CHAPTER 11

Straight Lines

Some persons have contended that mathematics ought to be taught by making the illustrations obvious to the senses. Nothing can be more absurd or injurious; it ought to be our never-ceasing effort to make people think, not feel.

- Coleridge

11.0 Introduction :

Application of algebra to study the geometrical figures in plane was first initiated by the French mathematician Rene - Descartes (1596 - 1665). In 1619, Descartes announced that by using algebra to study the geometry of plane figures, he had discovered an entirely new science which would help to solve all the problems of geometry. This new science he named as Analytical Geometry. Since then, the subject of Analytic Geometry has been an important and powerful tool for both physicists and the mathematicians as well.

11.1 Fundamental Concepts

The fundamental concepts of plane geometry which form a basis of introduction of coordinate geometry have been presented in article 4.01 of chapter-4. The reader is also advised to go through the rectangular Cartesian Co-ordinate System from article 4.1 of that chapter.

The essence of co-ordinate geometry is the interpretation of a geometrical phenomenon in terms of algebra. Consequently, a question on geometry turns out to be a question involving certain algebraic equations.

After going through the introduction of the Cartesian coordinates, one can easily notice that the set of points on the co-ordinate plane can be taken as the Cartesian product R × R where R is the set of real numbers. The geometry that is based on co-ordinates of points on a plane is, therefore, known as co-ordinate geometry or the geometry of R^2.

Division :

Definitions :

Internal Division : If A – P – B, then P is said to divide the line segment $\overline{AB}$ internally into the segments $\overline{PA}$ and $\overline{PB}$; the ratio of internal division being given by either PA : PB or PB : PA.

(In case of internal division, PA + PB = AB)

External Division : If P – A – B or A – B – P, then P is said the divide the line segment $\overline{AB}$ externally into the segments $\overline{PA}$ and $\overline{PB}$; the ratio of external division being given by either PA : PB or PB : PA.

(In case of external division, | PA – PB | = AB.)

Note : (a) A statement like 'P divides $\overline{AB}$ internally / externally in ratio $m : n$ is vague in as much as it does not clarify which of the ratios $\frac{PA}{PB}$ or $\frac{PB}{PA}$ is equal to $\frac{m}{n}$. To clarify the meaning of a statement as the above one we mention below a convention which is usually followed.

Convention :

I. We write 'P divides $\overline{AB}$ or the line segement joining A and B in ratio $m : n$' to mean that $\frac{PA}{PB} = \frac{m}{n}$.

II. We write 'P divides $\overline{BA}$ or line segment joinging B and A in ratio $m : n$' to mean that $\frac{PB}{PA} = \frac{m}{n}$.

[Note that $\overline{AB}$ and $\overline{BA}$ never have different meanings. Both of them mean the same line segment. The different interpretations of the ratio $m : n$, depending on the mention of $\overline{AB}$ or $\overline{BA}$ in a statement, is merely a matter of convention.

The convention is applicable to both external and internal division only when the ratio PA : PB or PB : PA of the segments is not explicitly mentioned.]

(b) In case of external division,

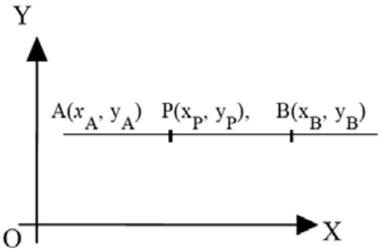
$$\frac{PA}{PB} < 1 \Rightarrow P - A - B \text{ and } \frac{PA}{PB} > 1 \Rightarrow A - B - P.$$

Alternatively,

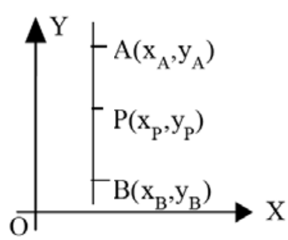
$$\frac{PB}{PA} > 1 \Rightarrow P - A - B \text{ and } \frac{PB}{PA} < 1 \Rightarrow A - B - P.$$

Internal Division Formula : If P (x_P, y_P) divides $\overline{AB}$, the line segment joining A (x_A, y_A) and B (x_B, y_B) internally so that

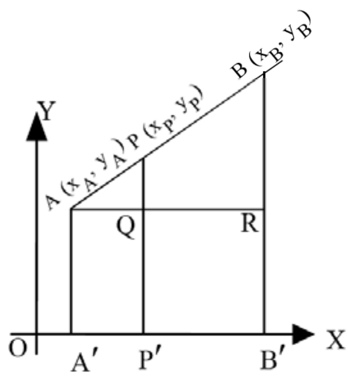
$$\frac{PA}{PB} = \frac{m}{n}, \text{ then } x_P = \frac{mx_B + nx_A}{m+n} \text{ and } y_P = \frac{my_B + ny_A}{m+n}.$$



(a)



(b)



(c)

Proof :

(a) If $\overline{AB}$ is parallel to x – axis [Fig (a)], then PA = $|x_P - x_A|$ and PB = $|x_B - x_P|$.

$$\frac{PA}{PB} = \frac{m}{n} \Rightarrow \frac{|x_P - x_A|}{|x_B - x_P|} = \frac{x_P - x_A}{x_B - x_P} = \frac{m}{n}.$$

$\Rightarrow mx_B - mx_P = nx_P - nx_A$

$\Rightarrow mx_B + nx_A = (m + n)\, x_P.$

$$\therefore\ x_P = \frac{mx_B + nx_A}{m+n}.$$

In this case, $y_A = y_P = y_B$

$\therefore\ my_B + ny_A = (m + n)\, y_P$

$$\Rightarrow y_P = \frac{my_B + ny_A}{m+n}.$$

(b) If $\overline{AB}$ is parallel to y – axis [Fig (b)], then

$x_A = x_P = x_B$.

$\therefore\ mx_B + nx_A = (m + n)\, x_P$

$$\Rightarrow x_P = \frac{mx_B + nx_A}{m+n}.$$

Also, PA = $|y_A - y_P|$, PB = $|y_P - y_B|$.

$$\frac{PA}{PB} = \frac{m}{n} \Rightarrow \frac{|y_A - y_P|}{|y_P - y_B|} = \frac{y_A - y_P}{y_P - y_B} = \frac{m}{n}.$$

$$\therefore\ y_P = \frac{my_B + ny_A}{m+n}.$$

(c) If $\overline{AB}$ is not parallel to any of the axes [Fig (c)], let $\overline{AA'}$, $\overline{PP'}$ and $\overline{BB'}$ be perpendicular to x - axis, meeting it at A′, P′ and B′ respectively. Let $\overline{AR}$ be perpendicular to $\overline{BB'}$, meeting it at R. Let $\overline{AR}$ intersect $\overline{PP'}$ at Q.

Obviously ΔAPQ and Δ ABR are similar.

$$\therefore \frac{AP}{PB} = \frac{AQ}{AR} = \frac{PQ}{BR}.$$

It is easy to see that,

AQ = A′P′ = | OP′ – OA′ | = $|x_P - x_A|$,

AR = A′B′ = | OB′ – OA′ | = $|x_B - x_A|$,

PQ = | PP′ – QP′ | = | PP′ – AA′ | = $|y_P - y_A|$,

BR = | BB′ – RB′ | = | BB′ – AA′ | = $|y_B - y_A|$.

Also, $\dfrac{PA}{PB} = \dfrac{m}{n} \Rightarrow \dfrac{AP}{AB} = \dfrac{m}{m+n}$.

$$\frac{AP}{AB} = \frac{AQ}{AR} \Rightarrow \frac{m}{m+n} = \frac{|x_P - x_A|}{|x_B - x_A|} = \frac{x_P - x_A}{x_B - x_A}$$

$\Rightarrow mx_B - mx_A = (m + n)\, x_p - mx_A - nx_A$

$\Rightarrow (m + n)\, x_P = mx_B + nx_A$

$\Rightarrow x_P = \dfrac{mx_B + nx_A}{m+n}$.

Similarly, $\dfrac{AP}{AB} = \dfrac{PQ}{BR} \Rightarrow \dfrac{m}{m+n} = \dfrac{|y_P - y_A|}{|y_B - y_A|} = \dfrac{y_P - y_A}{y_B - y_A}$

$y_P = \dfrac{my_B + ny_A}{m+n}$.

Corollary (Midpoint formula) :

If P (x, y) is the mid point of the line segment joining A(x_1, y_1) and B (x_2, y_2), then

x $= \dfrac{x_1 + x_2}{2}$, $y = \dfrac{y_1 + y_2}{2}$. (Taking PA : PB = 1 : 1)

External Division Formula :

If P(x_P, y_P) divides $\overline{AB}$, the line segment joining A(x_A, y_A) and B (x_B, y_B) externally so that $\dfrac{PA}{PB} = \dfrac{m}{n}$, then

$x_P = \dfrac{mx_B - nx_A}{m-n}$ and $y_P = \dfrac{my_B - ny_A}{m-n}$.

Proof : Suppose A – B – P.

Then AB + BP = AP. (refer definition of betweenness)

$\therefore \dfrac{PA}{PB} = \dfrac{m}{n} \Rightarrow \dfrac{PB + BA}{PB} = 1 + \dfrac{BA}{BP} = \dfrac{m}{n}$

$\Rightarrow \dfrac{BA}{BP} = \dfrac{m}{n} - 1 = \dfrac{m-n}{n}$.

Thus B divides $\overline{AP}$ internally, the ratio of internal division being given by $\dfrac{BA}{BP} = \dfrac{m-n}{n}$. Therefore, by internal division formula, we have

$x_B = \dfrac{(m-n)x_p + nx_A}{m-n+n} = \dfrac{(m-n)x_p + nx_A}{m}$

and $y_B = \dfrac{(m-n)y_p + ny_A}{m-n+n} = \dfrac{(m-n)y_p + ny_A}{m}$.

It now easily follows that

$x_P = \dfrac{mx_B - nx_A}{m-n}$ and $y_P = \dfrac{my_B - ny_A}{m-n}$.

Similarly, if P – A – B, we can apply internal division formula to the co-ordinates of A and obtain the above expressions. Note that in case of external division $m - n \neq 0$. (Why ?)

Coordinates of any point on the line joining the distinct points A (x_1, y_1) and B (x_2, y_2).

The ratio m: n of internal or external division of the line segment $\overline{\text{AB}}$ by P(x, y) can also be expressed as $\frac{m}{n}$: 1 or μ : 1 $\left(\mu = \frac{m}{n}\right)$.

$$\therefore\ x = \frac{x_1 + \mu x_2}{1+\mu},\ y = \frac{y_1 + \mu y_2}{1+\mu} \text{ (internal division)}$$

$$\text{and } x = \frac{x_1 - \mu x_2}{1-\mu},\ y = \frac{y_1 - \mu y_2}{1-\mu} \text{ (external division).}$$

In case of external devision $\mu \neq 1$ as, in this case $m \neq n$.

Both these results can be put together in the form :

If P (x, y) is a point (different from B) on $\overleftrightarrow{\text{AB}}$

$$\text{then } x = \frac{x_1 + \lambda x_2}{1+\lambda},\ y = \frac{y_1 + \lambda y_2}{1+\lambda} \quad (\lambda \in \text{R},\ \lambda \neq -1).$$

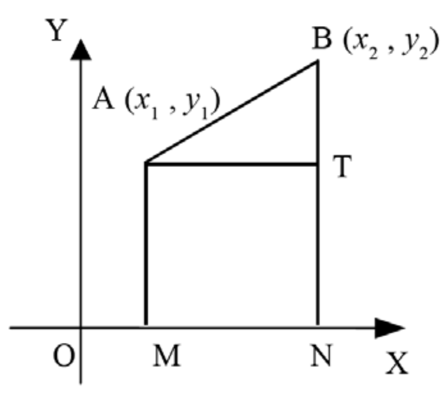
[$\lambda = 0$ means that P coincides with A which is not a case of internal or external division. If P coincides with B we cannot have the above expressions for its co-ordinates. Also $\lambda > 0 \Leftrightarrow$ A – P – B and $\lambda < 0 \Leftrightarrow$ P – A – B or A – B – P]

Some Useful Formulae

Distance Formula

If (x_1, y_1) and B (x_2, y_2) be two points in the plane then the distance AB or d (A, B) is given by, d (A, B) = $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$.

Proof : Supposing $\overleftrightarrow{\text{AB}}$ not parallel to either of the axes, let $\overline{\text{AM}}$ and $\overline{\text{BN}}$ be prependiculars drawn from A and B to x - axis; and $\overline{\text{AT}}$ be the perpendicular from A to $\overleftrightarrow{\text{BN}}$. (Fig. 2) Then, from the right angled triangle ABT, (By Pythagoras Theorem),



$$\text{AB}^2 = \text{AT}^2 + \text{BT}^2 = \text{MN}^2 + \text{BT}^2$$

$$= (x_2 - x_1)^2 + (y_2 - y_1)^2$$

$$\Rightarrow d\text{ (A, B)} = \text{AB} = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

The result is trivaial if $\overleftrightarrow{\text{AB}}$ is parallel to either of the axes.

Corollary : The distance of a point P (x, y) from the origin is OP = $\sqrt{x^2 + y^2}$.

Formula for Area of a Triangle :

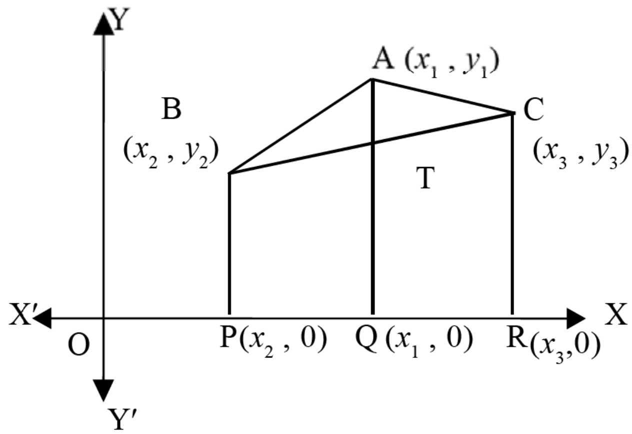
The area of a triangle with vertices at A (x_1, y_1), B (x_2, y_2) and C(x_3, y_3) is $|\Delta|$, where,

$$\Delta = \frac{1}{2}\{x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\}$$

$$= \frac{1}{2}\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{vmatrix}.$$

(After knowing diterminants in Vol-II, this expression will be convenient to use. For the time being, you may skip this expression)

Proof : Consider the triangle ABC with vertices at A(x_1, y_1), B (x_2, y_2) and C(x_3, y_3). Drop perpendiculars $\overline{BP}$, $\overline{AQ}$ and $\overline{CR}$ on x – axis.



If $|\Delta|$ denotes the area of the triangle ABC, then

Δ = area of trapezium ABPQ + area of trapezium AQRC – area of trapezium BPRC.

$$= \frac{1}{2}(BP + AQ)(PQ) + \frac{1}{2}(AQ + CR).(QR) - \frac{1}{2}(BP + CR).(PR)$$

$$= \frac{1}{2}\{(y_1 + y_2)(x_1 - x_2) + (y_3 + y_1)(x_3 - x_1) + (y_3 + y_2)(x_3 - x_2)\}$$

$$= \frac{1}{2}\{x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\}.$$

Since, at times Δ turns out to be negative, we take the area as $|\Delta|$.

Corollary : (Collinearity of three points)

The three points A(x_1, y_1), B (x_2, y_2) and C (x_3, y_3) will be collinear if and only if the expression for area of the triangle ABC is zero, which implies

$x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) = 0$

or $\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{vmatrix} = 0.$

Example 1 :

The distance between two points P (3, –1) and Q (–1, 1) is

$$PQ = \sqrt{\{3 - (-1)\}^2 + (-1 - 1)^2} = \sqrt{4^2 + 2^2} = \sqrt{20}.$$

Example 2 :

Show that the triangle with vertices A (–3, 1), B (5, 4) and C (0, –7) is isosceles and right angled, having area equal to $\frac{73}{2}$ sq. units.

By distance formula,

$$AB = \sqrt{(-3-5)^2+(1-4)^2} = \sqrt{64+9} = \sqrt{73}.$$

$$BC = \sqrt{(5-0)^2+\{4-(-7)\}^2} = \sqrt{25+121} = \sqrt{146}.$$

and, $AC = \sqrt{(-3-0)^2+\{1-(-7)\}^2} = \sqrt{9+64} = \sqrt{73}.$

Clearly, AB = AC, which shows that the triangle is isosceles. Also,

$$AB^2 + AC^2 = 73 + 73 = 146 = BC^2.$$

So m∠A = 90° i.e. the triangle is right angled.

Now, area of ABC = $\frac{1}{2}$ AB. AC = $\frac{1}{2}\sqrt{73}\,.\,\sqrt{73}$ sq. units = $\frac{73}{2}$ sq. units.

Example 3 :

Show that the point $\left(\frac{7}{4},-\frac{1}{4}\right)$ divides the line segment joining the points (1, 2) and (2, –1), internally in ratio 3 : 1. Find the co-ordinates of the point which divides the line segment joining the two given points externally in ratio 3 : 1.

Let the given points be A (1, 2) and B (2, – 1). If (x, y) divides $\overline{AB}$ internally in ratio 3 : 1,

then $x = \frac{3.2+1.1}{4} = \frac{7}{4}$,

$$y = \frac{3(-1)+1.2}{4} = -\frac{1}{4}\,. \quad \square$$

If Q (x_1, y_1) divides $\overline{AB}$ externally in ratio 3 : 1, then by external division formula,

$$x_1 = \frac{3.2-1.1}{3-1} = \frac{5}{2},\ y_1 = \frac{3(-1)-1.2}{3-1} = -\frac{5}{2}.$$

Hence the point Q is $\left(\frac{5}{2},-\frac{5}{2}\right)$. $\square$

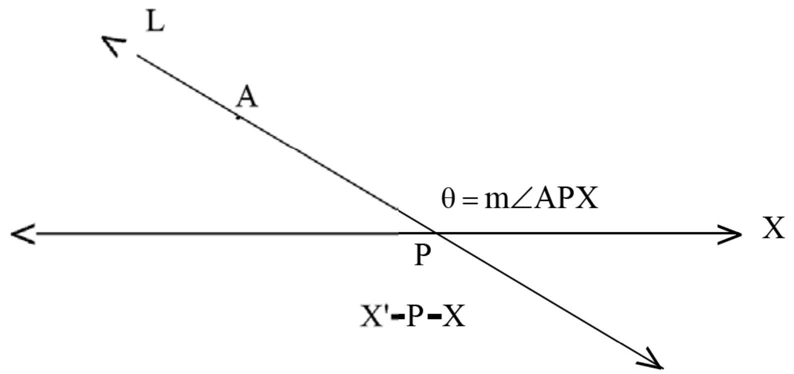
Inclination and Slope of a line :

Indination of a line is a real number θ as defined below :

Definition :

1. If a line L is parallel to x - axis or coincides with x- axis, then $\theta = 0$.
2. If L is not parallel to x-axis, then let it interesect x-axis at a point P. Take X and X' respectively on positive and negative x-axis such that X'-P-X (P is between X' and X) and take a point A on L so that A is on the upper half plane of x-axis.

Inclination θ of L is given by $\theta = m\angle APX$.



3. Inclination of $\overline{AB}$, $\overrightarrow{AB}$ or $\overrightarrow{BA}$ (i,e. line segment $\overline{AB}$, ray $\overrightarrow{AB}$ or $\overrightarrow{BA}$) is defined as the inclination of $\overleftrightarrow{AB}$.

Note :

1. If θ is the inclination of a line, then $0 \le \theta < \pi$.
2. Parallel lines have the same inclination, and conversely.
3. Inclinations of perpendicular lines differ by $\frac{\pi}{2}$.
4. Inclination is essentially an angle - measure. The only difference is that inclination can be zero, where as angle - measure cannot be zero.

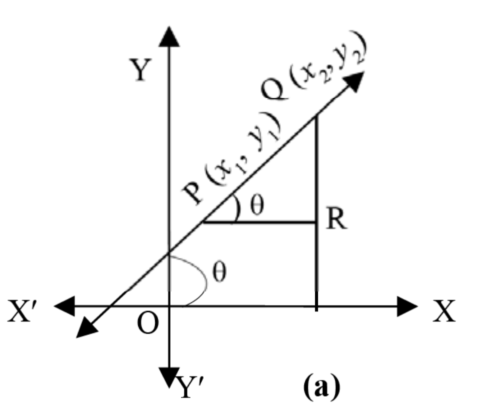
Slope (Gradient) of a nonvertical line :

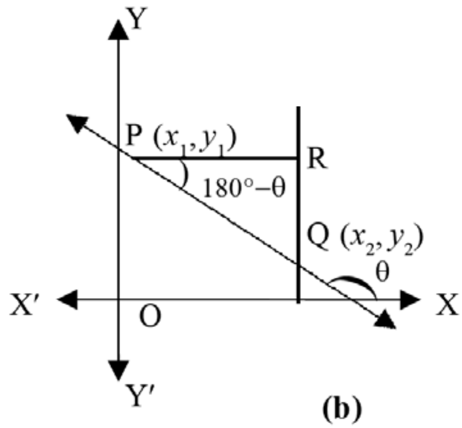
Definition : The slope of a nonvertical line (i.e. not perpendicular to the x- axis) is given by

$\mathbf{m = \tan\theta}$; where θ is the inclination of the line.

N.B. Inclination $\in [0, \pi)$, but slope $\in$ R. **Slope of a vertical line is not difined.**

Theorem : If P (x_1, y_1) and Q (x_2, y_2) are two distinct points on a nonvertical line, then slope of the line $\overleftrightarrow{PQ}$ is given by $m = \dfrac{y_2 - y_1}{x_2 - x_1}$.





Proof : Let $\overleftrightarrow{QR} \perp x-$ axis, $\overline{PR} \perp \overleftrightarrow{QR}$ [See figure (a) or (b)]

If m is the slope of $\overleftrightarrow{PQ}$, having inclination θ, then

(i) For $\theta < \frac{\pi}{2}$ [Fig (a)], $m = \tan\theta = \frac{QR}{PR} = \frac{|y_2 - y_1|}{|x_2 - x_1|}$.

(ii) For $\theta > \pi/2$ [Fig. (b)], $m = \tan\theta = -\tan(180° - \theta) = -\frac{QR}{PR} = -\frac{|y_2 - y_1|}{|x_2 - x_1|}$

But $y_2 - y_1$ and $x_2 - x_1$ have the same or opposite signs according as $\theta < \pi/2$ or $\theta > \pi/2$. So, removing the modulus sign in both cases, we get

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

Note :

1. Slope of a line is positive, zero or negative according as the inclination of the line is less than $\pi/2$, equal to zero or greater than $\pi/2$.
2. Slope of a vertical line is not defined.
3. It is obvious that slope of a line is the ratio of its rise or fall $(y_2 - y_1)$ to its run $(x_2 - x_1)$. A line with positive slope [fig. (a)] looks like rising and a line with negative slope [fig.(b)] looks like falling, as we move from left to right along the line.

Parallel and Perpendicular lines.

Theorem : If a pair of lines L_1 and L_2 have slopes m_1 and m_2 respectively, then

(i) $m_1 = m_2 \Leftrightarrow L_1 \parallel L_2$ or $L_1 = L_2$ (coincident)

(ii) $m_1 m_2 = -1 \Leftrightarrow L_1 \perp L_2$.

Proof : Taking θ_1 and θ_2 to be inclinations of L_1 and L_2 respectively, $m_1 = \tan\theta_1$ and $m_2 = \tan\theta_2$.

Slopes m_1 and m_2 being definite (as they are given numbers), none of θ_1 and θ_2 is $\frac{\pi}{2}$ i.e. none of L_1 and L_2 is vertical.

Suppose $m_1 = m_2$. Then $\tan\theta_1 = \tan\theta_2$.

θ_1 and θ_2 being inclinations, each different from $\frac{\pi}{2}$, we have $\theta_1, \theta_2 \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right)$.

Since $\tan x$ is a one -to- one function on $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right)$, we have

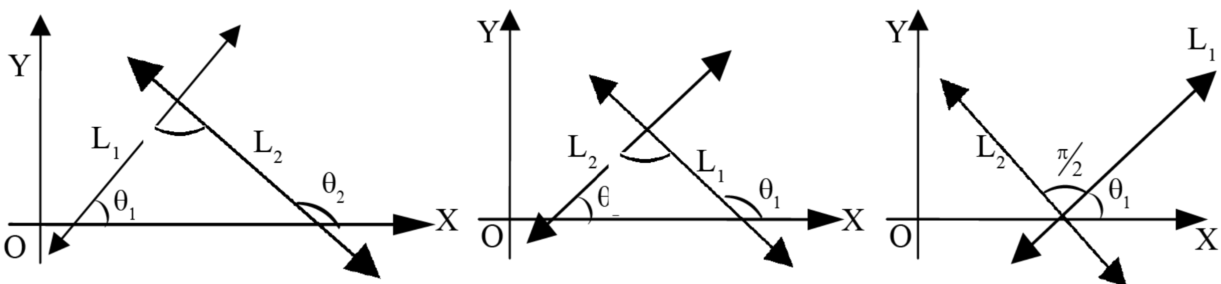
$\theta_1, \theta_2 \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right)$, $\tan\theta_1 = \tan\theta_2$.

$\Rightarrow \theta_1 = \theta_2$.

$\therefore$ L_1 and L_2 have the same inclination and are parallel or coincident.

Conversely suppose $L_1 \parallel L_2$ or $L_1 = L_2$ Then L_1 and L_2 have the same inclination. So, by definition, they have the same slope. Thus $m_1 = m_2$.

II.



(a) $\theta_2 = \theta_1 + \frac{\pi}{2}$ (b) $\theta_1 = \theta_2 + \frac{\pi}{2}$ (c) $\theta_2 = \theta_1 + \pi/2$

Suppose $m_1 m_2 = -1$, So one of m_1, m_2 is positive and the other, negative. Let us take $m_1 > 0$ and $m_2 < 0$. Taking inclinations of L_1 and L_2 as θ_1 and θ_2 resepectively we have,

$$m_1 = \tan\theta_1 > 0 \Rightarrow 0 < \theta_1 < \frac{\pi}{2},$$

$$m_2 = \tan\theta_2 < 0 \Rightarrow \theta_2 > \frac{\pi}{2}.$$

$$m_1 m_2 = -1 \Rightarrow m_2 = -\frac{1}{m_1}$$

$$\Rightarrow \tan\theta_2 = -\frac{1}{\tan\theta_1} = -\cot\theta_1 = \tan\left(\frac{\pi}{2} + \theta_1\right).$$

$$0 < \theta_1 < \frac{\pi}{2} \Rightarrow \frac{\pi}{2} < \frac{\pi}{2} + \theta_1 < \pi.$$

Thus θ_2 and $\frac{\pi}{2} + \theta_1$ both belong to $\left(\frac{\pi}{2}, \pi\right)$.

Since $\tan x$ is a one-to - one function on $\left(\frac{\pi}{2}, \pi\right)$, it follows from $\tan\theta_2 = \tan\left(\frac{\pi}{2} + \theta_1\right)$ that

$\theta_2 = \frac{\pi}{2} + \theta_1$ i.e. $\theta_2 - \theta_1 = \frac{\pi}{2}$.

If we take $m_1 < 0$ and $m_2 > 0$, it similarly follows that $\theta_1 - \theta_2 = \frac{\pi}{2}$.

Thus, in any case, the difference of the inclinations of L_1 and L_2 is $\frac{\pi}{2}$ and hence $L_1 \perp L_2$.

Conversely suppose that $L_1 \perp L_2$. We have already observed that none of L_1 and L_2 is vertical. Since a line perpendicular to a horizontal line must be vertical, it follows from $L_1 \perp L_2$ that none of L_1 and L_2 is also horizontal. Therefore θ_1 and θ_2 are each different from 0 and $\frac{\pi}{2}$. Again lines L_1 and L_2 being perpendicular the difference of their inclinations must be $\frac{\pi}{2}$. Hence $\theta_1 = \frac{\pi}{2} + \theta_2$ or $\theta_2 = \frac{\pi}{2} + \theta_1$.

Taking $\theta_1 = \frac{\pi}{2} + \theta_2$, $\tan\theta_1 = \tan\left(\frac{\pi}{2} + \theta_2\right)$

$= -\cot\theta_2 = -\dfrac{1}{\tan\theta_2}$. [$\tan\theta_1$ and $\cot\theta_2$ are defined as $\theta_1 \neq \frac{\pi}{2}$ and $\theta_2 \neq 0$]

$\therefore m_1 = -\dfrac{1}{m_2}$ or $m_1 m_2 = -1$.

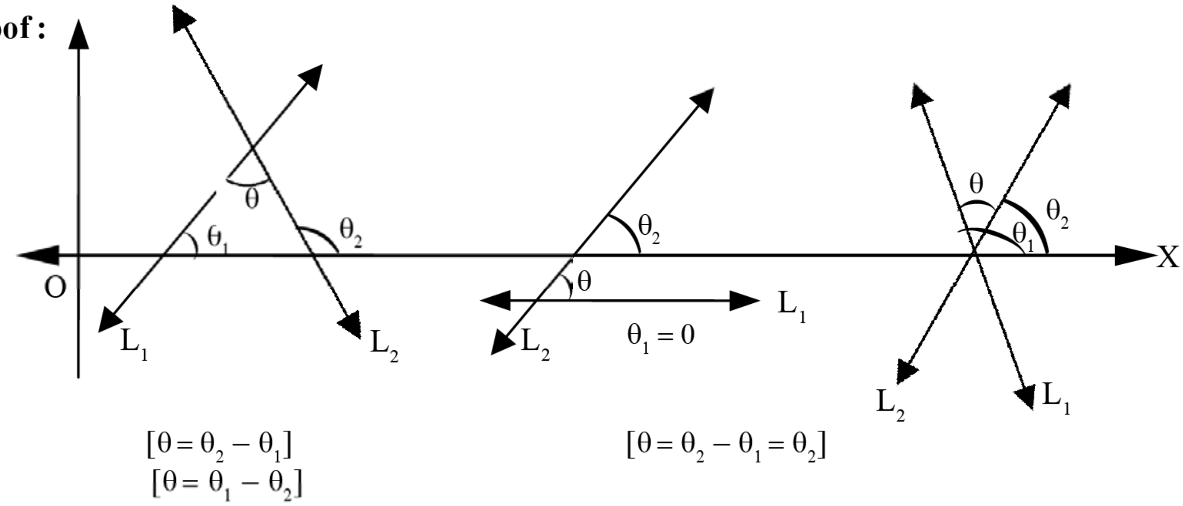
Similarly taking $\theta_2 = \frac{\pi}{2} + \theta_1$ we can deduce the same result. #

Angle between a pair of intersecting lines.

Theorem : If ϕ measures an angle between the intersecting lines L_1 and L_2, with slopes respectively m_1 and m_2 then

$$\tan\phi = \pm \frac{m_1 - m_2}{1 + m_1 m_2}$$

Proof :



$[\theta = \theta_2 - \theta_1]$
$[\theta = \theta_1 - \theta_2]$

$[\theta = \theta_2 - \theta_1 = \theta_2]$

If θ_1 and θ_2 are inclinations of L_1 and L_2 respectively, then obviously measures of the angles between L_1 and L_2 are given by θ and $\pi - \theta$, where $\theta = |\theta_1 - \theta_2|$.

Taking $\theta = \theta_1 - \theta_2$,

$$\tan\theta = \tan(\theta_1 - \theta_2) = \frac{\tan\theta_1 - \tan\theta_2}{1 + \tan\theta_1 \tan\theta_2} = \frac{m_1 - m_2}{1 + m_1 m_2}.$$

$$\tan(\pi - \theta) = -\tan\theta = -\frac{m_1 - m_2}{1 + m_1 m_2}.$$

Hence in general, measure of an angle, ϕ (either θ or $\pi - \theta$) between L_1 and L_2 is given by

$$\tan\phi = \pm\frac{m_1 - m_2}{1 + m_1 m_2}$$

The same conclusion can also be obtained by taking $\theta = |\theta_1 - \theta_2| = \theta_2 - \theta_1$.

Note :

The positive value of $\tan\phi$ gives the acute angle and the negative value gives the obtuse angle between L_1 and L_2.

Example 4 :

The slope of the line joining the points (1, 4) and (3, 5) is given by,

$$m = \frac{5-4}{3-1} = \frac{1}{2}.$$

Example 5 :

The line through (–1, –2) and (2, 2) is not parallel to the line through (6, 5) and (1, 1), since the slope of the first line is $m_1 = \dfrac{2-(-2)}{2-(-1)} = \dfrac{4}{3}$

and the slope of the second line is $m_2 = \dfrac{1-5}{1-6} = \dfrac{4}{5}$. $(m_1 \neq m_2)$

Example 6 :

The line through (1, 5) and (4, 4) is perpendicular to the line through (2, 1) and (3, 4).

The slope of the first line is $m_1 = \dfrac{4-5}{4-1} = -\dfrac{1}{3}$.

The slope of the second line is $m_2 = \dfrac{4-1}{3-2} = 3$

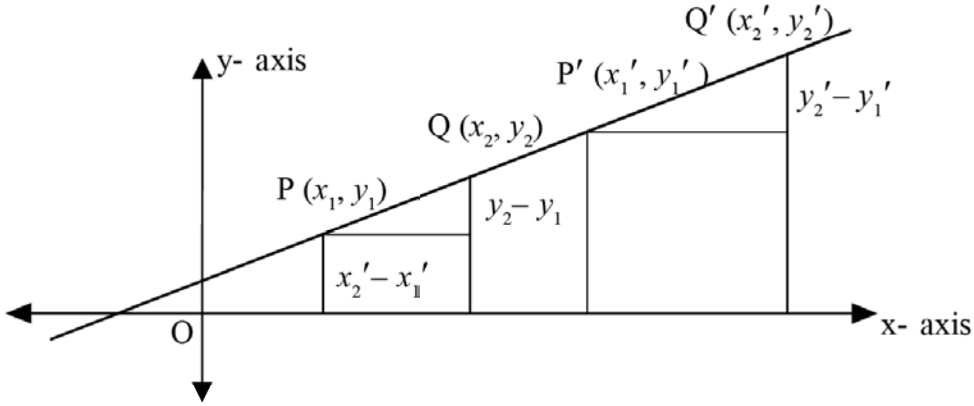
and so $m_1 \; m_2 = -1$.

Remarks :

The order of the points does not affect the slope since, $\dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{y_1 - y_2}{x_1 - x_2}$

The slope of the line $\overleftrightarrow{PQ}$ does not depend on the pair of points P (x_1, y_1) and Q (x_2, y_2) selected on it. If P′ (x_1', y_1') and Q′ (x_2', y_2') is another pair of points, then

$\dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{y_2' - y_1'}{x_2' - x_1'}$ (Fig. 8)



Example 7 :

Show that the points A (– 1, 4), B (0, 2) and C (2, – 2) are collinear.

m_1 = The slope of $\overleftrightarrow{AB} = \dfrac{2-4}{0-(-1)} = -2$,

m_2 = The slope of $\overleftrightarrow{BC} = \dfrac{-2-2}{2-0} = -2$

$\Rightarrow m_1 = m_2$.

Hence, the lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{BC}$ are parallel or coincident. But since B is a point common to both it follows that the two lines are same, i.e., the three points A, B, C are collinear.

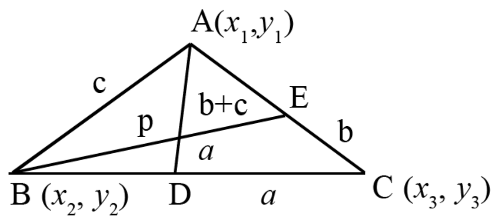
Remark : The collinearity of the above three points A, B, C can be established by showing that the expression for area of the triangle ABC is zero, i.e., by proving

$$\frac{1}{2}\begin{vmatrix} -1 & 0 & 2 \\ 4 & 2 & -2 \\ 1 & 1 & 1 \end{vmatrix} = 0.$$

(For the time being, you may skip expressions in terms of diterminants and use the other expression for area of a traingle; because determinants shall be discussed in Vol-II)

Example 8 :

Prove analytically that the internal bisectors of angles of a triangle are concurrent, and hence find the co- ordinates of their common point.



Let A(x_1,y_1), B (x_2, y_2) and C (x_3, y_3) be vertices of a triangle and let *a, b, c* be the lengths of the sides, $\overline{BC}$, $\overline{CA}$ and $\overline{AB}$ respectively. (Fig. 9)

Let the internal bisectors $\overline{AD}$ and $\overline{BE}$ of angles A and B meet at P. Since $\overline{AD}$ is internal bisector, we have, (by a result of Euclidean Geometry)

$$\frac{BD}{DC} = \frac{AB}{AC} = \frac{c}{b},$$

i.e. D divides $\overline{BC}$, in the ratio $c : b$.

Hence by internal division formula, co-ordinates of D are

$$\left(\frac{cx_3+bx_2}{b+c}, \frac{cy_3+by_2}{b+c}\right).$$

Again, $1 + \frac{DC}{BD} = 1 + \frac{b}{c}$

$$\Rightarrow \frac{BD+DC}{BD} = \frac{c+b}{c} \quad \Rightarrow \frac{BC}{BD} = \frac{c+b}{c}$$

$$\Rightarrow BD = \frac{c.BC}{b + c} = \frac{ac}{b+c}.$$

Since $\overline{BE}$ is internal bisector of $\angle$B and meets $\overline{AD}$ at P, we have from ΔABD,

$$\frac{AP}{PD} = \frac{AB}{BD} = \frac{c}{\frac{ac}{b+c}} = \frac{b+c}{a}$$

i.e. P divides $\overline{AD}$ in ratio $b + c : a$ and hence the co- ordinates of P are

$$\left(\frac{ax_1+bx_2+cx_3}{a+b+c}, \frac{ay_1+by_2+cy_3}{a+b+c}\right).$$

Similarly it can be shown that, if the internal bisectors of $\angle$B and $\angle$C meet at Q, then proceeding as above, the co-ordinates of Q will be same as those of P. Hence the three internal bisectors of meet at a point whose co-ordinates are

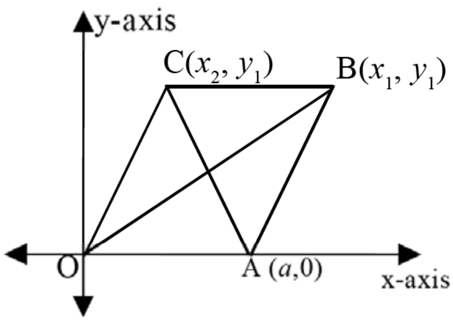
$$\left(\frac{ax_1+bx_2+cx_3}{a+b+c}, \frac{ay_1+by_2+cy_3}{a+b+c}\right).$$

This point is called the incentre of ΔABC.

Example 9 :

Prove analytically that the diagonals of a rhombus bisect at right angles.

Consider a rhombus with one vertex at origin O and side OA along x- axis. If the rhombus is of side-length a, then the vertex A has co- ordinates $(a, 0)$. Let the co-ordinates of B and C be (x_1, y_1) and (x_2, y_1) respectively. Then,



$BC = a = x_1 - x_2 \Rightarrow x_1 = a + x_2$.

Now, co-ordinates of the mid point of $\overline{OB}$ are $\left(\frac{x_1}{2}, \frac{y_1}{2}\right)$ and those of $\overline{AC}$ are $\left(\frac{a+x_2}{2}, \frac{y_1}{2}\right) = \left(\frac{x_1}{2}, \frac{y_1}{2}\right)$.

Hence, the two diagonals bisect each other.

Now, slope of $\overleftrightarrow{OB} = \frac{y_1}{x_1} = \frac{y_1}{a+x_2} = m_1$ and slope of $\overleftrightarrow{AC} = \frac{y_1}{x_2 - a} = m_2$.

Then, $m_1.m_2 = \frac{y_1}{x_2 + a} \cdot \frac{y_1}{x_2 - a} = \frac{y_1^2}{x_2^2 - a^2}$.

But, OA = OC = a

$\Rightarrow a = \sqrt{x_2^2 + y_1^2} \Rightarrow a^2 = x_2^2 + y_1^2 \Rightarrow y_1^2 = a^2 - x_2^2$.

Hence, $m_1 . m_2 = \frac{y_1^2}{x_2^2 - a^2} = \frac{a^2 - x_2^2}{x_2^2 - a^2} = -1$

which shows that $\overline{OB}$ is prependicular to $\overline{AC}$.

EXERCISES 11 (a)

1. Find the distance between the following pairs of points.
   (i) (3, 4) (–2, 1): (ii) (–1, 0), (5, 3).
2. If the distance between the points $(3, a)$ and $(6, 1)$ is 5, find the value of a.
3. Find the co-ordinates of the point which divides the line segment joining the points A(4, 6), B (–3, 1) in the ratio 2 :3 internally.

   Find also the co-ordinates of the point which divides $\overline{AB}$ in the same ratio externally.

4. Find the coordinates of the mid-point of the following pairs of points.

(i) $(-7, 3)$, $(8, -4)$; (ii) $\left(\frac{3}{4}, -2\right)$, $\left(-\frac{5}{2}, 1\right)$.

5. Find the area of the triangle whose vertices are $(1, 2)$, $(3, 4)$, $\left(\frac{1}{2}, \frac{1}{4}\right)$.

6. If the area of the triangle with vertices $(0, 0)$, $(1, 0)$, $(0, a)$ is 10 units, find the value of a.

7. Find the value of a so that the points $(1, 4)$, $(2, 7)$, $(3, a)$ are collinear.

8. Find the slope of the lines whose inclinations are given.

(i) $30°$, (ii) $45°$, (iii) $60°$, (iv) $135°$.

9. Find the inclination of the lines whose slopes are given below.

(i) $\frac{1}{\sqrt{3}}$, (ii) 1, (iii) $\sqrt{3}$, (iv) -1.

10. Find the angle between the pair of lines whose slopes are;

(i) $\frac{1}{\sqrt{3}}$, 1 (ii) $\sqrt{3}$, -1.

11. (a) Show that the points $(0, -1)$, $(-2, 3)$, $(6, 7)$ and $(8, 3)$ are vertices of a rectangle.

(b) Show that the points $(1, 1)$, $(-1, -1)$ and $(-\sqrt{3}, \sqrt{3})$ are the vertices of an equilateral triangle.

12. Find the co-ordinates of the point $P(x, y)$ which is equidistant from $(0, 0)$, $(32, 10)$ and $(42, 0)$

13. If the points (x, y) are equidistant from the points $(a + b, b - a)$ and $(a - b, a + b)$, prove that $bx = ay$.

14. The co-ordinates of the vertices of a triangle are (α_1, β_1), (α_2, β_2) and (α_3, β_3). Prove that the co-ordinates of its centroid are $\left(\frac{\alpha_1 + \alpha_2 + \alpha_3}{3}, \frac{\beta_1 + \beta_2 + \beta_3}{3}\right)$.

15. Two vertices of a triangle are $(0, -4)$ and $(6, 0)$. If the medians meet at the point $(2, 0)$, find the co-ordinates of the third vertex.

16. If the point $(0, 4)$ divides the line segment joining $(-4, 10)$ and $(2, 1)$ internally, find the point which divides it externally in the same ratio.

17. Find the ratios in which the line segment joining $(-2, -3)$ and $(5, 4)$ is divided by the co-ordinate axes and hence find the co-ordinates of these points.

18. In a triangle one of the vertices is at $(2, 5)$ and the centroid of the triangle is at $(-1, 1)$. Find the co-ordinates of the mid-point of the side opposite to the given angular point.

19. Find the co-ordinates of the vertices of a triangle whose sides have mid -points at $(2, 1)$, $(-1, 3)$ and $(-2, 5)$.

20. If the vertices of a triangle have their coordinates given by rational numbers, prove that the triangle cannot be equilateral.
21. Prove that the area of any triangle is equal to four times the area of the triangle formed by joining the mid points of its sides.
22. Find the condition that the point (x, y) may lie on the line joining (1, 2) and (5, –3).
23. Show that the three distinct points (a^2, a), (b^2, b) and (c^2, c) can never be collinear.
24. If A, B, C are (–1, 2), (3, 1) and (–2, –3) respectively, then show that the points which divide $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ in the ratios (1 : 3), (4 : 3) and (–9: 4) respectively are collinear.
25. Prove analytically :
    (a) The line segment joining the mid points of two sides of a triangle is parallel to the third and half of its length.
    (b) The altitudes of a triangle are concurrent.
    (c) The perpendicular bisectors of the sides of a triangle are concurrent.
    (d) An angle in a semicircle is a right angle.

11.2 Locus and its Equation

Locus : A set of points satisfying certain condition or conditions is called a locus.

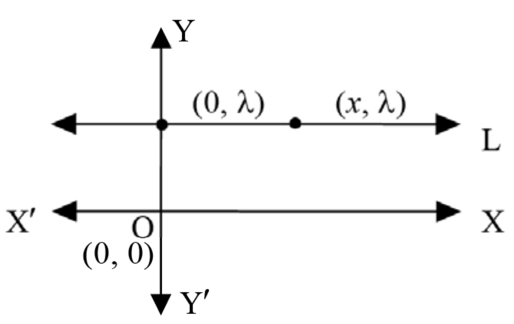
A point belonging to the locus is called a 'point on the locus'.

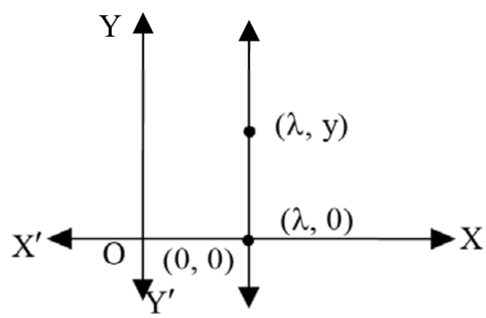
Chord of a locus : A line–segment joining two points on a Locus is a chord of the locus.

Equation of a Locus : The equation satisfied by the co-ordinates of all the points on a locus and by no others, is called the equation of / to the locus.

Equation of a Straight Line :

Equation of a line L parallel to x - axis is of the form $y = \lambda$ where $\lambda \in$ R. Any point P (x, y) shall lie on the line iff $y = \lambda$ (Fig. 11), that is $(x, \lambda) \in$ L, $\forall\ x \in$ R.





Similarly a line parallel to y - axis is of the form $x = \lambda$ for $\lambda \in$ R. A point P (x, y) shall lie on the line if $x = \lambda$ i.e. $(\lambda, y) \in$ L $\forall\ y \in$ R.

Now we proceed to obtain the equations of lines which are not parallel to co-ordinate axes.

(a) Slope - Intercept Form :

Intercepts (Definition) : The x - intercept of a line is x - co-ordinate of the point where it intersects the x - axis and its y - intercept is the y - co-ordinate of the point where it intersects the y - axis.

A line parallel to x - axis (horizontal line) has no x - intercept and a line parallel to y - axis (vertical line) has no y - intercept.

The term 'intercept' in 'Slope-Intercept Form' means the y - intercept of a line.

Theorem : If a line has slope m and y - intercept c then its equation is $y = mx + c$.

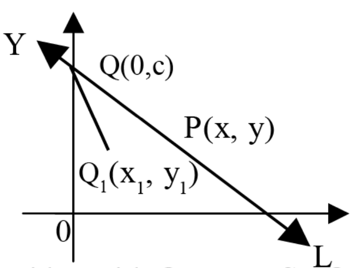
Proof : Let L be the line. Since y - intercept is c, L intersects the y - axis at $(0, c)$. Let us call it the point Q. It is easy to see that the co-ordinates of Q i.e. $x = 0$, $y = c$ satisfy the equation $y = mx + c$. We shall prove that co-ordinates of all other points on L also satisfy the given equation.

Let P (x, y) be any point on L, other than Q. Therefore L is the line $\overleftrightarrow{PQ}$.

Since $\overleftrightarrow{PQ}$ intersects y – axis, it cannot be vertical and its slope is given by $\dfrac{y-c}{x-0}$.

$\therefore\ m = \dfrac{y-c}{x}$ and we get $y = mx + c$.

Thus, co-ordinates of all the points on L satisfy the equation $y = mx + c$. We now show that co-ordinates of any other point, not on L, cannot satisfy the given equation $y = mx + c$.



Suppose $Q_1\ (x_1, y_1) \notin$ L and $y_1 = mx_1 + c$.

We claim : $x_1 \neq 0$.

For if $x_1 = 0$, $y_1 = mx_1 + c \Rightarrow y_1 = c$.

But $x_1 = 0$, $y_1 = c \Rightarrow Q_1\ (x_1, y_1)$ coincides with Q (0, c). So $Q_1 \in$ L contrary to our assumption.

Now, $y_1 = mx_1 + c \Rightarrow m = \dfrac{y_1 - c}{x_1} = \dfrac{y_1 - c}{x_1 - 0}$

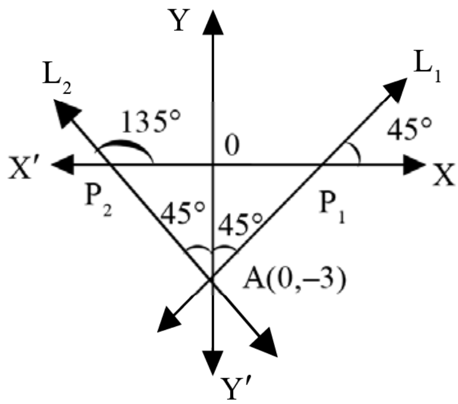
$\Rightarrow m$ = slope of $\overleftrightarrow{QP}$ = slope of $\overleftrightarrow{QQ_1}$. Since $\overleftrightarrow{QP}$ and $\overleftrightarrow{QQ_1}$ have the common point Q, it follows from equality of their slopes, that they must be coincident lines. So $Q_1\ (x_1, y_1) \in$ L, which is a contradiction.

This contradiction makes it clear that if a point lies outside the line L, its co-ordinates cannot satisfy the equation $y = mx + c$. Therefore, by definition of equation to a locus, it is proved that $y = mx + c$ is the equation of the given line.

Note : Hereafter we shall make no distinction between a line (in general, locus) and its equation. Phrases like, 'the line $y = mx + c$' shall mean 'the line whose equation is given by $y = mx + c$'.

Example 10 :

Obtain the equation of a line which meets the y - axis at $(0, -3)$ and makes an angle measuring $45°$ with the y - axis.



Solution : There can be two positions of the line, shown as L_1 and L_2 in figure. In both positions, the y - intercept of the line is given by $c = -3$.

The inclination of $L_1 = 45°$ and the inclination of $L_2 = 135°$.

So slope of $L_1 = m_1 = \tan 45° = 1$

and slope of $L_2 = m_2 = \tan 135° = -1$.

Therefore, equation of L_1 is given by $y = m_1x + c$

$= x - 3$ or $x - y - 3 = 0$ (i)

Equation of L_2 is given by $y = m_2x + c$

$= -x - 3$ or $x + y + 3 = 0$ .... (ii)

(i) and (ii) give the equations corresponding to the different positions of the line.

Corollary 1 :

(b) Slope - point Form

Let a line have slope m and let it pass through a point Q (x_1, y_1). Then its equation is given by $\boxed{(y - y_1) = m(x - x_1)}$. (4)

If the y - intercept of the line is c, then in slope - intercept form its equation will be

$y = mx + c$ (5)

Since it passes through (x_1, y_1) we have

$y_1 = mx_1 + c.$ (6)

From (5) and (6), we get $y - y_1 = m(x - x_1)$.

Corollary 2 :

(c) Two - point form

Let a line pass through two given points Q (x_1, y_1) and R (x_2, y_2). Then the equation of the line is given by

$$\boxed{y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)}. \quad (7)$$

Since the line passes through Q (x_1, y_1) and R (x_2, y_2) its slope is

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

Also the line passes through the point (x_1, y_1) and has slope m. So its equation is

$$(y - y_1) = m(x - x_1) = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1).$$

(by substituting the value of m)

Corollary - 3 :

(d) Intercept Form

Let a line have x - intercept a and y - intercept b. Then its equation is

$$\boxed{\frac{x}{a}+\frac{y}{b}=1}.$$

The line has x - intercept a and y - intercept b. So it passes through $(a, 0)$ and $(0,b)$ and hence by two - point form, its equation is given by

$$(y-0) = \frac{b-0}{0-a}\ (x-a),$$

or, $y = -\dfrac{b}{a}\ (x-a)$,

or, $\dfrac{y}{b} = -\dfrac{x}{a} + 1$

or, $\dfrac{x}{a} + \dfrac{y}{b} = 1$.

(e) Equation of a line in normal form :

Theorem : Let a line L be at a distance p from the origin. and let the line through origin, perpendicular to L, meet it at P. If P is the point (p, α) in polar coordinates (obviously $p = OP$), then

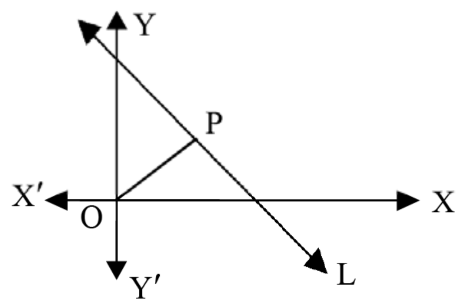
(1) the equation of L (normal form) is given by $x \cos\alpha + y \sin \alpha = p$, provided L does not pass through origin.

(2) If L passes through origin (i.e. $p = 0$)

$x \cos \alpha + y \sin \alpha = p$

is the equation of L, provided $\alpha = \dfrac{\pi}{2} + \theta$

where θ is the inclination of L.



Proof : (1) L does not pass through the origin.

We suppose that L is oblique (neither vertical nor horizontal).

So, $\overleftrightarrow{OP}$ is also oblique and therefore slopes of both L and $\overleftrightarrow{OP}$ are determinable.

Since P has polar coordinates (p, α) it has cartesian co-ordinates $(p \cos\alpha, p \sin\alpha)$.

Also origin $(0, 0)$ is on $\overleftrightarrow{OP}$

$$\therefore \text{ Slope of } \overleftrightarrow{OP} = \frac{p\sin\alpha - 0}{p\cos\alpha - 0} = \tan\alpha.$$

$\Rightarrow$ Slope of L = $-\cot\alpha$ $\quad (\because L \perp \overleftrightarrow{OP})$ .... (i)

Let Q (x, y) be any point on L, different from P.

$\therefore$ Slope of L (i,e $\overleftrightarrow{QP}$) = $\dfrac{y - p\sin\alpha}{x - p\cos\alpha}$ .... (ii)

(i) and (ii) imply

$$\frac{y - p\sin\alpha}{x - p\cos\alpha} = -\frac{\cos\alpha}{\sin\alpha}$$

$\Rightarrow x\cos\alpha + y\sin\alpha = p$ .... (iii)

The cartesian co-ordinates of P also satisfy (iii). (The polar coordinates satisfy some other equation, known as polar equation of a line, which is beyond our scope). Thus co-ordinates of every point on L, satisfy equation (iii).

Now suppose $Q_1 (x_1, y_1) \notin L$

and $x_1 \cos\alpha + y_1 \sin\alpha = p$ .... (iv)

Supposing $Q_2 (x_2, y_2) \in L$, we have

$x_2 \cos\alpha + y_2 \sin\alpha = p$ .... (v)

(iv) and (v) imply $\dfrac{y_2 - y_1}{x_2 - x_1} = -\cot\alpha.$

$\Rightarrow$ Slope of $\overrightarrow{Q_1Q_2}$ = slope of L

Since $Q_2 \in L$, it follows that $Q_1 \in L$, which is impossible. So co-ordinates of any point, not on L, cannot satisfy (iii). So equation of L is given by

$x\cos\alpha + y\sin\alpha = p.$

If L is either vertical or horizontal then it is easy to check that its equation is $x\cos\alpha + y\sin\alpha = p$.

(2) L passes through origin :

In this case $p = 0$. So P coincides with origin which is the pole. So polar co-ordinates of P are given by $(0,\alpha)$ where α is arbitrary and consequently $x\cos\alpha + y\sin\alpha = p$ $(p = 0)$, does not represent any particular line. It represents the family of lines passing through origin.

Now, suppose inclination of L is θ.

If $\theta \neq \dfrac{\pi}{2}$, slope of L is given by

$m = \tan\theta.$

The equation of L is given by (slope - intercept form)

$y = mx = x \tan\theta$ (Taking y - intercept = 0)

Or $y \cos\theta - x \sin\theta = 0 = p$

Or $x \cos(\frac{\pi}{2} + \theta) + y \sin(\frac{\pi}{2} + \theta) = p$

Or $x \cos\alpha + y \sin\alpha = p$; where $\alpha = \frac{\pi}{2} + \theta$.

If $\theta = \frac{\pi}{2}$, then L is the y - axis, having equation $x = 0$, which is same as

$x \cos\alpha + y \sin\alpha = p$ $(p = 0)$

where $\alpha = \frac{\pi}{2} + \theta$ i.e. π.

Thus the equation of L, when it passes through origin is given by

$x \cos\alpha + y \sin\alpha = \text{p}$ (p = 0), $\alpha = \frac{\pi}{2} + \theta$

where θ = inclination of L.

This completes the proof of the theorem

(f) General Form

From the above discussions we see that the equation of a straight line in different forms is a first degree equation in x and y and contains a maximum of two arbitrary real constants.

Consider the slope - intercept form of equation of the line $y = mx + c$, which contains two arbitrary constants m and c. If we fix the constant c i.e., the line is made to pass through the fixed point $(0, c)$ then for different values of m we get different lines. Similarly, if we fix m and vary c we get a family of parallel lines for different values of c. The line will be uniquely determined if both m and c are fixed.

Similarly, the intercept form of equation of a line contains two arbitrary constants 'a' and 'b' whereas the normal form contains two arbitary constants p and α. If the two arbitrary constants are fixed then the line is unique.

Let us consider the general equation of first degree in x and y given by

$$\text{A}x + \text{B}y + \text{C} = 0. \qquad \text{............. (II)}$$

If A = 0 = B, C = 0, no equation is obtained.

If $\text{A} \neq 0$ and $\text{B} = 0$, then (11) reduces to $x = -\frac{\text{C}}{\text{A}}$

which is the equation to a line parallel to y – axis.

If $\text{B} \neq 0$ and $\text{A} = 0$, then (11) reduces to $y = -\frac{\text{C}}{\text{B}}$

which represents a line parallel to x-axis. If $\text{A} \neq 0$, $\text{B} \neq 0$, then (11) can be reduced to slope intercept form as $y = \left(-\frac{\text{A}}{\text{B}}\right) x + \left(-\frac{\text{C}}{\text{B}}\right)$

with slope $m = -\frac{A}{B}$ and intercept $c = -\frac{C}{B}$.

Also (11) can be re-written in intercept form as $\frac{x}{\left(-\frac{C}{A}\right)} + \frac{y}{\left(-\frac{C}{B}\right)} = 1$

with x - intercept $-\frac{C}{A}$ and y - intercept $-\frac{C}{B}$.

The above discussion determines that every equation of first degree in x and y represents a straight line.

Example 11

(i) Equation of a line with slope 2 and passing through a point (1, – 1) (by slope - point form) is given by $y - (-1) = 2\,(x - 1)$

i.e. $y + 1 = 2x - 2$ or $2x - y - 3 = 0$.

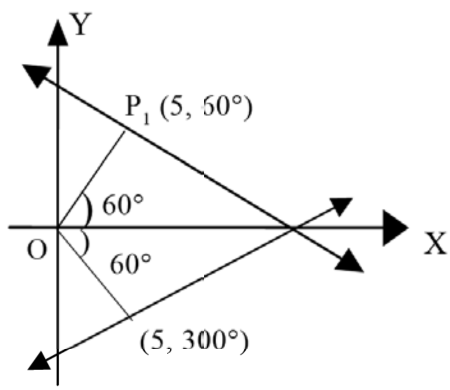
(ii) Equation of a line passing through points (2, 3) and (–1, 0) (by two-point form) is given by $y - 3 = \frac{0-3}{-1-2}\,(x - 2)$

or $y - 3 = (x - 2)$ or $x - y + 1 = 0$.

(iii) Equation of a line with x - intercept – 2 and y - intercept – 3 (by intercept form) is given by $\frac{x}{-2} + \frac{y}{-3} = 1$ or $3x + 2y + 6 = 0$.

(iv) Equation of a line at a distance 5 units from origin and its normal making an angle measuring 60° with positive direction of x- axis (in normal form) is givin by $x \cos 60° \pm y \sin 60° = 5$.

Solution :



The figure tells that there can be two positions of such a line, the foot of the perpendicular (normal) from origin upon the line, being given by P_1 (5, 60°) and P_2 (5, 300°) or, for that matter, P_2 (5, – 60°).

Hence equations of the line in both cases are given by

$x \cos(\pm 60°) + y \sin(\pm 60°) = 5$

or $x \cos 60° \pm y \sin 60° = 5$ or $x \pm \sqrt{3}y - 10 = 0$.

(v) If a line is given by $3x - 2y + 7 = 0$, then it has

slope $m = -\frac{3}{-2} = \frac{3}{2}$ and y - intercept c $= \frac{-7}{-2} = \frac{7}{2}$.

(vi) The line $\frac{x}{a} + \frac{y}{b} = 1$ passes through a fixed point, whatever be the values of a, b ; subject to the condition $\frac{2}{a} + \frac{3}{b} = 1$.

Obvious, since the point (2, 3) satisfies the equation, whatever be the values of a and b, under the given condition.

Here there is nothing special about 2 and 3 in the given condition. You can choose any other pair of numbers, say m and n and state the condition as $\frac{m}{a} + \frac{n}{b} = 1$. The fact can also be stated as :

The system of lines $\frac{x}{a} + \frac{y}{b} = 1$; a, b $\in$ R, pass through a fixed point (m, n) under the condition $\frac{m}{a} + \frac{n}{b} = 1$.

Example 12 :

A line intersects x and y – axes at A and B respectively.P (a,b) divides $\overline{AB}$ internally/ externally in ratio given by $\frac{PA}{PB} = \frac{m}{n}$. Find equation of $\overleftrightarrow{AB}$ in each case.

Solution : Soppose the coordinates of A and B are given by (p, o) and (o, q) respectively. Consider first the case of internal division. Since $\frac{PA}{PB} = \frac{m}{n}$, we have, by internal division formula, taking $x_P = a$, $y_P = b$, $x_A = p$, $y_A = 0$, $x_B = 0$, $y_B = q$

$$a = \frac{m.o + np}{m+n} = \frac{np}{m+n},$$

$$b = \frac{mq + no}{m+n} = \frac{mq}{m+n}.$$

$$\text{so } p = \frac{m+n}{n} a,\ q = \frac{m+n}{m} b.$$

Here x - intercept and y - intercept of $\overleftrightarrow{AB}$ are p and q respectively. So equation of $\overleftrightarrow{AB}$, in ‘intercept form’ is given by $\frac{x}{p} + \frac{y}{q} = 1$.

Putting values of p and q and simplifying, we get the equation of $\overleftrightarrow{AB}$, in case of internal division, as

$$\frac{nx}{a} + \frac{my}{b} = n + m.$$

Next consider external division.

By the external division formula

$$a = \frac{m.o - n.p}{m-n} = -\frac{np}{m-n},$$

$$b = \frac{mq - n.o}{m-n} = \frac{mq}{m-n}.$$

$$\therefore \text{ p} = -\frac{m-n}{n}a \text{ , q} = \frac{m-n}{m}\text{b.}$$

Putting these values in the 'intercept form' $\frac{x}{p} + \frac{y}{q} = 1$

we get, after simplification

$$\frac{nx}{a} - \frac{my}{b} = n - m.$$

Lines continued :

Consider the equations of lines L_1 and L_2 given by

$L_1 : a_1x + b_1y + c_1 = 0$

$L_2 : a_2x + b_2y + c_2 = 0.$

Slopes m_1 and m_2 of L_1 and L_2 are given by $m_1 = -\frac{a_1}{b_1}$ and $m_2 = -\frac{a_2}{b_2}$ respectively; supposing the lines to be nonvertical i.e. b_1 and b_2 are different from zero. We also suppose $a_1 \neq 0$, $a_2 \neq 0$ i.e. neither of the lines is horizontal.

Case of parallel lines :

The lines L_1 and L_2 are parallel if and only

if $m_1 = m_2$ or $\frac{a_1}{b_1} = \frac{a_2}{b_2}$ or $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{1}{\lambda}$, say.

$\therefore a_2 = a_1\lambda$, $b_2 = b_1\lambda$. (λ is obviously nonzero)

Therefore we can write equation of L_2 as

$\lambda(a_1x + b_1y) + c_2 = 0$ or $a_1x + b_1y + c_3 = 0$. (Taking $c_3 = \frac{c_2}{\lambda}$)

Thus we have the following

Working Rule : (Parallel lines) :

If a line L is represented by $ax + by + c = 0$, then the equation of a line L′, paralled to L, is given by $ax + by + d = 0$.

If, in the equation of L, either a or b is zero, then also the equation of L′, paralled to L, is given by $ax + by + d = 0$, since, in this case, both $ax + by + c = 0$ and $ax + by + d = 0$ represent lines parallel to the same coordinate axis.

A test for parallel lines :

The lines L_1 and L_2 represented by

$L_1 : a_1x + b_1y + c_1 = 0$ and

$L_2 : a_2x + b_2y + c_2 = 0$ are parallel (or coincident) if and only if $a_1b_2 - a_2b_1 = 0$.

When none of a_1, a_2 b_1, b_2 is zero, we get $a_1b_2 - a_2b_1 = 0$ from $\frac{a_1}{a_2} = \frac{b_1}{b_2}$.

The test also applies even when any one of a_1, b_1 and any one of a_2, b_2 is zero. [Both of a_1, b_1 and / or both of a_2, b_2 cannot be zero as, in that case we do not get any equation]

As an illustration, let us consider $a_1 = 0$.

Then $a_1b_2 - a_2b_1 = 0 \Rightarrow a_2\, b_1 = 0$

$\Rightarrow a_2 = 0$ (a_1 and b_1 cannot be both zero)

$\Rightarrow$ L_1 and L_2 are horizontal lines and hence parallel (or coincident).

Conversely suppose $L_1 \parallel L_2$ and $a_1 = 0$.

Now $a_1 = 0 \Rightarrow L_1$ is horizontal

So, $L_2 \parallel L_1 \Rightarrow L_2$ is also horizontal $\Rightarrow a_2 = 0$.

Therefore $a_1b_2 - a_2b_1 = 0$.

Similar arguments show the validity of the test in all other cases.

Case of perpendicular Lines.

The lines represented by $x + a = 0$ and $y + b = 0$ being respectively vertical and horizontal, are mutually perpendicular.

Now consider a line L with its equation as follows :

$L : ax + by + c = 0.$ $(a \neq 0, b \neq 0)$

We can write the equation of a line L',

Perpendicular to L, as

$L' : bx - ay + d = 0.$

The slopes of L and L', are respectively $-\frac{a}{b}$ and $\frac{b}{a}$. The product of the slopes being -1, the lines are mutually perpendicular. Thus, we have the following.

Working Rule (Perpendicular lines) :

If a line L is represented by $ax + by + c = 0$, then the equation of line L', perpendicular to L, can be written as $bx - ay + d = 0$.

To write the equation of L', perpendicular to L, just interchange the coefficients of x and y in the equation of L and write one of the cofficients by reversing its sign.

Example : If a line L is given by $2x + 3y + 5 = 0$ then $3x - 2y + k = 0$, $k \in R$, represents the family of lines L' such that $L' \perp L$.

Note : The rule applies even when one of the coefficients (i.e. cofficients of x or y) in the equation of Lis equal to zero.

Condition for coincidence of Lines :

Consider lines L_1 and L_2 given by equations,

$L_1 : a_1x + b_1y + c_1 = 0$

$L_2 : a_2x + b_2y + c_2 = 0.$

We suppose that a_i, b_i and c_i are all different from zero, for $i = 1, 2$.

We have seen that L_1 is parallel to L_2 if and only if $\frac{a_1}{a_2} = \frac{b_1}{b_2}$.

What happens if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$?

Writing $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} = \lambda$, the equation of L_1 can be written as $\lambda(a_2x + b_2y + c_2) = 0$.

Therefore coordinates of the set of points satisfying the second equation also satisfy the first equation and vice versa. Hence L_1 and L_2 are identical sets of points and are, therefore, coincident lines.

Conversely suppose that

$a_1x + b_1y + c_1 = 0$ .... (1)

and $a_2x + b_2y + c_2 = 0$ ...... (2); $a_i \neq 0$, $b_i \neq 0$, $c_i \neq 0$ for $i = 1, 2$.

represent the same line L.

From equation (1) we get

x – intercept of L $= -\frac{c_1}{a_1}$, y – intercept of L $= -\frac{c_1}{b_1}$.

Also, from equation (2),

x – intercept of L $= -\frac{c_2}{a_2}$, y – intercept of L $= -\frac{c_2}{b_2}$.

[Remember, x – intercept of line is the x – co-ordinate of the point where it intersects the x – axis and its y - intercept is the y - co-ordinate of the point where it intersects the y – axis]

Since x and y - intercepts of a line are each unique, we have

$$\frac{c_1}{a_1} = \frac{c_2}{a_2} \text{ and } \frac{c_1}{b_1} = \frac{c_2}{b_2} \text{ or } \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}.$$

We summarise the above discussion as :

The lines represented by

$a_1x + b_1y + c_1 = 0$

and $a_2x + b_2y + c_2 = 0$ ($a_i \neq 0$, $b_i \neq 0$, $c_i \neq 0$ for $i = 1, 2$)

are coincident if, and only if

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}.$$

Note : (1) Lines represented by

$a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ (a_i, b_i and c_i are nonzero for $i = 1, 2$) are parallel and different if, and only if

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}.$$

(2) $a_1x + c_1 = 0$ and $a_2x + c_2 = 0$ represent coincident lines if and only if $\dfrac{a_1}{a_2} = \dfrac{c_1}{c_2}$ and

$b_1y + c_1 = 0$ and $b_2y + c_2 = 0$ represent coincident lines if, and only if $\dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}$. (Prove)

Length of perpendicular from origin on a line :

For a vertical line $x = a$ or a horizontal line $y = b$, the length of the perpendicular from origin is given by $|a|$ and $|b|$ respectively.

Now consider a line L represented by $ax + by + c = 0$ with $a \neq 0$ and $b \neq 0$.

If $c = 0$, then obviously origin is a point on the line and therefore length of the perpendicular from origin on L is taken as zero.

Suppose that $c \neq 0$. Let p denote the length of the perpendicular from origin on L. Then $p > 0$.

Now suppose that the equation of L in normal form is given by

$$x \cos \alpha + y \sin \alpha = p$$

or $x \cos \alpha + y \sin \alpha - p = 0$.

Since $x \cos \alpha + y \sin \alpha - p = 0$

and $ax + by + c = 0$

represent the same line, we have by the condition for coincidence.

$$\frac{\cos\alpha}{a} = \frac{\sin\alpha}{b} = \frac{-p}{c} = \frac{\sqrt{\cos^2\alpha + \sin^2\alpha}}{\pm\sqrt{a^2+b^2}} = \frac{1}{\pm\sqrt{a^2+b^2}}.$$

$$\therefore \cos\alpha = \frac{a}{\pm\sqrt{a^2+b^2}}, \sin\alpha = \frac{b}{\pm\sqrt{a^2+b^2}}, p = \frac{-c}{\pm\sqrt{a^2+b^2}}.$$

p being positive, the sign of the radical in the denominator has to be + or – according as c < 0 or c > 0. After choosing the sign of the radical we can easily evaluate α.

$\boxed{p = \dfrac{-c}{\pm\sqrt{a^2+b^2}}}$, with proper choice of the sign of the radical, gives the length of the perpendicular from origin on L.

Exercise :

Find the length of the perpendicular from origin on the line $x + \sqrt{3}\,y + 4 = 0$. Also express the equation of the line in normal form.

Solution :

Here $a = 1$, $b = \sqrt{3}$, $c = 4$.

If $x \cos \alpha + y \sin \alpha = p$ is the equation of the given line in normal form, then

$$\cos \alpha = \frac{a}{\pm\sqrt{a^2+b^2}} = \frac{1}{\pm\sqrt{1+3}}$$

$$\sin \alpha = \frac{b}{\pm\sqrt{a^2+b^2}} = \frac{\sqrt{3}}{\pm\sqrt{1+3}},\ p = \frac{-c}{\pm\sqrt{a^2+b^2}} = \frac{-4}{\pm\sqrt{1+3}}.$$

In order to make p positive we choose '–' sign for the radical $\pm\sqrt{1+3}$.

$\therefore$ $p = 2$ and $\cos \alpha = -\frac{1}{2}$, $\sin \alpha = -\frac{\sqrt{3}}{2}$.

(choosing '–' sign for the radical in the expressions for $\cos \alpha$ and $\sin \alpha$)

so $\tan \alpha = \sqrt{3} = \tan \frac{\pi}{3}$, giving the general solution $\alpha = n\pi + \frac{\pi}{3}$, $n \in z$. But as $\sin\alpha$ and $\cos \alpha$ are both negative, α must be in the third quadrant. So we take $n = 1$ and obtain the value of α as $\pi + \frac{\pi}{3} = 4\frac{\pi}{3}$.

$\therefore$ Length of the perpendicular from origin on the given line is 2 (in the chosen unit of distance) and 'normal form' of the given equation is :

$$x \cos \frac{4\pi}{3} + y \sin \frac{4\pi}{3} = 2.$$

A word of caution :

In working out exercises as the above one, writing $p = \frac{c}{\mp\sqrt{a^2+b^2}}$ or $\frac{c}{\pm\sqrt{a^2+b^2}}$ in place of the expression $p = \frac{-c}{\pm\sqrt{a^2+b^2}}$ and determining the sign of the radical so as to make $p > 0$, does not pose any difficulty in determination of the length of the perpendicular from origin; but it certainly leads to erroneous conclusion while evaluating α for the normal form of the equation. Any manipulation of signs in the expression $p = \frac{-c}{\pm\sqrt{a^2+b^2}}$ affects the expressions $\cos \alpha = \frac{a}{\pm\sqrt{a^2+b^2}}$ and $\sin \alpha = \frac{b}{\pm\sqrt{a^2+b^2}}$. Therefore, the form of the expression $p = \frac{-c}{\pm\sqrt{a^2+b^2}}$ as such, must be adhered to, while determining α from $\cos\alpha = \frac{a}{\pm\sqrt{a^2+b^2}}$ and $\sin\alpha = \frac{b}{\pm\sqrt{a^2+b^2}}$, to obtain the normal form of the equation of the line.

For example, in the above exercise, if one writes $p = \frac{-4}{\pm\sqrt{1+3}}$ as $p = \frac{4}{\mp\sqrt{1+3}}$, then '+' sign of the radical has to be taken to make $p > 0$. But with the '+' sign of the radical, one gets $\cos \alpha$

$= \frac{1}{2}$ and $\sin\alpha = \frac{\sqrt{3}}{2}$ which lead to a 'normal form' that is erroneous.

Point of intersection :

Theorem : If two distinct lines L_1 and L_2, represented by the equations :

$L_1 : a_1x + b_1y + c_1 = 0$

$L_2 : a_2x + b_2y + c_2 = 0$

intersect at P (h, k), then $h = \dfrac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}$, $k = \dfrac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}$.

Proof : Since the lines intersect at P, P is the point common to L_1 and L_2.

$\therefore a_1h + b_1k + c_1 = 0$ .... (i)

and $a_2h + b_2k + c_2 = 0$ .... (ii)

Obviously L_1 and L_2 are not parallel and therefore, $a_1b_2 - a_2b_1 \neq 0$.

Solving (i) and (ii) for h and k,

we get

$$h = \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1},\ k = \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1} \ \square$$

Concurrent lines :

If P is the point common to distinct lines L_1, L_2, ... L_n, then the lines are said to concur / be concurrent at P.

Theorem : If three lines L_1, L_2 and L_3 be represented by the equations :

$L_1 : a_1x + b_1y + c_1 = 0$

$L_2 : a_2x + b_2y + c_2 = 0$ (The lines are supposed to be distinct)

$L_3 : a_3x + b_3y + c_3 = 0$, then

$$a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1) = 0 \text{ i.e. } \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

$\Leftrightarrow L_1, L_2, L_3$ are all parallel or concurrent.

[In the left hand side equality above, the suffices 1, 2, 3 occur in cyclic order in subsequent terms. It is abbreviated as $\sum_{1,2,3} a_1(b_2c_3 - b_3c_2) = 0$]

Proof : Suppose $\sum_{1,2,3} a_1(b_2c_3 - b_3c_2) = 0$

We have to show that the lines are all parallel or concurrent. Given three lines L_1, L_2, L_3, exactly one of the following must be true :

(i) L_1, L_2, L_3 are all parallel i.e. parallel to one another.

(ii) L_1, L_2, L_3 are not all parallel.

So it is enough to show that the second possibility leads to concurrence under the assumption

$$\sum_{1,2,3} a_1 (b_2c_3 - b_3c_2) = 0.$$

If L_1, L_2, L_3 are not all parallel, then there is at least one pair of intersecting lines, say L_1 and L_2. (One can also take any other pair.)

Let (h, k) be the point of intersection of L_1 and L_2.

Therefore $h = \dfrac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}$, $k = \dfrac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}$.

$[L_1 \parallel L_2 \Rightarrow a_1b_2 - a_2b_1 \neq 0]$

It is easy to see that

$a_3 (b_1c_2 - b_2c_1) + b_3 (c_1a_2 - c_2a_1) + c_3 (a_1b_2 - a_2b_1)$ is same as

$a_1 (b_2c_3 - b_3c_2) + a_2 (b_3c_1 - b_1c_3) + a_3 (b_1c_2 - b_2c_1)$.

So it follows from $\sum_{1,2,3} a_1 (b_2c_3 - b_3c_2) = 0$

that $a_3 (b_1c_2 - b_2c_1) + b_3 (c_1a_2 - c_2a_1) + c_3 (a_1b_2 - a_2b_1) = 0$.

Dividing by $a_1b_2 - a_2b_1$, which is nonzero,

$$a_3 \cdot \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1} + b_3 \cdot \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1} + c_3 = 0$$

$\Rightarrow a_3h + b_3k + c_3 = 0$.

Therefore the point of intersection of L_1 and L_2 lies on L_3 and consequently L_1, L_2, L_3 are concurrent.

Conversely suppose that L_1, L_2, L_3 are all parallel or concurrent. We have to show that

$$\sum_{1,2,3} a_1 (b_2c_3 - b_3c_2) = 0.$$

We first take up the case of parallelism.. Supposing all parallel.

$L_1 \parallel L_2 \Rightarrow a_1b_2 - a_2b_1 = 0$ .... (i)

$L_2 \parallel L_3 \Rightarrow a_2b_3 - a_3b_2 = 0$ .... (ii)

$L_3 \parallel L_1 \Rightarrow a_3b_1 - a_1b_3 = 0$ .... (iii)

(i), (ii) & (iii) $\Rightarrow c_1 (a_2b_3 - a_3b_2) + c_2 (a_3b_1 - a_1b_3) + c_3 (a_1b_2 - a_2b_1) = 0$

$\Rightarrow a_1 (b_2c_3 - b_3c_2) + a_2 (b_3c_1 - b_1c_3) + a_3 (b_1c_2 - b_2c_1) = 0$.

Next suppose that L_1, L_2, L_3 are concurrent, say at the point (h, k), Then

$a_1h + b_1k + c_1 = 0$ .... (1)

$a_2h + b_2k + c_2 = 0$ .... (2)

$a_3h + b_3k + c_3 = 0$ .... (3)

Solving (2) & (3), $h = \dfrac{b_2c_3 - b_3c_2}{a_2b_3 - a_3b_2}$, $k = \dfrac{c_2a_3 - c_3a_2}{a_2b_3 - a_3b_2}$.

Putting these values in (1),

$$a_1 \cdot \frac{b_2c_3 - b_3c_2}{a_2b_3 - a_3b_2} + b_1 \frac{c_2a_3 - c_3a_2}{a_2b_3 - a_3b_2} + c_1 = 0$$

or $a_1 (b_2c_3 - b_3c_2) + b_1 (c_2a_3 - c_3a_2) + c_1 (a_2b_3 - a_3b_2) = 0$

[$a_2 b_3 - a_3b_2 \neq 0$ as $L_2 \parallel L_3$]

or $a_1 (b_2c_3 - b_3c_2) + a_2 (b_3c_1 - b_1c_3) + a_3 (b_1c_2 - b_2c_1) = 0$ $\square$

Corollary (Condition for concurrence) :

If the set of lines $\{L_1, L_2, L_3\}$ contains an intersecting (nonparallel) pair,

then $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$ or $a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1) = 0$

turns out to be the condition for concurrence of L_1, L_2 and L_3 as, in this case, the possibility of all the lines being parallel does not arise.

Corallary (condition for parallelism) :

If the set of lines $\{L_1, L_2, L_3\}$ contains a parallel pair,

then $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$ or $a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1) = 0$

turns out to be the condition for parallelism (all the lines being parallel) as, in this case, the possibility of concurrence does not arise.

In otherwords, if the condition is satisfied and two of the lines are parallel, then the third line must also be parallel to the first two.

N.B. If the condition is not satisfied then the system (set) of lines $\{L_1, L_2, L_3\}$ neither constitutes a parallel system (all parallel) nor a concurrent system.

Exercise : Test for concurrence

$2x - y - 3 = 0$, $x - 3y + 1 = 0$, $4x - 5y - 3 = 0$.

Solution : $\begin{vmatrix} 2 & -1 & -3 \\ 1 & -3 & 1 \\ 4 & -5 & -3 \end{vmatrix} = 2\,(9 + 5) + 1\,(-3 - 4) - 3\,(-5 + 12) = 0.$

[Those, not familar with determinants, may calculate $\sum_{1,2,3} a_1 (b_2c_3 - b_3c_2)$, taking values of the coefficients and constant terms as a, b, c respectively with subscripts 1,2,3 for 1st, 2nd and 3rd lines]

Slope of first line = 2, slope of second line = $\frac{1}{3}$.

Since slopes are not equal, this is a non parallel pair in the set of lines. [You can also test any other pair].

$\therefore$ The set of lines are concurrent.

N.B. You can also find the point of intersection of one pair. If the coordinates of the point of intersection satisfy the equation of the remaining line, then the lines are concurrent.

*** Example 13 :** (You may skip this example or use alternative expression in place of the determinant)

If $\{a, b, c\}$ and $\{p, q, r\}$ are sets of distinct real numbers, prove that the lines

$(a-b)x + (p-q)\,y - 2 = 0,\ (b-c)x + (q-r)\,y + 1 = 0$

and $(c-a)x + (r-p)y + 1 = 0$ are all parallel or concurrent according as $a\,(q-r) + b\,(r-p) + c\,(p-q)$ is or is not zero.

Solution : $\begin{vmatrix} a-b & p-q & -2 \\ b-c & q-r & 1 \\ c-a & r-p & 1 \end{vmatrix} = 0$ ($\because$ The rows add upto zero)

So, the lines are all parallel or concurrent.

To ascertain parallelism or concurence we have to look for a parallel or intersecting pair in the given set of lines.

The first two lines are parallel if, and only if,

$(a-b)\,(q-r) - (b-c)\,(p-q) = a\,(q-r) + b(r-p) + c\,(p-q) = 0.$ (A)

By the cyclic order the coefficients of x and y in the given equation, the condition for parallelism of any other pair of lines is also given by (A).

Therefore there is parallel or intersecting pair in the given set of lines according as

$a\,(q-r) + b\,(r-p) + c\,(p-q) = 0$ or $\neq 0$.

$\therefore$ The given lines are all parallel or intersecting according as

$a\,(q-r) + b\,(r-p) + c\,(p-q) = 0$ or $\neq 0$.

Note : By assigning numerical values to a, b, c, p, q, r you can obtain numerous triads of lines all of which are either parallel or concurrent. To ascertain parallelism / concurrence assign numerical values to any five of the unknowns a, b, c, p, q, r arbitrarily and obtain the values of the remaining unknown by solving $a\,(q-r) + b\,(r-p) + c\,(p-q) = 0$. If s is the solution, then the lines will be all parallel or concurrent according as you take the remaining unknown equal to or different from s.

11.3 Family of lines through the point of intersection of two lines

The Lines : $a_1x + b_1y + c_1 + \lambda\,(a_2x + b_2y + c_2) = 0,\ \lambda \in \mathbf{R}$:

Let the lines L_a and L_b be given by

$L_a : a_1x + b_1y + c_1 = 0$ .... (i)

$L_b : a_2x + b_2y + c_2 = 0$ .... (ii)

Now consider the equation

$a_1x + b_1y + c_1 + \lambda\,(a_2x + b_2y + c_2) = 0$ .... (iii)

Equation (iii) being of first degree in x and y, represents a line for every value of λ in R.

So, for all $\lambda \in$ R, we get a set of lines (also called a family or system of lines) given by equations (iii).

Further informations about this family of lines is given by :

Theorem :

Given two lines, L_a: $a_1x + b_1y + c_1 = 0$ and L_b : $a_2x + b_2y + c_2 = 0$, the equations $a_1x + b_1y + c_1 + \lambda (a_2x + b_2y + c_2) = 0$, for $\lambda \in$ R represent the family of lines.

(i) through the point of intersection of L_a and L_b, if they intersect;

(ii) parallel to L_a and L_b if they are parallel.

Proof : Two lines L_a and L_b in a plane are either parallel or intersecting. [The lines L_a and L_b with the given equations contain points (x, y) which all lie in the cartesian plane. So L_a and L_b are necessarily coplanar.]

Let L_λ be the line represented by $a_1x + b_1y + c_1 + \lambda (a_2x + b_2y + c_2) = 0$

or $(a_1 + \lambda a_2)x + (b_1 + \lambda b_2)y + (c_1 + \lambda c_2) = 0$.

It is obvious from the theory of determinants

that $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_1+\lambda a_2 & b_1+\lambda b_2 & c_1+\lambda c_2 \end{vmatrix} = 0$. [Without using determinants alternative condition for concurrence/ Parallelism may be used.]

So L_a, L_b, L_λ are either concurrent or parallel.

(i) If L_a and L_b intersect, then L_a, L_b, L_λ must be concurrent. In otherwords, L_λ passes through the point of intersection of L_a and L_b.

(ii) If L_a and L_b are parallel, then L_a, L_b, L_λ must be parallel so that L_λ is parallel to L_a or L_b.

Note : (1) For $\lambda = 0$, L_λ coincides with L_a.

(2) In case L_a and L_b are coincident, then for every λ , L_λ is a line coincident with L_a or L_b.

Alternative Proof :

(i) Suppose L_a and L_b intersect at (h, k).

Then $a_1h + b_1k + c_1 = 0$ and $a_2h + b_2k + c_2 = 0$.

$\therefore$ For every $\lambda \in$ R, $a_1h + b_1k + c_1 + \lambda (a_2h + b_2k + c_2) = 0$.

Therefore L_λ passes through the point of intersection of L_a and L_b, for every $\lambda \in$ R. $\square$

(ii) Suppose $L_a \parallel L_b$.

Then $a_1b_2 - a_2b_1 = 0$.

Consider L_a : $a_1x + b_1y + c_1 = 0$

and L_λ : $(a_1 + \lambda a_2)\, x + (b_1 + \lambda b_2)\, y + (c_1 + \lambda c_2) = 0$.

$a_1 (b_1 + \lambda b_2) - (a_1 + \lambda a_2)\, b_1 = \lambda (a_1b_2 - a_2b_1) = 0 \quad (\because\ a_1b_2 - a_2b_1 = 0)$

$\Rightarrow L_\lambda \parallel L_a$.

Since $L_a \parallel L_b$, it follows that L_λ is parallel to L_a, L_b, for every $\lambda \in$ R $\square$

Example 14

(i) The two line $x + y - 3 = 0$, $3x + 3y - 9 = 0$ are the same.

(ii) The two lines $x - y + 1 = 0$, $3x - 3y + 2 = 0$ are parallel, but different as the constants are not in the same proportion as coefficient of x and y.

(iii) The two lines $x - y + 5 = 0$, $2x + 2y - 1 = 0$ are perpendicular, since
$m_1 \,.\, m_2 = 1.\,(-1) = -1$
and their point of intersection is given by

$$\left(\frac{(-1)\times(-1)-5\times 2}{1\times 2-(-1)\times 2}, \frac{5\times 2-1\times(-1)}{1\times 2-(-1)\times 2}\right) = \left(\frac{-9}{4}, \frac{11}{4}\right).$$

(iv) Obtain the equation of the line passing through the intersection of $3x - 2y + 7 = 0$, $x + 3y + 3 = 0$ and the point $(1, -1)$.

The equation of the line passing through the intersection of
$L_1 : 3x - 2y + 7 = 0$
$L_2 : x + 3y + 3 = 0$, is given by
$3x - 2y + 7 + \lambda\,(x + 3y + 3) = 0$, $(\lambda \in R)$ (17)
Since it passes through $(1, -1)$, we have $[3.\,1 - 2\,(-1) + 7] + \lambda\,[1 + 3\,(-1) + 3] = 0$

i.e. $\lambda = -\dfrac{12}{1} = -12.$

Thus, the equation of the required line is (from 17)
$(3x - 2y + 7) + (-12)\,(x + 3y + 3) = 0$
i.e. $-9x - 38y - 29 = 0$
or $9x + 38y + 29 = 0$.

(v) Obtain the equation of the line passing through the intersection of $x = 0$, $y = 0$ and perpendicular to $x + y + 1 = 0$.
Any line passing through the intersection of $x = 0$ and $y = 0$ is given by $x + \lambda y = 0$

whose slope $m_1 = -\dfrac{1}{\lambda}$.

If this is perpendicular to $x + y + 1 = 0$, whose slope is -1, then
$m_1\,(-1) = -1 \Rightarrow \lambda = -1$.
Hence the required line is $x + \lambda y = 0$, or $x = y$.

(vi) Obtain the quation of the line passing through the intersection of the lines $2x + y = 0$, $y + 2 = 0$ and whose coefficients are in A.P. with common difference 4.

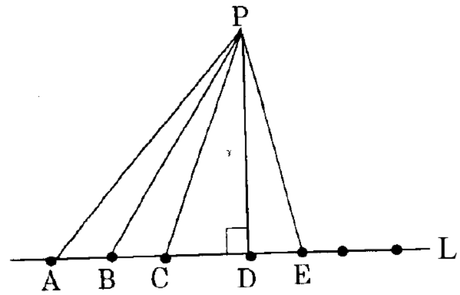
The equation of the line through intersection of $2x + y = 0$ and $y + 2 = 0$ is given by
$(2x + y) + \lambda\,(y + 2) = 0$ i.e. $2x + (1 + \lambda)\,y + 2\lambda = 0$.
The coefficients, 2, $1 + \lambda$, 2λ are in A.P. with common difference 4.
Hence, $1 + \lambda - 2 = 2\lambda - (\lambda + 1) = 4$
$\Rightarrow \lambda - 1 = 4$ or $\lambda = 5$.
$\therefore$ Equation of the required line is $(2x + y) + 5(y + 2) = 0$
or $2x + 6y + 10 = 0$.

11.4 Distance of a point from a line

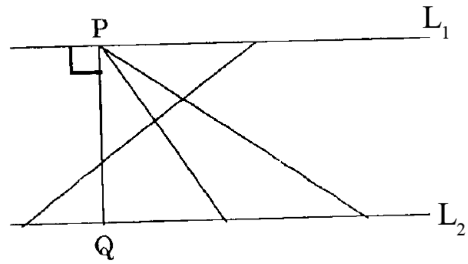
Before obtaining a formula to find the distance of a point from a line, we shall obtain some preliminary results.

We have already discussed about distance btween two points. If P and Q be two given points, then there are infinitely many paths (curves) from P to Q, out of which the path along $\overleftrightarrow{PQ}$ is the path of shortest distance and distance from P to Q is defined as d (P, Q) = PQ.

Similarly, let P be a given point and L be a given line. Let A, B, C, D, E, ... etc be different points of L (fig. 15). Then the distance AP, BP, CP, DP, EP... can be calculated by using formula for distance between two points. Of all such distances, PD (the perpendicular distance) is the shortest distance (from Euclidean Geometry). This PD is defined as the distance of P, from the line L, that is d (P, L) = PD. Clearly, the distance of any point P, (situated on L), from L is zero.



Similarly, the distance between two lines, is the length of their common perpendicular. Hence distance between two parallel lines, L_1 and L_2, is the distance of any point P on L_1 from the line L_2 (Fig. 16) i.e, d (L_1, L_2) = d (P, L_2).



Position of a point with respect to a line

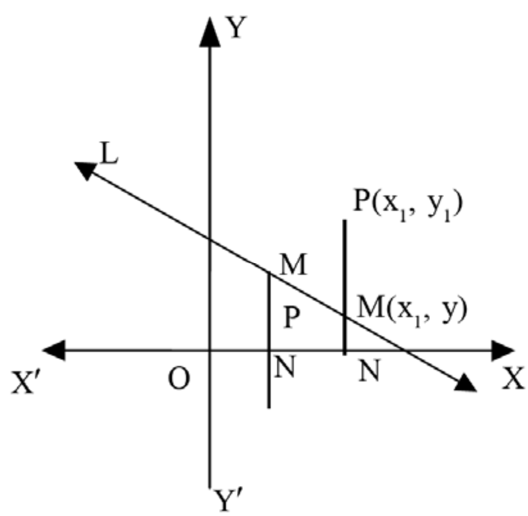
Consider a nonvertical line L, given by $ax + by + c = 0$ and a point P (x_1, y_1) . (Fig. – 17).

Let a line through P(x_1, y_1) parallel to the y – axis, interesect L at M (x_1, y). We define :

(i) P(x_1, y_1) is above L $\Leftrightarrow y_1 > y$ i.e. $y_1 - y > 0$.

(ii) P(x_1, y_1) is below L $\Leftrightarrow y_1 < y$ i.e. $y_1 - y < 0$.

Since the point M (x_1, y) is on the line L, we have $ax_1 + by + c = 0$



or $y = -\frac{(ax_1 + c)}{b}$ (b ≠ 0, since L is nonvertical)

So, $(y_1 - y) = y_1 + \frac{ax_1 + c}{b} = \frac{ax_1 + by_1 + c}{b}$.

Hence, the point P (x_1, y_1) will lie above or below the line $ax + by + c = 0$, if $\frac{ax_1 + by_1 + c}{b}$ is positive or negative.

i.e. $(ax_1 + by_1 + c)$ has same sign or opposite sign as that of b, respectively.

In particular, the origin will lie above or below the line, $ax + by + c = 0$, according as c and b have the same sign or opposite sign.

It follows from the above considerations that if both (x_1, y_1) and (x_2, y_2) lie above or below the line, $ax + by + c = 0$, then both $ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ have the same or opposite signs as that of b. Therefore, if one of (x_1, y_1) and (x_2, y_2) is above and the other, below the line $ax + by + c = 0$, then $ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ have opposite signs.

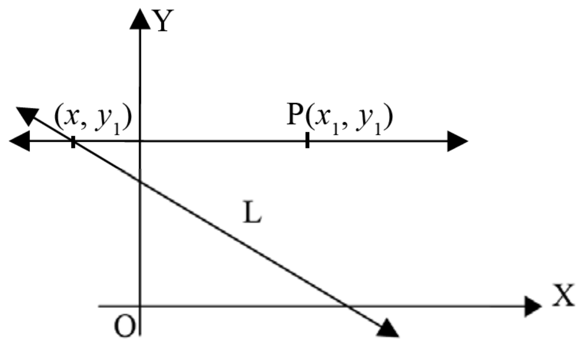
In other words, **the points (x_1, y_1) and (x_2, y_2) lie on the same side or opposite sides of the line $ax + by + c = 0$ according as $ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ have the same or opposite signs.**

Note : For a vertical line we do not have the concepts of 'above' or 'below'. However the half-planes corresponding to such line can be designated as 'right' or 'left'. We now define these concepts for all but horizontal lines.

Consider a nonhorizontal line L, given by : $ax + by + c = 0$ and a point P (x_1, y_1).

Let a line through $P(x_1, y_1)$, parallel x - axis, intersect L at the point (x, y_1).

We define :



(1) $P(x_1, y_1)$ is to the right of L
$\Leftrightarrow x_1 > x$ i.e. $x_1 - x > 0$.

(2) $P(x_1, y_1)$ is to the left of L
$\Leftrightarrow x_1 < x$, i.e. $x_1 - x < 0$.

Since $(x, y_1) \in$ L, $ax + by_1 + c = 0$.

$\therefore x = -\frac{by_1 + c}{a}$ ($a \neq 0$, as L is nonhorizontal)

Therefore $x_1 - x = \frac{ax_1 + by_1 + c}{a}$.

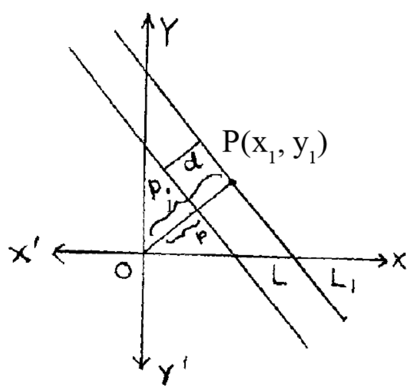
So $P(x_1, y_1)$ is to the right or left of the line $ax + by + c = 0$ according as $\frac{ax_1 + by_1 + c}{a}$ is positive or negative i.e. $ax_1 + by_1 + c$ has the same or opposite sign as that of a.

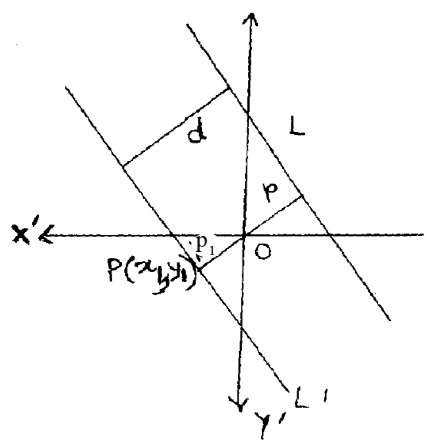
The origin is to the right or left of the line ax + by + c = 0 according as *c* and *a* have the same or opposite signs.

As before it can be proved that the points (x_1, y_1) and (x_2, y_2) lie on the same or opposite sides (right, left) of $ax + by + c = 0$, according as $ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ have the same or opposite signs.

Note : For a horizontal line we do not have the concepts of 'right' or 'left'. For oblique lines both the concepts, 'right and left', 'above and below' are valid.

Coming to the main result, let $P(x_1, y_1)$ be a given point and $ax + by + c = 0$ be a given line L. Draw a line L_1 through P parallel to L. Let p and p_1, be distances of lines L and L_1 from origin, respectively. If origin lies on the same side of L and L_1, then the distance between them is $d = |p - p_1|$, and if origin lies on opposite sides of Land L_1, then $d = |p + p_1|$





We know from 'normal form' that the distance of origin from the line L : $ax + by + c = 0$, is $p = \frac{-c}{\pm\sqrt{a^2 + b^2}}$, with proper choice of the sign of the radical $\pm\sqrt{a^2 + b^2}$ so as to make p positive.

Since L_1 is parallel to L, equation of L_1 can be written as $ax + by + c_1 = 0$.

$$\therefore p_1 = \frac{-c_1}{\pm\sqrt{a^2 + b^2}}.$$

But c and c_1 have the same or opposite signs according as the origin lies on the same side or opposite sides of L and L_1. (i.e. origin is to the left/right or origin is above/ below of both L and L_1)

Therefore (I) $p = \dfrac{-c}{\pm\sqrt{a^2+b^2}}$, $p_1 = \dfrac{-c_1}{\pm\sqrt{a^2+b^2}}$, if the origin is on the same side of L and L_1 and

(II) $p = \dfrac{-c}{\pm\sqrt{a^2+b^2}}$, $p_1 = \dfrac{-c_1}{\mp\sqrt{a^2+b^2}}$, if the origin is on opposite sides of L and L_1.

[If origin lies on the same side of L and L_1 i.e. c and c_1 have the same sign, we have to choose the same sign for the radical to make both p and p_1 positive. Therefore we write $p = \dfrac{-c}{\pm\sqrt{a^2+b^2}}$

and $p_1 = \dfrac{-c_1}{\pm\sqrt{a^2+b^2}}$ to mean that if '+' sign is chosen for the radical in case of p then '+' sign is also chosen for the radical in case of p_1 and, if '–' sign is chosen for the radical in case of p then '–' sign is also chosen in case of p_1. But if origin lies on opposite sides of L and L_1, in which case c and c_1 have opposite signs, opposite signs for the radical have to be chosen to make both p and p_1 positive i.e. if we choose '+' sign for the radical in case of p then we choose '–'sign for the radical in case of p_1 and viceversa. To indicate this mode of choice of '+'

and '–' signs we write $p = \dfrac{-c}{\pm\sqrt{a^2+b^2}}$ and $p_1 = \dfrac{-c_1}{\mp\sqrt{a^2+b^2}}$; in case the origin lies on opposite sides of L and L_1]

$$(x_1, y_1) \in L_1 \Rightarrow ax_1 + by_1 + c_1 = 0 \Rightarrow c_1 = -(ax_1 + by_1).$$

In case I, $d = |p - p_1| = \left|\dfrac{-c}{\pm\sqrt{a^2+b^2}} - \dfrac{-c_1}{\pm\sqrt{a^2+b^2}}\right|$

$$= \left|\frac{c_1 - c}{\pm\sqrt{a^2+b^2}}\right| = \left|\frac{-ax_1 - by_1 - c}{\pm\sqrt{a^2+b^2}}\right| = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2+b^2}}.$$

In case II,

$$d = |p + p_1| = \left|\frac{-c}{\pm\sqrt{a^2+b^2}} + \frac{-c_1}{\mp\sqrt{a^2+b^2}}\right| = \left|\frac{-c}{\pm\sqrt{a^2+b^2}} + \frac{c_1}{\pm\sqrt{a^2+b^2}}\right|$$

$$= \left|\frac{c_1 - c}{\pm\sqrt{a^2+b^2}}\right| = \left|\frac{-ax_1 - by_1 - c}{\pm\sqrt{a^2+b^2}}\right| = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2+b^2}}.$$

Thus, in any case, the distance of a point $P(x_1, y_1)$ from the line L : $ax + by + c = 0$ is given by $d = \dfrac{|ax_1 + by_1 + c|}{\sqrt{a^2+b^2}}$.

Applications :

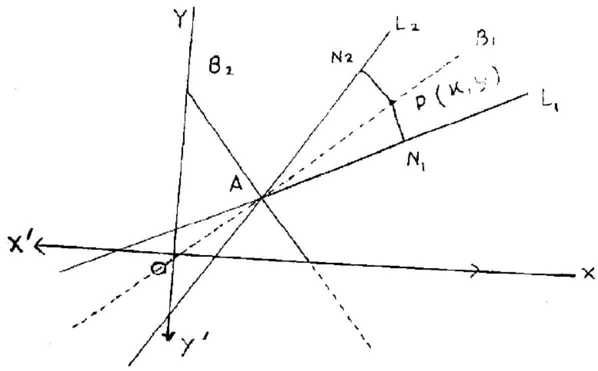
Equations of bisectors of angle between two (intersecting) Lines

Let the two given lines be L_1 and L_2 and the two bisectors be B_1 and B_2 (Fig. 20). Suppose L_1 and L_2 intersect at A. Let the equations of L_1 and L_2 be,

$$a_1x + b_1y + c_1 = 0$$

$$a_2x + b_2y + c_2 = 0$$

respectively. Consider a point P (x, y) on the bisector B_1.



Drop perependiculars $\overline{PN_1}$ and $\overline{PN_2}$ on L_1 and L_2 rspectively. Then tirangles APN_1 and APN_2 are congruent. Hence, $PN_1 = PN_2$

$$\Rightarrow \frac{|a_1x+b_1y+c_1|}{\sqrt{a_1^2+b_1^2}} = \frac{|a_2x+b_2y+c_2|}{\sqrt{a_2^2+b_2^2}} \Rightarrow \frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}} = \pm \frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}$$

are the equations of the two bisectors.

Note : Out of the two bisectors, one which makes an angle measuring less than 45° with any of the intersecting lines, is the internal bisector. Thus if θ is the angle which one of the bisectors makes with any one of the lines, then $|\tan\theta| < 1$, gives the internal bisector i.e. bisector of the acute angle between the lines L_1 and L_2.

Bisector of the angle containing a given point (h, k) in its interior,

Actually bisector of an angle is a ray. However, by ‘bisector of angle between two intersecting lines’, we mean the union of two opposite rays which bisect two opposite angles formed by the intersection of the lines. Therefore, in case of two intersecting lines, we get two lines, (obviously perpendicular to each other) as the bisectors of the angles.

Therefore, the bisector of the angle containing (h, k) in its interior, is a union of two opposite rays, their common vertex being the point of intersection of the given intersecting lines. One ray bisects the angle containing (h. k)in its interior, the other ray bisects the oppsite angle.

Let the equations of two lines L_1 and L_2 be given as $L_1 : a_1x + b_1y + c_1 = 0$, $L_2 : a'_2x + b'_2y + c'_2 = 0$.

To obtain the equation of the bisector of the angle containing (h, k) in its interior, proceed as follows :

1. First see if $a_1h + b_1 k + c_1$ and $a'_2 h + b'_2 k + c'_2$ have the same or opposite signs.

2. (i) If they have the same sign, write $a_2 = a_2', b_2 = b_2', c_2 = c_2'$.

   (ii) If they have opposite signs, write $a_2 = -a_2', b_2 = -b_2', c_2 = -c_2'$.

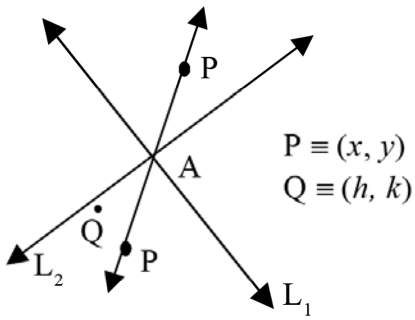
3. Write the bisector of the angle containing (h, k) in its interior as :

$$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}.$$

Justification of the Procedure :

After the second step, we can write the equations of L_1 and L_2 as :

$L_1 : a_1x + b_1 y + c_1 = 0$ and $L_2 : a_2x + b_2 y + c_2 = 0$, such that $a_1h + b_1 k + c_1$ and $a_2h + b_2 k + c_2$ have the same sign. [obviously none of $a_1h + b_1 k + c_1$ and $a_2h + b_2 k + c_2$ is zero, since (h, k) $\in$ interior of an angle between L_1 and L_2 $\Rightarrow$ (h, k) $\notin L_1$ and (h, k) $\notin L_2$. An angle and its interior are disjoint sets.]



If P $(x, y) \in$ the ray, bisecting the angle containing (h, k) in its interior,

then (x,y) and (h, k) are on the same side of L_1 as well as L_2. So the pairs of numbers $a_1x + b_1 y + c_1$, $a_1h + b_1 k + c_1$ and $a_2x + b_2 y + c_2$, $a_2h + b_2 k + c_2$ have the same sign.

Since $a_1h + b_1 k + c_1$ and $a_2h + b_2 k + c_2$ have the same sign, it follows that $a_1x + b_1 y + c_1$ and $a_2x + b_2 y + c_2$ have the same sign.

$\therefore$ From the equations of bisectors, we get

$$\frac{|a_1x + b_1y + c_1|}{\sqrt{a_1^2 + b_1^2}} = \frac{|a_2x + b_2y + c_2|}{\sqrt{a_2^2 + b_2^2}}$$

$$\Rightarrow \frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}. \qquad ......(A)$$

If P $(x, y) \in$ the ray bisecting the angle, opposite to that containing (h, k) in its interior, then (x, y) and (h, k) lie on opposite sides of L_1 as well as L_2. So the pairs of numbers $a_1x + b_1 y + c_1$, $a_1h + b_1 k + c_1$ and $a_2x + b_2 y + c_2$, $a_2h + b_2 k + c_2$ have opposite signs. But since $a_1h + b_1 k + c_1$ and $a_2h + b_2 k + c_2$ have the same sign, it follows that $a_1x + b_1 y + c_1$ and $a_2x + b_2 y + c_2$ have the same sign. Therefore,

$$\frac{|a_1x + b_1y + c_1|}{\sqrt{a_1^2 + b_1^2}} = \frac{|a_2x + b_2y + c_2|}{\sqrt{a_2^2 + b_2^2}}$$

$$\Rightarrow \frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}. \qquad \text{......(B)}$$

(A) and (B) imply that

the bisector of the angle, containing (h, k) in its interior, is given by

$\dfrac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \dfrac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}$, provided $a_1h + b_1k + c_1$ and $a_2h + b_2k + c_2$ have the same sign.

Corollary : Bisector of the angle, not containing (h, k) in its interior, is given by

$\dfrac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = (-)\dfrac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}$, provided $a_1h + b_1k + c_1$ and $a_2h + b_2k + c_2$ have the same sign.

Example 15 :

(i) The length of the perpendicular drawn from the point (0, 0) to the line $2x - 3y - 3 = 0$, is $p = \left|\dfrac{-3}{\sqrt{2^2 + (-3)^2}}\right| = \dfrac{3}{\sqrt{13}}$.

(ii) The length of the perpendicular from the point (1, 1) to the line $3x - y + 1 = 0$, is

$$d = \left|\frac{3.1 - 1 + 1}{\sqrt{3^2 + (-1)^2}}\right| = \frac{3}{\sqrt{10}}.$$

(iii) The distance between two parallel lines
$2x + y - 3 = 0$, $4x + 2y - 1 = 0$, is

$$d = \left|\frac{-3 - \left(-\frac{1}{2}\right)}{\sqrt{2^2 + 1^2}}\right| \qquad \text{(dividing equation of second line by 2)}$$

$= \dfrac{5}{2\sqrt{5}} = \dfrac{\sqrt{5}}{2}$ [Notice that the origin lies below both the given lines. So, the distance between these parallel lines is the absolute value of the difference of their distances from the origin.]

(iv) The distance between the two lines $2x + y - 3 = 0$, $x + 2y - 7 = 0$, is zero as the lines are intersecting.

(v) Find the distance of the point (–2, 3) from the line $x - 3y - 2 = 0$, measured parallel to the line $2x + y - 1 = 0$.

Let L be a line through the point (–2, 3) and parallel to the line $2x + y - 1 = 0$. (1)

Then slope of L, is $m = -2$, and hence its equation is $y - 3 = -2(x + 2)$

or, $2x + y + 1 = 0$.

(2)

Now the point of intersection of $x - 3y - 2 = 0$ (3)

with (2) is given by $x = -\frac{1}{7}$ and $y = -\frac{5}{7}$.

Hence, the distance of the point (–2, 3) from the line $x - 3y - 2 = 0$, measured parallel to $2x + y - 1 = 0$ is the distance between the points (–2, 3) and $\left(-\frac{1}{7}, -\frac{5}{7}\right)$, i.e.

$$d = \sqrt{\left(2-\frac{1}{7}\right)^2 + \left(-3-\frac{5}{7}\right)^2} = \frac{13\sqrt{5}}{7}.$$

(vi) The equations of bisectors of the angles between the lines $x + 2y - 1 = 0$ and $4x + 3y + 1 = 0$, are

$$\frac{x + 2y - 1}{\sqrt{5}} = \pm\frac{4x + 3y + 1}{5}$$

or, $\sqrt{5}(x + 2y - 1) = \pm(4x + 3y + 1)$,

or, i.e. $(4 + \sqrt{5})x + (3 + 2\sqrt{5})y + (1 - \sqrt{5}) = 0$,

and, $(4 - \sqrt{5})x + (3 - 2\sqrt{5})x + (1 + \sqrt{5}) = 0$.

Pair of Straight Lines

Consider the equation, $x^2 - y^2 = 0$

which can be written as $(x - y)(x + y) = 0$.

The co-ordinates of any point on the line $x - y = 0$, or $x + y = 0$, satisfy the equation (i). Hence the equation (i) represents a pair of lines, $x - y = 0$ and $x + y = 0$.

Consider the equation $ax^2 + 2hxy + by^2 = 0$

(1)

which is a second degree homogeneous equation in x and y. This can be written as

$$b\left(\frac{y^2}{x^2}\right) + 2h\left(\frac{y}{x}\right) + a = 0$$

which is a quadratic in $\frac{y}{x}$. This gives

$$\frac{y}{x} = \frac{-2h \pm\sqrt{4h^2 - 4ab}}{2b} = \frac{-h \pm\sqrt{h^2 - ab}}{b} \text{ (considering } b \neq 0)$$

or, $y = \left(\dfrac{-h + \sqrt{h^2 - ab}}{b}\right) x$, and $y = \left(\dfrac{-h - \sqrt{h^2 - ab}}{b}\right) x$, which represents a pair of lines through the origin, with slopes

$$\left.\begin{aligned} m_1 &= \left(\frac{-h+\sqrt{h^2-ab}}{b}\right) \\ m_2 &= \left(\frac{-h-\sqrt{h^2-ab}}{b}\right) \end{aligned}\right\}$$

provided $h^2 \geq ab$.

If $h^2 > ab$, the lines are different and if $h^2 = ab$, the lines are coincident, since their equations are identical in this case.

We have $m_1 + m_2 = -\frac{2h}{b}$ and $m_1 m_2 = \frac{a}{b}$. So lines are perpendicular when $a + b = 0$. (2)

Similarly, if we consider a pair of lines through the origin given by $y = m_1 x$, and $y = m_2 x$, then their combined equation is $(y - m_1 x)(y - m_2 x) = 0$

i.e., $y^2 - (m_1 + m_2)xy + m_1 m_2 x^2 = 0$

which is a homogeneous second degree equation in x and y.

Thus, every homogeneous equation of second degree in x and y i.e. $ax^2 + 2hxy + by^2 = 0$ represents a pair of lines through the origin if $h^2 - ab > 0$ and conversely.

Now consider the pair of lines represented by $ax^2 + 2hxy + by^2 = 0$. Their slopes are given above in (2). If the lines are not coincident (i.e. $h^2 - ab > 0$) and θ measures an angle between them, then

$$\theta = n\pi + \tan^{-1}\left(\pm\frac{m_1 - m_2}{1 + m_1 m_2}\right) \tag{3}$$

$n \in z$ being chosen so that $\theta \in (0, \pi)$.

It will be proved in chapter – 10 (art 10.5) that the quadratic expression $ax^2 + 2hxy + by^2 + 2gx + 2fy + c$ can be factored into first degree ploynomials under the condition $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$.

The factor polynomial have real coefficients when $a = b = 0$. But when

(i) $a \neq 0$, the factor - polynomials have real coefficients if $h^2 \geq ab$ and $g^2 \geq ac$

(ii) $a = 0$, but $b \neq 0$ then the factor - polynomials have real coefficients if $f^2 \geq bc$.

Therefore $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ (4)

represents a pair of lines under the conditions :

$abc + 2fgh - af^2 - bg^2 - ch^2 = 0$ and

(i) $h^2 > ab$, $g^2 > ac$ if $a \neq 0$

(ii) $f^2 > bc$ if $a = 0$, but $b \neq 0$.

If the lines represented by (4) are intersecting and θ measures the angle between them, then it can be shown by considering the linear factors of the quadratic expression that

$\tan\theta = \pm\frac{2\sqrt{h^2 - ab}}{a + b}$. The lines will be perpendicular if $a + b = 0$, and if $h^2 = ab$, it can be shown that the lines are parallel.

Example 16 :

Find the value of k for which, the equation $(k-3)x^2 + 4xy + y^2 = 0$ represents a pair of lines perpendicular to each other.

By condition of perpendicularity, the equation will represent a pair of perpendicular lines, if $(k-3) + 1 = 0$, or $k = 2$.

Example 17 :

Obtain the equation of the pair of lines which are perpendicular to the pair of lines $ax^2 + 2hxy + by^2 = 0$ and pass through the origin.

Let the pair of lines represented by $ax^2 + 2hxy + by^2 = 0$ be $y = m_1 x$ and $y = m_2 x$.

Then the pair of lines through origin, perpendicular to these two have slopes $-\frac{1}{m_1}$ and $-\frac{1}{m_2}$, and their equations are $y = -\frac{1}{m_1}x$ and $y = -\frac{1}{m_2}x$.

Hence the combined equation is $\left(y + \frac{1}{m_1}x\right)\left(y + \frac{1}{m_2}x\right) = 0$

or, $(m_1 m_2)y^2 + (m_1 + m_2)\,xy + x^2 = 0$

or, $\left(\frac{a}{b}\right) y^2 + \left(\frac{-2h}{b}\right)xy + x^2 = 0$

or, $ay^2 - 2hxy + bx^2 = 0$.

Example 18 :

Show that the equation of the pair of bisectors of the angle between the pair of lines, $ax^2 + 2hxy + by^2 = 0$ is $\dfrac{x^2 - y^2}{a-b} = \dfrac{xy}{h}$.

Let the pair of lines be $y = m_1 x$ and $y = m_2 x$.

Then, $m_1 + m_2 = -\frac{2h}{b}$ and $m_1 m_2 = \frac{a}{b}$.

The equation of pair of bisectors for $m_1 x - y = 0$ and $m_2 x - y = 0$

is $\dfrac{m_1 x - y}{\sqrt{m_1^2 + 1}} = \pm \dfrac{m_2 x - y}{\sqrt{1 + m_2^2}}$

whose combined equations is $\left(\dfrac{m_1 x - y}{\sqrt{1+m_1^2}} - \dfrac{m_2 x - y}{\sqrt{1+m_2^2}}\right)\left(\dfrac{m_1 x - y}{\sqrt{1+m_1^2}} + \dfrac{m_2 x - y}{\sqrt{1+m_2^2}}\right) = 0$

which on simplification gives $\dfrac{x^2 - y^2}{a - b} = \dfrac{xy}{h}$.

11.5 Change of Axes (Shifting of origin)

The origin and the axes in a co-ordinate system are chosen arbitrarily. In two different co-ordinate systems, the co-ordinates of a point are different. If the equation of a curve in one co-ordinate system (S) is known, then its equation in the second co-ordinate system (S′) can be determined provided there is a relation between the co-ordinates of the two systems; and this is studied under the transformation of co-ordinates :

Here, we consider the transformation of co-ordinates in the following two cases :

(a) The origins O and O′ of the systems S and S′ are different but the respective co-ordinate axes of the two systems are parallel.

(b) Systems S and S′ have the common origin. The x'– axis of S′ is the set of points with polar co-ordinates (r, α); $r \in R$ and α fixed; with respect to origin as pole and positive x – axis of S as the initial ray. [the set of points with polar co-ordinates (r, α), $r \in R$ and α fixed, can be easily seen to be a line.]

In (a), the co-ordinate system S′ is called a translation of the system S. In (b), the system S′ is called a rotation of the system S; $|\alpha|$ being the measure of rotation. The rotation is said to be be'anticlockwise' or 'clockwise' according as $\alpha > 0$ or $\alpha < 0$.

The following two theorems give the formulae for transformation of coordinates of one system into another.

Theorem

(i) O′ (h, k) is the origin of the system S′ with respect to the origin O $(0, 0)$ of the system S.

(ii) S′ is the translation of S.

(iii) (x,y) and (x', y') are the co-ordinates of a point P in system S and S′ respectively.
$\Rightarrow x' = x - h,\ y' = y - k.$

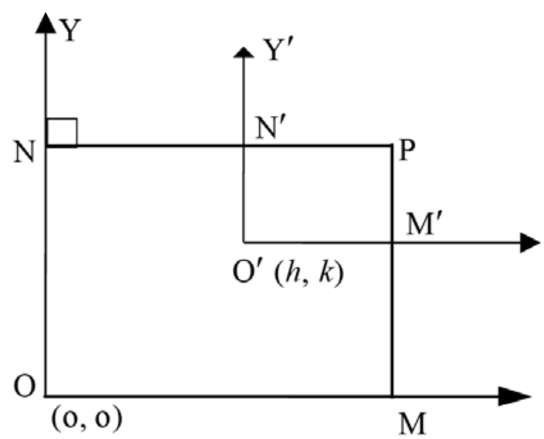
Proof :

From the Fig. 21 we note that

$x' = N'P = NP - NN' = x - h,$

$y' = M'P = MP - MM' = y - k.$

This completes the proof of the theorem.



Example 19 :

Transform the equation : $2x^2 + 3y^2 - 8x + 8y - 34 = 0$

when referred to parallel axes through the point $(2, -3)$.

Solution :

Here : $h = 2$, $k = -3 \Rightarrow$ (i) $x = x' + 2$, $y = y' - 3$

and the equation of the curve in terms of x', y' becomes $2\,(x' + 2)^2 + 3\,(y' - 3)^2$

$- 8\,(x' + 2) + 8\,(y' - 3) - 34 = 0.$

This after simplification becomes $2x^2 + 3y^2 = 39$;

dropping the primes *n*. Therefore the given equation represents an ellipse (art. 8.18) which is difficult to recognize unless we make use of the transformation (i).

Theorem

(i) S' is a rotation of S

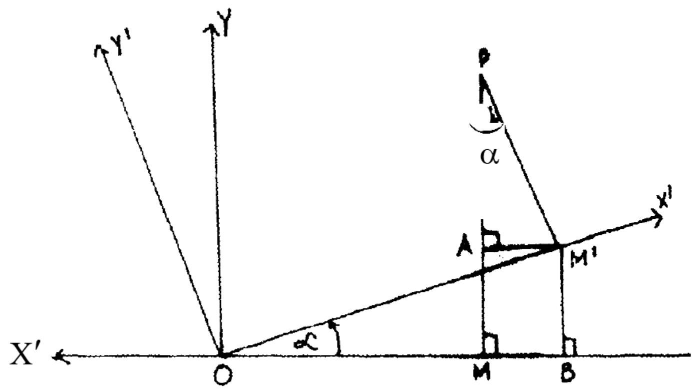
(ii) α is the measure of rotation.

(iii) (x,y) and (x', y') are the co-ordinates of a point P with respect to S and S'.

$\Rightarrow x = x' \cos\alpha - y' \sin\alpha$ and

$y = x' \sin\alpha + y' \cos\alpha.$

Proof :



From the Fig, 22

$x = OM = OB - MB = OB - AM'$,

$OB = OM' \cos\alpha = x' \cos\alpha$.

$AM' = PM' \sin\alpha = y' \sin\alpha$.

$\therefore\ x = x' \cos\alpha - y' \sin\alpha.$

Similarly it can be shown that $y = x' \sin\alpha + y' \cos\alpha$ and with this the proof is complete.

N.B. These formulae for transformation of coordinates hold for all α such that $0 \le \alpha < 2\pi$.

Corollary (1) If S' is roation of S with measure of rotation α then

$$x' = x\cos\alpha + y\sin\alpha,\ y' = y\cos\alpha - x\sin\alpha.$$

Corollary (2) If S' is a combination of a translation followed by a rotation (α being the measure of rotation) then $x = h + x' \cos\alpha - y' \sin\alpha$, $y = k + x' \sin\alpha + y' \cos\alpha$.

Example 20 :

How should the axes be rotated so that the transformed equation of

$ax^2 + 2hxy + by^2 = 0$

shall have no xy term ?

Solution :

The transformed equation will not contain $x'y'$ – term if its coefficient is zero.
Now, under the transformation :

$x = x' \cos\alpha - y' \sin\alpha$, $y = x' \sin\alpha + y' \cos\alpha$,

$o = ax^2 + 2hxy + by^2$

$= a\,(x' \cos\alpha - y' \sin\alpha)^2 + 2h\,(x' \cos\alpha - y' \sin\alpha)\,(x' \sin\alpha + y' \cos\alpha) + b\,(x' \sin\alpha + y' \cos\alpha)^2$

$= (..........)x'^2 + (..........)\,y'^2 + (-a \sin 2\alpha + 2h \cos 2\alpha + b \sin 2\alpha)\,x'y'$; and if the transformed equation does not contain $x'y'$ – term then we must have :

$(b - a) \sin 2\alpha + 2h \cos 2\alpha = 0$

$\Rightarrow \tan 2\alpha = \dfrac{2h}{a-b} \Rightarrow \alpha = n\dfrac{\pi}{2} + \dfrac{1}{2}\tan^{-1}\dfrac{2h}{a-b}$.

EXERCISES 11 (b)

1. Fill in the blanks in each of the following, using the answers given against each of them :

(a) The slope and x - intercept of the line $3x - y + k = 0$ are equal if $k =$ —— $(0, -1, 3, -9)$

(b) The lines $2x - 3y + 1 = 0$ and $3x + ky - 1 = 0$ are perpendicular to each other if $k =$ —— $(2, 3, -2, -3)$

(c) The lines $3x + ky - 4 = 0$ and $k - 4y - 3x = 0$ are coincident if $k =$ —— $(1, -4, 4, -1)$

(d) The distance between the lines $3x - 1 = 0$ and $x + 3 = 0$ is —— units. $\left(4, 2, \dfrac{8}{3}, \dfrac{10}{3}\right)$

(e) The angle between the lines $x = 2$ and $x - \sqrt{3}\,y + 1 = 0$ is —— $(30°, 60°, 120°, 150°)$

2. State with reasons which of the following are *true* or *false* :

(a) The equation $x = k$ represents a line parallel to x - axis for all values of k.

(b) The line, $y + x + 1 = 0$ makes an angle 45° with y - axis.

(c) The lines represented by $2x - 3y + 1 = 0$ and $3x + 2y - k = 0$ are perpendicular to each other for positive values of k only.

(d) The lines represented by $px + 2y - 1 = 0$ and $3x + py + 1 = 0$ are not coincident for any value of p.

(e) The equation of the line whose x – and y – intercepts are 1 and – 1 respectively is $x - y + 1 = 0$.

(f) The point $(-1, 2)$ lies on the line $2x + 3y - 4 = 0$.

(g) The equation of line through $(1, 1)$ and $(-2, -2)$ is $y = -2x$.

(h) The line through $(1, 2)$ perpendicular to $y = x$ is $y + x - 2 = 0$.

(i) The lines $\frac{x}{a}+\frac{y}{b}=1$ and $\frac{y}{a}-\frac{x}{b}=1$ are intersecting but not perpendicular to each other.

(j) The points (1, 2) and (3, – 2) are on the opposite sides of the line $2x + y = 1$.

3. A point P (x, y) is such that its distance from the fixed point $(\alpha, 0)$ is equal to its distance from y – axis. Prove that the equation of the locus is $y^2 = \alpha(2x - \alpha)$.

4. Find the locus of the point P(x, y) such that the area of the triangle PAB is 5, where A is the point (1, –1) and B is the point (5, 2).

5. A point is such that its distance from the point (3, 0) is twice its distance from the point (–3, 0). Find the equation of the locus.

6. Obtain the equation of straight lines :

(a) passing through (1, – 1) and having inclination 150°**.**

(b) passing through (–1, 2) and making intercept 2 on the y – axis.

(c) passing through the points (2, 3) and (–4, 1).

(d) passing through (–2, 3) and sum of whose intercepts in 2.

(e) whose perpendicular distance from origin is 2 such that the perpendicular from origin has inclination 150°.

(f) bisecting the line segment joining (3, –4) and (1, 2) at right angles.

(g) bisecting the line segment joining $(a, 0)$ and $(0, b)$ at right angles.

(h) biscting the line segments joining (a, b), (a^1, b^1) and $(-a, b)$, $(a^1, -b^1)$,

(i) passing through origin and the points of trisection of the portion of the line $3x + y - 12 = 0$ intercepted between the co-ordinate axes.

(j) passing through (–4, 2) and parallel to the line $4x - 3y = 10$.

(k) passing through the point $(a\cos^3\theta, a\sin^3\theta)$ and perpendicular to the straight line $x\sec\theta + y\operatorname{cosec}\theta = a$.

(l) which passes through the point (3, – 4) and is such that its portion between the axes is divided at this point internally in the ratio 2 : 3.

(m) which passes through the point (α, β) and is such that the given point bisects its portion between the co-ordinate axes.

7. (a) Find the equation of the lines which is parallell to the line $3x + 4y + 7 = 0$ and is at *a* distance 2 from it.

(b) Find the equations of the diagonals of the parallelogram formed by the lines $ax + by = 0$, $ax + by + c = 0$, $lx + my = 0$ and $lx + my + n = 0$. What is the condition that this will be a rhombus ?

(c) Find the equation of the line passing through the intersection of $2x - y - 1 = 0$ and $3x - 4y + 6 = 0$ and parallel to the line $x + y - 2 = 0$.

(d) Find the equation of the line passing through the point of intersection of lines $x + 3y + 2 = 0$ and $x - 2y - 4 = 0$ and perpendicular to the line $2y + 5x - 9 = 0$.

(e) Find the equation of the line passing through intersection of the lines $x + 3y - 1 = 0$ and $3x - y + 1 = 0$ and the centroid of the triangle whose vertices are the points (3, –1), (1, 3) and (2, 4).

8. If $lx + my + 3 = 0$ and $3x - 2y - 1 = 0$ represent the same line, find the values of l and m.

9. Find the equation of sides of the a triangle whose vertices are at (1, 2),(2, 3) and (–3, –5).

10. Show that the origin is within the triangle whose sides are given by the equations, $3x - 2y = 1$, $5x + 3y + 11 = 0$ and $x - 7y + 25 = 0$.

11. (a) Find the equations of straight lines passing through the point (3, – 2) and making angle 45° with the line $6x + 5y = 1$.

(b) Two straight lines are drawn through the point (3, 4) inclined at an angle 45° to the line $x - y - 2 = 0$. Find their equations and obtain the area included by the above three lines.

(c) Show that the area of the triangle formed by the lines given by the equations $y = m_1x + c_1$, $y = m_2x + c_2$ and $x = 0$ is $\frac{1}{2}\ \frac{(c_1 - c_2)^2}{|m_2 - m_1|}$.

12. Find the equation of the lines passing through the origin and perpendicular to the lines $3x + 2y - 5 = 0$ and $4x + 3y = 7$. Obtain the co-ordinates of the points where these perpendiculars meet the given lines. Prove that the equation of line passing through these two points is $23x + 11y - 35 = 0$.

13. (a) Find the length of perpendicular drawn from the point (–3, –4) to the straight line whose equation is $12x - 5y + 65 = 0$.

(b) Find the perpendicular distances of the point (2,1) from the parallel lines $3x - 4y + 4 = 0$ and $4y - 3x + 5 = 0$. Hence find the distance between them.

(c) Find the distance of the point (3, 2) from the line $x + 3y - 1 = 0$, measured parallel to the line $3x - 4y + 1 = 0$.

(d) Find the distance of the point (–1, –2) from the line $x + 3y - 7 = 0$, measured parallel to the line $3x + 2y - 5 = 0$.

(e) Find the distance of the line passing through the points ($a \cos \alpha$, $a \sin \alpha$) and ($a \cos \beta$, $a \sin \beta$) from the origin.

14. Find the length of perpendiculars drawn from the origin on the sides of the triangle whose vertices are A (2, 1), B (3, 2) and C (–1, –1).

15. Show that the product of perpendiculars from the points $\left(\pm\sqrt{a^2 - b^2}, 0\right)$ upon the straight line $\frac{x}{a}\cos\theta + \frac{x}{b}\sin\theta = 1$, is b^2.

16. Show that the lengths of perpendiculars drawn from any point of the straight line $2x + 11y - 5 = 0$ on the lines $24x + 7y - 20 = 0$ and $4x - 3y - 2 = 0$ are equal to each other.

17. If p and p' are the lengths of perpendiculars drawn from the origin upon the lines $x \sec \alpha + y \operatorname{cosec} \alpha - a = 0$ and $x \cos\alpha - y \sin \alpha - a \cos 2\alpha = 0$, prove that $4p^2 + p'^2 = a^2$.

18. Obtain the equation of the lines passing through the foot of the perpendicular from (h, k) on the line $Ax + By + C = 0$ and bisecting the angle between the perpendicular and the given line.

19. Find the direction in which a stright line must be drawn through the point (1,2) such that its point of intersection with the line $x + y - 4 = 0$, is at distance $\frac{1}{3}\sqrt{6}$ from this point.

20. A tringle has its three sides formed by the lines $x + y = 3$, $x + 3y = 3$ and $3x + 2y = 6$. Whithout solving for the vertices, find the equation of its altitudes and also calculate the angles of the traingle.

21. A tringle has its vertices at P(1, –1), Q(3, 4) and R (2, 5). Find the equation of altitudes through P and Q and obtain the co-ordinates of their point of intersection. (This point is called *orthocentre* of the traingle).

22. (a) Show that the line passing through (6,0) and (–2, –4) is concurrent with the lines $2x - 3y - 11 = 0$ and $3x - 4y = 16$.

    (b) Show that the lines $lx + my + n = 0$, $mx + ny + l = 0$ and $nx + ly + m = 0$ are concurrent, if $l + m + n = 0$.

23. Obtain the equation of the bisector of the acute angle between the pair of lines :

    (a) $x + 2y = 1$, $2x + y + 3 = 0$;

    (b) $3x - 4y = 5$, $12y - 5x = 2$.

24. (a) Find the co-ordinates of the centre of the inscribed circle of the triangle formed by the line $x \cos \alpha + y \sin \alpha = p$ with co-ordinate axes.

    (b) Find the co-ordinates of the circumcentre and incentre of the traingle formed by the line $3x - y = 5$, $x + 2y = 4$ and $5x + 3y + 1 = 0$.

25. The vertices B, C of a triangle ABC lie on the lines $3y = 4x$ and $y = 0$ respectively and the side $\overline{BC}$ passes through the point (2/3, 2/3). If ABOC is a rhombus, where O is the origin, find the equation of $\overline{BC}$ and also the co-ordinates of A.

26. Find the equation of the lines represented by the following equations :

    (a) $4x^2 - y^2 = 0$ (b) $2x^2 - 5xy - 3y^2 = 0$

    (c) $x^2 + 2xy \sec\theta + y^2 = 0$ (d) $3x^2 + 4xy = 0$.

27. Form the equations which represent the following pair of lines :

    (a) $y = mx$; $y = nx$ (b) $y - 3x = 0$; $y + 3x = 0$

    (c) $2x - 3y + 1 = 0$; $2x + 3y + 1$ - 0 (d) $x = y$, $x + 2y + 5 = 0$.

28. Which of the following equations represent a pair of lines ?

    (a) $2x^2 - 6y^2 + 3x + y + 1 = 0$ (b) $10x^2 - xy - 6y^2 - x + 5y - 1 = 0$

    (c) $xy + x + y + 1 = 0$.

29. For what value of λ do the following equations reprsent a pair of straight lines ?

(a) $\lambda x^2 + 5xy - 2y^2 - 8x + 5y - \lambda = 0$ (b) $x^2 - 4xy - y^2 + 6x + 8y + \lambda = 0$.

30. (a) Obtain the value of λ for which the pair of straight lines represented by $3x^2 - 8xy + \lambda y^2 = 0$ are perpendicular to each other.

(b) Prove that a pair of lines through origin perpendicular to the pair of lines represented by $px^2 + 2qxy + ry^2 = 0$ is given by $rx^2 - 2qxy + py^2 = 0$.

(c) Obtain the condition that a line of the pair of lines $ax^2 + 2hxy + by^2 = 0$ (i) coincides with, (ii) is perpendicular to, a line of the pair of lines $px^2 + 2qxy + ry^2 = 0$.

31. Find the acute angle between the pair of lines given by :

(a) $x^2 + 2xy - 4y^2 = 0$ (b) $2x^2 + xy - 3y^2 + 3x + 2y + 1 = 0$

(c) $x^2 + xy - 6y^2 - x - 8y - 2 = 0$.

32. Write down the equation of the pair of bisectors of the followingn pair of lines :

(a) $x^2 - y^2 = 0$ (b) $4x^2 - xy - 3y^2 = 0$

(c) $x^2 \cos\theta + 2xy - y^2 \sin\theta = 0$ (d) $x^2 - 2xy \tan\theta - y^2 = 0$.

[**Hint :** Use the formula that the equation of bisectors of angles between the pair of lines $ax^2 + 2hxy + by^2 = 0$ is $\dfrac{x^2 - y^2}{a-b} = \dfrac{xy}{h}$]

33. If the pair of lines represented by $x^2 - 2pxy - y^2 = 0$ and $x^2 - 2qxy - y^2 = 0$ be such that each pair bisects the angle between the other pair, then prove that $pq = -1$.

34. Transform the equation

$x^2 + y^2 - 2x - 4y + 1 = 0$ by shifting the origin to (1, 2) and keeping the axes parallel.

35. Transform the quation

$2x^2 + 3y^2 + 4xy - 12x - 14y + 20 = 0$, when referred to parallel axes through (2, 1).

36. Find measure of rotation so that the equation

$x^2 - xy + y^2 = 5$ when transformed does not contain xy - term.

37. What does the equation $x + 2y - 10 = 0$ become when the origin is changed to (4, 3) ?

●

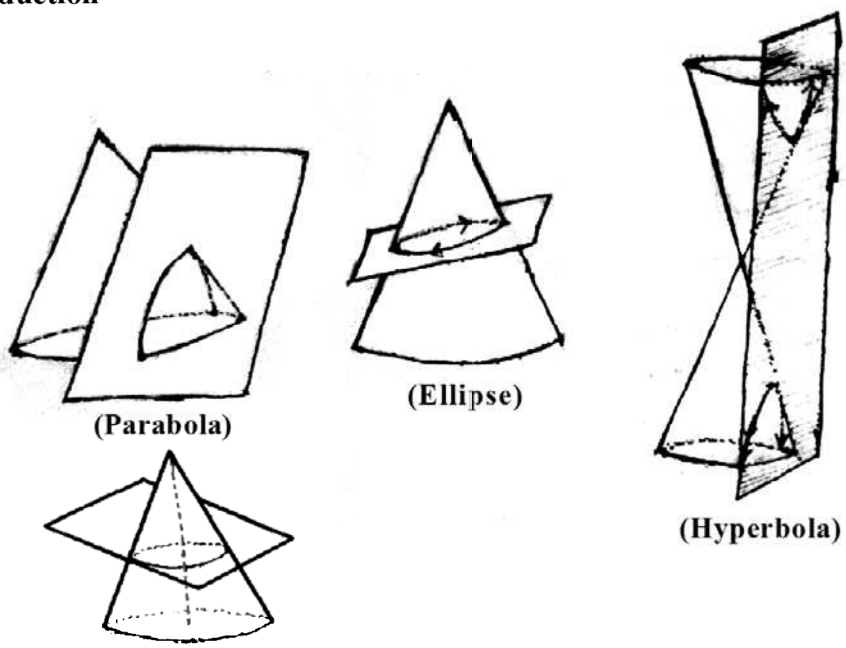
CHAPTER 12

Conic Sections

ମିଶୁ ମୋର ଦେହ ଏ ଦେଶ ମାଟିରେ, ଦେଶବାସୀ ଚାଲି ଯାଆନ୍ତୁ ପିଠିରେ
ଦେଶର ସ୍ୱରାଜ୍ୟ ପଥେ ଯେତେ ଗାଡ଼, ପୂରୁ ତହିଁ ପଡ଼ି ମୋର ମାଂସ ହାଡ଼ ।

- ଉତ୍କଳମଣି ପଣ୍ଡିତ ଗୋପବନ୍ଧୁ ଦାସ

12.0 Introduction



(Parabola) (Ellipse) (Hyperbola) (Circle)

Sections of a Cone :

When a plane cuts a cone, we get various types of plane sections depending upon the positions of the plane. As shown in the figures above, we get curves known as parabola, elipse, hyperbola and also circle. There are cases when we even get a single point (when the plane passes only through the vertex of the cone); a pair of lines when the plane passes through the axis of the cone which hapen to be two generators of the cone intersecting at the vertex.

The parabola, ellipse and hyperbola are called the main conic sections. They have well defined directrices, which will be defined later in the chapter. But directrix of a circle is not defined. For this reason, though a circle comes as a section of a cone when the axis of the cone is normal to the plane cutting it. A circle is, however, called a conic section of the fourth type.

Since cone is beyond the scope of this book we develop the geometry of circles and all other conic sections independent of the properties of a cone.

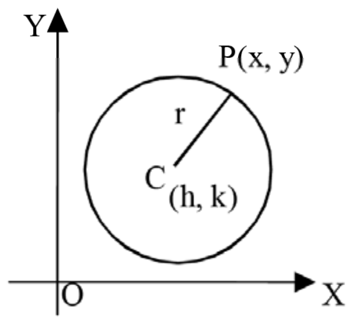
12.01 Circle :

Definition : A circle is the set (locus) of all points in a plane which are equidistant from a given point in that plane.

The given point is called the 'centre' of the circle. A linesegment joining the centre to a point on the circle, as well as its length, is known as the 'radius' of the circle.

Note that the term 'radius' means both a segment and its length. Its meaning will be clear from the context.

Equation of Circle with given centre and radius :



Let C (h, k) be the centre of the given circle and r be its radius. Then for any point (x, y) on the circle (Fig. 23) we have, $CP^2 = r^2$

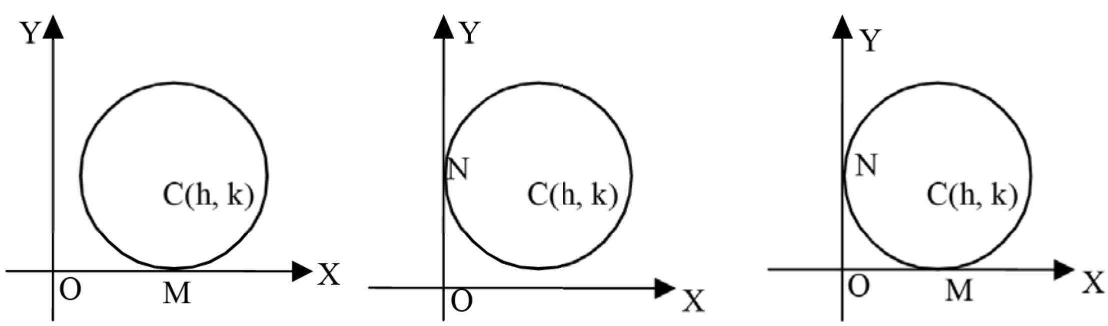
or $\boxed{(x-h)^2+(y-k)^2=r^2}$ (1)

which is the required equation. In particular, if we put $h = 0$ and $k = 0$ then equatin (1) reduces to,

$\boxed{x^2+y^2=r^2}$ (2)

which is the equation of a circle with centre at origin and radius r.

Note I : If the centre of the circle be C (h, k) and radius be r and if it touches the x - axis at M, then



$k = CM = r$ and the equation of the circle is $(x-h)^2 + (y-k)^2 = k^2$

or, $x^2 + y^2 - 2hx - 2ky + h^2 = 0$.

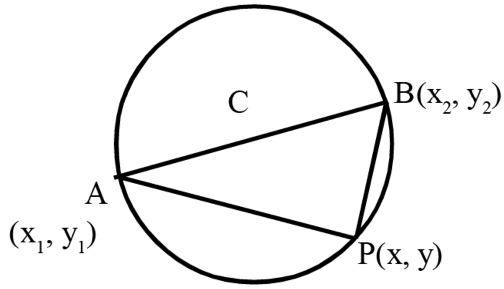
(II) Similarly, if the above circle touches y - axis at N. (fig. 25), then $h = CN = r$ and the equation of circle becomes.

$(x-h)^2 + (y-k)^2 = h^2$, or, $x^2 + y^2 - 2hx - 2ky + k^2 = 0$.

(III) If the above circle touches both the axes at M and N (Fig. 26), then CN = $h = k$ = CM = r, and hence the equation of circle becomes,

$$(x-h)^2 + (y-h)^2 = h^2, \text{ or, } x^2 + y^2 - 2hx - 2hy + h^2 = 0.$$

Circle on a given diameter :



Let A (x_1, y_1) and B (x_2, y_2) be the end points of a given diameter of a circle. Then for any point P (x, y) on the circle (fig. 27), we have m$\angle$APB = 90° i.e. $\overline{PA}$ and $\overline{PB}$ are perpendicular to each other and hence, (slope $\overleftrightarrow{PA}$). (slope $\overleftrightarrow{PB}$) = – 1

or, $\left(\dfrac{y-y_1}{x-x_1}\right) \cdot \left(\dfrac{y-y_2}{x-x_2}\right) = -1$

or, $\boxed{(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0}$ is the equation of the circle.

General Equation of a Circle : Conditions for the General Equation of second degree to represent a circle.

As discussed earlier the equation of the circle with centre at (h, k) and radius r, is given by, $(x - h)^2 + (y - k)^2 = r^2$

or, $x^2 + y^2 - 2hx - 2ky + (h^2 + k^2 - r^2) = 0$

which is of the form $x^2 + y^2 + 2gx + 2fy + c = 0$.

Conversely, if we consider an equation of the form $x^2 + y^2 + 2gx + 2fy + c = 0$ then it can be written as, $(x^2 + 2gx + g^2) + (y^2 + 2fy + f^2) = g^2 + f^2 - c$

or, $(x + g)^2 + (y + f)^2 = (g^2 + f^2 - c)$

which represents a circle with centre $(-g, -f)$ *and radius* $r = \sqrt{g^2+f^2-c}$, provided $g^2 + f^2 > c$. If $g^2 + f^2 < c.$, $r = \sqrt{g^2+f^2-c}$ becomes imaginary and is not acceptable as a distance since 'distance' is a function from R × R onto the set of nonnegative real numbers.

If $g^2 + f^2 - c = 0$, the circle reduces to a point.

If $a = b \neq 0$, $h = 0$ the equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ can be written as $x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$; where $g_1 = \dfrac{g}{a}, f_1 = \dfrac{f}{a}, c_1 = \dfrac{c}{a}$. It represents a circle if $g_1^2 + f_1^2 > c_1$ i-e. $g^2 + f^2 > ac$.

Thus we see that the general equation of second degree in x and y i.e. $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a circle subject to

$$a = b \neq 0,\ h = 0,\ g^2 + f^2 > ac.$$

Position of a point with respect to a Circle.

Let S (α, β) be a given point and $x^2 + y^2 + 2gx + 2fy + c = 0$ be a given circle. Then the co-ordinates of the centre C of the circle are $(-g, -f)$. The point S will lie outside the circle if CS $> r$; on the circle if CS $= r$ and inside the circle is CS $< r$. But

$r = \sqrt{g^2 + f^2 - c}$ and $CS^2 = (\alpha + g)^2 + (\beta + f)^2$.

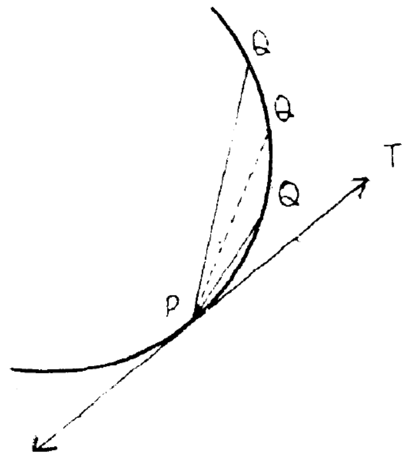
Hence the point S will be outside, on or inside the circle, according as, $CS^2 > = < r^2$

i.e.$(\alpha + g)^2 + (\beta + f)^2 > = < g^2 + f^2 - c$

i. e. $\alpha^2 + \beta^2 + 2g\alpha + 2f\beta + c > = < 0$.

12.02 Equation of Tangent and Normal

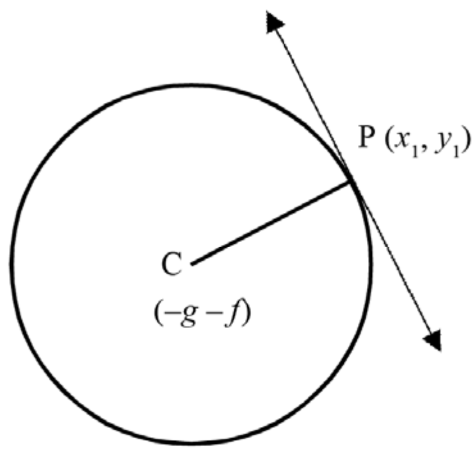
Let P be a given on a curve and Q be any other point on it. Then the line segment $\overline{PQ}$ is called a chord of the curve. As Q tends to P. i.e. we take Q closer and closer to P on the curve, then in the limiting poistion $\overleftrightarrow{PQ}$ assumes the position $\overleftrightarrow{PT}$ and is called the tangent line or simply *tangent* to the curve at the point P. A line through P, perpendicular to the tangent, is called the *normal* to the curve at the point P. [Since the concept of tangent involves limiting case it is best tackled in differential calculus]



Consider the circle given by the general equation.

$x^2 + y^2 + 2gx + 2fy + c = 0$

and let P (x_1, y_1) be a point on it. If C is the centre of the circle then the co-ordinates of C are $(-g - f)$ and hence the slope of $\overline{CP}$ is $\frac{y_1 + f}{x_1 + g}$. Since the tangent at any point to a circle is perpendicular to the line joining the point of contact to its centre, the slope of the tangent at P is given by $-\left(\frac{x_1 + g}{y_1 + f}\right)$; hence the equation of tangent through (x_1, y_1) is given by,



$$(y - y_1) = -\frac{x_1 + g}{y_1 + f}(x - x_1)$$

or, $(y - y_1)(y_1 + f) + (x_1 + g)(x - x_1) = 0$

or, $yy_1 + fy + xx_1 + gx - y_1^2 - y_1 f - x_1^2 - gx_1 = 0$

or, $xx_1 + yy_1 + gx + fy = x_1^2 + y_1^2 + gx_1 + fy_1$

$= (x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c) - (gx_1 + fy_1 + c)$.

Since, P (x_1, y_1) is a point on the circle, we have, $x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0$. Hence the equation of tangent is $xx_1 + yy_1 + gx + fy = -(gx_1 + fy_1 + c)$

or, $\boxed{xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0}$

In particular, if we have the circle $x^2 + y^2 = r^2$, then substituting $g = f = 0$ and $c = -r^2$ in the above equation, we get the equation of tangent at (x_1, y_1) as $xx_1 + yy_1 - r^2 = 0$

or, $\boxed{xx_1 + yy_1 = r^2}$

To get the equation of normal, we see that the slope of tangent to the circle

$x^2 + y^2 + 2gx + 2fy + c = 0$

at the point (x_1, y_1) is $-\dfrac{x_1 + g}{y_1 + f}$, and hence the slope of normal to the circle at (x_1, y_1) is $\dfrac{y_1 + f}{x_1 + g}$. So the equation of normal at (x_1, y_1) is, $y - y_1 = \dfrac{y_1 + f}{x_1 + g}(x - x_1)$

Or, $(y - y_1)(x_1 + g) = (y_1 + f)(x - x_1)$

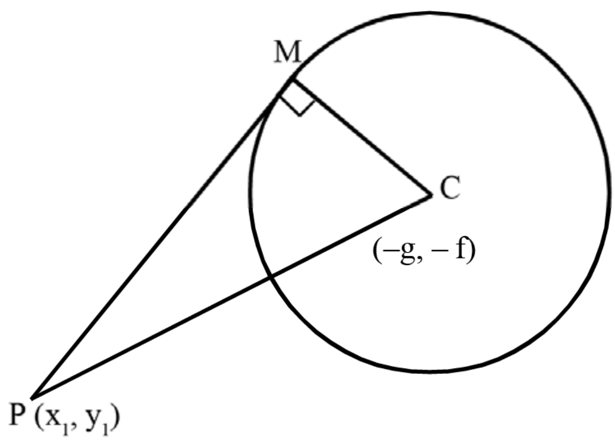
Or, $\boxed{(y_1 + f)x - (x_1 + g)y - x_1 f + y_1 g = 0}$

If instead, we have the circle $x^2 + y^2 = r^2$, then subsituting, $g = f = 0$ in the above equation of normal, we get the equation of normal at (x_1, y_1) as,

$\boxed{x_1 y = xy_1}$

12.03 Length of tangent drawn from an external point (x_1, y_1) to a given circle

Let P (x_1, y_1) be a given point and $x^2 + y^2 + 2gx + 2fy + c = 0$ be a given circle. Then the centre C of the circle has co-ordinates $(-g, -f)$ and radius $r = \sqrt{g^2 + f^2 - c}$.



If we draw a tangent to the circle from the point P and if M be the point of contact then PM = length of tangent drawn from P. Further, triangle PMC being a right angled triangle, we have, $PC^2 = PM^2 + MC^2$

or, $PM^2 = PC^2 - MC^2$

$= (x_1 + g)^2 + (y_1 + f)^2 - (g^2 + f^2 - c)$

$= x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$

or PM $= \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}$.

12.04 Points of intersection of a line and a circle and condition of tangency

Consider the circle given by the equation $x^2 + y^2 = a^2$, (i)

and the straight line, $y = mx + c$. (ii)

At the point of intersection of the circle and the straight line, both the equations (i) and (ii) will be satisfied. Hence the points of intersection are found out by solving (i) and (ii). Now eliminating y from (i) and (ii) we obtain, $x^2 + (mx + c)^2 = a^2$

or, $(1 + m^2)x^2 + 2mcx + (c^2 - a^2) = 0$ (iii) which is a quadratic in x and therefore has two roots, real and different, real and equal, or complex. Hence the straight line (ii) meets the circle (i) in two distinct or coincident points if the roots of equation (iii) are real; otherwise it does not meet the circle. The coordinates of the points at which it meets the circle are (x_1, y_1) and (x_2, y_2), where x_1, x_2 are root of the equation (iii) and y_1, y_2 are the corresponding values of y.

Now, the line (ii) will be tangent to the circle (i), if it meets the circle at two coincident points, i.e. if the two roots of (iii) are real and equal, i.e. if the discriminant of (iii) is zero, i.e. $4m^2c^2 - 4(1 + m^2)(c^2 - a^2) = 0$,

or, $m^2c^2 - (c^2 - a^2 + m^2c^2 - m^2a^2) = 0$

or, $\boxed{c^2 = a^2(1+m^2)}$

N.B. 1. It follows that

(i) $c^2 < a^2(1 + m^2) \Leftrightarrow$ the line $y = mx + c$ is a secant of the circle $x^2 + y^2 = a^2$ i.e. intersects the circle at two distinct points.

(ii) $c^2 > a^2(1 + m^2) \Leftrightarrow$ the line does not intersects the circle.

2. The equation $y = mx \pm a\sqrt{1+m^2}$ always represents a tangent to the circle $x^2 + y^2 = a^2$

If we consider the general equation of the circle, $x^2 + y^2 + 2gx + 2fy + c = 0$ (iv) and the line is given by $lx + my + n = 0$........................(v)

then the points of intersection of the line and the circle are obtained by solving (iv) and (v). Now eliminating y from (iv) and (v) we get,

$$x^2 + \left(\frac{-lx-n}{m}\right)^2 + 2gx + 2f\left(\frac{-lx-n}{m}\right) + c = 0$$

or, $m^2x^2 + (lx + n)^2 + 2gm^2x - 2mf(lx + n) + m^2c = 0$ (supposing $m \neq 0$)

or, $(m^2 + l^2)x^2 + (2ln + 2gm^2 - 2flm)x + (n^2 + m^2c - 2fnm) = 0$ (vi)

which is a quadratic in x giving two values of x, say α_1 and α_2 and correspondingly we obtain two values of y, say β_1 and β_2. Thus the points of intersection may be distinct or coincident according as the discriminat of the quadratic equation (vi) is positive or zero. If it is less than zero, then the line does not intersect the circle. As before the condition of tangency can be obtained by equating the discriminant of (vi) to zero.

The condition of tangency of line (v) to the circle (iv) can also be obtained as follows :

The line L given by $lx + my + n = 0$ will be a tangent to the circle

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

if the distance d of the line from the centre of the circle is equal to its radius r. Now the centre of the circle is C(–g, –f) and radius = $\sqrt{g^2 + f^2 - c}$. Hence line L will be tangent to the circle, if $\dfrac{|-lg-mf+n|}{\sqrt{l^2+m^2}} = \sqrt{g^2+f^2-c}$

or, $\dfrac{(lg+mf-n)^2}{l^2+m^2} = (g^2+f^2-c)$

or, $\boxed{(lg+mf-n)^2 = (l^2+m^2)(g^2+f^2-c)}$

N.B The condition for the line to be a secant or not intersecting the circle can be seen to be

$(lg+mf-n)^2 < (l^2+m^2)\,(g^2+f^2-c)$ and $(lg+mf-n)^2 > (l^2+m^2)\,(g^2+f^2-c)$ respectively.

12.05 System of Circles : Condition that two circles may touch or intersect :

Consider two circles S_1 and S_2 given by,

$$S_1 \equiv x^2+y^2+2g_1x+2f_1y+c_1=0 \qquad (1)$$

$$S_2 \equiv x^2+y^2+2g_2x+2f_2y+c_2=0 \qquad (2)$$

having centres at $C_1\,(-g_1 -f_1)$ and $C_2\,(-g_2, -f_2)$ with radii $r_1 = \sqrt{g_1^2+f_1^2-c_1}$ and

$r_2 = \sqrt{g_2^2+f_2^2-c_2}$ respectively (Fig.31). If neither of S_1 or S_2 lies completely in the interior of the other, nor touches it internally, then these two circles intersect, touch externally or, neither intersect nor touch each other if and only if

$C_1C_2 < r_2 + r_2$, $C_1C_2 = r_1 + r_2$, or $C_1C_2 > r_1 + r_2$, respectively; i.e.

$$\sqrt{(g_2-g_1)^2+(f_2-f_1)^2} \;<, = \text{ or } > r_1 + r_2.$$

If the circles touch each other internally, then $C_1C_2 = |\,r_1 - r_2|$. If either of the circles is in the interior of the other, then they cannot intersect.

Angle between two circles :

Angle between two intersecting circles is the angle between their tangents (or equivalently their normals) at a point of intersection. Since tangent at a point on a circle is perpendicular to the radius joining the point to the centre, we may take angle between the intersecting circles S_1 and S_2 in the accompanying figure in the next page given by equation (1) and (2) as $\angle C_1PC_2$, which is the angle between their normals. If Q is another point of intersection it is easy to see that $m\angle C_1PC_2 = m\angle C_1QC_2$.

Taking $m\angle C_1PC_2 = \theta$,

$$\cos\theta = \frac{C_1P^2 + C_2P^2 - C_1C_2^2}{2C_1P.C_2P}$$

$$= \frac{(g_1^2+f_1^2-c_1)+(g_2^2+f_2^2-c_2)-\{(g_1-g_2)^2+(f_1-f_2)^2\}}{2\sqrt{g_1^2+f_1^2-c_1}\sqrt{g_2^2+f_2^2-c_2}}$$

$$= \frac{2(g_1g_2 + f_1f_2) - c_2 - c_2}{2\sqrt{g_1^2 + f_1^2 - c_1}\sqrt{g_2^2 + f_2^2 - c_2}}.$$

$$\text{So } \theta = \cos^{-1} \frac{2(g_1g_2 + f_1f_2) - c_1 - c_2}{2\sqrt{g_1^2 + f_1^2 - c_1}\sqrt{g_2^2 + f_2^2 - c_2}}$$

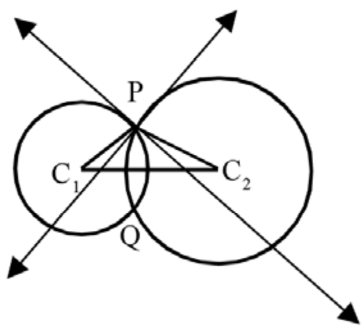
Cor. The circles cut each other **orthogonally** if and only if $\theta = \frac{\pi}{2}$,

i.e, iff $\cos\theta = \cos\frac{\pi}{2} = 0$

$$\text{i.e, iff } \frac{2(g_1g_2 + f_1f_2) - c_2 - c_2}{2\sqrt{g_1^2 + f_1^2 - c_1}\sqrt{g_2^2 + f_2^2 - c_2}} = 0$$

i.e. $\boxed{2(g_1g_2 + f_1f_2) - c_1 - c_2 = 0}$

The condition for orthogenality can be obtained in an easier way by application of pythagoras theorem.



Circles S_1 and S_2 are orthogonal, $\Leftrightarrow m\angle C_1PC_2 = 90°$.

$\Leftrightarrow C_1P^2 + C_2P^2 = C_1C_2^2$

$\Leftrightarrow (g_2 - g_1)^2 + (f_2 - f_1)^2 = g_1^2 + f_1^2 - c_1 + g_2^2 + f_2^2 - c_2$ or $2(g_1g_2 + f_1f_2) - c_1 - c_2 = 0.$

Circles through points of intersection of two given circles :

Consider the equation $S_1 + kS_2 = 0$ (circles S_1 and S_2 are supposed to intersect), for any real number k. This equation is equivalent to

$(1 + k)x^2 + (1 + k)\, y^2 + 2\,(g_1 + kg_2)\, x + 2(f_1 + kf_2)\, y + (c_1 + kc_2) = 0$ (3)

which is a second degree equation in x and y with xy - term absent and hence represents a circle if and only if $k + 1 \neq 0$ and

$(g_1 + kg_2)^2 + (f_1 + kf_2)^2 > (1 + k)\,(c_1 + kc_2)$.

Using the condition for intersection of two circles it can be proved that the second condition in the above is satisfied for all real values of $k \neq -1$. Therefore equation (3) represents a circle for every real number $k \neq -1$.

Again if (α, β) be a point of intersection of S_1 and S_2, then (α, β) satisfies both the equations (1) and (2) and hence,

$\alpha^2 + \beta^2 + 2g_1\alpha + 2f_1\beta + c_1 = 0$ and $\alpha^2 + \beta^2 + 2g_2\alpha + 2f_2\beta + c_2 = 0$.

Hence, for any real number k,

$(\alpha^2 + \beta^2 + 2g_1\alpha + 2f_1\beta + c_1) + k\,(\alpha^2 + \beta^2 + 2g_2\alpha + 2f_2\beta + c_2) = 0$

or $(1 + k)\,\alpha^2 + (1 + k)\beta^2 + 2\,(g_1 + kg_2)\alpha + 2\,(f_1 + kf_2)\beta + (c_1 + kc_2) = 0$

which shows that (α, β) satisfies equation (3).

Hence, $S_1 + kS_2 = 0$ (4)

always represents a circle passing through intersection of S_1 and S_2 for every real value of $k \neq -1$ and therefore represents the system of circles through intersection of S_1 and S_2.

If $k = -1$, then the above equation (4) becomes, $S_1 - S_2 = 0$

or, $2(g_1 - g_2)x + 2(f_1 - f_2)y + (c_1 - c_2) = 0$ (5)

which is linear in x and y and therefore represents a straight line, provided S_1 and S_2 are not concentric. This line is known as the **Radical Axis** of S_1 and S_2.

From equation (5) it is seen that the slope of the radical Axis is $-\left(\frac{g_1 - g_2}{f_1 - f_2}\right)$. (if $f_1 \neq f_2$)

Also, the centres of the circles S_1 and S_2 being at $C_1(-g_1, -f_1)$ and $C_2(-g_2, -f_2)$, the slope of the line joining the centres, $\overleftrightarrow{C_1C_2}$ is $\left(\frac{f_1 - f_2}{g_1 - g_2}\right)$ which shows that the *radical axis* of two circles is *perpendicular* to the *line joining their centres.* This conclusion also holds if $f_1 = f_2$ as in that case radical axis is vertical and $\overleftrightarrow{C_1C_2}$ is horizontal.

Properties of Radical Axis

The radical axis of two circles S_1 and S_2 whose equations are given by

$S_1 \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ and

$S_2 \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$

is $S_1 - S_2 \equiv 2(g_1 - g_2)x + 2(f_1 - f_2)y + c_1 - c_2 = 0$. (The circles may or may not be intersecting. In order that $S_1 - S_2 = 0$ gives equation of radical axis, the expressions for S_1 and S_2 must be such that the coefficients of x^2 and y^2 must be same in both S_1 and S_2)

Few interesting properties of radical axis are :

1. The radical axis of two circles is perpendicular to the line joining their centres.

2. In case of intersecting circles, the radical axis passes through their points of intersection.

3. The lengths of the tangents drawn to the circles from any point on their radical axis, lying in the exterior of both the circles, are equal.

4. The radical axes of three circles (with non collenear centres) taken in pairs, are concurrent.

Properties (1) and (2) are obvious from previous discussions. We only prove (3) and (4).

Proof of (3) :

Radical axis of circles S_1 and S_2 is given by

$S_1 - S_2 \equiv 2(g_1 - g_2)x + 2(f_1 - f_2)y + c_1 - c_2 = 0.$

Let P (h, k) be any point on the radical axis, Such that P(h, k) is in the exterior of both S_1 and S_2.

Let *a* tangent from *P* to S_1 touch it at T_1 and *a* tangent from P to S_2 touch it at T_2. Then, by the formula for length of tangent from an external point to a circle, we have

$PT_1^2 = h^2 + k^2 + 2g_1 h + 2f_1 k + c_1$, $PT_2^2 = h^2 + k^2 + 2g_2 h + 2f_2 k + c_2$.

$\therefore PT_1^2 - PT_2^2 = 2(g_1 - g_2)h + 2(f_1 - f_2)k + c_1 - c_2 = 0$, since P (h, k) is a point on the radical axis.

So $PT_1 = PT_2$ □

Proof of(4) : Let the circles be S_1, S_2 and S_3 with centres C_1, C_2 and C_3 respectively, given by equations

$S_1 \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$, $S_2 \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$ and
$S_3 \equiv x^2 + y^2 + 2g_3x + 2f_3y + c_3 = 0$.

Since C_1, C_2 and C_3 are noncollinear, no two of the lines $\overleftrightarrow{C_1C_2}$, $\overleftrightarrow{C_2C_3}$ and $\overleftrightarrow{C_3C_1}$ are parallel.

The radical axes of the circles, taken in pairs, are given by

$S_1 - S_2 \equiv 2(g_1 - g_2)x + 2(f_1 - f_2)y + c_1 - c_2 = 0$,
$S_2 - S_3 \equiv 2(g_2 - g_3)x + 2(f_2 - f_3)y + c_2 - c_3 = 0$ and
$S_3 - S_1 \equiv 2(g_3 - g_1)x + 2(f_3 - f_1)y + c_3 - c_1 = 0$;

no two of which are parallel, since no two of $\overleftrightarrow{C_1C_2}$, $\overleftrightarrow{C_2C_3}$ and $\overleftrightarrow{C_3C_1}$ parallel, (This follows from : Radical axis is perpendicular to the line joining the centres.)

It now follows from

$S_3 - S_1 = S_1 - S_2 + S_2 - S_3$ that the radical axis $S_3 - S_1 = 0$ passes through the intersection of the radical axes $S_1 - S_2 = 0$ and $S_2 - S_3 = 0$. [Remember the discussions on the lines $a_1x + b_1y + c_1 + \lambda(a_2x + b_2y + c_2) = 0$, $\lambda \in R$. Here $\lambda = 1$.]

So the radical axes are concurrent □

This point is known as the **Radical centre** of the three circles S_1, S_2 and S_3

Coaxal Circles :

Definition : A system of circles is said to be coaxal if every pair of circles in the system has the same radical axis.

Equation of a coaxal system of circles :

Since radical axis of a pair of circles is perpendicualr to the line joining their centres, it follows that the centres of all the circles forming a coaxal system must be on a line perpendicular to the radical axis.

Let us take the line containing the centres as the x – axis and the radical axis as the y – axis. Thus any circle of the system has its centre on the x–axis and hence has its equation in the form

$x^2 + y^2 + 2gx + c = 0.$ (A)

Radical axis of the system being the y – axis, its equation is given by $x = 0$.

Let P (o, k) be a point on the radical axis, supposed to be in the exterior of all the circles of the system. So the length of the tangent from P on any circle of the system which is $\sqrt{k^2 + c}$, must be the same for all the circles of the system. [Property (3) of radical axis]

Therefore c is the same for all circles given by (A). Thus a coaxal system of circles is represented by (A), where c is a constant for all circles and g is a parameter. By varying g in R we get different circles of the system, provided $g^2 > c$; which is the condition that equation (A) should represent a circle.

Note : By taking radical axis as the x –axis and centres of the circles on the y –axis, the coaxal system can be given by $x^2 + y^2 + 2fy + c = 0$; $f^2 > c$.

Limiting points of a coaxal system :

In the coaxal system (A), radius of any circle is $\sqrt{g^2 - c}$. Hence, if $c > 0$ and $g^2 = c$, we get two points at ($\pm\sqrt{c}$, 0), situated on both sides of the radical axis. These points are known as the limiting points of the coaxal system.

Limiting points do not exist if $c < 0$ as, in this case, $\sqrt{c}$ is not a real number.

Note : If the coaxal system is given by $x^2 + y^2 + 2fy + c = 0$, $f^2 > c$ the limiting points are (0, $\pm\sqrt{c}$); provided $c > 0$.

Intersecting and non intersecting coaxal systems :

Any circle of the coaxal system $x^2 + y^2 + 2gx + c = 0$ intersects the radical axis $x = 0$ at points given by $y^2 + c = 0$ i.e. $y = \pm\sqrt{-c}$.

So, if $c < 0$, the points of intersection exist and are $(0, \pm\sqrt{-c}\,)$. Such a system is called an **intersecting coaxal system.**

But if $c > 0$, the points of intersection do not exist and we call the system, a **non intersecting system of coaxal circles.** However, for a nonintersecting system, the limiting points exist and are given by $(\pm\sqrt{c}\ , 0)$.

Note : For the coaxal system $x^2 + y^2 + 2fy + c = 0, f^2 > c$ limiting points are $(0, \pm\sqrt{c}\,)$ if $c > 0$ and points of intersection are $(\pm\sqrt{-c}\ , 0)$ if $c < 0$.

2. In an intersecting system of coaxal circles, all the circles interesect one another and the radical axis as well, exactly at two points.

3. Limiting points and points of intersection cannot simultaneously exist if $c \neq 0$. (In $x^2 + y^2 + 2gx + c = 0$ or $x^2 + y^2 + 2fy + c = 0$)

However if $c = 0$, then all the circles touch one another at (0, 0) which is also the only limiting point of the system.

Example : Prove that each of the followings are coaxal systems of circles. Determine their points of intersection or limiting points as the case may be

(i) $x^2 + y^2 + kx - 5 = 0$, (ii) $x^2 + y^2 + ky + 1 = 0$, (iii) $x^2 + y^2 + 2gx + 7 = 0$,
(iv) $x^2 + y^2 - 2fy - 3 = 0$.

Solution :

(i) Radical axis of any pair is given by $(k_1 - k_2)\ x = 0$
i.e. $x = 0$. So the system is coaxal.
$c = -5 < 0$. $\Rightarrow$ Circles are intersecting.
Putting $x = 0$, we get $y^2 - 5 = 0$.
So the points of intersection are $(0, \pm\sqrt{5})$.

(ii) Radical axis is given by $y = 0$. So the system is coaxal.
$c = 1 > 0 \Rightarrow$ circles are nonintersecting.

$$r = \sqrt{\left(\frac{k}{2}\right)^2 - 1} = 0 \Rightarrow k = \pm 2.$$

$\therefore$ Limiting points are $(0, \pm 2)$.

(iii) Radical axis is $x = 0$. So the system is coaxal.
$c = 7 > 0 \Rightarrow$ Circles are non intersecting.

$$r = \sqrt{g^2 - 7} = 0 \Rightarrow g = \pm\sqrt{7}$$

$\therefore$ Limiting points are $(\pm\sqrt{7}, 0)$.

(iv) Radical axis is $y = 0$; hence the system coaxal.
$c = -3 < 0 \Rightarrow$ circles are intersecting.
Putting $y = 0$, we get $x^2 - 3 = 0$.
So points of intersection are $(\pm\sqrt{3}, 0)$.

Solved Examples :

Example 1 :

Find the equation of the circle with centre at $(-1, 2)$ which passes through the point $(3, 1)$.

Solution :

The centre of the circle is $(-1, 2)$. Hence its equation is, $(x + 1)^2 + (y - 2)^2 = r^2$, where r is the radius.

Since the circle passes through the point $(3, 1)$
we have, $(3 + 1)^2 + (1 - 2)^2 = r^2$
or, $r^2 = 17$.
Hence the required equation is $(x + 1)^2 + (y - 2)^2 = 17$.

Example 2 :

Find the equation of the circle which passes through the points $(0,1)$, $(1, 0)$ and $(2, 1)$.
Find its radius and co-ordinates of the centre.

Solution :

Let the equation of the circle be $x^2 + y^2 + 2gx + 2fy + c = 0$.

Since it passes through the point (0, 1), we have, $1 + 2f + c = 0$. (i)

Also, it passes through (1, 0), Hence, $1 + 2g + c = 0$. (ii)

From (i) and (ii) we get $f = g$.

Again (2, 1) is also a point on the circle. So putting $x = 2$ and $y = 1$ in the equation of the circle, we get

$4g + 2f + 5 + c = 0$ or $6f + c + 5 = 0$. (iii)

Now, from (i) & (iii) we get, $c = 1, f = -1, g = -1$.

So the required equation is $x^2 + y^2 - 2x - 2y + 1 = 0$.

The co-ordinates of the centre are $(-g, -f)$ and the radius is $\sqrt{g^2 + f^2 - c}$, thus the centre is(1, 1) and radius $r = 1$.

Example 3 :

Find the radius and co-ordinates of the centre of the circle $2x^2 + 2y^2 + 14x - 2y + 7 = 0$.

Find also the equation of the circle which is concentric with this circle and is of radius 5.

Solution :

The given equation of the circle can be written as $x^2 + y^2 + 7x - y + \frac{7}{2} = 0$ which gives us on comparision with the circle, $x^2 + y^2 + 2gx + 2fy + c = 0$;

$g = \frac{7}{2}, f = -\frac{1}{2}$ and $c = \frac{7}{2}$.

Hence the co-ordinates of the centre are $(-g, -f) = \left(-\frac{7}{2}, \frac{1}{2}\right)$ and the radius is

$$r = \sqrt{g^2 + f^2 - c} = \sqrt{\frac{49}{4} + \frac{1}{4} - \frac{7}{2}} = 3.$$

The equation of the concentric circle of radius 5 is

$$\left(x + \frac{7}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 = 25$$

or, $(2x + 7)^2 + (2y - 1)^2 = 100$

or, $4x^2 + 4y^2 + 28x - 4y - 50 = 0$.

Example 4 :

Find the equation of the circle whose diameter is a diagonal of the rectangle formed by the lines $x = 4$, $x = -4$, $y = 2$ and $y = -3$.

Solution :

The four vertices of the given rectangle are A (4, 2), B (–4, 2) C (– 4, –3) and D (4 – 3), Hence the equation of the circle with $\overline{AC}$ as diameter is

$(x-4)(x+4)+(y-2)(y+3)=0$

or, $x^2+y^2-y-22=0$.

Example 5 :

Find the length of the tangent drawn from the point (–2, 3) to the circle, $x^2+y^2-4x+6y+4=0$.

Solution :

The co-ordinates of the centre C of the circle are (2, –3) and the radius $r=\sqrt{4+9-4}=3$. So the length of the tangent is givenby,

$$\sqrt{(2+2)^2+(-3-3)^2-9}=\sqrt{16+36-9}=\sqrt{43}.$$

Example 6 :

Find the equation of the circle which has its centre on x – axis and which passes through the points (4, 7) and (12, 9). Prove that the straight line $9x-2y=5$ is a tangent to this circle.

Solution :

Let the equation of the circle be $x^2+y^2+2gx+2fy+c=0$.

Since the centre is on x – axis, we have $f=0$. Again this circle passes through the points (4, 7) and (12, 9). Hence putting the respective values of x and y, we get

$8g+14f+c=-65$, $24g+18f+c=-225$.

On subtraction, we get (putting $f=0$) $16g=-160$ or, $g=-10$.

Hence, $c=15$.

So the required equation is $x^2+y^2-20x+15=0$.

The centre of this circle is (10, 0) and radius $\sqrt{100-15}=\sqrt{85}$.

Now, the perpendicular distance of the point (10, 0) from the line $9x-2y-5=0$ is

$$\frac{90-5}{\sqrt{81+4}}=\sqrt{85}.$$

which is equal to the radius of the circle. Hence the line $9x-2y=5$ is a tangent to the circle.

Example 7 :

Find the equation of the circle which touches the y-axis at the point (0, 4) and passes through the point (2, 0). Find the equation of tangents at the points where this circle meets the x- axis.

Solution :

The y-axis is a tangent to the circle at (0, 4). Hence the y-co-ordinate of the centre is 4. If (h, 4) be the centre of the circle, then its equation is, $(x-h)^2+(y-4)^2=h^2$.

Since this circle passes through the point (2, 0), we have $(2-h)^2 + 16 = h^2$;

or, $h = 5$.

Hence, the equation of the cirlce is, $(x-5)^2 + (y-4)^2 = 25$

or, $x^2 + y^2 - 10x - 8y + 16 = 0$.

The point of intersection of this circle with x - axis is obtained by putting $y = 0$, which gives, $x^2 - 10x + 16 = 0$

or, $(x-8)(x-2) = 0$ or, $x = 2, 8$.

Therefore, the circle meets the x- axis at (2, 0) and (8, 0). The tangents to the circle,

$x^2 + y^2 - 10x - 8y + 16 = 0$

at (2, 0) and (8, 0) respectively are given by, $2x - 5(x+2) - 4(y+0) + 16 = 0$ and, $8x - 5(x+8) - 4(y+0) + 16 = 0$ i.e. $-3x - 4y + 6 = 0$; and $3x - 4y - 24 = 0$.

Example 8 :

Obtain the condition that the line $lx + my + n = 0$ will be a tangent to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$.

Solution :

A line is a tangent to a circle if the length of the perpendicular drawn from the centre of the circle to the given line, is equal to the radius of the circle.

Now, the given circle has its centre at $(-g, -f)$ and radius $\sqrt{g^2 + f^2 - c}$. The length of perpendicular from $(-g, -f)$ to the line $lx + my + n = 0$ is $\dfrac{lg + mf - n}{\pm\sqrt{l^2 + m^2}}$ and hence the line $lx + my + c = 0$ is a tangent to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$,

if $\dfrac{lg + mf - n}{\pm\sqrt{l^2 + m^2}} = \sqrt{g^2 + f^2 - c}$

or, $(lg + mf - n)^2 = (l^2 + m^2)(g^2 + f^2 - c)$.

12.06 Parametric form of equations of a circle

Let (0, 0) be the centre of the circle of radius a.

Let P (x, y) be any point on this circle.

If P$(x$, y) is the point P (a, θ) in polar coordinates with respect to the centre of the circle as pole and $\overrightarrow{OX}$ as the initial ray (see definition of polar co-ordinates), then $x = a \cos 0$ and $y = a \sin \theta$.

Here θ is a parameter with $0 \leq \theta < 2\pi$. For different values of θ lying in $[0, 2\pi)$ we can get different points on the circle.

Thus we see that the **parametric form** of equations of the circle with centre at origin and radius equal to a, is given by

$\boxed{x = a \cos \theta \text{ and } y = a \sin \theta}$ where θ is a parameter with $0 \leq \theta < 2\pi$.

The parametric form of equations of the circle with centre at (h, k) and radius equal to a , is given by

$x = h + a\ \cos\theta$ and $y = k + a\ \sin\theta$ where θ is a parameter, $0 \le \theta < 2\pi$.

EXERCISES 12 (a)

1. Fill in the blanks by choosing the correct answer from the given alternatives :
   (a) The centre of the circle $x^2 + y^2 + 2x - 6y + 1 = 0$ is ——. [(2, – 6),(–2, 6), (–1, 3), (1, –3)]
   (b) The equation $2x^2 - ky^2 - 6x + 4y - 1 = 0$ represents a circle if k = ——. [2, –2, 0,1]
   (c) The point (–3, 4) lies — the circle $x^2 + y^2 = 16$. [outside, inside, on]
   (d) The line $y = x + k$ touches the circle $x^2 + y^2 = 16$, if k = — [$\pm 2\sqrt{2}$, $\pm 4\sqrt{2}$, $\pm 8\sqrt{2}$, $\pm 16\sqrt{2}$]
   (e) The radius of the circle $x^2 + y^2 - 2x + 4y + 1 = 0$ is —— [1, 2, $4\sqrt{19}$]
2. State (with reasons), which of the follwing are *true* or *false* :
   (a) Every second degree equation in x and y represents a circle.
   (b) The circle $(x - 1)^2 + (y - 1)^2 = 1$ passes through origin.
   (c) The line $y = 0$ is a tangent to the circle $(x + 1)^2 + (y - 2)^2 = 1$.
   (d) The radical axis of two circles always passes through the centre of one of the circles.
   (e) The circles $x^2 + (y - 3)^2 = 4$ and $(x - 4)^2 + y^2 = 9$ touch each other.
3. Find the equation of circles determined by the following conditions.
   (a) The centre at (1, 4) and passing through (–2, 1).
   (b) The centre at (–2, 3) and passing through origin.
   (c) The centre at (3, 2) and the circle is tangent to x-axis.
   (d) The centre at (–1, 4) and the circle is tangent to y-axis.
   (e) The ends of diameter are (–5, 3) and (7, 5).
   (f) The radius is 5 and circle is tangent to both the axes.
   (g) The centre is on the x-axis and the circle passes through the origin and the point (4, 2).
   (h) The centre is on the line $8x + 5y = 0$ and the circle passes through the points (2, 1) and (3, 5).
   (i) The centre is on the line $2x + y - 3 = 0$ and the circle passes through the points (5, 1) and (2 – 3).
   (j) The circle is tangent to the line $x + 2y - 9 = 0$ at (5, 2) and also tangent to the line $2x - 3y - 7 = 0$ at (2, –1)
   (k) The circle touches the axis of x at (3, 0) and also touches the line $3y - 4x = 12$.
   (l) The circle is tangent to x-axis and passes through (1, –2) and (3, – 4).
   (m) The circle passes through origin and cuts off intercepts a and b from the axes.
   (n) The circle touches the axis of x at a distance 3 from origin and intercepts a distance of 6 on the y-axis.

4. Find the centre and radius of the following circles :

(a) $x^2 + y^2 + 6x - 4y - 12 = 0$ (b) $ax^2 + ay^2 + 2gx + 2fy + k = 0$

(c) $4x^2 + 4y^2 - 4x + 12y - 15 = 0$ (d) $a(x^2 + y^2) - bx - cy = 0$.

5. Obtain the equation of circles passing through the following points and determine the co-ordinates of centre and radius of the circle in each case :

(a) the points (3, 4) (4, – 3) and (–3, 4),

(b) the points (2, 3) (6, 1) and (4, – 6),

(c) the points $(a, 0)$, $(-a, 0)$ and $(0, b)$,

(d) the points (–3, 1), (5, –3) and (–3, 4).

6. Find the equation of the circles circumscribing the triangles formed by the lines given below :

(a) the lines $x = 0$, $y = x$, $2x + 3y = 10$;

(b) the lines $x = 0$, $4x + 5y = 35$, $4y = 3x + 25$;

(c) the lines $x = 0$, $y = 0$, $3x + 4y - 12 = 0$;

(d) the lines $y = x$, $y = 2$ and $y = 3x + 2$;

(e) the lines $x + y = 6$, $2x + y = 4$ and $x + 2y = 5$.

7. Find the co-ordinates of the points where the circle $x^2 + y^2 - 7x - 8y + 12 = 0$ meets the coordinate axes and hence find the intercepts on the axes.

[**Hint :** If a circle intersects a line L at points A and B, then the length, AB is its intercepts on the line L]

8. Find the equation of the circle passing through the point (1, –2) and having its centre at the point of intersection of lines $2x - y + 3 = 0$ and $x + 2y - 1 = 0$.

9. Find the equation of the circle whose ends of a diameter are the points of intersections of the lines $x + y - 1 = 0$, $4x + 3y + 1 = 0$ and $4x + y + 3 = 0$, $x - 2y + 3 = 0$.

10. Find the equation of the circle inscribed inside the triangle formed by the line $\frac{x}{4} + \frac{y}{3} = 1$ and the co- ordinate axes.

11. (a) Find the equation of the circle with its centre at (3, 2) and which touches the line $x + 2y - 4 = 0$.

(b) The line $3x + 4y + 30 = 0$ is a tangent to the circle whose centre is at $\left(-\frac{12}{5}, -\frac{16}{5}\right)$. Find the equation of the circle.

(c) Prove that the points (9, 7), (11, 3) lie on a circle with centre at the origin. Find the equation of the circle.

(d) Find the equation of the circle which touches the line $x = 0$, $x = a$ and $3x + 4y + 5a = 0$.

(e) If a circle touches the co-ordinate axes and also touches the straight line $\frac{x}{a} + \frac{y}{b} = 1$, and has its centre in the first quadrant, find its equation.

12. ABCD is a square of side a. $\overleftrightarrow{AB}$ and $\overleftrightarrow{AD}$ are taken as co-ordinate axes, prove that the equation of the circle circumscribing the square is $x^2 + y^2 = a(x + y)$.

13. (a) Find the equation of the tangent and normal to the circle $x^2 + y^2 = 25$ at the point $(3, -4)$.

(b) Find the equation of the tangent and normal to the circle $x^2 + y^2 - 3x + 4y - 31 = 0$ at the point $(-2, 3)$.

(c) Find the equation of the tangents to the circle $x^2 + y^2 + 4x - 6y - 16 = 0$ at the point where it meets the y-axis.

(d) Find the condition under which the tangents at (x_1, y_1) and (x_2, y_2) to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ are perpendicular.

(e) Calculate the radii and distance between the centres of the circles, whose equations are $x^2 + y^2 - 16x - 10y + 8 = 0$; $x^2 + y^2 + 6x - 4y - 36 = 0$.

Hence or otherwise prove that the tangents drawn to the circles at their points of intersection are perpendicular.

14. (a) Find the equation of the tangents to the circle $x^2 + y^2 = 9$, perpendicular to the line $x - y - 1 = 0$.

(b) Find the equation of the tangents to the circle $x^2 + y^2 - 2x - 4y = 40$, parallel to the line $3x - 4y = 1$.

(c) Show that the line $x - 7y + 5 = 0$ is a tangent to the circle $x^2 + y^2 - 5x + 5y = 0$. Find the point of contact. Find also the equation of tangent parallel to the given line.

(d) Prove that the line $ax + by + c = 0$ will be a tangent to the circle $x^2 + y^2 = r^2$, if $r^2 (a^2 + b^2) = c^2$.

(e) Prove that the line $2x + y = 1$ is a tangent to the circle $x^2 + y^2 + 6x - 4y + 8 = 0$.

(f) If the line $4y - 3x = \mathrm{k}$ is a tangent to the circle $x^2 + y^2 + 10x - 6y + 9 = 0$, find k.

Also find the co-ordinates of the point of contact.

15. (a) Find the length of the tangent drewn to the circles $x^2 + y^2 + 10x - 6y + 8 = 0$ from the centre of the circle $x^2 + y^2 - 4x = 0$.

(b) Find the length of the tangent drawn from the point $(2, -1)$ to the circle, $x^2 + y^2 - 6x + 10y + 18 = 0$.

(c) Find the length of the tangent drawn from the point $(4, 7)$ to the circle $x^2 + y^2 = 15$.

16. (a) Prove that the circles given by the equations $x^2 + y^2 + 2x - 8y + 8 = 0$ and $x^2 + y^2 + 10x - 2y + 22 = 0$ touch each other externally. Find also the point of cantact.

(b) Prove that the circles given by the equations $x^2 + y^2 = 4$ and $x^2 + y^2 + 6x + 8y - 24 = 0$, touch each other and find the equation of the common tangent.

(c) Prove that the two circles $x^2 + y^2 + 2by + c^2 = 0$ and $x^2 + y^2 + 2ax + c^2 = 0$, will touch each other if $\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{c^2}$.

(d) Prove that the circles given by $x^2 + y^2 + 2ax + 2by + c = 0$, and $x^2 + y^2 + 2bx + 2ay + c = 0$ touth each other if $(a + b) = 2c$.

17. Find the equation of the circle through the point of intersection of circles $x^2 + y^2 - 6x = 0$ and $x^2 + y^2 + 4y - 1 = 0$ and the point $(-1, 1)$.

18. Find the equation of the circle passing through the intersection of the circles $x^2 + y^2 - 2ax = 0$

and $x^2 + y^2 - 2by = 0$ and having the centre on the line $\frac{x}{a} - \frac{y}{b} = 2$.

19. Find the radical axis of the circles $x^2 + y^2 - 6x - 8y - 3 = 0$ and $2x^2 + 2y^2 + 4x - 8y = 0$.
20. Find the radical axis of the circles $x^2 + y^2 - 6x + 8y - 12 = 0$, and $x^2 + y^2 + 6x - 8y + 12 = 0$. Prove that the radical axis is perpendicular to the line joining the centres of the two circles.
21. If centre of one circle lies on or inside another, prove that the circles cannot be orthogonal.
22. If a circle S intersects circles S_1 and S_2 orthogonally, prove that the centre of S lies on the radical axis of S_1 and S_2.

    Hints : Take the line of centres of S_1 and S_2 as x - axis and the radical axis as y - axis. Use conditions for otrhogonal intersection of S, S_1 and S, S_2 simultaneously and prove that S is centred on the y - axis.
23. R is the radical centre of circles S_1, S_2 and S_3. Prove that if R is on / inside / outside one of the circles then it is similarly situated with respect to the other two.
24. Determine a circle which cuts orthogonally each of the circles, $S_1 : x^2 + y^2 - 4x - 6y + 12 = 0$, $S_2 : x^2 + y^2 + 4x + 6y + 12 = 0$, $S_3 : x^2 + y^2 - 4x + 6y + 12 = 0$.

    Hints : The centre of the required circle S must be the radical centre R (why ?), which lies outside all the circles. Then show that radius of S must be the length of the tangent from R to any circle of the system.
25. Prove that no pair of concentric circles can have a radical axis.

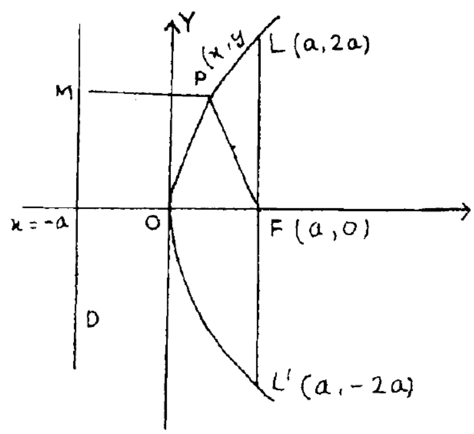
12.1 The Parabola

Definitions :

A parabola is a set (locus) of all points in a plane such that the distance of every point of the set from a given point is equal to its distance from a given line in that plane.

The given point is called the **'focus'** and the given line is called the **'directrix'** of the parabola. The line through focus perpendicular to the directrix is called the **'axis'** of the parabola. If the axis intersects the directrix at the point A and the focus is the point F, then the midpoint of $\overline{AF}$ is called the **'vertex'** of the parabola. Any chord of the parabola passing through the focus is called a **'focal chord'**. The focal chord perpendicular to the axis is called the **'latus rectum'** of the parabola.

Equation of Parabola :



Consider a parabola with its vertex at origin and focus F on the x - axis. If F be the point $(a, 0)$, then the derectrix D is the line given by $x = -a$.

Then by definition, for any point $P(x, y)$ on the parabola, we have

PF = PM

where M is the foot of the perpendicular from P on the directrix. Thus $PF^2 = PM^2$

which gives,

$(x-a)^2+(y-0)^2=(x+a)^2$

or, $y^2=(x+a)^2-(x-a)^2=4ax$

Hence, $\boxed{y^2=4ax}$

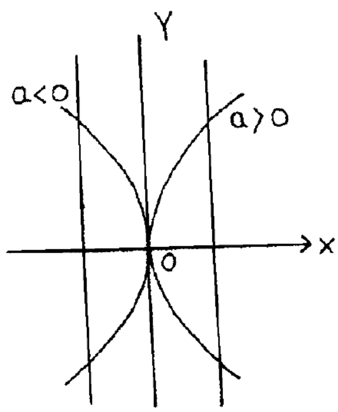
is the equation of the parabola with vertex at (0,0) and axis along x – axis, with focus at $(a, 0)$

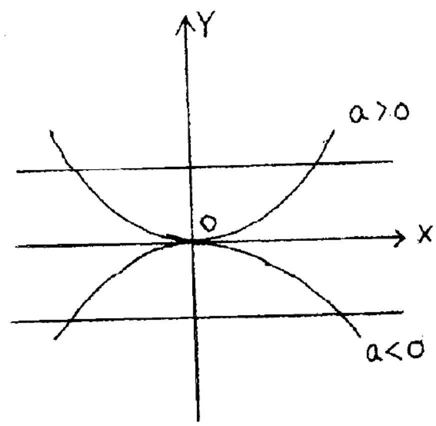
From the equation $y^2=4ax$ we see that if $a>0$, then $x<0$ and if $a<0$, then $x>0$ which shows that if a is positive, the parabola is open to the right of y – axis and if $a<0$, the parabola is open to left.

Proceeding likewise, we can show that if the vertex is at origin and the axis is along y – axis, then the equation of the parabola will be

$$\boxed{x^2=4ay} \tag{2}$$

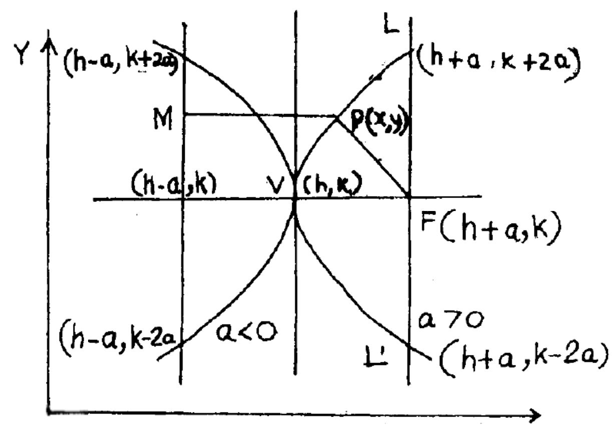
where F $(0, a)$ is the focus. The parabola will be open upward if $a>0$ and the downwards if $a<0$. (Fig. 35)





Instead of taking the vertex of the origin, let the vertex of the parabola be at V (h, k) and the axis be parallel to x - axis. If we consider the focus to be F $(h+a, k)$, then the directrix D is the line $x=h-a$. Now, for any point P(x,y) on the parabola on the RHS

PM = PF where M is the foot of the perpendicular drawn from P on the directrix.



Since, PM = $|x-(h-a)|$

$=|x-h+a|$

we have

$PM^2=PF^2$

or, $\{(x-h)+a\}^2 = (x-h-a)^2 + (y-k)^2$

or, $(y-k)^2 = \{(x-h)+a\}^2 - \{(x-h)-a\}^2$

$= 4a.(x-h).$

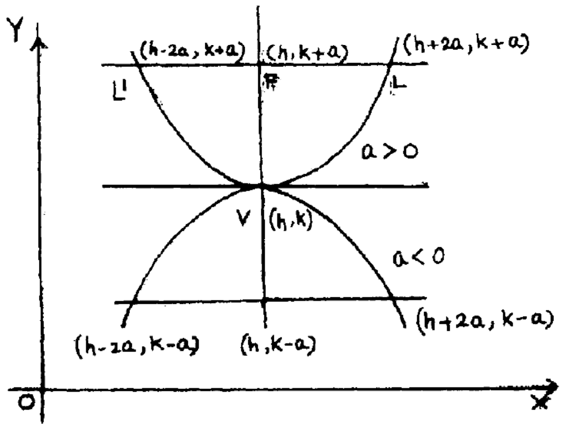
Hence $\boxed{(y-k)^2 = 4a(x-h)}$ (3)

is the equation of the parabola whose vertex is at (h, k) and axis parallel to x – axis. As before, if $a > 0$, the parabola is open to the right and if $a < 0$, the parabola is open to the left .(Fig.36)

Similarly, if the vertex is at (h, k) and the axis is parallel to y - axis, then it can be shown that the equation of the parabola is

$\boxed{(x-h)^2 = 4a(y-k)}$. (4)

and if a > 0, the parabola is open upwards and for a < 0, the parabola is open downwards. (Figure on RHS)



The above equations (3) and (4) are equations of parabola in *standard form*.

Note 1 : The parametric equations of the parabola $y^2 = 4ax$. are given by $x = at^2$, $y = 2at$, and thus any point on the parabola $y^2 = 4ax$ can be taken as $(at^2, 2at)$ for some value of the parameter t.

Note 2 : The general equation of the parabola with its axis parallel to one of the co-ordinate axes, is a quadratic in x and linear in y, or quadratic in y and linear in x. Thus the equations $y = ax^2 + bx + c$ and $x = \mathrm{A}y^2 + \mathrm{B}y + \mathrm{C}$ are respectively the equations of parabolas with their axes parallel to y – axis and x - axis and vertex being at some point (h, k) for suitable values of h and k.

Note 3 : The latus rectum of the parabola being the chord passing through focus and perpendicular to the axis, is the line segment $\overline{\mathrm{LL'}}$ (in the figure demonstrating derivication of the equation $y^2=4ax$). Since the equation of the parabola is $y^2 = 4ax$, for $x = a$, we have, $y = \pm 2a$. Hence the end points of the latus rectum $\overline{\mathrm{LL'}}$ have co-ordinates $(a, \pm 2a)$. This gives us that for the parabola $y^2 = 4ax$, the length of the *latus rectum* is $|4a|$. Similarly, for the parabola $x^2 = 4ay$, the length of the latus rectum is also $|4a|$ and the co-ordinates of the end points of the latus rectum are $(\pm 2a, a)$.

Note 4 : In general, if the parabola has the vertex at any point (h, k), then the co-ordinates of the focus and end points of latus rectum are as shown in the respective figures.

Equations of Tangent and Normal

Consider the parabola given by $y^2 = 4ax$

and let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be any two points on it. Then the equation of the chord $\overline{PQ}$ is given by $(y - y_1) = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$.

$P(x_1, y_1)$ and $Q(x_2, y_2)$ being points on the parabola $y^2 = 4ax$, we have $y_1^2 = 4ax_1$ and $y_2^2 = 4ax_2$.

Hence, $y_2^2 - y_1^2 = 4a(x_2 - x_1)$

or, $\frac{y_2 - y_1}{x_2 - x_1} = \frac{4a}{y_2 + y_1}$.

So, the equation of the chord $\overline{PQ}$ is given by $(y - y_1) = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) = \frac{4a}{y_2 + y_1}(x - x_1)$.

$\overleftrightarrow{PQ}$ becomes the tangent at (x_1, y_1) if $y_2 \to y_1$ and $x_2 \to x_1$ along the curve.

Hence taking the limit, the equation of tangent at (x_1, y_1) is

$$(y - y_1) = \frac{4a}{2y_1}(x - x_1)$$

or, $(y - y_1) = \frac{2a}{y_1}(x - x_1)$ or, $yy_1 - y_1^2 = 2ax - 2ax_1$

or, $yy_1 = 2ax + (y_1^2 - 2ax_1) = 2ax + (4ax_1 - 2ax_1)$,

or, $\boxed{yy_1 = 2a(x + x_1)}$ (1)

If instead of $y^2 = 4ax$ we consider the parabola $x^2 = 4ay$, then proceeding as above, it can be shown that the equation of tangent at (x_1, y_1) is

$$\boxed{xx_1 = 2a(y + y_1)} \quad (2)$$

Next, to find the equation of normal at the point (x_1, y_1) to the parabola $y^2 = 4ax$, we see that the tangent at (x_1, y_1) is $yy_1 = 2a(x + x_1)$,

or, $2ax - yy_1 + 2ax_1 = 0$

whose slope is $\frac{2a}{y_1}$.

Hence the slope of normal is $-\frac{y_1}{2a}$. So the equation of normal at (x_1, y_1) is

$$(y - y_1) = \frac{-y_1}{2a}(x - x_1).$$

If we denote the slope of normal by $m = \frac{-y_1}{2a}$, then $y_1 = -2am$ and hence $y_1^2 = 4ax_1$ gives $x_1 = \frac{y_1^2}{4a} = am^2$. Hence any point (x_1, y_1) on the parabola is given by $(am^2, -2am)$ with m being a parameter and the equation of normal becomes $(y + 2am) = m(x - am^2)$.

Thus, $\boxed{y = mx - 2am - am^3}$ is the normal at $(x_1\ y_1)$ with slope $m = \frac{-y_1}{2a}$.

Point of intersection of the parabola $y^2 = 4ax$ and the line $y = mx + c$ and condition of tangency.

The only vertical line which is a tangent to the parabola $y^2 = 4ax$ is $x = 0$ (the y – axis) which touches the parabola at its vertex. No other vertical or horizontal line can be a tangent. So, to explore the condition of tangency, we need only consider lines of the form : $y = mx + c$; $m \neq 0$.

Let $y^2 = 4ax$ be a given parabola and $y = mx + c$ be a given line. Then eliminating y between the above two equations, we obtain $y^2 = (mx + c)^2 = 4ax$

or, $m^2x^2 + (2mc - 4a)x + c^2 = 0$,

which is a quadratic equation in x and hence gives two values of x, say x_1 and x_2. If y_1 and y_2 be the corresponding values of y obtained from $y = mx + c$, then the two points of intersection are (x_1, y_1) and (x_2, y_2). This line becomes a tangent to the parabola if the two points of intersection coincide, i,e., the above quadratic in x has both its roots equal i.e., the discriminant of the equation is zero, i.e., $(2mc - 4a)^2 = 4c^2m^2$

or, $16a^2 = 16amc$

or, $\boxed{c = \frac{a}{m}}$

So the line $y = mx + \frac{a}{m}$; $m \neq 0$ is always a tangent to the parabola $y^2 = 4ax$.

Similarly it can be shown that the line $y = mx + c$ will be a tangent to the parabola $x^2 = 4ay$, if $\boxed{c = -am^2}$. The only horizontal tangent is $y = 0$ (the x - axis). No other horizontal or vertical line can be a tangent to this parabola.

Corrollary : No two tangents of a parabala can be parallel. (Prove)

Parametric form of equations of a parabola :

It is convenient to express the co-ordinates of any point on the parabola $y^2 = 4ax$ in terms of one variable.

It is clear that $x = at^2$ and $y = 2at$ always satisfy the equation $y^2 = 4ax$ for all real values of t.

Thus $\boxed{x = at^2 \text{ and } y = 2at}$, where t is a real parameter represent the **parametric form** of equations of the parabola $y^2 = 4ax$. Similarly, $x = 2at$, $y = at^2$ are the parametric equations of the parabola $x^2 = 4ay$.

The point whose co-ordinates are $(at^2, 2at)$ or $(2at, at^2)$ for given real value of t may, for brevity, be called the point 't' on the parabola $y^2 = 4ax$ or $x^2 = 4ay$ respectively.

SOLVED EXAMPLES :

Example 1 :

Obtain the equations of the following parabolas :

(a) focus at (2, 0) and directrix $x + 2 = 0$,

(b) focus at (1, 1) and directrix $y = 0$,

(c) focus at (1, 2) and directrix $x + y = 2$.

Solution :

(a) We know that if a parabola has the focus at $(a, 0)$ and the vertex at (0, 0), then its equation is $y^2 = 4ax$.

Since the focus of the parabola is at (2, 0) and the directrix is the line $x + 2 = 0$ the vertex is at (0, 0) and $a = 2$. So the equation of the parabola is $y^2 = 4.2x = 8x$.

(b) The focus of the parabola is at (1, 1). The directrix being the line $y = 0$ (i.e. x- axis), the axis is parallel to y-axis passing through (1,1). The vertex being the mid point of the perpendicular from focus on the directrix, is the point $\left(1, \frac{1}{2}\right)$.

We know that the equation of the porabola with vertex at (h, k) and axis parallel to y-axis is $(x - h)^2 = 4a\,(y - k)$

Here, $h = 1$, $k = \frac{1}{2}$ and $a = \frac{1}{2}$.

Hence the required equation is $(x - 1)^2 = 4.\ \frac{1}{2}\left(y - \frac{1}{2}\right)$

or, $(x - 1)^2 = 2y - 1$, or, $x^2 - 2x - 2y + 2 = 0$.

(c) The focus is at (1, 2) and directrix is the line $x + y - 2 = 0$.

By definition, the parabola is the locus of a point equidistant from the focus and the directrix. If P (x, y) is any point on the parabola, then

$$(x - 1)^2 + (y - 2)^2 = \left(\frac{|x + y - 2|}{\sqrt{2}}\right)^2$$

or, $2\,(x - 1)^2 + 2\,(y - 2)^2 = (x + y - 2)^2$

or $x^2 + y^2 - 2xy - 4y + 6 = 0$ is the required equation.

Example 2 :

Find the co-ordinates of the vertex, the focus, the length of latus rectum, the equation of the directrix, of the parabola $3x^2 + 12x - 8y = 0$.

Solution :

The given equation of the parabola can be rewritten as $3(x^2 + 4x + 4) = 8y + 12$

or, $(x + 2)^2 = \frac{8}{3}\left(y+\frac{3}{2}\right)$.

This on comparision with $(x-h)^2 = 4a\ (y-k)$ gives $h = -2$, $k = -\frac{3}{2}$ and $a = \frac{2}{3}$.

Hence the given parabola has the vertex at $\left(-2, \frac{-3}{2}\right)$ and the focus is at $(h, k + a) \equiv \left(-2, -\frac{5}{6}\right)$. The length of latus rectum is $4a = \frac{8}{3}$. The equation of the directrix is $y = k - a = \frac{-3}{2} - \frac{2}{3} = -\frac{13}{6}$.

Example 3 :

Find the equation of the parabola passing through the points (1, 2), (–2, 3) and (2, –1) and the axis being parallel to x – axis.

Solution :

We know that a parabola with its axis parallel to x - axis is given by

$x = Ay^2 + By + C$. .... (1)

Since this passes through the points (1, 2), (–2, 3) and (2, –1), putting the respective values of x and y in (1), we have

$4A + 2B + C = 1$, $9A + 3B + C = -2$, $A - B + C = 2$.

Solving these equations, we get $A = -\frac{2}{3}$, $B = \frac{1}{3}$ and $C = 3$.

Hence, the equation of the parabola is $x = -\frac{2y^2}{3} + \frac{y}{3} + 3$

or, $2y^2 + 3x - y - 9 = 0$.

Example 4 :

Find the equations of the tangent and normal to the parabola $y^2 = 4ax$ at the points $(at^2, 2at)$.

Solution :

The equation of tangent to the parabola $y^2 = 4ax$ at the point (x_1, y_1) is,
$yy_1 = 2a\ (x + x_1)$ and the equation of the normal at (x_1, y_1) is
$2a\ (y - y_1) + y_1\ (x - x_1) = 0$.

In the present case $x_1 = at^2$, $y_1 = 2at$ and hence the equation of tangent at $(at^2, 2at)$ is, $2\ ayt = 2a\ (x + at^2)$

or, $x - yt + at^2 = 0$.

The equation of the normal is, $2a\ (y - 2at) + 2at\ (x - at^2) = 0$

or, $y + xt - 2at - at^3 = 0$.

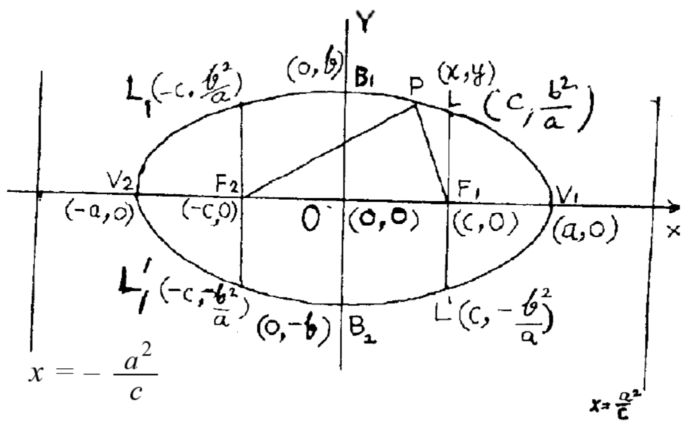
12.2 The Ellipse

Definitions : An ellipse is a set (locus) of all points in a plane such that the sum of the distances of any point of the set from two given points in that plane is a constant.

The two given points are called **foci** (plural of focus). The mid point of the line segment joining the foci is called the **centre** and the line through foci is called the **axis** of the ellipse. The points where the ellipse cuts the axis are called its **vertices**.

Equation of an Ellipse :

Let us consider an ellipse with its centre at origin and the foci along x-axis (in the figure on RHS). Then the foci will be given by F_1 $(c, 0)$ and F_2 $(-c, 0)$, for some real number $c > 0$. If we denote the constant sum of the distances of any point on the ellipse from the foci by $2a$, then for any point $P(x, y)$ on the ellipse, we have



$PF_1 + PF_2 = 2a$

or $PF_1 = 2a - PF_2$

Squaring both sides, we get,

$PF_1^2 = 4a^2 + PF_2^2 - 4a\, PF_2$

or, $(x - c)^2 + (y - 0)^2 = 4a^2 + (x + c)^2 + (y - 0)^2 - 4a\, PF_2$

or, $4a\, PF_2 = 4a^2 + (x + c)^2 - (x - c)^2 = 4a^2 + 4xc$,

or, $a\, PF_2 = a^2 + xc$

or, $a\sqrt{(x + c)^2 + y^2} = a^2 + xc$ squaring again, we have $a^2\{(x + c)^2 + y^2\} = a^4 + x^2c^2 + 2a^2xc$

or, $a^2x^2 + a^2c^2 + 2a^2xc + a^2y^2 = a^4 + x^2c^2 + 2a^2xc$

or, $(a^2 - c^2)x^2 + a^2y^2 = a^2(a^2 - c^2)$,

or, $\dfrac{x^2}{a^2} + \dfrac{y^2}{a^2 - c^2} = 1.$

Now, in the triangle PF_1F_2, $PF_1 + PF_2 > F_1F_2$

which gives $2a > 2c$, or $a > c$. So writing $b^2 = a^2 - c^2$ we have the equation of the ellipse as

$$\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1} \qquad (1)$$

Similarly, if we consider the centre of the ellipse to be at (0, 0) and foci at $(0, \pm c)$ i.e. F_1 $(0, c)$ and F_2 $(0, -c)$, then proceeding exactly as above, it can be shown that, the equation of the ellipse is,

$$\boxed{\frac{y^2}{a^2} + \frac{x^2}{b^2} = 1} \qquad (2)$$

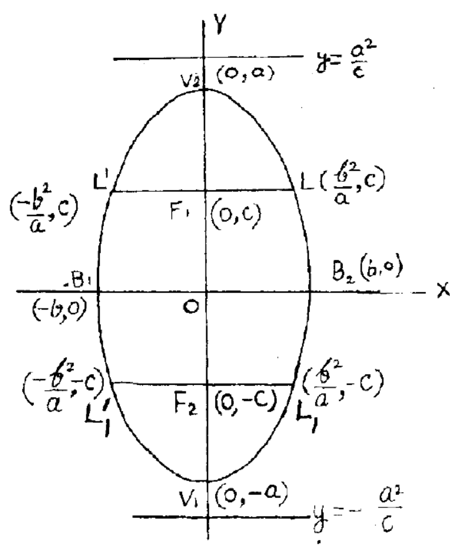
where $b^2 = a^2 - c^2$.

Note 1 : If we consider the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

then putting $y = 0$, we get $x = \pm a$ and $x = 0$ gives us $y = \pm b$. Hence the ellipse meets the x-axis at **the points** $(a, 0)$ and $(-a, 0)$ and y-axis at $(0, b)$ and $(0, -b)$. The line segment $\overline{V_1V_2}$ joining $(a, 0)$ and $(-a, 0)$ is called the **major axis**. The line segment $\overline{B_1B_2}$ joining $(0, b)$ and $(0, -b)$ is called the **minor axis**. Hence the major axis is of length $2a$ and minor axis is of length $2b$. The end points of major axis are the *vertices* of the ellipse having co-ordinates $(\pm a, 0)$.

Note2: If we consider the ellipse $\frac{y^2}{a^2} + \frac{x^2}{b^2} = 1$.

then the points with co-ordinates $(0, a)$ and $(0, -a)$ are end points of major axis and the points with co-ordinates $(b, 0)$ and $(-b, 0)$ are end points of the minor axis.



Note 3 : The chords $\overline{LL'}$ and $\overline{L_1L_1'}$ of the ellipse through the foci and perpendicular to the axis are called **Latera recta** *(Plural of latus rectum)*. Now, if we consider the ellipse,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \text{ then } y^2 = \frac{b^2}{a^2}(a^2 - x^2) \text{ or, } y = \pm \frac{b}{a}\sqrt{(a^2 - x^2)}.$$

If $x = \pm c$ then $y = \pm \frac{b}{a}\sqrt{(a^2 - c^2)} = \pm \frac{b^2}{a}$.

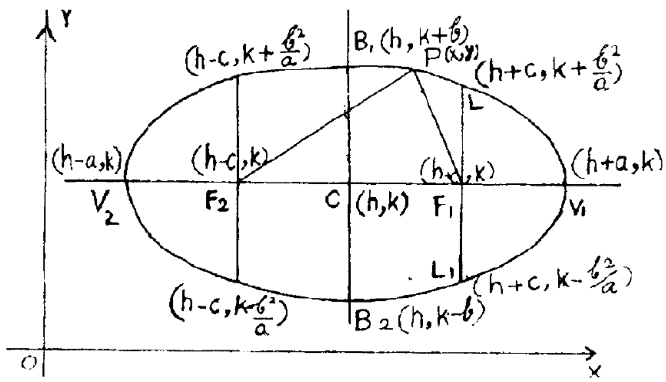
Hence, the end points of latera recta for this ellips are given by $(\pm c, \pm b^2/a)$ and each latus rectum is of length $\frac{2b^2}{a}$.

Similarly, we can show that the end points of latera racta of the ellipse $\dfrac{y^2}{a^2} + \dfrac{x^2}{b^2} = 1$ are given by $\left(\pm\dfrac{b^2}{a}, \pm c\right)$ and each one is also of length $\dfrac{2b^2}{a}$.

Note 4 : The ratio $e = \dfrac{c}{a}$ is called the **eccentricity** of the ellipse, and it always lies between 0 and 1 (as $0 < c < a$).

Equation of Ellipse with centre at (*h, k*) and axis parallel to co-ordinate axes :

Let us consider an ellipse with its centre at C (h, k) and let the axis of the ellipse be parallel to the x-axis. If F_1 $(h + c, k)$ and F_2 $(h - c, k)$ be the foci of the ellipse then for any point P (x, y) on the ellipse, we have,



$PF_1 + PF_2$ = constant $= 2a$, Say.

or, $PF_1 = 2a - PF_2$

or, $\sqrt{(x-h-c)^2 + (y-k)^2}$

$= 2a - \sqrt{(x-h+c)^2 + (y-k)^2}$

which on simplification gives,

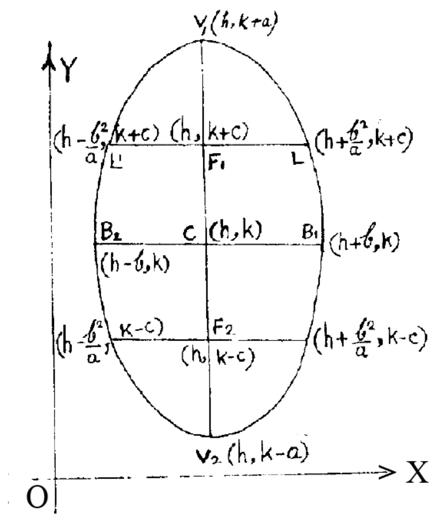
$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1 \qquad (1)$$

where $b^2 = a^2 - c^2$.

Similarly, if the centre of the ellipse is at (h, k) and the axis is parallel to y - axis, then proceeding as above, it can be seen that the equation of ellipse is,

$$\frac{(y-k)^2}{a^2} + \frac{(x-h)^2}{b^2} = 1 \qquad (2)$$

with $b^2 = a^2 - c^2$



the equations (1) and (2) are called equations of ellipse in **standard forms.**

Note (1) In these case the co-ordinates of the end points of major and minor axes, foci, the latera recta and the centre are as shown in the respective figures.

Equation of Tangent and Normal :

Consider the ellipse, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. (1)

If $P(x_1, y_1)$ and $Q(x_2, y_2)$ are two points on the ellipse, then the equation of the line $\overleftrightarrow{PQ}$ is given by $(y - y_1) = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1)$. (2)

Now, $P(x_1, y_1)$ and $Q(x_2, y_2)$ lie on (1). Hence,

$$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1 \text{ and } \frac{x_2^2}{a^2} + \frac{y_2^2}{b^2} = 1.$$

So subtracting we get, $\frac{x_1^2 - x_2^2}{a^2} + \frac{y_1^2 - y_2^2}{b^2} = 0$

or, $\frac{x_1^2 - x_2^2}{a^2} = \frac{y_2^2 - y_1^2}{b^2}$

or, $\frac{y_2 - y_1}{x_2 - x_1} = -\frac{b^2(x_1 + x_2)}{a^2(y_1 + y_2)}$.

So the equation (2) becomes $(y - y_1) = -\frac{b^2}{a^2}\left(\frac{x_1 + x_2}{y_1 + y_2}\right)(x - x_1)$. (3)

$\overleftrightarrow{PQ}$ becomes a tangent at $P(x_1, y_1)$ if $Q \to P$ along the curve, i.e. $x_2 \to x_1$ and $y_2 \to y_1$. So, taking the limit, from equation (3) we get

$$(y - y_1) = -\frac{b^2}{a^2}\left(\frac{2x_1}{2y_1}\right)(x - x_1),$$

or, $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1$.

Hence, the required equation of tangent at (x_1, y_1) is

$$\boxed{\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1} \quad \ldots\ldots (4)$$

From equation (4) it is seen that the slope of the tangent at (x_1, y_1) is, $-\left(\frac{b^2 x_1}{a^2 y_1}\right)$ and the slope of normal a (x_1, y_1) is $\frac{a^2 y_1}{b^2 x_1}$. So the equation of normal at (x_1, y_1) is given by

$$(y - y_1) = \frac{a^2 y_1}{b^2 x_1}(x - x_1)$$

or, $\boxed{\frac{x - x_1}{x_1 b^2} = \frac{y - y_1}{y_1 a^2}}$ (5)

Note : Similarly equations for tangent and normal can also be deduced for the ellipse,

$$\frac{y^2}{a^2} + \frac{x^2}{b^2} = 1$$

Condition that the line $y = mx + c$ is a tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

The only vertical tangents are : $x = \pm$ a. Now, consider a non vertical line $y = mx + c$.

The line $y = mx + c$ intersects the ellipse, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

at two points whose x – coordiantes are given by the equation $\frac{x^2}{a^2} + \frac{(mx + c)^2}{b^2} = 1$,

i.e. $(b^2 + m^2a^2)\, x^2 + 2mca^2x + (c^2a^2 - a^2b^2) = 0$

which is a quadratic in x having two roots, say x_1 and x_2. If y_1 and y_2 are the corresponding values of y, then (x_1, y_1) and (x_2, y_2) are the points of intersection. The line becomes a tangent if the two points of intersection are coincident i.e., if the roots of the above quadratic are equal i.e.,

$4m^2c^2a^4 - 4\,(b^2 + m^2a^2)\,(c^2 - b^2)\,a^2 = 0$

i.e. $\boxed{c^2 = a^2m^2 + b^2}$ (1)

So the lines : $y = mx \pm \sqrt{a^2m^2 + b^2}$ always represent a pair of parallel tangents to the given ellipse.

Focus - directrix property of Ellipse

While deriving the equation of the ellipse with centre at (0, 0) and foci at $(\pm c, 0)$ in 8.18 we obtained that for any point P(x, y) on the ellipse, if $2a$ be the constant sum of its distances from the foci, then $a\sqrt{(x+c)^2 + y^2} = a^2 + xc = c\left(x + \frac{a^2}{c}\right)$

or, $\sqrt{(x+c)^2 + y^2} = \frac{c}{a}\left(x + \frac{a^2}{c}\right)$,

which shows that the distance of P(x, y) from the point $(-c, 0)$ bears a constant ratio $\frac{c}{a}$ to its distance from $x = -\frac{a^2}{c}$ (the ratio being < 1). Hence the ellipse can be defined as a locus such that the distance of every point on it from a given point bears a constant ratio, less than 1, to its distance from a given line. The line $x = -\frac{a^2}{c}$ is a **directrix** for the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Similarly from the symmetry of the figure, we also obtain that the line $x = \frac{a^2}{c}$ is another directrix for the ellipse. Similarly the directrices of the ellipse

$$\frac{y^2}{a^2} + \frac{x^2}{b^2} = 1 \text{ are } y = \pm\frac{a^2}{c}$$

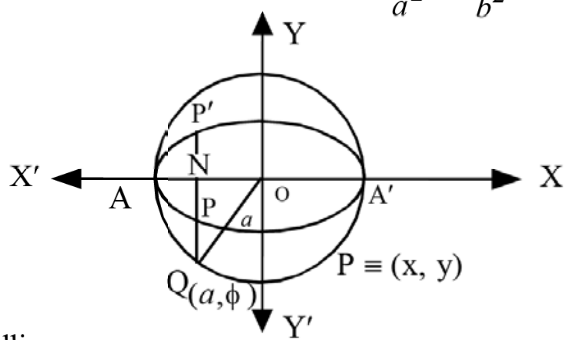
Parametric form of equations of an ellipse :

The circle which is described on the major axis of an ellipse as diameter, is called the **auxiliary circle** of the ellipse.

If we consider the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a > b$, then the length of the major axis of the ellipse is $2a$. The origin is the mid-point 0 of the major axis.

So the centre of the auxiliary circle of the ellipse is (0, 0) and the radius is a.

Hence the equation of the auxiliary circle of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $x^2 + y^2 = a^2$.



$\overline{AA'}$ is the major axis of the ellipse.

Let P (x, y) be any point on the ellipse. Let the line through P perpendicular to the major axis meet it at N.

Let $\overrightarrow{NP}$ meet the auxilary circle at Q.

Then OQ = a, the radius of the auxiliary circle.

If Q is the point (a, ϕ) in polar coordinates with respect to the centre O as pole and $\overrightarrow{OX}$ as initial ray, then ϕ is known as the 'eccentric angle' of the point P on the ellipse. [Note that although the name stands as such, the term 'eccentric angle' is actually not an angle. It is essentially an arc - measure, which is a real number lying in the interval $(0, 2\pi)$, $\phi = 0$ for A′]

We now proceed to obtain the cartesian coordinates of P i.e. x and y in terms of a, b and ϕ.

Let Q have cartesian coordinates (x, y'). [Since $\overleftrightarrow{PQ}$ is parallel to y - axis, P and Q have the same x - coordinate.] Thus we have Q (a, ϕ) in polar system and Q (x, y') in Cartesian system. By the relation between cartesian and polar co-ordinates, $x = a \cos\phi$ and $y' = a \sin\phi$.

Since P(x, y) is a point on the ellipse, we have :

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \Rightarrow \frac{a^2\cos^2\phi}{a^2}+\frac{y^2}{b^2}=1 \Rightarrow y^2=b^2\sin^2\phi \Rightarrow y=\pm b\sin\phi.$$

$y = -b\sin\phi$ corresponds to the y - co-ordinate of P′, the point of intersection of $\overrightarrow{PN}$ and the ellipse. Therefore, for the point P(x, y),

we have $x = a\cos\phi$ and $y = b\sin\phi$.

The equations $\boxed{x = a\cos\phi,\ y = b\sin\phi}$, $0 \le \phi < 2\pi$,

give the **paramateric form** of equations of the ellipse

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1;\ a>b$. Here ϕ is the parameter.

Similarly the parametric form of equations of the ellipse

$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$, $a > b$ are given by

$\boxed{x = h + a\cos\phi,\ y = k + b\sin\phi}$, $0 \le \phi < 2\pi$.

Note : 1. The parametric form of equations of the ellipse

$\frac{y^2}{a^2}+\frac{x^2}{b^2}=1$, $a > b$ can be similarly obtained as

$x = b\cos\phi,\ y = a\sin\phi;\ 0 \le \phi < 2\pi.$

2. The ellipse $\frac{(y-k)^2}{a^2}+\frac{(x-h)^2}{b^2}=1$, $a > b$ in parametric form becomes

$x = h + b\cos\phi,\ y = k + a\sin\phi;\quad 0 \le \phi < 2\pi.$

SOLVED EXAMPLES

Example 1 :

Find the equation of the ellipse with its center at origin, axes along the co-ordinate axes and which passes through the points (2, 2) and (3, 1).

Solution :

Let the equation of the ellipse be

$Ax^2 + By^2 = 1.$

This passes through the points (2, 2) and (3, 1). Hence, $4A + 4B = 1$ and $9A + B = 1$.

Solving these two equations, we get $A = \frac{3}{32}$ and $B = \frac{5}{32}$.

Hence, the equation of the ellipse is $\frac{3}{32}x^2 + \frac{5}{32}y^2 = 1$ or, $3x^2 + 5y^2 = 32$.

Example 2 :

Find the equation of the ellipse whose latus rectum is 5 and eccentricity $\frac{2}{3}$.

Solution :

Let the equation of the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Then the length of latus rectum is $\frac{2b^2}{a}$; hence, $\frac{2b^2}{a} = 5$ or $b^2 = \frac{5a}{2}$.

Again, the eccentricity of the ellipse is $\frac{2}{3}$.

So, $e = \frac{c}{a} = \frac{2}{3}$ or, $c = \frac{2a}{3}$. Hence, $b^2 = a^2 - c^2 = a^2 - \frac{4a^2}{9} = \frac{5a^2}{9}$.

Thus, we get, $\frac{5a^2}{9} = \frac{5a}{2}$ or, $a = \frac{9}{2}$

and hence $b^2 = \frac{5a}{2} = \frac{45}{4}$. So the equation of the ellipse is $\frac{x^2}{\left(\frac{9}{2}\right)^2} + \frac{y^2}{\frac{45}{4}} = 1$

or, $\frac{4x^2}{81} + \frac{4y^2}{45} = 1$ or, $20x^2 + 36y^2 = 405$.

Example 3 :

Reducing to standard from, obtain the co-ordinates of the centre, the foci the end points of minor and major axis, the length of latera recta and the eccentricity of the ellipse

$x^2 + 4y^2 - 6x - 16y + 21 = 0$.

Solution :

The equation of the ellipse can be rewritten as $(x^2 - 6x + 9) + 4(y^2 - 4y + 4) = 25 - 21 = 4$

or, $\frac{(x-3)^2}{4} + \frac{(y-2)^2}{1} = 1$

which is the equation of the ellipse in the standard form $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$.

On comparision, we get $h = 3$, $k = 2$, $a^2 = 4$ and $b^2 = 1$. Hence $c^2 = a^2 - b^2 = 3$. So,

(i) the co-ordinates of the center are $(h, k) = (3, 2)$,

(ii) the co-ordinates of the foci are $(h \pm c, k) = \left(3 \pm \sqrt{3}, 2\right)$;

(iii) the co-ordinates of the end points of the minor axis are, $(h, k \pm b) = (3, 2 \pm 1) = (3, 3)$ and $(3, 1)$

(iv) the co-ordinates of the end points of major axis are, $(h \pm a, k) = (3 \pm 2, 2) = (5,2)$ and $(1, 2)$

(v) The length of latus rectum = $\dfrac{2b^2}{a} = 1,$

(vi) the eccentricity $e = \dfrac{c}{a} = \dfrac{\sqrt{3}}{2}.$

Example 4 :

Prove that the straight line $y = x + \sqrt{5}$ touches the ellipse $x^2 + 4y^2 = 4$ and find the point of contact.

Solution :

The line $y = mx + c$ is a tangent to the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, if $c^2 = a^2m^2 + b^2$.

Now, the given line is $y = x + \sqrt{5}$, and the given ellipse is $x^2 + 4y^2 = 4$

or, $\dfrac{x^2}{4} + \dfrac{y^2}{1} = 1.$

So, we have $c = \sqrt{5}$, $m = 1$, $a^2 = 4$ and $b^2 = 1$ and hence $c^2 = a^2m^2 + b^2$**.**

So the condition of tangency is satisfied. Hence the given line touches the given ellipse.

To find the point of contact, we have $y = x + \sqrt{5}$, and $x^2 + 4y^2 = 4$

or, $x^2 + 4\,(x + \sqrt{5}\,)^2 = 4$, or $5x^2 + 8\sqrt{5}x + 16 = 0$

or, $x = \dfrac{-8\sqrt{5}}{2\times 5} = \dfrac{-4}{\sqrt{5}}$, and $y = x + \sqrt{5} = \dfrac{-4}{\sqrt{5}} + \sqrt{5} = \dfrac{1}{\sqrt{5}}$,

Hence, the point of contact is $\left(\dfrac{-4}{\sqrt{5}}, \dfrac{1}{\sqrt{5}}\right)$.

12.3 The Hyperbola

Definitions :

A hyperbola is a set (locus) of all points in a plane such that the difference of distances of any point of the set from two given points in that plane is a constant.

The two given points are called **foci** and the line through the foci is called the **axis.** The midpoint of the line segment joining the foci is called the **centre** of the hyperbola. The points where the hyperbola cuts the axis, are called its **vertices**.

Equation of a hyperbola with centre at (0, 0) and foci along x - axis.

Let the centre of the hyperbola be O (0,0) and the foci be F_1 $(c, 0)$ and F_2 $(-c, 0)$. Then for any point $P(x, y)$ on the hyperbola (figure on next page)

$PF_1 - PF_2 = \text{a constant} = \pm 2a$

i.e. $\sqrt{(x-c)^2+(y-0)^2}$

$-\sqrt{(x+c)^2+(y-0)^2}$

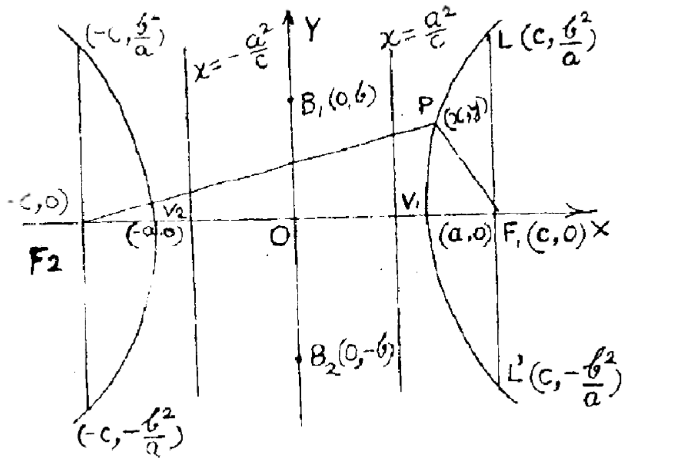
$= \pm 2a$, or $\sqrt{(x-c)^2+y^2}$

$= \sqrt{(x+c)^2+y^2} \pm 2a$

which on simplification gives,

$$\boxed{\frac{x^2}{a^2}-\frac{y^2}{b^2}=1} \quad (1)$$

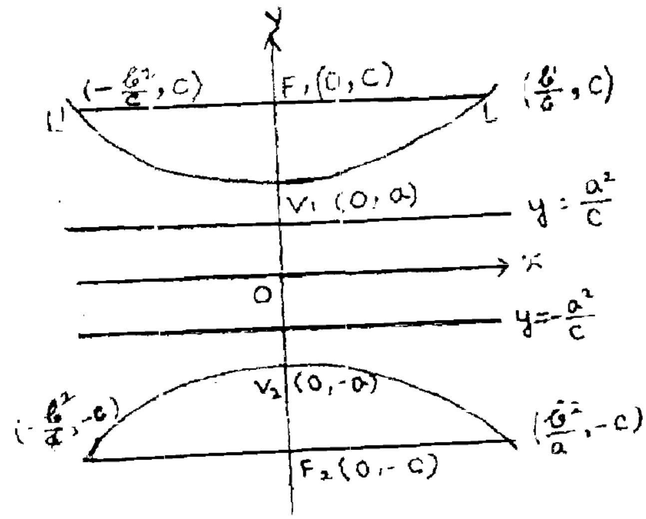
where $b^2 = c^2 - a^2$, which is the equation of the hyperbola. [In ΔPF_2F_1, $|PF_1 - PF_2| < F_1F_2 \Rightarrow 2a < 2c \Rightarrow c^2 - a^2 > 0$ so we can write $b^2 = c^2 - a^2$]



If the hyperbola has its centre at (0,0) and foci on y - axis at $(0,\pm c)$, then proceeding as above the equation can be obtained as

$$\boxed{\frac{y^2}{a^2}-\frac{x^2}{b^2}=1} \quad (2)$$

where $b^2 = c^2 - a^2$.



Now, consider the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2} = 1.$$

On solving, we get $y=\pm\frac{b}{a}\sqrt{x^2-a^2}$ which shows that $y = 0$ for $x = \pm a$. If $|x| < a$ i.e. $-a < x < a$, then y becomes imaginary. Hence we see that there is no part of the hyperbola between the lines $x = a$ and $x = -a$.

The hyperbola extends to infinity on both the sides for $x > a$ and $x < -a$, which are called the **two branches** of the hyperbola. The points V_1 $(a, 0)$ and V_2 $(-a, 0)$ are called the **vertices**. (Fig. 42) of the hyperbola and the line segment $\overline{V_1V_2}$ is called the **transverse axis** of the hyperbola. The line segment joining the points B_1 $(0, b)$ and B_2 $(0, -b)$ is called the **conjugate axis** of the hyperbola. It is to be noted that the conjugate axis has no point common with the hyperbola.

The chords through the foci, perpendicular to the axis, are called the **latera recta** of the hyperbola. The ratio $e = \frac{c}{a}$ is called the **eccentricity** of the hyperbola, and $e > 1$, since $c > a$.

Again, $y = \pm\frac{b}{a}\sqrt{x^2 - a^2}$, gives us that for $x = \pm c$, $y = \pm\frac{b^2}{a}$ (as $c^2 - a^2 = b^2$) and hence the end points of the latera recta are $\left(\pm c, \pm\frac{b^2}{a}\right)$ and each latus rectum is of length $\frac{2b^2}{a}$.

Similar consideration can also be made for the hyperbola $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$.

Note : If $a = b$, the hyperbola is called a **rectangular hyperbola**.

Equation of a hyperbola with centre (h, k) and axis parallel to x - axis or y - axis.

Consider the hyperbola with centre at C (h, k) and let its axis be parallel to the x- axis. Let F_1 $(c + h, k)$ and F_2 $(-c + h, k)$ be the foci (Fig. 44). Then for any point P (x, y) on the hyperbola, we have

$$PF_1 - PF_2 = \pm 2a$$

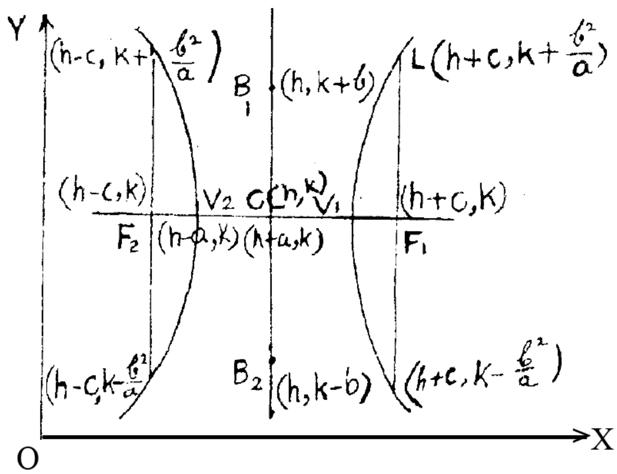
or, $\sqrt{(x-c-h)^2 + (y-k)^2}$

$= \sqrt{(x+c-h)^2 + (y-k)^2} \pm 2a$

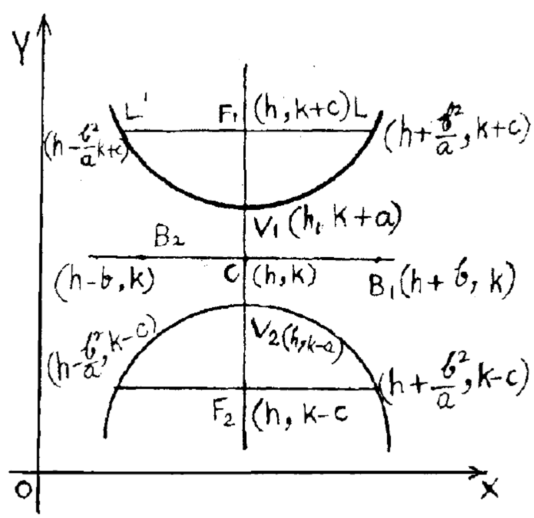
Now, squaring and simplifying, we finally obtain,

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1; \quad (1)$$

where $b^2 = c^2 - a^2$.



Instead of axis being parallel to x-axis, if it is parallel to y-axis, then the equation of hyperbola will be $\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1. \quad (2)$



The equations (1) and (2) are called the equations of hyperbola in **standard form**. The co-ordinates of the foci, the end points of latera recta, the vertices and the end points of conjugate axis are as shown in the accompanying figures on the RHS.

Equations of tangent and normal

Consider the hyperbola given by $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Let a line meet a branch of the hyperbola at P (x_1, y_1) and Q(x_2, y_2). Then equation of line is

$$(y - y_1) = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1). \quad (1)$$

Since P(x_1, y_1) and Q(x_2, y_2) are points on the hyperbola, we have $\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} = 1$ and $\frac{x_2^2}{a^2} - \frac{y_2^2}{b^2} = 1$.

Subtracting we get $\frac{x_1^2 - x_2^2}{a^2} = \frac{y_1^2 - y_2^2}{b^2}$

or, $\frac{y_1 - y_2}{x_1 - x_2} = \frac{b^2}{a^2}\left(\frac{x_1 + x_2}{y_1 + y_2}\right)$.

So equation (1) becomes $(y - y_1) = \frac{b^2}{a^2}\left(\frac{x_1 + x_2}{y_1 + y_2}\right)(x - x_1)$. (2)

$\overleftrightarrow{PQ}$, will be a tangent at P (x_1, y_1) if Q $\to$ P along the curve i.e., $y_2 \to y_1$ and $x_2 \to x_1$. Taking the limit, equation (2) becomes

$$y - y_1 = \frac{b^2}{a^2}\cdot\frac{x_1}{y_1}(x - x_1) \quad \text{or,} \quad \frac{xx_1}{a^2} - \frac{yy_1}{b^2} = \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} = 1$$

So the equation of tangent at (x_1, y_2) is,

$$\boxed{\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1} \quad (3)$$

From equation (3) it is seen that the slope of the tangent at (x_1, y_1) is $\frac{b^2x_1}{a^2y_1}$.Hence the slope of the normal at (x_1, y_1) will be $-\left(\frac{a^2y_1}{b^2x_1}\right)$ and so the equation of normal at (x_1, y_1) is $(y - y_1) = -\frac{a^2y_1}{b^2x_1}(x - x_1)$

or, $$\boxed{\frac{x_1y - x_1y_1}{a^2} = -\frac{(xy_1 - x_1y_1)}{b^2}} \quad (4)$$

N.B. Similarly, the equation of the tangent and the normal at a point (x_1, y_1) to the hyperbola $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$ are $\frac{yy_1}{a^2} - \frac{xx_1}{b^2} = 1$ (Tangent) and

$$\frac{x_1y - x_1y_1}{b^2} = -\frac{xy_1 - x_1y_1}{a^2} \quad \text{(Normal).}$$

Points of intersection of the line $y = mx + c$ with the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

The only vertical tangents are $x = \pm a$, which touch the curve at its vertices.

For a nonvertical line y = mx + c, the x-coordinates of the points of intersection are determined from $\frac{x^2}{a^2} - \frac{(mx+c)^2}{b^2} = 1$. This quardratic equation in x has discriminant $4a^2b^2(c^2 - a^2m^2 + b^2)$. So the given line intersects the hyperbola exactly at two points or has no point in common with it according as $c^2 > a^2m^2 - b^2$ or $c^2 < a^2m^2 - b^2$, respectively.

The condition for tangency turns out to be

$$\boxed{c^2 = a^2m^2 - b^2}$$

Corollary : The equations $y = mx \pm \sqrt{a^2m^2 - b^2}$ always represent a pair of parallel tangents to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$;

provided $a^2m^2 - b^2 > 0$.

If $a^2m^2 - b^2 = 0$, then the lines $y = mx \pm \sqrt{a^2m^2 - b^2}$ coincide with $y = mx$; $m = \pm\frac{b}{a}$.

But if m = $\pm\frac{b}{a}$, then the line y = mx has no point in common with the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ as, in this case $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$.

Note : The hyperbola $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$ has no vertical tangent. It can be proved, as above, that the line y = mx + c intersects the curve at two points or has no common point with it according as $c^2 > a^2 - b^2m^2$ or $c^2 < a^2 - b^2m^2$ respectively. The condition of tangency is given by

$$\boxed{c^2 = a^2 - b^2m^2}$$

Thus, $y = mx \pm \sqrt{a^2 - b^2m^2}$ represent a pair of parallel tangents to the hyperbola : $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$ provided $a^2 > b^2m^2$.

Focus - directrix property

Consider the hyperbola with centre at (0, 0) and foci at $(\pm c, 0)$. Then for any point $P(x, y)$ on the hyperbola

$PF_1 - PF_2 = \pm 2a$,

where $F_1(c, 0)$ and $F_2(-c, 0)$ are the two foci. This gives $PF_1 = PF_2 \pm 2a$

or, $PF_1^2 = PF_2^2 + 4a^2 \pm 4a\, PF_2$

or, $(x-c)^2 + y^2 = (x+c)^2 + y^2 + 4a^2 \pm 4a\sqrt{(x+c)^2+y^2}$

or, $\pm 4a\sqrt{(x+c)^2+y^2} = 4a^2 + 4cx = 4(a^2 + cx)$

or, $\sqrt{(x+c)^2+y^2} = \frac{c}{a}\left|x+\frac{a^2}{c}\right|$ (Taking the absolute values)

which shows that the distance of any point P(x, y) on the hyperbola from the point ($-c$, 0) bears a constant ratio, grater than 1, to its distance from the line $x = -\frac{a^2}{c}$. In the same way, starting from $PF_2 = PF_1 \pm 2a$ we get $\sqrt{(x-c)^2+y^2} = \frac{c}{a}\left|x-\frac{a^2}{c}\right|$ i.e. the distance of any point on the curve from (c, 0) bears a constant ratio (> 1) to its distance from the line $x = \frac{a^2}{c}$.

Therefore a hyperbola may be alternatively defined as a locus such that the distance of every point on it from a given point is in a constant ratio, greater than 1, to its distance from a given line.

The line $\boldsymbol{x = \pm \frac{a^2}{c}}$ are the **directrices** of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Similarly it can be shown that the directrices of the hyperbola $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$are $\boldsymbol{y = \pm \frac{a^2}{c}}$.

Parametric form of equations of hyperbola

1. In parametric form, the equation of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ can be written as

   $x = a \sec\phi$, $y = b \tan\phi$;

   where ϕ is the parameter and $\phi \in [0, 2\pi) - \left\{\frac{\pi}{2}, \frac{3\pi}{2}\right\}$.

2. In parametric form, the equations of the hyperbola $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$ can be written as

   $x = b \cot\phi$, $y = a \operatorname{cosec}\phi$;
   where ϕ is the parameter and $\phi \in (0, 2\pi) - \{\pi\}$

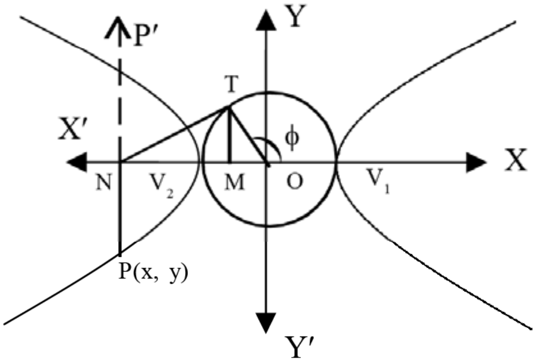
Derivation of the parametric equations :

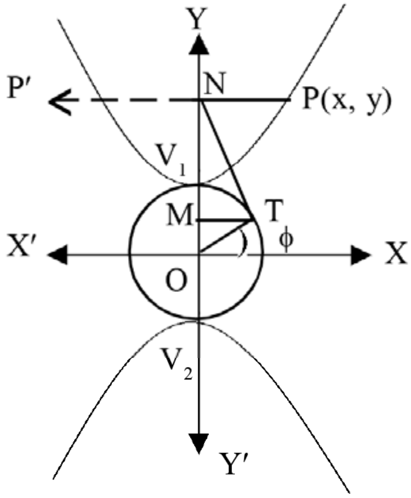
The following geometric considerations reveal how we arrive at these parametric equations.

The circle having the transverse axis as a diameter is called the **auxiliary circle** of a hyperbola.

Let P (x, y) be a point on any one of the hyperbolas $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

or $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1.$





Let the line through P, perpendicular to the axis of the hyperbola, meet it at N. Let the line through N, tangent to the auxiliary circle, touch it at T in such a way that

(i) in case of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, P and T (if different) lie in same or different quadrants according as P(x, y) is to the right or left of y - axis respectively. [We say, a point P(x, y) is to right of y - axis if $x > 0$ and to left if $x < 0$]

(ii) In case of the hyperbola $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$, P and T (if different) lie in same or different quadrants according as P(x, y) is above or below x – axis respectively. [We say, a point P(x, y) is above x - axis of $y > 0$ and below if $y < 0$].

Also observe that P and T always lie in the same side of y-axis in case of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and P and T always lie in the same side of x - axis in case of the hyperbola $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1.$

Now, let the line through T, perpendicular to the axis of the hyperbola meets the axis at M. (Description applicable to both the hyperbolas)

The centre of the auxiliary circle is (0, 0) and its radius is a. Let T be (x_1, y_1) in cartesian coordinates and (a, ϕ) in polar co-ordinates, the polar co-ordinates being referred to (0,0) as pole and $\overrightarrow{OX}$ as the initial line. Then, we have

$x_1 = a \cos\phi,\ y_1 = a \sin\phi$.

We now proceed to obtain the point P(x, y) in terms of a, b and ϕ for each of the hyperbolas.

Soppose P(x, y) is on $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. We observe that if P coincides with a vertex, then the points N, T and M also coincide with that vertex so that OM = ON = a.

$\therefore$ OM . ON = a^2 or $|x_1| . |x| = a^2$.

But if P does not coincide with a vertex, then Δ OTN is right – angled at T.

So $\overline{TM} \perp \overline{ON}$ $\Rightarrow$ OM . ON = OT^2 i.e. $|x_1| . |x| = a^2$.

Thus, irrespective of the position of P on the hyperbola, we have $|x||x_1| = a^2$. Since T and P lie on the same side of y - axis, their x - co-ordinates i.e. x and x_1 have the same sign. So $xx_1 > 0$. (Neither of x, y can be Zero. Why ?)

$\therefore xx_1 = |xx_1| = |x|\ |x_1| = a^2 \Rightarrow x = \frac{a^2}{x_1} = \frac{a^2}{a\cos\phi} = a\sec\phi$.

But we have, $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

$\therefore y = \pm b\sqrt{\frac{x^2}{a^2} - 1} = \pm b\tan\phi$.

$y = -b\tan\phi$ corresponds to the y – co-ordinate of P^1, the point of intersection of $\overrightarrow{PN}$ and the hyperbola. So we take $y = b\tan\phi$.

Thus we get x = a sec ϕ, y = b tan ϕ.

Depending on the positions of T in different quadrants or on the x - axis (i.e. at V_1 or V_2). ϕ takes its values in $[0, 2\pi) - \left\{\frac{\pi}{2}, \frac{3\pi}{2}\right\}$. ϕ cannot be either $\frac{\pi}{2}$ or $\frac{3\pi}{2}$ since T can never be on y - axis.

Similarly the parametric equations of $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$ can be derived with the restriction on values of ϕ as stated earlier. The only difference is that, in this case, OM. ON = $a^2 \Rightarrow |y| . |y_1| = a^2$.

Rectangular hyperbolas :

If $a = b$ in the equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, the equation becomes $x^2 - y^2 = a^2$ which is called a rectangular or an equilateral hyperbola.

For any hyperbola we have $b^2 = c^2 - a^2$

$$= a^2\left(\frac{c^2}{a^2} - 1\right) = a^2(e^2 - 1). \quad \left(\because c = \frac{c}{a}\right)$$

$\therefore$ For a rectangular hyperbola, $a = b \Rightarrow a^2 = a^2(e^2 - 1)$

$\Rightarrow 1 = e^2 - 1$

$\therefore\ e^2 = 2$.

So $e = \sqrt{2}$. ($\because$ e > 0)

Thus the eccentricity of any rectangular hyperbola is $\sqrt{2}$.

Conjugate hyperbolas :

Definition :

If transverse and conjugate axes of one are respectively the conjuagate and transverse axes of the other, then the hyperbolas, so related, are said to be conjugate to each other.

The hyperbola $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$ is conjugate to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Let e_1 and e_2 be the eccentricities of the hyperboals $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$ respectively.

Then $b^2 = a^2(e_1^2 - 1)$ ....(i)

and $a^2 = b^2 (e_2^2 - 1)$ ....(ii)

From (i) and (ii), we get

$$e_1^2 - 1 = \frac{1}{e_2^2 - 1}.$$

This implies that $(e_1^2 - 1)(e_2^2 - 1) = 1$

$\Rightarrow e_1^2 + e_2^2 = e_1^2 e_2^2$

$$\Rightarrow \frac{1}{e_1^2} + \frac{1}{e_2^2} = 1.$$

SOLVED EXAMPLES

Example 1 :

Obtain the equation of the hyperbola with eccentricity $\frac{3}{2}$ and foci at $(\pm 2, 0)$.

Solution :

Since the foci are at $(\pm 2, 0)$, the centre of the hyperbola is at $(0, 0)$ and the transverse axis is along the x– axis. Hence the equation of the hyperbola is, $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

The focii of the hyperbola is at $(\pm 2, 0)$, gives $c = 2$. Now $e = \frac{c}{a} = \frac{2}{a}$ gives $\frac{2}{a} = \frac{3}{2}$, so $a = \frac{4}{3}$.

Hence, $b^2 = c^2 - a^2 = 4 - \frac{16}{9} = \frac{20}{9}$.

$\therefore$ The equation of the hyperbola is $\frac{x^2}{\frac{16}{9}} - \frac{y^2}{\frac{20}{9}} = 1$

or, $\frac{9x^2}{16} - \frac{9y^2}{20} = 1$, or $45x^2 - 36y^2 = 80$.

Example 2 :

Find the co-ordinates of the vertices and ends of latera recta of the hyperbola with foci at (0, ± 4) and $y = \frac{3}{2}$ as one of the directrices.

Solution :

Since the foci are on the y - axis at (0, ± 4) the equation of the hyperbola is $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$.

The equation of the directrices are therefore $y = \pm \frac{a^2}{c}$.

It is given that $y = \frac{3}{2}$ is one of the directirces.

Hence, $\frac{a^2}{c} = \frac{3}{2}$. Since the foci are at (0, ± 4). We have $c = 4$, and therefore, $\frac{a^2}{4} = \frac{3}{2}$ or $a^2 = 6$.

Now, $b^2 = c^2 - a^2 = 16 - 6 = 10$. So the equation of the hyperbola is, $\frac{y^2}{6} - \frac{x^2}{10} = 1$.

The co-ordinates of the vertices are $(0, \pm a) = (0, \pm \sqrt{6})$.

The ends of latera recta have co-ordinates = $\left(\pm \frac{b^2}{a}, \pm c\right) = \left(\pm \frac{10}{\sqrt{6}}, \pm 4\right)$.

Example 3 :

Reducing to standard from, find the co-ordinates of the centre, the foci, the vertices and the equation of directrices of the hyperbola.

$9x^2 - 4y^2 - 36x + 16y - 16 = 0$.

Solution :

The given equation can be written as, $9(x^2 - 4x + 4) - 4(y^2 - 4y + 4) = 36$

or, $\frac{(x-2)^2}{4} - \frac{(y-2)^2}{9} = 1$, which is a hyperbola with centre (2, 2). Again $a^2 = 4$ and $b^2 = 9$ gives $c^2 = a^2 + b^2 = 13$; $c = \pm \sqrt{13}$.

Hence the co-ordinates of the foci are the points $(2 \pm \sqrt{13}, 2)$ and the co-ordinates of the vertices are (2, ± 2, 2) i.e. (4, 2) and (0, 2).

The equation of the directrices are $x = h \pm \frac{a^2}{c} = 2 \pm \frac{4}{\sqrt{13}}$.

Example 4 :

Find the equation of the tangent to the hyperbola $2x^2 - 3y^2 = 1$, which is parallel to the line $3x - y + 1 = 0$.

Solution :

The equation of a line parallel to the line $3x - y + 1 = 0$ is $3x - y + c = 0$ or, $y = 3x + c$.

Again, the line $y = mx + c$ is a tangent to the hyperbola, $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

if $c^2 = a^2m^2 - b^2$.

Hence, $y = 3x + c$, will be a tangent to the hyperbola

$2x^2 - 3y^2 = 1$ or, $\frac{x^2}{\frac{1}{2}} - \frac{y^2}{\frac{1}{3}} = 1$,

if, $c^2 = \left(\frac{1}{2}\right) .3^2 - \frac{1}{3} = \frac{9}{2} - \frac{1}{3} = \frac{25}{6}$ or, $c = \pm\frac{5}{\sqrt{6}}$.

Hence, the equation of tangent is $y = 3x \pm\frac{5}{\sqrt{6}}$.

EXERCISES 12 (b)

1. Fill in the blanks by choosing the correct answer from the given ones :

(a) The equation of the directrix to the parabola $x^2 = -6y$ is ——.
[$y + 6 = 0$, $2y - 3 = 0$, $y - 6 = 0$, $2y + 3 = 0$]

(b) The eccentricity of the parabola $y^2 = 8x$ is ——. [2, 8, 0, 1]

(c) The line $y + x = k$ is a tangent to the parabola $y^2 + 12x = 0$ if $k =$ —— (–3, 3, 6, –6)

(d) The latus rectum of the parabola $(y - 2)^2 = 8(x + 3)$ is ——. (2, 4, 8, 16)

(e) The equation of tangent to the parabola $x^2 = 6y$ at its vertex is ——.
($x = 0$, $y = 0$, $x = \frac{-3}{2}$, $y = \frac{-3}{2}$)

(f) The equation of the axis of the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ is ——.($x = 4$, $y = 3$, $x = 0$, $y = 0$)

(g) The equation of the major axis of the ellipse $\frac{(x+1)^2}{16} + \frac{(y-2)^2}{25} = 1$ is ——.
($x = 4$, $x = -1$, $y = 5$, $y = 2$)

(h) The distance between the foci of the ellipse $3x^2 + 4y^2 = 1$ is ——. $\left(1, \frac{1}{\sqrt{3}}, \frac{2}{\sqrt{3}}, \frac{1}{2\sqrt{3}}\right)$

(i) The eccentricity of the ellipse $\frac{x^2}{16} + \frac{y^2}{25} = 1$ is ——. $\left(\frac{4}{5}, \frac{5}{4}, \frac{3}{5}, \frac{16}{25}\right)$

(j) The line $y = 2x + k$ is a tangent to the ellipse $5x^2 + y^2 = 5$ if k = ——. $(2, 5, 3, \sqrt{21})$

(k) The length of latus rectum of the ellipse $\frac{(x-2)^2}{4} + \frac{(y+3)^2}{25} = 1$ is ——. $\left(\frac{4}{25}, \frac{2}{5}, \frac{5}{2}, \frac{8}{5}\right)$

(l) The equation of the conjugate axis of the hyperbola $\frac{x^2}{9} - \frac{(y+2)^2}{16} = 1$ is ——.
$(x = 0, x = 3, y = -3, y = 4)$

(m) The hyperbola $\frac{y^2}{16} - \frac{x^2}{12} = 1$ interesects x-axis at ——. $[(0, \pm 4), (\pm 2\sqrt{3}, 0), (2, 0)$, no where$]$

(n) The eccentricity of the hyperbola $4x^2 - 3y^2 = 1$ is ——. $\left(\frac{4}{3}, \frac{3}{4}, \frac{\sqrt{21}}{3}, \frac{\sqrt{7}}{2\sqrt{3}}\right)$

(o) The latus rectum of the hyperbola $\frac{x^2}{9} - \frac{y^2}{16} = 1$ is ——. $\left(\frac{16}{9}, \frac{9}{16}, \frac{1}{9}, \frac{32}{9}\right)$

(p) The line $y = 3x - k$ is a tangent to the hyperbola $6x^2 - 9y^2 = 1$ if k = ——.
$\left(1, \frac{5}{3\sqrt{2}}, \frac{1}{\sqrt{6}}, \frac{2}{3}\right)$

2. Mention which of the following statements are *true* (T) or *false* (F) :

(a) The equation $y = x^2 + 2x + 3$ represents a parabola with its axis parallel to y-axis.

(b) The latus rectum of the parabola $y^2 = -8x$ is 2.

(c) The eccentricity of the parabola $(y-1)^2 = 4(x+3)$ is $\frac{1}{3}$.

(d) The line $y = 3$ is a tangent to the parabola $(x+2)^2 = 6(y-3)$.

(e) The equation $Ax^2 + By^2 = 1$ represents an ellipse with its axis parallel to x-axis If $A > B > 0$.

(f) The foci of the ellipse $\frac{x^2}{3} + \frac{y^2}{2} = 1$ are the points $(\pm 1, 0)$.

(g) The equation of the ellipse with foci at $(0, \pm 4)$ and vertices $(0, \pm 7)$ is $\frac{x^2}{16} + \frac{y^2}{49} = 1$.

(h) The length of the latera recta of the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ and $\frac{(x+2)^2}{4} + \frac{(y-1)^2}{9} = 1$ are equal.

(i) The equation of the latera recta of the ellipse $\frac{(x-4)^2}{16} + \frac{(y-1)^2}{9}$ are $x = 4 \pm \sqrt{7}$.

(j) The line $y = x + 2$ is a tangent to the ellipse $\frac{x^2}{2} + \frac{y^2}{1} = 1$.

(k) The conjugate axis of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ meets teh hyperbola at two points which are at a distance $2b$ from each other.

(l) The conjugate axis of the hyperbola $\frac{(y-3)^2}{9} - \frac{(x+2)^2}{3} = 1$ is parallel to the line $x = 4$.

(m) The length of the transverse axis of the hyperbola with foci at $(\pm 5, 0)$ and vertices at $(\pm 2, 0)$ is 10.

(n) The latera recta of the ellipse $\frac{x^2}{25} - \frac{y^2}{16} = 1$ are the same.

(o) The y-axis is tangent to the hyperbola $ay^2 - bx^2 = 1$.

3. Find the equation of the parabola in each of the following cases.

(a) the vertex at (0, 0) and focus at (0, 3),
(b) the vertex at (0, 0) and directrix $x - 2 = 0$,
(c) the vertex at (6, –2) and focus at (–3, –2),
(d) the vertex at (–2, 1) and focus at (–2, 4),
(e) the length of the latus rectum is 6 and the vertex is at (0, 0), the parabola opening to the right,
(f) the vertex is at (0, 0), the parabola opening to the left and passing through (–1, 2),
(g) the vertex at (0, 0) the parabola opens downwards and the latus rectum is of length 10,
(h) the axis is vertical and the parabola passes through the points (0, 2), (–1, 1), (2, 10),
(i) the axis is horizontal and the parabola passes through the points (2, –1), (–2, –4) and (–1, 3),
(j) vertex at (1, 3) and directrix $x + 3 = 0$,
(k) Vertex at (1, –1) and directrix $y - 2 = 0$,
(l) the focus at (–2, 3) and directrix $3x + 4y - 2 = 0$.

4. Find the equation of the ellipse in each of the following cases :

(a) centre at (0, 0), one vertex at (0, –5) and one end of the minor axis is (3, 0),
(b) centre at (0, 0), one vertex at (7, 0) and one end of the minor axis is (0, –5)
(c) foci at $(\pm 5, 0)$ and length of the major axis is 12,
(d) vertices at $(\pm 5, 0)$ and length of latus rectum is $\frac{8}{5}$,
(e) centre at (5, 4) and the major axis is of length 16 and the minor axis is of length 10.
(f) centre at (–3, 3), vertex at (–3, 6) and one end of minor axis at (0, 3).

(g) centre at (0, 0), axes parallel to co-ordinate axes, eccentricity is $\frac{1}{\sqrt{2}}$ and the minor axis is of length 5,

(h) centre at (0, 0) axis parallel to co-ordinate axes, eccentricity is $\frac{\sqrt{3}}{2}$ and the ellipse passing through the point $\left(\sqrt{3}, \frac{1}{2}\right)$,

(i) centre at (0, 0) one end of the major axis is (–5, 0) and eccentricity $\frac{3}{5}$,

(j) axis parallel to co-ordinate axes, the centre at (0, 0) and the ellipse passing through (3, –2) and (–1, 3),

(k) centre at (3, 4), axis parallel to x-axis and passing through (6, 4) and (3, 6),

(l) centre at (–2, 1), axis parallel to y-axis, eccentricity is $\frac{\sqrt{7}}{4}$ and the ellipse passing through (–2, 5).

5. Obtain the equation of the hperbola in each of the following cases :

(a) foci at (± 4, 0) and vertices (±2, 0),

(b) foci at (0, ± √2) and vertices (0, ±1),

(c) centre at (0, 0) transverse axis along x-axis of length 4, and focus at (2√5, 0)

(d) centre at (0, 0), conjugate axis along x-axis of length 6 and eccentricity 2,

(e) focii at (±2 √3, 0) and eccentricity √3,

(f) centre at (0, 0) transverse axis is along y-axis, the distance between the foci is 14 and distances between the vertices is 12,

(g) centre (1, –2), transverse axis parallel to x-axis of length of 6 and conjugate axis of length 10,

(h) centre at (2, –3), eccentricity $\frac{5}{3}$ and hyperbola passing through (5, –3),

(i) centre at origin, axis perpendicular to y - axis, the hyperbola passes through the points (3, –2) and (5, –7).

(j) The transverse axis parallel to y - axis, the hyperbola passes through the points $\left(\frac{11}{3}, 0\right)$, (1, 2) and its centre is the intersection of the lines $x + y - 6 = 0$, $4x - y + 1 = 0$

6. Reducing to standard from, obtain the co-ordinates of the vertex, focus, end points of the latus rectum, the length of latus rectum, the equation of axis and directrix of the following parabolas;

(a) $y^2 - 4x + 4y - 1 = 0$ (b) $2x^2 - 4y + 6x - 3 = 0$

(c) $x^2 + x + y + 1 = 0$ (d) $y^2 + 14y - 3x + 1 = 0$.

7. Reducing to standard form, obtain the co-ordinates of centre, the foci, the vertices, the end points of minor -axis, the end points of latera recta, the equation of the directrices and the eccentricity of the following ellipse :
   (a) $3x^2 + 4y^2 + 6x + 8y - 5 = 0$ (b) $4x^2 + 8y^2 + 4x + 24y - 13 = 0$
   (c) $2x^2 + 3y^2 - 12x + 24y + 60 = 0$ (d) $9x^2 + 4y^2 + 36x - 8y + 4 = 0$.

8. Reducing to standard form, obtain the co-ordinates of the centre, the verties, the foci, the end points of conjugate axis, the end points of latera recta, the equation of directrices and the eccentricity of the following hyperbolas :
   (a) $x^2 - 2y^2 - 6x - 4y + 5 = 0$ (b) $9y^2 - 4x^2 - 90y + 189 = 0$
   (c) $49x^2 - 4y^2 - 98x + 48y - 291 = 0$ (d) $3x^2 - 2y^2 - 4y - 26 = 0$.

9. Prove that the equation of the parabola whose vertex and focus are at distances α and β from origin on x-axis respectively is $y^2 = 4(\beta - \alpha)(x - \alpha)$.

10. Find the locus of the points of trisection of a double ordinate of the parabola $y^2 = 4ax$.

11. (a) Prove that a double ordinate of the parabola $y^2 = 4ax$ of length $8a$ subtends a right angle at its vertex.
    (b) Find the angle which a double ordinate of length $2a$ subtends at its vertex and focus.

12. (a) Obtain the equations of the tangent and normal of the parabola $y^2 = 4ax$ at a point where the ordinate is equal to three times the abscissa.
    (b) Find the equation of tangents and normals to the parabola $y^2 = 4ax$ at the ends of its latus rectum.
    (c) Find the equations of tangents and normals to the parabola $y^2 = 4ax$ at the points where it is cut by the line $y = 3x - a$.
    (d) Show that the tangent to the parabola $y^2 = 4ax$ at the point (a^1, b^1) is perpendicular to the tangent at the point $\left(\frac{a^2}{a^1}, \frac{-4a^2}{b^1}\right)$.
    (e) A tangent to the parabola $y^2 = 8x$ makes an angle 45° with the line $3x - y + 5 = 0$. Find the equation and the point of contact.
    (f) Prove that for all values of k, the line $y = k(x + a) + \frac{a}{k}$ is a tangent to the parabola $y^2 = 4a(x + a)$.
    (g) Obtain the condition that the line $lx + my + n = 0$ will touch the parabola $y^2 = 4ax$.
    (h) Prove that the line $4x - 2y - 1 = 0$ touches the parabola whose focus it at (0, 0) and directrix is the line $y = 2x - 1$.

13. (a) If (–2, 0) and (2, 0) are the two vertices of a triangle with perimeter 16, then obtain the locus of the third vertex.
    (b) A point in a plane is such that the sum of its distances from the point (2, 2) and (6, 2) is 12. Find the locus of the point.
    (c) Obtain the equation of the ellipse which has its centre at origin, a focus at (2, 0) and the corresponding directrix is the line $2x = 7$. Calculate the length of the latus rectum.

(d) Find the equation of the ellipse which has its centre at (–1, 4), eccentricity $\frac{1}{\sqrt{3}}$ and the ellipse passes through the point (3, 2).

14. (a) Find the equation of tangent and normal to the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$ at the point $\left(\frac{8}{3},\sqrt{5}\right)$.

(b) Find the equations of tangents and normals to the ellipse $2x^2 + 3y^2 = 6$, at the end points of the latera recta.

(c) Prove that the line $y = 2x + 5$ is a tangent to the ellipse $9x^2 + 4y^2 = 36$ and find the point of contact.

(d) Find the equation of the tangents to the ellipse $4x^2 + 5y^2 = 20$ which are parallel to the line $x - y = 2$.

(e) Find the equation of the tangent to the ellipse $4x^2 + 9y^2 = 1$, which are perpendicular to $2ax + y - 1 = 0$.

(f) Prove that the line $x \cos \alpha + y \sin \alpha = p$ touches the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. if $p^2 = a^2 \cos^2 \alpha + b^2 \sin^2\alpha$.

(g) Prove that the product of the distances of the foci from any tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is equal to b^2.

15. (a) Find the equation of the hyperbola which has its foci at (0, 0) and (0, 4) and which passes through the point (12, 9).

(b) Find the equation of the hyperbola with foci at (±3, 0) and directrices $x = \pm 2$.

(c) Find the foci and latus rectum of the hyperbola whose transverse and conjugate axes are 6 and 4 and centre is at (0, 0).

16. (a) Find the equation of tangent and normal to the hyperbola $x^2 - 6y^2 = 3$ at the point (–3, –1).

(b) Find the equations of the tangent to the hyperbola $4x^2 - 11y^2 = 1$ which are parallel to the straight line $20x - 33y = 13$.

(c) Find the equation of tangents to hyperbola $9x^2 - 16y^2 = 144$ which are perpendicular to the line $2x + 3y = 4$.

(d) Prove thhat the line $x + y + 2 = 0$ touches the hyperbola $3x^2 - 5y^2 = 30$ and find the point of contact. Find also the equation of normal at that point.

(e) Prove that the line $x \cos \alpha + y \sin \alpha = p$ touches the hperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ if $p^2 = a^2 \cos^2 \alpha - b^2 \sin^2\alpha$.

●

CHAPTER 13

Introduction to Three-dimensional Geometry

There is no branch of mathematics, however abstract, which may not some day be applied to phenomena of the real world.

- Lobachevsky

13.1 SPACE AND ITS DIMENSION :

In plane geometry we have taken point, line and plane as undefined terms. We have axioms introducing line and plane as sets of points. Our geometrical universe, hitherto, has been a plane. We have discussed the geometry of lines, circles, parabolas, ellipses and hyperbolas. All of these are subsets of a plane.

In highschool-mensuration you have studied cuboids, cubes, spheres, cylinders and cones under application of geometry. None of these is a subset of a plane. To bring them into the theoretical fold of geometry we require a geometrical universe that would contain points not all of which are coplanar (lie on a plane).

For this purpose we bring another undefined term, '**Space**' into our axiom-system.

Axiom - 1 : A space is a nonempty set of points.

Axiom - 2 : If P and Q are distinct points in space S, then $\overleftrightarrow{PQ} \subset S$ and if π is a plane containing P and Q, then $\overleftrightarrow{PQ} \subset \pi \subset S$.

Axiom - 3 : Given any three noncollinear points P, Q, R in space S, there is exactly one plane π such that $\{P, Q, R\} \subset \pi \subset S$

Axiom - 4 : If π_1 and π_2 are two distinct planes in a space and P is a point such that $P \in \pi_1 \cap \pi_2$, then there exists another point Q different from P such that $Q \in \pi_1 \cap \pi_2$.

Definition (convex set) : A subset A of a space is said to be a convex set if, for all P, Q $\in$ A, $\overline{PQ} \subset A$.

Axiom - 5 : If π is a plane in space S then the points of S not contained in π are divided into two disjoint nonempty convex sets S_1 and S_2 such that

$P \in S_1, Q \in S_2 \Rightarrow \overline{PQ} \cap \pi \neq \phi$.

(S_1 and S_2 are called the two sides of the plane).

Note : If a point belongs to a space we also say that the point lies on/in that space.

Dimension of a space :

Definition : A space is said to be of dimension zero, one, two or three according as it comprises of a single point, line, plane or contains points not all of which are coplanar.

Note : Unless stated otherwise, the term, 'Space' shall mean a three dimensional space.

13.2 Lines and planes in space :

Position of a line in relation to a plane :

Definitions :

1. A line L is said to lie in/on a plane π if $L \subset \pi$.

2. A line L is said to intersect a plane π if it does not lie on π and has a point in common with the plane i.e. $L \not\subset \pi$ and $L \cap \pi \neq \phi$.

N.B. : It can be proved that $L \cap \pi$ contains exactly one point if $L \cap \pi \neq \phi$ and $L \not\subset \pi$ i.e. if a line intersects a plane, it does so at exactly one point. For, if P and Q are distinct points in $L \cap \pi$, then clearly $L = \overleftrightarrow{PQ}$

$\subset \pi$, by axiom-2. Therefore L lies on π, which is impossible by the definition of intersection. The definition demands that the line must not lie on the plane.

3. A line L, not lying on a plane π, is parallel to it, written as $L \| \pi$, if it has no point in common with the plane i.e. $L \cap \pi = \phi$.

Notes :

(i) If follows from the definitions of intersection and parallelism that a line must either intersect a plane or be parallel to it if it does not line on the plane.

(ii) If a point P or a line L lies on a plane, it is also said that the plane passes through P or L.

Lines in Space

1. Intersecting lines : Lines L_1 and L_2 intersect each other if they are distinct (not coincident) and $L_1 \cap L_2 \neq \phi$.

N.B. : It is proved in plane geometry (by taking, 'there is exactly one line containing two distinct points' as an axiom) that there is just one point common to two intersecting lines.

2. Parallel lines : Distinct lines L_1 and L_2 are called parallel if they are coplanar and have no point in common.

3. Skew lines : A pair of non-coplanar lines are called skew.

Notes :

(i) It follows from axioms - 2 and 3 that there is exactly one plane passing through two intersecting lines.

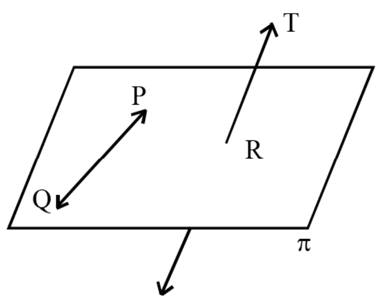
(ii) We shall explain later that there is exactly one plane passing through two parallel lines.

(iii) An example of skew lines :

There is no concept of skew lines in plane-geometry as, the universe being a single plane, we cannot talk of noncoplanarity.

Skew lines occur only in a three dimensional space.

Consider noncollinear points P, Q, R in a three dimensional space. By axiom-3 there is only one plane, say π, passing through P, Q and R. [The picture shows only a fragment of this plane] Since the space is three dimensional, we can find another point (in fact, infinitely many), say T which does not lie on the plane π.



$\overleftrightarrow{PQ}$ and $\overleftrightarrow{RT}$ are noncoplanar, hence skew.

Planes in Space :

Parallel planes : Distinct planes π_1 and π_2 are called parallel, written as $\pi_1 \| \pi_2$, if they have no point in common i.e. $\pi_1 \cap \pi_2 = \phi$.

Intersecting planes : Planes π_1 and π_2 are said to be intersecting if they are distinct and $\pi_1 \cap \pi_2 \neq \phi$.

Let us have a closer look at this definition. Since $\pi_1 \cap \pi_2 \neq \phi$, there is a point P in the space such that $P \in \pi_1 \cap \pi_2$. By axiom-4, there exists another point Q, different from P such that $Q \in \pi_1 \cap \pi_2$. Hence by axiom-2, $\overleftrightarrow{PQ} \subset \pi_1 \cap \pi_2$. Further, no point other than those on $\overleftrightarrow{PQ}$ can belong to $\pi_1 \cap \pi_2$. For if $R \notin \overleftrightarrow{PQ}$ and $R \in \pi_1 \cap \pi_2$, we get three noncollinear points P, Q, R such that they lie in π_1 as well as π_2. So by axiom-3, π_1 and π_2 must be one and the same plane, which is impossible as π_1 and π_2 are supposed to be distinct planes. Thus it follows that *if two planes intersect, they must intersect along exactly one line.*

In other words, there cannot be more than one lines common to two different planes.

Note : *There is exactly one plane containing two distinct parallel lines,* for otherwise, there shall be two lines common to more than one number of planes.

Perpendicular (Normal) to a plane :

Definition : A line intersecting a plane at a point P is said to be perpendicular to the plane at P if it is perpendicular to every line lying on the plane which passes through P.

13.3 Properties of lines and planes in space

The following facts regarding properties of lines and planes in space form the basis of further discussions. These can be proved by application of elementary methods of plane-geometry with the help of the axioms already stated in 15.10.

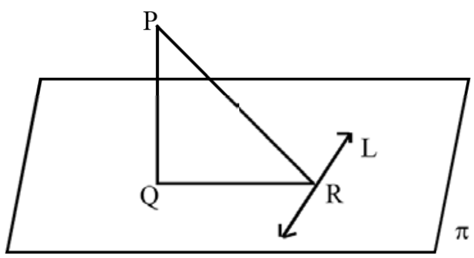
Fact-1 : If a line L is perpendicular to two intersecting lines L_1 and L_2 at their point of intersection, say P, then L is perpendicular to the plane of L_1 and L_2 at P.

P is called the foot of the perpendicular upon the plane.

Fact - 2 : There is exactly one line perpendicular to a plane at a given point on it. Also there is exactly one line perpendicular to a plane from an external point.

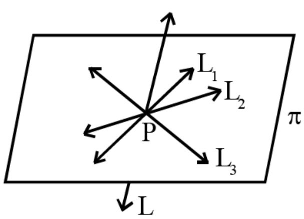
Fact - 3 : (Three perpendiculars)

$\overleftrightarrow{PQ}$ is perpendicular to a plane π at a point Q on it. L is a line on plane π and it does not pass through Q. If $\overleftrightarrow{QR}$ is perpendicular to L at R, then $\overleftrightarrow{PR}$ is also perpendicular to L.



In plane-geometry it follows from the angle-construction axiom that there is exactly one perpendicular to a line at a given point on it. But in a three dimensional space, as you have just seen, there can be several perpendiculars to a line at a given point on it. However, at a given point there is just one plane perpendicular to a line.

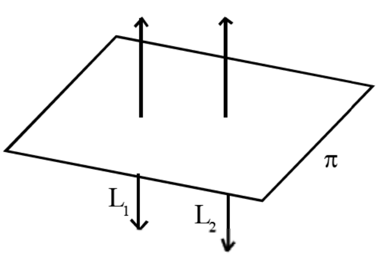
Fact - 4 : The perpendiculars in space to a line at a given point on it lie on exactly one plane which is perpendicular to the line at that point. (If a line is perpendicular to a plane, the plane is also called the perpendicualr-plane to the line)



(The perpendiculars L_1, L_2, L_3 etc. to the line L at P lie on the plane π.)

Fact - 5 : Any two perpendiculars to a plane are parallel.

Fact - 6 : Of two parallel lines, if one is perpendicular to a plane, then the other is also perpendicular to the same plane.



($L_1 \parallel L_2$, $L_1 \perp \pi$ $\Rightarrow L_2 \perp \pi$)

Set of parallel lines in space : Definition :

A set of lines in space are called parallel if any two of them are parallel.

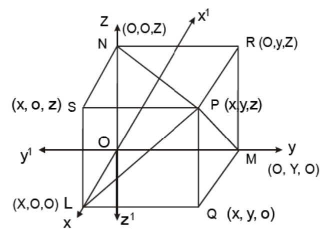
Common Transversal : If a line L intersects a set of lines $\{L_1, L_2, L_3, \ldots\ldots\ldots L_n\}$ then L is called a common transversal of $L_1, L_2, L_3, \ldots, L_n$.

Fact - 7 : A set of parallel or concurrent lines in space, having a common transversal, are coplanar.

Fact - 8 : If L_1, L_2 and L_1', L_2' are intersecting pairs of lines in space such that $L_1 \parallel L_1'$ and $L_2 \parallel L_2'$, then the angle between L_1, L_2 and L_1', L_2' are of equal measure.

13.4 : Rectangular Cartesian Coordinates in space :

Let $\overleftrightarrow{X'X}$, and $\overleftrightarrow{Z'Z}$ be mutually perpendicular lines in space intersecting one another at the point O. The point O is called **origin** and the lines $\overleftrightarrow{X'X}$, $\overleftrightarrow{Y'Y}$ and $\overleftrightarrow{Z'Z}$ are named respectively as X, Y and Z-axes. These lines are also known as ***coordinate axes***. The planes containing the pairs of intersecting lines ($\overleftrightarrow{X'X}$, $\overleftrightarrow{Y'Y}$), ($\overleftrightarrow{Y'Y}$, $\overleftrightarrow{Z'Z}$) and ($\overleftrightarrow{Z'Z}$, $\overleftrightarrow{X'X}$) are called ***coordinate planes*** named respectively as XY, YZ and ZX-planes. The X, Y and Z-axes are respectively perpendicular to the YZ, ZX and XY-planes. (Fact-1). The system of axes thus obtained is known as ***rectangular*** or ***othrogonal*** system of coordinate axes.



Let P be a point in space. There can be exactly one line through P perpendicular to the XY-plane. Let this perpendicular intersect the XY-plane at Q. Let the lines through Q perpendicular to the X and Y - axes intersect them respectively at L and M. Let the plane through P perpendicular to Z-axis intersect it at N.

We now associate an ordered triple of real numbers (x, y, z) with the point P by the following definition.

Definition : The point P-has coordinates (x, y, z)

where $x = \begin{cases} OL, \text{if } L \in \overrightarrow{OX} \\ -OL, \text{if } L \in \overrightarrow{OX'} \end{cases}$, $y = \begin{cases} OM, \text{if } M \in \overrightarrow{OY} \\ -OM, \text{if } M \in \overrightarrow{OY'} \end{cases}$

and $z = \begin{cases} ON, \text{if } N \in \overrightarrow{OZ} \\ -ON, \text{if } N \in \overrightarrow{OZ'} \end{cases}$

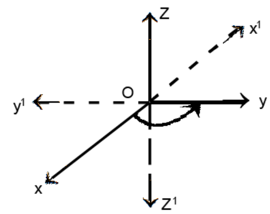
We choose the same scale to measure distances on the coordinate axes. The real numbers x, y, z thus associated with P, are respectively known as the x, y and z-coordinates of P.

Notes :

(i) It is obvious from the definition that the origin has coordinates (0, 0, 0).

(ii) $\overrightarrow{OX}$, $\overrightarrow{OY}$ and $\overrightarrow{OZ}$ are called the nonnegative X, Y and Z-axes and $\overrightarrow{OX'}$, $\overrightarrow{OY'}$ and $\overrightarrow{OZ'}$, nonpositive X, Y and Z-axes respectively. $\overrightarrow{OX}$, $\overrightarrow{OY}$ and $\overrightarrow{OZ}$ are also known as the positive directions and $\overrightarrow{OX'}$, $\overrightarrow{OY'}$ and $\overrightarrow{OZ'}$ are known as the negative directions of the X, Y and Z - axes.

(iii) The coordinate axes as shown in the figure constitute a **right handed system**. As an informal aid to understanding, it may be observed that if a right handed screw placed at origin is turned as shown in figure 15.6, it will move along $\overrightarrow{OZ}$.



Observe that if the points in any one of the pairs (X, X′), (Y, Y′), (Z, Z′) are reversed in position or any one pair of axes are interchanged, we get a system which is known as a **left handed system**.

(iv) The above system of coordinates is known as the rectangular cartesian coordinate system, named after the French mathematician Rene Descartes (1596-1650). By taking oblique axes an oblique coordinate system can also be obtained. There are also certain other types of coordinate systems such as the cylindrical and the spherical coordinates. But we shall stick to the rectangular Cartesian system.

One-one correspondence between a 3-dimensional space and $R^3 = \{(x, y, z)|x, y, z \in R\}$.

It can be argued from axioms - 2 and 3 regarding space and the axioms of plane geometry that a three dimensional space is a continuum of points which intuitively means that any point we can think of, belongs to this space. There is no gap in this space. For this reason a three dimensional space is also called a solid space and the geometry concerned is known as **solid geometry**.

From the uniqueness of perpendicular to a plane (either from an external point or at a point on it), it follows that the real numbers x, y, z associated with a point are unique and conversely, given any ordered triple of real numbers (x, y, z) we can get a unique point in space.

Thus the points in a three-dimensional space are in 1-1 correspondence with the set $R^3 = \{(x, y, z)|x, y, z \in R\}$. So we identify a point in space as an ordered triple of real numbers. The geometry that is based on identification of a three dimensional space with R^3 is known as **coordinate/analytical geometry in three dimensions or analytical solid geometry.**

Alternative ways of determining space coordinates :

1. See figure 15.5. It follows from fact-3 (three perpendiculars) that $\overline{PL}$ and $\overline{PM}$ are respectively perpendiculars from P on X and Y-axes. So each of the coordinates x, y, z of P can be regarded as the 'signed' or 'directed' distances from origin to the foot of the perpendicualr from P on the respective coordinate axes. By **signed/directed distance** we mean distance accompanied by a positive or negative sign. While determining coordinates the distance from origin to the foot of the perpendicular from P on an axis is given a positive or negative sign according as the foot of the perpendicular falls on the positive or negative axis. If it falls on the origin then the concerned coordinate becomes zero.

2. By passing planes through P parallel to the coordinate planes we can construct a cuboid as shown in figure 15.5. The coordinates (x, y, z) of P are the perpendicular distances of P from the YZ, ZX and XY-plane accompanied by positive or negative sign according as P lies in the side of YZ, ZX and XY-planes containing respectively the positive or negative X, Y and Z-axes.

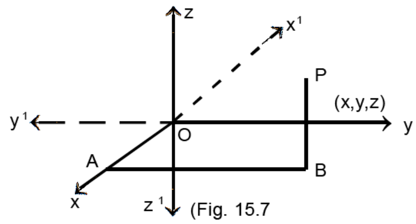
For example in figure 15.5, x = OL = MQ = PR = perpendicular distance of P from the YZ-plane. Here positive sign accompanies the distance PR as P is in the side of YZ-plane containing $\overrightarrow{OX}$ or the positive X-axis.

How to locate a point with given coordinates in space :

Given the coordinates (x, y, z) we can locate the point, say P having these coordinates by the following procedure described in intuitive language.

First proceed a distance $OA = |x|$ along $\overrightarrow{OX}$ if $x > 0$ or along $\overrightarrow{OX'}$ if $x < 0$. If $x = 0$, remain at the origin i.e. the point A coincides with the origin.

Then proceed a distance AB = |y| along a line parallel to Y-axis in the positive direction if y > 0 or in the negative direction if y < 0. If y = 0, the point B coincides with A i.e. do not move at all. Finally, from B proceed a distance BP = |z| along a line parallel to Z-axis in the positive direction if z > 0 or in the negative direction if z < 0. If z = 0, P coincides with B.



(Fig. 15.7

The point P has coordinates (x, y, z) which we express by writing P(x, y, z).

The Octants : It follows from axiom - 5 that each coordinate plane divides the set of points not lying on it into two disjoint convex sets. Thus the three coordinate planes intersecting at origin divide the set of points not lying on any of the coordinate planes into eight disjoint convex sets known as octants. Taking into account the identification of a three dimensional space with R^3 we can describe the eight octants as follows :

1. OXYZ = $\{(x, y, z) \in R^3 \mid x > 0, y > 0, z > 0\}$
2. OX′YZ = $\{(x, y, z) \in R^3 \mid x < 0, y > 0, z > 0\}$
3. OX′Y′Z = $\{(x, y, z) \in R^3 \mid x < 0, y < 0, z > 0\}$
4. OXY′Z = $\{(x, y, z) \in R^3 \mid x > 0, y < 0, z > 0\}$
5. OXYZ′ = $\{(x, y, z) \in R^3 \mid x > 0, y > 0, z < 0\}$
6. OX′YZ′ = $\{(x, y, z) \in R^3 \mid x < 0, y > 0, z < 0\}$
7. OX′Y′Z′ = $\{(x, y, z) \in R^3 \mid x < 0, y < 0, z < 0\}$
8. OXY′Z′ = $\{(x, y, z) \in R^3 \mid x > 0, y < 0, z < 0\}$

Projection of a point on a line or a plane

Definition : The projection of a point on a given line or plane is the foot of the perpendicular from that point on the given line or plane. If the point lies on the given line or plane then its projection is the point itself.

Illustration : 1. See figure 15.5. Q is the foot of the perpendicular from P on the XY-plane. By definition of coordinates, Q has coordinates (x, y, 0). So projection of P(x, y, z) on the XY-plane is given by Q (x, y, 0). Similarly projections of P on the YZ and ZX-planes are given by R(0, y, z) and S(x, 0, z) respectively.

2. Since $\overline{PL} \perp \overleftrightarrow{X'X}$ and $\overline{PM} \perp \overleftrightarrow{Y'Y}$, projections of P(x, y, z) on the X and Y-axes are given by L(x, 0, 0) and M(0, y, 0) respectively. Also the projection of P(x, y, z) on the Z-axis is given by N(0, 0, z).

Exercise : Find projections of the point (2, 3, –6) on the coordinate axes and the coordinate planes.

Answer : Projections of (2, 3, –6) on X, Y and Z-axes are given by (2, 0, 0), (0, 3, 0) and (0, 0, –6) respectively.

Projections of (2, 3, –6) on the XY, YZ and ZX-planes are (2, 3, 0), (0, 3, –6) and (2, 0, –6) respectively.

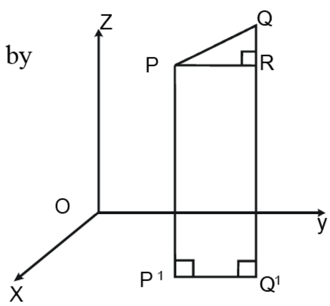
Projection of a segment on a line or a plane :

Definition : If P′ and Q′ are projections of P and Q on a given line or a plane, then $\overline{P'Q'}$ is the projection of the segment $\overline{PQ}$ on the given line or plane.

13.5. Distance between two points in space :

The distance between the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is given by

$$PQ = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$



Proof : Let $\overline{P'Q'}$ be the projection of $\overline{PQ}$ on the XY-plane.

$\overline{PP'}$ and $\overline{QQ'}$ are parallel (Fact-5). So $\overline{PP'}$ and $\overline{QQ'}$ are coplanar and PP′Q′Q is a plane quadrilateral.

Let R be a point on $\overleftrightarrow{QQ'}$ so that $\overline{PR} \parallel \overline{P'Q'}$.

Since $\overleftrightarrow{P'Q'}$ lies on the xy-plane and $\overleftrightarrow{PP'}$ is perpendicular to this plane, it follows from the difinition of perpendicular to a plane that $\overline{PP'} \perp \overline{P'Q'}$. Similarly $\overline{QQ'} \perp \overline{P'Q'}$. $\overline{PR}$ being parallel to $\overline{P'Q'}$, it follows from plane-geometry that PP′Q′R is a rectangle. So PR = P′Q′ and ∠PRQ is a right angle.

P′ and Q′ being projections of $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ on the xy-plane, they are given by $P'(x_1,y_1,0)$ and $Q'(x_2, y_2, 0)$. Therefore by the distance formula in the geometry of R^2,

$$P'Q' = \sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$$

In the rectangle PP′Q′R,

P′P = Q′R

Therefore $QR = |z_2 - z_1|$

In the right angled triangle PRQ, $PQ^2 = PR^2 + RQ^2$

$$= (x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2$$

$$\therefore PQ = \sqrt{(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2}$$

Notes :

1. Distance of P(x, y, z) from origin is given by

$$OP = \sqrt{x^2+y^2+z^2}$$

2. If L, M, N are projections of P(x, y, z) on x, y and z-axes respectively, then

$$PL = \sqrt{y^2+z^2},\ PM = \sqrt{z^2+x^2},\ PN = \sqrt{x^2+y^2}$$

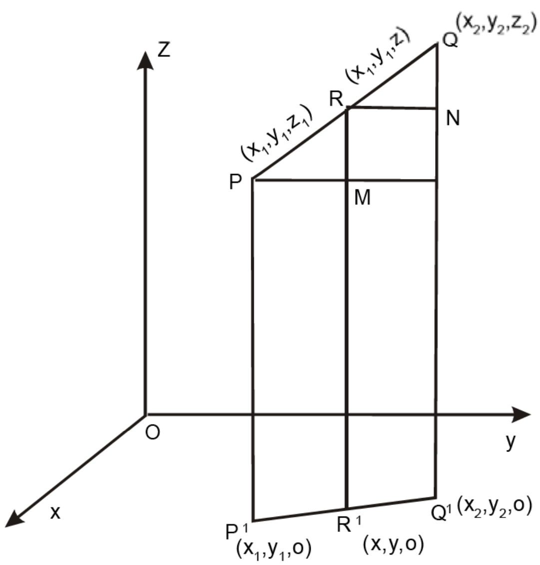
13.6 Division of a line segment in a given ratio. Internal and External Division formula :

If R(x, y, z) divides the segment joining $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ internally in ratio m : n i.e.

$$\frac{PR}{QR} = \frac{m}{n},\ \text{then}\ x = \frac{mx_2+nx_1}{m+n},\ y = \frac{my_2+ny_1}{m+n}$$

and $$z = \frac{mz_2+nz_1}{m+n}$$

Prof – Let P′, Q′ and R′ be the feet of the perpendiculars from P, Q and R on the xy-plane. (Fig 15.9) Being perpendiculars on the same plane $\overleftrightarrow{PP'}$, $\overleftrightarrow{QQ'}$ and $\overleftrightarrow{RR'}$ are parallel lines (Fact-5). Since these parallel lines have a common transversal $\overleftrightarrow{PQ}$, they are coplanar. (Fact-7) Let M and N be points on $\overleftrightarrow{RR'}$ and $\overleftrightarrow{QQ'}$ such that $\overleftrightarrow{PM} \perp \overleftrightarrow{RR'}$ and $\overleftrightarrow{RN} \perp \overleftrightarrow{QQ'}$. Since P′, R′ and Q′ are common to the xy-plane and the plane of $\overleftrightarrow{PP'}$, $\overleftrightarrow{QQ'}$ and $\overleftrightarrow{RR'}$, they must be collinar because two planes intersect along a line.



It follows from the definition of perpendicular to a plane that ∠PP′R′, ∠RR′Q′ and ∠QQ′R′ are all right angles. It now follows from plane-geometry that PP′R′M and RR′Q′N are rectangles. Also triangles RPM and QRN are similar. Hence $\frac{m}{n} = \frac{PR}{RQ} = \frac{PM}{RN} = \frac{P'R'}{R'Q'}$

($\because$ PM = P′R′ and RN = R′Q′ in the corresponding rectangles)

Thus the point R′ divides the segment $\overline{P'Q'}$ internally in the ratio m:n.

P′,R′ and Q′ being projections of P(x_1, y_1, z_1), R (x, y, z,) andQ (x_2, y_2, z_2) on the xy-plane have coordinates respectively (x_1,y_1,0), (x,y,0) and (x_2,y_2,0).

If we restrict our considerations to the xy-plane only, we can regard the points P′, R′ and Q′ as having coordinates (x_1, y_1), (x, y) and (x_2, y_2).

Thus, on the xy-plane the point R′(x,y) divides the segment joining P′(x_1,y_1) and Q′(x_2,y_2) internally in ratio given by $\frac{P'R'}{Q'R'}=\frac{m}{n}$.

Therefore, it follows from the internal division formula of the geometry of R^2 that $x=\frac{mx_2+nx_1}{m+n}$ and

$y=\frac{my_2+ny_1}{m+n}$

Similarly considering projections of P,Q, R on another coordinate plane, say YZ - plane we can prove

$y=\frac{my_2+ny_1}{m+n}, z=\frac{mz_2+nz_1}{m+n}$

Thus we have

$$x=\frac{mx_2+nx_1}{m+n}, y=\frac{my_2+ny_1}{m+n}, z=\frac{mz_2+nz_1}{m+n}$$

External Division Formula

If R (x,y, z) divides the segment $\overline{PQ}$ joining P(x_1,y_1,z_1,) and Q(x_2,y_2,z_2) externally in ratio m :n

i.e. $\frac{PR}{QR}=\frac{m}{n}$, then

$$x=\frac{mx_2-nx_1}{m-n}, y=\frac{my_2-ny_1}{m-n}, z=\frac{mz_2-nz_1}{m-n}$$

The above formulae can be proved by taking projections on any two coordinate planes and applying the external division formula of the geometry of R^2.

Notes :

1. The midpoint of $\overline{PQ}$ is given by

$$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$$

2. If R divides $\overline{PQ}$ in ratio λ : 1 then coordinates of R are given by

$$x=\frac{x_1+\lambda x_2}{1+\lambda}, y=\frac{y_1+\lambda y_2}{1+\lambda}, z=\frac{z_1+\lambda z_2}{1+\lambda}$$

in case of internal division and by

$x=\frac{x_1-\lambda x_2}{1-\lambda}, y=\frac{y_1-\lambda y_2}{1-\lambda}, z=\frac{z_1-\lambda z_2}{1-\lambda}$; in case of external division, $\lambda \neq 1$.

3. If P(x_1, y_1, z_1) and Q(x_2, y_2, z_2) are distinct points then the coordinates of any point on $\overleftrightarrow{PQ}$ except Q are given by $\left(\frac{x_1+\mu x_2}{1+\mu}, \frac{y_1+\mu y_2}{1+\mu}, \frac{z_1+\mu z_2}{1+\mu}\right)$; $\mu \in R, \mu \neq -1$.

Solved Examples

Example - 1

Find the distance of the point P (x_0, x_0, z_0) from

(i) the coordinate axes

(ii) the coordinate planes

Solution

Projection of $P(x_0, x_0, z_0)$ on x, y and z axes are reopectively given by P_1 $(x_0, 0, 0)$, $P_2(0, y_0, 0)$ and

P_3 $(0, 0, z_0)$.

So distance of P from x-axis

$= PP_1 = \sqrt{(x_0 - x_0)^2 + (y_0 - 0)^2 + (z_0 - 0)^2} = \sqrt{y_0^2 + z_0^2}$

Similarly the distances of P from y and z - axes are given by

$PP_2 = \sqrt{x_0^2 + z_0^2}$ and $PP_3 = \sqrt{x_0^2 + y_0^2}$

Projection of $P(x_0,y_0,z_0)$ on xy - plane is the point $Q_1(x_0,y_0,0)$. Similaryly is projections on yz and zx-planes are given by $Q_2(\mathbf{0},y_0,z_0)$ and $Q_3(x_0,\mathbf{0},z_0)$.

So the distances of P from xy, yz and zx- Planes are given by

$PQ_1 = \sqrt{(x_0 - x_0)^2 + (y_0 - y_0)^2 + (z_0 - 0)^2} = \sqrt{z_0^2} = |z_0|$

and $PQ_2 = |x_0|$ and $PQ_3 = |y_0|$ respectively.

Example - 2

Prove that the points A(2,3,2), B(5,5,6) and C(-4,-1,-6) are collinear.

Solution

As in case of two dimensional geometry, here also we have to show that out of the three distances AB, BC and CA, sum of two of them should be equal to the third.

We have

$AB = \sqrt{(5-2)^2 + (5-3)^2 + (6-2)^2} = \sqrt{29}$

$BC = \sqrt{(-4-5)^2 + (-1-5)^2 + (-6-6)^2} = 3\sqrt{29}$

$CA = \sqrt{(-4-2)^2 + (-1-3)^2 + (-6-2)^2} = 2\sqrt{29}$

We observe that CA+AB = BC and hence conclude that the points are collinear.

Example -3

The vertices of a triangle ABC are given by A(x_1,y_1,z_1), B(x_2,y_2,z_2) and C (x_3,y_3,z_3), If D and E are mid points of the sides $\overline{AB}$ and $\overline{AC}$ respectively, show that DE = $\frac{1}{2}$BC.

Solution

By the formula for internal division mid points of $\overline{AB}$ and $\overline{AC}$ are given by

$$D\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2},\frac{z_1+z_2}{2}\right) \text{ and } E\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2},\frac{z_1+z_2}{2}\right)$$

By distance - formula,

$$DE = \frac{1}{2}\sqrt{(x_2-x_3)^2+(y_2-y_3)^2+(z_2-z_3)^2} \quad \dots \quad \dots \quad \textbf{(i)}$$

Now

$$BC = \sqrt{(x_2-x_3)^2+(y_2-y_3)^2+(z_2-z_3)^2} \quad \dots \quad \dots \quad \textbf{(ii)}$$

(i) and (ii) imply

$$DE = \frac{1}{2}BC.$$

Example 4 :

Prove that the points A (–1, 6, 6), B (–4, 9, 6), C (0, 7, 10) form the vertices of a right angled isosceles triangle.

Solution :

By distance formula,

$AB^2 = (-4+1)^2 + (9-6)^2 + (6-6)^2 = 9+9 = 18$

$BC^2 = (0+4)^2 + (7-9)^2 + (10-6)^2 = 16 + 4 + 16 = 36,$

$AC^2 = (0+1)^2 + (7-6)^2 + (10-6)^2 = 1+1+16 = 18$

which gives $AB^2 + AC^2 = 18 + 18 = 36 = BC^2$

Hence,

ABC is a right angled isosceles triangle.

Example 5 :

Find the ratio in which the line segment joining the points (4, 3, 2) and (1, 2, –3) is divided by the co-ordinate planes.

Solution :

Let the given points be denoted by A (4, 3, 2) and B (1, 2, –3). If Q is the point where the line through A and B is met by the xy - plane, then the co-ordinates of Q are $\left(\frac{k+4}{k+1},\frac{2k+3}{k+1},\frac{-3k+2}{k+1}\right)$,

since Q divides $\overline{AB}$ in a ratio k : 1 for some real value of k. But being a point on the xy-plane, its z- co-ordinate is zero.

Hence,

$$\frac{-3k+2}{k+1} = 0 \text{ or } k = 2/3$$

$\therefore$ Q divides $\overline{AB}$ in a ratio 2 : 3. Similarly, the point where $\overline{AB}$ meets the xz - plane has its y-coordinate zero. Hence, equating the y-co-ordinate to zero, we get

$$\frac{2k+3}{k+1} = 0 \text{ or, } k = -3/2$$

i.e. the xz - plane divides $\overline{AB}$ externally in a ratio 3 : 2. Equating the x - co-ordinate to zero, we get,

$$\frac{k+4}{k+1} = 0 \text{ or } k = -4$$

i.e. the yz - plane divides $\overline{AB}$ externally in a ratio 4 : 1.

EXERCISE – 13

1. Fill in the blanks in each of the following questions by choosing the appropriate answer from the given ones.

   (a) The distance of the point P (x_0, y_0, z_0) from z - axis is –

   $$\left[\sqrt{x_0^2+y_0^2},\ \sqrt{y_0^2+z_0^2},\ \sqrt{x_0^2+z_0^2},\ \sqrt{(x-x_0)^2+(y-y_0)^2}\right]$$

   (b) The length of the projection of the line segment joining (1, 3, –1) and (3, 2, 4) on z axis is — .

   [1, 3, 4, 5]

   (c) The image of the point (6, 3, –4) with respect to yz - plane is ——.

   [(6, 0, –4), (6, –3, 4), (–6, –3, –4), (–6, 3, –4)]

   (d) If the distance between the points (–1, –1, z) and (1, –1, 1) is 2 then z = ——.

   [1, $\sqrt{2}$, 2, 0]

2. (a) Identify the axes on which the given points lie :

   (1,0,0), (0,1,0), (0,0,1)

   (b) Identify the planes containing the points !

   (7,0,4), (2,-5,0), (0, $\sqrt{2}$, -3)

3. (a) Determine, which of the following points have the same projection on x-axis.

   (2,-5,7), (2, $\sqrt{2}$, -3), (-2,1,1), (2, -1, 3)

   (b) Find the projection of the point (7,-5,3) on :

   (i) xy-plane, (ii) yz-plane, (iii) zx-plane

   (iv) x-a$x$iz, (v) y-a$x$is, (vi) z-a$x$is.

4. When do you say two lines in space are skew ?

Do they interset ?

5. From the three pairs of lines given below, identify those which uniquely determine a plane: (i) intersecting pair, (ii) parallel pair, (iii) a pair of skew lines.

6. Determine the unknown coprdinates of the following points if

(i) P(a,2,-1) $\in$ yz - plane (iv) S(7,y,z) $\in$ x- axis

(ii) Q (-1,y,3) $\in$ zx - plane (v) T (x, 0, z) $\in$ y- axis

(iii) R($\sqrt{2}$,-3,c) $\in$ xy - plane (vi) V(a, b,-3) $\in$ z - axis

7. Which axis is determined by the intersection of

(i) xy - plane and yz - plane

(ii) yz - plane and zx - plane

(iii) zx - plane and xy - plane

8. Which axis is represented by a line passing through origin and normal to

(i) xy - plane, (ii) yz-plane, (iii) zx - plane

9. What are the coordinates of a point which is common to all the coordinate planes.

10. If A, B, C are projections of P(3,4,5) on the coordinate planes, find PA, PB and PC.

11. (a) Find the perimeter of the triangle whose vertices are (0, 1, 2) (2, 0, 4) and (–4, –2, 7).

(b) Show that the points (a, b, c), (b, c, a) (c, a, b) form an equilateral triangle.

(c) Show that the points (3, –2, 4), (1, 1, 1) and (–1, 4, –1) are collinear.

(d) Show that points (0, 1, 2), (2, 5, 8), (5, 6, 6) and (3, 2, 0) form a parallelogram.

(e) Show that the line segment joining (7, –6, 1) (17, –18, –3) intersects the line segment joining (1, 4, –5), (3, –4, 11) at (2, 0, 3).

(f) Find the locus of the points which are equidistant from the points (1, 2, 3) and (3, 2, –1).

12. (a) Find the ratio in which the line segment through (1, 3, –1) and (2, 6, –2) is divided by zx–plane.

(b) Find the ratio in which the lines segment through(2, 4, 5),(3, 5,–4) is divided by xy- plane.

(c) Find the co-ordinates of the centroid of the triangle with its vertices at (a_1, b_1, c_1), (a_2, b_2, c_2) and (a_3, b_3, c_3).

(d) If A (1, 0, –1), B (–2, 4, –2) and C (1, 5, 10) be the vertices of a triangle and the bisector of the angle BAC, meets $\overline{BC}$ at D, then find the co-ordinates of the point D.

(e) Prove that the points P (3, 2, –4), Q(5, 4, –6) and R(9, 8, –10) are collinear. Find the ratio in which the point Q divides the line segment $\overline{PR}$.

●

CHAPTER - 14

Limit and Differentiation

The most suggestive and notable achievement of last century is the discovery of Non-Euclidean geometry. **- Hilbert**

14.1 Introduction :

The concept of limit and convergence is fundamental in the study of calculus and Analysis. The fragments of this concept are evident in **the method of exhaustion** formulated by ancient Greeks and used by Archimedes (287–212 B.C.) in obtaining a formula for the area of the circular region conceived as successive approximation of areas of inscribed polygons with increased number of sides. But somehow this concept could not be persued. It was the genius of Bhaskarachaya (1150 A.D.) who achieved a break-through in the invention of infinitesimal (anantaksudra) and instantaneous velocity (tatkalika gati) for his astronomical calculations. Madhava (1340-1425) a talented mathematician from Cochin refined the above concepts.

Three centuries later Fermat (1608) too dealt with the idea of rate of change and drawing of tangents but often due to ignorance, the credit of developing these concepts goes to Newton (1642–1727) and Leibnitz (1646-1716). In fact Newton has acknowledged and expressed his indebtedness to Fermat to further his ideas. It was mainly Cauchy (1789-1857) and Weierstrass (1815-1897) who put the limiting process on sound foundation.

Before discussion of the subject matter of this chapter we pose a simple problem. The problem reads as follows :

"A tortoise and a hare start a 100 metre race from A towards a palm tree. The tortoise runs at a uniform speed of 1 meter per minute. The hare runs 50 m. in the first minute. At the end of the first minute observing that the tortoise is at a distance of 49 m. behind the hare runs 25 m. in the next minute. After the lapse of 2 minutes again observing that the tortoise is at a distance of 73 m. behind the hare runs 12.5 m. in the next minute. Thereafter the hare reduces the distance covered in each minute to the half of the distance covered in the preceding minute with an impression that the tortoise can never overtake her."

We now ask the following question :

"Who reaches first at the foot of the palm tree ?

Since the tortoise runs at a uniform speed of 1 meter per minute it takes 100 minutes to reach the foot of the palm tree.

Distance covered by the hare at the lapse of 1 minute = 50 m.

Distance covered at the lapse of 2 minutes = (50 + 25)mt. = $50\left(1+\frac{1}{2}\right)$ m.

Distance covered at the lapse of 3 minutes = (50 + 25 + 12.5) mt. = $50\left(1+\frac{1}{2}+\frac{1}{2^2}\right)$ m.

Distance covered at the end of n minutes, where $n \in N$

$$= 50\left(1 + \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^{n-1}}\right) \text{m.}$$

$$= 50\left(\frac{1 - \frac{1}{2^n}}{1 - \frac{1}{2}}\right) \text{mt.} \quad = 100\left(1 - \frac{1}{2^n}\right) \text{m.}$$

We have $0 < 1 - \frac{1}{2^n} < 1 \; \forall \, n \in \mathrm{N}$. So $100\left(1 - \frac{1}{2^n}\right) < 100 \; \forall \, n \in \mathrm{N}$.

Thus the hare cannot reach the foot of the palm tree.

We observe the following facts :

The distance covered by the hare after n minutes = $100\left(1 - \frac{1}{2^n}\right)$ m.

So the distance covered by the hare after 1 minute, after 2 minutes, after 3 minutes,… are 100 $\left(1 - \frac{1}{2}\right)$ m., $100\left(1 - \frac{1}{2^2}\right)$ m., $100\left(1 - \frac{1}{2^3}\right)$ m. … respectively.

These numbers are obtained easily by putting n = 1, 2, 3, … respectively in $100\left(1 - \frac{1}{2^n}\right)$.

Here we say that the numbers $100\left(1 - \frac{1}{2}\right)$, $100\left(1 - \frac{1}{2^2}\right)$, $100\left(1 - \frac{1}{2^3}\right)$,…, $100\left(1 - \frac{1}{2^n}\right)$, … form a sequence.

In this case we understand intuitively what a sequence is by keeping in view the literal meaning of a sequence.

We see that there is a certain law of correspondence which assigns to every natural number n a number $100\left(1 - \frac{1}{2^n}\right)$.

The term sequence in mathematics is an important concept. So we introduce the idea of a sequence through the language of mathematics.

Definition :

A real **Sequence** is a function whose domain is the set of positive integers and range is a subset of the set of real numbers, that is $f : \mathrm{N} \rightarrow \mathrm{R}$.

Example 1 :

Let $f(n) = \frac{1}{n} \; \forall \, n \in \mathrm{N}$. Then f is a sequence in R.

$f(1), f(2), f(3), \ldots\ldots\ldots\ldots, f(n), \ldots\ldots\ldots\ldots\ldots$ are respectively called the 1st, 2nd, 3rd, …………… nth, ………………… terms of the sequence f.

Here $f(1) = \frac{1}{1} = 1, f(2) = \frac{1}{2}, f(3) = \frac{1}{3}$, ……………… and so on.

The sequence f is also represented by writing $\{f(n)\}$, that is $\left\{\frac{1}{n}\right\}$. Some authors also use the symbol $(f(n))$ to represent the sequence f. A few examples of sequences are :

$\{n^2\}$, $\left\{\frac{1}{2n-1}\right\}$, $\{2^n\}$, $\{(-1)^n\}$.

Subsequence :

Let $\{x_n\}$ be a sequence in R. $x_1, x_2, x_3, \ldots\ldots\ldots\ldots$ are the terms of this sequence. We can choose a new sequence $\{x_{n_k}\}$ consisting of some terms of the sequence $\{x_n\}$ only where the index n_k runs through an increasing sequence of natural numbers $n_1 < n_2 < n_3 \ldots\ldots\ldots\ldots$ The sequence $\{x_{n_k}\}$ is called a subsequence of the original sequence $\{x_n\}$.

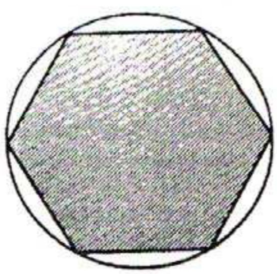
Example 2 :

$\left\{\frac{1}{n}\right\}$ is a sequence in R.

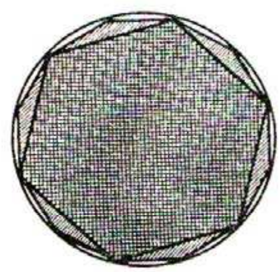
$\left\{\frac{1}{2n-1}\right\}$, $\left\{\frac{1}{2n}\right\}$, $\left\{\frac{1}{3n}\right\}$ are some of the sub-sequences of the sequence $\left\{\frac{1}{n}\right\}$

Limit of Sequence :

As we introduce the idea of limit let us describe a process by which we find the area of a curvilinear figure e.g. a circular region. Imagine we have a circle of unit radius and we want to find the area of the region bound by it.



First Approximation



Second Approximation

(Fig. 11.01)

Let us inscribe a regular hexagon in the circle. We may say that the area A_1 enclosed by the hexagon is an approximation of the area A enclosed by the circle. But then we clearly see that this leaves out a certain portion of the circle. To compensate for this we inscribe isosceles triangles inside the region bounded by the sides of the hexagon and the circle and add the new areas to A_1 to get a better approximation A_2. No doubt this is a better approximation but still we are missing a part of the area in the segments bounded by the sides of triangles and the circle. We can attempt to compensate for this by adding new isosceles triangles inscribed in the segments left out to get a still better approximation A_3, say.

We realize we can continue this process getting better and better approximations. We can now say $A_1, A_2, A_3, \ldots\ldots\ldots, A_n, \ldots\ldots$ are the first, second, third….., nth .. approximations of the area. That is to say we now have a scheme of approximation of area A by a sequence $(A_n)_{n=1}^{\infty}$. But the idea of approximate value is of little importance unless we know how far it is from the "exact" value. Let us recall how we use approximations in our daily life. We say Bhubaneswar is about 300 Kilometres from Sambalpur. This suggests that the distance measured is not quite equal to 300 but it is nearly so. We could even say that it is approximately 299 kms. But if we said it is approximately 200 Kilometers then nobody would think we are talking sense. Because with every approximation there is associated a notion called error. So when we are talking of approximation we must tell what is the error associated with it. But one can ask if we knew what the error was then why not the exact value, as the difference between the exact and approximate value is what we call as error.

We agree, but it is not that we want to know what exactly is the error when we have an approximation, but rather we would like to know what the maximum value of the error can be. We know context decides what is the maximum tolerable error when we approximate for the exact value. Indeed when we buy shirtlengths in a shop the owner does not mind if the measured length is a fewer milimeters extra where as measuring distance between two cities we can afford to ignore even a few meters. So the context tells us what is the maximum error allowed. So if the sequence $(A_n)_{n=1}^{\infty}$ represents a scheme of approximation of a value A then once the error $\varepsilon > 0$ is prescribed we should be able to say at what stage of approximation is the error less than ε. Otherwise this scheme of approximation is not of much worth. To be precise, given $\varepsilon > 0$ we should be able to find a positive integer n_0, depending on ε of course such that $|A - A_{n_0}| < \varepsilon$.

This tells at what stage the desired accuracy is achieved or the price one has to pay to achieve the desired accuracy. But this is not yet a good enough scheme unless it is **stable**. What it means is that the value of A_n coming close to the value of A with the accuracy desired should not be a chance event. An accuracy once achieved should remain so at subsequent stages. That is to say that given $\varepsilon > 0$ we should be able to find an n_0 such that

$$|A - A_{n_0}| < \varepsilon \text{ for all } n \geq n_0.$$

This is what is meant by stability of a scheme of approximation. In this case the value of A_n stabilizes around the value of A within the limit of accuracy prescribed. We observe that $(A_n)_{n=1}^{\infty}$ is a sequence whose terms can be brought as near A as we please by choosing n large enough. We have the

Definition :

We say that a sequence of real numbers $(a_n)_{n=1}^{\infty}$ converges to a real number l if for every $\varepsilon > 0$ there is an n_0 such that $|a_n - l| < \varepsilon \quad \forall\, n > n_0$.

Now the question arises : Can a sequence converge to two different real numbers? In other words can the terms of a sequence be brought arbitrarily close to two different real numbers. Answer is obviously **no**. We invite the reader to find a formal proof of this.

We see that if a sequence $(a_n)_{n=1}^{\infty}$ converges at all it has to converge to a unique real number l which we call its limit. We express this by

$$\lim_{n \to \infty} a_n = l. \qquad \text{(read : limit of } a_n \text{ as } n \text{ goes to infinity is } l\text{)}$$

We observe that

if $\lim\limits_{n \to \infty} a_n = l_1$ and $\lim\limits_{n \to \infty} b_n = l_2$, then

$$\lim_{n \to \infty} (a_n + b_n) = l_1 + l_2 = \lim_{n \to \infty} a_n + \lim_{n \to \infty} b_n.$$

Similarly we can show that

$$\lim_{n \to \infty} (a_n - b_n) = \lim_{n \to \infty} a_n - \lim_{n \to \infty} b_n,$$

$$\lim_{n \to \infty} (a_n\, b_n) = \lim a_n \lim b_n,$$

$$\lim \frac{a_n}{b_n} = \frac{\lim a_n}{\lim b_n} \text{ if } \lim_{n\to\infty} b_n \neq 0.$$

We discuss a few examples.

Example 2(a) :

Let $a_n = \frac{1}{n}$. We claim that $(a_n)_{n=1}^{\infty}$ conveges to 0. Now for $\varepsilon > 0$ let $n_o = \left[\frac{1}{\varepsilon}\right] + 1$.

So we have $n_o > \frac{1}{\varepsilon}$. Hence for $n > n_o$ we have $0 < \frac{1}{n} < \frac{1}{n_o} < \varepsilon$. Also see Example 16.

Example 2(b) :

$\left(\frac{1}{2^n}\right)_{n=1}^{\infty}$ conveges to 0. Indeed $2^n > n$. So for $\varepsilon > 0$ choose n_o such that $\frac{1}{n} < \varepsilon$ for $n > n_o$.

Thus we have $0 < \frac{1}{2^n} < \frac{1}{n} < \varepsilon$ for $n > n_o$.

Example 2(c) :

Madhava (13th century A.D.) wrote

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \ldots..$$

What he really meant was if we write

$P_1 = 1$

$P_2 = 1 - \frac{1}{3}$

$P_3 = 1 - \frac{1}{3} + \frac{1}{5}$

$P_4 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7}$

$P_n = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\ldots + \frac{(-1)^{n-1}}{2n-1}$

then the sequence $(P_n)_{n=1}^{\infty}$ conveges to $\frac{\pi}{4}$.

It can be shown that

$$\left|\frac{\pi}{4} - P_n\right| < \frac{1}{2n+1}$$

which shows the maximum error committed if we stopped summing the series at the nth stage and what is the maximum error we are liable to commit, or to achieve the desired accuracy we should sum the series to how many terms in the least.

Find out how far we need to sum by using the above formula if we want to achieve correct value to the third place of decimal.

Example 2(d) :

Let $b > 1$ and $a_n = b^{\frac{1}{n}}$. We claim that $(a_n)_{n=1}^{\infty}$ converges to 1. Indeed

$a_n > 1$. Let us write $a_n = 1 + \gamma_n$ such that $\gamma_n > 0$.

So we have $b^{\frac{1}{n}} = 1 + \gamma_n$.

We have then by Binomial Theorem

$b = (1+\gamma_n)^n = 1 + n\gamma_n + \dfrac{n(n-1)}{2!}\ \gamma_n^2 + \ldots\ldots + \gamma_n^n$.

This gives $b > n\,\gamma_n > 0$.

We get

$$\gamma_n < \frac{b}{n}.$$

So if we choose $n_o = \left[\dfrac{b}{\varepsilon}\right] + 1$, we get $\dfrac{b}{n} < \dfrac{b}{n_0} < b.\dfrac{\varepsilon}{b} = \varepsilon$ for $n > n_o$.

Hence $0 < \gamma_n < \varepsilon$ for $n > n_o$.

In other words, we have

$0 < a_n - 1 < \varepsilon$ for $n > n_o$.

Using the last inequality of Example-2(b) show that for $b > 1$, $b^{\frac{1}{n}}$ converges to 1.

Take your pocket calculator and take any number larger than 1 say 10. Take its square root by pressing $\sqrt{\ }$. Take the square root of the number obtained by pressing the key $\sqrt{\ }$ again. Continue this process at least 15 times. What do you see ? Why did you see what you saw?

Hitherto we have talked of convergence of a sequence which is a function of natural numbers. Many times we also encounter in the context of function of real numbers, the notion of limit. Suppose a function f is defined on a **deleted neighbourhood** of a point x_o. What we mean by that is that $f(x)$ is defined for all x satisfying $0 < |x - x_o| < a$. (*i.e.* for $x_o < x < x_o + a$ or $x_o - a < x < x_o$.). We observe that $f(x)$ is defined for all values of x in the interval $(x_o - a, x_o + a)$ **except** the value $x = x_o$. Now what would happen to $f(x)$ as x comes close to x_o from either side of x_o ? Does it come close to a real number? There are many situations in which it does not. For example for $f(x) = \sin\dfrac{1}{x}$, $x \neq 0$, the function f is defined for every real number except the real number 0. We observe that $f(x)$ does not come close to any value as x comes close to 0. For example when $x = \dfrac{1}{100\pi}$, $f(x) = 0$ where as for $x = \dfrac{1}{\left(100+\frac{1}{2}\right)\pi}$, $f(x) = 1$ and for $x = \dfrac{1}{\left(100-\frac{1}{2}\right)\pi}$ $f(x) = -1$. This oscillation goes on as x goes through the values $\dfrac{1}{2n\pi}$, $\dfrac{1}{\left(2n+\frac{1}{2}\right)\pi}$, $\dfrac{1}{\left(2n-\frac{1}{2}\right)\pi}$. $f(x)$ takes the values 0, 1 and -1 though x comes progressively near to 0 as n increases.

But there could be situations where $f(x)$ could come close to a number l as x comes close to x_0. For example if we define $f(x) = \dfrac{\sin x}{x}$ for $x \neq 0$ we see that

x	1.0001	.9006	.8011	.7016	.6004	.4014	.3002	.2007	.1012
$\sin x$	.8415	.9837	.7181	.8455	.5650	.3907	.2957	.1994	.1011

$f(x)=\frac{\sin x}{x}$	.8414	.8701	.8976	.9200	.9414	.9702	.9850	.9960	.9990

We see that $f\ (1.0001) = 0.8414$, $f\ (.9006) = 0.8701$, $f\ (.8011) = 0.8976$, $f\ (.7016) = 0.9200$, $f\ (.6004) = 0.9414$, $f\ (.4014) = 0.9702$, $f\ (.3002) = 0.9950$, $f\ (.2007) = 0.996$, $f\ (.1012) = 0.9990$. Radian measure of $2^0 54'$ is .0560 (correct to 4 places) and $\sin 2^0 54' = .0560$ (correct to 4 place). This seems a plausible that the value of $\frac{\sin x}{x}$ progressively comes close to 1 as x comes close to 0. We shall in fact demonstrate later that $\lim\limits_{n \to \infty} \frac{\sin x}{x} = 1$.

Now we study about limit of a function in greater detail.

14.2 Limit of a function (Including polynomials and rational functions)

The limit of a function is the most fundamental concept in the study of calculus as well as of mathematical analysis. In this section we shall be concerned with the limit of a real-valued .function of a real variable. For this purpose we make a very simple beginning. Our approach to the subject matter is based on our intuition. In the later stage we shall discuss the subject matter in logical and rigorous way. At present we must understand what is meant by the limit of a function at a point. We begin with a simple illustration.

Example 3 :

Consider a function f defined by $f\ (x) = 2x + 1$.

The domain of this function is the set of real numbers. $1 \in \text{dom} f$.

It is natural to ask whether or not $2x + 1$ becomes closer and closer to 3 as x becomes closer and closer to 1 but not equal to 1.

We observe that :

$f\ (0.9) = 2.8$	$f\ (1.1) = 3.2$
$f\ (0.99) = 2.98$	$f\ (1.01) = 3.02$
$f\ (0.999) = 2.998$	$f\ (1.001) = 3.002$
$f\ (0.9999) = 2.9998$	$f\ (1.0001) = 3.0002$
$f\ (0.99999) = 2.99998$	$f\ ((1.00001) = 3.00002$
and so on.	and so on.

We see intuitively that

(1) $2x + 1$ becomes closer and closer (or nearer and nearer) to 3, as x becomes closer and closer to 1,

(2) $2x + 1$ can be made as close to 3 as desired by taking x sufficiently close to 1.

The above facts are expressed by writing

$$\lim_{x \to 1} (2x + 1) = 3,$$

i.e. $2x + 1 \to 3$ as $x \to 1$.

The symbol $\to$ stands for **approaches** or **tends to.**

$x \to 1$ is read as x approaches (or tends) to 1 and $x \neq 1$.

$\lim\limits_{x \to 1} (2x + 1)$ is read as limit of $(2x + 1)$ as x approaches (or tends) to 1.

$\lim\limits_{x \to 1} (2x + 1) = 3$ is read as limit of $2x + 1$ as x tends to 1 is equal to 3.

The last statement is also expressed by saying that limit of $(2x + 1)$ at $x = 1$ is equal to 3.

Example 4 :

Let a real function f be defined by

$$f(x) = \frac{x^2 - 4}{x - 2}, \; x \in \text{R}, x \neq 2.$$

Clearly the domain of this function is R — { 2 }.

This function is undefined at $x = 2$.

But in this case we raise the question :

"Does $\dfrac{x^2 - 4}{x - 2}$ approach a definite real number as $x \to 2$? "

When $x \to 2$, there are two types of influences acting on f. The numerator approaches zero. The denominator also approaches zero by pushing the function to take large values. How these two opposing influences balance out ?

Since $x - 2 \neq 0$,

$$\frac{x^2 - 4}{x - 2} = \frac{(x + 2)(x - 2)}{(x - 2)} = x + 2.$$

So the behaviour of the function f near 2 but not equal to 2 is the same as the behaviour of $(x + 2)$ near 2 but not equal 2.

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2} (x + 2) = 4.$$

In this case $2 \notin$ dom f, but there are points in the domain that are arbitrarily close to 2. Intuitively we see that $\lim\limits_{x \to 2} f(x)$ is equal to 4.

In view of the preceding facts it is desirable to study the concept of limit in such a way that the limit of a function at a point, if it exists, can be established mathematically in a precise and logical manner.

The idea of **closeness** or **nearness** plays an important role in the discussion.

By writing $x \to a$, we mean that x is arbitrarily close to a and $x \neq a$.

We understand this only through our intuition. Mathematically we can not decide whether a point $x \neq a$ is close to a or not. This is because of the fact that there are infinitely many real numbers between x and a, and there is no objective standard which determines whether a point x is close to a or not.

Since mathematics is a language that gives clarity and precision to an expression with perfection, mathematically the expression

" x is arbitrarily close to a "

is not meaningful as yet. But if we say that **"a point x is at a distance less than δ ($\delta > 0$) from the point a**, then mathematically our expression is meaningful. So **the set of all $x \neq a$ such that x is at a distance less than any fixed $\delta > 0$ from a** serves the purpose of conveying the meaning of the expression

"x is arbitrarily close to a and $x \neq a$ ".

From the preceding discussion we conclude that perhaps the best way of exhibiting the idea of closeness is through neighbourhoods.

If **$a \in$ R, then any open interval containing the point a is called a neighbourhood of the point a.**

The **open interval (a —δ, $a + \delta$) is called the δ–neighbourhood of the point** a where $\delta > 0$.

(i) The set $(a - \delta, a + \delta) = \{ x \in R \mid a - \delta < x < a + \delta \}$

$$= \{ x \in R \mid \mid x - a \mid < \delta \}.$$

(ii) $(a - \delta, a + \delta) - \{ a \} = \{x \in R \mid a - \delta < x < a + \delta \text{ and } x \neq a \}$

$$= \{ x \in R \mid 0 < \mid x - a \mid < \delta \}.$$

(iii) $(a - \delta, a) = \{ x \in R \mid a - \delta < x < a \}$.

(iv) $(a, a + \delta) = \{ x \in R \mid a < x < a + \delta \}$.

The above discussion enables us to convey the concept of limit through the following definition.

Definition :

A number l is called the **limit** of a function f as x tends to a or simply $\lim_{x \to a} f(x) = l$ if for any $\in > 0$ there exists $\delta > 0$ depending on $\in$ such that

$0 < \mid x - a \mid < \delta \Rightarrow \mid f(x) - l \mid < \in$,

i.e. $a - \delta < x < a + \delta$ and $x \neq a \Rightarrow l - \in < f(x) < l + \in$.

Note : The choice of the number $\in$ is arbitrary, but as small as one wishes. The number $l = \lim_{x \to a} f(x)$ has nothing to do with the value of f at the point $x = a$ itself. Moreover the function f may not even be defined at $x = a$, i.e. the point a may or may not belong to the domain of f, but what is essential is that the function be defined in some deleted neighbourhood of the point a.

The number l provides some information on the behaviour of the function in a deleted neighbourhood of the point a. It conveys that when x assumes values different from a and approaches a according to an arbitrary law, $f(x)$ becomes arbitrarily close to l.

If there exists $\delta > 0$ such that a function f is not defined on $(a - \delta, a + \delta) - \{ a \}$, then the question of limit of the function as $x \to a$ does not arise.

We illustrate the preceding definition by some examples below.

Example 5:

$$\lim_{x \to a} x = a$$

Solution:

Let $\in$ be any positive number.

We take $f(x) = x$

Then $| f(x) - a | < \in$ if $| x - a | < \in$.

Taking $\delta = \in$ we see that there exists $\delta > 0$ depending on $\in$ such that

$$| x - a | < \delta \Rightarrow | f(x) - a | < \in$$

So $0 < | x - a | < \delta \Rightarrow | f(x) - a | < \in$.

Hence $\lim_{x \to a} f(x) = a$,

i.e. $\lim_{x \to a} x = a$.

Example 6:

Show that $\lim_{x \to 1} (2x + 1) = 3$.

Solution :

Let $\in$ be any positive number. Then $| (2x + 1) - 3 | < \in$ if $2 | x - 1 | < \in$,

i.e. if $| x - 1 | < \frac{\in}{2}$.

We set $\delta = \frac{\in}{2}$. Thus there exists $\delta > 0$ depending on $\in$ such that $|x - 1| < \delta \Rightarrow |(2x + 1) - 3| < \in$.

So $\lim_{x \to 1} (2x + 1) = 3$.

Example 7 :

Prove that $\lim_{x \to 4} \sqrt{x} = 2$.

Solution :

For the evaluation of this limit we must keep x within the domain of definition of $\sqrt{x}$.

The domain of $\sqrt{x}$ is the set of non-negative real numbers.

Since the distance between the point $x = 0$ and the point $x = 4$ is 4 we restrict x within 4 units of the point $x = 4$. Then $0 < x < 8$.

This implies that $-4 < x - 4 < 4$

$\Rightarrow |x-4| < 4.$

Given $\in > 0$ we seek for $\delta > 0$ depending on $\in$ such that

$0 < |x-4| < \delta \Rightarrow | \sqrt{x} - 2 | < \in.$

Since $|x-4| < 4$ we must take $0 < \delta \leq 4$. We have

$$|\sqrt{x}-2| = \left|\frac{(\sqrt{x}-2)(\sqrt{x}+2)}{\sqrt{x}+2}\right| = \frac{|x-4|}{\sqrt{x}+2}.$$

Since $\sqrt{x} + 2 \geq 2$, $\dfrac{1}{\sqrt{x}+2} \leq \dfrac{1}{2}$.

So $|\sqrt{x} - 2| = \dfrac{|x-4|}{\sqrt{x}+2} < \dfrac{\delta}{2}$.

In order to have $|\sqrt{x} - 2| < \in$ we choose $\dfrac{\delta}{2} < \in$, i.e. $\delta < 2\in$.

But $0 < \delta \leq 4$. So if we take $\delta = \min(4, 2\in)$, then

$$|x-4| < \delta \Rightarrow |\sqrt{x} - 2| < \in.$$

Thus given $\in > 0$ there exists $\delta > 0$ depending on $\in$ such that

$$|\sqrt{x} - 4| < \delta \Rightarrow |\sqrt{x} - 2| < \in.$$

Hence $\lim\limits_{x\to 4} \sqrt{x} = 2.$

EXERCISE 14 (a)

Use intuition and then $\in$–δ technique to obtain the following limits.

1. $\lim\limits_{x\to 3} (x+4)$
2. $\lim\limits_{x\to 1} (4x-1)$
3. $\lim\limits_{x\to 1} \left(\sqrt{x}+3\right)$
4. $\lim\limits_{x\to 0} (x^2+3)$
5. $\lim\limits_{x\to 0} 7$
6. $\lim\limits_{x\to 1} \dfrac{(x-1)^3}{(x-1)^3}$
7. $\lim\limits_{x\to 3} \dfrac{x^3-9}{x-3}$
8. $\lim\limits_{x\to 1} \dfrac{3x+2}{2x+3}$
9. $\lim\limits_{x\to 0} |x|$
10. $\lim\limits_{x\to 2} (|x|+3)$

Note: In the above exercises, you have been asked to compute the limit by determining δ whenever $\in$ is given. But this is not always easy. In such cases it is advisable to use certain theorems to determine the limits.

Laws of limits :

The following theorem is useful for evaluation of limits. Different parts of this theorem can be proved by using the $\in - \delta$ definition of the limit.

Theorem 1 :

If the functions f and g are such that $\lim\limits_{x \to a} f(x) = l$ and $\lim\limits_{x \to a} g(x) = m$, then

(i) $\lim\limits_{x \to a} \{f(x) + g(x)\} = \lim\limits_{x \to a} f(x) + \lim\limits_{x \to a} g(x) = l + m,$

i.e. the limit of a sum is equal to the sum of the limits.

(ii) $\lim\limits_{x \to a} \{f(x) - g(x)\} = \lim\limits_{x \to a} f(x) - \lim\limits_{x \to a} g(x) = l - m,$

i.e. the limit of a difference is equal to the difference of the limits.

(iii) $\lim\limits_{x \to a} \{\mathrm{k} f(x)\} = \mathrm{k} \lim\limits_{x \to a} f(x) = kl$

if k is a constant.

(iv) $\lim\limits_{x \to a} \{f(x)\, g(x)\} = \lim\limits_{x \to a} f(x) \,.\, \lim\limits_{x \to a} g(x) = lm,$

i.e. the limit of a product is equal to the product of the limits.

(v) $\lim\limits_{x \to a} \left\{\dfrac{f(x)}{g(x)}\right\} = \dfrac{\lim\limits_{x \to a} f(x)}{\lim\limits_{x \to a} g(x)} = \dfrac{l}{m}$ provided $m \neq 0,$

i.e. the limit of a quotient is equal to the quotient of the limits when the limit of the denominator is nonzero.

Proof of (i)

Let $\in$ be any positive numbers.

Since $\lim\limits_{x \to a} f(x) = l$, there exists $\delta_1 > 0$ depending on $\in$ such that

$$0 < |x - a| < \delta_1 \Rightarrow |f(x) - l| < \frac{\in}{2}.$$

Similarly since $\lim\limits_{x \to a} g(x) = \mathrm{m}$, there exists $\delta_2 > 0$ depending on $\in$ such

$$0 < |x - a| < \delta_2 \Rightarrow |g(x) - m| < \frac{\in}{2}.$$

So if $\delta = \min(\delta_1, \delta_2)$, then $|\{f(x) + g(x)\} - (l + m)|$

$$= |f(x) - l + g(x) - m|$$

$$\leq |f(x) - l| + |g(x) - m|$$

$$< \frac{\in}{2} + \frac{\in}{2} = \in \text{ for } 0 < |x - a| < \delta.$$

Thus $\lim_{x \to a} \{f(x) + g(x)\} = l + m = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$.

The proof of (i) is complete. Other parts are left as exercises.

Example 8 :

(i) $\lim_{x \to 1} \left(\sqrt{x} + x + \frac{1}{\sqrt{x}}\right)$

$= \lim_{x \to 1} \left(\sqrt{x}\right) + \lim_{x \to 1} (x) + \lim_{x \to 1} \frac{1}{\sqrt{x}}$ by theorem 1 (i)

$= 1 + 1 + 1 = 3.$

(ii) $\lim_{x \to 1} (17\sqrt{x}) = 17 \lim_{x \to 1} \sqrt{x} = 17.$ by theorem 1 (iii)

(iii) $\lim_{x \to 0} (2x + 1)(\sqrt{x} + 5)$

$= \lim_{x \to 0} (2x + 1) \lim_{x \to 0} \left(\sqrt{x} + 5\right)$ by theorem 1 (iv)

$= 1 . 5 = 5.$

(iv) $\lim_{x \to 1} \left(\frac{x + \sqrt{x}}{2x + 1}\right) = \left(\frac{\lim_{x \to 1}\left(x + \sqrt{x}\right)}{\lim_{x \to 1}(2x + 1)}\right) = \frac{1 + 1}{2 + 1} = \frac{2}{3}.$ by theorem 1 (v)

14.3 Left hand and right hand limit of a function.

From the definition for the limit of a function f at a point a we observe that if for any $\in > 0$ there exists $\delta > 0$ depending on $\in$ such that $a - \delta < x < a + \delta$ and $x \neq a \Rightarrow l - \in < f(x) < l + \in$, then $\lim_{x \to a} f(x) = l$.

$a - \delta < x < a + \delta$ and $x \neq a$ means

$a - \delta < x < a$ or $a < x < a + \delta$.

So if (i) $a - \delta < x < a \Rightarrow l - \in < f(x) < l + \in$

and (ii) $a < x < a + \delta \Rightarrow l - \in < f(x) < l + \in$,

then $\lim_{x \to a} f(x) = l$.

Proceeding from the previous discussions we introduce the notion of left-hand and right-hand limits of a function f at $x = a$.

The symbol $\lim_{x \to a-} f(x)$ stands for the left-hand limit of $f(x)$ at $x = a$. It means the value to which $f(x)$ approaches as x approaches a through the values of x less than a, i.e. from the left- hand side of a.

Similarly the symbol $\lim_{x \to a+} f(x)$ stands for the right-hand limit of $f(x)$ at $x = a$. It means the value to which $f(x)$ approaches as x approaches a through the values of x greater than a, i.e. from the right-hand side of a.

Definition :

A number l_1 is called the left-hand limit of $f(x)$ at $x = a$ or simply $\lim_{x \to a-} f(x) = l_1$ if for any $\in > 0$ there exists $\delta > 0$ depending on $\in$ such that

$a - \delta < x < a \Rightarrow |f(x) - l_1| < \in.$

Definition :

A number l_2 is called the right-hand limit of $f(x)$ at $x = a$ or simply $\lim\limits_{x \to a+} f(x) = l_2$ if for any $\in > 0$ there exists $\delta > 0$ depending on $\in$ such that

$a < x < a + \delta \Rightarrow |f(x) - l_2| < \in.$

Using the last two definitions and the definition of the limit of $f(x)$ at $x = a$ it is easy to see that

(1) if $l_1 = l_2 = l$,

i.e. $\lim\limits_{x \to a-} f(x) = \lim\limits_{x \to a+} f(x) = l$, then

$\lim\limits_{x \to a} f(x)$ exists and is equal to l,

i.e. $\lim\limits_{x \to a} f(x) = l$.

(2) If $l_1 \neq l_2$, then $\lim\limits_{x \to a} f(x)$ does not exist.

(3) For a function f to have a limit at a point $a \in R$ it is necessary and sufficient that both $\lim\limits_{x \to a-} f(x)$ and $\lim\limits_{x \to a+} f(x)$ exist, and these limits coincide.

Example 9 :

Examine $\lim\limits_{x \to 0}$ Sgn x.

Solution :

We have $\text{Sgn } x = \begin{cases} -1 \text{ if } x < 0 \\ 0 \text{ if } x = 0 \\ 1 \text{ if } x > 0. \end{cases}$

From the definition it follows that

$$\lim_{x \to 0-} \text{Sgn } x = \lim_{x \to 0-} (-1) = -1$$

and $$\lim_{x \to 0+} \text{Sgn } x = \lim_{x \to 0+} 1 = 1$$

since $\lim\limits_{x \to 0-} \text{Sgn } x \neq \lim\limits_{x \to 0+} \text{Sgn } x$, $\lim\limits_{x \to 0}$ Sgn x does not exist.

Example 10 :

Prove that $\lim\limits_{x \to n} [x]$ does not exist for any $n \in Z$.

Solution :

$$\lim_{x \to n-} [x] = n - 1$$

and $$\lim_{x \to n+} [x] = n.$$

Since $n - 1 \neq n$ for any $n \in Z$,

$\lim\limits_{x \to n-} [x] \neq \lim\limits_{x \to n+} [x]$ for any $n \in Z$

So $\lim\limits_{x \to n} [x]$ does not exist for any $n \in Z$.

14.4 Infinite limits and limits at infinity.

The special symbols ∞ and $-\infty$, called infinity and minus infinity respectively are used to deal with problems in mathematics under special situations. It should be borne in mind that ∞ and $-\infty$ are not real numbers. We illustrate below the concepts of ∞ and $-\infty$ by means of an example.

Let us consider the function f defined by $f(x) = \dfrac{1}{x}$, x is real, $x \neq 0$.

Here f (0) is undefined.

But intuitively we observe that when $x \to 0+$, i.e. x approaches 0 through positive values, $\dfrac{1}{x}$ becomes indefinitely larger and larger. In fact the value of $f(x)$ can be made larger than any positive number by taking positive values of x sufficiently close to zero.

Under this situation we write

$$\lim_{x \to 0+} \frac{1}{x} = \infty.$$

Similarly by applying our intuition we see that when $x \to 0-$, i.e. x approaches 0 through negative values, $\dfrac{1}{x}$ decreases indefinitely through negative values. In this case the value of $f(x)$ can be made smaller than any negative number by choosing negative values of x sufficiently close to 0.

This situation is described by writing

$$\lim_{x \to 0-} f(x) = -\infty.$$

Note that the notation $\lim\limits_{x \to 0+} f(x) = \infty$ does not mean that $\lim\limits_{x \to 0+} f(x)$ exists.

Similarly the notation $\lim\limits_{x \to 0-} f(x) = -\infty$ does not mean that $\lim\limits_{x \to 0-} f(x)$ exists.

We are led to the following definition.

Definition :

$\lim\limits_{x \to a} f(x) = \infty$ if given $G > 0$, there exists $\delta > 0$ depending on G such that $0 < |x - a| < \delta \Rightarrow f(x) > G$.

Definition :

$\lim\limits_{x \to a} f(x) = -\infty$ if given $G > 0$, there exists $\delta > 0$ depending on G such that $0 < |x - a| < \delta \Rightarrow f(x) < -G$.

Note that in the preceding definitions we can choose G as large as we please.

Example 11 :

Show that $\lim_{x \to 0+} \dfrac{1}{x\,(x+1)} = \infty$ and $\lim_{x \to 0-} \dfrac{1}{x\,(x+1)} = -\infty$.

Solution :

Let G be any positive number, however large.

Then given G > 0, there exists $\delta > 0$ depending on G such that

$$\text{if } 0 < x < x\;(1+x) < \frac{1}{G}, \text{ then } \frac{1}{x\,(1+x)} > G.$$

We take $\delta = \dfrac{1}{G}$.

Thus given G > 0, there exists $\delta > 0$ depending on G such that

$$0 < x < \delta \Rightarrow \frac{1}{x\,(1+x)} > G.$$

So $\lim_{x \to 0+} \dfrac{1}{x\,(1+x)} = \infty$.

Again for large G, $-\dfrac{1}{G} < x < 0$ implies $0 < x + 1 < 1$. So

$$-\frac{1}{G} < x < (x+1)\,x < 0 \text{ so that } \frac{1}{x\,(x+1)} < -G.$$

Hence $\lim_{x \to 0-} \dfrac{1}{x\,(x+1)} = -\infty$.

Example 12 :

Show that $\lim_{x \to 0} \dfrac{1}{x^2} = \infty$.

Solution :

Let G be any positive number, however large.

Then $\dfrac{1}{x^2} > G$ if $x^2 < \dfrac{1}{G}$, $x \neq 0$,

i.e. if $-\dfrac{1}{\sqrt{G}} < x < \dfrac{1}{\sqrt{G}}$, $x \neq 0$,

i.e. if $0 < |\,x\,| < \dfrac{1}{\sqrt{G}}$.

We take $\delta = \dfrac{1}{\sqrt{G}}$.

Thus given G > 0, there exists $\delta > 0$ depending on G such that

$$0 < |\,x\,| < \delta \Rightarrow \frac{1}{x^2} > G.$$

So $\lim\limits_{x \to 0} \dfrac{1}{x^2} = \infty$.

Definition :

$\lim\limits_{x \to \infty} f(x) = l$ if given $\in > 0$, there

exists $G > 0$ depending on $\in$ such that

$x > G \Rightarrow |f(x) - l| < \in$.

Definition :

$\lim\limits_{x \to -\infty} f(x) = l$ if given $\in > 0$, there

exists $G > 0$ depending on $\in$ such that

$x < -G \Rightarrow |f(x) - l| < \in$.

Definition :

$\lim\limits_{x \to \infty} f(x) = \infty$ if given $G > 0$, there

exists $k > 0$ such that

$x > k \Rightarrow f(x) > G$.

The concepts $\lim\limits_{x \to \infty} f(x) = -\infty$ and $\lim\limits_{x \to -\infty} f(x) = \infty$ can be defined similarly.

Examples 13 :

Show that $\lim\limits_{x \to \infty} \dfrac{1}{x} = 0$.

Solution :

We choose $\in > 0$.

Then $\left| \dfrac{1}{x} - 0 \right| = \left| \dfrac{1}{x} \right| = \dfrac{1}{x} < \in$ if $x > \dfrac{1}{\in}$. $\quad (\because x > 0)$

We take $G = \dfrac{1}{\in}$.

Thus given $\in > 0$, there exists $G > 0$ depending on $\in$ such that

$x > G \Rightarrow \left| \dfrac{1}{x} - 0 \right| < \in$.

So $\dfrac{1}{x} \to 0$ as $x \to \infty$,

i.e. $\lim\limits_{x \to \infty} \dfrac{1}{x} = 0$.

Note: If x increases indefinitely through positive values, $\dfrac{1}{x}$ remains positive and decreases indefinitely.

So intuitively we see that $\dfrac{1}{x} \to 0$ as $x \to \infty$.

Remark :

If $\lim\limits_{x \to \infty} f(x) = l$ and $\lim\limits_{x \to \infty} g(x) = m$, then all the conclusions of Theorem–I hold if a is replaced by ∞. Similar is the case if a is replaced by $-\infty$.

Example 14 :

Evaluate $\lim\limits_{x \to \infty} \dfrac{3x^2 + 4x - 1}{2x^2 - 3x + 5}$.

Solution :

$$\lim_{x \to \infty} \frac{3x^2 + 4x - 1}{2x^2 - 3x + 5}$$

$$= \lim_{x \to \infty} \frac{3 + \frac{4}{x} - \frac{1}{x^2}}{2 - \frac{3}{x} + \frac{5}{x^2}}$$

$$= \frac{\lim\limits_{x \to \infty}\left(3 + \frac{4}{x} - \frac{1}{x^2}\right)}{\lim\limits_{x \to \infty}\left(2 - \frac{3}{x} + \frac{5}{x^2}\right)}$$

$$= \frac{\lim\limits_{x\to\infty} 3 + \lim\limits_{x\to\infty} \frac{4}{x} - \lim\limits_{x\to\infty} \frac{1}{x^2}}{\lim\limits_{x\to\infty} 2 - \lim\limits_{x\to\infty} \frac{3}{x} + \lim\limits_{x\to\infty} \frac{5}{x^2}}$$

$$= \frac{3 + 0 - 0}{2 - 0 + 0} = \frac{3}{2}.$$

We state below the following theorems without proof. These theorems are useful for solving some problems on limits.

Theorem 2 :

(i) If a function f satisfies the inequality $f(x) > M > 0$ in a neighbourhood of a and a function g is such that $\lim\limits_{x \to a} g(x) = 0$, $g(x) \neq 0$ for $x \neq a$ and $g(x) > 0$, then $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)} = \infty$.

(ii) If $f(x) > M > 0$ in a neighbourhood of the point a and if g is such that $\lim\limits_{x \to a} g(x) = 0$, $g(x) \neq 0$ for $x \neq a$ and $g(x) < 0$, then $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)} = -\infty$.

Theorem 3 :

(Sandwich theorem or squeezing theorem)

If $\lim\limits_{x \to a} f(x) = \lim\limits_{x \to a} g(x) = l$ and a function ϕ is such that $f(x) \leq \phi(x) \leq g(x)$ for all x in a deleted neighbourhood of a, then $\lim\limits_{x \to a} \phi(x) = l$.

Example 15 :

Find the limit as $x \to 0$ of the following function on R :

$$\phi(x) = \begin{cases} x \sin \dfrac{1}{x} & (x \neq 0) \\ 0 & (x = 0). \end{cases}$$

Solution :

Since $\left| \sin \dfrac{1}{x} \right| \leq 1$, it follow that

$$0 \leq |\phi(x)| = \left| x \sin \frac{1}{x} \right| \leq |x|.$$

With $f(x) = 0$, $g(x) = |x|$ we have $\lim\limits_{x \to 0} f(x) = 0$, $\lim\limits_{x \to 0} g(x) = 0$

and so by squeezing theorem $\lim\limits_{x \to 0} \phi(x) = 0$.

14.5 Further results on limits and their applications

Convergence of a sequence:

Definition:

A number l is called the limit of a sequence $\{x_n\}$ if given $\in > 0$, there exists $m \in N$ such that

$$|x_n - l| < \in \text{ for } n > m.$$

If l is the limit of a sequence $\{x_n\}$, then we write $\lim\limits_{n \to \infty} x_n = l$.

Definition:

A sequence $\{x_n\}$ is said to be convergent if $\lim\limits_{n \to \infty} x_n$ exists; otherwise it is called divergent.

If $\lim\limits_{n \to \infty} x_n = l$, then we say that the sequence $\{x_n\}$ converges to l.

Example 16 :

Prove that the sequence $\left\{ \dfrac{1}{n} \right\}$ is convergent.

Solution :

Let $\in > 0$.

Then $\dfrac{1}{n} < \in$ if $n > \dfrac{1}{\in}$

We can choose $m \in N$ such that $n > m > \dfrac{1}{\in}$.

Thus there exists $m \in N$ such that

$$\left| \frac{1}{n} - 0 \right| = \frac{1}{n} < \in \text{ for } n > m.$$

So the sequence $\left\{ \dfrac{1}{n} \right\}$ converges to 0;

i.e. $\lim_{n \to \infty} \frac{1}{n} = 0$.

Examples 17 :

Prove that the sequence $\{ n^2 \}$ is divergent.

Solution :

Let G be any +ve number, however large.

Then $n > \sqrt{G} \Rightarrow n^2 > G$.

We can find $m \in N$ such that

$n > m \geq \sqrt{G}$.

So $n > m \Rightarrow n^2 > G$.

Hence $\lim_{n \to \infty} n^2 = \infty$.

Thus the sequence $\{ n^2 \}$ is divergent.

Theorem 4 : (Without proof)

A sequence $\{ x_n \}$ converges to l if and only if every sub-sequence of $\{ x_n \}$ converges to l.

Example 18 :

The sequence $\left\{(-1)^n\right\}$ is divergent.

Solution :

Write $x_n = (-1)^n$.

Then $x_1 = -1$, $x_2 = 1$, $x_3 = -1$, $x_4 = 1$, ……………………

i.e. $x_{2k} = (-1)^{2k} = 1$, $x_{2x+1} = (-1)^{2k+1} = -1$.

Then the value of x_n is 1 if n is even, -1 if n is odd. In other words this sequence **oscillates**. We may see that

$$\lim_{k \to \infty} x_{2k} = \lim_{k \to \infty} 1 = 1$$

$$\lim_{k \to \infty} x_{2k+1} = \lim_{k \to \infty} (-1) = -1.$$

So by Theorem 4, the sequence is divergent.

Now we show how limit of a function can be decided through limit of a sequence.

Theorem 5: (Without Proof)

$$\lim_{x \to a} f(x) = l \Leftrightarrow \lim_{n \to \infty} f(x_n) = l$$

for every real sequence $\{ x_n]$ with $x_n \neq a$

for any $n \in N$ and $\lim_{n \to \infty} x_n = a$.

N.B. (This result will be proved in higher classes. It has wide applications. The next example is an illustration)

Example 19:

Prove that $\lim\limits_{x \to 0} \sin\frac{1}{x}$ does not exist.

Solution :

Let $f(x) = \sin\frac{1}{x}$, $x \neq 0$.

Let $\{x_n\}$ be a sequence defined by

$$x_n = \frac{2}{(2n+1)\pi} \quad \forall\, n \in N.$$

clearly $x_n \neq 0$ for any $n \in N$ and

$$\lim_{n \to \infty} x_n = \lim_{n \to \infty} \frac{2}{(2n+1)\pi} = 0.$$

We have $f(x_n) = \sin\frac{1}{x_n}$

$$= \sin(2n+1)\frac{\pi}{2}$$

$$= (-1)^n \;\forall\, n \in N.$$

$\{f(x_n)\}$ is a sequence in R.

Consider the subsequences $\{f(x_{2n})\}$ and $\{f(x_{2n-1})\}$ of the sequence $\{f(x_n)\}$.

$$\lim_{n \to \infty} f(x_{2n-1}) = \lim_{n \to \infty} (-1)^{2n-1} = \lim_{n \to \infty} (-1) = -1.$$

$$\lim_{n \to \infty} f(x_{2n}) = \lim_{n \to \infty} (-1)^{2n} = \lim_{n \to \infty} 1 = 1.$$

Since $\lim\limits_{n \to \infty} f(x_{2n-1}) \neq \lim\limits_{n \to \infty} f(x_{2n})$, $\lim\limits_{n \to \infty} f(x_n)$ does not exist, by Theorem 4.

So by Theorem 5, $\lim\limits_{x \to 0} \sin\frac{1}{x}$ does not exist.

EXERCISE 14 (b)

1. Using the $\in - \delta$ definition prove that

(i) $\lim\limits_{x \to 0} (2x+3) = 3$ (ii) $\lim\limits_{x \to 1} (2x-1) = 1$

(iii) $\lim\limits_{x \to -2} (3x+8) = 2$ (iv) $\lim\limits_{x \to 3} (x^2+2x-8) = 7$

(v) $\lim\limits_{x \to 9} \sqrt{x} = 3$ (vi) $\lim\limits_{x \to a} \sqrt{x} = \sqrt{a}$, $a > 0$.

(vii) $\lim\limits_{x \to 1} |3x+2| = 5$ (viii) $\lim\limits_{x \to 2} |5x-7| = 3$.

2. If $\lim_{x \to a} f(x) = l$, then prove that $\lim_{x \to a} |f(x)| = |l|$.

Is the converse true ? Justify your answer with reasons.

3. (i) Prove that $\lim_{x \to a} x = a$.

(ii) Using (i) and the laws of limits prove that

$$\lim_{x \to a} x^n = a^n, \text{ where } n \text{ is an integer.}$$

(iii) Using (ii) and the laws of limits prove that

$$\lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1} \text{ where } n \text{ is an integer.}$$

(iv) Using (iii), the laws of limits and assuming that

$$\lim_{x \to a} x^{\frac{1}{m}} = a^{\frac{1}{m}}$$

where m is a non-zero integer prove that for any rational number n,

$$\lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}.$$

4. Evaluate the following :

(i) $\lim_{x \to 1} (1 + 2x - 3x^2 + 4x^3 - 5x^4)$

(ii) $\lim_{x \to 0} (3x^2 + 4x - 1)(x^4 + 2x^3 - 3x^2 + 5x + 2)$

(iii) $\lim_{x \to 2} \dfrac{x^2 + 3x - 9}{x + 1}$

(iv) $\lim_{x \to 3} \dfrac{x^2 - 9}{x - 3}$

(v) $\lim_{x \to 1} \dfrac{x^3 - 1}{x - 1}$

(vi) $\lim_{x \to 2} \dfrac{x - 2}{x^4 - 16}$

(vii) $\lim_{x \to 2} \dfrac{x^3 - 8}{x^5 - 32}$

(viii) $\lim_{x \to 3} \dfrac{x^2 + 2x - 15}{x^2 - x - 6}$

(ix) $\lim_{x \to 0} \dfrac{(3 + x)^3 - 27}{x}$

(x) $\lim_{x \to 2} \dfrac{\dfrac{1}{x^2} - \dfrac{1}{4}}{x - 2}$

(xi) $\lim\limits_{x\to 1} \dfrac{1}{(x-1)} \left\{\dfrac{1}{x+3} - \dfrac{2}{3x+5}\right\}$

(xii) $\lim\limits_{h\to 0} \dfrac{(x+h)^3 - x^3}{h}$

(xiii) $\lim\limits_{h\to 0} \dfrac{(x+h)^4 - x^4}{h}$

(xiv) $\lim\limits_{x\to 1} \dfrac{x^m - 1}{x^n - 1}$, where m, n are integers.

(xv) $\lim\limits_{x\to 1} \dfrac{x^2 - 2x + 1}{x^2 - x}$

(xvi) $\lim\limits_{x\to 1} \dfrac{x^2 + x - 2}{x^3 - x^2 - x + 1}$.

5. Evaluate the following :

(i) $\lim\limits_{x\to\infty} \dfrac{2x+1}{3x-2}$

(ii) $\lim\limits_{x\to\infty} \dfrac{3x^2 + x - 1}{2x^2 - 7x + 5}$

(iii) $\lim\limits_{x\to\infty} \dfrac{x^3 + 2x^2 + 3}{x^4 - 3x^2 + 1}$

(iv) $\lim\limits_{x\to\infty} \dfrac{x^4 - 5x + 2}{x^3 - 3x + 1}$

(v) $\lim\limits_{x\to\infty} \left(\dfrac{x^3}{2x^2 - 1} - \dfrac{x^2}{2x+1}\right)$

(vi) $\lim\limits_{n\to\infty} \dfrac{n}{n+1}$

(vii) $\lim\limits_{n\to\infty} \dfrac{n^2 + n + 1}{5n^2 + 2n + 1}$

(viii) $\lim\limits_{n\to\infty} \left(\dfrac{\sqrt{n} - 1}{\sqrt{n} + 1}\right)$

(ix) $\lim\limits_{n\to\infty} \left(\dfrac{6n^5 + 2n + 1}{n^5 + n^4 + 3n^3 + 2n^2 + n + 1}\right)$

(x) $\lim\limits_{n\to\infty} \dfrac{1 + 2 + 3 + \ldots\ldots\ldots + n}{n^2}$

(xi) $\lim\limits_{n\to\infty} \dfrac{1^2 + 2^2 + 3^2 + \ldots\ldots\ldots + n^2}{n^3}$

(xii) $\lim\limits_{n\to\infty} \dfrac{1^3 + 2^3 + 3^3 + \ldots\ldots\ldots + n^3}{n^4}$

(xiii) $\lim\limits_{n\to\infty} \dfrac{1 + \frac{1}{2} + \frac{1}{2^2} + \ldots\ldots + \frac{1}{2^n}}{1 + \frac{1}{3} + \frac{1}{3^2} + \ldots\ldots + \frac{1}{3^n}}$

(xiv) $\lim\limits_{n\to\infty} \dfrac{\lfloor n}{\lfloor (n+1) - \lfloor n}$.

6. Examine the existence of the following limits.

(i) $\lim\limits_{x\to\sqrt{3}} [\,x\,]$

(ii) $\lim\limits_{x\to 0} [\,x\,]$

(iii) $\lim\limits_{x\to -2} [\,x\,]$

(iv) $\lim\limits_{x\to 0} \dfrac{|\,x\,|}{x}$

(v) $\lim\limits_{x \to 2} \dfrac{x - 2}{|x - 2|}$ (vi) $\lim\limits_{x \to \frac{1}{2}} \dfrac{|2x - 1|}{2x - 1}$

(vii) $\lim\limits_{x \to 1} [2x + 3]$ (viii) $\lim\limits_{x \to \infty} \dfrac{x}{[x]}$

(ix) $\lim\limits_{x \to \infty} \dfrac{x^2 - x}{[x^2 - x]}$ (x) $\lim\limits_{x \to 1} \dfrac{|x^2 - 3x + 1|}{x^2 - 3x + 2}$

(xi) $\lim\limits_{x \to \infty} (-1)^{[x]}$ (xii) $\lim\limits_{x \to \infty} \sin x$

(xiii) $\lim\limits_{x \to \infty} \cos x$ (xiv) $\lim\limits_{x \to 0} \cos \dfrac{1}{x}$

(xv) $\lim\limits_{x \to 0} \sin \dfrac{1}{x}$

(xvi) $\lim\limits_{x \to 1} f(x)$ if $f(x) = \begin{cases} 2x - 1, & x \le 1 \\ 2x + 1, & x > 1 \end{cases}$

(xvii) $\lim\limits_{x \to 0} f(x)$ and $\lim\limits_{x \to 1} f(x)$ if $f(x) = \begin{cases} 0, & x \le 0 \\ 1 - 2x, & 0 < x \le 1 \\ 3 - 4x, & x > 1. \end{cases}$

7. Let $f(x) = \begin{cases} 1 \text{ if } x \text{ is rational} \\ 0 \text{ if } x \text{ is irrational.} \end{cases}$

Then show that $\lim\limits_{x \to a} f(x)$ does not exist for any $a \in R$.

14.6 Limits of trigonometric functions

We need the following inequality for the study of limits of sine and cosine functions. We state below this inequality in the form of a theorem.

Theorem 6:

For any $\theta \in R$, $|\sin\theta| \le |\theta|$.

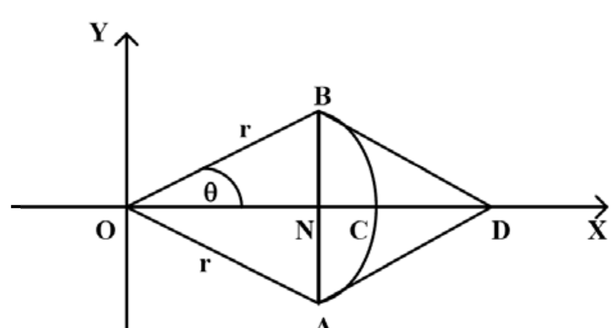
Proof:

For $\theta = 0$, $|\sin\theta| = |\theta|$ is quite trivial.

Let the arc AB of a circle with centre at O and radius r be bisected at the point C by the x-axis.

Let the radian measure of $\angle COB = \theta$, where $0 < \theta < \dfrac{\pi}{2}$.

Clearly x-axis is the perpendicular bisector of the chord AB.



Since the length of a chord subtending a circular arc is less than the length of the arc, the length of the chord AB is less than the length of the arc AB.

So half the length of chord AB is less that half the length of the arc AB,

i.e. $r \sin\theta < r\theta$

or, $\dfrac{\sin\theta}{\theta} < 1.$

Since $\dfrac{\sin\theta}{\theta}$ is an even function of θ this inequality holds not only for the positive values of θ but also for negative values of θ satisfying the inequalities $0 < |\theta| < \dfrac{\pi}{2}$.

For $0 < |\theta| < \dfrac{\pi}{2}$ that is $-\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$ and $\theta \neq 0$,

$\dfrac{\sin\theta}{\theta} > 0.$

So $\dfrac{|\sin\theta|}{|\theta|} = \left|\dfrac{\sin\theta}{\theta}\right| = \dfrac{\sin\theta}{\theta} < 1.$

This gives $|\sin\theta| < |\theta|$.

Let $|\theta| \geq \dfrac{\pi}{2}$.

Since the maximum value of $|\sin\theta|$ $\forall\, \theta \in R$ is 1 and$1 < \dfrac{\pi}{2}$ we must have $|\sin\theta| < |\theta|$.

Thus $|\sin\theta| \leq |\theta| \;\; \forall\, \theta \in R$.

Theorem 7 :

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1.$$

Proof :

We refer to the figure given for the proof of the last theorem, where $0 < \theta < \dfrac{\pi}{2}$. In that figure AD and BD are tangents to the arc ACB at A and B respectively. Since half the length of the chord subtending a circular arc is less then half the length of the arc and half the length of the arc is less than BD (=AD) we must have

NB < length of arc CB < BD,

i.e. $r \sin\theta < r\theta < r \tan\theta$ $\qquad \left(\because m\ \angle OBD = \dfrac{\pi}{2}\right)$

or, $\sin\theta < \theta < \tan\theta$

or, $1 < \dfrac{\theta}{\sin\theta} < \dfrac{1}{\cos\theta}$ $\qquad (\because \sin\theta > 0)$

or, $\cos\theta < \dfrac{\sin\theta}{\theta} < 1.$

Since cosθ is a continuous function of θ (see the next example),

$\lim\limits_{\theta \to 0+} \cos\theta = \cos 0 = 1.$

[Note : We use the property of a function which will be proved while discussing continuity in Vol-II next year. However you can convince yourself of this property of a continuous function that its graph does not have any gaps/breaks. So $\cos\theta$, when $\theta \to 0$-, shall approach $\cos 0$ which is equal to 1. In general, a continuous function, in the limit, approaches its functional value.]

So by sandwich theorem $\lim\limits_{\theta \to 0+} \dfrac{\sin\theta}{\theta} = 1.$

Let $-\dfrac{\pi}{2} < \theta < 0.$

If we take $\theta = -\varphi$, then $0 < \varphi < \dfrac{\pi}{2}$ and $\varphi \to 0+$ as $\theta \to 0-$.

So $\lim\limits_{\theta \to 0-} \dfrac{\sin\theta}{\theta} = \lim\limits_{\varphi \to 0+} \dfrac{\sin(-\varphi)}{-\varphi} = \lim\limits_{\varphi \to 0+} \dfrac{-\sin\varphi}{-\varphi} = \lim\limits_{\varphi \to 0+} \dfrac{\sin\varphi}{\varphi} = 1.$

Thus $\lim\limits_{\theta \to 0-} \dfrac{\sin\theta}{\theta} = \lim\limits_{\theta \to 0+} \dfrac{\sin\theta}{\theta} = 1.$

So $\lim\limits_{\theta \to 0} \dfrac{\sin\theta}{\theta}$ exists and is equal to 1,

i.e. $\lim\limits_{\theta \to 0} \dfrac{\sin\theta}{\theta} = 1.$

14.7 Limit of Exponential and Logarithmic functions

For this purpose we develop the concept of

Monotonic Sequences.

Definition :

Let $\{x_n\}$ be a real sequence.

The sequence $\{x_n\}$ is said to be monotonic increasing or non-decreasing if $x_n \le x_{n+1}$ $\forall\, n \in$ N. If $x_n < x_{n+1}$ $\forall\, n \in$ N, then the sequence is strictly increasing.

If $x_n \ge x_{n+1}$ $\forall$ n $\in$ N, then the sequence is monotonic decreasing.

If $x_n > x_{n+1}$ $\forall\, n \in$ N, then the sequence is strictly decreasing.

Bounded Sequence

A real sequence $\{x_n\}$ is said to be bounded above if there exists a real number M such that $x_n \le$ M $\forall\, n \in$ N. The number M is called an upper bound of the sequence.

If there exists a real number m such that $x_n \ge$ m $\forall\, n \in$ N, then m is called a lower bound of the sequence.

If there exists a real number K > 0 such that $|x_n| \le$ K $\forall\, n \in$ N, then the sequence $\{x_n\}$ is said to be bounded.

A sequence $\{x_n\}$ is bounded iff it is bounded below and bounded above, We need the following theorems for our purpose. The proofs of these theorems appear in higher mathematics.

Theorem 8:

(a) A monotonic increasing sequence $\{ x_n \}$ bounded above is convergent, i.e. x_n tends to a limit as $n \to \infty$.

(b) Monotonic decreasing sequence $\{ x_n \}$ bounded below is convergent, i.e. x_n tends to a limit as $n \to \infty$.

With the preceding information in our hand we discuss below the $\lim\limits_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$, where $n \in \mathbb{N}$.

Example 20

Consider convergence of $x_n = \left(1 + \frac{1}{n}\right)^n \ \forall\, n \in \mathbb{N}$.

Using binomial expansion, we have

$$x_n = \left(1 + \frac{1}{n}\right)^n = 1 + n \cdot \frac{1}{n} + \frac{n(n-1)}{\lfloor 2} \cdot \frac{1}{n^2} + \ldots\ldots\ldots\ldots + \frac{1}{n^n}$$

$$= 1 + 1 + \frac{1}{\lfloor 2}\left(1 - \frac{1}{n}\right) + \frac{1}{\lfloor 3}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) + \ldots\ldots \text{ to } (n+1) \text{ terms.}$$

$$< 1 + 1 + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} + \ldots\ldots\ldots\ldots + \frac{1}{\lfloor n}$$

$$< 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \ldots\ldots\ldots\ldots\ldots + \frac{1}{2^{n-1}}$$

$$= 1 + \frac{1 - \frac{1}{2^n}}{1 - \frac{1}{2}}$$

$$= 1 + 2\left(1 - \frac{1}{2^n}\right) < 1 + 2 = 3 \ \forall\, n \in \mathbb{N}.$$

So the sequence $\{ x_n \}$ is bounded above.

Further $x_n = 1 + 1 + \frac{1}{\lfloor 2}\left(1 - \frac{1}{n}\right) + \frac{1}{\lfloor 3}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) + \ldots\ldots\ (n+1)$ terms

$$< 1 + 1 + \frac{1}{\lfloor 2}\left(1 - \frac{1}{n+1}\right) + \frac{1}{\lfloor 3}\left(1 - \frac{1}{n+1}\right)\left(1 - \frac{2}{n+1}\right) + \ldots \text{ to } (n+2) \text{ terms}$$

$$= x_{n+1} \ \forall\, n \in \mathbb{N}.$$

Thus the sequence $\{ x_n \}$ is an increasing sequence bounded above.

So x_n tends to a limit as $n \to \infty$. This limit is denoted by the symbol e.

So $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e$.

$$\left(1+\frac{1}{n}\right)^n = 1 + 1 + \frac{1}{\lfloor 2}\left(1-\frac{1}{n}\right) + \ldots\ldots\ldots\ldots\ldots \text{ to } (n+1) \text{ terms} > 2 \ \forall\ n \geq 2.$$

Thus $2 < \left(1+\frac{1}{n}\right)^n < 3 \ \forall\ n \in \mathrm{N}.$

So $2 < \lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^n < 3,$

i.e. $2 < e < 3$.

Note : *e* is an irrational number and equals 2.71818284 … *e* is used as base of natural logarithm.

Example 21

Prove that $\lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e.$

Solution :

Since $x \to \infty$, we may assume that $x > 1$.

So there exists *a* +ve integer n such that $n \leq x < n+1$.

This implies that $\frac{1}{n} \geq \frac{1}{x} > \frac{1}{n+1}$.

Hence $1+\frac{1}{n} \geq 1+\frac{1}{x} > 1+\frac{1}{n+1}$.

So $\left(1+\frac{1}{n}\right)^{n+1} > \left(1+\frac{1}{x}\right)^x > \left(1+\frac{1}{n+1}\right)^n$.

$x \to \infty \iff n \to \infty.$

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n+1} = \lim_{n\to\infty}\left\{\left(1+\frac{1}{n}\right)\left(1+\frac{1}{n}\right)^n\right\}$$

$$= \lim_{n\to\infty}\left(1+\frac{1}{n}\right) . \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

$$= 1.e. = e.$$

$$\lim_{n\to\infty}\left(1+\frac{1}{n+1}\right)^n = \lim_{n\to\infty}\frac{\left(1+\frac{1}{n+1}\right)^{n+1}}{1+\frac{1}{n+1}}$$

$$= \lim_{m\to\infty}\frac{\left(1+\frac{1}{m}\right)^m}{1+\frac{1}{m}}, \text{ where } m = n+1$$

$$= \frac{\operatorname{Lim}\limits_{m\to\infty}\left(1+\frac{1}{m}\right)^m}{\operatorname{Lim}\limits_{m\to\infty}\left(1+\frac{1}{m}\right)} = \frac{e}{1} = e.$$

So using the sandwich theorem in the last inequalities we get

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e.$$

Corollary 1 : $\lim\limits_{x\to-\infty}\left(1+\frac{1}{x}\right)^x = \lim\limits_{y\to\infty}\left(1-\frac{1}{y}\right)^{-y}$

$$= \lim_{y\to\infty}\left(\frac{y}{y-1}\right)^y$$

$$= \lim_{y\to\infty}\left(1+\frac{1}{y-1}\right)^y$$

$$= \lim_{y\to\infty}\left\{\left(1+\frac{1}{y-1}\right)^{y-1}\left(1+\frac{1}{y-1}\right)\right\}$$

$$= \lim_{z\to\infty}\left\{\left(1+\frac{1}{z}\right)^z\left(1+\frac{1}{z}\right)\right\}, \text{ where } z = y-1$$

$$= \lim_{z\to\infty}\left(1+\frac{1}{z}\right)^z \times \lim_{z\to\infty}\left(1+\frac{1}{z}\right)$$

$$= e \times 1 = e.$$

Corollary 2 : $\lim\limits_{x\to 0}(1+x)^{\frac{1}{x}} = e.$

Proof :

$$\lim_{x\to 0+}(1+x)^{\frac{1}{x}} = \lim_{y\to\infty}\left(1+\frac{1}{y}\right)^y, \text{ where } y = \frac{1}{x}$$

$$= e.$$

$$\lim_{x\to 0-}(1+x)^{\frac{1}{x}} = \lim_{y\to-\infty}\left(1+\frac{1}{y}\right)^y, \text{ where } y = \frac{1}{x}$$

$$= e.$$

So $\lim\limits_{x\to 0}(1+x)^{\frac{1}{x}} = e.$

Example 22

Prove that $\lim\limits_{x\to 0}\frac{\log_a(1+x)}{x} = \frac{1}{\ln a}$. $(a > 0)$.

Proof :

$$\lim_{x\to 0}\frac{\log_a(1+x)}{x} = \lim_{x\to 0}\log_a(1+x)^{\frac{1}{x}}$$

$$= \log_a\left\{\lim_{x\to 0}(1+x)^{\frac{1}{x}}\right\}$$

[$\because$ Logarithmic function is 'continuous' on its domain: refer to the note attached to theorem – 7, i.e. proof of $\lim\limits_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$]

$$= \log_a e$$

$$= \frac{1}{\log_e a}$$

$$= \frac{1}{\ln a}.$$

Corollary : $\lim_{x \to 0} \frac{ln\,(1 + x)}{x} = 1.$ $(\because\ lne = 1)$

Example 23

Prove that $\lim_{x \to 0} \frac{a^x - 1}{x} = \ln a.\ (a > 0)$

Proof :

Let $a^x - 1 = y$

Then $y \to 0$ as $x \to 0$.

$a^x - 1 = y \ \Rightarrow\ a^x = 1 + y$

$\Rightarrow\ x = \log_a (1 + y)$

So $\lim_{x \to 0} \frac{a^x - 1}{x} = \lim_{y \to 0} \frac{y}{\log_a (1 + y)}$

$$= \lim_{y \to 0} \frac{1}{\frac{\log_a (1 + y)}{y}}$$

$$= \frac{1}{\frac{1}{\ln a}} = \ln a.$$

Cor. $\lim_{x \to 0} \frac{e^x - 1}{x} = 1.$

Example 24

Prove that $\lim_{x \to a} \frac{x^\alpha - a^\alpha}{x - a} = \alpha\, a^{\alpha - 1}, \forall \alpha \in \mathrm{R}\ (x > 0, a > 0)$

Proof :

Let $\alpha \neq 0$.

Let $x = a\,(1 + y)$.

Then $y \to 0$ as $x \to a$.

So $\lim_{x \to a} \frac{x^\alpha - a^\alpha}{x - a} = a^{\alpha - 1} \lim_{y \to 0} \frac{(1 + y)^\alpha - 1}{y}$

$$= a^{\alpha-1} \lim_{y\to 0} \left\{ \frac{\ln(1+y)}{y} \cdot \frac{(1+y)^{\alpha}-1}{\ln(1+y)} \right\}$$

$$= a^{\alpha-1} \lim_{y\to 0} \cdot \frac{\ln(1+y)}{y} \cdot \lim_{y\to 0} \frac{(1+y)^{\alpha}-1}{\ln(1+y)}$$

$$= a^{\alpha-1} \lim_{y\to 0} \frac{(1+y)^{\alpha}-1}{\ln(1+y)}$$

$$= a^{\alpha-1} \lim_{y\to 0} \frac{e^{\ln(1+y)^{\alpha}}-1}{ln(1+y)}$$

$$= a^{\alpha-1} \lim_{y\to 0} \frac{e^{\alpha\ln(1+y)}-1}{\ln(1+y)}$$

$$= a^{\alpha-1} \lim_{t\to 0} \frac{e^{\alpha t}-1}{t}, \text{ where } t = ln(1+y)$$

$$= \alpha a^{\alpha-1}$$

If $\alpha = 0$, the result is trivial as $x^{\alpha} - a^{\alpha} = 0$.

Example 25

Find $\lim_{x\to 0} \frac{\sin 2x}{\sin 3x}$.

Solution :

$$\lim_{x\to 0} \frac{\sin 2x}{\sin 3x} = \lim_{x\to 0} \frac{2\left(\frac{\sin 2x}{2x}\right)}{3\left(\frac{\sin 3x}{3x}\right)}$$

$$= \frac{2}{3} \lim_{x\to 0} \frac{\left(\frac{\sin 2x}{2x}\right)}{\left(\frac{\sin 3x}{3x}\right)}$$

$$= \frac{2}{3} \frac{\lim_{x\to 0} \frac{\sin 2x}{2x}}{\lim_{x\to 0} \frac{\sin 3x}{3x}} = \frac{2}{3} \times \frac{1}{1} = \frac{2}{3}.$$

Example 26

Evaluate $\lim_{x\to 0} \frac{1-\cos x}{x^2}$.

Solution :

$$\lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1-\cos^2 x}{x^2(1+\cos x)}$$

$$= \lim_{x\to 0} \frac{\sin^2 x}{x^2(1+\cos x)}$$

$$= \lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2 \times \lim_{x\to 0}\frac{1}{1+\cos x}$$

$$= 1 \times \frac{1}{2} = \frac{1}{2}.$$

Example 27

Prove that $\lim_{h\to 0} \frac{\sec(x+h) - \sec x}{h} = \sec x \tan x.$

Solution :

$$\lim_{h\to 0} \frac{\sec(x+h) - \sec x}{h} = \lim_{h\to 0} \frac{\frac{1}{\cos(x+h)} - \frac{1}{\cos x}}{h}$$

$$= \lim_{h\to 0} \frac{\cos x - \cos(x+h)}{h\cos x\cos(x+h)}$$

$$= \lim_{h\to 0} \frac{2\sin\left(x+\frac{h}{2}\right)\sin\frac{h}{2}}{h\cos x\cos(x+h)}$$

$$= \lim_{h\to 0}\sin\left(x+\frac{h}{2}\right) \times \lim_{h\to 0}\frac{\sin\frac{h}{2}}{\frac{h}{2}} \times \lim_{h\to 0} \cdot \frac{1}{\cos x\cos(x+\mathrm{h})}$$

$$= \sin x \times 1 \times \frac{1}{\cos^2 x} = \sec x \tan x.$$

Example 28

Find $\lim_{x\to\infty} x\sin\frac{1}{x}$.

Solution :

$$\lim_{x\to\infty} x\sin\frac{1}{x} = \lim_{y\to 0+}\frac{\sin y}{y}, \text{ where } y = \frac{1}{x}$$

$$= 1.$$

Example 29

Evaluate $\lim_{x\to 0}\frac{e^{3x} - e^x}{x}$

Solution :

$$\lim_{x\to 0}\frac{e^{3x} - e^x}{x} = \lim_{x\to 0}\frac{e^x(e^{2x}-1)}{x}$$

$$= 2\lim_{x\to 0} e^x \times \lim_{x\to 0}\frac{e^{2x}-1}{2x}$$

$$= 2 \times 1 \times 1 = 2.$$

Example 30

Evaluate $\lim\limits_{x\to 2} \dfrac{\log_e(3x-5)}{x-2}$

Solution :

$$\lim_{x\to 2} \frac{\log_e(3x-5)}{x-2} = \lim_{y\to 0} \frac{\log_e(1+3y)}{y}, \text{ where } y = x-2.$$

$$= 3\lim_{y\to 0} \frac{\log_e(1+3y)}{3y}$$

$$= 3\lim_{y\to 0} \log_e (1+3y)^{\frac{1}{3y}}$$

$$= 3\log_e\left\{\lim_{y\to 0}(1+3y)^{\frac{1}{3y}}\right\}$$

$$= 3\log_e e$$

$$= 3.$$

Example 31

Evaluate $\lim\limits_{x\to 2} \dfrac{\sqrt{x}-\sqrt{2}}{x^4-16}$

Solution :

$$\lim_{x\to 2} \frac{\sqrt{x}-\sqrt{2}}{x^4-16} = \lim_{x\to 2} \frac{\dfrac{\sqrt{x}-\sqrt{2}}{x-2}}{\dfrac{x^4-16}{x-2}}$$

$$= \lim_{x\to 2} \frac{\dfrac{x^{\frac{1}{2}}-2^{\frac{1}{2}}}{x-2}}{\dfrac{x^4-2^4}{x-2}} = \frac{\lim\limits_{x\to 2} \dfrac{x^{\frac{1}{2}}-2^{\frac{1}{2}}}{x-2}}{\lim\limits_{x\to 2} \dfrac{x^4-2^4}{x-2}}$$

$$= \frac{\dfrac{1}{2}\times 2^{\frac{1}{2}-1}}{4\times 2^{4-1}}$$

$$= \frac{1}{64\sqrt{2}}.$$

EXERCISE 14 (c)

1. Evaluate the following limits :

(i) $\lim\limits_{x\to 0} \dfrac{x}{\sin 2x}$ (ii) $\lim\limits_{x\to 0} \dfrac{\sin 3x}{\sin 5x}$

(iii) $\lim_{x \to 0} \dfrac{\sin mx}{\sin nx}$ (iv) $\lim_{x \to 0} \dfrac{\tan \alpha x}{x}$

(v) $\lim_{x \to 0} \dfrac{1 - \cos x}{x^2}$ (vi) $\lim_{x \to 0} \dfrac{\sin x^0}{x}$

(vii) $\lim_{x \to \pi} \dfrac{\sin x}{\pi - x}$ (viii) $\lim_{x \to \frac{\pi}{2}} \dfrac{1 - \sin x}{\left(\frac{\pi}{2} - x\right)^2}$

(ix) $\lim_{x \to 0} \dfrac{1 - \cos^3 x}{x \sin 2x}$ (x) $\lim_{x \to 0} \dfrac{1 + \sin x - \cos x}{1 - \sin x - \cos x}$

(xi) $\lim_{x \to 0} \dfrac{\tan x - \sin x}{x^3}$ (xii) $\lim_{x \to 0} \dfrac{(1 - \cos x)^2}{\tan^3 x - \sin^3 x}$

(xiii) $\lim_{x \to \frac{\pi}{2}} \left(\dfrac{\pi}{2} - x\right) \tan x$ (xiv) $\lim_{x \to \frac{\pi}{4}} \dfrac{\cos x - \sin x}{\cos 2x}$

(xv) $\lim_{x \to 0} \dfrac{x - x\cos 2x}{\sin^3 2x}$ (xvi) $\lim_{x \to 0} \dfrac{\sqrt{1 + \sin x} - \sqrt{1 - \sin x}}{\tan x}$

(xvii) $\lim_{x \to 0} \dfrac{2\sin x - \sin 2x}{x^3}$ (xviii) $\lim_{x \to 0} \dfrac{\cos x - \cos 5x}{\cos 2x - \cos 6x}$

(xix) $\lim_{x \to 0} \dfrac{\sin^{-1} x}{x}$.

2. Evaluate (i) $\lim_{x \to \alpha} \dfrac{x\sin\alpha - \alpha\sin x}{x - \alpha}$ (ii) $\lim_{x \to 0} x \sin \dfrac{1}{x}$.

3. Evaluate the following limits :

(i) $\lim_{h \to 0} \dfrac{\sin (x + h) - \sin x}{h}$ (ii) $\lim_{h \to 0} \dfrac{\cos (x + h) - \cos x}{h}$

(iii) $\lim_{h \to 0} \dfrac{\tan (x + h) - \tan x}{h}$ (iv) $\lim_{h \to 0} \dfrac{\operatorname{cosec} (x + h) - \operatorname{cosec} x}{h}$

(v) $\lim_{h \to 0} \dfrac{\sec (x + h) - \sec x}{h}$ (vi) $\lim_{h \to 0} \dfrac{\cot (x + h) - \cot x}{h}$

(vii) $\lim_{h \to 0} \dfrac{\sqrt{x + h} - \sqrt{x}}{h}$ (viii) $\lim_{h \to 0} \dfrac{\log_a (x + h) - \log_a x}{h}$

(ix) $\lim_{h \to 0} \dfrac{\ln (x + h) - \ln x}{h}$ (x) $\lim_{h \to 0} \dfrac{a^{x+h} - a^x}{h}$

(xi) $\lim_{h \to 0} \dfrac{e^{x+h} - e^x}{h}$ (xii) $\lim_{h \to 0} \left\{\dfrac{1}{(x + h)^3} - \dfrac{1}{x^3}\right\}$

(xiii) $\lim_{h \to 0} \dfrac{\sin (x + h) - \sin (x - h)}{h}$ (xiv) $\lim_{h \to 0} \dfrac{1}{h} \left\{\dfrac{1}{\sqrt{x + h}} - \dfrac{1}{\sqrt{x}}\right\}$.

4. Evaluate the following :

(i) $\lim\limits_{x\to 0} \dfrac{\log_e\left(1+\dfrac{x}{2}\right)}{x}$ (ii) $\lim\limits_{x\to 1} \dfrac{x-1}{\log_e x}$

(iii) $\lim\limits_{x\to 1} \dfrac{\log_e(2x-1)}{x-1}$ (iv) $\lim\limits_{x\to 0} \dfrac{\log_e(x+1)}{\sqrt{x+1}-1}$

(v) $\lim\limits_{x\to 2} \dfrac{\log_e(x-1)}{x^2-3x+2}$ (vi) $\lim\limits_{x\to 0} \dfrac{e^{ax}-1}{x}$

(vii) $\lim\limits_{x\to 0} \dfrac{e^{ax}-e^{-ax}}{x}$ (viii) $\lim\limits_{x\to 0} \dfrac{e^{3x}-e^{2x}}{e^{4x}-e^{3x}}$

(ix) $\lim\limits_{x\to 0} \dfrac{a^{2x}-1}{x}$ (x) $\lim\limits_{x\to 0} \dfrac{a^x-b^x}{x}$

(xi) $\lim\limits_{x\to 1} \dfrac{2^{x-1}-1}{x-1}$ (xii) $\lim\limits_{x\to 0} \dfrac{a^x-a^{-x}}{x}$

(xiii) $\lim\limits_{x\to 1} \dfrac{3^x-3}{x-1}$ (xiv) $\lim\limits_{x\to 0} \dfrac{3^x-2^x}{4^x-3^x}$

(xv) $\lim\limits_{x\to 1} \dfrac{2^{x-1}-1}{\sqrt{x}-1}$.

5. Evaluate the following :

(i) $\lim\limits_{x\to 0} \dfrac{\sqrt{x+1}-1}{x}$ (ii) $\lim\limits_{x\to 0} \dfrac{\sqrt{x+2}-\sqrt{2}}{x}$

(iii) $\lim\limits_{x\to 5} \dfrac{\sqrt{x}-\sqrt{5}}{x-5}$ (iv) $\lim\limits_{x\to 0} \dfrac{\sqrt{3-2x}-\sqrt{3}}{x}$

(v) $\lim\limits_{x\to 5} \dfrac{\sqrt{x-1}-2}{x-5}$ (vi) $\lim\limits_{x\to 1} \dfrac{x^2-\sqrt{x}}{\sqrt{x}-1}$

(vii) $\lim\limits_{x\to a} \dfrac{\sqrt{x-b}-\sqrt{a-b}}{x^2-a^2}$, $(a > b)$ (viii) $\lim\limits_{x\to 1} \dfrac{x^{\frac{1}{m}}-1}{x^{\frac{1}{n}}-1}$, ($m, n$ are integers)

(ix) $\lim\limits_{x\to 0} \dfrac{\sqrt{x^2+1}-1}{\sqrt{x^2+4}-2}$ (x) $\lim\limits_{x\to \infty} \left(\sqrt{x+1}-\sqrt{x}\right)$

(xi) $\lim\limits_{x\to \infty} \left(\sqrt{x^2+1}-\sqrt{x^2-1}\right)$ (xii) $\lim\limits_{x\to 0} \dfrac{\sqrt[3]{1+x}-\sqrt[3]{1-x}}{x}$

(xiii) $\lim\limits_{x\to 0} \dfrac{(x+9)^{\frac{3}{2}}-27}{x}$ (xiv) $\lim\limits_{x\to 0} \dfrac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt[3]{1+x}-\sqrt[3]{1-x}}$

(xv) $\lim_{x \to \infty} \dfrac{a_0 + a_1 x + a_2 x^2 + \ldots\ldots + a_m x^m}{b_0 + b_1 x + b_2 x^2 + \ldots\ldots + b_n x^n}$.

6. Evaluate the following limits :

(i) $\lim_{x \to \infty} \dfrac{\sin x}{x}$

(ii) $\lim_{x \to \infty} x\left(a^{\frac{1}{x}} - 1\right)$, $a > 0$.

(iii) $\lim_{x \to 0} \dfrac{x^{\frac{1}{2}} + 2x + 3x^{\frac{3}{2}}}{2x^{\frac{1}{2}} - 2x^{\frac{5}{2}} + 4x^{\frac{7}{2}}}$

(iv) $\lim_{x \to \infty} \sqrt{x}\left\{\sqrt{x + 1} - \sqrt{x}\right\}$

(v) $\lim_{x \to \infty} x^2\left\{\sqrt{x^4 + a^2} - \sqrt{x^4 - a^2}\right\}$

(vi) $\lim_{x \to 0} \cos(\sin x)$

(vii) $\lim_{x \to 0} \log_e \dfrac{\sqrt{1 + x} - 1}{x}$

(viii) $\lim_{x \to 2} \log_2 \dfrac{x^2 - 4}{\sqrt{3x - 2} - \sqrt{x + 2}}$

(ix) $\lim_{x \to \infty} \log_e \left(1 + \dfrac{a}{x}\right)^x$

(x) $\lim_{x \to 0} \log_e (1 + bx)^{\frac{1}{x}}$

(xi) $\lim_{x \to \frac{\pi}{4}} \dfrac{\text{Sin}\left(\dfrac{1 - \tan x}{1 + \tan x}\right)}{\dfrac{\pi}{4} - x}$

(xii) $\lim_{x \to \frac{\pi}{2}} \log \dfrac{1 - \sin^3 x}{\cos^2 x}$

(xiii) $\lim_{x \to \infty} e^x \left(a^{\frac{1}{x}} - 1\right)$

(xiv) $\lim_{x \to 0} \dfrac{x\left(e^{\frac{\sqrt{1 + x^2 + x^4} - 1}{x}} - 1\right)}{\sqrt{1 + x^2 + x^4} - 1}$

(xv) $\lim_{x \to 0} \dfrac{b \tan x \left(e^{\frac{\sin ax}{\tan bx} - \frac{a}{b}}\right)}{b \sin ax - a \tan bx}$.

7. Examine the existence of the following limits :

(i) $\lim_{x \to 0+} \log_a x$

(ii) $\lim_{x \to \frac{\pi}{2}} \tan x$

(iii) $\lim_{x \to 0} \operatorname{cosec} x$

(iv) $\lim_{x \to 0-} e^{\frac{1}{x}}$

(v) $\lim_{x \to 0+} e^{\frac{1}{x}}$

(vi) $\lim_{x \to 0} \dfrac{1}{e^{\frac{1}{x}} - 1}$.

8. Find the value of a if,

(i) $\lim_{x \to \alpha} \frac{\tan a(x - \alpha)}{x - \alpha} = \frac{1}{2}$ (ii) $\lim_{x \to 0} \frac{\tan ax}{\sin 2x} = 1$

(iii) $\lim_{x \to 0} \frac{e^{ax} - e^{x}}{x} = 2$ (iv) $\lim_{x \to 1} \frac{5^{x} - 5}{(x - 1)\log_e a} = 5$

(v) $\lim_{x \to 2} \frac{\log_e (2x - 3)}{a(x - 2)} = 1.$

14.8 Introduction to differentiation

The study of differential calculus originated in the process of solving the following three problems :

1. from the astronomical considerations particularly involving an attempt to have a better approximation of π as developed by Bhaskaracharya, Madhava and Nilakantha,
2. finding the tangent to any arbitrary curve as developed by Fermat and Leibnitz,
3. finding the rate of change as developed by Fermat and Newton.

(See the introductory remark in Chapter 14)

That the differential calculus is a very important tool in every branch of science can be realized from the apt remark of G.F.B. Riemann : "Physics became a science only after the invention of differential calculus." We begin the subject with the following fundamental concept.

14.9 Instantaneous Rate of Change and differentiability :

When a quantity undergoes change, the change in the quantity is always associated, in almost every field of human endeavour, with the **rate** at which the quantity changes. The rate of change in the quantity may be observed with respect to another quantity such as time, distane or position in space etc. In some problems, it is required to find not only the average rate of change but the **instantaneous** rate of change, i.e. the rate at which the quantity changes at any particular time. This requirement is observed in Physics, Chemistry, Economics and many social sciences.

Let us examine a case of instantaneous rate of change by means of the following example.

Example 32

A person drives a car from a point A through the point B, a distance of 1.3 km. He starts recording time in seconds every tenth of a kilometer as shown in the following table. How fast is he travelling at the point B ? In other words, what is the rate of change of position of the person at the **instant** when he is at B ?

The data are tabulated below.

Distance travelled in km.	**Time recorded in seconds**
0 (position A)	$0 = t_A$
0.1	$16 = t_1$
0.2	$31 = t_2$
0.3	$43 = t_3$
0.4	$54 = t_4$
0.5	$64 = t_5$
0.6	$73 = t_6$

Distance	**Time recorded**

travelled in km.	**in seconds**
0.7	$82 = t_7$
0.8	$90 = t_8$
0.9	$98 = t_9$
1.0	$105 = t_{10}$
1.1	$112 = t_{11}$
1.2	$118 = t_{12}$
1.3 (position B)	$124 = t_B$

Let S(t) denote the distance travelled in time t. Then t is the input (independent) variable and distance S(t) is the output (dependent) variable. The average rate at which the person changes his position, i.e. average speed, is given by the change in the output variable S(t) divided by the change in the input variable t.

Let us calculate (with the help of a calculator) the various average speeds as the person approaches the point B.

Average speed at the point B as measured over the time interval [t_A, t_B]

$$= \vartheta_0 = \frac{\text{Distance covered}}{\text{Time taken}} = \frac{S(t_B) - S(t_A)}{t_B - t_A}$$

$$= \frac{1.3 - 0}{124 - 0} \text{ km/sec.} = 37.7419 \text{ km/hr.}$$

That measured over the interval [t_1, t_B]

$$= \vartheta_1 = \frac{1.3 - 0.1}{124 - 16} \text{ km/sec.}$$

$$= 39.9999 \text{ km/hr.}$$

In similar fashion $\vartheta_2 = \dfrac{S(t_B) - S(t_2)}{t_B - t_2} = 42.5806$ km/hr.

The entire sequence of average speeds as calculated successively over smaller and smaller time intervals can be seen to be as follows.

Average speed in km/hr.

$\vartheta_0 = 37.7419$	$\vartheta_7 = 51.4285$
$\vartheta_1 = 39.9999$	$\vartheta_8 = 52.9422$
$\vartheta_2 = 42.5806$	$\vartheta_9 = 55.3846$
$\vartheta_3 = 44.4444$	$\vartheta_{10} = 65.8481$
$\vartheta_4 = 46.2857$	$\vartheta_{11} = 59.9999$
$\vartheta_5 = 47.9999$	$\vartheta_{12} = 59.9999$
$\vartheta_6 = 49.4117$	

It is reasonable to say that the person is travelling at the rate of 60km/hr. at B. Notice that every average reading comes closer and closer to this number and still closer reading near the point B would increase the accuracy to determine the instantaneous rate of change of position at B.

Of course when we say that the person is travelling at the rate of 60km/hr at B it does not mean that he will travel 60kms in the next hour or for that matter 0.0166km in the next second. All that we mean is that the rate with which the person is changing his position at the **instant when he is at B** is 60km/hr.

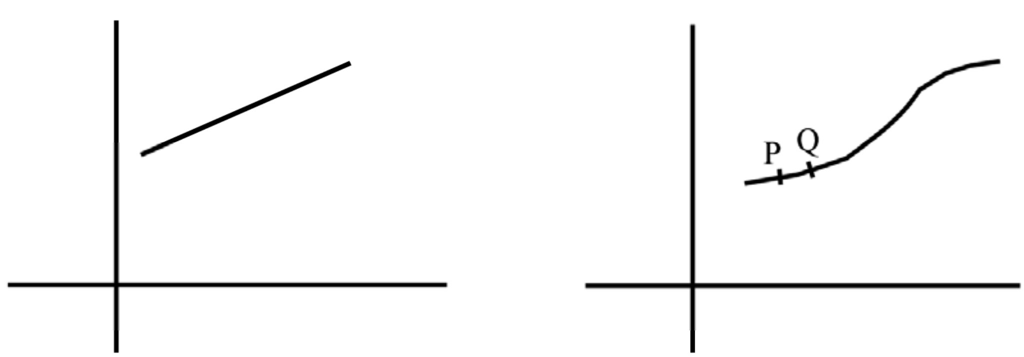
Notice further that if the person travels at **uniform speed** i.e. if there is no change in the rate at which he is changing his position, then the instantaneous rate of change of position at any point would remain a **constant**. In this case we do not have to calculate such averages as ϑ_1, ϑ_2 etc. to find the rate of change at B.

With this background we now develop the principle of differentiability.

Differentiability

One of the most popular methods in arithmetic is unitary method. A typical problem reads like "If 6 people can finish a piece of work in 14 days working for 5 hours a day then how many people would be needed to finish it in 12 days if each of them works 7 hours a day ?" While solving this problem using unitary method we tacitly assume that the amount of work turned out is proportional to the number of people working, is also proportional to the number of hours they work per day and also to the number of people working. This, what is called in another language is that the dependence of the above parameters is linear.

But this happy situation is not always true in every case. For example the distance covered by a body falling freely under gravity is **not** proportional to time. Nor is the velocity attained by a body is proportional to the distance through which it has fallen. We can multiply examples where dependence is not linear. In other words the graph of the function involved is not necessarily a straight line. The researchers like Bhaskara, Fermat, Newton and Leibnitz propose that even if the whole graph does not look like a straight line a small part of it looks nearly like a straight line.



(A Straight line) (The portion PQ is nearly a straight line)

Figure 12.0

If f is a function whose graph is a straight line then $\frac{f(t)-f(t_0)}{t-t_0}$ is a constant. The propounders of calculus proposed why not consider functions such that $\frac{f(t)-f(t_0)}{t-t_0}$ is nearly a constant when $t-t_0$ is small. What it means is that there is a constant m such that

$$\frac{f(t)-f(t_0)}{t-t_0} = m+r$$

where r can be made small by taking t close enough to t_0. This means $\lim_{t\to t_0}\frac{f(t)-f(t_0)}{t-t_0} = m$.

Is this not our experience where we observe the speedometer of a moving vehicle over an interval of time ? If we look at the speedometer for about half an hour it is likely that it was pointing to different numbers at different time most of the times. May be for some moments it was pointing to a fixed number over an interval. This means the speed was nearly constant over that interval i.e. it covered equal distance in equal time interval, which is the same thing as saying that its average speed over that interval was equal to its instantaneous velocity at every instant in the interval. We also see that the fluctuation in speed is small over a small interval of time. That is to say that the speed was nearly uniform. In other words average speed in a small interval is nearly equal to its instantaneous speed at the beginning of the interval. If $x(t)$ is the distance covered by the time t then

$$\frac{x(t)-x(t_0)}{t-t_0} = \nu_o+\alpha$$

where α is small when $t-t_0$ is small and ν_o is called its instantaneous speed at t_0. All this means is that $\lim_{t\to t_0}\frac{x(t)-x(t_0)}{t-t_0} = \nu_o$

We make extensive use of this fact in numerical calculations. The **principle of proportional parts** states that the increment in value of the function can be taken to be proportional to increment in the value of the argument when the increment in the argument is small enough. Observe that sin $30^0 = 0.5000$ and sin $31^0 = 0.5150$. So there is an increase in value of sin x by 0.0150 for an increase in value of x by 1^0 . If we assume that over the interval $[30^0, 31^0]$ the sine graph is nearly a straight line then $\sin(30^0 + \theta) = \sin 30^0 + \frac{0.0150}{60}\times\theta$ when θ is measured in minutes. We see in fact this gives $\sin 30^0\,6' = 0.5015$, $\sin 30^0\,12' = 0.5030$ which agrees with the value of sine for those angles correct upto 4 places of decimal. If we apply the same principle for an increment of 2^0 then we would get $\sin 32^0 - \sin 30^0 = 0.0299$. By this principle stated above we would get $\sin 31^0 = 0.51495$ (which deviates from the tabular value at the fourth place). Reader is encouraged to try the same thing for an increment of 5^0 at 30^0 and see what happens.

From the above discussion we see that the concept of instantaneous rate of change involves a process of finding the limit to which the quotient of average changes approaches when the averages are taken over arbitrarily smaller and smaller intervals. We now formalise this process in what is known as 'derivative' in mathematics.

14.10 Derivative :

Let f be a function of a variable x over an interval (a, b). Let a variable quantity y be given by the rule $y = f(x)$. Suppose now that x changes to $x+\delta x$, $(\delta x \neq 0)$ and correspondingly y changes from $f(x)$ to $f(x + \delta x)$ where δx (read 'delta x') is a small **increment** in x. We continue to call δx an increment even when it is negative. The corresponding increment δy in y is given by

$$\delta y = f(x + \delta x) - f(x)$$

and the increment ratio (sometimes called Newton quotient) is given by

$$\frac{\delta y}{\delta x} = \frac{f(x+\delta x) - f(x)}{\delta x}.$$

Thus $\delta y/\delta x$ is the average rate of change of y. The instantaneous rate of change of y at the value x is then given by the limit of $\delta y/\delta x$ as $\delta x \to o$, provided that the limit exists.

Definition :

A function $f : (a, b) \to R$ is said to be **derivable** or **differentiable** at $x \in (a, b)$ if

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x} \text{ exists.} \qquad \text{.......... (1)}$$

This limit is called the **derivative of f with respect to (*w.r.t.*) x.**

The derivative of f is denoted by f' or $\frac{dy}{dx}$ or sometimes by Dy or Df.

We can use an alternative notation. Let $c \in (a, b)$ and f be derivable at c, then using the above definition $\left.\frac{dy}{dx}\right]_{x=c} = f'(c) = \lim_{h \to 0} \frac{f(c + h) - f(c)}{h}$. .... (2)

(We prefer the symbol h to δx in this case.) If the limit on the right of (2) does not exist, the function is not derivable at c. Let us analyze the limiting process of (1). How does $\delta x \to 0$? It can do so from the left i.e. by taking negative values or it can take only positive values (from the right). It leads to the concept of **Left Hand Derivative and Right Hand Derivative**.

If the limit in (2) exists when $h \to 0+$ (i.e. when $h \to 0$ through positive values only) the limit is called the **right hand derivative** of f at c and is denoted by $f'(c+)$. Similarly we define the **left hand derivative** at c and denote it by $f'(c-)$.

Thus $$f'(c+) = \lim_{h \to 0} \frac{f(c + h) - f(c)}{h}, \; h > 0 \qquad \text{......... (3)}$$

$$f'(c-) = \lim_{h \to 0} \frac{f(c - h) - f(c)}{-h}, \; h > 0. \qquad \text{......... (4)}$$

It is easily seen that the (unique) derivative of $f(x)$ exists at $x = c$ if and only if $f'(c+)$ and $f'(c-)$ both exist and are equal and is denoted by (2) irrespective of sign of h.

Differentiability of a function on/in an interval (a,b)

Definition

If f has a derivative at every point of the interval (a, b) then it is said to be differentiable **on** (a, b). **The process of finding the derivative of a function is known as differentiation.**

Example 33

Differentiate (i) x, (ii) $x + 3$, (iii) $3x + 4$.

Solution :

(i) Let $y = x$

Then $\dfrac{dy}{dx} = \lim\limits_{\delta x \to 0} \dfrac{(x + \delta x) - x}{\delta x} = \lim\limits_{\delta x \to 0} \dfrac{\delta x}{\delta x} = 1.$

(ii) Let $y = x + 3$

Then $\dfrac{dy}{dx} = \lim\limits_{\delta x \to 0} \dfrac{(x + \delta x + 3) - (x + 3)}{\delta x} = \lim\limits_{\delta x \to 0} \dfrac{\delta x}{\delta x} = 1.$

(iii) Let $y = 3x + 4$

Then $\dfrac{dy}{dx} = \lim\limits_{\delta x \to 0} \dfrac{3(x + \delta x) + 4 - (3x + 4)}{\delta x} = \lim\limits_{\delta x \to 0} \dfrac{3\delta x}{\delta x} = 3.$

In the above examples, the derivatives are the slopes of respective lines $y = x$, $y = x + 3$ and $y = 3x + 4$.

Example 34

Differentiate x^2.

Solution :

Let $y = x^2$.

$$\text{By definition, } \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{(x + \delta x)^2 - x^2}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\delta x\,(2x + \delta x)}{\delta x}$$

$$= \lim_{\delta x \to 0} (2x + \delta x) = 2x.$$

Example 35

Find the derivative of $\sqrt{x + 1}$ at $x = 2$.

Solution :

Let $y = \sqrt{x+1}$.

$$\text{By (2) } \left.\frac{dy}{dx}\right]_{x=2} = \lim_{h \to 0} \frac{\sqrt{(2+h)+1} - \sqrt{2+1}}{h}$$

$$= \lim_{h \to 0} \frac{\left(\sqrt{3+h} - \sqrt{3}\right)}{h} \cdot \frac{\sqrt{3+h} + \sqrt{3}}{\sqrt{3+h} + \sqrt{3}}$$

$$= \lim_{h \to 0} \frac{(3+h) - 3}{h\left(\sqrt{3+h} + \sqrt{3}\right)}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{3+h} + \sqrt{3}} = \frac{1}{2\sqrt{3}}.$$

Example 36

Differentiate a constant function.

Solution :

Let $y = c$ (constant)

$$\text{By definition } \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{0}{\delta x} = 0.$$

We observe that if δx is any increment in x then $\delta y = 0$ because there is no change in y.

Example 37

Test the differentiability of

$f(x) = x + [x]$ at $x = 2$.

Solution :

Here $f(2) = 2 + [2] = 4$.

$$f'(2+) = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h}, \; h > 0$$

$$= \lim_{h \to 0} \frac{2 + h + [2+h] - 4}{h}$$

$$= \lim_{h \to 0} \frac{2 + h + 2 - 4}{h} = 1.$$

$$f'(2-) = \lim_{h \to 0} \frac{f(2-h) - f(2)}{-h}, \; h > 0$$

$$= \lim_{h \to 0} \frac{2 - h + [2-h] - 4}{-h}$$

$$= \lim_{h \to 0} \frac{2 - h + 1 - 4}{-h}$$

$$= \lim_{h \to 0} \frac{-h - 1}{-h} = \infty.$$

So $f'(2+) \neq f'(2-)$. Thus $f'(2)$ does not exist. Hence f is not differentiable at $x = 2$.

EXERCISE 14 (d)

1. Find the derivative of the following functions 'ab initio', that is, using the definition.

(i) $2x^3$ (ii) x^4

(iii) $x^2 + 1$ (iv) $\frac{1}{x}$

(v) $\frac{1}{3x + 2}$ (vi) $\frac{1}{x^2}$

(vii) $\frac{x}{x + 1}$ (viii) $t(t - 1)$

(ix) $s^2 - 6s + 5$ (x) $\sqrt{z}$

(xi) $\tan\theta$ (xii) $\cos 2\theta$

(xiii) $x \sin x$.

2. Find the derivative of the following functions from definition at the indicated points.

(i) x^4 at $x = 2$ (ii) $2x^2 + x + 1$ at $x = 1$

(iii) $x^3 + 2x^2 - 1$ at $x = 0$ (iv) $\tan x$ at $x = \frac{\pi}{3}$

(v) $\sqrt{3x + 2}$ at $x = 0$ (vi) $\ln x$ at $x = 2$

(vii) e^x at $x = 1$ (viii) $\sin 2\theta$ at $\theta = \frac{\pi}{4}$.

Test whether the following functions are differentiable at the indicated points. If so find the derivative.

3. $\frac{x + 1}{x - 1}$ at $x = -1$.

4. $\sqrt{x}$ at $x = 0$.

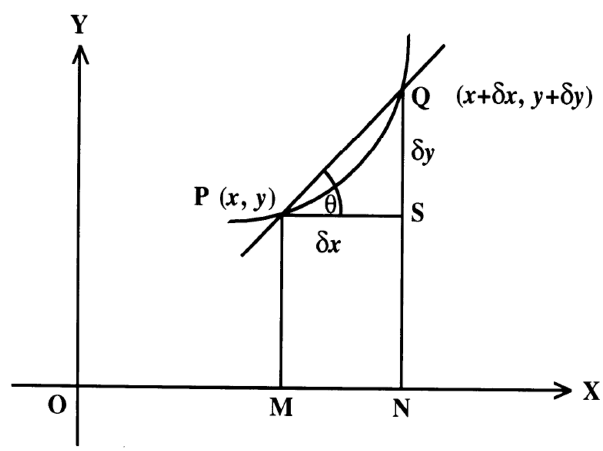
5. $f(x) = \begin{cases} 1 - x, & x \leq \frac{1}{2} \\ x, & x > \frac{1}{2} \end{cases}$ at $x = \frac{1}{2}$.

6. $f(x) = \begin{cases} \sin\dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ at $x = 0$.

7. $f(x) = \begin{cases} x^2 \sin\dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ at $x = 0$.

14.11 Tangent line of a graph at a point :

We are already familiar with the concept of tangent line to circles and other elementary figures. Intuitively, by tangent line to a graph (or curve) at a point P on it, we understand that it is a straight line which is the "limiting position" of secant $\overleftrightarrow{PQ}$ of the graph when Q lies on the curve and approaches P indefinitely. We have not yet developed any technique by which such a limiting position of secants can be determined nor is there any reason to suppose that this limiting position always exists. Consider, however, the graph given in the diagram below representing the function $y = f(x)$.



Let $P(x, y)$ and $Q(x+\delta x, y + \delta y)$ be two points on the curve $y=f(x)$ and suppose that the secant $\overleftrightarrow{PQ}$ has an inclination measuring θ with x-axis. Then the figure shows that $\tan\theta = \dfrac{\delta y}{\delta x} = \dfrac{f(x+\delta x) - f(x)}{\delta x}$.

We may suppose that f is continuous at x, i.e. the graph of $f(x)$ does not have any gap or break at x, so that $f(x + \delta x) \to f(x)$ as $\delta x \to 0$, that is, $(x + \delta x, y + \delta y) \to (x, y)$ as $\delta x \to 0$, or in other words, Q approaches P.

If also it happens that $\lim\limits_{\delta x \to 0} \dfrac{\delta y}{\delta x}$ exists, this limit equals $\dfrac{dy}{dx} = f'(x)$ which is the derivative of f at x and, in this limiting position, the secant $\overleftrightarrow{PQ}$ is called the tangent line to the curve at P. The inclination of the tangent line to x-axis is usually written as ψ (Psi) (In fact $\theta \to \psi$ as $Q \to P$)

Note that the proof of this result i.e. is based on figure 12.1 and no conclusion can, therefore, be drawn from it. It however, suggests the following definition of the tangent line to a graph at a point.

Definition :

The tangent line to the graph of a function f at the point $P = (x_0, f(x_0))$ is

(i) the line through P with slope $f'(x_0)$ if $f'(x_0)$ exists; which is also given by $f'(x_0) = \tan\psi$

(ii) the line $x = x_0$ if $\lim\limits_{x \to x_0} \left| \frac{f(x) - f(x_0)}{x - x_0} \right| = \infty$.

The tangent line at $(x_0, f(x_0))$ does not exist if neither (i) nor (ii) holds.

If (i) holds, then clearly the equation of the required tangent line is :

$$y - f(x_0) = f'(x_0)\,(x - x_0).$$

If (ii) holds the equation of the tangent line is $x = x_0$.

Example 38

Find the equation of the tangent to the parabola $y = x^2$ at the point of the curve whose x–coordinate is -1.

Solution :

If $x = -1$, then $y = 1$, so that tangent line is to be determined at the point $(-1, 1)$.

Now $y = x^2 \Rightarrow \frac{dy}{dx} = 2x$.

$$\Rightarrow \left.\frac{dy}{dx}\right]_{x=-1} = 2(-1) = -2.$$

Hence the equation of the tangent line to the given curve at $(-1, 1)$ is

$$y - 1 = -2(x + 1)$$

or $\quad y + 2x + 1 = 0$.

14.12 Derivatives of some Standard Functions :

We shall find out derivatives of some standard functions from definition i.e. 'ab initio' and use them to find out derivatives of other functions. We observe that finding derivative 'ab initio' is essentially finding certain limits. In the following computations of derivatives, we suppose that the respective functions are defined and differentiable at the points at which derivatives are calculated.

Example 39

If $y = x^\alpha$, $\alpha \in R$, then $\frac{dy}{dx} = \alpha x^{\alpha-1}$.

Proof :

Let δx be a small increment in x and δy be the corresponding increment in y.

Then $\frac{\delta y}{\delta x} = \frac{(x + \delta x)^\alpha - x^\alpha}{\delta x} = \frac{(x + \delta x)^\alpha - x^\alpha}{(x + \delta x) - x} = \frac{z^\alpha - x^\alpha}{z - x}$

writing $z = x + \delta x$. As $\delta x \to 0$, $z \to x$.

Hence $\frac{dy}{dx} = \lim\limits_{\delta x \to 0} \frac{\delta y}{\delta x}$

$$= \lim_{z \to x} \frac{z^\alpha - x^\alpha}{z - x} = \alpha x^{\alpha-1}$$

using a result we have learnt earlier. (See Example 24, Chapter 14)

Note that we tacitly assume that $x > 0$ when α is a non integer in order to have y well defined.

Example 40

If $y = \sin x$ then $\frac{dy}{dx} = \cos x$.

Proof :

Without repeating the preliminary explanations every time we differentiate a given function from definition; we proceed as follows.

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{\sin(x + \delta x) - \sin x}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{2\cos\left(x + \frac{\delta x}{2}\right)\left(\sin\frac{\delta x}{2}\right)}{\delta x}$$

$$\left[\because \sin C - \sin D = 2\cos\frac{C+D}{2}\sin\frac{C-D}{2}\right]$$

$$= \lim_{\delta x \to 0} \cos\left(x + \frac{\delta x}{2}\right) \cdot \lim_{\delta x \to 0} \frac{\sin\left(\frac{\delta x}{2}\right)}{\left(\frac{\delta x}{2}\right)}$$

$$= \cos x \,.\, 1 = \cos x.$$

Example 41

If $y = \cos x$ then $\frac{dy}{dx} = -\sin x$.

Proof :

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{\cos(x + \delta x) - \cos x}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{-2\sin\left(x + \frac{\delta x}{2}\right)\left(\sin\frac{\delta x}{2}\right)}{\delta x}$$

$$= -\lim_{\delta x \to 0} \sin\left(x + \frac{\delta x}{2}\right) \cdot \lim_{\delta x \to 0} \frac{\sin\left(\frac{\delta x}{2}\right)}{\left(\frac{\delta x}{2}\right)}$$

$$= -\sin x \,.\, 1 = -\sin x.$$

Example 42

If $y = \tan x$ then $\frac{dy}{dx} = \sec^2 x$.

Proof :

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\tan(x + \delta x) - \tan x}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\sin(x + \delta x)\cos x - \sin x \cos(x + \delta x)}{\cos(x + \delta x) \, . \, \cos x \, . \, \delta x}$$

$$= \lim_{\delta x \to 0} \frac{\sin\{(x + \delta x) - x\}}{\cos(x + \delta x) \, . \, \cos x \, . \, \delta x}$$

$$= \lim_{\delta x \to 0} \frac{1}{\cos(x + \delta x) \, . \, \cos x} \, . \, \lim_{\delta x \to 0} \frac{\sin \delta x}{\delta x}$$

$$= \frac{1}{\cos^2 x} \, . \, 1 = \sec^2 x.$$

Similarly we can find that

$$\frac{d}{dx} \cot x = -\operatorname{cosec}^2 x.$$

Example 43

If $y = \sec x$ then $\dfrac{dy}{dx} = \sec x \tan x$.

Proof :

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\sec(x + \delta x) - \sec x}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\cos x - \cos(x + \delta x)}{\cos x \, . \, \cos(x + \delta x) \, . \, \delta x}$$

$$= \lim_{\delta x \to 0} \frac{-2\sin\left(x + \frac{\delta x}{2}\right)\sin\left(-\frac{\delta x}{2}\right)}{\cos x \, . \, \cos(x + \delta x) \, . \, \delta x}$$

$$= \lim_{\delta x \to 0} \frac{\sin\left(x + \frac{\delta x}{2}\right)}{\cos(x + \delta x)\cos x} \, . \, \lim_{\delta x \to 0} \frac{\sin\left(\frac{\delta x}{2}\right)}{\left(\frac{\delta x}{2}\right)}$$

$$= \frac{\sin x}{\cos^2 x} \, . \, 1 = \sec x \tan x.$$

Example 44

If $y = \operatorname{cosec} x$ then $\dfrac{dy}{dx} = -\operatorname{cosec} x \cot x$.

Proof :

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\operatorname{cosec}(x + \delta x) - \operatorname{cosec} x}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\sin x - \sin(x + \delta x)}{\sin x \cdot \sin(x + \delta x) \cdot \delta x}$$

$$= \lim_{\delta x \to 0} \frac{2 \cos\left(x + \frac{\delta x}{2}\right) \sin\left(-\frac{\delta x}{2}\right)}{\sin x \cdot \sin(x + \delta x) \cdot \delta x}$$

$$= \lim_{\delta x \to 0} \frac{\cos\left(x + \frac{\delta x}{2}\right)}{\sin(x + \delta x) \cdot \sin x} \cdot \lim_{\delta x \to 0} \frac{-\sin\left(\frac{\delta x}{2}\right)}{\left(\frac{\delta x}{2}\right)}$$

$$= \frac{\cos x}{\sin^2 x} \cdot (-1) = -\operatorname{cosec} x \cot x.$$

Example 45

Find derivative of $\sqrt{x}\, \cos x$ from definition.

Solution :

Let $y = \sqrt{x}\, \cos x$.

Then $\dfrac{dy}{dx} = \lim\limits_{\delta x \to 0} \dfrac{\sqrt{x + \delta x} \cos(x + \delta x) - \sqrt{x} \cos x}{\delta x}$

$$= \lim_{\delta x \to 0} \frac{1}{\delta x} \left[\sqrt{x + \delta x}\, \left(\cos(x + \delta x) - \cos x\right) + \cos x \left(\sqrt{x + \delta x} - \sqrt{x}\right) \right]$$

$$= \lim_{\delta x \to 0} \sqrt{x + \delta x} \cdot \frac{\cos(x + \delta x) - \cos x}{\delta x} + \lim_{\delta x \to 0} \cos x \frac{\sqrt{x + \delta x} - \sqrt{x}}{\delta x}$$

$$= \lim_{\delta x \to 0} \sqrt{x + \delta x} \cdot \lim_{\delta x \to 0} \frac{\cos(x + \delta x) - \cos x}{\delta x} + \cos x \lim_{\delta x \to 0} \frac{x + \delta x - x}{\delta x\left(\sqrt{x + \delta x} + \sqrt{x}\right)}$$

$$= \sqrt{x}\, (-\sin x) + \cos x \cdot \frac{1}{2\sqrt{x}} \qquad \text{(Ref. Example 41 for the second limit)}$$

$$= -\sqrt{x}\, \sin x + \frac{\cos x}{2\sqrt{x}}.$$

EXERCISE 14 (e)

1. Find derivatives of the following functions from definition.

(i) $3x^2 - \dfrac{4}{x}$ (ii) $(4x - 1)^2$

(iii) $2 + x + \sqrt{x^3}$ (iv) $x - \sqrt{x^2 - 1}$.

(v) $\dfrac{1}{x^{\frac{2}{5}}} + 1$

2. (i) $\cos(ax + b)$ (ii) $x^2 \sin x$

(iii) $\sqrt{\tan x}$ (iv) $\cot x^2$

(v) cosec$3x$.

3. (i) $\sqrt{x}\sin x$ (ii) $\sqrt{x^2+1}\cos x$

(iii) $\tan x - x^2 - 2x$.

14.13 Algebra of Derivatives :

Before we find out derivatives of other standard functions we shall learn some rules for obtaining derivatives of sum, difference, product and quotient of two derivable functions.

Theorem 1 :

Let u and v be two derivable functions of x. Then (using ʹ notation for derivatives)

(i) $(u+v)' = u' + v'$

(ii) $(u-v)' = u' - v'$

(iii) $(uv)' = u'v + uv'$

(iv) $\left(\dfrac{u}{v}\right)' = \dfrac{u'v - uv'}{v^2}$, provided $v(x) \neq 0$.

Proof :

(i) Let $y = u + v$.

If δx is a small increment in x and δu, δv and δy are corresponding increments in u, v and y respectively; then

$$y + \delta y = (u + \delta u) + (v + \delta v)$$

$$\Rightarrow \delta y = \delta u + \delta v$$

$$\Rightarrow \frac{\delta y}{\delta x} = \frac{dv}{dx} + \frac{\delta v}{\delta x}$$

Taking limit as $\delta x \to 0$ we have

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{\delta u}{\delta x} + \lim_{\delta x \to 0} \frac{\delta v}{\delta x} = \frac{du}{dx} + \frac{dv}{dx}$$

or $(u+v)' = u' + v'$ .... **(1)**

Similarly we can prove (ii)..

(iii) Let $y = uv$.

Let δx be a small increment in x and δu, δv, δy be corresponding increments in u, v and y respectively.

Then $y + \delta y = (u + \delta u)(v + \delta v)$

$$= uv + u\delta v + v\delta u + \delta u\delta v$$

$$\Rightarrow \delta y = u\delta v + v\delta u + \delta u\delta v$$

$$\Rightarrow \frac{\delta y}{\delta x} = \frac{\delta u}{\delta x} v + u\frac{\delta v}{\delta x} + \delta u \frac{\delta v}{\delta x}.$$

[Remark : Now, we make use of an important property of differentiable functions which will be dealt with in Vol-II, which states 'every differentiable functions is continuous. For this reason, u being a derivable (i.e. differentiable) function of x, is necessarily continuous. So $u(x+\delta x) \to u(x)$ as $\delta x \to 0$.

$\therefore$ $\delta u = u(x+\delta x) - u(x) \to u(x) - u(x) = 0$, when $\delta x \to 0$. Thus $\delta x \to 0 \Leftrightarrow \delta u \to 0$.

For the same reason, $\delta x \to 0 \Leftrightarrow \delta v \to 0$]

Hence we have from above

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{\delta u}{\delta x} \cdot v + u \lim_{\delta x \to 0} \frac{\delta v}{\delta x} + \lim_{\delta u \to 0} \delta u \cdot \lim_{\delta x \to 0} \frac{\delta v}{\delta x}$$

$$= \frac{du}{dx} v + u \frac{dv}{dx} + 0 \cdot \frac{du}{dx}$$

$$= \frac{du}{dx} v + u \frac{dv}{dx}.$$

i.e. $(uv)' = u'v + uv'$. .... **(2)**

(iv) Let $y = \frac{u}{v}$,

where u and v are derivable functions of x such that $v(x) \neq 0$. With usual interpretation of notations.

$$\delta y = \frac{u + \delta u}{v + \delta v} - \frac{u}{v}$$

$$= \frac{v\,\delta u - u\delta v}{v\,(v + \delta v)}$$

$$\Rightarrow \frac{\delta y}{\delta x} = \frac{v \frac{\delta u}{\delta x} - u \frac{\delta v}{\delta x}}{v\,(v + \delta v)}$$

So $$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \left[v \lim_{\delta x \to 0} \frac{\delta u}{\delta x} - u \lim_{\delta x \to 0} \frac{\delta v}{\delta x} \right] / \, v \lim_{\delta v \to 0} (v + \delta v)$$

$$= \frac{vu' - uv'}{v^2}$$ [Refer remark in proof of (iii) .... **(3)**

Note: (1) The rule $(u \pm v)' = u' \pm v'$ can be generalized to the case of more than two functions. Thus if

$y = u \pm v \pm \omega \pm \ldots\ldots\ldots\ldots\ldots\ldots \pm t$

where $u, v, w, \ldots\ldots\ldots\ldots, t$ are derivable functions of x then

$$\frac{dy}{dx} = \frac{du}{dx} \pm \frac{dv}{dx} \pm \frac{dw}{dx} \pm \ldots\ldots\ldots\ldots \pm \frac{dt}{dx}$$

i.e. $(u \pm v \pm w \pm \ldots\ldots\ldots \pm t)' = u' \pm v' \pm w' \pm \ldots\ldots\ldots\ldots \pm t'$. .... **(1′)**

(2) The rule $(uv)' = u'v + uv'$ can also be generalized to

$(uv\omega \ldots\ldots\ldots\ldots t)' = u'vw \ldots t + uv'w \ldots t + uvw' \ldots t + \ldots\ldots + uvw \ldots t'$ .... **(2′)**

where$(uvw \ldots t)$ is a product of finite numbers of derivable functions of x.

(3) It follows from the result (2) by taking $v = c$, that if $y = cu$ where c is a constant and u is a derivable function of x then

$$\frac{dy}{dx} = c\,\frac{du}{dx}.$$ i.e. $(cu)' = cu'$ .... **(4)**

Example 46

Find $\frac{dy}{dx}$ if $y = x^3 - x^2 + 6$.

Solution :

$$\frac{dy}{dx} = \frac{d}{dx}(x^3 - x^2 + 6)$$

$$= \frac{dx^3}{dx} - \frac{dx^2}{dx} + \frac{d6}{dx}$$

$$= 3x^2 - 2x + 0$$

$$= 3x^2 - 2x.$$

Example 47

Find $\frac{dy}{dx}$ if $y = \sqrt{x} + x^2(1 - x) + \sin x$.

Solution :

$$\frac{d}{dx}\left[\sqrt{x} + x^2(1 - x) + \sin x\right]$$

$$= \frac{d}{dx}x^{\frac{1}{2}} + \frac{d}{dx}x^2 - \frac{d}{dx}x^3 + \frac{d}{dx}\sin x$$

$$= \frac{1}{2}x^{-\frac{1}{2}} + 2x - 3x^2 + \cos x.$$

Example 48

Find derivative of

(i) $x^2 \sin x$, (ii) $5\tan x$

Solution :

(i) $\frac{d}{dx}(x^2 \sin x) = \frac{d}{dx}x^2 \sin x + x^2 \frac{d}{dx}\sin x$

$= 2x \sin x + x^2 \cos x.$

(ii) $\frac{d}{dx}5\tan x = 5\frac{d}{dx}\tan x$, using (4)

$= 5\sec^2 x.$

Example 49

Find derivative of

$f(x) = x^2 \sin\frac{1}{x}, x \neq 0, f(x) = 0$ at $x = 0$.

Solution :

(Refer to problem 7 of exercise 12 (a) .

Here we cannot apply the rule (2) to $f(x)$ since $\sin\frac{1}{x}$ is not derivable at $x = 0$. So we proceed as follows. By definition,

$$f'(o) = \lim_{h \to 0} \frac{f(0+h) - f(o)}{h}$$

$$= \lim_{h \to 0} \frac{h^2 \sin\frac{1}{h} - 0}{h}$$

$$= \lim_{h \to 0} h \sin\frac{1}{h}$$

Now $0 < \left|h \sin\frac{1}{h}\right| = |h| \left|\sin\frac{1}{h}\right| \le |h|. \left(\text{since} \left|\sin\frac{1}{h}\right| \le 1\right)$

If $\in > 0$ then $\left|h \sin\frac{1}{h}\right| < \in$ whenever $|h| < \delta \,(= \in)$.

Thus $\lim_{h \to 0} h \sin\frac{1}{h} = 0 \Rightarrow f'(0) = 0.$

Example 50

Find derivatives of

(i) $\frac{2x+1}{x^2+1}$, (ii) $\tan x$, (iii) $\frac{1}{f(x)}$.

Solution :

(i) $$\frac{d}{dx}\left(\frac{2x+1}{x^2+1}\right) = \frac{\frac{d}{dx}(2x+1).(x^2+1) - (2x+1)\frac{d}{dx}(x^2+1)}{(x^2+1)^2}$$

$$= \frac{2(x^2+1) - (2x+1)\,2x}{(x^2+1)^2}$$

$$= \frac{2(1 - x - x^2)}{(x^2+1)^2}.$$

(ii) $$\frac{d}{dx}\tan x = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right)$$

$$= \left[\frac{d}{dx}\sin x . \cos x - \sin x \frac{d}{dx}\cos x\right] / \cos^2 x$$

$$= \left[\cos x . \cos x - \sin x\,(-\sin x)\right] / \cos^2 x$$

$$= \frac{1}{\cos^2 x} = \sec^2 x.$$

This result has already been obtained 'ab initio' earlier.

(iii) $$\frac{d}{dx}\left(\frac{1}{f(x)}\right) = \frac{0 . f(x) - 1 . f'(x)}{\{f(x)\}^2}$$

$$= -\frac{f'(x)}{\{f(x)\}^2} \quad \text{......... (5)}$$

This is a useful formula.

We emphasize that the rules (1) – (5) should be committed to memory and unless specifically asked to find derivatives from the first principle (or from 'ab initio' or from definition) one should always use these rules (and some more to follow) to find out the derivative of a given function.

EXERCISE 14 (f)

Differentiate.

1. $x^8 + x^7$
2. $x^{5/3} - x^{1/2}$
3. $x^3 - 5x$
4. $\sqrt{x} + \dfrac{1}{\sqrt{x}} - \sqrt[3]{x^2}$
5. $x^2 + 2x - \sin x + 5$
6. $\dfrac{1}{2}\,x^{\frac{1}{2}} + \dfrac{1}{3}\,x^{\frac{1}{3}}$
7. $ax^2 + b\tan x + \ln x^3$
8. $\sqrt{x}\,\left(\sqrt{x} + 1\right)$
9. $(x - 1)^2$
10. $(x^2 - x + 2)^2$
11. $x\sin x - \dfrac{e^x}{1 + x^2}$
12. $\tan 2x + \sec 2x$
13. $\dfrac{x^2}{x + 1} - \dfrac{x}{1 - x}$
14. $\dfrac{\sqrt{x} - 1}{\sqrt{x} + 1}$
15. $\dfrac{\tan x - \cos x}{\sin x \cos x}$
16. $\left(\dfrac{x - 1}{x + 1}\right)^2$
17. $x^2(1 + x)(2 - x)$
18. $x^3 \sin x\, e^{4\ln x}$
19. $\dfrac{1}{\sqrt{x}} + x\ln x^3$
20. $x^2 \log_2 x + \sec x$
21. $\dfrac{x^2 - 1}{x^3 + 1}$
22. $(x^3 + 1)(3x^2 + 2x - 7)$
23. $\cot x - \sec x - \log_{10} x$
24. $\dfrac{1 - \cos x}{1 + \cos x}$
25. $\dfrac{1 - \tan x}{1 + \tan x}$
26. $\left[x^{3/5} - 2e^{2\ln x} + \ln x^{2/3}\right] / (1 + x)$
27. $\operatorname{cosec} x + \cot x$
28. $\tan^2 x + \sec^2 x$
29. $\tan^2 x + a^x$
30. $\sin^2 x - x\ln x$
31. $\cos^2 x + e^x \cos x$
32. $\dfrac{a^x - b^x}{x}$
33. $\dfrac{e^x + e^{-x}}{x^2 + 1}$
34. $\dfrac{\ln x}{x^2}$
35. Show that $f(x) = \begin{cases} x\sin\dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ is not differentiable at $x = 0$.

CHAPTER 15

Statistics

Strong minds discuss ideas, average minds discuss events, weak minds discuss people.

- Socrates

15.1 Introduction :

The phenomenal world around us is a storehouse of knowledge. Not only human beings, birds and animals also have wonderful ability to feel the ways things happen all around.

Through prolonged observation we have been able to develop the science of music, architecture, medicines and a host of several others.

Observation can be both qualitative and quantitative. It is quantitative observations that will be our sole concern in this chapter and we shall discuss how qualitative informations can be drawn out of quantitative observations which are in the form of numbers better known as data. So, naturally, our aim is to discuss the collection, organisation and interpretation of data. The science that develops as a result of such type of endeavour is known as 'statistics' which derives its name from the latin word 'status' from which there originated the term 'statist' and the present day 'statistics'.

Statistical observations come into use in planning for social devlopment, analysis of trends in various social sciences such as finance, economics, commerce and even natural sciences.

The relative exactness of a statistical information depends upon the prudence that an investigator exercises in the collection and analysis of the data using appropriate methods.

On this score, the inferences of statistics differ from the laws of nature. Despite all measurs, there is some amount of subjectivity in a statistical observation unlike the laws of natural science.

15.2 Measures of Dispersion

In a statistical investigation, the initial numerical observations, known as raw data, are organised and what the organised collection of data or more technically the statistical distribution points to, is known as its central tendency. For example, the central tendency of the temperature distribution across the state of Odisha during the hottest summer is around 42^0C. However, temperature readings of individual places may differ from 42^0C; some may be even more and some may be less.

To study howfar the individual items of a distribution differ from central tendency or central value we develop the concept of 'dispersion'.

Dispersion literally means 'scatteredness', that is, the variation of a set of observations from its central value. We may recall that the central values or the measures of central tendency of observations are usually given by

(a) mean (b) median (c) mode.

By 'mean' we shall always mean, the arithmetic mean. For example if x_1, x_2,....x_n are the observations, then the mean is given by

$$\bar{x} = \frac{x_1 + x_2 + \ldots + x_n}{n}.$$

Median is the middle most observation when the data is arranged in either **ascending** or **descending** order of magnitude. If there are n observations and n is odd, then the median is the $\left(\frac{n+1}{2}\right)^{\text{th}}$ observation in the arranged order.

If n is even, the median is taken as the mean of the $\left(\frac{n}{2}\right)^{\text{th}}$ and $\left(\frac{n}{2}+1\right)^{\text{th}}$ observations.

For example the median of daily earnings of 15 workers: 16, 8, 16, 7, 12, 6, 13, 14, 16, 17, 19, 20, 9, 12, 10 is calculated as 8^{th} observation in the arranged order and it is 13.

Mode is the value of the observation that occurs most.

For example in the data given above, 16 occurs most often and so mode is 16.

It is possible that a set of data may have more than one mode if certain observation occurs most often and equal number of times. If each observation occurs equal number times, the data is said to have no mode. For example the data : 1, 2, 1, 2, 1, 2, has no mode while the data : 1, 2, 1, 1, 2, 1, 2, 3, 2 has two modes : 1, 2.

Such type of data with two modes is called a 'bimodal' distribution. A distribution with two or more than two modes is called 'multimodal' and, analogously, one with a single or unique mode is called 'unimodal'.

Measres of Central tendency for grouped distribution

First let us clarify discrete and continuous frequency distributions:

Hitherto, we have discused discrete distributions, which are arangements of sets of observations stated with their individual frequencies mentioned against each. We shall now consider-

Continuous distribution : If observed values are classified into class-intervals without gaps between them, with frequencies mentioned interval-wise, then such a classification of data is known as a continuous frequency distribution.

More specifically, in an internal (l_{i-1}, l_i), l_{i-1} is called the lower limit and l_i, the upper limit. If, eventually, it happens that an observation equals l_{i-1}, then it is placed in the interval (l_{i-1}, l_i). But if it equals l_i, then it is placed in (l_i, l_{i+1}). Ofcourse, depending upon situation, there may be exception to this convention. This all depends upon the investigator who classifies the data.

In your highschool curriculum there have been detailed discussion on computation of mean and median for a grouped frequency distribution. However, we present a brief recapituation of the formulae.

Mean

(i) Direct method : If x_i are the mid values $\frac{1}{2}(l_{i-1} + l_i)$ of the intervals (l_{i-1}, l_i) with frequencies f_i, i = 1, 2, ..., n ; then the mean $\bar{x}$ is given by

$$\boxed{\bar{x} = \frac{1}{N}\sum_{i=1}^{n} f_i x_i \text{, where } N = \sum_{i=1}^{n} f_i}$$

(ii) Short cut methods :

(*a*) Taking a working mean :

Usually, to help out the computation of mean from being tedious, a working mean A is taken, presumably close to the actual mean. The choice of A is actually made out of the experience gained from organising data. However, it can be easily proved from the formulae for computation of mean that $\bar{x}$ is independent of the choice of A.

With working mean A, $\bar{x} = \text{A} + \frac{1}{\text{N}}\sum_{i=1}^{n} f_i d_i$, where $\text{d}_\text{i} = x_i - \text{A}$, $\text{N} = \sum_{i=1}^{n} f_i$.

(b) Step-deviation method :

It can be easily proved that

$\bar{x} = \text{A} + \frac{\text{c}}{\text{N}}\sum_{\text{i}=1}^{\text{n}} f_i u_i$; $u_i = \frac{x_i - \text{A}}{\text{c}}$ and c is the uniform width $(\text{l}_\text{i} - \text{l}_{\text{i-1}})$ of the class-intervals and all other symbols have usual meaning.

Median

Median for a groped frequency distribtion is computed through the following steps :

(i) Determine the position m of the median as $\frac{\text{N}}{2}$ or $\frac{\text{N}+1}{2}$ according as N, the total frequency, is even or odd.

(ii) Determine the cumulative frequencies (c.f) of the intervals

$(x_{\text{i-1}}, x_1)$ as $\sum_{\text{j}=1}^{i} f_i$; for $i = 1,2,\ldots, n$.

As on illustration, c.f of (x_0, x_1) is f_1 and those of (x_1, x_2), (x_2, x_3) are $f_1 + f_2$, $f_1 + f_2 + f_3$ and so on.

(iii) Determine the highest cumlative freqency c, which is not greater than m, the position of the median, i.e. $\text{c} \leq \text{m}$.

(iv) Identify the interval corresponding to the cumulative frequency c as the 'class preceeding median-class' and the next interval as the 'median-class'.

(v) Compute median applying the formula :

Median $= l + \frac{m-c}{fm} \times h$, where h = width of median-class,

l= lower limit of the median-class and

fm = freqency (not c.f.) of the median-class

Mode

For a grouped distribtion the mode is given by the empirical formula :

Mode = 3 Median-2 Mean.

N.B. Mark the adjective 'empirical' before 'mode'. 'Empirical' is something which is established through observations, not argument.

The above formula gives the model values in most grouped distributions, but applied to discrete, i.e. ungrouped distributions, there is likelihood of weird results.

Consider the data : 1, 1, 2, 5, 6.

Obviously : Median = 2, Mode =1 (it occurs maximm, 2 times) and mean = 3.

But '3 Median-2 Mean' gives 0 as mode which is not the case !

Another formula for mode

Mode of a grouped frequency distribution can also be conveniently computed without computing mean and median ≠ by the formula :

$$\text{Mode} = l + \frac{f_m - f_1}{2f_m - f_1 - f_2} \times c,$$

Where l = lower limit of the class with maximum frequency fm

f_1= frequency just before f_m

f_2= frequency just after f_m

N.B. The class-interval with frequency f_m is called the modal class.

Two sets of observations may have the same central value, yet their "scatteredness" may differ. Consider for example the scores of three cricket players A, B, C in five consecutive innings.

A : 48, 49, 50, 51, 52
B : 30, 40, 50, 60, 70
C : 50, 90, 10, 70, 30

All these three players have a mean score of 50 (and median 50), but it is clear even to a layman that the three sets of scores differ remarkably from each other. For A, the range is 48 - 52 which is within ± 2 points of the mean value. For B, the range is 30 - 70, which is within ± 20 points of the mean whereas for C, the range is within ± 40 points of the mean.

Definition (Dispersion) : The variability or the scatter in the values of a set of observations is known as dispersion.

There are several measures of dispersion. In this section we shall confine our discussion to the following measures :

(i) Range (ii) Mean deviation

(iii) Variance (iv) Standard deviation.

Range : It is the simplest measure of dispersion of data. This gives the extent upto which the data fluctate in general. It is defind as

Range = Maximum vale-Minimum value

Though range does not give us any information regarding the fluctuation of data about any central value, it comes to use in quality control in production and prices, fluctuations in share prices and, to a measure extent on weather forecast and also medical treatment.

Example-1

From the list of body temperatures of a patient, recorded at different hours, determine the range of fluctuation:

Time	6a.m.	10a.m.	2 p.m.	6p.m.	10p.m.
Temperature in degrees Fahrenheit (^{0}F)	99	100	104	102	101

Solution : Range = 104^0F - 99^0F = 5^0F

Mean Deviation :

Definition : The mean deviation from the mean is defined as the mean value of the absolute deviations from the mean. Mean deviation from the mean is generally called mean deviation. Thus

$$\text{Mean deviation} = \frac{1}{n}\sum_{i=1}^{n}|x_i - \bar{x}|, \text{ where } \bar{x} = \frac{1}{n}(x_1 + x_2 + \ldots + x_n) \qquad \textbf{(1)}$$

This definition can be adapted to a more general situaion as follows. Suppose that the observed values are $x_1, x_2 ..., x_n$ with frequencies $f_1, f_2, ...f_n$ respectively, then

the mean value $\bar{x}$ is given by, $\bar{x} = \dfrac{f_1x_1 + f_2x_2 + ...+f_nx_n}{f_1 + f_2 + ...f_n}$.

The mean deviaion (M.D. in short) is then defined by

$$\text{M.D.} = \frac{\sum_{i=1}^{n} f_i|x_i - \bar{x}|}{f_1 + f_2 + ...f_n} \qquad \textbf{(2)}$$

Both(1) and (2) above can be slightly generalized as follows. Let a be a real number; if x_1, x_2,.....x_n are the values of an observation then the mean deviaion from a is defined as

$$\sum_{i=1}^{n} \frac{|x_i - a|}{n} \qquad \textbf{(3)}$$

Similarly, if the observed values $x_1, x_2, \ldots\ldots x_n$ occur with frequencies $f_1, f_2, \ldots\ldots f_n$ respectively, then the mean deviation from a is defined as the value

$$\frac{\sum_{i=1}^{n} f_i |x_i - a|}{\sum_{i=1}^{n} f_i}. \qquad \textbf{(4)}$$

Example 2 (Mean deviation from mean) :

The scores of a cricket player in 10 consecutive innings are

28, 20, 52, 90, 37, 68, 51, 62, 0, 17.

Find the mean deviation from the mean.

Clearly, $\bar{x} = \dfrac{28+20+52+90+37+68+51+62+0+17}{10} = 42.5$.

Therefore mean deviation from the mean =

$$\frac{14.5+22.5+9.5+47.5+5.5+25.5+8.5+19.5+42.5+25.5}{10} = \frac{221}{10} = 22.1 .$$

Example 3 (Mean deviation from median):

Five fair coins are tossed and the number of heads noted. This experiment is repeated 20 times. From the results tabulated below, compute the mean deviation from the median and from the mean.

Number of heads :	0	1	2	3	4	5	(x_i)
Frequency :	1	3	7	6	2	1	(f_i).

It is clear that the median is 2. Hence the mean deviation from 2 is given by the formula

$$\frac{\sum_{i=0}^{5} f_i |x_i - 2|}{\sum_{i=0}^{5} f_i} = \frac{2+3+0+6+4+3}{20} = 0.9 .$$

The mean value $\bar{x} = \dfrac{0+3+14+18+8+5}{20} = \dfrac{48}{20} = 2.4$.

Hence the mean deviation from the mean = $\dfrac{2.4+4.2+2.8+3.6+3.2+2.6}{20} = 0.94$.

Variance : Let $x_1, x_2, \ldots, x_n$ be a set of observations with mean $\bar{x}$. Then their **variance,** denoted by σ^2, is given by $\sigma^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2$. **(5)**

In other words, variance can be defined as the arithmetic meanof squares of the deviations from the mean.

When the observed values $x_1, x_2, \ldots, x_n$ occur with frequencies $f_1, f_2, \ldots f_n$ respectively their **variance**, σ^2, is defined by

$$\sigma^2 = \frac{\sum_{i=1}^{n} f_i (x_i - \bar{x})^2}{\sum_{i=1}^{n} f_i}. \quad \textbf{(6)}$$

Observe that if $f_1 = f_2 = \ldots = f_n = 1$ then the definition given in (6) is same as the one given in (5). To shorten writing, let N be the total frequency; that is,

$N = f_1 + f_2 + \ldots + f_n$.

Then $\sigma^2 = \frac{1}{N}\sum_{i=1}^{n} f_i (x_i^2 - 2\bar{x}x_i + \bar{x}^2)$ [By (6)]

$$= \frac{1}{N}\sum_{i=1}^{n} f_i x_i^2 - \frac{2\bar{x}}{N}\sum_{i=1}^{n} f_i x_i + \frac{\bar{x}^2}{N}\sum_{i=1}^{n} f_i = \frac{1}{N}\sum_{i=1}^{n} f_i x_i^2 - \bar{x}^2.$$

$$\boxed{\sigma^2 = \frac{1}{N}\sum_{i=1}^{n} f_i x_i^2 - \bar{x}^2} \quad \textbf{(7)}$$

This formula is often used in computations.

Standard Deviation

Definition : The positive square root of the variance is called the **standard deviation.** It is denoted as σ.

Example 4

A student scores the following marks in six tests.

45, 54, 41, 57, 43, 48.

Find the standard deviation of his scores.

The mean score $\bar{x} = \frac{45 + 54 + 41 + 57 + 43 + 48}{6} = 48$.

Therefore, $\sigma^2 = \frac{1}{6}\{(-3)^2 + 6^2 + (-7)^2 + 9^2 + (-5)^2 + 0\}$

$= \frac{1}{6}\{9 + 36 + 49 + 81 + 25\} = \frac{100}{3}$.

$\therefore \sigma = 5.7735$.

Example 5

The marks obtained by a group of 50 students are as follows :

Marks :	0 – 10	10 – 20	20 – 30	30 – 40	40 – 50
Number of students :	6	12	18	9	5

Find the standard deviation.

We can tabulate the above marks as follows :

Marks	**Number of students (f_i)**	**Mid values (x_i)**	$f_i x_i$	$f_i x_i^2$
0 – 10	6	5	30	150
10 – 20	12	15	180	2,700
20 – 30	18	25	450	11,250
30 – 40	9	35	315	11,025
40 – 50	5	45	225	10,125
	N=50		1200	35,250.

We thus have

$$N = 50,\ \Sigma f_i x_i = 1200,\ \Sigma f_i x_i^2 = 35{,}250$$

$$\bar{x} = \frac{1}{N}\Sigma f_i x_i = 24.$$

$$\sigma^2 = \frac{1}{N}\Sigma f_i x_i^2 - \bar{x}^2 = 705 - 576 = 129\ .$$

Standard deviation $\sigma \simeq 11.35$.

Variance of grouped frequency distribution:

We have already given two formulae for computing variance,

(i) $\sigma^2 = \dfrac{1}{N}\sum_{i-1}^{n} f_i (x_i - \bar{x})^2$, $N = \sum_{i=1}^{n} f_i$

This is as per the definition of variance.

(Taking $f_1 = f_2 = \ldots f_n = 1$, we get $\sigma^2 = \dfrac{1}{n}\sum_{i-1}^{n} f_i (x_i - \bar{x})^2$

(ii) The above formula was modified as

$$\sigma^2 = \frac{1}{N}\sum_{i-1}^{n} f_i x_i^2 - \bar{x}^2 ,$$

Where summation extends over $f_i x_i^2$ only. This formula is often used in computations.

Both the above formulae are suitable for computing variance in case of grouped frequency distribution by taking x_i as the mid values $\frac{1}{2}(l_{i-1}+l_i)$ of the class-intervals.

(iii) To make the computations easy in case of comparatively larger data we now modify the second formula by step-deviation :

Lit $u_i = \dfrac{x_i - A}{c}$, where A = assumed mean and c = uniform class interval, then

$$\sigma^2 = \frac{1}{N}\sum f_i x_i^2 - \bar{x}^2$$

$$= \frac{1}{N}\sum f_i (A + cu_i)^2 - (A + \frac{c}{N}\sum f_i u_i)^2$$

$$= \frac{1}{N}\sum f_i (A^2 + c^2 u_i^2 + 2cAu_i) - \left\{A^2 + c^2\left(\frac{1}{N}\sum f_i u_i\right)^2 + 2cA\frac{1}{N}\sum f_i u_i\right\}$$

$$= \left[\left(\frac{1}{N}\sum f_i\right)A^2 + \frac{c^2}{N}\sum f_i u_i^2 + 2\frac{c}{N}A\sum f_i u_i - A^2 - c^2\left(\frac{1}{N}\sum f_i u_i\right)^2 - 2\frac{c}{N}A\sum f_i u_i\right]$$

$$= A^2 + \frac{c^2}{N}\sum f_i u_i^2 - A^2 - c^2\left(\frac{1}{N}\sum f_i u_i\right)^2 \quad (\because N = \sum f_i)$$

$$= c^2\left[\frac{1}{N}\sum f_i u_i^2 - \left(\frac{1}{N}\sum f_i u_i\right)^2\right]$$

Therefore

$$\boxed{\sigma^2 = c^2\left[\frac{1}{N}\sum_{i=1}^{n} f_i u_i^2 - \left(\frac{1}{N}\sum_{i=1}^{n} f_i u_i\right)^2\right]} \qquad \textbf{(8)}$$

Accordingly standard deviation is given by

$$\sigma = c\sqrt{\frac{1}{N}\sum_{i=1}^{n} f_i u_i - \left(\frac{1}{N}\sum_{i=1}^{n} f_i u_i\right)^2} \qquad \textbf{(9)}$$

Example 6

Find the mean deviation from the following distribution of age :

Age groups	0-10	10-20	20-30	30-40	40-50	50-60	60-70	70-80	80-90	90-100
No. of persons	8	10	20	25	24	28	17	9	7	2

Solution :

Intervals	**No. of persons** f_i	**Mid points** x_i	$u_i \frac{x_i - 45}{10}$	$f_i u_i$	$\lvert x_i - \bar{x}\rvert$	$f_i \lvert x_i - \bar{x}\rvert$
0-10	8	5	-4	-32	40	320
10-20	10	15	-3	-30	30	300
20-30	20	25	-2	-40	20	400
30-40	25	35	-1	-25	10	250
40-50	24	45	0	0	0	0
50-60	28	55	1	28	10	280
60-70	17	65	2	34	20	340
70-80	9	75	3	27	30	270
80-90	7	85	4	28	40	280
90-100	2	95	5	10	50	100
	$N = 150 = \sum f_i$			$\Sigma f_i u_i = 0$		$\sum f_i \lvert x_i - \bar{x}\rvert = 2540$

$$\bar{x} = A + \frac{c}{N}\sum_{i=1}^{10} f_i u_i = 45 + \frac{10}{150} \times 0 = 45$$

$$\text{Therefore, Mean Deviation} = \frac{1}{N}\sum_{i=1}^{10} f_i\,(x_i - \bar{x}) = \frac{2540}{150} = 16.93$$

Example 7

Given below is the distribution of marks of 120 students in mathematics.

Range	20-30	30-40	40-50	50-60	60-70	70-80	80-90	90-100
No. of Students	3	5	15	18	22	30	18	9

Compute the variance and standard deviation.

Solution :

We apply the step-deviation method to find variance.

Intervals	Frequency f_i	Mid value x_i	ui=$\frac{x_i - 65}{10}$	u_i^2	$f_i u_i$	$f_i u_i^2$
20-30	3	25	-4	16	-12	48
30-40	5	35	-3	9	-15	45
40-50	15	45	-2	4	-30	60
50-60	18	55	-1	1	-18	18
60-70	22	65	0	0	0	0
70-80	30	75	1	1	30	30
80-90	18	85	2	4	72	72
90-100	9	95	3	9	27	81
	N=120				$\sum f_i u_i$=18	$\sum f_i u_i^2$=354

$$\text{Variance } \sigma^2 = c^2\left[\frac{1}{N}\sum_{i=1}^{n} f_i u_i^2 - \left(\frac{1}{N}\sum_{i=1}^{n} f_i u_i\right)^2\right]$$

$$= 100\times\left[\frac{354}{120}-\left(\frac{18}{120}\right)^2\right]=292.75$$

Standard diviation $\sigma=\sqrt{292.75} \simeq 17.11$

15.3 Analysis of frequency distributions

We now come to the point which we indicated in the beginning of the chapter, i.e. drawing qualitative informations from quantitative observations.

For this purpose we define :

Coefficient of variation (C.V.)

$$= \frac{\text{Standard Deviation}}{\text{Mean}} \times 100, \text{ Provided mean} \neq 0.$$

$$\boxed{\text{C.V.}=\frac{\sigma}{\bar{x}} \times 100}$$

Observe that for a given distribution, standard deviation σ and mean $\bar{x}$ are expressed in the same unit in which the data of the distribution are given. While finding $\frac{\sigma}{\bar{x}}$, we simply take into account the numbers expressing σ and $\bar{x}$, not their units, otherwise, such a division will be meaningless.

This C.V. is just a number devoid of any units.

The coefficient of variance (or variation) C.V. expresses the measure of scatter or dispersion or variability of the data from their central value. Being devoid of any unit C.V. facilitates comparison between data expressed in different units.

The distribution with greater C.V. is said to be more variable or dispersed than that with lesser C.V. To speak the other way, **the distribution with lesser C.V. is said to be more consistent than that with greater C.V.**

Example - 8 :

The following is a distribution of weight against height of seven persons.

Height in meter (x_1)	1.50	1.56	1.60	1.62	1.63	1.70	1.72
Weight in Kg.	50	57	60	65	65	67	75

Which one shows less variability, height or weight ?

Solution :

$$\bar{x}_1 \text{ (Mean height)} = \frac{1.50+1.56+1.60+1.62+1.63+1.70+1.72}{7} \simeq 1.62 \text{ meter}$$

$$\sigma_1^2 = \text{Variance of height}$$

$$= \frac{1}{7}\{(1.50-1.62)^2+(1.56-1.62)^2+(1.60-1.62)^2+(1.62-1.62)^2+(1.63-1.62)^2+$$

$$(1.70-1.62)^2+(1.72-1.62)^2\}$$

$$\simeq 0.005 \text{ meter}^2$$

$$\sigma_1 = \text{S.D. of height} = \sqrt{.005} \simeq 0.71 \text{ meter}$$

$$\text{C.V. of height} = \frac{\sigma_1}{\bar{x}_1} \times 100 = \frac{0.71}{1.62} \times 100 \simeq 43.82 \qquad (i)$$

$$\bar{x}_2 = \text{Mean weight} = \frac{50+57+60+65+65+67+75}{7} \simeq 63 \text{ kg.}$$

$$\sigma_2^2 = \text{Variance of weight} = \frac{1}{7}\{(50-63)^2+(57-63)^2+(60-63)^2+$$

$$(65-63)^2+(65-63)^2+(67-63)^2+(75-63)^2\} = 54.57 \text{ kg}^2$$

$$\sigma_2 = \text{S.D. of weight} = \sqrt{54.57} \text{ kg} \simeq 7.39 \text{ kg.}$$

$$\text{C.V. of weight} = \frac{\sigma_2}{\bar{x}_2} \times 100 = \frac{7.39}{63} \times 100 \simeq 11.73 \qquad (ii)$$

From (i) and (ii) we conclude that weight shows less variability than height.

Example 9 :

Two players A and B have the following scores in the last 10 innings they have played.

A : 83, 0, 41, 45, 1, 103, 55, 21, 49, 22.

B : 5, 0, 7, 14, 10, 11, 12, 3, 11, 7.

Find out who is a better player and who is a more consistent player.

For A : Mean score $\bar{x} = \frac{420}{10} = 42$.

Variance $\sigma^2 = \frac{1}{10}\{41^2 + 42^2 + 1^2 + 3^2 + 41^2 + 61^2 + 13^2 + 21^2 + 7^2 + 20^2\} = \frac{9916}{10} = 991.6$.

Standard deviaion $\sigma = 31.6$.

$$\text{C.V.} = \frac{\sigma}{\bar{x}} \times 100 = \frac{31.6}{42} \times 100 \simeq 75.24$$

For B : Mean socre $\bar{x} = \frac{80}{10} = 8$

Variance $\sigma^2 = \frac{1}{10}\{9 + 64 + 1 + 1 + 36 + 4 + 9 + 16 + 25 + 9\} = \frac{174}{10} = 17.4$

Standard deviation $\sigma \simeq 4.17$.

$$\text{C.V.} = \frac{\sigma}{\bar{x}} \times 100 = \frac{4.17}{8} \times 100 \simeq 52.13$$

These calculations show that A is a better player (Since the mean of A′s scores 42 is greater than that of B′s) whereas B is a more consistent player. (Since the C.V. of B′ is scores is much less than the C.V. of A′ s scores).

15.4 Analysis of frequency distribution with equal means but different variances

In case of equal means there is no necessity of finding out coefficient of variation (C.V) of the data.

The distribution with lesser standard deviation is said to be more consistant or less variable than that with greater standard deviation.

In otherwards, the distribution with greater standard deviation is more dispersed or scattered than that with lesser standard deviation.

Example - 10

A student takes four successive tests in literature and mathematics. From the follwoing table of his marks, comment on his average and consistency of performance is literature versus mathematics.

Literature (x)	74	76	67	71
Mathematics (y)	80	61	57	90

Solution :

Average marks in literature

$$\bar{x} = \frac{74+76+67+71}{4} = 72$$

Average marks in mathematics

$$\bar{y} = \frac{80+61+57+90}{4} = 72$$

Variance (literature) =

$$\sigma_1^2 = \frac{(74-72)^2+(76-72)^2+(67-72)^2+(71-72)^2}{4} = 11.5$$

$\therefore$ S.D. (literature) $\sigma_1 = \sqrt{11.5} \simeq 3.39$

Variance (mathematics) =

$$\sigma_2^2 = \frac{(80-72)^2+(61-72)^2+(57-72)^2+(90-72)^2}{4} = 183.5$$

$\therefore$ S.D. (mathematics) $\sigma_2 = \sqrt{183.5} \simeq 13.55$

Comment : Average performance in both the subjects is same, but his performance shows more consistency in literature than in mathematics.

EXERCISES 15

1. If the values observed are 1, 2, ..., n each with frequency 1, find
   (i) the mean value
   (ii) the mean deviation from the mean separately for two cases when n is odd and when n is even.
2. For the same set of values as in (1) above, find the variance and standard deviation.
3. From the table below, find the mean value and the variance.

Values	:	1	2	3 .....n
Frequency	:	1	2	3 ....n

4. From the tables below, find the mean and the variance.

(a)	Values	:	1	3	5 ....$(2n-1)$	
	Frequency	:	1	1	1	1

(b) Values	:	2	4	6 ....$2n$
Frequency	:	1	1	1 1

5. From the table below, calculate the mean and the variance.

Values	:	0	1	2	....	r	...	n
Frequency	:	${}^nC_0$	${}^nC_1$	${}^nC_2$	....	${}^nC_r$	....	${}^nC_n$

(Those interested may take up this exercise after learning differentication from Volume-II- not examineble in 1st year)

Hints $\Sigma f_r = \sum_{r=0}^{n} {}^nC_r = 2^n.$

On the other hand, $(1+x)^n = \sum_{r=0}^{n} {}^nC_r x^r$.

Differentiating both sides with respect to x, we have $n(1+x)^{n-1} = \sum_{r=0}^{n} r.\, {}^nC_r x^{r-1}$.

Putting $x = 1$ in this equation, we obtain $n.\, 2^{n-1} = \sum_{r=0}^{n} r.\, {}^nC_r$ (*).

Thus the mean value $\bar{x} = \dfrac{n.2^{n-1}}{2^n} = \dfrac{n}{2}$

For the standard deviation and variance, we have $\sigma^2 = \dfrac{1}{2^n} \Sigma r^2\, {}^nC_r - \bar{x}^2$

To compute $\Sigma r^2\, {}^nC_r$. we again employ the Binomial Theorem.

$$(1+x)^n = \Sigma\, {}^nC_r . x^r .$$

Differentiating twice, we obtain $n(n-1)(1+x)^{n-2} = \Sigma r(r-1)\, {}^nC_r x^{r-2}$

Put $x = 1$ in this equation : $n(n-1)2^{n-2} = \Sigma(r^2 - r)\, {}^nC_r$

$$= \Sigma r^2\, {}^nC_r - \Sigma r\, {}^nC_r = \Sigma r^2\, {}^nC_r - n.2^{n-1} \text{ (from (*))}$$

Therefore, $\Sigma r^2\, {}^nC_r = n(n-1)2^{n-1} = n(n+1)2^{n-2}$

$$\therefore \sigma^2 = \frac{n(n+1)2^{n-2}}{2^n} - \left(\frac{n}{2}\right)^2 = \frac{n(n+1)-n^2}{4} = \frac{n}{4}$$

and $= \sigma + \sqrt{n}/2$.

6. From the following table calculate the mean, mean deviation from the mean and variance.

Marks Number of students

30 – 35	5
35 – 40	7
40 – 45	8
45 – 50	20
50 – 55	16
55 – 60	12
60 – 65	7
65 – 70	5.

7. In a soccer league, two teams A and B have the following records :

A :	Goals scored :	0	1	2	3	4	
	Number of matches :	11	18	8	6	2	
B :	Goals scored :	0	1	2	3	4	5
	Number of matches :	5	20	10	6	3	1.

Which team is more consistent ? Which is a better team ?

8. Find the coefficient of variation c.v. for each of the following set of observations.

(i) 2, 3, 4, 2, 5, 7, 8, 9

(ii) 5, 7, 9, 10, 7, 5, 8, 9, 3

(iii) 3, 3, 3, 4, 4, 4, 5, 5, 5.

9. Suppose the values $x_1, x_2, ... x_n$ having frequency $f_1, ... f_2 ... f_n$ respectively have mean value $\bar{x}$ and variance σ^2. Let a be a fixed real number. Show that the values $x_1 + a, x_2 + a, ..., x_n + a$ with frequency $f_1, f_2, ..., f_n$ respectively will have mean value $\bar{x} + a$ and variance σ.

10. Find the mean deviation from the mean and the standard deviation of $a, a + d, a + 2d, ..., a +$ $2nd$; assume that $d > 0$.

11. Let $x_1, x_2, ..., x_n$ be a set of observations with mean value 0 and variance σ_x^2 and $y_1, y_2 ... y_m$ be another set of observation with mean value 0 and variance σ_y^2. Find the mean value and variance of the set of observations $x_1, x_2 .... x_n, y_1, y_2, ... y_m$ combined.

12. Find which group of the following data is more dispersed :

	Range	10-20	20-30	30-40	40-50	50-60
(Group A)	Frequency	5	1	3	2	1
(Group B)	Frequency	1	3	2	3	1

13. The price of land per square meter and that of gold per ten grams over five consecutive years is given below. Decide, which price mantains a better stability.

(Hint : Stability $\Leftrightarrow$ Consistency)

Price of land/Sq. meter (₹)	1500	2500	2600	3000	4000
Price of Gold/10gms (₹)	2500	2600	2750	2900	2850

●

CHAPTER 16

Probability

A reasonable probability is the only certainty.

– E.W. Howe

He who has heard the same thing told by 12,000 eye-witnesses has only 12,000 probabilities, which are equal to one strong probability, which is far from certainty.

– Voltaire

16.0 Historical Introduction :

The foundations of the theory of probability are believed to have been laid by French mathematicians Fermat (1604 – 1665); (the name is pronounced as 'Ferma'), Pascal (1623-1662) and Laplace (1949–1827), Italian mathematician Bernoulli (1654 – 1705) and a host of others. However, long before those celebrated people arrived on the scene, an Italian mathematician Jerome Cardan (1501 – 1576) in his little book 'Liber de Ludo Aleae', considered to be a gambler's manual, gave most of the laws of probability. Of course, Cardan's work never came to the notice of the international mathematical community. A solid mathematical foundation to the modern theory of probability was given by the Russian mathematician A.N.Kolmogorov.

The theory of probability had its origin in the exchange of a series of letters between Pascal and Fermat in the year 1654; it involved a very simple question posed by a gambler named 'Chevalier de Meve': how fairly the stakes at a game of dice were to be distributed if the game was abruptly halted at some point before completion. The answer to this question involved a sample space which is not uniform.

16.1 Basic Concepts

Probability is the study of random or nondeterministic events. For example, if we toss a fair coin, then it results either in a head or a tail; of course we can't be sure of the outcome in advance. Suppose we repeat this process of tossing a coin a large number of times and keep a record of the number of heads we get in the process. To be more precise, let

N = the number of tosses

n = the number of heads obtained,

What then do you expect of the ratio n/N as N becomes large ? If N = 1000, n may not be exactly 500, but you may convince yourself by doing this experiment if necessary that

$n/\mathrm{N} \rightarrow \frac{1}{2}$ as $\mathrm{N} \rightarrow \infty$. That this limit exists is provided by a theorem known as the **Law of large numbers**. In this book, we shall not *go* to such lengths; instead, we will make a simple and straight forward approach. To get started, let us first define some words which will be used throughout this chapter.

Coming back to the toss of a coin again :

If we toss a coin, we are zsure to get either a head (h) or a tail (t); that is either 'h' occurs or 't' occurs. In the language of probability, the set

S = $\{h, t\}$

is called the ''sample space'' of the experiment (of tossing a coin). Similarly, if we throw a die, then we are sure to get one of these results : 1, 2, 3, 4, 5, 6. The set

S = {1, 2, 3, 4, 5, 6}

is the sample space of this experiment (of throwing a die). With these two examples in mind, we are now ready to define the terms **experiment** and **sample space.**

Definition :

A random or statistical experiment is one in which

(i) all possible outcomes of the experiment are known in advance

(ii) a performance of an experiment results in an outcome which is not known in advance.

(iii) the experiment can be repeated under identical conditions.

Definition :

The **sample space** of an experiment is the set of all possible outcomes of the experiment.

Example 1 :

Toss a coin twice. The possible outcomes are hh, ht, th, tt where h = head and t = tail.

Therefore the sample space is

S = $\{hh, ht, th, tt\}$.

Example 2 :

Toss a coin thrice. In this case

S = {hhh, hth, thh, tth, hht, htt, tht, ttt}.

Example 3 :

Throw a die twice. There are 36 possible outcomes

S = {(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6),
(2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6),
(3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6),
(4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6),
(5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6)
(6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)}.

The reader should not be led to believe that all sample spaces are finite.

Example 4 :

One is allowed to go on tossing a coin until a head is obtained. In this case the sample space is S = {h, th, tth, $ttth$, ...} which is not finite.

Example 5 :

Pick up an integer at random from the set of positive integers. In this case

S = {1, 2, 3, ...}

which is an infinite set.

Definition : An element of the sample space is called an **elementary event.**

Definition : A subset of a sample space S (of an experiment) is called an **event**. An event is said to occur if an element of the event occurs.

Remark :

This definition applies to all sample spaces which are finite or countable infinity. For uncountable sample spaces, a technical definition will apply which is beyond the scope of this book. According to this definition.

(i) S itself is an event.

(ii) The empty set ϕ is an event.

(iii) If A and B are events (that is $A \subset S$, $B \subset S$) then $A \cup B$ is an event (that occurs if A occurs or B occurs).

(iv) Similarly, $A \cap B$ is an event (that occurs when both A and B occur)

(v) If A is an event A^c (= S – A) is an (event that occurs when A does not occur.)

Sometimes, events (that are subsets of the sample space) can be described in words.

Example 6 :

Consider the sample space of Example 3 above.

Let A = {(1, 6), (2, 5), (3, 4), (5, 2), (6, 1)} which is a subset of S. Clearly, A can be described as the event that sum of points obtained in two throws of a die is 7.

Example 7 :

In the same sample space, let B = {(6, 5), (5, 4), (4, 3), (3, 2), (2, 1)}.

B can then be described as the event that the result of the first throw of the die exceeds the result of the second throw by 1.

Example 8 :

Consider Example 2 above; let C = {hht, hth, thh}.

C can then be described as the event of getting exactly 2 heads in tossing a coin three times.

Example 9 :

In the same example, if D = $\{hhh, ttt\}$ then D can be described as the event of getting either only heads or only tails in tossing a coin 3 times.

We are now in a position to define the probability of an event. But before we do so, we urge the reader to consider the examples above once again a little more carefully. While tossing a coin once

$$S = \{h, t\}.$$

If the coin is unbiased (that is, it is loaded neither way), then both head and tail are equally likely to occur. This means that if the experiment is repeated a large number of times, then half of the times heads will occur and half of the times tails will occur; we say that such sample spaces are **equiprobable spaces** or **uniform spaces**. We also say that the probability of getting a head is $\frac{1}{2}$ and the probability of getting a tail is $\frac{1}{2}$. In short,

$$P(h) = \frac{1}{2}, P(t) = \frac{1}{2}.$$

Again, if we are tossing a fair coin twice, the sample space is S = $\{hh, ht, th, tt\}$.
Let A = $\{ht, th\}$, that is the event of getting one head and one tail. Since each of those four elementary events are equally likely to occur, A will occur twice out of four times; thus

$$P(A) = \frac{2}{4} = \frac{\text{size of A}}{\text{size of S}}.$$

We are intentionally using the word 'size' instead of the phrase 'number of elements'; the reason for this usage will appear a little later.

In the same way, let us cosider the experiment of throwing a die twice. The sample space S in this case has 36 outcomes. If the die is a fair one, each elementary event is equally likely to happen, and so, each will have probability of $\frac{1}{36}$. Thus, for example, if B is the event of getting a total of 8 in two throws of a die, then

$$B = \{(6, 2), (5, 3), (4, 4), (3, 5), (2, 6)\}.$$

Therefore, $P(B) = 5 \times \frac{1}{36} = \frac{5}{36} = \frac{\text{size of B}}{\text{size of S}}$.

Thus for all experiments having finite sample spaces, we can make the following definition.

Definition :

If the sample space S is finite, the probability of an event A denoted by P (A) is defined as $P(A) = \frac{\text{size of A}}{\text{size of S}}$.

It will now be easy to prove the following rules of probability by using the definition given above.

Rule 1 : For any event A, $0 \le P(A) \le 1$.

Rule 2 : $P(S) = 1$.

Rule 3 : If A, B are mutually exclusive events, that is, when $A \cap B = \phi$, then

$P(A \cup B) = P(A) + P(B)$.

We can now prove a few propositions, using these elementary rules.

Theorem 1 : $P(\phi) = 0$, where ϕ is the empty set.

Proof : Let A be any event (that is, any subset of S). Since $A \cap \phi = \phi$, A and ϕ are mutually exclusive events. Therefore, $P(A) = P(A \cup \phi) = P(A) + P(\phi)$ and hence $P(\phi)$ must be zero.

Theorem 2 : $P(A^c) = 1 - P(A)$, where A^c is the complement of A in S.

Proof : Since A and A^c are mutually exclusive, we have by Rule 3,

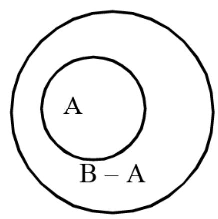
$$P(A) + P(A^c) = P(A \cup A^c) = P(S) = 1.$$

Hence the result.

Theorem 3 : If A, B are any two events and $A \subset B$, then $P(A) \le P(B)$.

Proof : Since $A \subset B$, B can be written as the union of two disjoint sets :

$$B = A \cup (B - A).$$



We can now apply Rule 3 to obtain

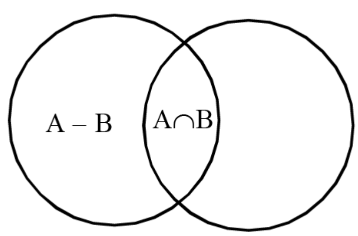
$$P(B) = P\big(A \cup (B - A)\big) = P(A) + P(B - A).$$

Since $P(B - A) \ge 0$ (by Rule 1), it follows that $P(B) \ge P(A)$.

Theorem 4 : For any two events A and B, $P(A - B) = P(A) - P(A \cap B)$.

Proof : As is evident from the accompanying diagram, A can be written as a union of two disjoint sets :

$$A = (A - B) \cup (A \cap B).$$



By applying Rule 3, we have

$$P(A) = P(A - B) + P(A \cap B).$$

Thus $P(A - B) = P(A) - (A \cap B)$.

Theorem 5 : If A and B are two events, then $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

Proof : From the diagram accompanying theorem 4, we can write $A \cup B$ as a union of two disjoint set : $A \cup B = (A - B) \cup B$.

By Rule 3 again, we have $P(A \cup B) = P(A - B) + P(B)$.

By substituting the value of $P(A - B)$ as obtained in Theorem 4, we have

$$P(A \cup B) = P(A) - P(A \cap B) + P(B) = P(A) + P(B) - P(A \cap B).$$

Theorem 5 can be used repeatedly to prove the following.

Corollary : If A, B, C are events, then

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C).$$

Proof : Let $A \cup B = D$. Then

$$P(A \cup B \cup C) = P(D \cup C) = P(D) + P(C) - P(D \cap C)$$

$$= P(A \cup B) + P(C) - P(A \cup B) \cap C)$$

$$= [P(A) + P(B) - P(A \cap B)] + P(C) - P((A \cap C) \cup (B \cap C)). \quad (1)$$

If we let $E = A \cap C$ and $F = B \cap C$ then the last term becomes :

$$P((A \cap C) \cup (B \cap C)$$

$$= P(E \cup F) = P(E) + P(F) - P(E \cap F)$$

$$= P(A \cap C) + P(B \cap C) - P(A \cap B \cap C).$$

(2)

Putting (2) in (1), we get the desired result.

We will now illustrate the use of these theorems.

Example 10 :

Find the probability of getting an even number in throwing a die. As has been said earlier, the sample space in this case S = {1, 2, 3, 4, 5, 6} and the event A (of getting an ever number) is A = {2, 4, 6}.

So the probability of getting an even number is $\dfrac{\text{size of A}}{\text{size of S}} = \dfrac{3}{6} = \dfrac{1}{2}$.

Example 11 :

A die is thrown twice. Find the probability that the sum of points obtained is 8.

As has been indicated earlier in Example 3, the sample space of this experiment has 36 elements. You don't have to write all the points of the sample space to arrive at this conclusion. When you throw a die once, you have 6 possible outcomes; another throw you have 6 possible outcomes. Hence the total number of possible outcomes will be $6 \times 6 = 36$. If A is the event of getting a sum of 8 points then

A = {(6, 2), (5, 3), (4, 4), (3, 5), (2, 6)}.

Thus the size of A is 5; so the required probability is $\dfrac{5}{36}$.

Example 12 :

Two cards are drawn from a standard pack of 52 cards. Find the probability that both cards are aces.

Now 2 cards can be drawn from a pack of 52 cards in ${}^{52}C_2$ ways. The size of the sample space, therefore, is ${}^{52}C_2$. On the other hand there are only 4 aces. One can draw 2 aces in ${}^{4}C_2$ ways; thus the required probability is $\dfrac{{}^{4}C_2}{{}^{52}C_2} = \dfrac{4!}{2!2!} \times \dfrac{2!50!}{52!} = \dfrac{1}{221}$.

Example 13 :

A bag contains 5 white and 3 black balls. If a ball is drawn at random, find the probability that it is white.

Let the 5 white balls be W_1, W_2, W_3, W_4, and W_5 and let the black balls be B_1, B_2 and B_3. Since only one ball is drawn, the sample space is clearly given by

$S = \{W_1, W_2, W_3, W_4, W_5, B_1, B_2, B_3\}$.

Thus the required probability is $\frac{5}{8}$.

Remark

The problem of picking up a ball of a perticular colour from a bag will occur frequently in many problems. It is not, however, necessary to write the sample space in detail as written above. The reader is urged to understand underlying principle.

Example 14 :

A bag contains 6 white and 7 black balls. If 2 balls are drawn at random, find the probability that both balls are white.

The sample space in this case is of size $^{13}C_2$, and the event A of drawing 2 white balls is of size $^6C_2$. The required probability is $\frac{^6C_2}{^{13}C_2} = \frac{5}{26}$.

Example 15 :

Two balls are drawn from a bag containing 4 white and 6 black balls. Find the probability that atleast one of the balls is white.

Let A be the event that one of the drawn balls is white the another is black; let B be the event that both the balls are white; clearly A and B are mutually exclusive. We are interested in finding the probability of A or B. We apply Rule 3 :

$$P(A \cup B) = P(A) + P(B)$$
$$= \frac{^4C_1 \times {^6C_1}}{^{10}C_2} + \frac{^4C_2}{^{10}C_2} = \frac{2}{3}.$$

One can also look at this problem in another way. Let C represent the event of drawing at least one white ball. Then what is C^c ? Clearly, C^c is the event that none of the two balls is white; in other words, both the balls are black. So

$$P(C^c) = P(\text{both balls are black}) = \frac{^6C_2}{^{10}C_2} = \frac{1}{3}.$$

By Theorem 2, $P(C) = 1 - P(C^c) = 1 - \frac{1}{3} = \frac{2}{3}$.

Remark

The reader is urged to look at this problem carefully. There are many situations in which it will be more convenient to compute the probability of A^c than to directly compute that of A.

EXERCISES 16 (a)

1. A coin is tossed twice. Find the probability of getteing
   (i) exactly one head (ii) at least one head (iii) at most one head.
2. A coin is tossed three times. Find the probability of getting
   (i) all heads (ii) at most 2 heads (iii) at least 2 heads.
3. List all possible outcomes when a die is rolled twice (or a pair of dice is rolled once). Then find the probability that
   (i) sum of points is 10 (ii) sum of points is ataleast 10
   (iii) sum of points is at most 10.

Hint : For part (iii) : Let A be the event that the sum of points is at most 10; then A^c is the event that the sum of points is either 11 or 12. Now compute the probabilities of getting a sum of 11, and 12; their sum is $P(A^c)$; now apply the rule $P(A) + P(A^c) = 1$.

4. A die is rolled twice. Find the probability that the result of the first roll exceeds the result of the second roll by
   (i) 3 (ii) at least 3 (iii) at most 3.
5. A card is selected from 100 cards numbered 1 to 100. If the card is selected at random, find the probability that the number on the card is
   (i) divisible by 5 (ii) divisible by 2
   (iii) divisible by both 2 and 5(iv) divisible by either 2 or 5.

Hints : There are just 20 numbers, namely 5, 10 ....., 100 which are divisible by 5; so the probability $= \frac{20}{100}$. For part (iii) : If a number is divisible by both 2 and 5, then it is divisible by 10. For part (iv) : Let A be the event that the number is divisible by 2 and B be the event that the number is divisible by 5. Find the probability of the event $A \cup B$.

6. Eight persons stand in a line at random. What is the probability that two persons X and Y don't stand together.

Hints : The sample space in this case is of size 8 ! Let A be the event that X and Y stand together. You can consider X and Y tied together as one person so that altogether you have 7! arrangements; but then in each such arrangement X and Y can interchange their places still staying together; thus the total number of ways in which X and Y can stand together is 2×7 !.

7. What is the probability that four aces appear together when a pack of 52 cards is shuffled completely ?

Hints : Try the same trick as in problem 6 above.

8. If 8 persons are to sit around a table, what is the probability that two personsX and Y don't sit together.
9. A die is rolled three times. Find the probability that the numbers obtained are in strictly increasing

order.

Hints : As has been remarked earlier, the sample space has $6 \times 6 \times 6 = 216$ points. The question then is : how many of these 216 triples are in strictly increasing order that is like (1, 2, 3) or (1, 2, 5) etc ? If you want to count then you must do so carefully by following the procedure given below. Keep the first entry fixed and enumerate all cases satisfying the given condition (of increasing order). For example, we keep our first entry fixed,say 1, then all cases with first entry 1 are

(i)	First Entry	Second Entry	Third Entry	
	1 (fixed)	2	3	
		2	4	
		2	5	
		2	6	
		3	4	10 cases
		3	5	
		3	6	
		4	5	
		4	6	
		5 $\frac{1}{3}$	6	

(ii) Now take the first entry as 2 and enumerate all cases, then 3 and finally 4.

Check that you have 20 cases in all so that the required probability is $\frac{20}{216}$.

A more elegant way of looking at this problem : Let x_1, x_2, x_3 be triple with $x_1 < x_2 < x_3$ from $\{1, 2, 3, 4, 5, 6\}$, the number of such triples is ${}^6C_3 = 20$.

Author's remark : I have asked this question in numerous class examinations and have got the right answer in a variety of ways; one student even wrote all 216 triples of the sample space in the centre-spread of his examination booklet. He then ticked all triples satisfying $x_1 < x_2 < x_3$, and in the process missed only one !

10. Three phonorecords are removed from their jackets, played and then returned to the jackets at random. Find the probability that (i) none of the records goes to the right jacket (ii) just one record goes to the right jacket (ii) just two records go to the right jackets (iii) all three of them go to the right jackets.

Hints : Let the records be numbered 1, 2, 3 and let their jackets be similarly numbered 1, 2, 3. The first record can be put in any one of the three jackets; the second one then can be put in any one of the two remaining jackets; for the third record there is just one jacket left. Thus the sample space in this case has $3 \times 2 \times 1 = 6$ points, these can be explicitly written down :

$\begin{pmatrix}1 & 2 & 3\\1 & 2 & 3\end{pmatrix}, \begin{pmatrix}1 & 2 & 3\\2 & 3 & 1\end{pmatrix}, \begin{pmatrix}1 & 2 & 3\\3 & 1 & 2\end{pmatrix}, \begin{pmatrix}1 & 2 & 3\\1 & 3 & 2\end{pmatrix}, \begin{pmatrix}1 & 2 & 3\\3 & 2 & 1\end{pmatrix}, \begin{pmatrix}1 & 2 & 3\\2 & 1 & 3\end{pmatrix}.$

In each of these matrices, the first row represents the records and the second row represents the jackets in which they are put. For example, $\begin{pmatrix}1 & 2 & 3\\2 & 3 & 1\end{pmatrix}$

means that record 1 is put in jacket 2, record 2 is the jacket 3 and record 3 is jacket 1.

11. Four records are taken out of their jackets, played and returned to the jackets at random. Find the probability that
    (i) none of the records goes into the right jacket.
    (ii) at least one record is put in the right jacket.

12. Let A and B be events with $P(A) = \frac{3}{8}$, $P(B) = \frac{1}{2}$ and $P(A \cap B) = \frac{1}{4}$. Find
    (i) $P(A \cup B)$ (ii) $P(A^c)$ and $P(B^c)$ (iii) $P(A^C \cup B^C)$
    (iv) $P(A^C \cap B^C)$ (v) $P(A \cap B^c)$ (vi) $P(A^c \cap B)$.

13. Let A and B be events with $P(A) = \frac{1}{3}$, $P(A \cup B) = \frac{3}{4}$ and $P(A \cap B) = \frac{1}{4}$. Find
    (i) $P(A)$ (ii) $P(B)$ (iii) $P(A \cap B^c)$
    (iv) $P(A \cup B^c)$.

14. There are 20 defective bulbs in a box of 100 bulbs. If 10 bulbs are chosen at random what is the probability that (i) there are just 3 defective bulbs (ii) there are at least 3 defective bulbs.

15. A pair of dice is rolled once. Find the probability that the maximum of the two numbers (i) is greater than 4 (ii) is 6.

16. 4 girls and 4 boys sit in a row. Find the probability that (i) the four girls are together (ii) the boys and girls sit in alternate seats.

17. A committee of 3 is to be chosen from among 10 people including X and Y. Find the probability that
    (i) X is in the committee (ii) X or Y belongs to the committee
    (iii) X and Y belong to the committee.

18. A class consists of 25 boys and 15 girls. If a committee of 6 is to be chosen at random, find the probability that
    (i) all members of the committee are girls.
    (ii) all members of the committee are boys.
    (iii) there are exactly 3 boys in the committee.
    (iv) there are exactly 4 girls in the committee.
    (v) there is at least one girl in the committee.

19. There are 20 boys and 10 girls in a class. If a committee of 6 is to be chosen at random having

at least 2 boys and 2 girls, find the probability that

(i) there are 3 boys in the committee.

(ii) there are 4 boys in the committee.

20. There are 120 students in a class who have opted for the following MIL. English 20, Oriya 70, Bengali 30. If a student is chosen at random, find the probability that the student is studying.

(i) Bengali or English.

(ii) neither Bengali nor English.

21. Sometimes, probability of an event A is expressed as follows. We say that odds in favour of A are x to y if P(A) = $\frac{x}{x+y}$. Similarly, we say that odds against A are x to y if P(A) = $\frac{y}{x+y}$. Find P(A) and P(A^c) if.

(i) odds in favour of A are 2 to 5.

(ii) odds against A are 4 to 3.

22. Six dice are rolled. Find the probability that all six faces show different numbers.

23. There are 60 tickets in a bag numbered 1 through 60. If a ticket is picked at random, find the probability that the number on it is divisible by 2 or 5 and is not divisible by any of the numbers 3, 4, 6.

24. Compute P($A \Delta B$) in term of P(A), P(B) and P($A \cap B$) where $A \Delta B$ denotes the symmetric difference of A and B.

25. Three volumes of a book and five volumes of another book are placed at random on a book shelf. Find the probability that all volumes of both the books will be found together.

26. 2 black cards and 2 red cards are laying face down on a table. If you guess their colours find the probability that you get

(i) none of them right (ii) two of them right (iii) all four of them right.

Hints : Denote the cards by B B R R (B = black, R = red). The four cards can be kept face down in any manner. You can guess their colours in $\frac{4!}{2!\ 2!}$ = 6 ways.

Cards kept face down :		B	B	R	R
Your guess	(i)	B	B	R	R
	(ii)	B	R	B	R
	(iii)	B	R	R	B
	(iv)	R	R	B	B
	(v)	R	B	R	B
	(vi)	R	B	B	R

One can now answer the questions.

16.2 Non - uniform Spaces

As remarked earlier, there are sample space which are not equiprobable. For example if we have a biased coin (a coin with manufacturing defect), then the sample space S = $\{h, t\}$ need not be equiprobable. Suppose, for example, the coin is such that the head turns up twice as often as the tail; in other words the head turns up two-thirds of the times whereas the tail turns up one-third of the times. We thus assign probabilities 2/3 to head and 1/3 to the tail.

In general if a finite sample space is not equiprobable, then probabilities can be assigned to the elementary events and the probability of any event A is the sum of the probabilities of all points of A.

Example 16 :

Only three horses A, B and C are in arena. B is twice as likely to win as A and C is 3/2 times as likely to win as B. Find their respective chances (probabilities) of winning the race.

Let p = probability that A wins'

Then 2p = probability that B wins;

and 3p = probability that C wins.

Since sum of all these probabilities must be 1, it follows than $p + 2p + 3p = 1$ and $p = \frac{1}{6}$.

Example 17 :

A die is so weighted that even numbers have the same chance of appearing, odd numbers have the same chance of appearing and each odd number is twice as likely to appear as an even number. Find the probability that (i) and odd number appears (ii) an even number appears (iii) a prime number appears.

Let P (n) be the probability that the number n appears when the die is rolled. We then have $P(2) = P(4) = P(6) = p$ and $P(1) = P(3) = P(5) = 2p$

Since $P(1) + P(2) + P(3) + P(4) + P(5) + P(6) = 1$, it follows that $9p = 1$ and $p = \frac{1}{9}$.

(i) Probability of getting an odd number

$= P(\{1, 3, 5\})$

$= P(1) + P(3) + P(5)$

$= 2p + 2p + 2p = \frac{6}{9} = \frac{2}{3}$.

(ii) The probability of getting an even number $= 1 - \frac{6}{9} = \frac{3}{9} = \frac{1}{3}$.

(iii) The probability of getting a prime number

$= P(\{2, 3, 5\}) = P(2) + P(3) + P(5)$

$= p + 2p + 2p = 5p = \frac{5}{9}$.

EXERCISES 16 (b)

1. A school has six classes 1, 2, 3, 4, 5 and 6. Classes 2, 3, 4, 5 and 6 each have the same number of students, but there are twice this number in calss 1. If a student in selected at random from the school, what is the probability that he (she) will be in (i) class 1 (ii) calss 2.
2. Let a die be weighted in such a way that the probability of getting a number n is propertional to n.
   (i) Find the probability of each elementary event.
   (ii) Find the probability of getting an even number in a single roll of the die.
   (iii) Find the probability of getting an odd number in a single roll of the die.
   (iv) Find the probability of getting a prime number in a single roll of the die.
3. Five boys and three girls are playing a chess tournament. All boys have the same probability p of winning the tournament and all the girls have the same probability q of winning. If $p = 2q$, find the probability that
   (i) a boy wins the tournament
   (ii) a girl wins the tournament.

●

Real Number System

Cogito ergo sum (I think, therefore I am)

- Rene Descartes

1.0 Historical Introduction

Numbers perhaps were the earliest mathematical abstraction by man. How man first came to use numbers is still shrouded in mystery. But by all evidence it seems it was the earliest attempt by man to keep count of his livestock. Often he placed a stone against every animal that left his cave in the morning, when the cattle returned to his cave after a day's grazing he casts away a stone, that he collected in the morning against his cattles, for every animal that entered the cave. If some stones remained in the heap without being cast for an animal then he understood that some of his animals are still at large. Lines drawn on a bone of a young wolf dated 30, 000 years back discovered in Czechoslovakia in groups of five suggest man perphaps had started using his figures for counting. It is most likely that number **signs** were discovered before the number **words** as it is easier to draw a line on stone, cut a notch on a piece of wood; rather than creat a well modulated sound to identify numbers.

All these suggest that man discovered numbers to keep count of things. In a way it was the "cardinal" aspect of numbers. But anthropological studies suggest an alternative ordinal approach. It is suggested that order of precedence during ancient religious rituals led to invention of numbers. This is an intersting theory but yet to be established. Till now we do not know for certain how it all began.

Once it began, the mankind must have experienced enormous difficulty in writing and pronouncing the symbols. Egyptians, Bablyonians, Mayans, Romans, Chinese, Indians developed these method of symbols of numeration. The **concept of zero originated in India along with decimal system of enumeration and powerful place value system.** The bulk of Indian mathematics were generated and perfected largely by the following Indian mathematicians :

	Name of the Texts	
Aapastava Katyayana Batyayana	Shulba Sutras	800 BC – 500 BC
Pingala	Chhaanda Sutra	476 A.D.
Baraha Mihira	Pancha Sidhhanta Tika Bruha Samhita Bharata Samhita	505 A.D.
Bhaskar (1st)		522 A.D.
Brahma Gupta	Brahma Siddhanta	598 A.D.
Sridhar (1st)	Trishatika	750 A.D.
Mahavira	Ganita Saara Sangraha	850 A.D.
Aryabhatta (2nd)	Maha Saara Sangraha	950A.D.

Sridhar (2nd)		1025 A.D.
Sri Pati	Ganita Tilaka	1039 A.D.
	Siddhanta Shekhara	
Bhaskara (2nd)	Siddhanta Shiromani	1114 – 1185 A.D.
Madhaba		1340 – 1425 A.D.
Narayana		1350 A.D.
Nilakantha	Tanta Sangraha	1500 A.D.
Ramanujan		1887 – 1919 A.D.

Natural Numbers and Integers :

However we have been using numbers ourselves since we are children. One of them is counting. But if we ask ourselves what numbers are, perhaps many of us would have no immediate answers. Well, it is really hard to define numbers. It will take quite some mathematical maturity and preparation to define numbers without being circular. So what we propose is to try to analyse how we use numbers and agree to these rules in all our dealings. We would like the set of numbers we used in counting as the set of **natural numbers** and denote it by N*.

We agree that :

1. Every natural number n has a **successor** which we tentatively denote by n^+. In such a case we call n a **predecessor of** n^+.
2. There is a unique number in N* denoted by 0, which has no predecessor. Every other element of N* has a predecessor.
3. For $m, n \in$ N, $n^+ = m^+$ if and only if $m = n$.
4. For A $\subset$ N* if (i) $0 \in$ A and (ii) $n \in$ A imply $n^+ \in$ A then A = N* (This is called the principle of Mathematical Induction).

With these rules of the game called the **Peano Axioms** as described above we can define mathematical operation on N* and develop the arithmetic of natural numbers with which we are familiar.

Addition

(i) $m + 0 = m \quad \forall\ m \in$ N* (ii) $m + n^+ = (m + n)^+$

So by principle of mathematical induction stated above one sees that the sum $m + n$ is defined for every $m, n \in$ N*

Using the above definition it is not hard to show that

(i) $m + n = n + m \qquad \forall\ m, n \in$ N*

(ii) $(m + n) + p = m + (n + p) \qquad \forall\ m, n, p \in$ N*

(iii) $m + 0 = m \qquad \forall\ m \in$ N*

(iv) $m + p = n + p \Rightarrow m = n \qquad \forall\ m, n, p \in$ N*

Order in N*. There is an ordering in N* which is easily discernible :

(i) $0 < n$ if $n \in N^* - \{0\}$ (ii) $n < n^+ \ \forall \ n \in N^*$

(iii) $m < n \Rightarrow m^+ < n$ or $m^+ = n, \ \forall \ m, n \in N^*$

Subtraction :

Given $m, n \in N^*$ and $m \geq n$ there is a unique element $d \in N^*$ such that $n + d = m$
We write $d = m - n$. With this new notation we can now answer the **restricted inverse** problem : which number added to m gives n ? This has an answer if $m \leq n$.

Multiplication

For $m \in N^*$ define

$m \,.\, 0 = 0$

$m \,.\, n^+ = m \,.\, n + m$

Additive inverse problem can be solved if we extend the natural numbers to the set of integers $Z = \{0, \pm1, \pm2, ...\}$.

1.1 Arithmetic of Integers

The place value system is better understood with the help of :

Division Algorithm (Euclidean Algorithm) Given a natural number $p > 1$, every integer can be expressed uniquely in the form.

$s = q\,p + r,\ 0 \leq r < p,\ q \in Z.$

Using the above result which has been stated without proof, we can prove tha following theorem.

Theorem 1 :

Given a natural number $p > 1$, every natural number can be written with the base p.

Proof :

Suppose X is a natural number. Using division algorithm write

$$X = x_0 + q_1 p,\ 0 \leq x_0 < p,\ q_1 < X \qquad (1)$$

$$q_1 = x_1 + q_2\,p,\ 0 \leq x_1 < p,\ q_2 < q_1 \qquad (2)$$

$$q_2 = x_2 + q_3 p,\ 0 \leq x_2 < p,\ q_3 < q_2 \qquad (3)$$

$$q_{n-1} = x_{n-1} + q_n p,\ 0 \leq x_{n-1} < p,\ q_n < q_{n-1} \qquad (4)$$

Discontinue the process as soon as $q_n < p$ and write $q_n = x_n$. Substitute the value of q_1 of (2) in (1), then the value of q_2 of (3) in (1) and so on. These gradual substitutions give

$X = x_0 + q_1 p = x_0 + p\,(x_1 + q_2 p) = x_0 + x_1 p + q_2 p^2$

$= x_0 + x_1 p + p^2\,(x_2 + q_3 p) = x_0 + x_1 p + x_2 p^2 + q_3 p^3$

$= \ldots\ldots\ldots = x_0 + x_1 p + x_2 p^2 + \ldots.. + x_n p^n.$

We have proved that X can be written with the base p; Written symbolically

$X = < x_n, x_{n-1}, \ldots\ldots., x_1, x_0 > p.$

Moreover, in the division algorithm, $x_0 \ldots.. x_n$ are unique. Therefore, the expansion (5) of x in base p is unique.

Example 1 :

Express $<2721>_{10}$ in the base 11.

Solution : $2721 = 247 \times 11 + 4, \quad x_0 = 4$

$247 = 22 \times 11 + 5, \quad x_1 = 5$

$22 = 2.11 + 0, \quad x_2 = 0$

The algorithm stops since $2 < 11$, So $<2721>_{10} = <2054>_{11}$

Example 2 :

Determine the number of digits of $<21210>_3$ when expressed in base 10.

Solution : $<21210>_3 = 0 + 1.3 + 2.3^2 + 1.3^3 + 2.3^4 = <210>_{10}$

It is a three digit number.

Divisibility and Primality

An integer b is said to be **divisible** by an iteger $a \neq 0$ if there exists ann interger c such that $b = ac$. Then we write $a \mid b$. Otherwise, we write $a \nmid b$.

If $a \mid b$, a is called a **factor** or **divisor** of b and b is called **multiple** of a.

If $a \mid b$ and $1 < |a| < |b|$, then a is called a **proper divisor** of b. clearly, $a \mid 0$ and $\pm a \mid a$ for each integer $a \neq 0$ and $\pm 1 \mid a$ for each integer a.

An integer a is called **even** if $2 \mid a$ and **odd** otherwise. Thus $0, \pm 2, \pm 4, \ldots$ are even while $\pm 1, \pm 3, \pm 5, \ldots$ are odd. Any even integer is of the form $2m$ for some $m \in Z$ and any odd integer is of the form $2m + 1$ for some $m \in Z$.

The following easily verifiable facts are left as exercises.

Proposition 1

Show that

(a) the sum of two even integers is even,

(b) the product of two even integers is even,

(c) the sum of two odd integers is even,

(d) the sum of an even and odd integer is odd,

(e) the product of two odd integers is odd.

(f) the product of an odd and an even integer is even,

(g) If a is odd, so is a^n for every positive integer n (particular case of (e)).

(h) If a is even, so is a^n for every positive integer n (particular case of (b)).

The following facts are also immediate consequences of definition and are left as exercises.

Proposition 2

Prove that

(i) $a \mid b \Rightarrow a \mid bc$ for each $c \in$ Z,

(ii) $a \mid b$ and $b \mid c \Rightarrow a \mid c$,

(iii) $a \mid b$ and $a \mid c \Rightarrow a \mid (bm + cn)$ for all $m, n \in$ Z,

(iv) $a \mid b$ and $b \mid a \Rightarrow a = \pm b$,

(v) $a \mid b$, $a > 0$, $b > 0 \Rightarrow a \le b$,

(vi) if m is a nonzero integer, than $a \mid b$ iff $am \mid mb$.

If m, n are positive integers, then in order to know if $m \mid n$, one way is to carry out actual division of n by m which can be quite formidable when the integers are large. There are some simple rules which settle the problem when m takes some specific values. It may not be out of place to list some such rules here.

We write $n = a_k\, a_{k-1} \,.....\, a_1\, a_0$

if $n = a_0 + 10a_1 + ... + 10^k a_k$.

(I) If r is an integer ≥ 1, then $n = a_k a_{k-1} \,....\, a_0$ is divisible by 2^r (supposing that $n > r$) if $2^r \mid a_r a_{r-1} \,......\, a_0$.

This test is useful for small values for r.

(II) If r is an integer ≥ 1, then $5^r \mid n = a_k a_{k-1} \,....\, a_0$ if $5^r \mid a_r a_{r-1} \,......\, a_0$.

The rules that follow use the idea of congruence modulo m which is dealt with in the chapter on Relations and Functions. We recall that n is congruent to r modulo m (or $n \equiv r \pmod{m}$) iff $m \mid (n - r)$.

(III) Divisibility by 3 and 9

Since $10^r \equiv 1 \pmod 9$ and hence $10^r \equiv 1 \pmod 3$ where r is any integer ≥ 1,

we have $n = a_k a_{k-1} ... a_0 \equiv a_k + a_{k-1} + ... + a_0 \pmod 9$ and $n \equiv a_k + \; ... + a_0 \pmod 3$.

Hence $3 \mid n$ iff $3 \mid a_0 + a_1 + ... + a_k$ and $a|n$ iff $9 \mid a_0 + a_1 + ... + a_k$.

(IV) Divisibility by 11

Since $10^r \equiv (-1)^r$ (mode 11), r being any integer ≥ 0

$n = a_k a_{k-1} ... a_0 \equiv a_0 - a_1 + a_2 \; ..... + (-1)^k a_k \pmod{11}$.

$11 \mid n$ iff $11 \mid a_0 - a_1 + ... = (a_0 + a_2 + ...) - (a_1 + a_3 + ...)$.

In other words $11|n$ iff $11 \mid (a - b)$, when a is the sum of the even place digits of n and b is that of the odd place digits of n.

For instance $11 \mid 2\,7\,3\,4\,5\,0\,2\,1$.

Remark

Whenever there is a confusion of mistaking divide sign | with integer 1 we use bold face to denote the divide sign.

The above rules settle divisibility of n by m = 2, 3, 4, 5, 8, 9, 11, to choose only some small values of m. Divisibility by 6 can be settled by considering divisibility by 2 and 3. Divisibility by 10, 12 can be similarly settled.These rules do not not cover the case of m = 7. There is, however, a curious test given below that settles divisibility by 7, 11 and 13 simultaneously.

(V) Divisibility by 7, 11, 13 simultaneously

We have 7 . 11 . 13 = 1001 and $10^3 \equiv -1 \pmod{1001}$, $10^6 \equiv 1 \pmod{1001}$ etc.

Let $n = 2\ 5\ 3\ 5\ 1\ 2\ 6\ 8\ 1\ 2\ 4$. Then

$n = 124 + 268.10^3 + 351.10^6 + 25.10^9$

$\equiv 124 - 268 + 351 - 25 \pmod{1001}$, that is, $n \equiv 475 - 293 \pmod{1001}$

or, $n \equiv 182 \pmod{7.11.13}$ or, $n \equiv 2.\ 7.\ 13 \pmod{7.\ 11.\ 13}$

Hence $7 \mid n$, $13 \mid n$, but $11 \nmid n$.

The general rule is obvious from this.

The above method can be used to obtain rules for determining divisibility by numbers such as 101, 10001 (= 73 × 137), 100001 ..... and their factors.

(VI) A general division criterion

If 10 and m have no common factors and $m \mid (10k_0 - 1)$ for some $k_0 \in Z$, then $m \mid n = a_k a_{k-1} \ldots\ldots a_0$ iff $m \mid a_k a_{k-1} \ldots\ldots a_1 + k_0 a_0 = n'$.

To prove this, we have $10n' - n = 10 \times a_k a_{k-1} \ldots\ldots a_1 + 10 \times k_0 a_0$

$$- (10 \times a_k a_{k-1} \ldots\ldots a_1 + a_0) = (10k_0 - 1)\, a_0$$

and since $m \mid (10k_0 - 1)$ by hypothesis, $m \mid n$ iff $m \mid 10n'$ or equivalently iff $m \mid n'$ (by Theorem 5 that is stated later).

This test covers divisibility by a large number of integers m. for instance, it covers the cases of m = 7, 11, 13, 17, 19, 23, 29 and so on. Note that m is necessarily odd. The test can be repeated sufficient number of times to settle the divisibility.

Consider divisibility by 29.

Here we take $k_0 = 3$ since $10 \times 3 - 1$ is divisible by 29. Let $n = 7\ 2\ 5\ 3\ 1\ 4\ 6\ 7\ 2$. The procedure is as follows.

$$
\begin{array}{rll}
29 \mid 72531467 \mid & 2 & \\
+6 & & \\
\hline
7253147 \mid & 3 & \because k_0 a_0 = 3.2 = 6 \\
+9 & & \\
\hline
725315 \mid & 6 & \\
+18 & & \\
\hline
72533 \mid & 3 & \\
+9 & & \\
\hline
7254 \mid & 2 & \\
+6 & & \\
\hline
726 \mid & 0 & \\
+0 & & \\
\hline
72 \mid & 6 & \\
+18 & & \\
\hline
90 & &
\end{array}
$$

But 29 $\nmid$ 90. Hence 29 $\nmid$ *n*.

For $m = 31$, we take $k_0 = -3$ since $10k_0 - 1 = -31$ which is divisible by 31.

Let $n = 5\ 3\ 2\ 6\ 9\ 1$.

$$
\begin{array}{l}
31 \mid 53269 \mid 1 \\
\quad\ \ \underline{\quad -3\quad} \\
\quad \mid 5326 \mid 6 \\
\quad\ \ \underline{-18\quad\ } \\
\quad \mid 530 \mid 8 \\
\quad\ \ \underline{-24\quad\ } \\
\quad\ \ \mid 50 \mid 6 \\
\quad\ \ \underline{-18\quad\ } \\
\qquad 32
\end{array}
\qquad
\begin{array}{l}
\text{Now } 31 \nmid 32 \\
\\
\text{Hence } 31 \nmid n.
\end{array}
$$

(Collected from Ganita Bichitra 1995)

Now 31 $\nmid$ 32 $\Rightarrow$ 31 $\nmid$ n.

(V) As a last rule for testing divisibility, we recall that any product of *r* consecutive natural numbers in divisible by r !. This has been proved in the chapter on permutations and combinations.

The reader is urged to discover more of such rules.

Primes

An intger $p > 1$ is called *a* **prime** (or *a* **Prime number**) if it has no proper divisor, that is, its only divisors are 1 and *p*.

Some of the initial primes are 2, 3, 5, 7, 11, 13, 17, 19, 23, ... How long is this list ? Indeed it was shown by the celebrated Greek mathematician Euclid (300 B. C) that the list is **infinite.** A proof was divised by him for this, to which we shall come later.

The condition $p > 1$ is a mathematical necessity that will be explained in due course. We now come to a useful definition.

It is plain that a nonzero integer has only a finite number of divisors and so, if m, n are integers, not both zero, then they can have only a finite number of common divisors, the largest of which is called the **greatest common divisor** (g, c, d) of *m* and *n* denoted by (m, n). Note that (*m, n*) always exists and is positive when *m, n* are not both zero.

If (*m, n*) = 1, we say that *m, n* are **relatively prime**. For instance, – 3 and 5 are relatively prime.

On the other hand, if *m, n* are both nonzero integers, then an integer *r* is called a **common multiple** of *m, n* if $m \mid r$, $n \mid r$. The least positive common multiple of two nonzero integer (show that this necessarily exists) *m, n* is called their **least common multiple (*l.c.m*),** denoted by [*m, n*].

The definitions of g. c. d and *l.c.m* can be extended to more than two integers in an obvious manner. There are a number of interesting results about the g.c.d. and *l.c.m*, but we shall not treat them here. Rather we shall continue the discussion on primes which form a very interesting subset of natural numbers.

Note that 2 is the **least** prime and is the **only even prime,** all other primes being odd.

Theorem 2 : Every integer > 1 (and hence every integer < – 1) has a prime factor.

Proof :

The result is proved by induction. It is true for $n = 2$. Assume that it holds for all integers between 2 and n (both 2 and n included). It will be shown that it holds for $n + 1$. If $n + 1$ is prime this is obvious. Otherwise, n + 1 is composite and so one can write $n + 1 = pq$ where $1 < p,\ q < n$. By the induction hypothesis, p has *a* prime factor which is, a fortiori, *a* factor of $n + 1$, as required.□

We use Theorem 2 to prove below the infinitude of primes.

Theorem 3 (Euclid). The number of primes is infinite.

Proof :

Suppose that the contrary is true, that is, there are only k primes, say, $2 = p_1,\ 3 = p_2, \ldots\ldots, p_{k-1}, p_k$. Now $n = p_1 p_2 \ldots\ldots p_k + 1 \in Z$ and hence by Theorem 2, $p_i \mid n$ for some $i \le k$. However, the division of n by p_i leaves a ramainder 1 which is a contradiction. The conclusion now follows.

The next theorem, popularly known as the **'fundamental theorem of arithmetic',** is of prime importance in Number theory. The proof can be found in standard texts and is omitted.

Theorem 4 Every integer> 1 can be expressed as a product of primes, uniquely apart from the order of its prime factors.

A representation $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k}$, where $p_1, p_2, \ldots\ldots, p_k$ are distinct primes in ascending order and $\alpha_i > 0\ \forall i$, is called **canonical (standard).** In view of the uniqueness requirement, 1 is excluded from the list of primes.

One consequence of Theorem 4 is that if m, n are integers > 1, then $(m, n).\ [m, n] = mn$.

The next theorem, whose proof is also omitted, is one of the oft-used results concerning divisibility.

Theorem 5 If $a \mid bc$ and $(a, b) = 1$, then $a \mid c$.

The above result appears to be an easy consequence of Theorem 4, but the fact is that the proof of Theorem 4 depends upon Theorem 5, so that these two results are equivalent.

The literature concerning divisibility and primality is alarmingly vast and anyone who pursues through it, stands face to face with the great mystery that numbers possess. A natural problem associated with any positive integer is whether it is a prime and if composite, what are its factors. Mathematicians like Fermat, Mersenne, Lucas, Sophie German, Wilson and many others discovered important classes of natural numbers that bear their names and studied them with respect to the problem just posed. This happened during the precomputer age. They discovered curious facts, committed errors that were rectified in due course and left more problems unsolved than they solved. Number theory is indeed replete with unsolved problems in spite of the abundance of jewel's in the form of rich results that it has been able to collect. With the advent of super computers, a new era has started. It has been possible to indentify larger and larger primes and factorize even larger composites. The largest prime known until 1991, discovered by six mathematicians in 1989, after several months of continuous computer work is $391581\times 2^{216193}-1$ and this number contains more than 65,000 (and less than 75,000) digits ! The second largest prime, known around that period, is $2^{216091}-1$ which is one of the Mersenne class number. Several other large primes are :

$235235\times 2^{70,000}-1$, $8423\times 2^{59877}+1$,

$150093\times 10^{8000}+1$, $289\times 2^{18502}+1$, $(6952)^4\times 2^{9952}+1$,

(collected from *P*. Ribenboim, The Little book of big primes)

The last quoted prime has 8006 digits and it was discovered by Dubner on March 12 in the year 1986 after about 60×24 hours of computing spread over a period of 7 months !

There are various tests for deciding the primality of an integer $n>1$ and it is not possible to discus all those here. The most primitive way of determining if $n>1$ is a prime is to perform actual division by all integers $k>1$ such that $k^2<n$. This is quite convenient when n is not quite large, using the divisibility tests already described.

Another procedure, known as the **sieve** of **Eratosthenes,** divised by the ancient Greek mathematician Eratosthenes (3rd century B.C.) consists in writing all integers from 2 to n in a table, striking off every second number after 2, every third number after 3 every fifth number after five and so on until the same is carried out for the largest prime p such that $p^2<n$. For example, let $n=50$. The table, after striking off, looks like this :

2	3	~~4~~	5	~~6~~	7	~~8~~	~~9~~	~~10~~	11	~~12~~	13
~~14~~	~~15~~	~~16~~	17	~~18~~	19	~~20~~	~~21~~	~~22~~	23	~~24~~	~~25~~
~~26~~	~~27~~	~~28~~	29	~~30~~	31	~~32~~	~~33~~	~~34~~	~~35~~	~~36~~	37
~~38~~	~~39~~	~~40~~	41	~~42~~	43	~~44~~	~~45~~	~~46~~	47	~~48~~	~~49~~
~~50~~											

It is needed to strike off all numbers multiples of 2, 3, 5, 7 (execpt these numbers) and the numbers that are left are

2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47

which are all the primes upto 50. During the precomputer era, this was the chief device to determine primality and surpisingly large primes could be discovered in this way. Even with a high speed computer, this is not altogether a bad test for integers of moderate size, as it involves a simple algorithm. On the contrary, the criterion of Wilson which states that n is a prime if $(n-1)\,!\not\equiv -1 \pmod{n}$, though theoretically quite gratifying, is not of much practical value since there is no known algorithm to compute $(n-1)\,!$ for large n in comparatively small number of steps.

Number of divisors :

To find the **number of divisors** of a composite number n, we write

$n = a^p b^q c^r .....$

where a, b, c are different prime numbers and p, q, r .... are positive integers. It is clear that each term of the product

$$(1 + a + a^2 ....+ a^p)\ (1 + b + b^2 ....+ b^q)\ (1 + c + c^2 ....+ c^r) \qquad ....(1)$$

is a divisor of the given number. Hence the number of divisors is the number of terms in the product, that is

$(p + 1)\ (q + 1)(r + 1)$ ....

To find the **sum of the divisors** of n, we note that since each term in (1) is a divisor, the sum of the divisors is equal to

$$\frac{a^{p+1}-1}{a-1}.\frac{b^{q+1}-1}{b-1}.\frac{c^{r+1}-1}{c-1}.......$$

Example 3:

Find the number and sum of divisors of 251680.

Solution : Since $251680 = 11^2 \times 13 \times 5^2 \times 3^3 \times 2^5$, it follows that the number of divisors

$= (2 + 1)(1 + 1)(2 + 1)(3 + 1)\ (5 + 1) = 432$

and the sum of divisors

$$= \frac{11^{2+1}-1}{11-1}.\frac{13^{1+1}-1}{13-1}.\frac{5^{2+1}-1}{5-1}.\frac{3^{3+1}-1}{3-1}.\frac{2^{5+1}-1}{2-1}.$$

Before we take up Fermat's theorem, by way of preparation we now take up :

Example 4 :

${}^pC_r$ is divisible by p for $1 \le r \le p$-1, if p is prime.

Proof :

Since ${}^pC_r = \dfrac{p!}{r!(p-r)!}$ and none of the factors of r!or $(p - r)$! can divide p for $r < p$, it is clear that the factor p in p! will persist after all the factors of r! and $(p$-$r)$! have been cancelled out.

Example 5 :

$n^p - n$ is divisible by p when p is a prime number and for every natural number n.

Proof : (By method of induction)

Observe that it is trivially true for $n = 1$. If we assume it to be true for n, then we see that $(n+1)^p - (n+1) = n^p + {}^pC_1 n^{p-1} + ...+ 1 - n - 1$

$= n^p - n + {}^pC_1 n^{p-1} - ....- {}^pC_{p-1}\, n$

But each of ${}^pC_r$ is divisible by p for $1 \le r \le p-1$. So $(n+1)^p - (n+1)$ is divisible by p. The result follows by induction.□□

Theorem 6 : (Fermat's Little Theorem)

For any prime number p and any number n, relatively prime to p, $n^{p-1} - 1$ is divisible by p.

Proof : Since $n^p - n = n\,(n^{\,p-1} - 1)$ and n is not divisible by p, it follows from Example 5 that $n^{p-1} - 1$ divisible by p.

Example 6 :

Show that $n^3 - n$ is divisible by 6.

Proof : Since $n^3 - n = n(n-1)(n+1)$, at least one of $n - 1, n + 1$ is an even number. So it is divisible by 2, but by Example 5, it is divisible by 3. Hence it is divisible by 6.

EXERCISES - 1 (a)

1. Which of the following statements are True or False.
   (a) In the following algorithm a quotient can not be equal to zero.
   (b) In the division algorithm $s = q.p+r$, the remainder r is always strictly less than p.
   (c) Eighty one in decimal scale when written in ternary scale has 1 in the unit place.
   (d) Difference of two natural numbers is always a natural number.
   (e) $a \in Z, b \in Z, b \neq 0 \Rightarrow a/b \in Z$
   (f) $a \in Z, b \in Z \Rightarrow a\,.\,b \in Z$
   (g) 7 is a composite number.
   (h) 111 is a composite number.
   (i) The canonical factrorization of 210 is equal to 5. 2. 7. 3
   (j) There are infinitely many integers.
   (k) There are only finitely many primes.
2. (i) Show that every integer can be written as either $3m$ or $3m + 1$ or $3m + 2$ for some integer m.
   (ii) Show that every integer can be written as either $4m$ or $4m + 1$ or $4m + 2$ or $4m + 3$ for some integer m.
3. Show that every integer can be written as either $5m$ or $5m + 1$ or $5m + 2$ or $5m + 3$ or $5m + 4$ for some integer m.
4. Show that if $d_1 \mid a$ and $d_2 \mid d_1$, $d_1 > 0$, $d_2 > 0$ then $d_2 \mid a$.
5. Show that if $d \mid a$, $d \neq 0$ and $k \neq 0$ then $kd \mid ka$.
6. Show that if $d \mid x$ and $d \mid y$ then for any a and b, $d \mid (ax + by)$.
7. Show that if $a \neq 0$, $b \neq a$,a $\mid b$ and $b \mid a$ then $a = \pm\, b$.
8. Write down all the primes less than 100 and count how many are there.
9. Find the only prime among 1, 11, 111 and 1111.
10. Write down the canonical factorisation of 832 and 420.

11. Decide if $2^{24} + 1$ is a prime or a composite number.

12. Determine the number of digits of 2^{100}.

13. Show that $1^{99} + 2^{99} + 3^{99} + 4^{99} + 5^{99}$ is divisible by 15.

14. Show that $n^5 - n$ is divisible by 30 and $n^7 - n$ is divisible by 42.

15. Test the divisibility of following numbers by 7, 11, 13, 17
(i) 5383912 (ii) 993407 (iii) 123456789

16. Find the number and sum of divisors of :
(i) 8064 (ii) 9725 (iii) 53236

17. Show that $2^{4n} - 1$ is divisible by 15.

18. Show that
(i) $n(n+1)(n+5)$ is divisible by 6
(ii) $n(n^2 - 1)(3n+2)$ is divisible by 24
(iii) $n^5 - 5n^3 + 4n$ is divisible by 120 if $n > 2$.

19. Prove that
(i) $3^{2n} + 7$ is divisible by 8

1.2 Rational Numbers

Integers are called whole numbers because these are used as measures of " whole" quantities e.g. one man, two cows, area of unit square, credit of hundred rupees etc. However, for measuring "parts of the whole" the **rational** or **fractional** numbers become necessary. Numbers of the form $\frac{p}{q}$, where $p \in Z$, $q \in Z$ and $q \neq 0$ are called rational numbers. $\frac{2}{3}, \frac{10}{15}, \frac{7}{120}, \frac{25}{5}, \frac{-39}{11}$ are examples of rational numbers. Every integer can be written as $\frac{p}{1}$. We denote by Q the set of all rational numbers.

We know from elementary arithmetic that sum, difference, product, and quotient of two rational numbers are rational numbers provided that in the case of division, the divisor has to be non-zero.

It would be borne in mind that the set Q of rational numbers enjoys a property which is called **density property** not shared by N and Z.

It states that **between any two rational numbers there are infinitely many rational numbers.**

For example if $a, b \in$ Q then $\frac{a+b}{2}$ is a rational number and $a < \frac{a+b}{2} <$ b.
In fact $a + \frac{b-a}{2^n} \in$ Q for $n = 1, 2, 3, ...$ and $a < a + \frac{b-a}{2^n} < b$.

It is well known that the rational numbers can be expressed as decimals. For example $= \frac{1}{5} = .2, \frac{5}{6} = .8\overline{3}, \frac{1}{3} = .\overline{3}$ where the last two decimals are recurring. In fact **every recurring decimal is a rational number.** For example, suppose that $x = .\overline{7}$. Then $10x = 7.\overline{7}$. Subtracting, we get $10x - x = 9x = 7$ so that so that $x = \frac{7}{9} = .\overline{7}$.

1.3 Inadequacy of rationals

We know from the theorem of Pythagoras that the area of the square on the hypotenuse of a right angled triangle is equal to the sum of the areas of squares on the other two sides of the triangle. So for a square of sides of unit length the square of the length of its diagonal is equal to 2 square units. So the question is what is the length of the diagonal of the square in terms of the sides of the square ? Pythagoras thought that if the sides are divided into large enough equal parts then the diagonal of the square could be expressed as an integral multiple of those parts. In other words we can divide the sides into q equal parts (for some integer q) such that the length of the diagonal is p times the length of these qth parts for some integer q. In modern parlance it means that the length of the diagonal of a square is a rational multiple of its sides. But it was demonstrated by Greeks themselves that there cannot be two non zero integral numbers p and q satisfying the equation $p^2 = 2q^2$. It has not been determined when really a fool proof demonstration of nonexistence of solution of the above equation was given. But Euclid's Elements, which are extant even today, does have a proof of nonexistence of solution which means that the equation $x^2 = 2$ does not have a rational solution.

As the readers are already acquainted with the proof of this, we leave the proof.

In fact, we have a more general result : **given a prime number p, there cannot be non zero integers m and n such that $pn^2 = m^2$.**

This puts into jeopardy the scheme of arithmetization of geometry. The idea is that every geometrical entity could be expressed in terms of numbers. But if our scheme of numbers contain only rationals then this grand hope comes to nought even in the simple case of diagonal of a square not to mention of other lengths (like circumference of a circle in terms of its diameter).

1.4 Real Numbers :

Since it was assumed that every length corresponds to a "number", those lengths which did not Correspond to a rational number were to correspond to an "irrational number". Mathematicians accepted these for a long time. Not until late ninteenth century a formal theory of real numbers was developed. We won't develop them here. But rather give the rules of the game in dealing with real numbers. What we need now is to expand the realm of numbers which has all the properties of Q including ordering but which accommodates such numbers which would correspond to length of diagonals of square etc. We list the rules to be followed : First of all we shall call this new realm of numbers **real numbers** and shall denote it by the symbol R.

Axioms of Addition :

RA -1. R is closed under addition. viz for every pair of numbers a and b in R there is a unique element $a + b \in$ R.

$a + b$ is called the sum of a and b. This axiom is commonly known as the **closure axiom.**

RA -2 Addition in R is **commutative** i.e. a, b $\in$ R $\Rightarrow a + b = b + a$

RA- 3 Addition is **associative** in R i.e. for every $a, b, c \in$ R, $a + (b + c) = (a + b) + c$.

RA - 4 There is an element in R called the **additive identity** and denoted by 0, such that for $a \in$ R, $a + 0 = a$.

RA -5 For every $a \in$ R, there is a unique element in R, called the **additive inverse** of a and written as $-a$, such that $a + (-a) = 0$.

The subtraction or difference of a and b is by definition $a + (-b)$ and this is written as $a - b$. We shall write $a + a = 2a$ and in general $a + a + ... + a$ (added n terms) as na. We have immediately

(i) **Cancellation Law :** $a, b, c \in$ R, $a + b = a + c \Rightarrow b = c$

(ii) **Uniqueness of additive identity :** $a, b \in$ R, $a + b = a \Rightarrow b = 0$

(iii) **Uniqueness of additive inverse :** $a + b = 0 \Rightarrow a = -b$

(iv) The additive inverse of additive inverse of a real number a is a itself. $a \in$ R $\Rightarrow -(-a) = a$.

These can be proved easily as we proved in Z.

Proof of (i) If $a + b = a + c$ adding (–a) to both sides we gets $(-a) + (a + b) = (-a) + (a + c)$ which by associativity yields

$((-a) + a) + b = ((-a) + a) + c$

$\Rightarrow 0 + b = 0 + c \Rightarrow b = c$.

It is clear also from commutativity of addition that right cancellation law also holds : $a, b, c \in$ R, $a + b = c + b \Rightarrow a = c$.

This facilitates the proof of uniqueness of additive identity.

Proof of (ii) $a + b = a \Rightarrow a + b = a + 0 \Rightarrow b = 0$.

Proving (iii) and (iv) are exactly like similar assertion we made about Z.

Now we describe multiplication.

Axioms of Multiplication :

RM - 1. R is closed under multiplication i.e. for a pair of real numbers $a, b \in$ R, there is a unique element, $a.b \in$ R, called the product of a and b. This is called the **closure** axiom of multiplication.

RM -2 Multiplication is commutative i.e. $a, b \in$ R $\Rightarrow a.\ b = b.\ a$

RM - 3 Multiplication is associative : $a, b, c \in$ R $a.\ (b.c) = (a.\ b)\ .c$

RM - 4 There is an element in R, called the **multiplicative identity** and denoted by 1, such that for every $a \in$ R, $a.\ 1 = a$.

RM - 5 For every nonzero element $a \in$ R, there is an element in R, called the multiplicative inverse of a and written a^{-1}, such that $a.a^{-1} = 1$.

The following axiom relates addition with multiplication :

RAM - 6 Multiplication is distributive over addition that is $a, b, c \in$ R $\Rightarrow a.\ (b + c) = a.\ b + a.\ c$. The axioms RA - 1 to RA - 5, RM - 1 to RM - 5 and RAM - 6 together are called field axioms. In fact any set with this structure is called a field. So R is a field. In fact as a field it contains the field of rational numbers.

We have already seen that for $a \in R$, $b \in R - \{0\}$ we can define ab^{-1} which is called the quotient of a and b. It is customary to denote this by $\frac{a}{b}$ or $a \div b$.

We write a^2 for $a.\ a$ In general $a......a$ (n times) is written a^n. We can prove the following as a consequence of the field axioms.

(i) **Cancellation law** : a, b, c R, $a \neq 0$, $ab = ac \Rightarrow b = c$.

(ii) **The multiplicative identity** is unique.

(iii) The **multiplicative inverse** of the multiplicative inverse of a $\neq 0$ is itself i.e. for $a \in R - \{0\}$, $(a^{-1})^{-1} = a$.

The proof of these are very much similar to those we proved earlier for addition. The reader is encouraged to work out the proofs.

In the axioms of multiplication we did not mention about multiplication by zero. We introduced zero as the additive identity. All the rules of arithmetic regarding multiplication by 0 follow easily from the distributive law :

We record $a\ .\ 0 = 0$

Indeed $0.a = (0 + 0)\ .a = 0\ .a + 0\ .a$

$\Rightarrow 0 + 0\ .a = 0\ .a + 0\ .a \Rightarrow 0 = 0\ .a$.

Recall that while defining multiplicative inverse we excluded 0. The reason is :

Zero has no multiplicative inverse :

If 0 had a multiplicative inverse a (say) then we would get $a.0 = 1$ which contradicts our contention (proved just now) that $a.0 = 0$.

It is also easy to prove :

If $a \neq 0$ then $a^{-1} \neq$, 0, for if $a^{-1} = 0$ then $a.\ a^{-1} = a.\ 0 = 0$ but $aa^{-1} = 1$.

We have also

If $a, b \in R$ and $a.\ b = 0$ then either $a = 0$ or $b = 0$

Its demonstration is straight forward.

$a \neq 0$ and $ab = 0$

$\Rightarrow (a^{-1}a)\ b = a^{-1}\ (ab) = 0$

But $a^{-1}\ a = 1$ so $b = 0$.

We can now prove the formula : $\boldsymbol{(-a).\ (-b) = a.\ b}$

which often baffled us during our High School days.

Since $(-a)\ .\ b + a\ .\ b = (-a + a)\ .b = 0.\ b = 0$, we get $(-a)\ .b = -(a.\ b)$ (additive inverse)

putting $-b$ for b, $(-a)\ (-b) = -(a\ (-b)) = -(-ab) = a.b$.

Binomial Theorem

We have proved Binomial Theorem in Chapter 3 by using combinatorial techniques. We shall now present an **alternative proof** which is based on algebraic properties of R and method of induction.

Theorem 7 :

$$(a + b)^n = a^n + {}^nC_1\, a^{n-1} b + \ldots\ldots + {}^nC_r\, a^{n-r} b^r + \ldots\ldots + {}^nC_n\, b^n$$

Proof :

Suppose the above statement is p_n. Then clearly it is true for $n = 1$. Suppose it is true for $n = k$. Then by using the algebraic properties of R, we obtain

$(a + b)^{k+1} = (a + b)^k (a + b) = (a^k + {}^kC_1\, a^{k-1} b + \ldots\ldots + b^k)(a + b)$

$= a^{k+1} + ({}^kC_1 + 1)\, a^k b + \ldots\ldots + ({}^kC_r + {}^kC_{r-1})\, a^{k-r+1} b^r + \ldots\ldots + b^{k+1}$

(by collecting the coefficients of like powers of a and b)

$= a^{k+1} + {}^{k+1}C_r\, a^k b + \ldots\ldots + {}^{k+1}C_r\, a^{k+1-r} b^r + \ldots\ldots + b^{k+1}$,

by using the fact that ${}^kC_r + {}^kC_{r-1} = {}^{k+1}C_r$ $(1 \leq, r \leq k)$

(proved in Chapter 3). Hence p_{k+1} is true. This proves that p_n is true for all n.□

Ordering in R :

Besides the axioms of field, R has an additional property which we introduce as follows :

P 1 Some numbers x in R are called positive, (and written as $x > 0$ or $0 < x$) we denote the set of such elements by P.

P 2 $x, y \in P \Rightarrow x + y \in P$ and $x.\, y \in P$

P 3 $x \in R \Rightarrow x \in P$ or $x = 0$ or $-x \in P$.

If we wirte $-P = \{-x : x \in P\}$ then P3 can be written as $R = P \cup \{0\} \cup (-P)$.

It is also clear that $P \cap (-P) = \phi$. Since if $x \in P \cap (-P)$ then $x + (-x) \in P$ and this is a contradiction.

If $x \in (-P)$ we call x a **negative** number and write this as $x < 0$ or $0 > x$.

Now we define ordering in R as follows.

For $a, b \in R$ by P3 either $a - b \in P$ or $a - b = 0$ or $b - a \in P$. In case $a - b \in P$. we write $a > b$ equivalently $b < a$. If $a - b = 0$ then $a = b$.

If $b - a \in P$ we write $b > a$ or equivalently $a < b$.

By P1, P2, P3 it clear that for $a, b \in R$

either $a < b$ or $a = b$ or $b < a$.

We have

Proposition : If $a \in R$ and $a \neq 0$ then $a^2 \in P$, that is , $a^2 > 0$. In particular $1 > 0$.

Proof : If $a \in R - \{0\}$ then either $a \in P$ or $(-a) \in P$. (By P2)

$a^2 \in P$ or $(-a)(-a) \in P$

But $(-a)(-a) = a^2$. This proves that $a^2 > 0$. Since $1 \neq 0$, it now follows that $1 > 0$.

Proposition : For $a, b, \in R$, (i) $ab > 0$ if both a and b are positive or both $(-a)$, $(-b)$ are positive.

(ii) if $ab < 0$, either $a < 0$ and $b > 0$ or $a > 0$ and $b < 0$.

(iii) $a < b,\ c > 0 \Rightarrow ac < bc$

(iv) $a < b,\ c < 0 \Rightarrow ac > bc$

Indeed $a < b \Rightarrow b - a > 0$, So if $c > 0$ then by P2

$(b - a).\ c > 0 \Rightarrow bc > ac$

If $a < b$ and $c < 0$ then $(b - a)(-c) > 0$

$\Rightarrow -bc + ac > 0$

$\Rightarrow ac > bc$

Rest of the proof are left as exercise.

Absolute value : If a is a real number, then the absolute value (modulus) of a, denoted by $|a|$, is defined as

$$|a| = \begin{cases} a \text{ if } a \geq 0 \\ -a \text{ if } a < 0. \end{cases}$$

For example : $|0| = 0$, $|2| = 2$, $|-3| = -(-3) = 3$, $|\pi - 3| = \pi - 3$ and $|3 - \pi| = \pi - 3$. One easily verifies the following properties regarding the absolute value.

Properties :

1. The absolute value of a real number a is always non negative. Moreover, the absolute value $|a| = 0$ if and only if $a = 0$.
2. $|a| = |-a| \ \forall \ a \in R$
3. $-|a| \leq a \leq |a|$ equality holding on one side at a time.
4. For $a > 0$, $|x| < a$ if and only if $-a < x < a$.

Proof : Let $|x| < a$. But we have by Property 3, $-x \leq |x|$ and $x \leq |x|$

So $-x < a$ and $x < a$. Combining these two we get $-a < x < a$.

Conversely suppose $-a < x < a$. In case $x \geq 0$ we have $|x| = x \Rightarrow |x| < a$ In case $x < 0$ we have $|x| = -x \Rightarrow -x < a \Rightarrow -a < x$.

5. **Triangle inequality ;**
   For $a, b \in R$ $|a + b| \leq |a| + |b|$.

Proof : We know as such that

$-|a| \leq a \leq |a|$ and $-|b| \leq b \leq |b|$

$\Rightarrow -(|a| + |b|) \leq a + b \leq |a| + |b|$

$\Rightarrow |a + b| \leq |a| + |b|$

6. $||a| - |b|| \leq |a - b| \leq |a| + |b|$
7. For $a, b \in R$ $|a.\ b| = |a|\ .\ |b|$

Proof : There can be four cases in all (i) $a \geq 0, b \geq 0$ (ii) $a, \leq 0, b \leq 0$, (iii) $a \leq 0, b \geq 0$, (iv) $a \geq 0, b \leq 0$. Verification in all these cases proves our result.

Some Inequalities :

Using the properties of order stated above we can obtain many important inequalities for real numbers.

(i) $a, b \in R \quad \Rightarrow a^2 + b^2 \geq 2ab$

Since $(a - b)^2 \geq 0$ the inequality follows where equality holds if and only if $a = b$.

(ii) $\frac{a+b}{2} \geq (ab)^{\frac{1}{2}} \geq \frac{2ab}{a+b}$ for $a, b \in R_+$

i.e Arithmetic Mean ≥ Geometric Mean ≥ Harmonic Mean.

Equality occurs when a = b.

Proof : Since $a, b \in R_+$, $\sqrt{a}$, $\sqrt{b}$ are real.

Now from (i) above

$$(\sqrt{a})^2 + (\sqrt{b})^2 \geq 2\sqrt{a}\sqrt{b} \Rightarrow \frac{a+b}{2} \geq (ab)^{\frac{1}{2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Taking $\frac{1}{a}, \frac{1}{b}$ in place of a and b we have

$$\frac{\frac{1}{a}+\frac{1}{b}}{2} \geq \sqrt{\frac{1}{a}}\sqrt{\frac{1}{b}} \Rightarrow (ab)^{\frac{1}{2}} \geq \frac{2ab}{a+b} \ldots\ldots\ldots\ldots \qquad (2)$$

From (1) and (2), (ii) follows

The above result can be generalised to a finite set of positive numbers $a_1, a_2, \ldots\ldots a_n$ as follows

A. M = $\frac{a_1 + a_2 + \ldots\ldots + a_n}{n}$

G. M = $(a_1\, a_2 \ldots\ldots\, a_n)^{1/n}$

H.M. = Reciprocal of the A. M of the reciprocals $= n \Big/ \left[\frac{1}{a_1}+\frac{1}{a_2}+\ldots\ldots+\frac{1}{a_n}\right]$

Then AM ≥ GM ≥ HM .... (3)

where equality holds iff the numbers are equal.

Proof of (3) in the general case is beyond the scope of the book.

(iii) **Weighted Means :**

If $a_1, a_2 \ldots\ldots a_n > 0$ and $m_1, m_2 \ldots\ldots m_n$ are positive rationals then

$$\frac{m_1a_1 + m_2a_2 + \ldots\ldots + m_na_n}{m_1 + m_2 + \ldots\ldots + m_n} \geq \left[a_1^{m_1}a_2^{m_2}\ldots\ldots a_n^{m_n}\right]^{1/(m_1+m_2+\ldots\ldots+m_n)} \quad \ldots\ldots\ldots (4)$$

Equality holds when $a_1 = a_2 \ldots = a_n$.

Proof : *Case I :* If $m_1, m_2 \ldots\ldots, m_n$ are positive integers, taking AM and GM of m_1 numbers each equal to a_1, m_2 numbers each equal to $a_2, \ldots, m_n$ numbers each equal to a_n, the inequality (4) follows from (3)

Case II : If $m_1, m_2, \ldots. m_n$ are positive rationals,

Let $m_i = \frac{p_i}{q_i}$, $i = 1, 2, \ldots n$, where $p_i, q_i \in N$

Let L = L. C. M of $\{q_1, q_2, \ldots\ldots q_n\}$ and $L_i = \frac{L}{q_i}$

$$\text{Then } \frac{m_1a_1+m_2a_2+\ldots..+m_na_n}{m_1+m_2+\ldots...+m_n} = \frac{L_1p_1a_1+L_2p_2a_2+\ldots..+L_np_na_n}{L_1p_1+L_2p_2+\ldots...+L_np_n}$$

$$\geq \left[a_1^{L_1P_1}a_2^{L_2P_2}\ldots\ldots..a_n^{L_nP_n}\right]^{1/(L_1P_1+\ldots\ldots.+L_nP_n)}$$

$$= \left[a_1^{L_1P_1/L}a_2^{L_2P_2/L}\ldots\ldots..a_n^{L_nP_n/L}\right]^{L/(L_1P_1+\ldots\ldots.+L_nP_n)} \qquad \text{(by case I)}$$

$$= \left[a_1^{m_1}a_2^{m_2}\ldots\ldots..a_n^{m_n}\right]^{1/(m_1+m_2+\ldots\ldots.+m_n)}$$

(iv) If $a_1, a_2, \ldots., a_n > 0$ and are unequal, then

$$\frac{a_1^m + a_2^m+\ldots..+a_n^m}{n} \quad \left(\frac{a_1 + a_2 + \ldots..+a_n}{n}\right)^m \quad \ldots\ldots\ldots\ldots\ldots..(5)$$

according as $m \notin (0, 1)$ or $m \in (0, 1)$.

The proof is omitted.

Corollary : From the inequality (3) It follows that if the **sum** of n positive numbers is **constant**, their **product** is **maximum** when they are equal and if the product is constant, the sum is **least** when they are equal.

Example 7 : Show that for positive numbers $a_1, a_2 \ldots..a_n$,

$$\frac{a_1}{a_2} + \frac{a_2}{a_3} + \frac{a_3}{a_4} + \ldots\ldots.+\frac{a_{n-1}}{a_n} + \frac{a_n}{a_1} \geq n$$

Proof : Use the result : AM $\geq$ GM for the numbers $\frac{a_1}{a_2}, \frac{a_2}{a_3}, \ldots\ldots. \frac{a_n}{a_1}$.

Example 8 : Show that for any non-zero real number r, $\left(\frac{1^r+2^r+3^r+\ldots\ldots..+n^r}{n}\right) > (\ \ n)^{\frac{r}{n}}$

Proof : Since 1, 2, 3, ...n are all unequal we use the result that AM > GM

that is, $\frac{1^r+2^r+\ldots\ldots.+n^r}{n} > \left(1^r, 2^r \ldots.n^r\right)^{\frac{1}{n}}$ whence the result follows.

Example 9 : If a, b, c are positive and unequal numbers show that $(a + b)(b + c)(c + a) > 8abc$

Proof : By inequality (3), we obtain $\frac{a+b}{2} > \sqrt{ab}$, $\frac{b+c}{2} > \sqrt{bc}$, $\frac{c+a}{2} > \sqrt{ca}$ whence (on multiplication)

$$\frac{a+b}{2}.\frac{b+c}{2}.\frac{c+a}{2} > \sqrt{ab}\sqrt{bc}\sqrt{ca}$$

The result is obtained on simplification.

Example 10 : If x and y are any two positive rational numbers then

$$x^xy^y \geq \left(\frac{x+y}{2}\right)^{x+y} \geq x^yy^x.$$

Proof : Since x, y are positive rationals, using (4)

We have $\dfrac{y.x+x.y}{y+x} \geq [x^y y^x]^{1/(x+y)}$

$\Rightarrow \left(\dfrac{2xy}{x+y}\right)^{x+y} \geq x^y y^x$ .... (i)

But AM ≥ HM Hence

$\dfrac{x+y}{2} \geq \dfrac{2xy}{x+y} \Rightarrow \left(\dfrac{x+y}{2}\right)^{x+y} \geq \left(\dfrac{2xy}{x+y}\right)^{x+y} \geq x^y y^x$ .... (ii)

Also from (i)

$$\frac{(xy)^{x+y}}{x^y y^x} \geq \frac{(x+y)^{x+y}}{2^{x+y}}$$

$\Rightarrow x^x y^y \geq \left(\dfrac{x+y}{2}\right)^{x+y}$ .... (iii)

From (ii) and (iii) the result follows.

Example 11 : Show that for $n \in N$

$$2\sqrt{n} - 2 < 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + ... + \frac{1}{\sqrt{n}} < 2\sqrt{n} - 1$$

Solution : For $m \in N$,

$2\sqrt{(m+1)} - 2\sqrt{m} = 2\{(\sqrt{m}+1) - \sqrt{m}\}$

$= \dfrac{2\{\sqrt{(m+1)} - \sqrt{m}\}\{\sqrt{(m+1)} + \sqrt{m}\}}{\sqrt{(m+1)} + \sqrt{m}}$

$= \dfrac{2}{\sqrt{(m+1)} + \sqrt{m}} < \dfrac{2}{2\sqrt{m}} = \dfrac{1}{\sqrt{m}}$

and similarly

$2 . \sqrt{m} - 2\sqrt{(m-1)} = \dfrac{2}{\sqrt{m} + \sqrt{(m-1)}} > \dfrac{1}{\sqrt{m}}$.

Hence combining the results we have

$$2\sqrt{(m+1)} - 2\sqrt{m} < \frac{1}{\sqrt{m}} < 2\sqrt{m} - 2\sqrt{(m-1)} \text{ .... (i)}$$

Putting $m = 2, 3, ..., n$ in (i) and adding columnwise we get

$$2\sqrt{(n+1)} - 2\sqrt{2} < \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + ... + \frac{1}{\sqrt{n}} < 2\sqrt{n} - 2$$

$$\Rightarrow 2\sqrt{(n+1)} - 3 < \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + ... + \frac{1}{\sqrt{n}} < 2\sqrt{n} - 2.$$

Adding 1 to each part we get the desired inequality.

Irrationals :

We have always seen that $1 \in R$, We write

$1 + 1 = 2 \quad 1 + 2 = 3 \; 1 + 3 = 4 \qquad 1 + n = n + 1$

But because $1 > 0$ we have $0 < 1 < 2 < 3 < 4 < .... < n < n + 1 <$

Similarly $-1 + (-1) = -2$ .... etc and

$.... -4 < -3 < -2 < -1 < 0.$

Thus we have $Z \subset R$. Since for $p, q \in Z, q \neq 0, \frac{p}{q} \in R$. we have $Z \subset Q \subset R$.

Are the inclusions proper ? We know $Z \subset Q$.

Unitil now we have not demonstrated a single element in R which is not in Q though there has been indication that there are numbers which cannot be accommodated in Q.

To do that we assume the following result of R.

For every $a > 0$, $n \in N$, there exists $x \in R$ such that $x^n = a$.

It is obvious that x so constructed is positive and we write this as $x = \sqrt[n]{a}$.

It follows from this theorem that as a solution of the equation $x^2 = 2$ we get a real number $\sqrt{2}$ but we know that $x^2 = 2$ has no rational solution. This proves that $\sqrt{2} \in R - Q$.

We define $R - Q$ to be the set of **irrational numbers.**

Hence $\sqrt{2}$ is an irrational number. Can you show that $\sqrt{3}$ is an irrational number ? Other examples of irrationals are $\sqrt{7}$, $\sqrt{11}$, $\sqrt{13}$, $\sqrt{17}$, etc.

It should not lead to the feeling that irrationals are obtained only as the solution of algebraic equations. In fact there are irrational numbers like π and e which are not obtained as roots of the algebraic equations.

We come across rationals more frequently than irrational. This leads to the misconception that there are more number of rationals than irrationals. In fact this is not true. The truth is that there are more number of irrationals than rationals and we cannot provide a proof of this within the scope of this book.

Theorem 7 :

Let a be rational and b be irrational. Then

(i) $a + b$ is irrational (ii) ab is irrational if $a \neq 0$.

Proof : (i) Suppose that $a + b$ is rational

then $(a + b) - a = b$ is rational as Q is closed under subtraction.

This is a contradiction. Hence $a + b$ is irrational.

(ii) Suppose that ab is rational.

Then $ab \div a = b$ is rational as $a \neq 0$ is rational. But this is a contradiction.

Hence ab is irrational.

It follows from the above theorem that number like $2 + \sqrt{3}$, $e + 5$, 5π are irrational numbers.

Now the question can be raised **if the set of irrational numbers form an algebra just as rationals do.** Answer is "no".

Let us consider the following example.

Example 7 :

Prove that $\sqrt{3} + \sqrt{5}$ is irrational.

Solution :

Suppose that $\sqrt{3} + \sqrt{5}$ is rational number and denote it by r. Then

$$r = \sqrt{3} + \sqrt{5}$$

$$\Rightarrow r^2 = 3 + 5 + 2\sqrt{3} \cdot \sqrt{5}$$

$$\Rightarrow \sqrt{3} \cdot \sqrt{5} = \frac{r^2 - 8}{2}.$$

Note that $\sqrt{3} . \sqrt{5} = \sqrt{15}$ is irrational (why ?) whereas $\frac{r^2 - 8}{2}$ is rational. This is a contradiction. This proves that $\sqrt{3} + \sqrt{5}$ is irrational.

Similarly the numbers like $\sqrt{2} + \sqrt{3}$, $e + \sqrt{7}$, $e + \pi$ are irrational numbers. It is wrong to think that the sum of two irrational numbers is irrational. For example

$$(2-\sqrt{3}) + (2+\sqrt{3}) = 4.$$

Similarly irrationals are also not closed under multiplication. For example

$$(2-\sqrt{3}) \times (2+\sqrt{3}) = 4-3 = 1$$

In fact one should note that irrational numbers are not **closed under addition, subtraction, multiplication and division.**

Decimal representation of reals

We have seen that rationals can be expressed as terminating or recurring decimals. Thus irrationals have non-recurring decimals. For example, the numbers like

.0100100010000100000l.....

.2122122212222122222l ....

5.32133213332133332l....

are non-recurring and hence are examples of irrationals. We cannot express these numbers fully as the process is unending. Similarly, if we proceed to extract the square root of 2, then the process continues endlessly and the decimal representation is bound to be non - recurring (otherwise it would be rational). So it may be necessary to determine a sequence of rationals that are colse to $\sqrt{2}$. since $1^2 < (\sqrt{2})^2 < 2^2$ it follows that $1 < \sqrt{2} < 2$.

This is a crude **approximation**. However a better approximation can be achieved.

Since $(1.4)^2 = 1.96$, $(1.5)^2 = 2.25$ it follows that $(1.4)^2 < (\sqrt{2})^2 < (1.5)^2$

that is, $1.4 < \sqrt{2} < 1.5$.

Similarly

$1.41 < \sqrt{2} < 1.42$

$1.414 < \sqrt{2} < 1.415$

$1.4142 < \sqrt{2} < 1.4143$

$1.41421 < \sqrt{2} < 1.41422$

This process can continue indefinitely. Thus we obtain a succession of rational number 1, 1.4, 1.41, 1.414, 1.4142, 1.41421 which come closer or nearer to $\sqrt{2}$ from the left. Similarly the succession of rational number 2, 1.5, 1.42, 1.415, 1.4143, 1.41422 come nearer to $\sqrt{2}$ from the right of $\sqrt{2}$.

One can verify in calculator that $\sqrt{2} = 1.414213562$ ....

This is the situation when we say that the sequence of rational numbers **approximate** the irrational number. Since the exact decimal representation of irrational numbers is not possible, it is necessary to remain contented with an approximate rational value of such number. For example, we may manage by saying that 1.41 is an approximate value of $\sqrt{2}$. When we do this we commit an error and that error is $\sqrt{2} - 1.41$. The less error we commit, the better.

Similar is the situation with regard to the values of $\sqrt{3}$, $\sqrt{5}$, $\sqrt{6}$, $\sqrt{7}$, $\sqrt{10}$, $\sqrt{11}$ etc.

Note that (verify these in calculators or computers)

$\sqrt{3} = 1.732050808$.....　　　$\sqrt{5} = 2.236067978$....

$\sqrt{6} = 2.449489743$ ....　　　$\sqrt{7} = 2.645751311$ .....

There is another irrational number which is very interesting. That number is e, which is taken as the base of the natural or Napier logarithm. In fact e is defined to be the real number which is approximated by the sequence of rationals

$$\left(1+\frac{1}{n}\right)^{n}.$$

We postpone the discussion of this to a later chapter. However it may be mentioned that it is an irrational number whose approximate value is given by

$e = 2.718281828459045$.

A number is called an **algebraic number** if it is a root of some algebraic equation i.e. polynomial equation with rational coefficient. Otherwise it is called a **transcendental number.** $\sqrt{2}$ is an algebric number, as it is the root of the equation $x^2 = 2$. But e is an irrational number which is not algebraic; it is a transcendental number.

The number like sin 3,cos 10 are examples of irrational numbers which are transcendental.

The most interesting, the most elusive, the most used and perhaps the most popular irrational number happens to be the number π which is **the ratio of the circumference with the diameter**

of any circle. This too is a transcendental number and its decimal representation correct upto a few decimal places is given as

$$\pi = 3.14159265358979323846264338527950288841971...$$

In fact in the year 1961, the value of π correct up to first 100265 places of decimal have been found out by means of a computer.

3000 years age, Babylonians seemed to have the knowledge of this number and they took $\frac{25}{8} = 3.125$ as the approximate value of π. By comparing this value with the one given above, one can see that this is incorrect at the second place of decimal. Archmedes (300 B.C.) took $\frac{22}{7} = 3.142857$ as the approximate value of π and this is incorrect at the third place of decimal. In Vedic period, Indians used $\sqrt{10}$ as its approximate value and this is incorrect at the second decimal place.

Indian mathematician Aryabhatta (*b*. 476 A.D.) put the approximate value of π as $\frac{62852}{20000} = 3.1416$ and this is incorrect at the fourth place. Bhaskar (1114 - 1185 A.D.) another notable Indian mathematician took the approximate value of π as $\frac{3927}{1250} = 3.1416$ and this too is incorrect at the fourth place. Madhaba (1340–1425) and also Gregory (1671), a German mathematician expressed π in terms of an infinite series as follows :

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - ... \quad \text{(Madhab – Gregory series)}$$

It should be mentioned that this series representation of π is known to have been worked out by Hindu mathematicians in Kerala during 14th and 15th century. There are many interesting series representation of π from which it would be possible to calculate the approximate value of π by computing the first few terms of the series.

Some examples of series representation of π are as follows :

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + ..., \qquad \frac{\pi^4}{90} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + ...$$

But the most notable contribution in the evalution of π is due to an Indian mathematician.

Guess who ?

Born 111 years ago (in December 1887 and died at the age of 32) the world famous Indian mathematician genius Srinivas Ramanujan left behind a strange collection of over 4000 formulas. We quote two of his famous formulas for π which are at once beautiful and obscure.

Ramanujan formula :

$$\frac{1}{\pi}=\frac{\sqrt{8}}{9801}\sum_{n=0}^{\infty}\frac{(4n)!}{(n!)^4}\frac{(1103+26390n)}{(396)^{4n}}$$

where $m! = 1.2.3.4 \ldots m$

$$\frac{1}{\pi}=2\sqrt{2}\sum_{n=0}^{\infty}\frac{\left(\frac{1}{4}\right)_n\left(\frac{1}{2}\right)_n\left(\frac{3}{4}\right)_n}{(1)_n(1)_n n!}\times(1103+26390n)\left(\frac{1}{99}\right)^{4n+2}$$

where $(a)_n = a\,(a+1)(a+2) \ldots (a+n-1)$.

These formulas provide an extremely rapid way to calculate the value of π. In the year 1986, an American computer scientist used one of the above versions of Ramanujan's formula to calculate π correct upto 17 million places of decimal.

If we take the first term of the first series of Ramajunan formula, then we get the approximate value of $\frac{1}{\pi}$ to be $\frac{\sqrt{8}\times 103}{9801}$ and this gives the approximate value of π to be 3.14159262180330...

which is incorrect only at the eighth place of decimal. If one considers two terms, this gives an approximation which is incorrect only at 16th place after decimal. Compare this fast series with the slow series as the one of Madhaba-Gregory. If one calculate upto 600 terms in Madhaba-Gregory series the error would be at the fourth place of decimal (as calculated by a computer).

Are these not nice and inscrutable formulas ! Should we not be proud that such a mind is an Indian !

Number Line :

We know how to represent rational numbers by points on a line. Let us recapitulate. Take a geometrical line. Pick up any point A and represent it by 0. Take any point B to the right of it and we call it 1. Thus the length of the segment $\overline{AB}$ is considered to be of unit length. Similarly, the natural numbers 2, 3, 4 ... can be dotted on the line which are equally spaced. Once this is done we do the same thing to the left of the point A and mark the points –1,–2, ... etc. Let n be any positive natural number. Dividing the segment $\overline{AB}$ into n equal parts we can represent the numbers

$$\frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \ldots\ldots \frac{n-1}{n}$$

on the segment $\overline{AB}$. By this way we can locate all rational points on the line.

What about the irrationls ? We also know how to plot the irrational numbers like $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$ etc. on the line by means of ruler and compass. Therefore these numbers are called constructible numbers.

But there are certain irrational numbers like π and e which are not constructible by means of ruler and compass.

Once rational points have been plotted on the line, we can find many points on the line which have not been represented by rational numbers. These gaps correspond to irrational numbers. To avoid any philosophical discussion about the representation of real numbers on a line, it is safe to assume the following :

Axiom : There is a one-to-one correspondence between the set of real numbers and the points on a line.

Once this is done, the line is called a number line and this serves as an infinite scale for measurement of distances (a discussion about this follows).

Intervals :

Using the concept of the number line and distance we can define four types of intervals :

Let $a, b \in$ R such that $a < b$. Then

$(a, b) = \{x \in \text{R} : a < x < b\}$

is called an **open interval.** The interval $[a, b]$, called **closed interval** is defined as

$[a, b] = \{x \in \text{R} : a \leq x \leq b\}$

The interval $[a, b) = \{x \in \text{R} : a \leq x < b\}$

is called an **open closed** (or **semi-open** interval). This is closed on the left and open on the right.

$(a, b] = \{x \in \text{R} : a < x \leq b]$

is too called a semi-closed (or semi-open interval) and this is open on the left and closed on the right. In this sense we can also express R as

$R = (-\infty, \infty) = \{x \in R : -\infty < x < \infty\}$

Similarly $[a, \infty) = \{x \in R : a \leq x < \infty\}$ and $(-\infty, a) = \{x \in R : -\infty < x < a\}$

Example 8 :

Find the interval for which

(i) $|2x + 7| < 10$ (ii) $|5x - 3| > 1$

Solution :

(i) $|2x + 7| < 10 \Leftrightarrow -10 < 2x + 7 < 10$

$$\Leftrightarrow -17 < 2x < 3 \Leftrightarrow -\frac{17}{2} < x < \frac{3}{2}.$$

Thus the required interval is $\left(-\frac{17}{2}, \frac{3}{2}\right)$.

(ii) $|5x - 3| > 7$

$\Leftrightarrow x$ is such that it does not satisfy $|5x - 3| \leq 7$.

Now $|5x - 3| \leq 7 \quad \Leftrightarrow -7 \leq 5x - 3 \leq 7$

$$\Leftrightarrow -4 \leq 5x \leq 10 \Leftrightarrow -\frac{4}{5} \leq x \leq 2 \Leftrightarrow x \in \left[-\frac{4}{5}, 2\right].$$

Thus if $x \notin \left[-\frac{4}{5}, 2\right]$ then $x \in R - \left[-\frac{4}{5}, 2\right]$

$$= \left(-\infty, -\frac{4}{5}\right) \cup (2, \infty) = \{x \in R : |5x - 3| > 7\}.$$

Quadratic Polynomial

An expression of the form $p(x) = ax^2 + bx + c$ is called a quadratic polynomial in $x \in R$ where it is assumed that the coefficients $a, b, c \in R$ and $a \neq 0$. We know that the roots of the quadratic equation $p(x) = 0$ are given by

$$\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad \beta = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

and the roots are real if the discriminant $D = b^2 - 4ac \geq 0$.

Since $p(x)$ can be factored as $p(x) = a(x - \alpha)(x - \beta)$ it follows that $p(x)$ **has the same sign as a if x does not lie between the two real roots α and β and $p(x)$ has the opposite sign as a if x lies between the two real roots.**

EXERCISES 1(b)

1. (i) Prove that the sum, difference, product of two rational numbers is a rational number.

   (ii) Prove that the quotient obtained by the division of a rational number by a non zero rational number is a rational number.

2. Prove that $\sqrt{5}$ is not a rational number.

3. Prove that $\sqrt{6}$ is an irrational number.

4. Prove that the following numbers are irrational.:

   (i) $2 + \sqrt{5}$ (ii) $2\sqrt{3}$ (iii) $\sqrt{2} + \sqrt{5}$ (iv) $\dfrac{\sqrt{3}}{\sqrt{2}}$

5. Give examples of two irrational numbers whose

   (i) sum is rational (ii) sum is irrational

   (iii) difference is rational (iv) product is rational

   (v) quotient is rational (vi) quotient is irrational

6. Prove the following statements stating the field axioms used at each step.

   (i) $a + b = a \Rightarrow b = 0$ (ii) $-(-a) = a$

   (iii) $ab = 1 \Rightarrow b = a^{-1}$ $(a \neq 0)$

   (iv) $(a^{-1})^{-1} = a$ $(a \neq 0)$

   (v) $(ab)^{-1} = b^{-1}a^{-1}$ $(a \neq 0, b \neq 0)$

   (vi) $-(a + b) = -a - b$.

7. Answer the following questions where it is meaningful. Otherwise say 'non-existent'.

   (a) What is the

(i) additive identity in the set of natural numbers ?

(ii) additive identity in the set of integers ?

(iii) additive identity in the set of rational numbers ?

(iv) additive identity in the set of real numbers

(v) multiplicative identity in the set of natural numbers ?

(vi) multiplicative identity in the set of integers ?

(vii) multiplicative identity in the set of rational numbers ?

(b) What is the

(i) additive inverse of natural numbers in N ?

(ii) aditive inverse of integers in Z ?

(iii) additive inverse of rational numbers in Q ?

(iv) additive inverse of real numbers in R ?

(v) multiplicative inverse of natural numbers in N ?

(vi) multiplicative inverse of integers in Z ?

(vii) multiplicative inverse of rational numbers in Q ?

(viii) multiplicative inverse of real numbers in R ?

(c) What is the additive inverse of (in the set R) ?

$2+\sqrt{3}$,π, $\pi - e$, -3, 0

(d) What is the multiplicative inverse of (in the set R)

$\sqrt{3}$, $2-\sqrt{3}$, $\frac{1}{\sqrt{5}}$ 1, 0

8. Prove that if $a > b$, $c > d$, then $a - d > b - c$.

9. Show that if $0 < a < b$, then $0 < \frac{1}{b} < \frac{1}{a}$.

10. If $0 < a < 1$, then show that $0 < a^2 < 1$, $0 < a^3 < 1$, $0 < a^4 < 1$.

11. Prove that there is no largest real number.

12. Let $a > 0$ and $b > 0$. Show that $a^2 > b^2$ if and only if $a > b$.

13. Let $a > b > 0$ and $c > d > 0$. Then show that $ac > bd$; and $\frac{a}{d} > \frac{b}{c}$.

14. Let a , $b \in$ R such that $a < b$ Then show that $a < a + \frac{b - a}{n}, < b$ for $n = 2, 3, 4, 5$.....

15. Prove that $1 > 0$.

16. Which of the following number are rational or irrational (why) ?

(i) .1011011101111011111 0....(ii) .123456789123456789.....

(ii) .9191191119111191111 1...

17. Give an approximate value of $\sqrt{3}$ which is correct up to third decimal place.

18. Which is bigger, π or $\frac{22}{7}$?

19. If $\sqrt{10}$ is taken as the approximate value of π then at what stage is the error committed ? Which is bigger : π or $\sqrt{10}$?

20. Show that for $x, y \in$ R,

(i) $|xy| = |x|\,|y|$ (ii) $|x^2| = |x|^2$

(iii) $|x^3| = |x|^3$ (iv) $|x^n| = |x|^n$, $n \in$ N

21. Show that for $x \in$ R, $|x|^2 = x^2$.

22. Prove that

(i) $\left|\frac{1}{x}\right| = \frac{1}{|x|}$ ($x \neq 0$) (ii) $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$ ($y \neq 0$).

23. What is the absolute value of $\frac{a}{|a|}$ ($a \neq 0$) ?

24. We can write the statement $-1 \leq x \leq 1$ by $x \in [-1, 1]$.

Similarly express the following intervals in the inequality form :

(i) $x \in [-2, 2]$ (ii) $x \in (-2, 2)$ (iii) $x \in [-2, 2)$

(iv) $x \in (-2, 2]$ (ii) $x \in (5, 10)$ (iii) $x \in [-3, \infty)$

25. Express the following inequalities in the interval form :

(i) $-1 \le x \le 5$ (ii) $-3 < x < 5$ (iii) $-\frac{1}{2} \le x < \frac{1}{2}$

(iv) $-\frac{1}{3} < x \le \frac{1}{3}$ (v) $-\infty < x < -1.$

26. Express the following subsets of R in the interval form.

(i) $\{x : |x| < 2\}$ (ii) $\{x : |x| \le 2\}$ (iii) $\{x : |x| < 2\} \cup \{-2\}$

(iv) $\{x : |x| < 2\} \cup \{2\}$ (v) $\{x : |x| \le 2 \text{ and } x \ne 2\}$ (vi) $\{x : |x| \ge 1\}$

(vii) $\{x : |x| > 5, x = 5\}$ (viii) $\{x : |3x + 1| \le 2\}$ (ix) $\{x : |5x - 7| \le 2\}$

(x) $\{x : |-3x + 5| < 2\}$ (xi) $\{x : |11x - 5| < 4\}$ (xii) $[x: |-7x + 3| > 5\}$

(xiii) $\{x: |-7x + 3| > 5 \text{ and } x = \frac{8}{7}\}$

(xiv) $\{x : |-7x + 3| > 5 \text{ and } x = 0, x \ne 3\}$.

27. Correct the mistakes if any

(i) $2 \in (2, 3)$ (ii) $2 \in (2, 3]$ (iii) $2 \in [2, 3)$

(iv) $0 \in (1, 5)$ (v) $5 \notin (1, 5)$ (vi) $5 \notin (0, 1)$

(vii) $-1 \notin (-\infty, -1)$ (viii) $3 \in [1, \infty)$ (ix) $3 \in [3, \infty)$

(x) $2 \in (-\infty, 2)$.

28. Prove the following inequalities where concerned numbers are all positives.

(i) $a^2 + b^2 + c^2 > ab + bc + ca$

(ii) $(a + b + c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) > 9$ unless $a = b = c$

(iii) $\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \ge 3$ (iv) $\frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a} \ge 4.$

State when strict inequality holds.

(v) $(1 - a)(1 - b)(1 - c) > 8abc$ provided that $a + b + c = 1$

(vi) $\sqrt{ab} \geq \dfrac{2ab}{a+b}$.

State when strict inequality holds, equality holds.

(vii) $\sqrt{a} - \sqrt{b} \leq \sqrt{|a-b|}$

(viii) Let $0 < a < b < c$. Then

(α) $3a < a + b + c < 3c$

(β) $3a^2 < a^2 + b^2 + c^2 < 3c^2$

(γ) $\dfrac{a^2}{c} < \dfrac{a^2+b^2+c^2}{a+b+c} < \dfrac{c^2}{a}$

(δ) $a^a b^b > a^b b^a$

29. Prove the following inequlities :

(i) $\sqrt{1} + \sqrt{2} + \ldots\ldots + \sqrt{n} < n\sqrt{\dfrac{n+1}{2}} < (n+1)^{\frac{3}{2}}$

(ii) $(\underline{|n})^2 > n^n$

(iii) $2.\ 4.\ 6 \ldots.\ 2n < (n + 1)^n$

(iv) $n^n > 1\ .\ 3\ .\ 5 \ldots. (2n - 1)$

(v) $(n + 1)^n > 2^n\ .\ n!$

(vi) $(\underline{|n})^3 > 8\,(1^3 + 2^3 + 3^3 + \ldots.+n^3)$, for $n \geq 3$

(vii) $n\,(n + 1)^3 < 8\,(1^3 + 2^3 + 3^3 + \ldots.+n^3)$

(viii) $\dfrac{1}{n+1} + \dfrac{1}{n+2} \ldots\ldots + \dfrac{1}{3n+1} > 1,\ n \in \mathrm{N}$

(ix) $1 < \dfrac{1}{1001} + \dfrac{1}{1002} + \dfrac{1}{1003} + \ldots.. + \dfrac{1}{3001} < \dfrac{4}{3}$.

Hints : For (i), (vii) use the fact that if $a_1, a_2, \ldots\ldots a_n$ are positive which are not equal, then

$$\frac{a_1^m + a_2^m + \ldots\ldots + a_n^m}{n} \gtrless \left(\frac{a_1 + a_2 + \ldots\ldots + a_n}{n}\right)^m$$ according as m does not or does lie between 0 and 1.

30. Find the greatest value of x^2y^3 where x and y are positive numbers satisfying $5x + 2y = 3$.

[Hint. x^2y^3 is greatest when $\left(\frac{5x}{2}\right)^2\left(\frac{2y}{3}\right)^3$ is greatest. Use the corollary before example 7]

31. If a, b, c are the sides of a tringle prove that abc $\geq (b + c - a)(c + a - b)(a + b - c)$.

[Hint. take $b + c - a = x,\ c + a - b = y \ \Rightarrow c = \frac{x + y}{2}$ etc]

32. If a and b are positive and unequal then show that $8(x^2 + y^2)(x^3 + y^3) > (x + y)^5$

●

Answers

Exercises 1 (a)

1. (i) B; (ii) A; (ii) A; (iv) D; (v) B, (vi) A; (vii) C; (viii) B; (ix) C; (x) B.

Exercises 2 (b)

1. (a) $(M\cap C) - P$ (b) $M \cap C \cap P$ (c) $M - (C \cup P)$ (d) $(P \cup C) - M$

   (e) $(P \cap C - M) \cup (P \cap M - C) \cup (C \cap M - P) \cup (P \cap C\cap M)$

2. (i) $A = B = \phi$ (ii) $A = B$ (iii) $A\subset B$ (iv) $A \cap B = \phi$ (v) $A = B = U$

3. $A - B \neq B - A$, $A \Delta B = B \Delta A$ 4. Not true.

9. 5; 10. 100; 11. 70; 12. 16; 13. 225; 14. 25, 35; 15. 11.

Exercises 3 (a)

1. (i) $\{(0, 0)\}$, (ii) $\{(a,a), (a, b), (a,c), (b,a), (b, b), (b,c)$; (iii) ϕ 2. (i) mn (ii) 2^{m+n} 3. (i) $x = -3, y=2$, (ii) $x = 1, y = 0$ (iii) $x = -1, y = 1$ 4. A=B; 5. $(-1, x), (0, x), (1,y)$.

Exercises 3 (b)

1. (a) Any subset of $\{(a, 1), (a, 2), (b, 1), (b, 2), (c, 1) (c, 2)\}$

   (b) Any subset of $\{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)$

   (c) Empty set, only one

   (d) $\{(a, 1), (b, 1), (c, 1)\}$

   $\{(a, 2), (b, 2), (c, 2)\}$ are some examples of many one relations.

   Similarly $\{(a, 1), (b, 1)\}$, $\{(a, 2), (b, 1) (c, 2)\}$

   $\{(a, 1), (a, 2)\}$

   $\{(b, 1), (b, 2)\}$

   $\{(c, 1), (c, 2)\}$ are some examples of one- many relations

   $\{(a, 1), (b, 2)\}$, $\{(a, 2), (c, 1)\}$

   $\{(a, 2), (b, 1)\}$, $(b, 1), (c, 2)\}$

   $\{(a, 1), (c, 2)\}$, $\{(b, 2), (c, 1)\}$ are some examples of one- one relations.

2. (i) $D_\phi = \phi = R_\phi$ (ii) $D_{A\times B} = A, R_{A\times B} = B$

(iii) – (vii)

$D_{A\times\phi} = D_\phi = D_{\phi\times B} = D_{\phi\times\phi}$

$R_{A\times\phi} = R_\phi = R_{\phi\times B} = R_{\phi\times\phi}$

3. (i) $f = \{(1, 1), (2, 2^2)...., (10, 10^2)\}$ (ii) $f = \{(1, 0), (0, 1), (-1, 0), (0, -1)\}$

(iii) $f = \{(1, 1), (1, 3), (1, 5), (2, 2), (2, 4), (3, 1), (3, 3), (3, 5), (4, 2), (4, 4)\}$

4. 2^{mn}

6. (i) {(2,8), (3,27), (5,125), (7,343); (ii) {2,3,5,7}; (iii) {8,27,125,343}; (iv) {(8,2),(27,3), (125,5), (343,7)}; (v) {8,27,125,343}; (vi) {2,3,5,7} .

7. (i) {(1,2), (2,4), (3,6); (ii) {1,2,3}; (iii) {2,4,6}; (iv) {(2,1), (4,2), (6,3); (v) {2,4,6}; (vi) {1,2,3}.

Exercises 3 (c)

2. n^m; 3. (i) [-3,3]; (ii) R; (iii) R; (iv) R-{-1,1}; (v) R-$\{x\in R)|\tan x = -1\}$; (vi) R-{0}; (vii) R-$\{x\in R|x<0\}$; (vii) [-3,4]-{0,1}; (ix) R; (x) (-1,1); (xi) $\{x\in R|\sin x>0\}$.

4. (i) [-1, 1], (ii) $[0,\infty)$, (iii) [-1,0]; (iv) R-{-1}; (v) $[-\frac{1}{2}, \frac{1}{2}]$; (vi) $[\frac{1}{3}\ 1]$; (vii) R; (viii) $[1,\infty)$.

5. domR=R, Range=[0,1); (ii) dom=$[\frac{3}{2}, \infty)$, Range= $[0,\infty)$; (iii) dom=R-{2}, Range=R.

7. (*a*) (i) No, dom$f \neq X$ (Also two ordered pairs contain same first component); (ii) Yes; (iii) No, Element '*a*' has two distinct images; (iv) Yes; (v) Yes; (vi) Yes.

(b) (ii) dom = *{a,b,c},* Range = {2,3,4}, (iv) dom= *{a,b,c},* Range = {1}; (v) dom=*X, Range={1,2}; (vi) dom= X=Range.*

(c) {(a,1), (b,1), (c,1)}; (d) {a,a), (b,b), (c,c)}; 8. $\frac{1}{\sqrt{2}}$, *3;*

9. (i) x=1 or x=3; (ii) x=2; 10. (i) x= 2; (ii) x=0; (iii) $\{x\in R|\sec x=0\}\ U\ \{x\in R \mid \cos x \leq 0\}$

11. For (i) , (ii) and (iii) dom = [-1,1], (iv) dom=[-1,1).

13. $2x^2-x+1$;

Exercises 4 (a)

1. (i) + ve, (ii) + ve, (iii) –ve, (iv) + ve (v) – ve, (vi) + ve, (vii) + ve, (viii) + ve

2. (i) cos 15°, (ii) tan 55°, (iii) – cos 3° (iv) tan 18°, (v) – cot 18°, (vi) – cos 60°
(vii) – sin 50°, (viii) sec 20°

3. $R - \left\{(2n+1)\frac{\pi}{2} | n \in Z\right\}$ $R - \{n\pi \mid n \in Z)\}$

4. [–1, 1], 5. A = 18°, 6. 0

7. $\frac{1}{2}$ 8. 1

Exercises 4 (b)

1. (i) F, (ii) F, (iii) T, (iv) T, (v) F, (vi) F, (vii) T, (viii) T, (ix) F, (x) F

2. (i) $\frac{1}{2\sqrt{3}} + \frac{\sqrt{2}}{3}$, (ii) $\frac{\pi}{4}$ (iii) $\sqrt{3}$ (iv) $\sqrt{2}-1$ (v) $-\frac{1}{4}$

(vi) $1+\frac{1}{\sqrt{2}}$ (vii) $\sqrt{3}\cos 25°$ (viii) $\frac{\sqrt{5}-1}{8}$ (ix) $\frac{1}{2}$

(x) 15° (xi) negative (xii) $\frac{-2}{\sqrt{5}}$ (xiii) sin 1° < sin1 (xiv) tan 1 > tan 2

7. (i) 13, (ii) 25, (iii) 7, (iv) 15 8. (ii) *a*

11. (i) cos (A + B + C) + cos (A + B – C) + cos (A – B + C) + cos (A – B – C)
(ii) 4 cos (B + C) cos (C + A) cos (A + B)

13. $$\sin 3° = \frac{(\sqrt{5}-1)\,(\sqrt{3}+1) - \sqrt{10+2\sqrt{5}}\,(\sqrt{3}-1)}{8\sqrt{2}}$$

$$\cos 3° = \frac{(\sqrt{3}+1)\,\sqrt{10+2\sqrt{5}} + (\sqrt{5}-1)\,(\sqrt{3}-1)}{8\sqrt{2}}$$

$$2\sin\frac{\pi}{32} = \sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2}}}}$$

Exercises 4 (c)

1. (i) infinite, (ii) 2π (iii) 0, (iv) 4th (v) $\frac{\pi}{10}$, (vi) $\frac{\pi}{4}$, (vii) 1, (viii) $\frac{\pi}{3}$, (ix) $\frac{3\pi}{4}$ (x) $\frac{\pi}{2}$

2. (i) $0, \pi, \frac{\pi}{3}, \frac{5\pi}{3}$ (ii) $\frac{2\pi}{3}$ (iii) $\frac{3\pi}{2}$ (iv) $0, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}$

(v) $\frac{7\pi}{12}$, $\frac{23\pi}{12}$.

3. (i) $(2n+1)\ \frac{\pi}{4}$ (ii) $180\,n - 40 + (-1)^n\, 45$

(iii) $\frac{n\pi}{5-3(-1)^n}$ (iv) $\frac{\pi\left(n+\frac{1}{2}\right)}{a+b}$ (v) $\frac{n\pi}{3} \pm \frac{\pi}{9}$

4. (i) $n\,\pi \pm \frac{\pi}{4}$ (ii) $n\,\pi \pm (-1)^n\ \frac{\pi}{4}$ (iii) $(-1)^n\ \frac{\pi}{2} + n\,\pi - \tan^{-1}\frac{4}{3}$

(iv) $2n\,\pi \pm \frac{\pi}{3}$ (v) $2n\,\pi + \frac{7\pi}{12}$, $2n\,\pi + \frac{\pi}{12}$ (vi) $(2n+1)\ \frac{\pi}{2}$, $n\,\pi + \frac{\pi}{4}$

(vii) $2n\pi + \frac{\pi}{6}$ (viii) $2n\pi$, $(4n+1)\ \frac{\pi}{6}$, (ix) $(2n+1)\frac{\pi}{2}$, $(2n+1)\pi$, $\frac{2}{5}\,n\pi$

(x) 360n (xi) $n\pi$, $\frac{n\pi}{2}$, $\frac{n\pi}{3}$ (xii) $n\pi - \frac{\pi}{4}$, $n\pi + \tan^{-1}(2 \pm \sqrt{3}\ )$

(xiii) $n\pi + \frac{\pi}{4}, \frac{n}{2} + \frac{\pi}{4}$ (xiv) $2n\pi \pm \frac{\pi}{3}$, $2n\pi \pm \frac{\pi}{4}$ (xv) $2n\pi$, $2n\,\pi \pm \frac{\pi}{4}$

(xvi) $n\pi$, $n\pi + (-1)^n\ \frac{\pi}{6}$ (xvii) $(2n+1)\ \frac{\pi}{8}$, $n\pi \pm \frac{\pi}{3}$ (xviii) $n\pi - \frac{\pi}{4}$, $2n\pi$, $n\pi\ (-1)^n\frac{\pi}{2}$

(xix) $2n\pi$, $\frac{2n\pi}{3} + \frac{\pi}{6}$ (xx) $\frac{n\pi}{12}$ (xxi) $(2n+1)\ \frac{\pi}{4}$ (xxii) $(-1)\frac{n\pi}{6} + n\pi$, $n\pi + (-1)^n\frac{\pi}{2}$

Exercises 4 (d)

1. (i) $b =$ c cos A + a cos C (ii) isoceles (iii) 1

(iv) equilateral (v) $\frac{1}{2}$ (vi) $\sqrt{6} : 2$

(vii) A (viii) $\frac{c}{2a}$ (ix) B (x) 21

3. (i) 5 : 12 (ii) 120° (iii) $\sqrt{\frac{5}{27}}$

(iv) 90° (v) 43

28. B = 30°, A = 90°, C = 60°

Exercises 6 (a)

2. $80 - 24\sqrt{14}$ 4. $x^2 - x + 1$ 8. $\frac{4ax}{a^2 + x^2} i$

10. $\frac{3a^2 - 1}{2a}$ 11. i; 12. 100 13. (i) $\frac{-9}{13} + i\ \frac{19}{13}$ (iv) $\frac{8(a^2 - b^2)ab}{a^2 + b^2} i$

16. $\sqrt{3}\ (\cos\theta + i\sin\theta)$, $\tan\theta = \frac{1}{\sqrt{2}}$. $\sqrt{3}\,(\cos\theta + i\sin\theta)$ where $\tan\theta = \sqrt{2}$. $i\sqrt{3}\sin\frac{\pi}{2}$

Exercises 6 (b)

3. (i) ellipse (ii) hyperbola (iii) for $|a| = c$ a straight line, for $|a| > c$ no representation

6. $\cos n\theta = \frac{1}{2}\,[(\cos\theta + i\sin\theta)^n + (\cos\theta + i\sin\theta)^{-n}]$

7. (i) $\pm(2 + 3i)$ (ii) $\pm(5 - 6i)$ (iii) $\pm\left[\left(\frac{\sqrt{2273} - 47}{2}\right)^{\frac{1}{2}} + i\left(\frac{\sqrt{2273} + 47}{2}\right)^{\frac{1}{2}}\right]$

(iv) $\pm\left[\left(\frac{\sqrt{65} - 8}{2}\right)^{\frac{1}{2}} + i\left(\frac{\sqrt{65} + 8}{2}\right)^{\frac{1}{2}}\right]$ (v) $\pm(a + i)$ (vi) $\pm[(a + b) + i\,(a - b)]$

9. $\frac{\sqrt{5} + 1}{4}$

11. (i) $\cos\frac{2n\pi}{7} + i\sin\frac{2n\pi}{7}$, $n = 0, 1, 2, 3, 4, 5, 6$

(ii) $\cos\frac{4n + 1}{6}\pi + i\sin\frac{4n + 1}{6}\pi$, $n = 0, 1, 2$

(iii) $z = \cos\frac{4n - 1}{12} + i\sin\frac{4n - 1}{12}\pi$, $n = 0, 1, 2, 3, 4, 5$

(iv) $z = 2^{\frac{1}{6}}\left[\cos\frac{8n + 1}{4}\pi + i\sin\frac{8n + 1}{4}\pi\right]$, $n = 0,1\ 2$

19. $\cos\frac{2n + 1}{4}\pi + i\sin\frac{2n + 1}{4}\pi$, $n = 0, 1, 2, 3$ and $\cos\frac{2n}{5}\pi + i\sin\frac{2n}{5}\pi$, $n = 0, 1, 2, 3, 4$

20. $\theta = \dfrac{4k\pi}{(n+1)\,(n+2)}$, $k = 0, 1, 2, 3.....$ 23. 0

Exercises 7 (a)

1. (i) Finite; (ii) Infinite; (iii) finite; (iv) Empty;

2. (i) $\{1,2,3,4\}$, $\{... -3, -2, -1, 0, 1, 2, 3, ...\}$

(ii) 7, 8, 9, ... (in both cases), No difference;

(iii) $\{... -4, -3, -2, -1, 0, 1, 2, 3, 4\}$, $\{0,1,2,3,4\}$;

(iv) $\{000, -12, -11, -10\}$; $(-\infty, -9)$

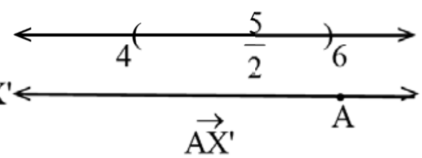
(v) $\{9,10,11,12\}$, $(8,14)$

3. (i) $(-10, \infty)$; (ii) $x > \dfrac{4}{3}$; (iii) $[0,10]$; (iv) $(-3,2)$; (v) $\{x \in Z \mid 0 \le x \le 15\}$; (vi) $\{x \in N \mid x > 1\}$;

(vii) $[4,5)$, $x = 4$; (viii) $(-\infty, \dfrac{8}{5})$, $\{x \in Z \mid x \le 1\}$

5. (i) $\{x \in R \mid 4 < x < 6\}$

(ii) $(-\infty, 15)$



(iii) $[-\dfrac{1}{2}, \infty)$

(iv) $(3,5]$

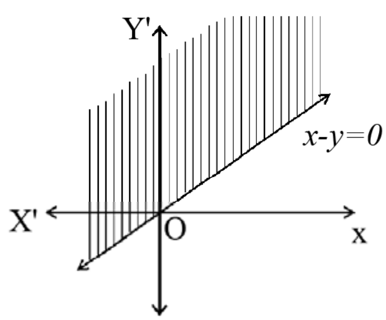
6. 1, 5, 5 units; 7. 6 and 7 units; 8. 5,6,10,15 units.

9. (21,23), (23,25), (25,27), (27,29).

10. (24,26), (26,28), (28,30), (30,32), (32,34)

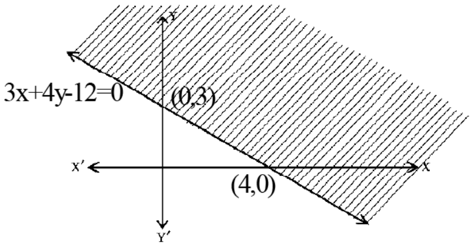
Exercises 7 (b)

1.



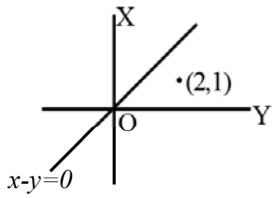
Half plane of $x-y=0$ containing $(0,1)$ or any pt.(x,y) with $x<y$.

2.



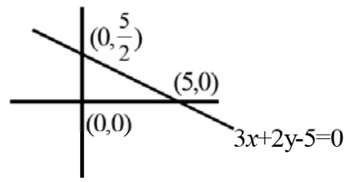
Half plane 3x+4y-12=0 containing (0,0)

3.



(Half plane of x-y=0 containing (2,1)

4.

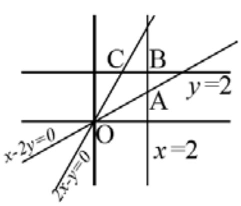


(Half plane of x+$2y$-5=0 containing (0,0) including the line.

5. Solution Set (S.S.) is the half plane 7x-14y=14 containing the origin.
(Hence onwards draw the figures yourself following the first four illustrations.
6. Half plane of x+8y+10=0 containing the origin.
7. Half plane of $5x$+$6y$-12=0 containing the origin.
8. Half plane of 3x-y = 0 containing (0,1).
9. Half plane of 3x+8y-24 = 0 not containing origin.
10. Half plane of x+y-1 = 0 not containing origin, including the line.
11. Half plane of y axis containing negative x axis.
12. Half plane of y = 5 not containing origin.

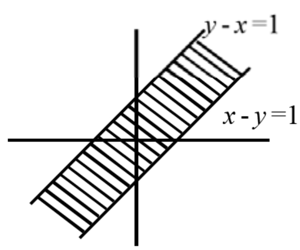
Exercises 7 (c)

1.



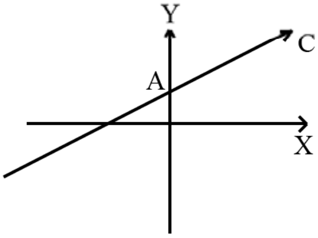
S.S. is interior of the quadrilateral OABC.

2.



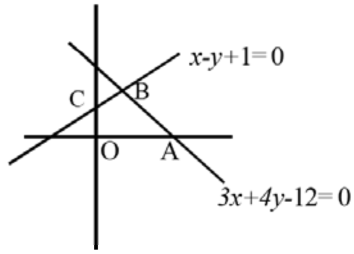
S.S. is intersection of half planes of both lines containing the origin.

3.



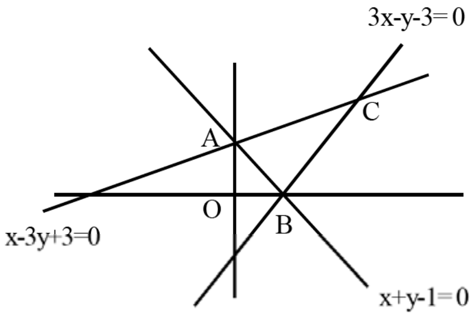
The S.S. is Interior of triangle $\angle YAC$

4.



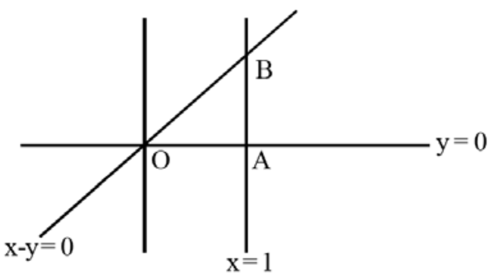
S.S = interior of the quadrilateral OABC

5.



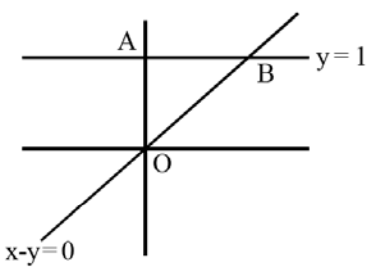
Interior of ΔABC.

6.



Interior of ΔOAB.

7.



Exercises 8 (*a*)

1. 9, 2. 36, 3. 10, 4. 5, 5. 6, 6. 256, 7. 1332, 8. 16, 9. 16.

Exercises 8 (b)

1. 120 2.(i) 1680, (ii) 720, (iii) 1 3. (i) 30240, (ii) 840, (iii) 144 (iv) $\frac{101}{3628800}$

(v) 64, (vi) 40320 5. 8 6. 6 7. 8 8. $m = 6, n = 2$

10. 8, 11. 36, 12. 150, 13. 420, 14. 60, 15. 216, 16. 12880

17. $\frac{1}{8}$ (12) !, 18. 480, 19. 12, 20. 27720, 21. (i) n !, (ii) 0, (iii) $\frac{\underline{|m}}{\underline{|m-n}}$,

22. 12

Exercises 8 (c)

1. (i) 220 (ii) 455 (iii) 252 (iv) 70 (v) 256

2. (i) 15 (ii) 17 3. $n = 8, r = 4$, 4. $\frac{1}{2} n (n-3)$ 5. $2^{mn} - 1$

6. 7200 7. 700 8. 84,50 9. 120 10. 32 11. 1770 12. $\underline{|10} \times \underline{|11}$

13. $(\underline{|7})^2$ 14. 7 (besides 1155) 15. 14 (besides 1 and 210) 16. 2^k–2 (besides 1 and n)

17. $(r_1 + 1)(r_2 + 1) \ldots (r_n + 1) - 1$ (including N), 18. 4, 19. 4, 20. 57, 21. ${}^7C_3$

22. 30 23. 78 24. $\frac{r}{2}(2n + r - 1) + 1$ 25. $\underline{|6} \times \underline{|5}$

Exercises 9 (a)

1. For $n = 6$; 1, 6, 15, 6, 1 ; for $n = 7$; 1, 7, 21, 35, 21, 7, 1

3. $24x^{-6}$ 5. 1·0510100501 6. (a) F, (b) F, (c) F, (d) T

7. (a) $\frac{1}{x}$ (b) $n + 1$ (c) $\frac{r}{n - r + 1}$ (d) 5 and 5, 8. (a) 20 (b) $126x$, $\frac{126}{x}$ (c) $70x^6y^6$

9. $252\, x^{10} y^{-10}\, a^{20}$ 10. (a) $210 \times 6^6 x^2 a^{12}$ (b) yes, $-252 \times 6^5\, a^{15}$ (sixth term)

11. (a) $252\, a^{35}$ (b) No. 12. (a) 2220, (b) 2352

Exercises 9 (b)

2. (i) $n.2^{n-1}$ (ii) $(n+2)\ 2^{n-1}$ 3. 1

5. (i) 0 (ii) $n(n-1)\ 2^{n-2}$ (iii) $n(n+1)2^{n-2}$ (iv) $\frac{1}{n+1}$ 9. $a = 9,\ b = 10$

EXERCISE - 10 (a)

1. (iii)
2. (i) No (ii) Yes (iii) No
3. $-1 < r < 1$
4. $t_n = 0$
5. (iv)
6. p+q-n
7. -(p+q)
8. $\sum_{n=1}^{\infty} 4(-\frac{1}{3})^{n-1}$
9. $\sum_{n=1}^{\infty} \frac{7}{2^{n-1}}$
10. (i) $\frac{1-a^n}{(1-a)^2} - \frac{na^n}{1-a}$ (ii) $\frac{1}{(1-y)(1-xy)}$ (iii) $\frac{25}{12}$ (iv) $\frac{1+x-x^2}{(1-x)^3}$

    (v) $-2 + 2^{n+1} - n2^{n+2}$
11. (i) 1, (ii) $\frac{1}{4}$ (iii) $\frac{1}{60}$ (iv) $\frac{1}{3}$ (v) 1
12. (i) $\frac{1}{12}n(n+1)(n+2)$ (ii) $\frac{1}{4}n(n+1)(n+2)(n+3)$

    (iii) $\frac{1}{5}n(n+1)(n+2)(n+3)(n+4)$ (iv) $\frac{1}{12}(3n-1)(3n+2)(3n+5)(3n+8) + \frac{20}{3}$

    (v) $\frac{1}{6}n(n+1)(2n+13)$ (vi) $\frac{1}{4}n^2(n+1)^2 + n(n+1) + \frac{1}{6}n(n+1)(2n+1) + 2n$

vii) $n^2(n+1)^2 + \frac{2}{3} n(n+1)(2n+1) + \frac{1}{2} n(n+1)$

13. (i) $2^n + 4n - 1$. (ii) $\frac{1}{6} n(2n^2 - 3n + 25)$

14. 20615 (ii) 4335

EXERCISE - 10 (b)

1. (i) $1 + \text{Log}_e 2 + \frac{x^x(\text{Log}_e 2)^2}{2!} + \frac{x^3\, \text{Log}_e 2)^3}{3!} + \ldots..$

(ii) $-1 + 5x - 4x^2 + \frac{56}{3}x^3 + 0.x^4 + \ldots..$

(iii) $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots..$

(iv) $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots..$

(v) $e\left[1 + x + x^2 + \frac{5}{6}x^3 + \frac{5}{8}x^4 + \ldots\ldots\right]$

2. $e^{x^2} - e^{y^2}$

Exercises 11 (a)

1. (i) $\sqrt{34}$ (ii) $3\sqrt{5}$ 2. $-3, 5$ 3. $\frac{6}{5}, 4$ 4. (i) $\left(\frac{1}{2}, \frac{-1}{2}\right)$ (ii) $\left(\frac{-7}{8}, \frac{-1}{2}\right)$

5. $\frac{5}{2}$ 6. 8 7. 10 8. (i) $\frac{1}{\sqrt{3}}$ (ii) 1 (iii) $\sqrt{3}$ (iv) -1

9. (i) 30° (ii) 45° (iii) 60° (iv) 135°

10. (i) $\tan^{-1}[\pm(2-\sqrt{3})]$ (ii) $\tan^{-1}[\pm(2+\sqrt{3})]$ 12. $(21, -11)$ 15. $(0, 4)$ 16. $(-10, 19)$

17. 2 : 5; 3 : 4 18. $\left(-\frac{5}{2}, -1\right)$ 19. (1, 3), (3, –1), (–5, 7) 21. $7x + 4y =$ 15

Exercises 11 (b)

1. (a) –9 (b) 2 (c) 4 (d) $\frac{10}{3}$ (e) 60°
2. (a) F (b) T (c) F (d) T (e) F (f) T (g) F (h) F (i) F (j) F
4. $3x - 4y + 3 = 0$ 5. $x^2 + y^2 + 10x + 9 = 0$
6. (a) $x + \sqrt{3}y + \sqrt{3} - 1 = 0$ (b) $y = 2$ (c) $x + 3y + 7 = 0$

   (d) $x + y = 1$, $3x - 2y + 12 = 0$ (e) $\sqrt{3}y + x \pm 4 = 0$ (f) $x - 3y - 5 = 0$ (g) $2ax - 2by = a^2 - b^2$ (h) $2b'x - 2a'y = a'b' - ab$ (i) $2y = 3x$, $y = 6x$ (j) $4x - 3y + 22 = 0$ (k) $y - a\cos^3\theta = (x - a\sin^3\theta)\cot\theta$ (l) $\frac{x}{5} - \frac{y}{10} = 1$ (m) $\frac{x}{l} - \frac{y}{\beta} = 2$
7. (a) $3x + 4y + 3 = 0$, $3x + 4y + 17 = 0$ (b) $(an - cl)x + (nb - cm)y = 0$

   $(an + cl)x + (bn + cm)y + cn = 0$, $(a^2 + b^2)n^2 = (l^2 + m^2)c^2$

   (c) $x + y = 5$ (d) $10x - 25y - 46 = 0$ (e) $8x - 11y + 6 = 0$
8. – 9, 6
9. (i) $x - y + 1 = 0$ (ii) $8x - 5y = 1$ (iii) $7x - 4y + 1 = 0$
11. (a) $11x - y = 35$, $x + 11y + 19 = 0$ (b) $x = 3$, $y = 4$, $\frac{9}{2}$
12. $2x - 3y = 0$, $3x - 4y = 0$, $\left(\frac{15}{13}, \frac{10}{13}\right)$, $\left(\frac{28}{25}, \frac{21}{25}\right)$
13. (a) $\frac{49}{13}$ (b) $\frac{6}{5}, \frac{3}{5}, \frac{9}{5}$ (c) $\frac{40}{13}$ (d) $2\sqrt{13}$ (e) $a\cos\frac{\alpha - \beta}{2}$
14. $\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}, \frac{1}{5}$
18. $Ax + By + C = \pm(Bx - Ay + Ak - Bh)$ 19. 15° or 75° to x - axis
20. $2x - y + 3 = 0$, $2x - 3y - 6 = 0$, $7x - 7y - 9 = 0$
21. $x - y = 2$, $x + 6x = 27$, $\left(\frac{31}{7}, \frac{25}{7}\right)$ 23. (a) $3x + 3y + 2 = 0$ (b) $64x - 112y = 55$

24. (a) $x = y = \dfrac{p}{1+\sin\alpha+\cos\alpha}$ (b) $\left(-\dfrac{6}{7}, \dfrac{2}{7}\right)$ 25. $2x + y - 1 = 0, \left(\dfrac{8}{5}, \dfrac{4}{5}\right)$

26. (a) $y = \pm 2x$ (b) $2x + y = 0, x - 3y = 0$

(c) $y = (\tan\theta - \sec\theta)x$, $(\tan\theta + \sec\theta)x + y = 0$ (d) $x = 0, 3x + 4y = 0$

27. (a) $(y - mx)(y - nx) = 0$ (b) $y^2 = 9x^2$ (c) $4x^2 - 9y^2 + 4x + 1 = 0$ (d) $(x - y)(x + 2y + 5) = 0$

28. (c) and (d) 29. (a) ± 3 (b) -11 30 (a) -3 (b) Follow 8.7 Example 16.

(c) $\dfrac{a}{r} = \dfrac{b}{p} = \dfrac{-h}{q}$, $\dfrac{h-\sqrt{h^2-ab}}{b} = \dfrac{q-\sqrt{q^2-pr}}{r}$

31. (a) $\tan^{-1}\dfrac{2\sqrt{5}}{3}$ (b) $\tan^{-1}5$ (c) $45°$

32. (a) $xy = 0$ (b) $x^2 - y^2 + 14xy = 0$ (c) $x^2 - y^2 = (\cos\theta + \sin\theta)xy$ (d) $x^2 - y^2 + \cot\theta\, xy = 0$

34. $x^2 + y^2 = 4$ 35. $2x^2 + 3y^2 + 4xy + 1 = 0$ 36. $45°$ 37. $x + 2y = 0$

Exercises 12 (a)

1. (a) $(-1, 3)$ (b) -2 (c) outside (d) $\pm 4\sqrt{2}$ (e) 2
2. (a) F (b) F (c) T (d) F (e) T
3. (a) $(x-1)^2 + (y-4)^2 = 18$ (b) $(x+2)^2 + (y-3)^2 = 13$

   (c) $(x-3)^2 + (y-2)^2 = 4$ (d) $(x+1)^2 + (y-4)^2 = 1$

   (e) $(x+5)(x-7) + (y-3)(y-5) = 0$ (f) $(x+5)^2 + (y \pm 5)^2 = 25$

   (g) $x^2 + y^2 - 5x = 0$ (h) $4x^2 + 4y^2 - 5x + 8y - 18 = 0$

   (i) $5(x^2 + y^2) - 11x - 8y - 67 = 0$ (k) $(x-3)^2 + (y+12)^2 = 144$ or $x^2 + y^2 - 6x - 6y + 9 = 0$

   (l) $(x-3)^2 + (y+2)^2 = 4$ (m) $x^2 + y^2 - ax - by = 0$ or $x^2 + y^2 + 10 + 20y + 25 = 0$

   (n) $(x-3)^2 + (y - 3\sqrt{2})^2 = 18$
4. (a) $(-3, 2)$, 5 (b) $\left(\dfrac{-g}{a}, \dfrac{-f}{a}\right)$, $\dfrac{\sqrt{q^2+f^2-ak}}{a}$ (c) $\left(\dfrac{1}{2}, -\dfrac{3}{2}\right)$, $\dfrac{5}{2}$
5. (a) $x^2 + y^2 = 25$, $(0, 0)$, 5 (b) $16(x^2 + y^2) - 69x + 54y - 232 = 0$; $\left(\dfrac{69}{32}, \dfrac{-27}{16}\right)$

   (c) $b(x^2 + y^2) + (a^2 - b^2)y - a^2b = 0$, $\left(0, \dfrac{b^2-a^2}{2b}\right)$, $\dfrac{a^2+b^2}{2b}$

   (d) $2(x^2 + y^2) - 11x - 10y - 43 = 0$, $\left(\dfrac{11}{4}, \dfrac{5}{2}\right)$

6. (a) $3x^2 + 3y^2 - 2x - 10y = 0$ (c) $x^2 + y^2 - 4x - 3y = 0$
(d) $x^2 + y^2 - 6x + 8y = 0$ (e) $98(x^2 + y^2) - 196x - 392y - 95 = 0$

7. (0, 2), (0, 6), 4, (3, 0), (4, 0); 1

9. $(x+1)(x+4) + (y-1)(y-5) = 0$ 10. $x^2 + y^2 - 2x - 2y + 1 = 0$

11. (a) $(x-3)^2 + (y-2)^2 = \frac{9}{5}$ (b) $(5x+12)^2 + (5y+16)^2 = 100$ (c) $x^2 + y^2 = 130$
(d) $(2x-a)^2 + 4(y+a)^2 = a^2$ or $16x^2 + 16y^2 - 16ax + 72ay + 81a^2 = 0$
(e) $x^2 + y^2 - 2kx - 2ky + k^2 = 0$ $[2k = a + b \pm \sqrt{a^2+b^2}]$

13. (a) $3x - 4y = 25,\ 4x + 3y = 0$ (b) $-7x + 10y - 44 = 0,\ 10x + 7y - 1 = 0$
(c) $2x + 5y - 40 = 0,\ 2x - 5y - 10 = 0$ (d) $(x_1 + g)(x_2 + g) + (y_1 + f)(y_2 + f) = 0$
(e) 9, 7, $\sqrt{130}$

14. (a) $x + y = \pm 3\sqrt{2}$ (b) $3x - 4y + 5 \pm 15\sqrt{5} = 0$
(c) (2, 1), $x - 7y - 45 = 0$ (*f*) $4y - 3x + 2 = 0,\ 4y - 3x + 52 = 0,\ k = 2, 5$

15. (a) $4\sqrt{2}$ (b) 1 (c) $5\sqrt{2}$

16. (a) $\left(-\frac{17}{5}, \frac{11}{5}\right)$ (b) $3x + 4y = 10$ 17. $3x^2 + 3y^2 + 30x + 32y - 8 = 0$

18. $x^2 + y^2 - 3ax + by = 0$ 19. $8x + 4y + 3 = 0$ 20. $3x - 4y + 6 = 0$

Exercises 12 (b)

1. (a) $2y - 3 = 0$ (b) 1 (c) 3 (d) 8 (e) $y = 0$
(f) $y = 0$ (g) $x + 1 = 0$ (h) $\frac{1}{\sqrt{3}}$ (i) $\frac{3}{5}$ (j) 3 (k) $\frac{8}{5}$ (l) $x = 0$
(m) no where (n) $\frac{\sqrt{21}}{3}$ (o) $\frac{32}{3}$ (p) $\frac{5}{3\sqrt{2}}$

2. (a) T (b) F (c) F (d) T (e) F (f) T (g) F (h) T (i) T (j) F (k) F (l) F (m) F (n) T (o) F

3. (a) $x^2 = 12y$ (b) $y^2 + 8x = 0$ (c) $(y+2)^2 = 36(x-6)$
(d) $(x+2)^2 = 12(y-1)$ (e) $y = 6x$ (f) $y^2 + 4x = 0$
(g) $x^2 + 10y = 0$ (h) $y = x^2 + 2x + 2$ (i) $25y^2 + 13y + 84x - 180 = 0$
(j) $(y-3)^2 = 12(x-1)$ (k) $(x-1)^2 + 12(y+1) = 0$

4. (a) $\frac{x^2}{9} + \frac{y^2}{2b} = 1$ (b) $\frac{x^2}{49} + \frac{y^2}{25} = 1$ (c) $\frac{x^2}{36} + \frac{y^2}{11} = 1$

(d) $\frac{x^2}{25}+\frac{y^2}{4} = 1$ (e) $\frac{(x-5)^2}{64}+\frac{(y-4)^2}{25} = 1$ (f) $(x+3)^2+(y-3)^2=9$

(g) $2x^2+4y^2=25$ or $4x^2+2y^2=25$ (h) $x^2+4y^2=4$ or $4x^2+y^2=4$

(i) $\frac{x^2}{25}+\frac{y^2}{16} = 1$ (j) $5x^2+8y^2=77$ (k) $\frac{(x-3)^2}{9}+\frac{(y-4)^2}{4} = 1$

5. (a) $\frac{x^2}{4}-\frac{y^2}{12} = 1$ (b) $y^2-x^2=1$ (c) $\frac{x^2}{4}-\frac{y^2}{16} = 1$

(d) $3y^2-x^2=9$ (e) $2x^2-y^2=8$ (f) $\frac{y^2}{36}-\frac{x^2}{13} = 1$

(g) $\frac{(x-1)^2}{9}-\frac{(y+2)^2}{25} = 1$ (h) $\frac{(x-2)^2}{9}-\frac{(y+3)^2}{16} = 1$ (i) $45x^2-16y^2=341$

(j) $\frac{(y-5)^2}{9}-\frac{(x-1)^2}{4} = 1$

6. (a) $\left(\frac{-5}{4},\ -2\right)$, $\left(\frac{-1}{4},\ -2\right)$, $\left(-\frac{1}{4},\ 0\right)$, $\left(-\frac{1}{4},\ -4\right)$, 4, $y+2=0$, $4x+9=0$

(b) $\left(-\frac{3}{2},\ -\frac{15}{8}\right)$, $\left(-\frac{3}{2},\ -\frac{11}{8}\right)$, $\left(-\frac{1}{2},\ -\frac{11}{8}\right)$, $\left(-\frac{5}{2},\ -\frac{11}{8}\right)$, $\left(-\frac{5}{2},\ -\frac{11}{8}\right)$, $2x+3=0$, $8y+9=0$

(c) $\left(-\frac{1}{2},\ -\frac{3}{4}\right)$, $\left(-\frac{1}{2},\ -1\right)$, $(0, -1)$, $(-1, -1)$, 1, $2x+1=0$, $2y+1=0$

(d) $(-16, -17)$, $\left(-\frac{61}{4},\ -7\right)$, $\left(-\frac{61}{4},\ -\frac{11}{2}\right)$, $\left(-\frac{61}{4},\ -\frac{17}{2}\right)$, $y+7=0$, $4x+67=0$

7. (a) $(-1,-1)$, $(-2,-1)$, $(0,-1)$, $(1,-1)$, $(-3,-1)$, $(-1,-1\pm\sqrt{3})$ $(-1\pm 1,-1\pm\frac{3}{2})$, $x=-1\pm 4$ $e=\frac{1}{2}$

(b) $\left(-\frac{1}{2},\ -\frac{3}{2}\right)$, $\left(-\frac{1}{2}\ \pm 2,\ -\frac{3}{2}\right)$, $\left(-\frac{1}{2}\ \pm 2\sqrt{2},\ -\frac{3}{2}\right)$, $\left(-\frac{1}{2},\ -\frac{3}{2}\pm 2\right)$,

$\left(-\frac{1}{2}\ \pm\ 2,\ \frac{-3}{2}\pm\sqrt{2}\right)$, $2x-7=0$, $2x+9=0$, $\frac{1}{\sqrt{2}}$

(c) $(3, -4)$, $(3\pm 1, -4)$, $(3\pm\sqrt{3}, -4)$, $(3, -4\pm\sqrt{2})$, $\left(3\pm\ 1, -\ 4\ \pm\frac{2}{\sqrt{3}}\right)$, $x=3\pm 3$, $e=\frac{1}{\sqrt{3}}$

(d) $(-2, 1)$, $(-2, 1\pm\sqrt{5})$, $(-2, 1\pm 3)$, $(-2\pm 2, 1)$ $\left(-2\pm\frac{4}{3},\ 1\pm\sqrt{5}\right)$, $y=1\pm\frac{9}{\sqrt{5}}$, $e=\frac{\sqrt{5}}{3}$

8. (a) $(3, -1)$, $(3\pm\sqrt{2}\ -1)$, $(3\pm\sqrt{3}, -1)$, $(3,\ -1\pm 1)$, $\left(3\ \pm\sqrt{3},\ -1\pm\frac{1}{\sqrt{2}}\right)$, $x=3\pm\frac{2}{\sqrt{3}}$, $e=\frac{\sqrt{3}}{2}$

(b) $(0, 5)$, $(0, 5\pm 2)$, $(0, 5, \pm\sqrt{13})$, $(\pm 3, 5)$ $\left(\pm\frac{9}{2},\ 5\pm\sqrt{13}\right)$, $y=5\ \pm\frac{4}{\sqrt{13}}$ $e=\frac{\sqrt{13}}{2}$

(c) $(1, 6),(1 \pm 2, 6), (1 \pm \sqrt{63}, 6)$ $(1, 6 \pm 7)$, $\left(1\pm \sqrt{53},\ 6\pm \frac{49}{2}\right)$, $x = 1 \pm \frac{4}{\sqrt{153}}$, $e = \frac{\sqrt{53}}{2}$

(d) $(0, -1), (\pm 2\sqrt{2}, -1)$ $(\pm 2\sqrt{5}, -1)$, $(0, -1 \pm 2\sqrt{3})$, $(\pm 2\sqrt{5}, -1 \pm 3\sqrt{2})$.

$x = \pm \frac{4}{\sqrt{5}}$, $e = \sqrt{\frac{53}{2}}$

10. $9y^2 = 4ax$ 11. (b) $\tan^{-1}\left(\frac{8}{15}\right), \tan^{-1}\left(\frac{24}{7}\right)$

12. (a) $9x - 6y + 4a = 0$, $18x + 27y - 44a = 0$
(b) $y - x - a = 0$, $x + y + a = 0$, $y + x - 3a = 0$, $y - x + 3a = 0$
(c) $x - y + a = 0$, $9x + 3y + a = 0$, $x + y - 3a = 0$, $9x - 27y - 19a = 0$

(e) $2x + y + 1 = 0$, $\left(\frac{1}{2}, -2\right)$ (g) $ln = am^2$

13. (a) $8x^2 + 9y^2 - 288 = 0$ (b) $\frac{(x-4)^2}{36} + \frac{(y-2)^2}{32} = 1$

(c) $\frac{x^2}{7} + \frac{y^2}{3} = 1, \frac{6}{\sqrt{7}}$ (d) $\frac{(x+1)^2}{22} + \frac{3(y-4)^2}{44} = 1$

14. (a) $3x + 2\sqrt{5}y - 18 = 0$, $9(y - \sqrt{5}) = 2\sqrt{5}(3x - 8)$

(b) $\pm x \pm \sqrt{3}y = 3$, $x \pm 3y = 1 + 2\sqrt{3}$, $x \pm 3y = 2\sqrt{3} - 1$

(c) $\left(\frac{-8}{5}, \frac{9}{5}\right)$ (d) $y = x \pm 3$ (e) $6x - 12y \pm 5 = 0$

15. (a) $3y^2 - 12y - x^2 + 9 = 0$ (b) $x^2 - 2y^2 = 6$ (c) $(\pm\sqrt{13}, 0)$, $\frac{8}{3}$

16. (a) $2y - x = 1$, $18x + y + 55 = 0$, (b) $33y - 20x = 11$

(c) $3x - 2y \pm 3\sqrt{3} = 0$ (d) $(-5, 3)$, $x - y + 8 = 0$

EXERCISE 13

1. (a) $\left(\frac{-7}{6}, \frac{77}{18}, \frac{4}{3}\right)$; (b) 5 ; (c) $(-6, 3, -4)$; (d) 1

2. (a) x, y and z axes respectively.
   (b) zx-plane, xy-plane and yz-plane

3. (a) 1st, 2nd and 4th.
   (b) (i) (7,-5,0); (ii) (0,-5,3); (iii) (7,0,3); (iv) (7,0,0); (v) (7,0,3); (vi) (7,-5,0)

4. (a) Non-coplanar, No

5. (i) and (ii)

6. (i) a=0, (ii) y=0, (iii) c=0, (iv) y=z=0, (v) x=z=0, (vi) a=b=0

7. (i) y-axis; (ii) z-axis; (iii) x-axis;

8. z, x and y axes

11. (a) $10+5\sqrt{2}$; (f) x-2z = 0

12. (a) -1:2; (b) 5:4 ; (c) $\left(\frac{a_1+a_2+a_3}{3}, \frac{b_1+b_2+b_3}{3}, \frac{c_1+c_2+c_3}{3}\right)$; (d) $\left(\frac{-7}{6}, \frac{77}{18}, \frac{4}{3}\right)$; (e) 1:2

EXERCISE 14 (a)

1. 7 2. 3 3. 4 4. 3 5. 7 6. 1 7. ∞ 8. 1 9. 0 10. 5

EXERCISE 14 (b)

2. False [since $|\,l\,| = |-l\,|$

4. (i) –1 (ii) –2 (iii) $\frac{1}{3}$ (iv) 6 (v) 3 (vi) $\frac{1}{32}$ (vii) $\frac{3}{20}$ (viii) 8/5 (ix) 27 (x) $-\frac{1}{4}$ (xi) $\frac{1}{32}$ (xii) $3x^2$ (xiii) $4x^3$ (xiv) $\frac{m}{n}$ (xv) 0 (xvi) ∞

5. (i) $\frac{2}{3}$ (ii) $\frac{3}{2}$ (iii) 0 (iv) ∞ (v) $\frac{1}{4}$ (vi) 1 (vii) $\frac{1}{5}$ (viii) 1 (ix) 6 (x) $\frac{1}{2}$ (xi) $\frac{1}{3}$ (xii) $\frac{1}{4}$ (xiii) $\frac{4}{3}$ (xiv) 0

6. (i) 1 (ii) - (vii) does not exist. (viii) 1 (ix) 1 (x) - (xvi) does not exist (xvii) $\underset{x\to 0}{\text{Lt}}\, f(x)$ does nto exist, $\underset{x\to 1}{\text{Lt}}\, f(x)$ = - 1

EXERCISE 14 (c)

1. (i) $\frac{1}{2}$ (ii) $\frac{3}{5}$ (iii) $\frac{m}{n}$ (iv) α (v) $\frac{1}{2}$ (vi) $\frac{\pi}{180}$ (vii) 1 (viii) $\frac{1}{2}$ (ix) $\frac{3}{4}$ (x) –1 (xi) $\frac{3}{4}$ (xii) ∞ (xiii) 1 (xiv) 1 (xv) $\frac{1}{4}$ (xvi) 1 (xvii) 1 (xviii) $\frac{3}{4}$ (xix) 1

2. (i) $\sin\alpha - \alpha\cos\alpha$ (ii) 0

3. (i) cos x (ii) – sin x (iii) $\sec^2 x$ (iv) – cosec x cot x (v) sec x tan x (vi) – $\text{cosec}^2 x$ (vii) $\frac{1}{2\sqrt{x}}$ (viii) $\frac{1}{x}\log_e e$ (ix) $\frac{1}{x}\log_a e$ (x) $a^x \log a$ (xi) e^x (xii) 0 (xiii) 2 cos x (iv) $-\frac{1}{2x^{3/2}}$

4. (i) $\frac{1}{2}$ (ii) 1 (iii) 2 (iv) 2 (v) 1 (vi) a (vii) 2a (viii) 1 (ix) 2 (x) $\log\frac{a}{b}$ (xi) log 2 (xii) 2 log a (xiii) 3 log 3 (xiv) $\frac{\log 3 - \log 2}{\log 4 - \log 3}$ (xv) 2 log 2

5. (i) $\frac{1}{2}$ (ii) $\frac{1}{2\sqrt{2}}$ (iii) $\frac{1}{2\sqrt{5}}$ (iv) $-\frac{1}{\sqrt{3}}$ (v) $\frac{1}{4}$ (vi) 3 (vii) $\frac{1}{4a\sqrt{a-b}}$ (viii) $\frac{n}{m}$ (ix) 2 (x) 0 (xi) 0 (xii) $\frac{2}{3}$ (xiii) $\frac{9}{2}$ (xiv) $\frac{3}{2}$ (xv) $m = n \Rightarrow \frac{a_m}{b_n}$, $m < n \Rightarrow 0$, $m > n \Rightarrow \infty$

6. (i) 0 (ii) log a (iii) $\frac{1}{2}$ (iv) $\frac{1}{2}$ (v) a^2 (vi) 1 (vii) $\log\frac{1}{2}$ (viii) 3 (ix) a (x) b (xi) 1 (xii) $\log\frac{3}{2}$ (xiii) ∞ (xiv) 1

7. (i) $-\infty$ (ii) does not exist (iii) does not exist (iv) 0 (v) ∞ (vi) does not exist

8. (i) $\frac{1}{2}$ (ii) 2 (iii) 3 (iv) 5 (v) 2

Exercises 15

1. (i) $\frac{(n+1)}{2}$

   (ii) If n is even, say $n = 2m$, then M.D. $= \frac{m}{2}$; if n is odd, say $n = 2m+1$,

   then M.D. $= \frac{m(m+1)}{(2m+1)}$

2. Variance $= \frac{(n^2-1)}{12}$

3. Mean value $= \frac{(2n+1)}{3}$; variance $= \frac{(n^2+n-2)}{18}$

4. (a) Mean value $= n$; Variance $= \frac{(n^2-1)}{3}$ (b) Mean value $= n+1$; Variance $= \frac{(n^2-1)}{3}$

6. Mean = 50.25; M D = 7.075; Variance = 82.42

7. For A : Mean $= \frac{4}{3}$, Variance $\simeq 1.244$

   For B : Mean $= \frac{5}{3}$; Variance $\simeq 1.377$

8. (i) Mean = 5; Variance = 6.5; c $\cong$ .51 (ii) Mean = 7; Variance = 4.67; c $\cong$ .32

   (iii) Mean = 4; Variance $= \frac{2}{3}$ c $\cong$ `.204

10. Mean $= a + nd$; M.D. $= \frac{dn(n+1)}{2n+1}$, Variance $= \frac{1}{3}\, n\,(n+1)d^2$

11. Mean value = 0; Variance = $\dfrac{(n\sigma^2{}_x + m\sigma^2{}_y)}{(m+n)}$

13. Gold

Exercises 16 (a)

1. (i) $\frac{1}{2}$ (ii) $\frac{3}{4}$ (iii) $\frac{3}{4}$

2. (i) $\frac{1}{8}$ (ii) $\frac{7}{8}$ (iii) $\frac{4}{8}$

3. (i) $\frac{3}{36}$ (ii) $\frac{6}{36}$ (iii) $1-\frac{3}{36}$

4. (i) $\frac{3}{36}$ (ii) $\frac{6}{36}$ (iii) $1-\frac{3}{36}$

5. (i) $\frac{1}{5}$ (ii) $\frac{1}{2}$ (iii) $\frac{1}{10}$ (iv) $\frac{3}{5}$

6. (i) $\frac{3}{4}$

7. $\frac{49!\ 4!}{52!}$

8. $\frac{5}{7}$

9. $\frac{20}{216}$

10. (i) $\frac{2}{6}$ (ii) $\frac{3}{6}$ (iii) $\frac{1}{6}$

11. (i) $\frac{9}{24}$ (ii) $\frac{15}{24}$

12. (i) $\frac{5}{8}$ (ii) $\frac{5}{8}, \frac{4}{8}$ (iii) $\frac{3}{4}$ (iv) $\frac{3}{8}$

(v) $P(A \cap B^c) = P(A - B) = P(A) - P(A \cap B) = \frac{1}{8}$ (vi) $\frac{2}{8}$

13. (i) $\frac{1}{3}$ (ii) $\frac{2}{3}$ (iii) $\frac{1}{12}$ (iv) $\frac{7}{12}$

14. (i) $\dfrac{{}^{20}C_3 \times {}^{80}C_7}{{}^{100}C_{10}}$ (ii) $1 - \dfrac{({}^{80}C_{10} + {}^{20}C_1 \times {}^{80}C_9 + {}^{20}C_2 \times {}^{80}C_8)}{{}^{100}C_{10}}$

15. (i) $\frac{20}{36}$ (ii) $\frac{11}{36}$

16. (i) $\frac{5!\ 4!}{8!}$

(ii) There are two cases : The row can start with a boy; the row can start with a girl answer :

$2\left(\frac{4!\ 4!}{8!}\right)$

17. (i) $\dfrac{^{9}C_2}{^{10}C_3}$ (ii) $\dfrac{(2 \times {}^{9}C_2 - {}^{8}C_1)}{^{10}C_3}$ (iii) $\dfrac{^{8}C_1}{^{10}C_3}$

18. (i) $\dfrac{^{15}C_6}{^{40}C_6}$ (ii) $\dfrac{^{25}C_6}{^{40}C_6}$ (iii) $\dfrac{(^{25}C_3 \times {}^{15}C_3)}{^{40}C_6}$

(iv) $\dfrac{(^{15}C_4 \times {}^{25}C_2)}{^{40}C_6}$ (v) $1 - \dfrac{^{25}C_6}{^{40}C_6}$

19. Since the committee is to have at least 2 boys and 2 girls, the sample space is of size $n = (^{20}C_2 \times {}^{10}C_4) + (^{20}C_3 \times {}^{10}C_3) + (^{20}C_4 \times {}^{10}C_2)$

(i) $^{20}C_3 \times \dfrac{^{10}C_3}{n}$ (ii) $\dfrac{^{20}C_4 \times {}^{10}C_2}{n}$

20. (i) $\dfrac{50}{120}$ (ii) $\dfrac{70}{120}$ 21. (i) $\dfrac{2}{7}, \dfrac{5}{7}$ (ii) $\dfrac{3}{7}, \dfrac{4}{7}$

22. $\dfrac{6!}{6^6}$ 23. $\dfrac{14}{60}$

24. $P(A \Delta B) = P(A) + P(B) - 2P(A \cap B)$ 25. $\dfrac{2 \times 3!\ 5!}{8!}$

Exercises 16 (b)

1. (i) $\dfrac{2}{7}$ (ii) $\dfrac{1}{7}$

2. (ii) $P(n) = \dfrac{n}{21}$ for $n = 1, 2, 3, 4, 5, 6$ (ii) $\dfrac{12}{21}$ (iii) $\dfrac{9}{21}$ (iv) $\dfrac{10}{21}$

3. (i) $\dfrac{10}{13}$ (ii) $\dfrac{3}{13}$

SYLLABUS

MATHEMATICS (+2 1st Year)

Course Structure

Unit	Topic	Marks	No. of Periods
I	Sets and Functions	29	60
II	Algebra	37	70
III	Co-ordinate Geometry	13	40
IV	Calculus	06	30
V	Mathematical Reasoning	03	10
VI	Statistics and Probability	12	30
	Total	**100**	**240**

UNIT - I : Sets and Functions

1. Sets

Sets and their representations. Empty set, Finite and Infinite sets, Equal sets, Subsets, Subsets of a set of real numbers especially intervals (with notations), Power set, Universal set, Venn diagrams, Union and Intersection of sets, Difference of sets, Complement of a set, Properties of Complement Sets, Practical Problems based on sets.

2. Relations & Functions

Ordered pairs, Cartesian product of sets. Number of elements in the Cartesian product of two finite sets. Cartesian product of the sets of real (upto $R \times R$). Definition of relation, pictorial diagrams, domain, co-domain and range of a relation. Function as a special kind of relation from one set to another. Pictorial representation of a function, domain co-domain and range of a function. Real valued functions, domain and range of these functions: Constant, identity, polynomial, rational, modulus, signum, exponential, logarithmic and greatest integer function, with their graphs, Sum, difference, product and quotients of functions.

3. Trigonometric Functions

Positive and negative angles. Measuring angles in radians and in degrees and conversion of one into other. Definition of trigonometric functions with the help of unit circle. Truth of $\sin^2 x + \cos^2 x = 1$, for all x. Signs of trigonometric functions. Domain and range of trigonometric functions and their graphs. Expressing $\sin(x \pm y)$ and $\cos(x \pm y)$ in terms of $\sin x$, $\sin y$, $\cos x$ & $\cos y$ and their simple application. Deducing identities like the following :

$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}, \cot(x \pm y) = \frac{\cot x \cot y \mp 1}{\cot y \pm \cot x}$$

$$\sin x + \sin y = 2 \sin \frac{x+y}{2} \cos \frac{x-y}{2}, \cos x + \cos y = \frac{x+y}{2} \cos \frac{x-y}{2},$$

$$\sin x - \sin y = 2 \cos \frac{x+y}{2} \sin \frac{x-y}{2}, \cos x - \cos y = -2 \sin \frac{x+y}{2} \sin \frac{x-y}{2},$$

Identities related to sin $2x$, cos $2x$, tan $2x$, sin $3x$, cos $3x$ and tan $3x$. Trigonometric equations Principal solution, General solution of trigonometric equations of the type $\sin x = \sin y$, $\cos x = \cos y$ and $\tan x = \tan y$. Proof and Simple applications of sine and cosine formula.

UNIT-II : Algebra

1. Principle of Mathematical Induction

Process of the proof by induction, motivation the application of the method by looking at natural numbers as the least inductive subset of real numbers. The principle of mathematical induction and simple applications.

2. Complex Numbers and Quadratic Equations

Need for complex numbers, especially $\sqrt{-1}$, to be motivated by inability to solve some of the quardratic equations; Algebraic properties of complex numbers. Argand plane and polar representation of complex numbers. Statement of Fundamental Theorem of Algebra, solution of quadratic equations in the complex system. Square root of a complex number.

3. Linear Inequalities

Linear inequalities. Algebraic solutions of linear inequalities in one variable and their representation on the number line. Graphical solution of linear inequalities in two variables. Graphical solution of system of linear inequalities in two variables.

4. Permutations and Combinations

Fundamental principle of counting, factorial n. (n!), Permutations and combinations, derivation of formulae and their connections, simple applications.

5. Binomial Theorem

History, statement and proof of the binomial theorem for positive integral indices. Pascal's triangle, General and middle term in binomial expansion, simple applications.

6. Sequence and Series

Sequence and Series, Arithmetic Progression (A.P.). Arithmetic Mean (A.M.) Geometric Progression (G.P.), general term of a G.P, sum of n terms of a G.P., Arithmetic and Geometric series, infinite G.P. and its sum, geometric mean (G.M.), Harmonic (mean) relation between A.M., G.M. and H.M., Formula for the following special sum :

Arithmetico-Geometric Series, Exponential Series, Logarithmic Series, Binomial Series.

UNIT - III : Co-ordinate Geometry

1. Straight Lines

Brief recall of two dimensional geometry from earlier classes. Slope of a line and angle between two lines. Various forms of equations of a line : parallel to axis, point-slope form, slope-intercept form, two-point form, intercept form and normal form. General equation of a line. Equation of family of lines passing through the point of intersection of two lines. Distance of a point from a line, Shifting of Origin.

2. Conic Sections

Sections of a cone : circles, ellipse, parabola, hyperbola; a point, a straight line and a pair of intersecting lines as a degenerated case of a conic section; Standard equations and simple properties of Circle, parabola, ellipse and hyperbola.

3. Introduction to Three-dimensional Geometry

Coordinate axes and coordinate planes in three dimensions. Coordinates of a point. Distance between two points and section formula.

UNIT-IV: Calculus

1. Limits and Derivatives

Derivative introduced as rate of change both as that of distance function and geometrically. Intuitive idea of limit. Limits of polynomials and rational functions, trigonometric, exponential and logarithmic functions. Definition of derivative, relate it to slope of tangent of a curve, derivative of sum, difference, product and quotient of functions. The derivative of polynomial and trigonometric functions.

UNIT-V : Mathematical Reasoning

1. Mathematical Reasoning

Mathematically acceptable statements. Connecting words/phrases-consolidating the understanding of "if and only if (necessary and sufficient) condition," "implies", "and/ or", "implied by", "and", "or", "there exists" and their use through variety of examples related to real life and Mathematics. Validating the statements involving the connecting words, difference between contradiction, converse and contrapositive,

UNIT-VI : Statistics and Probability

1. Statistics

Measures of dispersion; Range, mean deviation, variance and standard deviation of ungrouped/ grouped data. Analysis of frequency distributions with equal means but different variances.

2. Probability

Random experiments; outcomes, sample spaces (set representation). Events; occurrence of events, 'not', 'and' and 'or' events, exhaustive events, mutually exclusive events, Axiomatic (set theoretic) probability, connections with the theories of earlier classes. Probability of an event. Probability of 'not', 'and' 'or' events.

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