct odd digits ?

Solution :

The odd digits are 1, 3, 5, 7, 9. Our integers whose number is being sought, consists of two digits, since 100 is not of their type. We are, therefore, to choose two digits out of the five odd digits 1, 3, 5, 7, 9 and the answer in ${}^5P_2 = \frac{5!}{3!} = 4 \times 5 = 20$

Example 9 :

How many odd integers between 99 and 999 (both inclusive) have distinct digits ?

Solution :

An integer is odd if its unit's place has a digit chosen from 1, 3, 5, 7, 9. We wish to find the number of all 3 - digit integers whose digit at the hundredth place is one chosen from 1, 2, 3, 4, 5, 6, 7, 8, 9 that is different from its digit at the unit's place, that is, in 8 possible ways. The digit at the tenth place is chosen from the integers 0, 1, 2, ..., 9 that is distinct from the digits of the previous two places. The total number space of such integers is, therefore, equal to $5 \times 8 \times 8 = 320$

Example 10 :

In how many different ways, can the letters of the word SCHOOL be arranged ?

Solution :

The word SCHOOL contains six letters, the two 0's amongst them being identical. If we distiguished between these 0's designating them as 0_1, 0_2, say, then there would be 6! arrangements. An arrangement such as $SCHO_1LO_2$ would be decidedly different from the arrangement $SCHO_2LO_1$. This is not the case at present. Hence we shall admit only one arrangement in place of two, which implies that the required number of arrangements would be $\frac{1}{2} \times 6! = 360$.

One may easily extend the ideas embodied in example 10 to more general situations. For instance, in the word **ESTEEM**, there are three E's and three single letters S, T, M, If the E's were distinct, there would have been 6! arrangements in total, but under the present circumstance, we will admit one arrangement in place of 3! arrangements and so, in place of 6! arrangements, our required number of arrangements would be $\frac{6!}{3!}$.

In a general situation, several of the letters may be repeated, each being repeated several times. For instance, the seven - lettered word PREPARE contains two E's, two P's and two R's, so that the total number of different arrangements is $\frac{7!}{2!\,2!\,2!} = 630$

In general, the number of arrangements of p + q + ... + t things of which p things are of one kind, q things are of second kind and so on is $\frac{\underline{|p+q+\ \ldots\ldots+t}}{\underline{|p}\,\underline{|q}\ldots\ldots\underline{|t}}$

Example 11 :

A coin (unbiased) is tossed three times in succession. How many outcomes are possible ?

Solution :

If H denotes the outcome showing head and T denotes that showing tail, then {H, T} is the set of possible outcomes after a single toss. After the second toss, the set of outcomes is the cartesian product set $\{H, T\} \times \{H, T\}$
and after the third toss, the set of outcomes for all the three tosses is

$$\{H, T\} \times \{H, T\} \times \{H, T\}$$

which contains $2 \times 2 \times 2 = 8$ elements and is the required number of possible outcomes.

Example 12 :

Find n when

$P(n+1, 4) = 2\,P(n, 4)$

Solution :

$P(n+1, 4) = 2\,P(n, 4)$ is the same as $(n+1)\,n\,(n-1)\,(n-2) = 2n\,(n-1)\,(n-2)\,(n-3)$

$\Rightarrow n\,(n-1)\,(n-2)\,[n+1-2\,(n-3)] = 0$

$\Rightarrow n\,(n-1)\,(n-2)\,(7-n) = 0$

$\Rightarrow n = 0$ or 1 or 2 or 7.

But $n \geq 4$ as otherwise $P(n, 4)$ is meaningless.

Hence $n = 7$.

Example 13 :

Find r if $16\,P(15, r) = 13\,P(16, r)$

Solution :

$16\,P(15, r) = 13\,P(16, r)$

$$\Rightarrow \frac{16.15!}{(15-r)!} = \frac{13.16!}{(16-r)!} = \frac{13.16!}{(16-r)\,.\,(15-r)!}$$

$\Rightarrow 16 - r = 13$ (cancelling common factors).

$\Rightarrow r = 3$.

Example 14 :

Find n if

$P(n, 3) : P(n+2, 3) = 6 : 11$

$\Rightarrow 11\,P(n, 3) = 6\,P(n+2, 3)$

$$\Rightarrow 11\,.\,\frac{n!}{(n-3)!} = 6\,.\,\frac{(n+2)!}{(n-1)!} = \frac{6(n+2)\,(n+1)n!}{(n-1)\,(n-2)\,(n-3)!}$$

$\Rightarrow 11\,(n-1)\,(n-2) = 6\,(n+2)\,(n+1)$

$\Rightarrow 5n^2 - 51n + 10 = 0$

$\Rightarrow (n - 10)(5n - 1) = 0$

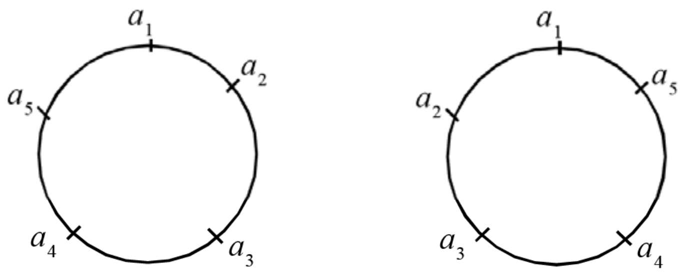
since n cannot be a fraction, $n = 10$.

Example 15 :

In how many ways can n persons sit at a round table instead of sitting in a row ?

Solution :

While sitting at a round table, the relative positions of the persons with respect to one another is to be considered. For instance, the two arrangements shown in the figure below are deemed identical.



Hence we may consider one of the persons fixed and count the number of ways in which the remaining $n - 1$ positions can be assigned to the remaining $n - 1$ persons. This can be done in ${}^{n-1}P_{n-1} = (n-1)!$ ways.. For instance, if four persons A, B, C, D are to be seated around a table, the various positions are ABCD, ABDC, ACBD, ACDB, ADBC, ADCB giving rise to $(4 - 1)! = 3! =$ arrangements.

EXERCISES 8 (b)

1. Find the number of ways in which 5 different books can arranged on a shelf.
2. Compute ${}^nP_r$ for

   (i) $n = 8, r = 4$ (ii) $n = 10, r = 3$ (iii) $n = 11, r = 0$
3. Compute the following :

   (i) $\dfrac{10!}{5!}$ (ii) $5! + 6!$ (iii) $3! \times 4!$

   (iv) $\dfrac{1}{8!} + \dfrac{1}{9!} + \dfrac{1}{10!}$ (v) $2!^{3!}$ (vi) $2^3!$
4. Show that

   $2.6.10 \text{ ....to } n \text{ factors} = \dfrac{(2n)!}{n!}$

5. Find r if P $(20, r) = 13$. P $(20, r-1)$
6. Find n if P $(n, 4) = 12$. P $(n, 2)$
7. If P $(n-1, 3)$: P $(n+1, 3) = 5 : 12$, find n.
8. Find m and n if P $(m+n, 2) = 56$, P $(m-n, 2) = 12$.
9. Show that
   (i) P (n, n) = P $(n, n-1)$ for all positive integers.
   (ii) P $(m, 1)$ + P $(n, 1)$ = P $(m+n, 1)$ for all positive integers m, n.
10. How many two digit even numbers of distinct digits can be formed with the digits 1, 2, 3, 4, 5 ?
11. How many 5-digit odd numbers with distinct digits can be formed with the digits 0, 1, 2, 3, 4 ?
12. How many numbers, each less than 400 can be formed with the digits 1, 2, 3, 4, 5, 6 if repetition of digits is allowed ?
13. How many four - digit even numbers with distinct digits can be formed out of the digits 0, 1, 2, 3, 4, 5, 6 ?
14. How many integers between 100 and 1000 (both inclusive) consists of distinct odd digits ?
15. An unbiased die of six faces, marked with the integers 1, 2, 3, 4, 5, 6, one on each face, is thrown thrice in succession. What is the total number of outcomes ?
16. What is the total number of integers with distinct digits that exceed 5500 and do not contain 0, 7 and 9 ?
17. Find the total number of ways in which the letters of the word PRESENTATION can be arranged.
18. Find the numbers of all 4-lettered words (not necessarily having meaning) that can be formed using the letters of the word BOOKLET.
19. In how many ways can 2 boys and 3 girls sit in a row so that no two girls sit side by side ?
20. Five red marbles, four white marbles and three blue marbles of the same shape and size are placed in a row. Find the total number of possible arrangements.
21. Solve Example 2.
22. In how many ways can three men and three women sit at a round table so that no two men can occupy adjacent positions ?

8.3 Combination

As we have just seen, a permutation is an ordered arrangement of objects. There are, however, situations in which order is not significant. For instance, suppose that in an examination, a student has to answer any three questions out of a total of five. In how many ways can this be done ? As another example, suppose that a college offers instruction in six subjects, namely, Physics, Chemistry, Mathematics, Statistics, Computer and Biology. A student seeking admission in the college is asked to choose any four subjects out of these. In how many ways can a student do it if he is not asked to indicate his choice in order of preference ? In each case, a selection is to be made without regard to order. We call it a **combination**. The number of combinations of r objects out of n objects is denoted by ${}^nC_r$ or C (n, r) or sometimes by, $\binom{n}{r}$ where $0 \le r \le n$.

We now proceed to compute a formula for C (n, r). This indeed can be derived from the formula for P(n, r). As explained above, to determine C (n, r), we count the number of ways in which r objects can be chosen out of n given objects. Any one of these choices is simply a collection of r objects which can be ordered in P$(r, r) = r!$ ways. A total of P(n, r) ordered selections is made after ordering each of the C(n, r) collections of r objects. Hence, one must have P $(n, r) = r$! C (n, r) and so

$$\mathbf{C}\ (\boldsymbol{n}, \boldsymbol{r}) = \frac{1}{r!}\ \mathrm{P}(n, r) = \frac{n!}{r!(n-r)!} = \frac{n(n-1).......(n-r+1)}{r\ (r-1)\ (r-2).....2.1} \qquad (3.3)$$

From (3.3), we immediately note that $\mathbf{C}\ (\boldsymbol{n}, \boldsymbol{r}) = \mathbf{C}\ (\boldsymbol{n}, \boldsymbol{n} - \boldsymbol{r})$. (3.4)

In particular, C $(n, 0)$ = C $(n, n) = 1$.

Let us apply (3.3) to the question of finding the number of ways in which three questions can be selected out of five. The answer is ${}^5C_3 = \frac{5!}{3!2!} = 10$. If the questions are numbered as a, b, c, d, e, the various combinations are *abc, abd, abe, acd, ace, ade, bcd, bce, bde, cde*.

Formula (3.3) has a curious number theoretic interpretation. It is undoubtedly a fact that C$(n$, r) is always a positve integer for all r such that $0 \le r \le n$ since one can then always select r objects out of n objects. It follows from (3.3) that $(n - r + 1)\ (n - r + 2)\ ....n$ which is a product of r consecutive positive integers, is always divisible by $r!$. This is an instance of a combinatorial proof of a number theoretic statement.

The following result, known **as Pascal's formula**, is also of this type, which states that

$${}^n\mathbf{C}_r = {}^{n-1}\mathbf{C}_r + {}^{n-1}\mathbf{C}_{r-1}. \qquad (3.5)$$

This can of course, be proved by using (3.3), but the following proof using general reasoning is worth considering.

From the list of n given objects, fix a particular object A. A combination of r objects out of these n objects may contain A or may not contain A. For forming combinations of r objects that contains A, one has to choose $r - 1$ objects from the $n - 1$ objects that do not include A and their total number is ${}^{n-1}C_{r-1}$. On the other hand, if a combination of r objects does not include A, it must be chosen from the $n - 1$ objects that do not include A, so that the total number in this case is ${}^{n-1}C_r$. Since the totality of all combinations is ${}^nC_r$ in number, Pascal's formula follows.

Let us try to illustrate our ideas by solving a few more examples including those given in section 8-1. Consider example 1 which deals with the question of determining the number of functions that can be defined from a set A having n elements to a set B having m elements. Let us take A = $\{a_1, ..., a_n\}$, B = $\{b_1, ..., b_m\}$. If f : A $\to$ B is a function, then f(a_1) can take any one of the m values $b_1, ..., b_m$. Similarly $f(a_2)$ may take any one of the m values $b_1 ..., b_m$ and so on.

Thus, there are exactly m^n such functions.

This principle answers the problem of finding the number of ways in which n distinguishable balls can be placed in m boxes. Since the first ball can be placed in any one of the m boxes, the second and others can also be placed similarly, the total number of choices is m^n. Thus if four balls *r, b, g, w* are to be placed in six boxes 1, 2, 3, 4, 5, 6, the total number of choices in $6^4 = 1296$.

Suppose that we wish to find the number of subsets that a set with n elements can have. A subset may consist of no elements, one element, two elements and so on up to n elements. It is, therefore, immediate that the total number of subsets is

$$^nC_0 + {}^nC_1 + ... + {}^nC_n. \tag{3.6}$$

But how much does the above sum add up to ? The problem can be looked at in a different way. Let the given set be $A = \{a_1, ..., a_n\}$. For each subset B of A, there is a function $f_B : A \to \{0, 1\}$ given by $f_B(a_1) = 1$ if $a_1 \in B$ and $f_B(a_1) = 0$ otherwise. Conversely, for every $f : A \to \{0, 1\}$, $B = \{a_1 : f(a_1) = 1\} \subset A$ and $f = f_B$. Thus there is a one - to - one correspondence between the subsets of A and the functions $f : A \to \{0, 1\}$. We have already seen that there are exactly 2^n such functions and hence we have the formula.

$$\mathbf{{}^nC_0 + {}^nC_1 + ... + {}^nC_n = 2^n} \tag{3.6}$$

for any non-negative integer n. There is a differnt proof of (3.6) by induction, which the reader may try. Formula (3.6) can be proved in yet another way by using the Binomial Theorem, dealt with in a subsequent section.

Now let us take the questions of distributing n objects in m boxes when the objects are **not** distinguishable.

	0	0		0	

0		00			

It is clear that there may be cases when some boxes are empty and there are cases when more than one object are filled in a single box.

Let us add on extra object to each box, so that the number that we wish to find out is the same as the number of ways in which $m + n$ objects can be distributed in m boxes when no box is empty. A typical distribution will look like the figure below when the boxes are placed in a row !

00	0	0000	...	00	000

There are $m - 1$ walls that separate the objects and these walls may occur at any one of the $m + n - 1$ gaps between the $m + n$ objects. So the number of distributions sought is exactly the number of ways in which $(m - 1)$ positions can be selected from $(m + n - 1)$ positions, that is,

$$^{m+n-1}C_{m-1} = {}^{m+n-1}C_n = \frac{(m+n-1)!}{(m-1)!\,n!}$$

Example 16 :

The number of combinations of n different objects taken r at a time in which

(i) one particular thing always occurs is $C(n-1, r-1)$

(ii) one particular thing never cocurs is C $(n-1, r)$

Since one particular thing is to occur in all the combinations we are to select $r-1$ things from the rest $n-1$ things and add with the particular thing. This can be done in C $(n-1, r-1)$ ways, which is the required number.

Since one particular thing is not to occur at all, we are to select r things from the rest n – 1 things in all possible ways. Hence the required number is C $(n-1, r)$.

Example 17 :

Find the value of n and r when P $(n, r) = 1680$ and C $(n, r) = 70$.

Solution : Now

$$\frac{P(n, r)}{C(n, r)} = \frac{1680}{70} = 24$$

or $r! = 24 = 4!$

so $r = 4$.

But P $(n, r) = 1680$

i. e. P $(n, 4) = 1680$

i.e. $n(n-1)(n-2)(n-3) = 8.7.6.5$

So $n = 8$.

Example 18 :

There are 8 boys and 6 girls. In how many ways can a committee of 4 boys and 2 girls be formed ? In how many different ways can they be arranged in a row ?

Solution :

4 boys out of 8 can be selected in C (8, 4) different ways.

2 girls out of 6 girls can be selected in C (6, 2) different ways.

Since each group of boys can be associated with each group of girls to form the committee, the total number of ways of selecting 4 boys and 2 girls is

$$C(8, 4).\ C(6, 2) = \frac{8.7.6.5}{1.2.3.4} \cdot \frac{6\ .\ 5}{1\ .\ 2} = 70\ .\ 15 = 1050.$$

Further, each of these groups contains 6 members who can be arranged among themselves in 6! ways. Hence the required number of arrangements in a row = 1050 (6!) = 756000.

Example 19 :

Find the number of diagonals of a polygon of n sides.

Solution :

A polygon of n sides has n vertices. The join of any two of its vertices is either a side or a diagonal. We can choose two vertices out of n in C(n, 2) different ways. So the number of diagonals + n = C (n, 2)

or, the number of diagonals = C(n, 2) – n

$$= \frac{n(n,-1)}{1.\ 2} - n = \frac{n^2 - 3n}{2} = \frac{n(n-3)}{2}.$$

Example 20 :

A bag contains 4 black and 5 white balls from which 6 balls are drawn.. Determine the number of ways in which at least 3 black balls can be drawn.

Solution :

Since atleast 3 black balls are to be drawn, the possibilities are :

(i) 3 blacks and 3 whites (ii) 4 blacks and 2 whites

Case(i) : 3 black balls from 4 black balls and 3 white balls from 5 white balls can be drawn in C (4, 3) . C (5, 3) different ways.

$$= 4 \times \frac{5\ .\ 4}{1\ .\ 2} = 40 \text{ different ways.}$$

Case (ii) : 4 black and 2 wihite balls can be drawn in C(4, 4) . C(5, 2) different ways

$$= 1 \times \frac{5\ \times\ 4}{1\ .\ 2} = 10 \text{ different ways.}$$

Hence the total number of ways = 40 + 10 = 50.

EXERCISES 8 (c)

1. Compute the following :

   (i) ${}^{12}C_3$ (ii) ${}^{15}C_{12}$ (iii) ${}^{9}C_4 + {}^{9}C_5$

   (iv) ${}^{7}C_3 + {}^{6}C_4 + {}^{6}C_3$ (v) ${}^{8}C_0 + {}^{8}C_1 + ... + {}^{8}C_8$.

2. Solve :

   (i) ${}^{n}C_4 = {}^{n}C_{11}$; (ii) ${}^{2n}C_3 : {}^{n}C_3 = 44 : 5$.

3. Find n and r if ${}^{n}P_r = 1680$, ${}^{n}C_r = 70$.

4. How many diagonals can an n – gon (a polygon with n sides) have ?

5. If a set A has n elements and another set B has m elements, what is the number of relations from A to B ?

6. From five consonants and four vowels, how many words can be formed consisting of three consonants and two vowels ?

7. In how many ways can a committee of four gentlemen and three ladies be formed out of seven gentlemen and six ladies ?

8. A bag contains 4 black and 5 white balls out of which 6 balls are drawn arbitrarily. In how many ways can this be done ? Find also the number of ways such that at least 3 black balls can be drawn.

9. How many triangles can be drawn by joining the vertices of a decagon ?

10. How many triangles can be drawn by joining the vertices and the centre of a regular hexagon ?

11. Sixty points lie on a plane, out of which no three points are collinear. How many straight lines can be formed by joining pairs of points ?

12. In how many ways can 10 boys and 10 girls sit in a row so that no two boys sit together ?

13. In how many ways can six men and seven girls sit in a row so that the girls always sit together ?

14. How many factors does 1155 have that are divisible by 3 ?

15. How many factors does 210 have ?

16. If n is a product of k distinct primes, what is the total number of factors of n ?

17. If m has the prime factor decomposition $p_1^{r_1} p_2^{r_2} \ldots\ldots p_n^{r_n}$, what is the total number of factors of m (excluding 1) ?

18. If 20 ! were multiplied out, how many consecutive zeros would it have on the right ?

19. How many factors of 10,000 end with a 5 on the right ?

20. A man has 6 friends. In how many ways can he invite two or more to a dinner party ?

21. In how many ways can a student choose 5 courses out of 9 if 2 courses are compulsory ?

22. In how many ways can a student choose five courses out of the courses C_1, C_2 .... C_9 if C_1, C_2 are compulsory and C_6, C_8 can not be taken together ?

23. A cricket team consisting of 11 palyers is to be chosen from 8 batsmen and 5 bowlers. In how many ways can the team be chosen so as to include at least 3 bowlers ?

24. There are $n + r$ points on a plane out of which n points lie on a straight line L and out of the remaining r points that lie outside L no three points are collinear. What is the number of straight lines that can be formed by joining pairs of these points ?

25. There are 10 books in a shelf with different titles; five of these have red cover and others have green cover. In how many way can these be arranged so that the red books are placed together ?

●

CHAPTER 9

Binomial Theorem

The essence of mathematics lies in its freedom.

- Sylvester

9.0 The Historical Background :

Binomial expansion for the case $n = 2$ was used by the Greek mathematician Euclid around 300 BC. During that period the Hindu Mathematicians also knew about the coefficients of the binomial expansion. However, the Arab mathematician Omar Khayyam (1048 – 1122 A.D) is credited with the binomial expansion for higher natural numbers. Some earlier work on binomial expansions was also done by the Indian and Chinese mathematicians. The German mathematician Michael Stifel (1486 – 1567) first introduced the term binomial coefficient sometimes around 1544 A.D. and the great British scientist Sir Isaac Newton (1642 – 1727 A.D.) generalized the binomial theorem for negative integral and fractional indices in 1665. Colin Maclaurin (1691 – 1746 A.D); Abraham G. Kastner (1719 – 1800AD); Leonhard Euler (1707 – 1783 A.D) and many others have given proofs of binomial theorem for real integral and fractional exponents. But binomial theorm for complex indices was given by Niels Henrik Abel (1802 – 1829 A. D).

The sum of two distinct terms, like $x + y$, is called a binomial. A formula for the power of binomial, i.e. $(x + y)^n$, $n \in N$ is known as the binomial theorem. We are already familiar with the binomial expansions in the particular cases $n = 1, 2, 3, 4, 5$ etc viz :

for $n = 1, (a + b)^1 = a + b$

$n = 2, (a + b)^2 = (a + b)(a + b) = aa + ab + ba + bb = a^2 + 2ab + b^2$

$n = 3, (a + b)^3 = (a + b)(a + b)(a + b)$

$= aaa + aab + aba + abb + baa + bab + bba + bbb = a^3 + 3a^2b + 3ab^2 + b^3$

$n = 4, (a + b)^4 = (a + b)(a + b)(a + b)(a + b) = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4.$

$n = 5, (a + b)^5 = (a + b)(a + b)(a + b)(a + b)(a + b)$

= --

$= a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$

We notice that when n becomes larger the actual process of multiplication becomes unmanageably tedius. Thus, before writing a general formula for $(a + b)^n$ let us have a fresh look at the process of multiplication carried out in the simple cases of $n = 2, 3$ etc.

The case $n = 2$. Suppose instead of multiplying $(a + b)(a + b)$ we were to start with $(a + b)(c + d)$, $a, b, c, d,$ all distinct. Then we shall get

$(a + b)(c + d) = ac + ad + bc + bd.$

There are four terms in the expansion. Each is a product of two symbols, one symbol taken from the first parenthesis and the other symbol taken from the second parenthesis. We know that there are exactly 2.2 = 4 ways to choose one symbol from one parenthesis and the other from the second. This also explains why there are 4 terms in the expansion. In the case of $(a + b)\ (a + b)$ only two symbols a and b are distinct. Thus, after multiplication, the terms such as ab and ba, are paired to make $2ab$. Finally we get

$$(a + b\ (a + b) = \cdots\cdots = a^2 + 2ab + b^2.$$

There are exactly three terms in the expansion.

The case n = 3. Suppose we were to start with $(a + b)\ (c + d)\ (e + f)$, a, b, c, d, e, f all distinct. We shall get $(a + b)\ (c + d)\ (e + f) = ace + acf + ade + adf + bce + bcf + bed + bdf.$

Each term in the expansion is a product of three symbols, one symbol taken from each parenthesis. We know that there are precisely 2 . 2 . 2 = 8 ways to choose three distinct symbols from three distinct binomials. We did not have to count the terms in the expansion but could say that there are eight terms. When we multiply (a + b) (a + b) (a + b), only two symbols i.e. a and b are distinct. So in the product some symbols are repeated. For example we get terms aab, aba, baa (a is taken from first and second parathesis, b from 3rd;

a is taken from first and third parenthesis, b from second;

a is taken from second and third parenthesis, b from 1st.)

We know that in a set of three symbols if 2 are alike (repeated), these can be arranged in ${}^3C_2 =$ 3 ways. Thus out of 8 terms in the final product there shall be 3 terms of the form a^2b.

For the same reason there shall be ${}^3C_0 = 1$ term of the form $a^3b^0 = a^3$; there shall be ${}^3C_1 = 3$ terms of the form ab^2 and ${}^3C_3 = 1$ term of the form $a^0b^3 = b^3$, Thus

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

All the 8 terms, finally reduce to 4 terms.

Case *n* = 4 The reader is invited to provide an explanation for this.

Case *n* = 5

$(a + b)^5 = (a + b)\ (a + b)\ (a + b)\ (a + b)\ (a + b)$

If we open up the brackets, there shall be 2^5 terms, each term is a product of 5 symbols but not all the symbols are distinct (a and b can occur repeatedly) :

In a term if a repeated 5 times b does not occur at all

if a is repeated 4 times b occurs only once

if a is repeated 3 times b is repeated twice

if a is repeated 2 times b is repeated thrice

if a occurs once b is repeated 4 times

if a doesn't occur at all b is repeated 5 times.

Thus if the symbol a is repeated r times $(0 \le r \le 5)$ in which case b occurs $5 - r$ times then the number of such terms in ${}^5C_r\ (0 \le r \le 5)$ and this contributes the term ${}^5C_r\, a^r b^{5-r}$ Now taking $r = 0, 1, 2, 3, 4, 5$, We get $(a + b)^5 = {}^5C_0\, a^5 + {}^5C_1\, a^4b + {}^5C_2\, a^3b^2 + {}^5C_3\, a^2b^3 + {}^5C_4\, a^1b^4 + {}^5C_5 b^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$.

Notice that finally there are exactly 6 terms in the expansion of $(a + b)^5$.

We now prove the general result.

9.1 BINOMIAL THEOREM (For positive integral index)

For n = 1, 2, 3, ... *and a, b* $\in$ R

$(a + b)^n = a^n + {}^nC_1\, a^{n-1}\, b + {}^nC_2\, a^{n-2}b^2 + ... + {}^nC_r a^{n-r}b^r + ... + {}^nC_{n-1}\, ab^{n-1} + b^n.$ ... ... **(1)**

Proof. By definition $(a + b)^n = (a + b)\,(a + b)\,.........\,(a + b)$ (n factors)

If the symbol a is chosen from all the factors then we get a^n and this can be done in ${}^nC_0 = {}^nC_n = 1$ way. So the coefficient of a^n is ${}^nC_0 = 1$.

If a is taken from $(n-1)$ factors then b automatically comes from one factor. This can be done in ${}^nC_1 = {}^nC_{n-1}$ ways. So the coefficient of $a^{n-1}b$ is ${}^nC_1$.

In general, if a is taken from $(n-r)$ $(0 \le r \le n)$ factors, b is chosen from the rest r factors, then this can be done in ${}^nC_r$ ways. Thus the coefficient of $a^{n-r}b^r$ is equal to ${}^nC_r$.

Finally, when only b is chosen from all the n factors then we get b^n. This can be done in ${}^nC_n = {}^nC_0$ = 1 way. So the coefficient of b^n is equal to 1.

Addition of all the terms gives : $(a + b)^3 = a^n + {}^nC_1 . a^{n-1}b + ... + {}^nC_r .\, a^{n-r}b^r + ... + b^n$

An alternative proof applying induction :

Proof :

Take $p_n : (a+b)^n = a^n + {}^nC_1 a^{n-1}b + ... + {}^nC_r\, a^{n-r}\, b^r ... + {}^nC_n b^n$

Then clearly it is true for n = 1.

Suppose it is true for n = k.

Then

$(a+b)^{k+1} = (a+b)^k\ (a+b) = (a^k + {}^kC_1\, a^{k-1}b + ... + b^k)\ (a + b)$

$= a^{k+1} + ({}^kC_1 + 1)\ a^k b + ...\ ({}^kC_r + {}^kC_{r-1})\ a^{k-r+1}\, b^r + ... + b^{k+1}$

(by collecting the coefficients of like powers of a and b)

$= a^{k+1} + {}^{k+1}C_1\, a^k\, b + ... + {}^{k+1}C_r\, a^{k+1-r}\, b^r + ... + b^{k+1}$

(by using the fact that ${}^kC_r + {}^kC_{r-1} = {}^{k+1}C_r\ (1 \le r \le k)$ (See Pascal's fomula in chapter- 8)

Hence p_{k+1} is true.

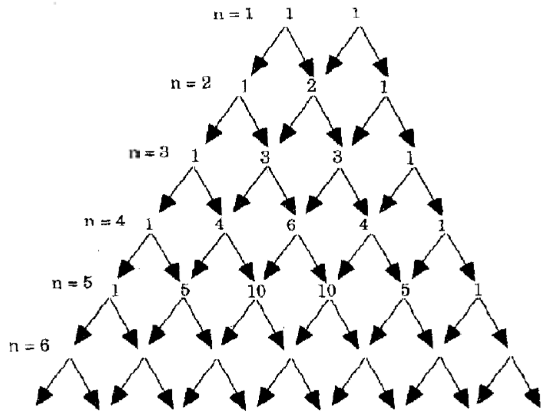
This proves that p_n is true for all n.

Remarks

(i) the general term in the binomial expansion is ${}^nC_r a^{n-r}b^r$ and all the terms are obtained by taking $r = 0, 1, 2, ... n$ giving rise to exactly $(n + 1)$ terms. So in each term sum of powers of a and b is always $n = r + (n - r)$.

(ii) **Pascal Triangle**

The cofficients with binomial expansion can be arranged in the following triangle pattern named after Blaise Pascal (1623 – 1662). See the following figure.



(Pascal's Triangle)

Any coefficient in a row in the triangle is obtained by adding the coefficients to its immediate left and the coefficient to the immediate right in the preceding row.

(iii) Using the above formula we can see the following expansions :

1. $(a - b)^n = (a + (-b))^n$
$= a^n - {}^nc_1\, a^{n-1} b + {}^nc_2\, a^{n-2}b^2 + \ldots + (-1)^r\, {}^nc_r\, a^{n-r}\, b^r + \ldots.$
$+ (-1)^{n-1}\, {}^nc_{n-1}\, a\, b^{n-1} + (-1)^n\, b^n.$ ... **(2)**

2. $(1 + x)^n = 1 + {}^nc_1 x + {}^nc_2 x^2 + \ldots + {}^nc_r\, x^r + \ldots + {}^nc_{n-1}\, x^{n-1} + x^n.$

3. $(1 - x)^n = 1 - {}^nc_1\, x + {}^nc_2 x^2 + \ldots + (-1)^r\, {}^nc_r\, x^r + \ldots. + (-1)^{n-1}\, {}^nc_{n-1}\, x^{n-1} + (-1)^n x^n.$

BINOMIAL COEFFICIENTS

The numbers ${}^nC_r$, r = 0, 1, 2 .... n, are called **binomial coefficients.** Note that although ${}^nC_r$ is expressed in the form of fractions, these are actually natural numbers (Why ?)

GENERAL TERM

If $t_1, t_2 \ldots. t_{r+1} \ldots. t_n\, t_{n+1}$ are the terms in the expansion of $(a + b)^n$ then the $(r + 1)^{th}$ term is called the general term.

$$\boxed{\mathbf{t_{r+1} = {}^nC_r\, a^{n-r}\, b^r.}} \quad ..\ \mathbf{(3)}$$

EQUIDISTANT TERMS

It can be verified that the $(r + 1)^{th}$ term form the beginning $= t_{r+1} = {}^nC_r.\, a^{n-r}\, b^r$ and the $(r + 1)^{th}$ term form the end which is equal to the $(n + 1 - r)^{th}$ term from the beginning

$= t_{n+1-r} = {}^nC_{n-r}\, a^r b^{n-r}$

Since ${}^nC_r = {}^nC_{n-r}$ the coefficients of the equidstant terms from beginning and end are equal.

MIDDLE TERMS

We know that in a collection of odd number of ordered terms there can be only one middle term. Similarly in a collection of even number of ordered terms there are always two middle terms.

In the binomial expansion of $(a + b)^n$ there are $(n + 1)$ terms. Thus, there is **only one middle term** if n **is even** and **there are two middle terms if n is odd.**

(a) Let n **be even** say $n = 2m$. Then the middle term is equal to

$$t_{m+1} = {}^{2m}C_m\, a^m b^m = {}^nC_{\frac{n}{2}}\, b^{\frac{n}{2}} b^{\frac{n}{2}} = t_{\frac{n}{2}+1}$$

(b) Let n **be odd** say $n = 2m + 1$, then the two middle terms t_{m+1} and t_{m+2} are given by

$$t_{m+1} = {}^{2m+1}C_m\, a^{m+1} b^m = {}^nC_{\frac{n-1}{2}}\, a^{\frac{n+1}{2}} b^{\frac{n-1}{2}} = t_{\frac{n+1}{2}}$$

$$t_{m+2} = {}^{2m+1}C_{m+1}\, a^m b^{m+1} = {}^nC_{\frac{n+1}{2}}\, a^{\frac{n-1}{2}} b^{\frac{n+1}{2}} = t_{\frac{n+3}{2}}.$$

The two middle terms happen to be equidistant terms also. hence their binomial coefficients must be equal. Verify that

$${}^nC_{\frac{n-1}{2}} = {}^nC_{\frac{n+1}{2}}.$$

9.2 Applications :

Example 1 :

Find the middle term in the expansion of $\left(x^2 + \frac{1}{x^3}\right)^{14}$

Solution :

n = 14, So there are 15 (odd) terms in the expansion and there is only one middle term i.e. $t_{\frac{n}{2}+1}$

$$= t_8 = {}^{14}C_7\, (x^2)^{14-7} \left(\frac{1}{x^3}\right)^7$$

$$= 2.\ 12.\ 11\ .\ 13\ x^{-7}$$

Example 2 :

Find the middle term in the expansion of $(3a + 9c)^{11}$.

Solution :

n = 11. So there are 12 (even) terms in the expansion and there are two middle terms :

$$t_{\frac{n+1}{2}} = t_6 \text{ and } t_{\frac{n+3}{2}} = t_7$$

$$t_6 = {}^{11}C_5 (3a)^{11-5} (9c)^5 = \frac{11!}{6!5!}\, 3^6 a^6 9^5 c^5 = 3^{16} \times 462\, a^6c^5$$

$$t_7 = {}^{11}C_6 (3a)^{11-6} (9c)^6 = 66 \times 7\ .\ 3^5.\ a^5\, 3^{12}\, c^6 = 3^{17} \times 462a^5\ .\ c^6.$$

Example 3 :

(a) Find the 7th term in the expansion of $\left(3x^3 - \frac{2}{3}\, \frac{1}{x^2}\right)^{15}$

(b) Is there a term independent of x in the above expansion ? If yes, find it.

Solution :

(a) $t_7 = {}^{15}c_6\,(3x^3)^9\,(-1)^6 \left(\frac{2}{3}\right)^6 \left(\frac{1}{x^2}\right)^6$

$= {}^{15}c_6\, 2^6 \,.\, 3^3 x^{27-12} = 2^6 \,.\, 3^3 \,.\, 5005\, x^{15}$

(b) Suppose t_{r+1} is the term independent of x.

Then $t_{r+1} = {}^{15}C_r \,.\, 3^{15-r}(x^3)^{15-r}\,(-1)^r \left(\frac{2}{3}\right)^r \frac{1}{x^{2r}}$

$= (-1)^r\, {}^{15}C_r 3^{15-r} \left(\frac{2}{3}\right)^r x^{45-5r}.$

This term is independent of $x \Rightarrow 45 - 5r = 0 \Rightarrow r = 9$.

Yes the tenth term, is independent of x and the term is

$- {}^{15}c_9\, 3^6 \frac{2^9}{3^9} = - {}^{15}c_9 \frac{2^9}{3^3} = - 5005 \times \frac{2^9}{3^3}$

Example 4 :

Using Binomial Theorem find the value of $(0.999)^4$ correct upto three places of decimal.

Solution :

$(0.999)^4 = (1 - 0.001)^4 = 1 - 4\,(0.001) + 6\,(0.001)^2 - 4\,(0.001)^3 + (0.001)^4$

Note that $(0.001)^2 = 0.000001$

$(0.001)^3 = 0.00\,00\,00\,00\,1$

$(0.001)^4 = 0.0 \ldots \ldots 0 \ldots 01$

Eleven zeros

So these numbers do not contribute up to third place of decimal in the above expansion.

$\therefore\ (0.999)^4 \simeq 1 - 0.004 = 0.996$

Example 5 :

If n is a positive integer and $(x + 1)^n$ is expanded in decreasing powers of x, the consecutive coefficients are in the ratio 2 : 15 : 70. Find n.

Solution :

$(x + 1)^n = x^n + {}^nC_1 x^{n-1} + \ldots\ldots + {}^nC_{r-1}\, x^{n-r+1} + {}^nC_r x^{n-r} + {}^nC_{r+1} x^{n-r-1} + \ldots. + 1$

By hypothesis

$$\frac{{}^nC_{r-1}}{{}^nC_r} = \frac{2}{15} \text{ and } \frac{{}^nC_r}{{}^nC_{r+1}} = \frac{15}{70}$$

$$\Rightarrow \frac{r}{n+1-r} = \frac{2}{15} \text{ and } \frac{r+1}{n-r} = \frac{15}{70}$$

$\Rightarrow 2n - 17r = -2$ and $3n - 17r = 14$

Solving for n we get $n = 16$.

Example 6 :

If n is a positive integer then show that $5^{2n} - 24n - 1$ is divisible by 576.

Solution :

$5^{2n} - 24n - 1 = 25^{n} - 24n - 1$

$= (1 + 24)^{n} - 24n - 1 = (1 + C(n, 1)\,24 + C(n, 2)\,24^{2} + ... + C(n, n)24^{n}) - 24n - 1$

$= 24^{2}\,(C(n, 2) + C(n, 3)\,24 + ... + C(n, n)\,24^{n-2})$

$= 576 \times$ a positive integer.

Hence $5^{2n} - 24n - 1$ is divisible by 576.

Example 7 : Find the value of

$(a + \sqrt{a^2 - 1})^7 + (a - \sqrt{a^2 - 1})^7$

Solution : $(a + \sqrt{a^2 - 1})^7 + (a - \sqrt{a^2 - 1})^7$

$= (a^7 + C(7, 1)\,a^6 \sqrt{a^2 - 1} + C(7, 2)\,a^5 (\sqrt{a^2 - 1})^2 + C(7, 3)\,a^4 (\sqrt{a^2 - 1})^3 + C(7, 4)$
$a^3 (\sqrt{a^2 - 1})^4 + C(7, 5)\,a^2 (\sqrt{a^2 - 1})^5 + C(7, 6)\,a\,(\sqrt{a^2 - 1})^6 + (\sqrt{a^2 - 1})^7\} + \{a^7 - C(7, 1)$
$a^6 (\sqrt{a^2 - 1}) + C(7, 2)\,a^5 (\sqrt{a^2 - 1})^2 - C(7, 3)\,a^4 (\sqrt{a^2 - 1})^3 + C(7, 4)\,a^3 (\sqrt{a^2 - 1})^4 - C(7, 5)$
$a^2 (\sqrt{a^2 - 1})^5 + C(7, 6)\,a\,(\sqrt{a^2 - 1})^6 - (\sqrt{a^2 - 1})^7\}$
$= 2\,\{a^7 + C(7,2)\,a^5 (a^2 - 1) + C(7, 4)\,a^3 (a^2 - 1)^2 + C(7, 6)\,a\,(a^2 - 1)^3\}$

$= 2\,\{a^7 + \frac{7\,.\,6}{7\,.\,2}(a^7 - a^5) + \frac{7\,.\,6\,.\,5}{7\,.\,2\,.\,3}a^3 (a^4 - 2a^2 + 1) + 7a\,(a^6 - 3a^4 + 3a^2 - 1)\}$
$= 2\,(a^7 + 21a^7 - 12a^5 + 35a^7 - 70a^5 + 35a^3 + 7a^7 - 21a^5 + 21a^3 - 7a)$
$= 2\,(64a^7 - 112a^5 + 56a^2 - 7a).$

EXERCISES 9 (a)

1. The rows $n = 6$ and $n = 7$ in Pascal triangle have been kept vacant. Fill in the gaps.
2. Write down the expansion of $(a + b)^8$ using Pascal's triangle.
3. Find the 3rd term in the expansion of $\left(2x^3 - \frac{1}{x^6}\right)^4$ using rules of Pascal Triangle.
4. Expand the following

   (a) $(7a + 3b)^6$ (b) $\left(\frac{-9}{2}a + b\right)^7$ (c) $\left(a - \frac{7}{3}c\right)^4$
5. Apply Binomial theorem to find the value of $(1.01)^5$
6. State true or false

(a) The number of terms in the expansion of $\left(x^2 - 2 + \frac{1}{x^2}\right)^6$ is equal to 7.

(b) There is a term independent of both x and y in the expansion of $\left(x^2 + \frac{1}{y^2}\right)^9$

(c) The highest power in the expansion of $x^{40}\left(x^2 + \frac{1}{x^2}\right)^{20}$ is equal to 40.

(d) The product of K consecutive natural numbers is divisible by K !

7. Answer the following

(a) If 6^{th} term in the expansion of $(x + *)^n$ is equal to ${}^nC_5\, x^{n-10}$ find *.

(b) Find the number of terms in the expansion of $(1+x)^n (1-x)^n$.

(c) Find the value of ${}^nC_{r-1} / {}^nC_r$.

(d) How many terms in the expansion of $\left(\frac{3}{a} + \frac{a}{3}\right)^{10}$ have positive powers of a, how many have negative powers of a. ?

8. Find the middle term (s) in the expansion of the following.

(a) $\left(\frac{a}{b} + \frac{b}{a}\right)^6$ (b) $\left(x + \frac{1}{x}\right)^9$ (c) $\left(x^{\frac{3}{2}} - y^{\frac{3}{2}}\right)^8$

9. Find the 6^{th} term in the expansion of $\left(x^2 + \frac{a^4}{y^2}\right)^{10}$.

10. (a) Find the fifth term in the expansion of $\left(6x - \frac{a^3}{x}\right)^{10}$.

(b) Is there a term independent of x ? If yes find it out.

11. (a) Find the coefficient of $\frac{1}{y^{10}}$ in the expansion of $\left(y^3 + \frac{a^7}{y^5}\right)^{10}$

(b) Does there exist a term independent of y in the above expansion ?

12. (a) Find the coefficient of x^4 in the expansion of $(1 + 3x + 10x^2)\left(x + \frac{1}{x}\right)^{10}$.

(b) Find the term independent of x in the above expansion.

13. Show that the coefficient of a^m and a^n in the expansion of $(1+a)^{m+n}$ are equal.

14. An expression of the form $(a+b+c+d+ ....)$ consisting of sum of many distinct symbols is called a multinomial. Show that $(a+b+c)^n$ is the sum of all terms of the form $\frac{n!}{p!\,q!\,r!}\, a^p b^q c^r$ where p, q and r range over all possible triples of non negative integer such that $p+q+r=n$..

15. State and prove a multinomial Theorem.

9.3 Further Applications :

IDENTITIES INVOLVING BINOMIAL COEFFICIENTS.

Throughout this section, for convenience of notation we shall use $C_0, C_1, C_2 .... C_n$ respectively in place of ${}^nC_0, {}^nC_1, {}^nC_2, ... {}^nC_n$. Infact, when the context is clear it saves extra effort by dropping n.

We know that

$$(1 + x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + ... + C_{n-1} x^{n-1} + C_nx^n. \qquad \textbf{(4)}$$

Putting x = 1and x = – 1 in (4) we have respectively

$$C_0 + C_1 + C_2 + ... + C_n = 2^n \qquad \textbf{(5)}$$

$$C_0 - C_1 + C_2 + ... + (-1)^n C_n = 0 \qquad \textbf{(6)}$$

Adding (5) and (6) and subtracting (6) from (5) we get respectively

$$C_0 + C_2 + C_4 + ... = 2^{n-1} \qquad \textbf{(7)}$$

$$C_1 + C_3 + C_5 + ... = 2^{n-1} \qquad \textbf{(8)}$$

We summarize :

For a given n, the sum of the even numbered binomial coefficients is equal to the sum of odd numbered binomial coefficients and both sums are equal to 2^{n-1}.

We know that

$${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r \qquad \textbf{(9)}$$

Again

$${}^{n+1}C_r = \frac{(n+1)!}{r!\,(n+1-r)!} = \frac{(n+1)n!}{(n+1-r)\,(n-r)\,!r\,!} = \frac{n+1}{(n+1-r)}\ {}^nC_r \qquad \textbf{(10)}$$

Therefore (9) can be written as

$$C_r + C_{r-1} = \frac{n+1}{(n+1-r)}\ C_r \qquad \textbf{(11)}$$

If we put r = 1, 2, 3, ... n successively in (11) we get

$$C_1 + C_0 = \frac{n+1}{n}\ C_1$$

$$C_2 + C_1 = \frac{n+1}{n-1}\ C_2$$

$$C_3 + C_2 = \frac{n+1}{n-2}\ C_3$$

$$C_n + C_{n-1} = (n + 1)C_n$$

Multiplying all the terms we get

$$(C_0 + C_1)\,(C_1 + C_2)\,(C_2 + C_3).....\,(C_{n-1} + C_n) = \frac{(n+1)....(n+1)}{n(n-1)....1}\ C_1C_2 .... C_n$$

Thus,

$$\left(1+\frac{C_1}{C_0}\right)\left(1+\frac{C_2}{C_1}\right)\left(1+\frac{C_3}{C_2}\right)\ldots\left(1+\frac{C_n}{C_{n-1}}\right)=\frac{(n+1)^n}{n!} \quad \ldots.(12)$$

Example 8 : Evaluate

C (10, 1) + C (10, 2) + ...+ C (10, 10)

Solution : $(1+x)^{10} = C_0 + C_1x + C_2x^2 + \ldots. + C_{10}x^{10}$

Putting $x = 1$, 2^{10} = C (10, 0) + C (10, 1) + ... + C (10, 10)

or C (10, 1) + C (10, 2) + ... + C (10, 10) = 2^{10} – C (10, 0) = $2^{10} - 1$.

Example 9 :

If P be the sum of the odd terms and Q be the sum of the even terms in the expansion of $(a+x)^n$, then prove that $(a^2 - x^2)^n = P^2 - Q^2$.

Solution : $(a + x)^n = C(n, 0)\, a^n + C(n, 1)\, a^{n-1} x + C(n, 2)\, a^{n-2} x^2 + \ldots + C(n, n)\, x^n = P + Q$

and $(a - x)^n = C(n, 0)\, a^n - C(n, 1)\, a^{n-1} x + C(n, 2)\, a^{n-2} x^2 - \ldots + (-1)^n C(n, n) x^n = P - Q$

Hence $(a^2 - x^2)^n = P^2 - Q^2$.

Example 10 :

Three consecutive coefficients in the expansion of $(1+x)^n$ are the ratio 2 : 4 : 5. Find n.

Solution : Let three consecutive coefficients be $C(n, r-1)$, $C(n, r)$ and $C(n, r+1)$

Then

$$\frac{C(n, r-1)}{2} = \frac{C(n, r)}{4} = \frac{C(n, r+1)}{5}$$

or $$\frac{n!}{2(r-1)!\,(n-r+1)!} = \frac{n!}{4r!\,(n-r)!} = \frac{n!}{5(r+1)!\,(n-r-1)!}$$

Now cancelling the non-zero factor $\dfrac{n!}{(r-1)!\,(n-r-1)!}$

from each we get $$\frac{1}{2(n-r)!\,(n-r+1)} = \frac{1}{4r\,(n-r)} = \frac{1}{5r\,(r+1)}.$$

Taking $\dfrac{1}{2\,(n-r)\,(n-r+1)} = \dfrac{1}{4r\,(n-r)}$ we get, $\dfrac{1}{n-r+1} = \dfrac{1}{2r}$ or $3r = n + 1$.

Taking $\dfrac{1}{4r(n-r)} = \dfrac{1}{5r(r+1)}$, we get $4n = 9r + 5 = 3\,(n + 1) + 5 = 3n + 8$,

$\Rightarrow n = 8$.

Example 11 :

Find the sum of $1 + \frac{1}{2} C_1 + \frac{1}{3} C_2 + \ldots. + \frac{1}{(n+1)} C_n$.

Solution : $1 + \frac{1}{2} C_1 + \frac{1}{3} C_2 + \frac{1}{4} C_3 + \ldots. + \frac{1}{(n+1)} C_n$

$$= 1 + \frac{1}{2}n + \frac{1}{3}\ \frac{n(n-1)}{2!} + \frac{1}{4}\ \frac{n(n-1)\ (n-2)}{3\ !} + ... + \frac{1}{n+1}$$

$$= \frac{1}{n+1}\left[(n+1) + \frac{(n+1)n}{2} + \frac{(n+1)(n)(n-1)}{3!} + ...+ 1\right]$$

$$= \frac{1}{(n+1)}\ [{}^{n+1}C_1 + {}^{n+1}C_2 + ... + {}^{n+1}C_{n+1}]$$

$$= \frac{1}{(n+1)}\ [{}^{n+1}C_0 + {}^{n+1}C_1 + {}^{n+1}C_2 .... + {}^{n+1}C_{n+1} - 1]$$

$$= \frac{1}{(n+1)}\ [2^{n+1} - 1] \text{ (use (5) here.)}$$

Example 12 :

Find the sum of $\frac{C_1}{C_0} + 2\ \frac{C_2}{C_1} + 3\ \frac{C_3}{C_2} + ... + n\ \frac{C_n}{C_{n-1}}$

Solution : We know $\frac{C_r}{C_{r-1}}\ \frac{n+1-r}{r}$

Putting r = 1, 2, ... n, we get

$$\frac{C_1}{C_0} + 2\ \frac{C_2}{C_1} + 3\ \frac{C_3}{C_2} + ... + n\frac{C_n}{C_{n-1}}$$

$$= n + 2\ \frac{n\ -1}{2} + 3\ \frac{n\ -2}{3} + ... + n\frac{1}{n}$$

$$= n + (n-1) + (n-2) + ... + 1 = \frac{n\ (n+1)}{2}$$

Example 13 : Show that

$$(C_0)^2 + (C_1)^2 + ... (Cn)^2 = \frac{(2n)!}{(n!)^2}$$

Solution : Since we are to sum the squares of binomial coefficients, by considering the coefficients of $(1 + x)^n$ alone we may not get the result. We apply a special trick :

$$(1 + a)^n = C_0 + C_1\ a + C_2\ a^2 + ... + C_n\ a^n \quad \textbf{(13)}$$

and

$$(a + 1)^n = C_0a^n + C_1a^{n-1} + C_2a^{n-2} + ... + C_n \quad \textbf{(14)}$$

If we multiply both sides of (13) and (14) we get

$$(1 + a)^{2n} = (C_0 + C_1a + C_2a^2 + ...+ C_na^n)\ (C_0a^n + C_1a^{n-1} + C_2a^{n-2} + ... + C_n) \quad \textbf{(15)}$$

On the other hand the binomial expansion of $(1 + a)^{2n}$ is given by

$$(1 + a)^{2n} = {}^{2n}C_0 + {}^{2n}C_1a + {}^{2n}C_2a^2 + ...+ {}^{2n}C_{n-1}\ a^{n-1} + {}^{2n}C_n\ a^n + {}^{2n}C_{n+1}\ a^{n+1} + ... + a^{2n} \quad \textbf{(16)}$$

Therefore, the coefficients of a^n in the righthand side of (15) and (16) must be equal.

Cofficient of a^n in (15) is equal to

$(C_0)^2 + (C_1)^2 + (C_2)^2 + \ldots\ldots (C_n)^2$

The coefficient of a^n in (16) is equal to ${}^{2n}C_n = \dfrac{(2n)!}{(n!)(n!)} = \dfrac{(2n)!}{(n!)^2}$.

We have thus got $(C_0)^2 + (C_1)^2 + \ldots. + (C_n)^2 = \dfrac{(2n)!}{(n!)^2}$.

Some problems involving Binomial coefficients can also be solved by using differentiation (to be dealt with in ch-14) and integration (Vol-II)

Example 14 : Prove that :

(This is not for examination Interested students may go through it after knowing integration in Vol-II)

(i) $C_0^2 - 2^2 C_1 + 3^2 C_2 - \ldots + (-1)^n (n+1)^2 C_n = 0 \quad n > 2$

(ii) $\dfrac{C_0}{2} + \dfrac{C_1}{3} + \dfrac{C_2}{4} + \ldots\ldots + \dfrac{C_n}{n+2} = \dfrac{n2^{n+1} + 1}{(n+1)(n+2)}$

(This part is not for examination. Interested students may cover it after getting through integration in Vol-II)

Solution :

(i) We have

$C_0 + C_1 x + C_2 x^2 + \ldots + C_n x^n = (1 + x)^n$ .... (a)

$\Rightarrow C_0 x + C_1 x^2 + C_2 x^3 + \ldots + C_n x^{n+1} = x(1 + x)^n$ ...... (b)

Now differentiating (b) we get

$$C_0 + 2C_1 x + 3C_2 x^2 + \ldots + (n + 1) C_n x^n = (1 + x)^n + nx(1 + x)^{n-1}$$

$\Rightarrow C_0 x + 2C_1 x^2 + 3C_2 x^3 + \ldots + (n + 1) C_n x^{n+1}$

$$= x(1 + x)^n + nx^2 (1 + x)^{n-1} \quad .....(c)$$

Differentiating (c), we have

$C_0 + 2^2 C_1 x + 3^2 C_2 x^2 + \ldots + (n+1)^2 C_n x^n = (1 + x)^n + 3nx(1+ x)^{n-1} + n(n - 1) x^2 (1 + x)^{n-2}$

Putting x = – 1 in the above identity we get

$$C_0 - 2^2 C_1 + 3^2 C_2 - \ldots + (-1)^n (n + 1)^2 C_n = 0$$

(ii) Integrating (b) above w.r.t x between 0 and 1 we get

$$C_0 \frac{x^2}{2} + C_1 \frac{x^3}{3} + C_2 \frac{x^4}{4} + \ldots + C_n \frac{x^{n+2}}{n+2} \Big]_0^1 = \int_0^1 x(1 + x)^n \, dx.$$

$$\text{L.H.S} = \frac{C_0}{2} + \frac{C_1}{3} + \frac{C_2}{4} + \ldots + \frac{C_n}{n+2}$$

$$\text{R.H.S} = \left.\frac{x(1+x)^{n+1}}{n+1}\right]_0^1 - \int_0^1 \frac{(1+x)^{n+1}}{n+1}\,dx$$

$$= \frac{2^{n+1}}{n+1} - \frac{1}{n+1}\left[\frac{(1+x)^{n+2}}{n+2}\right]_0^1$$

$$= \frac{2^{n+1}}{n+1} - \frac{2^{n+2}-1}{(n+1)(n+2)} = \frac{n2^{n+1}+1}{(n+1)(n+2)}$$

Hence proved.

EXERCISES - 9 (b)

1. Prove that

(i) ${}^{2n}C_0 + {}^{2n}C_2 + \ldots.. + {}^{2n}C_{2n} = 2^{2n-1}$

(ii) ${}^{2n}C_1 + {}^{2n}C_3 + \ldots\ldots + {}^{2n}C_{2n-1} = 2^{2n-1}$

2. Find the sum of

(i) $C_1 + 2C_2 + 3C_3 + \ldots.. + nC_n$

(ii) $C_0 + 2C_1 + 3C_2 + \ldots. + (n+1)\,C_n$

Hint : Write this as $(C_0 + C_1 + \ldots. + C_n) + (C_1 + 2C_2 + \ldots. + nC_n)$ use (5) and Exercise 1.

3. Compute $\dfrac{(1+k)\left(1+\frac{k}{2}\right)\ldots..\left(1+\frac{k}{n}\right)}{(1+n)\left(1+\frac{n}{2}\right)\ldots..\left(1+\frac{n}{k}\right)}$

4. Show that

(i) $C_0C_1 + C_1C_2 + C_2C_3 + \ldots.. + C_{n-1}C_n = \dfrac{(2n)!}{(n-1)!\,(n+1)!}$

(ii) $C_0C_r + C_1C_{r+1} + C_2C_{r+2} + \ldots.. + C_{n-r}C_n = \dfrac{(2n)!}{(n-r)!\,(n+r)!}$

Hint : Proceed as in Example -13. Compare the coefficient of a^{n-1} to get (i) and the coefficient of a^{n-r} to get (ii)

Write the result of (ii) for $r = 2$.

(iii) $3\,C_0 - 8C_1 + 13C_2 - 18\,C_3 + \ldots + (n+1)^{th}$ term $= 0$

(iv) $C_0n^2 + C_1(2-n)^2 + C_2(4-n)^2 + \ldots\ldots + C_n\,(2n-n)^2 = n2^n$

(v) $C_0 - 2C_1 + 3C_2 + \ldots + (-1)^n\,(n+1)\,C_n = 0$

(vi) $C_0 + 3C_1 + 5C_2 + \ldots. + (2n+1)\,C_n = (n+1)2^n$

5. Find the sum of the following

(i) $C_1 - 2C_2 + 3C_3 - .... + n(-1)^{n-1}C_n$

(ii) $1.2\, C_2 + 2\,.\,3\, C_3 + ... + (n-1)n\, C_n$

(iii) $C_1 + 2^2C_2 + 3^2C_3 + ... + n^2C_n$

(iv) $C_0 - \frac{1}{2}C_1 + \frac{1}{3}C_2 - ... + (-1)^n \frac{1}{n+1}C_n$

6. Show that

(i) $C_1^{\,2} + 2C_2^{\,2} + 3C_3^{\,2} + ... + nC_n^{\,2} = \dfrac{(2n-1)!}{\{(n-1)!\}^2}$

(ii) $C_2 + 2C_3 + 3C_4 + ... + (n-1)C_n = 1 + (n-2)\,2^{n-1}$

7. Prove that

$$C_1 - \frac{1}{2}C_2 + \frac{1}{3}C_3 - .... + (-1)^{n+1}\frac{1}{n}C_n = 1 + \frac{1}{2} + ... + \frac{1}{n}$$

8. Prove that

$$C_0C_1 + C_1C_2 + ... + C_{n-1}C_n = \frac{2^n.n.1.3.5...(2n-1)}{(n+1)!}$$

9. The sum $\frac{1}{1!9!} + \frac{1}{3!7!} + ... + \frac{1}{7!3!} + \frac{1}{9!1!}$ can be written in the form $\frac{2^a}{b!}$. Find a and b

10. (a) Using binomial theorem show that $1^{99} + 2^{99} + 3^{99} + 4^{99} + 5^{99}$ is divisible by 5

(Regional Mathematical Olympiad, Orissa – 1987)

(b) Using the same procedure show that $1^{99} + 2^{99} + 3^{99} + 4^{99} + 5^{99}$ is also divisible by 3 so that it is actually divisible by 15.

CHAPTER-10

Sequences and Series

In a conflict between the heart and the brain follow your heart.

- Swami Vivekananda

10.1 Introduction (Sequence and Series)

We come across the phrase 'one after another' on a variety of occasions. The days in a week, the months and seasons in a year, the festivals we observe all come oen after another.

Their occurrence one after another chains them in an order relation. We also fondly cherish such a chain of order in our minds, that is owned by none, yet belongs to all - the ordering of our counting numbers (up to a certain length ofcourse !) which we learn almost simultaneously with our mother tongue from the good old days of our forgotten infancy :

*	ଏକଦଲ	↓	କାଞ୍ଚି	↓	ଆମେସିନା	↓	ମିଠା
	ଦଦଲ		ବେଙ୍ଗୀ		ରଜା		ଠାର୍
	ଦଲାଉତ		ପିତଳ		ଦଧିବନ		ଠୁର୍
	ମହାଦଲ		ଖଡ଼ବଡ଼		କଉଡ଼ି		ଉଣେଇଶ୍
	କୁଞ୍ଚି		ଖଜା		ଦଧିସିନା		ବିଶ୍

* ଏକ ପୁତ, ଦୁଇ ସଙ୍ଗାତ, ତିନିତିନିତା,
ଚାରି ଚାଉଳ, ପା ବଉଳ, ଛଅ ପଇତା,
ସାତ ଭାତ, ଆଠ ହାତ,
ନଅର ହାଟ, ଦଶ କୁହାଟ,
ଏଗାର ଗାର, ବାର ଦୁଆର,
ତେର ତରକା, ଚଉଦ ଚକା
ପନ୍ଦର ପଡ଼ା, ଷୋଳ କୋରଡା,
ସତର ଭାଇ, ଅଠର ଗାଈ,
ଉଣେଇଶ ଗାଁ ହାକିମ ମୁହିଁ,
କୋଡି ପୂରିଲା,
ସଙ୍ଗାତ ସାଙ୍ଗରୁ କଥା ସରିଲା । (ଗଣନା ବୋଲି)

* One, two, buckle my shoe;
Three, four, open the door;
Five, six, pick up the stick;
Seven, eight, lay them straight;
Nine, ten, a good fat hen;
Eleven, twelve, dig and delve;
Thirteen, fourteen, maids a-counting;
Fifteen, sixteen, maids in the kitchen;
Seventeen, eighteen, maids a-waiting;
Nineteen, twenty, my plate's empty !
(Number-Rhyme)

The set of counting numbers N={1,2,3,...} is one of the foremost inventions of the human brain. This set is associated with entities of our thought or perception which occur in order so keenly that we go on to define :

Definitions :

Sequence :

A sequence is a function whose domain is N.

[It may be observed that a function whose domain is N* = {0,1,2,...} is also a sequence]

Real Sequence :

If the range of a sequence is a subset of R, it is a real sequence.

Terms of a Sequence :

The functional values *f(n)* of *a* sequence $f : N \rightarrow R$ are called the terms of the sequence.

Finite and Infinite Sequences :

A sequence with finite numbers of terms is called a finite sequence, otherwise it is called infinite seqence.

Example -1: $f : N \rightarrow R$ such that :

(i) $f(n) = 2n + 3$ (iii) $f(n) = (-1)^{n+1}$

(ii) $f(n) = \dfrac{1}{n(n+1)}$ (1v) $f(n) = (2n - 1)2^{-n}$

are examples of real sequences.

Recall that a sequence f is usually denoted in either of the forms $\{t_n\}_{n=1}^{\infty}$

or simply as (t_n) or $\{t_n\}$; where $f(n) = t_n$

t_n is called the n^{th} term of the sequence. Accordingly t_1, t_2, t_3 ... are termed as 1st, 2nd, 3rd... n^{th} terms of this sequence.

Series : A formal expression of the type

$t_1 + t_2 + t_3 + ...+ t_n + ...$ (or its abbreviation $\sum t_n$)

where t_n is the nth term of a sequence, is called a series. If the number of terms is finite, then it is called a finite series and if the number of terms is infinite, it is called an infinite series.

[It is customary to indicate an infinite series as $\sum\limits_{n=1}^{\infty} t_n$ or $\sum t_n$].

For a given $n \in N$ an expression like $\sum\limits_{k=1}^{n} t_k$ **$= t_1 + t_2 + t_3 + ...+ t_n$, is meanigful because we can alwaye add finitely many real numbers.**

But , an expression like $\sum\limits_{n=1}^{\infty} t_n$ **$= t_1 + t_2 + t_3 +$ ...... which involves addition of infinitely many real numbers, is nothing more than a symbol unless we clarify what it means to find the sum of infinitely many real numbers.** To do so, we proceed through the following definitions:

Definition (Partial sums of an infinite Series) :

For an infinite Series $\sum\limits_{n=1}^{\infty} t_n$ a sum $S_n = \sum\limits_{k=1}^{n} t_k$ is called the n^{th} partial sum

of the series ; for n = 1, 2, 3, ...

Thus $S_1 = t_1$, $S_2 = t_1 + t_2$, $S_3 = t_1 + t_2 + t_3$ and so on.

Consequently, given a sequence $(t_n)_{n=1}^{\infty}$, we obtain another related sequence $(S_n)_{n=1}^{\infty}$ which is the sequence of n^{th} partial sums of the infinite series $\sum_{n=1}^{\infty} t_n$

Definition (Sum of an infinite series) :

Let (S_n) be the sequence of partial sums of an infinite series $\sum t_n$. (S_n) is convergent, if there exists $s \in R$, such that $\lim_{n\to\infty} S_n = s$. **If (S_n) is convergent, then $\sum t_n$ is said to be convergent and s is called its sum. Thus $\sum t_n$ is convergent, if the sequence (S_n) of partial sums is convergent.**

If (S_n) does not converge, i.e. $\lim_{n\to\infty} S_n$ does not exist, then the series $\sum t_n$ is called a divergent series, and we say $\sum t_n$ diverges.

Let us now examine the infinite series corresponding to the sequences given in examples below, in the light of the above definition.

(i) $t_n = 2n+3$

$$\therefore S_n = \sum_{k=1}^{n} t_k = \sum_{k=1}^{n}(2k+3) = 2\sum_{k=1}^{n} k + 3n = n(n+1) + 3n = n^2+4n \to \infty \text{ as } n \to \infty$$

[It will be proved in the next section that $\sum_{k=1}^{n} k = \frac{1}{2}n(n+1)$]

This proves that $\sum_{n=1}^{\infty}(2n+3)$ diverges.

[Note that ' there exists $s \in R$ ' always means a definite real number s. The phrase '$S_n \to \infty$' means that there is no bound to the largeness of S_n. Given any number k however large, we can find an appropriate $n_0 \in N$ such that $S_n > k$, whenver $n > n_0$.]

(ii) $t_n = \dfrac{1}{n(n+1)} = \dfrac{1}{n} - \dfrac{1}{n+1}$

$$\Rightarrow S_n = \sum_{k=1}^{n} t_k = \sum_{k=1}^{n}\left(\frac{1}{k} - \frac{1}{k+1}\right) = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \ldots\ldots + \frac{1}{n} - \frac{1}{n+1}.$$

$$= 1 - \frac{1}{n+1} \to 1 \text{ as } n \to \infty$$

$\therefore \sum \dfrac{1}{n(n+1)}$ is a covergent series having sum = 1.

So we can write :

$$\Sigma t_k = \frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + ... + \frac{1}{n(n+1)} + ... = 1$$

(iii) $t_n = (-1)^{n+1}$

$\therefore$ $S_n = 1 - 1 + 1 - 1 + ... + (-1)^{n+1} = 0$, if n is even and 1 if n is odd.

So, $\lim_{n \to \infty} S_n$ does not exist.

Hence (S_n) does not converge and hence $\sum(-1)^{n+1}$ is a divergent series, (since the value of (S_n) oscillates between 0 and 1). The series 1-1+ 1- 1 - ..... is called an oscillatory series.

(iv) For $t_n = (2n-1)\, 2^{-n}$, the corresponding infinite series is ,

$$\frac{1}{2} + \frac{3}{2^2} + \frac{5}{2^3} + ...$$

It can be shown that the sum of this infinite series is 3. We defer the discussion until introduction of arithmetico-geometric series.

In the next section we expose the reader to the skill and ingenuity regarding determination of sum of an infinite series in the context of the well known progressions and some related series.

10.2. Arithmetic Progression (A.P.) and Arithmetic Mean (A.M.)

If $t_{n+1} - t_n = d$ (constant) , for n = 1, 2, 3, ..., then (t_n) is called an arithmetic sequence or an Arithmetic Progression (A.P.) and the series Σt_n is called an Arithmetic Series.

The constant d is known as the comon difference (c.d.) which may be positive or negative.

Partial sums of an Arithmetic, series :

If Σt_n is an Arithmetic series , then by definition $t_2 - t_1 = t_3 - t_2 = t_4 - t_3 = ...\ t_{n-2} - t_{n-3}$

$= t_{n-1} - t_{n-2} = t_n - t_{n-1}$

$\Rightarrow t_1 + t_n = t_2 + t_{n-1} = t_3 + t_{n-2} = ... = t_k + t_{n-k+1}$,

(for k = 1, 2, ... , n) . ...(1)

Writing $S_n = t_1 + t_2 + t_3 + ... + t_{n-2} + t_{n-1} + t_n$

and also $S_n = t_n + t_{n-1} + t_{n-2} + ... + t_3 + t_2 + t_1$,

We get $S_n = \frac{1}{2}\ [(t_1 + t_n) + (t_2 + t_{n-1}) + (t_3 + t_{n-2}) + ... + (t_n + t_1)]$

$= \frac{n}{2}\ (t_1 + t_n)$ [By (1)]

Thus, if $t_1 = a$ and common difference = d, then $t_n = a + (n-1)\, d$, so that

$\boxed{S_n = \frac{n}{2}\ [2a+(n-1)d]}$ which is same as $\boxed{S_n = \frac{n}{2}(a+l)}$, where a = first term t_1 and l = last term

$a+(n-1)d$ of the finite series $a+(a+d)+(a+2d)+\ldots+a+(n-1)d$.

Therefore $\lim_{n\to\infty} S_n = \infty$ or $-\infty$ according as $d > 0$ or $d < 0$.

If $d = 0$, $S_n = na \to \infty$ or $-\infty$ according as $a > 0$ or $a < 0$.

This shows that an arithmetic series always diverges, except the special case when $t_1 = 0 = d$. In this case $t_n = 0$ for all n so that sum of the corresponding arithmetic series is zero.

N.B. : Sum of first n counting numbers :

Taking $a = 1$, d= 1 in the A.P.(a_n) we get

$$\boxed{1+2+\ldots+n = \frac{n(n+1)}{2}}$$

Example- 2

In an A.P, $t_m = n$ and $t_n = m$, $m \neq n$. Prove that $t_{m+n} = 0$

Solution :

Taking 1st term = a, $c.d. = d$

$$\left.\begin{array}{l} t_m = n \Rightarrow a+(m-1)\,d = n \\ t_n = m \Rightarrow a+(n-1)\,d = m \end{array}\right\} \Rightarrow (m-n)\,d = n-m \Rightarrow d = -1$$

$a+(m-1)d = n \Rightarrow a = n-(m-1)d = n+m-1$

$\therefore\ t_{m+n} = a+(m+n-1)d = n+m-1-(m+n-1) = 0$

Example- 3

Find an A.P, sum of whose first n term is n^2.

Solution :

$S_n = t_1 + t_2 + \ldots + t_{n-1} + t_n = S_{n-1} + t_n$

$\Rightarrow t_n = S_n - S_{n-1} = n^2 - (n-1)^2 = 2n-1$

$\therefore$ The A.P. is given by (t_n) = where $t_n = 2n-1$

or explicitly 1, 3, 5, 7, ... , the sequence of odd natural numbers.

Example- 4

Let S_n, t_n and S'_n, t'_n denote respectively sum of first n terms and the n^{th} term of two arithmetic progressions,

Given, $\dfrac{S_n}{S'_n} = \dfrac{n}{n+1}$, find $\dfrac{t_7}{t'_7}$.

Solution :

Taking a, a' and d, d' respectively as first terms and c.d's

$$\frac{S_n}{S_n'} = \frac{\frac{n}{2}\{2a+(n-1)d\}}{\frac{n}{2}\{2a'+(n-1)d'\}} = \frac{2a+(n-1)d}{2a'+(n-1)d'} = \frac{n}{n+1} \quad (given)$$

Now $\dfrac{t_7}{t_7'} = \dfrac{a+6d}{a'+6d'} = \dfrac{2a+12d}{2a'+12d'} = \dfrac{13}{14}$

(Putting) n = 13).

Arithmetic Mean (A.M.)

When three numbers a, m and b are in A.P, then m is average of a and b.

This may be seen as follows :

Taking d as common difference $m = a+d,\ b = a+2d.$

Thus $\dfrac{a+b}{2} = \dfrac{a+(a+2d)}{2} = a+d = m.$

In general if a_1, a_2, ..., a_n are in A.P., their arithmetic mean A.M. is defined as

$$\boxed{\text{A.M.} = \frac{a_1+a_2+\ldots+a_n}{n} = \frac{1}{n}\sum_{k=1}^{n} a_k}$$

N.B. The terms a_2, a_3 ..., a_{n-1} are called the arithmetic means between a_1 and a_n.

Insertion of A.M's between given numbers :

Example - 5

Insert n number of A.M's between a, $b \in$R. Let the A.M's be m_1, m_2, ... m_n.

Then $a_1, m_1, m_2 ..., m_n$, b are in A.P.

Taking d as common difference

$m_1 = a+d$, $m_2 = a+2d$, ... , $m_n = a+nd$, $b = a+(n+1)d$

$\therefore$ d = $\dfrac{b-a}{n+1}$ which gives

$m_1 = a + \dfrac{b-a}{n+1}$, $m_2 = a+\dfrac{2(b-a)}{n+1}$, . . . $m_n = a+\dfrac{n(b-a)}{n+1}$.

10.3 Geometric Progression (G.P.) and Geometric Mean (G.M.)

If $\frac{t_{n+1}}{t_n} = r$ **(constant), for n = 1, 2, 3, ..., then (t_n) is called a geometric sequence or geometric progression (G.P)**, and the series $\sum_{n=1}^{\infty} t_n$ **is called a geometric series**.

The constant **r is known as the common ratio (c.r.).**

<u>**Note.**</u> **(1)** If t_1 (first term) = a, common ratio = r, then $\mathbf{t_n = a\, r^{n-1}}$.

(2) No term of a geometric sequence can be zero, for otherwise, $\frac{t_{n+1}}{t_n}$ will be meaningless for the corresponding value of n.

n^{th} partial sum of a geometric series :

For a geometric series with $t_n = a$ and common ratio = r,

$S_n = t_1 + t_2 + t_3 + ... + t_{n-1} + t_n = a + ar + ar^2 + ... + ar^{n-2} + ar^{n-1}$

$\therefore rS_n = ar + ar^2 + ... + ar^{n-1} + ar^n$

$\Rightarrow$ (subtracting) $(1-r)\, S_n = a(1-r^n)$

or $\boxed{\mathbf{S_n} = \frac{a(1-r^n)}{1-r} \text{ for } r \neq 1.}$

If r=1, then $t_n = a$, for every n, so that $S_n = na$.

Sum of a gemetric series :

If $|r| < 1$, i.e -1, < r <1 then $r^n \to 0$ when $n \to \infty$. So for the geometric series with $|r| < 1$

we have $\lim_{n\to\infty} S_n = \lim_{n\to\infty} a \frac{1-r^n}{1-r} = \frac{a}{1-r}$.

Therefore $\boxed{\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \;;\; if |r| < 1}$

and $\sum_{n=1}^{\infty} ar^{n-1}$ **diverges if | r | > 1.**

Hence the geometric series $\sum_{n=1}^{\infty} ar^{n-1}$ converges if $|r| < 1$ and diverges if $|r| \geq 1$.

Gemetric Mean :

If three terms *a, m and b* form a G.P. then m is said to be the Geometric Mean (G.M.) of *a and b*.

Here, $\frac{m}{a} = \frac{b}{m}$ = r (common ratio)

$\Rightarrow m^2 = ab \qquad \Rightarrow m = \pm\sqrt{ab}$

$m^2 = ab \Rightarrow ab$ is always positive.

Therefore Geometric Mean between a and b is $\sqrt{ab}$ or $-\sqrt{ab}$.

Geometric Mean of n terms in G.P.

If $a_1, a_2, \ldots a_n$ are n positive numbers in G.P. then their geometric mean is defined as :

$$\text{G.M.} = (a_1.a_2. \ldots a_n)^{\frac{1}{n}} = \sqrt[n]{a_1.a_2 \ldots a_n}\ .$$

N.B. : The terms $a_2, a_3, \ldots, a_{n-1}$ are called geometric means between a_1 and a_n.

Example- 6

Insert three geometric means between $\frac{1}{2}$ and 128.

Solution :

Suppose G.M's to be inserted are g_1, g_2, and g_3

$\Rightarrow \frac{1}{2}, g_1, g_2, g_3$, 128 are in G.P.

Taking r as common ratio, $128 = \frac{1}{2} r^4$

$\Rightarrow r^4 = 256 \Rightarrow r = 4$ or -4

Taking $r = 4$, $g_1 = \frac{1}{2} \times r = 2$, $g_2 = \frac{1}{2} r^2 = 8$, $g_3 = \frac{1}{2} r^3 = 32$

Taking $r = -4$, $g_1 = -2$, $g_2 = 8$, $g_3 = -32$

So, the inserted G.M's are either 2, 8, 32 or -2, 8, -32.

Example- 7

Find 3 +33+333+... up to the n^{th} term

Solution :

$$S_n = \frac{3}{9}[9+99+999+\ldots+n^{th}\text{ term}]$$

$$= \frac{1}{3}[(10-1)+(10^2-1)+(10^3-1)+\ldots+(10^n-1)]$$

$$= \frac{1}{3}[10(1+10+10^2+\ldots+10^{n-1})-n]$$

$$= \frac{1}{3}[10\times\frac{10^n-1}{10-1}-n] = \frac{10}{27}(10^n-1) - \frac{n}{3}$$

10.4 Harmonic Progression (H.P.) & Harmonic Mean (H.M.)

A number of terms $a_1, a_2, ..., a_n$ are said to be in harmonic progression (H.P.) if their reciprocals $\frac{1}{a_1}, \frac{1}{a_2}, ... \frac{1}{a_n}$ are in A.P.

Harmonic Mean (H.M.): If $a_1, a_2, ..., a_n$ are in H.P. then their harmonic mean (H.M.) H is given by :

$$\frac{1}{\mathrm{H}}=\frac{1}{n}\left(\frac{1}{a_1}+\frac{1}{a_2}+...+\frac{1}{a_n}\right).$$

This means that reciprocal of harmonic mean is arithmetic mean of the reciprocals.

Example - 8

If a, b, c are in harmonic progression, prove that $b = \frac{2ac}{a+c}$

Solution :

Obviously $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A.P. therefore $\frac{1}{b}-\frac{1}{a}=\frac{1}{c}-\frac{1}{b}$

$$\Rightarrow \frac{2}{b}=\frac{1}{a}+\frac{1}{c}=\frac{a+c}{ac}$$

Therefore $b = \frac{2ac}{a+c}$.

Example- 9

If $a,b \neq 0$ and H is the harmonic mean between them, then $\frac{1}{a}, \frac{1}{\mathrm{H}}, \frac{1}{b}$ are in A.P.

$$\therefore \frac{1}{\mathrm{H}}=\frac{1}{2}\left(\frac{1}{a}+\frac{1}{\mathrm{b}}\right)=\frac{a+b}{2ab}$$

$$\Rightarrow \mathrm{H}=\frac{2ab}{a+b}.$$

Relation between A.M., G.M. and H.M.

If a and b are two positive numbers then their Airthmetic Mean 'A', Geometric Mean 'G' and Harmonic Mean 'H' are given by

$$\mathrm{A}=\frac{a+b}{2},\ \mathrm{G}=\sqrt{\mathrm{ab}}\ \textit{and}\ \mathrm{H}=\frac{2ab}{a+b}$$

Then A-G = $\frac{a+b}{2} - \sqrt{ab} = \frac{1}{2}(a+b-2\sqrt{ab}) = \frac{1}{2}(\sqrt{a}-\sqrt{b})^2 \geq 0$

$\therefore$ A $\geq$ G

also G-H = $\sqrt{ab} - \frac{2ab}{a+b} = \frac{\sqrt{ab}}{a+b}(a+b-2\sqrt{ab}) = \frac{\sqrt{ab}}{\text{a+b}}\left(\sqrt{a}-\sqrt{b}\right)^2 \geq 0$

$\therefore$ G $\geq$ H

Combining the two inequalities, we have

A ≥ G ≥ H.

N.B. This relation between A.M., G.M. and H.M. can be generalised to n numbers, which is stated without proof :

If A, G, H are respectively arithmetic, geometric and harmonic mean of $a_1, a_2, \ldots a_n$

then **A ≥ G ≥ H.**

10.5 Arithmetico-geometric Series :

If (a_n is an arithmetic sequence and (b_n) is a geometric sequence, then (a_n b_n) is known as an Arithmetico - geometric sequence. Accordingly $\sum a_n b_n$ is called an arithmetico-geometric series.

(i) $\sum_{n=1}^{\infty} (2n-1)2^{-n}$

(ii) $1+\frac{3}{2}+\frac{5}{4}+\frac{7}{8}+\ldots\ldots$

(iii) $1 + 3x + 5x + 7x^3 + \ldots\ldots$

are examples of arithmetico-geometric series.

Partial Sums

For an arithmetico - geometric series $\sum a_n b_n$

taking $a_1 = a$, $a_n = a+(n-1)d$ and $b_1 = b$, $b_n = br^{n-1}$,

[where d and r are respectively common difference and common ratio corresponding to the arithmetic series $\sum a_n$ and the geometric series $\sum b_n$] , we have,

$a_k b_k = \{a+(k-1)d\}\ br^{k-1} = abr^{k-1}+db(k-1)r^{k-1}$

$\therefore\ S_n = \sum_{k=1}^{n} a_k b_k = ab\,(1+r+r^2+\ldots\ldots r^{n-2}+r^{n-1})$

$\quad +db\ \{r+2r^2+3r^3+\ldots\ldots+(n-2)r^{n-2}+(n-1)r^{n-1}\}$

$rS_n = ab\,(r+r^2+\ldots\ldots+r^{n-1}+r^n)$

$\quad +db\{r^2+2r^3+\ldots\ldots+(n-2)\,r^{n-1}+(n-1)\,r^n\}$

$\Rightarrow$ (Subtracting) $(1-r)\,S_n = ab\,(1-r^n) + db\,\{r+r^2+r^3+\ldots\ldots+r^{n-1}-(n-1)\,r^n\}$

$= ab\,(1-r^n) + db\left\{r.\dfrac{1-r^{n-1}}{1-r} - (n-1)\ r^n\right\}$ [By the formula for n^{th} partical sum of a geometric series]

$$= ab\,(1-r^n) + db\left(\frac{r}{1-r} - \frac{r^n}{1-r} - nr^n + r^n\right)$$

$$\Rightarrow S_n = ab\,\frac{1-r^n}{1-r} + db\left\{\frac{r}{(1-r)^2} - \frac{r^n}{(1-r)^2} + \frac{r^n}{1-r} - \frac{nr^n}{1-r}\right\}$$

Sum of an arithmetico-geometric series :

For the above series we derive expression for its sum by assuming $|r| < 1$, for which the series $\sum\limits_{n=0}^{\infty} r^n$ converges.

For $|r|<1$, $r^n \to 0$ and $nr^n \to 0$ when n approaches infinity. (For the latter limit see the note (2) that follows.) Therefore

$$\lim_{n\to\infty} S_n = \frac{ab}{1-r} + \frac{dbr}{(1-r)^2}$$

Thus we have,

$$ab + (a+d)\,br + (a+2d)\,br^2 + (a+3d)\,br^3 + \ldots\ldots = b\left[\frac{a}{1-r} + \frac{dr}{(1-r)^2}\right] \quad \text{for } |r| < 1.$$

Note (1) In solving problems involving arithmetico-geometric series it is aften advisable to proceed from first principles, rather than memorizing the above formulae. (See worked out example-8)

(2) **Proof that $\lim\limits_{n\to\infty} nr^n = 0$ for $|r| < 1$**

$\mathbf{0} < |r| < 1 \Rightarrow \dfrac{1}{|r|} > 1$

[Since no term of a geometric series is zero, there is no possiblity that r = 0]

$\therefore |nr^n| = n\,|r|^n = \dfrac{n}{b^n}$; taking $b = \dfrac{1}{|r|}$

As b > 1, We can take $b = 1 + a$; $a > 0$ so that

$b^n = (1+a)^n = 1 + na + \dfrac{n(n-1)}{2}a^2 + \ldots\ldots + a^n$ (by binomial expansion)

$$> \frac{n(n-1)a^2}{2}$$

$$\Rightarrow \frac{2}{(n-1)a^2} > \frac{n}{b^n} > 0 \text{ ; taking } n > 1$$

Taking limits as n approaches infinity and applying Sandwich theorem we get

$$\lim_{n \to \infty} \frac{n}{b^n} = \lim_{n \to \infty} n|r|^n = 0$$

$$\therefore \lim_{n \to \infty} nr^n = 0 \text{ for } |r| < 1.$$

[try to prove : $\lim_{n \to \infty} |x_n| = 0 \Rightarrow \lim_{n \to \infty} x_n = 0$. Begin from definition of limjit at infimity.

Observe that the implication is not true for a nozero limit in LHS, e.g. $x_n = (-1)^n$].

10.6 Applications (worked out examples on Sequences and Series) :

[Star marked examples are not to be set in examination]

Example 10 : For an arithmetic sequence if $S_m = S_n$ $(m \neq n)$ then prove that $S_{m+n} = 0$.

Solution :

Taking first term = a, common difference = d

$$S_m = S_n \Rightarrow \frac{m}{2}[2a + (m-1)d] = \frac{n}{2}[2a + (n-1)d]$$

$$\Rightarrow 2am + m(m-1)d = 2an + n(n-1)d$$

$$\Rightarrow 2a(m-n) + d(m^2 - m - n^2 + n) = 0$$

$$\Rightarrow 2a(m-n) + d(m-n)(m+n-1) = 0$$

$$\Rightarrow 2a + (m+n-1)d = 0 \quad (\because m \neq n)$$

$$\Rightarrow S_{m+n} = \frac{m+n}{2}\{2a + (m+n-1)d\} = 0$$

Example - 11 : For an arithmetic seris Σt_n if ,

$$\frac{S_m}{S_n} = \frac{m^2}{n^2} \ (m \neq n) \text{, then prove that } \frac{t_m}{t_n} = \frac{2m-1}{2n-1}$$

Solution : Taking $t_1 = a$, common difference = d

$$\frac{S_m}{S_n} = \frac{m^2}{n^2} \Rightarrow \frac{\frac{m}{2}\{2a+(m-1)d\}}{\frac{n}{2}\{2a+(n-1)d\}} = \frac{m^2}{n^2}$$

$$\Rightarrow \frac{2a+(m-1)d}{2a+(n-1)d} = \frac{m}{n}$$

$$\Rightarrow 2an + n(m-1)\,d = 2am + m\,(n-1)\,d$$

$$\Rightarrow d\,(m-n) = 2a\,(m-n) \Rightarrow d = 2a$$

$$\therefore \frac{t_m}{t_n} = \frac{a+(m-1)d}{a+(n-1)d} = \frac{a+2a(m-1)}{a+2a(n-1)} = \frac{2m-1}{2n-1}$$

[Here $a \neq 0$, for otherwise, d = 0 and $S_m = S_n = 0$ and $\frac{S_m}{S_n}$ will be meaningless, contrary to hypothesis]

Example - 12 : Sum the series $\frac{1}{1+\sqrt{x}}, + \frac{1}{1-x} + \frac{1}{1-\sqrt{x}} + ...$ up to n terms.

Solution : Here $t_1 = \frac{1}{1+\sqrt{x}}, t_2 = \frac{1}{1-x}, t_3 = \frac{1}{1-\sqrt{x}}$

$$\therefore t_1 + t_3 = \frac{1}{1+\sqrt{x}} + \frac{1}{1-\sqrt{x}} = \frac{1-\sqrt{x} + 1 + \sqrt{x}}{1-x} = \frac{2}{1-x} = 2t_2$$

Therefore the given series is an arithmetic series with first term $a = \frac{1}{1+\sqrt{x}}$ and common difference

$$d = \frac{1}{1-x} - \frac{1}{1+\sqrt{x}} = \frac{\sqrt{x}}{1-x}$$

Sum upto n terms is, $S_n = \frac{n}{2}\left[2a+(n-1)d\right]$

$$= \frac{n}{2}\left[\frac{2}{1+\sqrt{x}} + (n-1)\frac{\sqrt{x}}{1-x}\right]$$

$$= \frac{n}{2(1-x)}\left[2+(n-3)\sqrt{x}\right]$$

Example-13 (Method of Differences) :

If we can express t_n in the from $t_n = u_n - u_{n-1}$ or $t_n = u_{n-1} - u_n$, then obtaining partial sums, S_n of the series Σt_n becomes fairly easy.

In this case $S_n = \sum_{k=1}^{n} t_k = \sum_{k=1}^{n} (u_k - u_{k-1})$

$= u_1 - u_0 + u_2 - u_1 + ... + u_n - u_{n-1} = u_n - u_0$

Similarly if $t_n = u_{n-1} - u_n$, them $S_n = u_0 - u_n$.

The skill of writing t_n as a difference is illustrated through a variety of examples :

Example - 14 :

Sum upto n terms and obtain the sum of the series , if it is covergent :

(i) 1. 3.5 + 3. 5. 7 + 5. 7. 9 + ...

(ii) $\frac{1}{1.3.5} + \frac{1}{3.5.7} + \frac{1}{5.7.9} + ...$

(iii) $\frac{1}{1.4} + \frac{1}{2.5} + \frac{1}{3.6} + ...$

Soultion :

(i) $t_n = (2n-1)(2n+1)(2n+3)$

[observe that t_n is a product 3 successive terms of an A.P, begining with 2n-1]

$= \frac{1}{8}(2n-1)(2n+1)(2n+3)\{(2n+5) - (2n-3)\}$

[Multiply and divide by the difference of the factors following 2n+3 and preceeding 2n-1]

$= \frac{1}{8}(2n-1)(2n+1)(2n+3)(2n+5) - \frac{1}{8}(2n-3)(2n-1)(2n+1)(2n+3)$

$= u_n - u_{n-1}$; where $u_n = \frac{1}{8}(2n-1)(2n+1)(2n+3)(2n+5)$.

$\therefore S_n = u_n - u_o = \frac{1}{8}(2n-1)(2n+1)(2n+3)(2n+5) - \frac{1}{8}(-1)(1)(3)(5)$

$= \frac{1}{8}[(2n-1)(2n+1)(2n+3)(2n+5)+15]$.

Here $S_n \to \infty$ as $n \to \infty$, so the series is not convergent.

N.B. The above method applies when t_n can be first expresed as a product of a number of successive terms of an A.P.

(ii) $t_n = \frac{1}{(2n-1)(2n+1)(2n+3)}$

[observe that t_n is the reciprocal of the product of 3 sucessive terms of an A.P]

$$= \frac{(2n+3) - (2n-1)}{4(2n-1)(2n+1)(2n+3)}$$

[multiply and divide by the difference of the last and first factors in the denominator.]

$$= \frac{1}{4} \cdot \frac{1}{(2n-1)(2n+1)} - \frac{1}{4} \cdot \frac{1}{(2n+1)(2n+3)} = u_{n-1} - u_n ;$$

where $u_n = \frac{1}{4} \cdot \frac{1}{(2n+1)(2n+3)}$

$$\therefore S_n = \sum_{k=1}^{n} t_k = u_0 - u_n = \frac{1}{4.1.3} - \frac{1}{4(2n+1)(2n+3)}$$

Here $S_n \to \frac{1}{12}$ as $n \to \infty$, so the series Σt_n is convergent.

$$\therefore \frac{1}{1.3.5} + \frac{1}{3.5.7} + \frac{1}{5.7.9} + \ldots\ldots = \frac{1}{12}$$

[This method is applieable when t_n is the reciprocal of the product of a number of successive terms of an A.P.]

✱ (iii) $t_n = \frac{1}{n(n+3)} = \frac{(n+1)(n+2)}{n(n+1)(n+2)(n+3)}$

[First insert the missing factors so that the succession of the factors is in A.P]

$$= \frac{n^2 + 3n + 2}{n(n+1)(n+2)(n+3)} = \frac{n(n+3) + 2}{n(n+1)(n+2)(n+3)}$$

$$= \frac{1}{(n+1)(n+2)} + \frac{2}{n(n+1)(n+2)(n+3)}$$

(Now processed as in the previous example)

$$= \frac{(n+2)-(n+1)}{(n+1)(n+2)} + \frac{2\{(n+3)-n\}}{3n(n+1)(n+2)(n+3)}$$

$$= \frac{1}{(n+1)} - \frac{1}{(n+2)} + \frac{2}{3} \cdot \frac{1}{n(n+1)(n+2)} - \frac{2}{3} \cdot \frac{1}{(n+1)(n+2)(n+3)}$$

$$= \left[\frac{1}{n+1} + \frac{2}{3} \cdot \frac{1}{n(n+1)(n+2)}\right] - \left[\frac{1}{n+2} + \frac{2}{3} \cdot \frac{1}{(n+1)(n+2)(n+3)}\right]$$

$= u_{n-1} - u_n$; where $u_n = \frac{1}{n+2} + \frac{2}{3} \cdot \frac{1}{(n+1)(n+2)(n+3)}$

$$\therefore S_n = \sum_{k=1}^{n} t_k = u_o - u_n$$

$$= \frac{1}{2} + \frac{2}{3} \cdot \frac{1}{1.2.3} - \left\{ \frac{1}{n+2} + \frac{2}{3} \cdot \frac{1}{(n+1(n+2)(n+3)} \right\}$$

Here $S_n \to \frac{1}{2} + \frac{2}{3} \cdot \frac{1}{1.2.3} = \frac{11}{18}$ as $n \to \infty$, so $\sum t_n$ is convergent

$$\therefore \frac{1}{1.\ 4} + \frac{1}{2.\ 5} + \frac{1}{3.\ 6} + \ldots\ldots = \frac{11}{18}$$

Example - 15 [Sum of the special series $\sum_{k=1}^{n} k$, $\sum_{k=1}^{n} k^2$ *and* $\sum_{k=1}^{n} k^3$]

Sum upto n terms the following series :

(i) $1+2+3+ \ldots + n$

(ii) $1^2 + 2^2 + 3^2 + \ldots + n^2$

(iii) $1^3 + 2^3 + 3^3 + \ldots + n^3$

Solution :

(i) $1+2+3+ \ldots + n$ [See Section 10.2 (sum of first n counting numbers)]

(ii) $k^3 - (k-1)^3 = k^3 - (k^3 - 3k^2 + 3k-1) = 3k^2 - 3k + 1$

Taking k = 1, 2, 3, ..., n and adding

$$n^3 = 3 \sum_{k=1}^{n} k^2 - 3 \sum_{k=1}^{n} k + n = 3 \sum_{k=1}^{n} k^2 - \frac{3}{2} n(n+1) + n$$

It is now left to the reader to derive

$$\mathbf{1^2 + 2^2 + 3^2 + \ldots\ldots n^2 = \frac{1}{6} n(n+1)(2n+1).}$$

(iii) $k^4 - (k-1)^4 = k^4 - (k^4 - 4k^3 + 6k^2 - 4k + 1) = 4k^3 - 6k^2 + 4k - 1$

Taking k = 1, 2, 3, ..., n and adding

$$n^4 = 4 \sum_{k=1}^{n} k^3 - 6 \sum_{k=1}^{n} k^2 + 4 \sum_{k=1}^{n} k - n$$

$$= 4\sum_{k=1}^{n} k^3 - 6\cdot\frac{n(n+1)(2n+1)}{6} + 4\cdot\frac{n(n+1)}{2} - n$$

Now solve for $\sum_{k=1}^{n} k^3$ and derive

$$\mathbf{1^3 + 2^3 + 3^3 + \ldots\ldots + n^3 =} \left[\frac{n(n+1)}{2}\right]^2$$

Example - 16 :

Finding S_n for the series Σt_n, where $(t_{n+1} - t_n)$ is an arithmetic or a geometric sequence.

In this case we have

$0 = S_n - S_n = (t_1 + t_2 + t_3 \ldots\ldots t_n) - (t_1 + t_2 + \ldots\ldots + t_{n-1} + t_n)$

$= t_1 + (t_2 - t_1) + (t_3 - t_2) + \ldots\ldots + (t_n - t_{n-1}) - t_n$

$\Rightarrow t_n = t_1 + (t_2 - t_1) + (t_3 - t_2) + \ldots\ldots + (t_n - t_{n-1}) = t_1 + \sum_{k=1}^{n-1}(t_{k+1} - t_k)$

$\sum_{k=1}^{n-1}(t_{k+1} - t_k)$, being the sum of n - 1 terms in A.P or G. P, we can obtain t_n in terms of n..,

whereafter we can find $\sum_{k=1}^{n} t_k$ in the usual way.

As an illustration, consider the following series :

8+11 +20+47+....

Observe that the difference of the terms, i.e

11 - 8, 20-11, 47-20, ... form the G.P 3, 3^2,3^3 ....

So we apply the above technique.

$0 = S_n - S_n = 8+11+20+47+\ldots\ldots + t_n$

$-\{8+11+20+\ldots\ldots+ t_{n-1}\} - t_n$

(Note the style of writing S_n- S_n. This is for sake of our convenience.)

$= 8 + (11-8) + (20-11) + (47-20) + \ldots\ldots + (t_n - t_{n-1}) - t_n$

$\therefore\ t_n = 8 + \{3+3^2+3^3+\ldots\ldots + (t_n - t_{n-1})\}$

Since the bracketed expression is a sum upto n-1 terms of the G.P. 3, 3^2, 3^3..... ,

$(t_n - t_{n-1})$ is 3^{n-1}

(You can be otherwise sure that the terms in the bracket correspond to the terms 11, 20, 47, ... t_n of the series of n terms : 8 + 11 + 20 + 47 + ... t_n So their number must be n-1)

$$\therefore\ t_n = 8 + (3+3^2+3^3+......+3^{n-1}) = 8+3.\frac{3^{n-1}-1}{3-1}$$

$$= 8+\frac{1}{2}(3^n-3) = \frac{3^n}{2}+\frac{13}{2}$$

(You may put n=1, 2, 3, ... and check whether you obtain the terms of the given series : 8 +11+20 + 47 + ...)

Therefore sum upto n terms of the given series is given by

$$S_n = \sum_{k=1}^{n} t_k = \frac{1}{2}\sum_{k=1}^{n} 3^k + \frac{13n}{2}$$

$$= \frac{3}{2}.\frac{3^n-1}{3-1}+\frac{13n}{2} = \frac{3}{4}(3^n-1)+\frac{13n}{2}$$

Example - 17 : (Arithmetico-geometric series)

Find S_n and hence sum the infinite series if it is convergent.

(i) $\sum_{n=1}^{\infty}(2n-1)2^{-n}$ (ii) $\sum_{n=1}^{\infty}(2n+1)2^{n}$

Solution :

(i) $S_n = \sum_{k=1}^{n}(2k-1)2^{-k}$

$$\therefore\ S_n = \frac{1}{2}+\frac{3}{2^2}+\frac{5}{2^3}+......+\frac{2n-3}{2^{n-1}}+\frac{2n-1}{2^n}$$

$$\frac{1}{2}S_n = \frac{1}{2^2}+\frac{3}{2^3}+...... +\frac{2n-3}{2^n}+\frac{2n-1}{2^{n+1}}$$

$$\therefore (1-\frac{1}{2})S_n = \frac{1}{2}+\frac{3-1}{2^2}+\frac{5-3}{2^3}+.......+\frac{(2n-1)-(2n-3)}{2^n}-\frac{2n-1}{2^{n+1}}$$

$$= \frac{1}{2}+2\left(\frac{1}{2^2}+\frac{1}{2^3}+.......\frac{1}{2^n}\right)-\frac{2n-1}{2^{n+1}}$$

$$= \frac{1}{2}+\left(\frac{1}{2}+\frac{1}{2^2}+.......\frac{1}{2^{n-1}}\right)-\frac{2n-1}{2^{n+1}}$$

$$= \frac{1}{2} + \frac{1}{2} \cdot \frac{1 - \frac{1}{2^{n-1}}}{1 - \frac{1}{2}} - \frac{2n-1}{2^{n+1}}$$

$$= \frac{1}{2} + 1 - \frac{1}{2^{n-1}} - \frac{n}{2^n} + \frac{1}{2^{n+1}}$$

$$= \frac{3}{2} - \frac{1}{2^{n-1}} - \frac{n}{2^n} + \frac{1}{2^{n+1}}$$

$$\therefore S_n = 3 - \frac{1}{2^{n-2}} - \frac{n}{2^{n-1}} + \frac{1}{2^n}$$

$$\lim_{n \to \infty} S_n = 3$$

So the series is convergent and

$$\sum_{n=1}^{\infty} (2n-1)2^{-n} = 3$$

(ii) $S_n = \sum_{k=1}^{n} (2k+1)2^k = 3.2 + 5.2^2 + 7.2^3 + ... + (2n-1)2^{n-1} + (2n+1)2^n$

$2S_n = 3.2^2 + 5.2^3 + ... + (2n-1)2^n + (2n+1)2^{n+1}$

$\Rightarrow -S_n = 3.2 + 2.2^2 + 2.2^3 + ... 2.2^n - (2n+1)\, 2^{n+1}$

$= 6 + .2^3 + 2^4 + ... + 2^{n+1} - (2n+1)\, 2^{n+1}$

$$= 6 + 2^3 \cdot \frac{2^{n-1} - 1}{2-1} - (2n+1)\, 2^{n+1} = 6 + 2^{n+2} - 2^3 - (2n+1)\, 2^{n+1}$$

$\therefore S_n = (2n+1)2^{n+1} - 2^{n+2} + 2 = 2^{n+1}(2n-1) + 2 \rightarrow \infty$ $n \rightarrow \infty$

So the given series is not convergent.

Example - 18 :

(Series reducible to arithmetico-geometric series)

Find the sum upto infinity, assuming that the series $1 + 3x + 6x^2 + 10x^3 + ...$

has a sum for $|x| < 1$.

Solution : Let $S = 1 + 3x + 6x^2 + 10x^3 + ...$

$\Rightarrow xS = x + 3x^2 + 6x^3 + ...$

$\therefore$ $(1-x)S = 1 + 2x + 3x^2 + 4x^3 + ...$, which is an arithmetic - geometric series.

$(1-x)\,xS = x + 2x^2 + 3x^3 + ...$

$\therefore$ $(1-x)\,S-(1-x)\,xS = (1-x)^2\,S = 1 + x + x^2 + x^3 + ... = \dfrac{1}{1-x}$

[Geometric series with common ratio x, satisfying $|x| < 1$]

$\therefore$ $S = \dfrac{1}{(1-x)^3}$

Example - 19 (Application of geometic series) :

Express the recurring decimal $0.\dot{1}4285\dot{7}$ in the form $\dfrac{p}{q}$; p, q. $\in$. N.

Solution :-

$0.\dot{1}4285\dot{7} = 0.\,142857\,142857\,142857\,......$

$= 0.\,142857 + 0.\,000000\,142857 + 0.\,000000\,000000\;\,142857 + ...$

$$= \frac{142857}{10^6} + \frac{142857}{10^{12}} + \frac{142857}{10^{18}} + ...$$

$$= \frac{142857}{10^6} \cdot \left(\frac{1}{1 - \dfrac{1}{10^6}} \right) = \frac{142857}{10^6 - 1} = \frac{142857}{999999}.$$

$$= \frac{1}{7}$$

[N.B. (1) 142857 is an interesting number. Calculate $\dfrac{2}{7}, \dfrac{3}{7}$ ...... and see how its digits permute in the recurring block. If you bring 1 to the right, it becomes 3 times.]

EXERCISES - 10 (a)

(Starred exercises are not to be set in the examintion)

1. Which of the following is a sequence ?

(i) $f(x) = [x],\ x \in R$

(ii) $f(x) = |x|,\ x \in R$

(iii) $f(n) = \sqrt[n]{\pi},\ n \in N$

2. Determine whether (t_n) is an arithmetic sequence if :

(i) $t_n = an^2 + bn$

(ii) $t_n = an + b$

(iii) $t_n = an^2 + b$

3. If a geometric series converges, which of the following is true about its common ratio r ?

(i) $r > 1$

(ii) $r > 0$

(iii) $-1 < r < 1$

(iv) $r < -1$

4. If an arithmetic series Σt_n converges, which of the following is true about t_n ?

(i) $t_n < 1$

(ii) $1\ t_n\ 1 < 1$

(iii) $t_n = 0$

(iv) $t_n \rightarrow 0$

5. Which of the following is an arithmetico - geometric series ?

(i) $1 + 3x + 7x^2 + 15x^3 + ...$

(ii) $x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + ...$

(iii) $x + (1+2)x^2 + (1+2+3)x^3 + ...$

(iv) $x + 3x^2 + 5x^3 + 7x^4 + ...$

6. For an arithmetic sequence (t_n),

$t_p = q$ and $t_q = p$, $(p \neq q)$. Find t_n

7. For an arithmetic series Σa_n,

$S_q = q$ and $S_q = p$, $(p \neq q)$ Find S_{p+q} .

8. The sum of a geometric series is 3. The series of squares of its terms has sum 18. Find the series.

9. The sum of a geometric series is 14 and the series of cubes of its terms has sum 392. Find the series.

10. Find the sum as directed :

(i) $1+2a+3a^2+4a^3+......$ (first n terms $(a \neq 1)$

(ii) $1 + (1+x)y + (1+x+x^2)y^2 + (1+x+x^2+x^3)\,y^3 + \ldots..$ (to infinity),

assuming that this series has a sum for $|y| < 1$.

(iii) $1 + \frac{3}{5} + \frac{7}{25} + \frac{15}{125} + \frac{31}{625} + \ldots\ldots$ (to infinity)

(iv) $1 + 4x + 8x^2 + 13x^3 + 19x^4 + \ldots\ldots$ (to infinity) ;

assuming that the series has a sum, for $|x| < 1$.

(v) $3.2 + 5.2^2 + 7.2^3 + \ldots\ldots$ (first n terms)

11. Find the sum of the infinitie series :

(i) $\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \ldots..$

(ii) $\frac{1}{1.2.3} + \frac{1}{2.3.4} + \frac{1}{3.4.5} + \ldots..$

(iii) $\frac{1}{2.5.8} + \frac{1}{5.8.11} + \frac{1}{8.11.14} + \ldots..$

(iv) $\frac{3}{1^2.2^2} + \frac{5}{2^2.3^2} + \frac{7}{3^2.4^2} + \ldots\ldots$

[Hint : take $t_n = \frac{2n+1}{n^2(n+1)^2} = \frac{(n+1)^2 - n^2}{n^2(n+1)^2}$]

(v) $\frac{1}{1.5} + \frac{1}{3.7} + \frac{1}{5.9} + \ldots..$

[Hint : Take $t_n = \frac{1}{(2n-1)\,(2n+3)} = \frac{2n+1}{(2n-1)\,(2n+1)\,(2n+3)}$

$$= \frac{2n-1+2}{(2n-1)\,(2n+1)\,(2n+3)} = \frac{1}{(2n+1)\,(2n+3)} + \frac{2}{(2n-1)\,(2n+1)\,(2n+3)}$$

and apply the method of differences.

12. Find S_n for the series :

(i) $1.2 + 2.3 + 3.4 + \ldots\ldots$

(ii) $1.2.3 + 2.3.4 + 3.4.5 + \ldots..$

(iii) $2.5.8 + 5.8.11 + 8.11.14 + \ldots..$

(iv) $1.2.3.4 + 2.3.4.5 + 3.4.5.6 + \ldots..$

[Hint : t_n = (3n - 1) (3n+2) (3n+5)]

(v) 1.5 + 2.6 + 3.7 + ...

[Hint : t_n = n (n+4) is not a product of two successive terms of an A.P. for, the term following n should be n+1, not n+4. So method of previous exercises is not applicable. In stead, write $t_n = n^2 + 4n$ and find $S_n = \sum_{k=1}^{n} k^2 + 4\sum_{k=1}^{n} k$ applying formulae.]

(vi) 2.3 + 3.6 + 4.11 + ...

[Hint : Take t_n = (n+1 (n^2+2)]

(vii) $1.3^2 + 2.5^2 + 3.7^2$ + ...

13. Find the sum of first n terms of the series :

(i) 5 + 6 + 8 + 12 + 20 + ...

(ii) 4 + 5 + 8 + 13 + 20 + ...

14. Find the sum of the products of 1, 2 , 3 ....., 20 taken two at a time.

[Hint : Required sum = $\frac{1}{2}\left\{\left(\sum_{k=1}^{20} k\right)^2 - \sum_{k=1}^{20} k^2\right\}$

(ii) Do the same for 1, 3, 5, 7, ... , 19

15. If a = 1 + x + x^2+x^3+ ..... and b = 1+y+y^2+y^3+.....; $|x| < 1$ and $|y| < 1$, then prove that

$$1 + xy + x^2y^2+x^3y^3+ ...... = \frac{ab}{a+b-1}$$

16. If *a, b, c* are respectively the p^{th}, q^{th}, r^{th} terms of an A.P., then prove that

$a(q-r)+b(r-p)+c(p-q) = 0$

17. If $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A.P, and $a+b+c \neq 0$, prove that $\frac{b+c}{a}, \frac{c+a}{b}, \frac{a+b}{c}$ are in A.P.

18. If a^2, b^2, c^2 are in A.P., prove that $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ are in A.P.

19. If $\frac{b+c}{a}, \frac{c+a}{b}, \frac{a+b}{c}$ are in A.P., prove that $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A.P., given $a+b+c \neq 0$

20. If $(b-c)^2$, $(c-a)^2$, $(a-b)^2$ are in A.P., prove that $\frac{1}{b-c}, \frac{1}{c-a}, \frac{1}{a-b}$ are in A.P.

21. If *a, b, c* are respectively the sums of p, q, r terms of an A.P, prove that

$$\frac{a}{p}(q-r)+\frac{b}{q}(r-p)+\frac{c}{r}(p-q)=0.$$

22. If a, b, c, d are in G.P., prove that

$(a^2+b^2+c^2)\ (b^2+c^2+d^2) = (ab+bc+cd)^2$

10.7 Binomial, Exponential and Logarithmic series :

Binomial series :

We know from binomial theorem for positive integral index n, that

$$(1+x)^n = 1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+...+\frac{n(n-1)...2.1}{n!}x^n \quad (1)$$

The above expression (or expansion) of $(1+x)^n$ holds for any value of x.

We now state without proof that

Theorem (Binomial Theorem for a Real Index)

$$(1+x)^\alpha = 1+\alpha x+\frac{\alpha(\alpha-1)}{2!}x^2+\frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3+...+\frac{\alpha(\alpha-1)...(\alpha-n+1)}{n!}x^n+... \quad (2)$$

Where $\alpha \in$ R, provided $|x| < 1$.

(The proof is beyond the scope of the book)

The infinite series in (2) is known as the **Binomial Series** and it has the sum $\mathbf{(1+x)^\alpha}$ for $|x| < 1$.

Also observe that if $\alpha = n \in$ N, the binomial series in (2) becomes the finite series in (1) and has the sum $(1+x)^n$ without any restriction on x.

Application of Binomial Series

Applying the binomial series, you can now prove that :

For $|x| < 1$,

$(1-x)^{-1} = 1 + x + x^2 + x^3 + ...$ ....**(3)**

$(1+x)^{-1} = 1-x + x^2 - x^3 + ...$ ....*(4)*

$(1+x)^{-2} = 1- 2x +3x^2 - 4x^3 + ...$ ....**(5)**

$(1-x)^{-2} = 1+2x +3x^2 + 4x^3 + ...$ ....**(6)**

Series Expansion of a Function :

If a series $t_0 + t_1 + t_2 +$ ...... has a finite sum 's' we say that the series is convergent and write.

$s = t_0 + t_1 + t_2 + t_3 +$ ......

If f(x) is the sum of a series, the series is said to be an expansion of f(x). As seen eartier, $1+x+x^2+x^3+$...... is an expansion of $(1-x)^{-1}$. We now discuss expansions of exponential and logarithmic functions.

The Exponential Series : We know

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e,\ \ n\in N.$$

Now if x is any real number, by the Binomial Theorem for real index we get, for n > 1

$$\left(1+\frac{1}{n}\right)^{nx} = 1+nx\ .\ \frac{1}{n} + \frac{nx(nx-1)}{2\ !}.\frac{1}{n^2}+\frac{nx(nx-1)\ (nx-2)}{3\ !}.\frac{1}{n^3}+......$$

$$= 1 + x + \frac{x(x-\frac{1}{n})}{2\ !}. + \frac{x(x-\frac{1}{n})\ (x-\frac{2}{n})}{3\ !} +...... \qquad ....(7)$$

Since $(1+\frac{1}{n})^{nx} = \left\{(1+\frac{1}{n})^n\right\}^x$, taking limit as $n\to\infty$, in (8) we get

$$\boxed{e^x = 1+\frac{x}{1\ !}+\frac{x^2}{2!}+\frac{x^3}{3!}+......\ \text{for } x\in R} \qquad ....(8)$$

The series on the R.H.S is called the exponential series and also called the expansion of e^x.

Now let $a > 0$ and $b = \log_e a$, i.e $e^b = a$. Writing bx for x we have from (8)

$$e^{bx} = 1+bx+\frac{b^2x^2}{2\ !}+\frac{b^2x^3}{3!}+$$

$$\Rightarrow a^x = (e^b)^x = e^{bx} = 1 + bx + \frac{b^2x^2}{2\ !} + \frac{b^3x^3}{3!} +$$

$$\Rightarrow a^x = 1 + x\log_e a + \frac{x^2(\log_e a)^2}{2!} + \frac{x^3(\log_e a)^3}{3!} + \qquad ....(9)$$

The Logarithmic Series :

From (9) we have for $a > 0$ and $y \in R$

$$a^y = 1+ y\log_e a + \frac{y^2(\log_e a)^2}{2!} + \frac{y^3(\log_e a)^3}{3!} + .$$

[We may also use $l_n a$ for $\log_e a$]

Now let $a = 1+x > 0$ so that $x > -1$. Then

$$(1+x)^y = 1+ y \log_e (1+x) + \frac{y^2}{2!} \{\log_e (1+x)\}^2 + \quad \text{................(10)}$$

But futher restricting x so that x<1, i.e $|x| < 1$, we use (2) to obtain

$$(1+x)^y = 1+ yx + \frac{y(y-1)}{2!}x^2 + \frac{y(y-1)(y-2)}{3!}x^3 + \quad \text{................(11)}$$

The R.H.S of (10) and (11) being identical, the two series in powers of y must have identical coefficients of y, y^2 etc.

Coefficients of y on the R.H.S. of (11) is $x - \frac{x^2}{2} + \frac{x^3}{3} - \text{.............}$

Hence, equating coeflicients of y in (10) and (11), we have

$$\log_e (1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \text{.....}, \ |x| < 1 \quad \text{................(12)}$$

The series on the R.H.S of (12) is called the logarithmic series and also called the expansion of $\log_e (1+x)$. It is known that (12) is valied for x =1 also.

$$\text{Hence } \log_e (1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \text{.....}, \ -1 < x \leq 1 \text{ for } -1 < x \leq 1 \quad \text{................(13)}$$

$$\text{Putting } x = 1, \ \log_e 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \text{.....} \quad \text{................(14)}$$

Now writing - x for x in (12) we have

$$\log_e (1-x) = - x - \frac{x^2}{2} - \frac{x^3}{3} - \text{.....}, \ -1 \leq x < 1 \quad \text{................(15)}$$

From (13) and (15) it easily follows that

$$\Rightarrow \log_e \left(\frac{1+x}{1-x}\right) = 2 \left(x + \frac{x^3}{3} + \frac{x^5}{5} + \text{......}\right), \ |x| < 1 \quad \text{................(16)}$$

Example -20 Find the values of the following correct upto 3 places of decimal.

i) $\frac{1}{e}$ (ii) $\frac{1}{\sqrt[3]{e}}$

Solution : **We have for $x \in R$**

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \text{......}$$

Taking x = - 1,

$$\frac{1}{e} = e^{-1} = 1 - 1 + \frac{(-1)^2}{2!} + \frac{(-1)^3}{3!} + \frac{(-1)^4}{4!} + \frac{(-1)^5}{5!} + \frac{(-1)^6}{6!} + ...$$

$$= \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \frac{1}{720} - \frac{1}{5040}$$

$$= 0.5 - 0.1666 + 0.0416 - 0.0083 - .0001$$

$$= 0.368$$

Taking $x = \frac{-1}{3}$

$$\frac{1}{\sqrt[3]{e}} = e^{-\frac{1}{3}} = 1 - \frac{\left(\frac{1}{3}\right)}{1!} + \frac{\left(\frac{1}{3}\right)^2}{2!} - \frac{\left(\frac{1}{3}\right)^3}{3!} + \frac{\left(\frac{1}{3}\right)^4}{4!} - \frac{\left(\frac{1}{3}\right)^5}{5!} + ...$$

$$= 1 - \frac{1}{3} + \frac{1}{18} - \frac{1}{162} + \frac{1}{1944} - \frac{1}{29160} + ...$$

$$= 1 - 0.3333 + 0.0555 - 0.0061 + 0.0005$$

$$= 0.717$$

Example 21 Prove that

$$1 + \frac{2^3}{2!} + \frac{3^3}{3!} + \frac{4^3}{4!} + ... = 5e$$

Solution : The nth term of the given series is

$$t_n = \frac{n^3}{n!} = \frac{n^2}{(n-1)!} = \frac{(n-1)(n-2) + 3n - 2}{(n-1)!}$$

$$= \frac{1}{(n-3)!} + \frac{3(n-1) + 1}{(n-1)!}$$

$$= \frac{1}{(n-3)!} + \frac{3}{(n-2)!} + \frac{1}{(n-1)!}$$

Taking n = 3, 4, 5, ...

$$t_3 = 1 + \frac{3}{1!} + \frac{1}{2!}$$

$$t_4 = \frac{1}{1!} + \frac{3}{2!} + \frac{1}{3!}$$

$$t_5 = \frac{1}{2!} + \frac{3}{3!} + \frac{1}{4!}$$

Adding column wise

$$t_3 + t_4 + t_5 + \ldots\ldots = (1 + \frac{1}{1!} + \frac{1}{2!} + \ldots\ldots) + 3\,(\frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots\ldots)$$

$$+ (\frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots\ldots)$$

$$= e + 3\,(e-1) + (e-1 - \frac{1}{1!})$$

$$= e + 3e - 3 + e - 2 = 5e - 5$$

Hence the given series = $t_1 + t_2 + t_3 + \ldots\ldots$

$$= 1 + \frac{2^3}{2} + (5e - 5) = 5e$$

Example 22 : Sum the series

$$9 + \frac{16}{2!} + \frac{27}{3!} + \frac{42}{4!} + \ldots\ldots$$

Solution : Observe first the differences of numerators of consecutive terms are in A.P. . Let t_n denote the nth term of the numerators.

Let S = 9 + 16 + 27 + 42 + ... t_n

Also write S = 9 + 16 + 27 + ... + $t_{n-1} + t_n$

Subtracting

$0 = 9 + 7 + 11 + 15 + \ldots\ldots$ to n terms $- t_n$

or , $t_n = 9 + (7 + 11 + 15 + \ldots\ldots$ to n-1 terms)

or , $t_n = 9 + \dfrac{(n-1)}{2}\ \{2.7 + \overline{n-2}\,.4\} = 2n^2 + n + 6.$

$\therefore$ The nth term of the given series

$$T_n = \frac{t_n}{n!} = \frac{2n^2+n+6}{n!}$$

$$= \frac{2n(n-1) + 3n + 6}{n!}$$

$$= \frac{2}{(n-2)!} + \frac{3}{(n-1)!} + \frac{6}{n!}$$

Taking $n = 2, 3, 4, ...$

$$T_2 = = \frac{2}{0!} + \frac{3}{1!} + \frac{6}{2!}$$

$$T_3 = = \frac{2}{1!} + \frac{3}{2!} + \frac{6}{3!}$$

$$T_4 = \frac{2}{2!} + \frac{3}{3!} + \frac{6}{4!}$$

....

....

Adding column wise

$$\sum_{n=2}^{\infty} T_n = 2\left(1+\frac{1}{1!} + \frac{1}{2!} + ...\right) + 3\left(\frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + ...\right) + 6\left(\frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + ...\right)$$

$$= 2e + 3(e-1) + 6\left(e-1 - \frac{1}{1!}\right)$$

$$= 2e + 3e - 3 + 6e - 12 = 11e - 15.$$

Hence sum of the given series

$$\sum_{n=1}^{\infty} T_n = 9 + (11e - 15) = 11e - 6.$$

* **Example 23 :** Find the cuefficient of x^n in the expansion of

(i) $(1 - ax + x^2)/ e^x$

(ii) $\log_e (1 - x + x^2), |x| < 1$

Solution :

(i) $\dfrac{1-ax+x^2}{e^x} = (1 - ax + x^2)\, e^{-x}$

$$= (1 - ax + x^2)\left(1-x + \frac{x^2}{2!} - \frac{x^3}{3!} + ... + \frac{(-1)^n x^n}{n!} + ...\right)$$

Hence the coefficient of x^n

$$= \frac{(-1)^n}{n!} - \frac{a(-1)^{n-1}}{(n-1)!} + \frac{(-1)^{n-2}}{(n-2)!}$$

$$= \frac{(-1)^n}{n!}[1 + an + n(n-1)]$$

(ii) $\log_e(1 - x + x^2) = \log_e\left(\frac{1+x^3}{1+x}\right)$

$$= \log_e(1+x^3) - \log_e(1+x)$$

$$= \left\{x^3 - \frac{x^6}{2} + \frac{x^9}{3} - \ldots\ldots + (-1)^{r+1}\frac{x^{3r}}{r} + \ldots\ldots\right\}$$

$$- \left\{x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots\ldots + (-1)^{r+1}\frac{x^r}{r} + \ldots\ldots\right\}$$

If n is a mulfiple of 3, 3r (say), then the coefficient of x^n is $\frac{(-1)^{r+1}}{r}$ from the first series on the right and $\frac{-(-1)^{3r+1}}{3r}$ from the second series. Hence the coefficient is

$$\frac{(-1)^{r+1}}{r} - \frac{(-1)^{3r+1}}{3r} = \frac{(-1)^r}{3r}[(-1).\ 3 - (-1)^{2r+1}]$$

$$= \frac{-2(-1)^r}{3r} = \frac{-2(-1)^{n/3}}{n}$$

If r is not a multiple of 3 then x^n doesnot occur in the first series and hence the required coefficient is $\frac{(-1)^n}{n}$.

When working with series one should observe carefully the terms of the series. Look at the following example.

Example 24 : If $|x| < 1$ and $y = -x^3 - \frac{x^6}{2} - \frac{x^9}{3} - \ldots\ldots$ then prove that $x^3 = 1 - e^y$.

Solution : $y = (x^3) - \frac{(x^3)^2}{2} - \frac{(x^3)^3}{3} - \ldots\ldots$

$= \log_e (1-x^3)$

$\Rightarrow e^y = 1 - x^3 \Rightarrow x^3 = 1 - e^y$

* **Example 25 :** If $|x| < 1$, prove that

$$\log_e [(1+x)^{1+x}(1-x)^{1-x}] = 2\left(\frac{x^3}{1.2} + \frac{x^4}{3.4} + \frac{x^6}{5.6} + \ldots\ldots\right)$$

Solution :

L.H.S. $= (1+x)\log_e(1+x) + (1-x)\log_e(1-x)$

$= \log_e(1+x) + x\log_e(1+x) + \log_e(1-x) - x\log_e(1-x)$

$= [\log_e(1+x) + \log_e(1-x)] + x[\log_e(1+x) - \log_e(1-x)]$ ....(i)

Now

$$\log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots$$

and $$\log_e(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots$$

$$\Rightarrow \log_e(1+x) + \log_e(1-x) = -2\left[\frac{x^2}{2} - \frac{x^4}{4} - \frac{x^6}{6} + \ldots\ldots\right]$$

and $$\log_e(1+x) - \log_e(1-x) = 2.\left[x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots\ldots\right]$$

Substitution in (i) gives

$$\text{L.H.S} = -2\left[\frac{x^2}{2} + \frac{x^4}{4} + \frac{x^6}{6} + \ldots\ldots\right] + 2\left[x^2 + \frac{x^4}{3} + \frac{x^6}{5} + \ldots\ldots\right]$$

$$= 2\left[\left(1 - \frac{1}{2}\right)x^2 + \left(\frac{1}{3} - \frac{1}{4}\right)x^4 + \left(\frac{1}{5} - \frac{1}{6}\right)x^6 + \ldots\ldots\right]$$

$$= 2\left[\frac{x^2}{1.2} + \frac{x^4}{3.4} + \frac{x^6}{5.6} + \ldots\ldots\right] = \text{R.H.S}$$

EXERCISE - 10 (b)

[Starred problems are not to be set in the examination.]

Assume $|x| < 1$ whenever required

1. Expand in ascending powers of x

   (i) 2^x

   ✱ (ii) Cos x [Hint : Write $\cos x = \frac{1}{2}(e^{ix} + e^{-ix})$ and assume :

   $e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots\ldots$ when z is complex]

   (iii) Sin x

   (iv) $(x\, e^{7x} - e^{-x}) / e^{3x}$

   ✱ (v) e^{e^x} upto the term containing x^4

   [Hint : write $e^{e^x} = e.e^y$ where $y = x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots$]

2. If $x = y + \frac{y^2}{2!} + \frac{y^3}{3!} + \ldots\ldots$ show that $y = x - \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \ldots\ldots$

3. Find the value of $x^2 - y^2 + \frac{1}{2!}(x^4 - y^4) + \frac{1}{3!}(x^6 - y^6) + \ldots\ldots$

4. Show that

   (i) $2\left(\frac{1}{3!} + \frac{2}{5!} + \frac{3}{7!} + \ldots\ldots\right) = \frac{1}{e}$

   (ii) $\frac{9}{1!} + \frac{19}{2!} + \frac{35}{3!} + \frac{57}{4!} + \frac{85}{5!} + \ldots\ldots = 12e^{-5}$

   (iii) $1 + \frac{1+3}{2!} + \frac{1+3+3^2}{3!} + \ldots\ldots = \frac{1}{2}(e^3 - e)$

   (iv) $\frac{1.3}{1!} + \frac{2.4}{2!} + \frac{3.5}{3!} + \frac{4.6}{4!} + \ldots\ldots = 4e$

   ✱ (v) $\frac{1}{1.2} + \frac{1.3}{1.2.3.4} + \frac{1.3.5}{1.2.3.4.5.6} + \ldots\ldots = \sqrt{e}$ [Hint : $t_n = \frac{\left(\frac{1}{2}\right)^n}{n!}$

✶ 5. Prove that

(i) $\log_e (1+3x+2x^2) = 3x - \frac{5}{2}x^2 + \frac{9}{3}x^3 - \frac{17}{4}x^4 + \ldots\ldots \quad |x| < \frac{1}{2}$

(ii) $\log_e (n+1) - \log_e (n-1) = 2\left[\frac{a}{n} + \frac{a^3}{3n^3} + \frac{a^5}{5n^5} + \ldots\ldots\right]$

(iii) $\log_e (n+1) - \log_e n = 2\left[\frac{1}{2n+1} + \frac{1}{3(2n+1)^3} + \frac{1}{5(2n+1)^5} + \ldots\ldots\right]$

(iv) $\log_e m - \log_e n = \frac{m-n}{m} + \frac{1}{2}\left(\frac{m-n}{m}\right)^2 + \frac{1}{3}\left(\frac{m-n}{m}\right)^3 + \ldots\ldots, \ m, n > 0$

(v) $\log_e a - \log_e b = 2\left[\frac{a-b}{a+b} + \frac{1}{3}\left(\frac{a-b}{a+b}\right)^3 + \frac{1}{5}\left(\frac{a-b}{a+b}\right)^5 + \ldots\ldots\right] \ a > b.$

(vi) $\log_e n = \frac{n-1}{n+1} + \frac{1}{2}\frac{n^2-1}{(n+1)^2} + \frac{1}{3}\frac{n^3-1}{(n+1)^3} + \ldots\ldots$

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