Prescribed by CHSE, Odisha

BUREAU'S
Higher Secondary

Physics

Class-XI

Odisha State Bureau of Textbook
Preparation and Production
Pustak Bhavan, Bhubaneswar

BUREAU'S Higher Secondary Physics
s-XI

Bureau's

HIGHER SECONDARY

PHYSICS

CLASS - XI

Prescribed by the Council of Higher Secondary Education, Odisha

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FOREWORD

(New Edition)

It is heartening to find that Bureau's Higher Secondary Physics (Class XI & XII) are exhausted and very much appreciated by the students and teachers alike. This warm and oositive response had led the Bureau to plan for a new edition.

In the present edition, the volumes have been revised keeping in view the changes made in the CHSE Syllabus for Higher Secondary Science students. This revision work was done by a team of experts consisting of Dr. Nikhilananda Panigrahi, Dr. Kedarnath Biswal, Dr. B.M.Swain and Debasis Mohanty. The untiring efforts made by these teachers-cum-writers of this book deserves our heartfelt thanks. I am very positive that this book will receive appreciation of the students as well as teachers. I am thankful to Dr. Biswal for taking pain of doing the arduous task of proof reading of both the volumes before final printing.

I sincerely thank to all employees of the Bureau, in general and Sri Biraja Bhusan Mohanty, the Deputy Director, in Particular for their co-ordination in publication of this book (New Edition). I further thank M/s Sonali Prakashan, Bhubaneswar for DTP works on scheduled time.

Umakanta Tripathy

DIRECTOR

Odisha State Bureau of Textbook Preparation & Production

Pustak Bhavan, Bhubaneswar

FOREWORD

(First Edition)

The Council of Higher Secondary Education, Odisha has restructured the courses in Physics for its Examination 2005 A.D. and onwards. The Council has constituted a team of experienced and eminent teachers for writing textbooks in Physics in the light of the +2 revised syllabus. Physics provides the basis for the development of different branches of knowledge such as Astrophysics, Bio-Physics, Geophysics, Oceanography, Metallurgy, Climatology and even Chemistry and Engineering. Physics has chronologically developed by the pioneering work of Galileo, Newton, Maxwell, Einstein, Bohr and others.

The basic concepts of physics have been developed in a logical manner and the writers have taken utmost care in explaining the subject matter in simple and lucid English. The book contains quite a good number of numerical examples which have been worked out at the end of each section. Model questions (Multiple Choice, Very Short Answer, Short Answer, Conceptual, Numericals and Long Answer types) have been given at the end of each chapter. This book +2 Physics Vol. I has been written to fulfil the two-fold need of the students; as a textbook of physics and also as a source book for providing basic knowledge to appear for Entrance Examinations.

As Director of the Bureau I record my gratefulness to the Council of Higher Secondary Education, Odisha for allowing the Bureau to publish this book which has been recommended to be used by +2 students of Odisha as text book. I sincerely express my gratitude to the Board of Writers who have taken great pains in preparing this text book within a short span of time. I am specially thankful to Prof. (Dr.) Niranjan Barik of Utkal University for reviewing the book and guiding the Board of Writers at every step. I record herewith my greater sense of appreciation for Sri J. N. Pattanayak, Assistant Director of the Bureau and the Production Officers who have taken utmost care to see that printing and production are excellent by all standards.

Any suggestion from students and teachers for the improvement of this book will be acknowledged. Lastly, I hope that this textbook in Physics will serve the greater need of the +2 students of our State.

Jatindranath Mohanty

DIRECTOR
Odisha State Bureau of Textbook Preparation & Production
Pustak Bhavan, Bhubaneswar

Biswakarma Puja
17.09.03

PREFACE

The "Bureau's Higher Secondary Physics" is intended to provide a proper motivation to the students and teachers at large in the study of physics. The basic concepts have been developed in a logical manner and utmost care has been taken to impart various nuances of physics with clarity of physical assumptions, mathematical formulation and approximations. Wherever necessary, better and appropriate alternative approaches have been adopted to overcome the conceptual inadequacies. S.I. units have been consistently used and whenever necessary C.G.S. and other useful practical units have been mentioned. Representative diagrams have been given to facilitate easy understanding. At the end of each section numerical examples have been worked out and at the end of each chapter model questions (multiple choice, very short answer, short answer, conceptual, unsolved numericals and long answer type) have been given to foster and boost the understanding and ability of readers for applying various concepts developed.

The authors are very much thankful to their colleagues and students for their valuable discussions, suggestions and advice. The authors can never adequately express their gratitude to the Director Dr. J.N. Mohanty, Asst. Director and Production Officers and all other personnel of "Odisha Text Book Bureau" for their constant support and encouragement. The authors are also thankful to Prof. (Dr.) N. Barik whose timely review and advice has streamlined the process of writing.

Inspite of our sincere efforts typographical, conceptual, and factual errors might have crept in inadvertently. It is our sincere and humble request to all the readers to use the book with a receptive mind and point out our mistakes. Any suggestion for the improvement of the textbook shall be highly appreciated with regards.

Authors

PHYSICS

(Theory)
Class-XI
(Detailed Syllabus)

No. of Periods – 160

Unit-I Physical world and Measurement **(10 Periods)**

Physics and its scope, Physics, Technology and Society. Measurement, need for measurement, units of measurement, Fundamental and derived units, SI Units, accuracy and precision of measuring instruments, errors in measurement, absolute, relative error, percentage of error, Combination of errors, significant figures.

Dimensions of Physical quantities. Dimensional analysis and its applications. **(See Chapter-1)**

Unit-II Kinematics. **(24 Periods)**

1. Motion in a straight line:

Rest and motion, Frame of reference, motion in a Straight line, position _ time graph, speed and velocity. Concepts of differentiation and integration for describing motion(elementary idea), uniform and non-uniform motion, average speed and instantaneous velocity, uniformly accelerated motion, velocity _time and position_ time graph, Relations for uniformly accelerated motion (graphical treatment) **(See Chapter-3)**

2. Motion in a plane:

Scalars and vectors, general vectors and their notations, position and displacement vectors, equality of vectors, unit vectors, multiplication of vectors by a real number, addition and subtraction of vectors, relative velocity, resolution of a vector in a plane, rectangular components, Dot and Cross products of two vectors. **(See Chapter-2)**

Motion in a plane, cases of uniform velocity and uniform acceleration- projectile motion (equation of trajectory, range, time of flight, maximum height); **(See Chapter-8).** Uniform circular motion. **(See Chapter-5)**

Unit-III Laws of Motion **(14 Periods)**

Concept of force, Newton's first law, inertia, momentum and Newton's 2nd law, impulse, impulse-momentum theorem, Newton's 3rd law, Law of Conservation of linear momentum and its applications. Equilibrium of Concurrent forces, static and Kinetic friction, rolling friction, lubrication. **(See Chapter-4)**

Dynamics of uniform circular motion, Centripetal force, motion of a vehicle on a level circular road and vehicle on a banked road. **(See Chapter-5)**

Unit-IV Work, Energy and Power: **(12 Periods)**

Work done by a Constant force and variable force, kinetic energy, work- energy theorem, power. Notion of potential energy, potential energy of a spring, conservative and non-conservative forces, conservation of mechanical energy (Kinetic and Potential energies), motion in a vertical circle, elastic and in-elastic collisions in one and two dimensions. **(See Chapter-4)**

Unit-V Motion of System of Particles and Rigid bodies: **(18 Periods)**

System of Particles and Rotational Motion:

Centre of mass of a two-particle system, momentum conservation and centre of mass motion, centre of mass of rigid bodies, Centre of Mass of a uniform rod.

Moment of a force, torque, angular momentum, conservation of angular momentum with its applications.

Equilibrium of rigid bodies, equations of rotational motion, comparison of linear and rotational motions.

Moment of inertia, radius of gyration, moment of inertia of simple geometrical objects (no derivation). Parallel and perpendicular axes theorem and their applications. **(See Chapter-6)**

Unit-VI Gravitation **(12 Periods)**

Newton's law of gravitation, Kepler's law of planetary motion (only statements), Gravitational field and Potential, gravitational potential energy, acceleration due to gravity and its variation with altitude and depth, Escape velocity, orbital velocity of a satellite, Geostationary satellites. **(See Chapter-7)**

Unit-VII Properties of Bulk Matter **(24 Periods)**

1. Mechanical properties of Solids:

Elastic behaviour, Stress, Strain, Hookes' Law, Stress- Strain diagram, Young's modulus, Bulk modulus, Shear modulus of rigidity, Poison's ratio, elastic energy. **(See Chapter-12)**

2. Mechanical properties of fluids:

Pressure due to a fluid column, Pascal's law and its applications (hydraulic lift and hydraulic brakes) effect of gravity on fluid pressure.

Surface energy and surface tension, angle of contact, excess pressure across a curved surface, application of surface tension ideas to drops, bubbles and capillary rise. **(See Chapter-13)**

Viscosity, Stoke's law, terminal velocity, streamline and turbulent flow, critical velocity, Bernoulli's theorem and its application. **(See Chapter-15)**

3. Thermal properties of matter:

Concepts of heat and temperature, Thermal expansion of solids, liquids and gasses, anomalous expansion of water, specific heat capacity: Cp, Cv. Calorimetry, change of state, latent heat capacity. **(See Chapter-16)**

Heat transfer: Conduction, Convection and radiation, thermal conductivity, qualitative ideas of black body radiation, wien'sdisplacement law, Stefan's law, Greenhouse effect. **(See Chapter-18)**

Unit-VIII Thermodynamics **(12 Periods)**

Thermal equilibrium, definition of temperature (Zeroth law of thermodynamics) heat, work and internal energy. First law of thermodynamics, isothermal and adiabatic processes, second law of thermodynamics, reversible and irreversible processes, Carnot's engine and refrigerator, Efficiency of Carnot's engine (no derivation). **(See Chapter-19)**

Unit- IX Kinetic theory of gases: **(08 Periods)**

Equation of state of a perfect gas, work done in compressing a gas.

Kinetic theory of gases- Postulates, concept of pressure, kinetic interpretation of temperature, mean and RMS speed of gas molecules, degrees of freedom, law of equipartition of energy (statement only) and its applications to specific heat of gases, concept of mean freepath and Avogadro's number. **(See Chapter-19)**

1. Periodic motion: Period, frequency, displacement as a function of time, periodic function.

Simple harmonic motion and its equation, phase, oscillation of a spring, Restoring force and force constant, kinetic and potential energy in SHM, simple pendulum, derivation of expression for its time period.

Free, damped and forced oscillations (qualitative idea only), resonance. **(See Chapter-9)**

2. Waves:

Wave motion, transverse and longitudinal waves, speed of wave motion, displacement relation for a progressive wave, speed of longitudinal wave in an elastic medium and speed of transverse wave in a stretched string (qualitative idea only), principle of superposition of waves, reflection of waves, standing waves in strings and organ pipes, fundamental mode and harmonics, Beats, Doppler's effect. **(See Chapter-10 & 11)**

Books Recommended:

1. Physics Part-I and Part-II published by NCERT.
2. Bureau's Higher Secondary Physics Class-XI Vol.-I published by Odisha State Bureau of Text Book Preparation and Production; Bhubaneswar.

PRACTICALS

Total Periods 60

Section A

Experiments

1. To measure diameter of a small spherical/cylindrical body using Verniercallipers.
2. To measure internal diameter and depth of a given beaker/calorimeter using Vernier callipers and hence find its volume.
3. To measure diameter of a given wire using screw gauge.
4. To measure thickness of a given sheet using screw gauge.
5. To measure volume of an irregular lamina using screw gauge.
6. To determine radius of curvature of a given spherical surface by aspherometer.
7. To determine the mass of two different objects using a beam balance.
8. To find the weight of a given body using parallelogram law of vectors.
9. Using a simple pendulum, plot L-T and L-T2 graphs. Hence find theeffective length of a second'spendulum using appropriate graph.
10. To study the relationship between force of limiting friction and normal reaction and to find the coefficient of friction between a block and a horizontal surface.
11. To find the downward force, along an inclined plane, acting on a roller due to gravitational pull ofthe earth and study its relationship with the angle of inclination (?) by plotting graph between forceand*sin* ?.

Section B

Experiments

1. To determine Young's modulus of elasticity of the material of a given wire.
2. To find the force constant of a helical spring by plotting a graph between load and extension.
3. To study the variation in volume with pressure for a sample of air at constant temperature byplotting graphs between P and V, and between P and 1/V.
4. To determine the surface tension of water by capillary rise method.

5. To determine the coefficient of viscosity of a given viscous liquid by measuring the terminal velocityof a given spherical body.
6. To study the relationship between the temperature of a hot body and time by plotting a coolingcurve.
7. To determine specific heat capacity of a given (i) solid (ii) liquid, by method of mixtures.
8. (i) To study the relation between frequency and length of a given wire under constant tensionusingsonometer.

   (ii) To study the relation between the length of a given wire and tension for constant frequencyusingsonometer.
9. To find the speed of sound in air at room temperature using a resonance tube by two resonancepositions.

PHYSICS

(Theory)

Class - XII

(Detailed Syllabus)

No. of Periods yearly – 160

Unit-I Electrostatics **(22 Periods)**

1. Electric charges and fields:

Electric charge and its quantization, conservation of charge, Coulomb's law, force between two point charges, force between multiple charges, super position principle, Continuous charge distribution.

on a dipole in uniform electric field.

Electric flux, Gauss's theorem (statement only) and its applications to find field due to uniformly charged infinite plane sheet, infinitely long straight wire and uniformly charged thin spherical shell (field inside and outside).

2. Electrostatic potential and capacitance:

Electric potential, potential difference, electric potential due to a point charge, potential due to a dipole, potential due to a system of charges. Equipotential surfaces, electrical potential energy of a system of two point charges and of electric dipole in an electrostatic field.

Conductors, insulators, free charges and bound charges inside a conductor, Dielectrics and electric polarisation, capacitors and capacitance, capacitance of a parallel plate capacitor with and without dielectric medium between the plates, combination of capacitors in series and in parallel, energy stored in a capacitor.

Unit-II Current Electricity: **(20 Periods)**

Electric current, drift velocity, mobility and their relation with electric current, Ohm's law, electrical resistance, conductance, resistivity,conductivity, effect of temperature on resistance, V~I characteristics (linear and non-linear) electrical energy and power, carbon resistors, colour code of carbon resistors, combinations of resistors in series and parallel.

EMF and potential difference, internal resistance of a cell, combination of cells in series and parallel, Kirchhoff's laws and simple applications: Wheatstone bridge and meter bridge. Potentiometer-Principle and its applications to measure potential difference and for comparing EMF of two cells; measurement of internal resistance of a cell.

Unit-III Magnetic effect of Current and magnetism: (23 Periods)

1. Moving charges and magnetism:

Concept of magnetic field, Oesrted'sexperiment,Biot-Savart law and its application to find magnetic field on the axis and at the centre of a current carrying circular loop, Ampere's law and its application to infinitely long straight wire. Straight and toroidal solenoid (qualitative treatment only); Force on a moving charge inuniform magnetic and electric fields, Cyclotron.

Force on a current carrying conductor in a uniform magnetic field, force between two parallel current carrying conductors- definition of ampere, torque experienced by a current loop in uniform magnetic field, moving coil galvanometer- its current sensitivity and conversion to ammeter and voltmeter.

2. Magnetism and matter:

Current loop as a magnetic dipole and its magnetic dipole moment, magnetic dipole moment of a revolving electron, magnetic field intensity due to a magnetic dipole (bar magnet) along its axis and perpendicular to its axis, torque on a magnetic dipole (bar magnet) in a uniform magnetic field, bar magnet as an equivalent solenoid, magnetic field lines, earth's magnetic field and magnetic elements.

Para-, dia- and ferro- magnetic substances with examples, Electromagnets and factors affecting their strengths, permanent magnets.

Unit-IV Electromagnetic induction and Alternating current: (20 Periods)

1. Electromagnetic induction:

Faraday's laws of electromagnetic induction, inducedEMF and current, Lenz's law, Eddy currents, self and mutual induction.

2. Alternating current:

Alternating currents, peak and RMS value of alternating current/ voltage, reactance and impedance, LC oscillations (qualitative idea only), LCR series circuit, resonance, power in AC circuits, wattles current, A.C. generator and transformer.

Unit- V Electromagnetic waves: (04 Periods)

Basic idea of displacement current, qualitative idea about characteristics of electromagnetic waves, their transverse nature. Electromagnetic spectrum (radio waves, microwaves, infrared, visible, X-ray and gamma rays), including elementary ideas about their uses.

Unit-VI Optics (25 Periods)

1. Ray optics and optical instruments:

Reflection of light, spherical mirrors, mirror formula,lateral and longitudinal magnification, refraction of light, refractive index, its relation with velocity of light (formula only) total internal reflection and its applications, optical fibre, Refraction at spherical surfaces, thin lens formula, lens makers formula, magnification, power of lenses, combination of two this lenses in contact, combination of a lens and a mirror, refraction and dispersion of light through prism;

Scattering of light: blue colour of sky and reddish appearance of sun at sunset and sunrise.

Optical instruments: microscopes and telescopes (reflecting and refracting) and their magnifying powers.

2. Wave Optics:

Wave front, Huygen's principle, reflection and refraction of plane wave at a plane surface using wave fronts, proof of laws of reflection and refraction using Hygen's principle. Interference, Young's double slit experiment and expression for fringe width, coherent sources, sustained interference of light, diffraction due to a single slit, width of a central maximum, resolving power of microscope and astronomical telescope (qualitative idea) polarisation, plane polarised light, Brewster's law, uses of plane polarised light and polaroids.

Unit-VII Dual nature of Radiation and matter: (08 Periods)

Dual nature of radiation, Photoelectric effect, Hertz and Lenard's observations, Einstein's photoelectric equation, particle nature of light.

Matter waves- wave nature of particles, de-Broglie relation, Davisson- Germer experiment, (only conclusions should be explained).

Unit- VIII Atoms and Nuclei (14 Periods)

1. Atoms:

Alpha- particle scattering experiment, Rutherford's model of atom, its limitations, Bohr model, energy levels, hydrogen spectrum.

2. Nuclei:

Atomic nucleus, its composition, size, nuclear mass, nature of nuclear force, mass defect, binding energy per nucleon and its variation with mass number, nuclear fission, fusion, Radioactivity, alpha, beta and gamma particles/ rays and their properties, radioactive decay law, half life and decay constant.

Unit-IX Semiconductor electronics: (10 Periods)

Energy bonds in conductors, semiconductors and insulators (qualitative idea only), p-type, n-type semiconductors, semiconductor diode, V-I characteristics in forward and reverse bias, diode as a half and full wave rectifier (centre tap), efficiency (no derivation).

Special purpose p-n junction diodes: LED, photodiode, solar cell and Zener diode and their characteristics, Zener diode as a voltage regulator.

Junction transistor, transistor action, Characteristics of a transistor, transistor as an amplifier (CE configuration), basic idea of analog and digital signals, Logic gates (OR, AND, NOT, NAND, and NOR)

Unit-X Communication System: (10 Periods)

Elements of a communication system (block diagram only), bandwidth of signals (speech, TV and digital data), bandwidth of transmission medium, propagation of electromagnetic waves in the atmosphere, sky and space wave propagation, satellite communication, Need for modulation, qualitative idea about amplitude modulation and frequency modulation, advantages of frequency modulation over amplitude modulation, basic idea about internet, mobile telephony and global positioning system (GPS).

PRACTICALS

Total Periods 60

Section A
Experiments

1. To determine resistance per cm of a given wire by plotting a graph of potential difference versus current.
2. To find resistance of a given wire using metre bridge and hence determine the resistivity (specific resistance)of its material.
3. To verify the laws of combination (series) of resistances using a metre bridge.
4. To verify the laws of combination (parallel) of resistances using a metre bridge.
5. To compare the emf of two given primary cells using potentiometer.
6. To determine the internal resistance of given primary cell using potentiometer.
7. To determine resistance of a galvanometer by half-deflection method and to find its figure ofmerit.
8. To convert the given galvanometer (of known resistance and figure of merit) into avoltmeter of desired range and to verify the same.
9. To convert the given galvanometer (of known resistance and figure of merit) into an ammeter of desired range and to verify the same.
10. To find the frequency of the ac mains with a sonometer.

Section B
Experiments

1. To find the value of *v* for different values of *u* in case of a concave mirror and to find the focal length.
2. To find the focal length of a convex mirror, using a convex lens.
3. To find the focal length of a convex lens by plotting graphs between *u* and *v* or between $1/u$ and $1/v$.
4. To find the focal length of a concave lens, using a convex lens.
5. To determine angle of minimum deviation for a given prism by plotting a graph between the angleof incidence and the angle of deviation.
6. To determine refractive index of a glass slab using a travelling microscope.
7. To find refractive index of a liquid by using convex lens and plane mirror.
8. To draw the I-V characteristics curves of a p-n junction in forward bias and reverse bias.
9. To draw the characteristics curve of a zener diode and to determine its reverse break downvoltage.
10. To study the characteristics of a common-emitter npn or pnp transistor and to find out the values of current and voltage gains.

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MOLECULAR PROPERTIES OF MATTER

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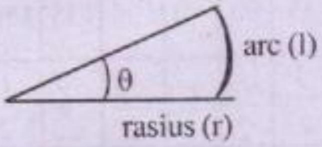
1

Physical World And Measurements

Mathematical Formula

1. $\theta = \frac{\text{length of arc}}{\text{radius}} = \frac{l}{r}$



2. π radian = 180 $\Rightarrow 1 = \frac{\pi}{180}$ $[\pi = 3.1416]$

3. a) Circle of radius r :

Circum ference = $2\pi r$

area = πr^2

b) Sphere of radius r :

Surface area = $4\pi r^2$

Volume = $\frac{4}{3}\pi r^3$

c) Right circular cylinder of radius r and length l :

Total length l :

Total surface area = $2\pi r^2 + 2\pi rl$

Volume = $\pi r^2 l$

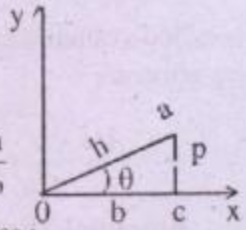
d) Triangle of base 'a' and height h :

Area = $\frac{1}{2}$ a x h

4. $\sin\theta = \frac{p}{h}$, $\cos\theta = \frac{b}{h}$

$\tan\theta = \frac{p}{b}$, $\cot\theta = \frac{b}{p}$

$\sec\theta = \frac{h}{p}$, $\mathrm{cosec}\,\theta = \frac{h}{b}$



5. Trigonometric identities :

a) $\sin(90° - \theta) = \cos\theta$

$\cos(90° - \theta) = \sin\theta$

$\sin(90° + \theta) = \cos\theta$

$\cos(90° + \theta) = -\sin\theta$

$\sin(180° - \theta) = \sin\theta$

$\cos(180° - \theta) = -\cos\theta$

$\sin(180° + \theta) = -\sin\theta$

$\cos(180° + \theta) = -\cos\theta$

b) $\sin(A \pm B) = \sin A.\cos B \pm \cos A.\sin B$

$\cos(A \pm B) = \cos A.\cos B \mp \sin A.\sin B$

$\sin A \pm \sin B = 2\sin\left(\frac{A \pm B}{2}\right).\cos\left(\frac{A \mp B}{2}\right)$

$\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right).\cos\left(\frac{A-B}{2}\right)$

$\cos A - \cos B = -2\sin\left(\frac{A+B}{2}\right).\sin\left(\frac{A-B}{2}\right)$

c) $\sin^2\theta + \cos^2\theta = 1$ · $\sin 2\theta = 2\sin\theta.\cos\theta$

$1 + \tan^2\theta = \sec^2\theta$

$1 + \cot^2\theta = \mathrm{cosec}^2\theta$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$
$$= 2\cos^2\theta - 1$$
$$= 1 - 2\sin^2\theta$$
$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$$

6. Quadratic equation :

An equation having the form

$$ax^2 + bx + c = 0,$$

is called a quadratic equation. Two roots of the equation are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

7. Binomial Theorem :

$$(1 \pm x)^n = 1 \mp \frac{nx}{1!} + \frac{n(n-1)\,x^2}{2!} + \ldots \qquad (x^2<1)$$

$$(1 \pm x)^{-n} = 1 \mp \frac{nx}{1!} + \frac{n(n+1)\,x^2}{2!} + \ldots \qquad (x^2<1)$$

8. Esponential, logarithmic and Trigonometric expansions :

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots \qquad (|x|<1)$$

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots$$

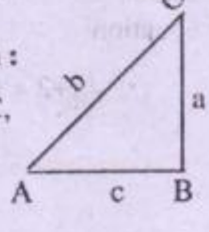
$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \ldots$$

$$\tan\theta = \theta + \frac{\theta^3}{3} + \frac{2\theta^5}{15} - \ldots$$

9. Pythagoream Theorem :

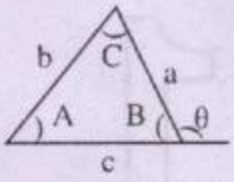
In the right triangle ABC,

$$a^2 + c^2 = b^2$$



10. Triangles :

If A, B and C are angles and a, b and c are opposite sides then :



i) $A + B + C = 180$

ii) $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$

iii) $c^2 = a^2 + b^2 + 2ab\cos C$

iv) Exterier angle $\theta = A + C$

11. Values of Trigonometric ratios of some standard angles :

T ratio \ angle	0	30	45	60	90	120	135	150	180
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$	-1
$\tan\theta$	0	1	1	$\sqrt{3}$	Not defend	$-\sqrt{3}$	-1	$-\frac{1}{\sqrt{3}}$	0

12. Differentiation :

Let $y = f(x)$

Then $\frac{dy}{dx}$ is called the differential coefficient of derivative of 'y' with respect to x. It is defined as

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$\frac{dy}{dx}$ of some common functions :

a) $\frac{d}{dx} x^n = nx^{n-1}$

b) $\frac{d}{dx} \sin x = \cos x$

c) $\frac{d}{dx} \cos x = -\sin x$

d) $\frac{d}{dx}\tan x = \sec^2 x$

e) $\frac{d}{dx}\cot x = -\operatorname{cosec}^2 x$

f) $\frac{d}{dx}(\sec x) = \sec x.\tan x$

g) $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x.\cot x$

h) $\frac{d}{dx}\ln x = \frac{1}{x}$

i) $\frac{d}{dx}e^x = e^x$

Basic Rules :

1. $\frac{d}{dx}c = 0$ c = constant

2. $\frac{d}{dx}cy = c\frac{dy}{dx}$

3. $\frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dy}$

4. $\frac{d}{dx}(u . v) = u\frac{dv}{dx} + v\frac{du}{dx}$

5. $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

6. $\frac{dy}{dx} = \frac{dy}{du} . \frac{du}{dx}$

using the rule (6) we can write

i) $\frac{d}{dx}u^n = nu^{n-1}.\frac{du}{dx}$

ii) $\frac{d}{dx}\sin u = \cos u.\frac{du}{dx}$ and so on.

13. Integration :

The function $g(x) = \int f(x)\,dx$ is read as the indefinite integral of f(x) with respect to x. The function f(x) which is to be integrated is called the integrand.

An integral with lower and upper limits is known as a definite integral.

A fundamental theorem of mathematics states that :

$$\int_a^b f(x)\,dx = [g(x)]_a^b = g(b) - g(a)$$

To evaluate definite integrals, we need to know the corresponding indefinite integrals.

So me common indefinite integrals are :

1) $\int x^n dx = \frac{x^{n+1}}{n+1}$ $(n \neq -1)$

2) $\int \frac{1}{x} dx = \ln x$ $(n < 0)$

3. $\int \sin x dx = -\cos x$

4. $\int \cos x dx = \sin x$

5. $\int \sec^2 x dx = \tan x$

6. $\int \operatorname{cosec}^2 x dx = -\cot x$

7. $\int \sin ax\, dx = -\frac{\cos ax}{a}$

8. $\int \cos ax\, dx = \frac{\sin ax}{a}$

9. $\int e^x\, dx = e^x$

Some useful rules for integration are :

1. $\int cu dx = c\int u dx$ (c = constant)

2. $\int (u \pm v) dx = \int u dx \pm \int v dx$

Examples : 1) Obtain the roots of the quadratic equation

$x^2-3x+2 = 0$

Soln : $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Here a = 1, b = -3, c = 2

$$\therefore x = \frac{3 \pm \sqrt{9 - 4 \times 1 \times 2}}{2}$$

Solving : $x_1 = 2$ and $x_2 = 1$

2. Evaluate the following, assuming x<<1.

$$\sqrt[3]{(1+x)^2}$$

soln : Using binomial expansion :

$$(1+x)^n = 1 + \frac{nx}{1!} + \frac{n(n-1)x^2}{2!} + ...$$

Since x<<1, we can neglect its square and other higher powers.

$$\therefore\ (1+x)^n = 1 + nx$$

Now the expression

$$\sqrt[3]{(1+x)^2} = (1+x)^{\frac{2}{3}} = 1 + \frac{2}{3}x$$

3. Find the derivative of the following with respect to x.

a) $3x^4 - 6x^{-2}$

b) $(\sqrt{x} + \dfrac{1}{\sqrt{x}})^3$

c) $\dfrac{\sin x}{x + \cos x}$

Solution

a) $\dfrac{d}{dx}(3x^4 - 6x^{-2})$

$$= 3\frac{d}{dx}x^4 - 6\frac{d}{dx}x^{-2}$$

$$= 3 \times 4x^3 - 6 \times (-2)x^{-3}$$

$$= 12x^3 + 12x^{-3}$$

b) $\dfrac{d}{dx}(\sqrt{x} + \dfrac{1}{\sqrt{x}})^3$

$$= 3(\sqrt{x} + \frac{1}{\sqrt{x}})^2 \frac{d}{dx}(\sqrt{x} + \frac{1}{\sqrt{x}})$$

$$= 3(\sqrt{x} + \frac{1}{\sqrt{x}})^2 \left\{\frac{1}{2}x^{-\frac{1}{2}} + (-\frac{1}{2})x^{-\frac{3}{2}}\right\}$$

$$= \frac{3}{2}(\sqrt{x} + \frac{1}{\sqrt{x}})^2(\frac{1}{\sqrt{x}} - \frac{1}{x^{3/2}})$$

c) $\dfrac{d}{dx}(\dfrac{\sin x}{x + \cos x})$

$$= \frac{(x + \cos x)\dfrac{d}{dx}\sin x - \sin x\dfrac{d}{dx}(x + \cos x)}{(x + \cos x)^2}$$

$$= \frac{(x + \cos x)\cos x - \sin x\,(1 - \sin x)}{(x + \cos x)^2}$$

4. Evaluate the following integral.

a) $\int_0^{\pi/4} \sin 4x\, dx$

b) $\int_R^{\infty} \dfrac{GMm}{r^2}\, dr$ [G, M and m are constants]

Soln : a) $\int_0^{\pi/4} \sin 4x\, dx$

$$= \left[-\frac{\cos 4x}{4}\right]_0^{\pi/4} = \left[-\frac{\cos 4\pi/4}{4} + \frac{\cos 0}{4}\right]$$

$$= \frac{-\cos\pi + \cos 0}{4}$$

$$= \frac{1+1}{4} = \frac{1}{2}$$

b) $\int_R^{\infty} \dfrac{GMm}{r^2}\, dr = GMm \int_R^{\infty} r^{-2} dr$

$$= GMm\left[\frac{r^{-1}}{-1}\right]_R^{\infty}$$

$$= GMm\left[\frac{1}{r}\right]_R^\infty$$

$$= GMm\left[\frac{1}{\infty}-\frac{1}{R}\right]$$

$$= \frac{GMm}{R}$$

Exercise

1. Obtain the roots of the quadratic equation $x^2-2x + 15 = 0$

2. Evaluate the following assuming $x<<1$.

$(1+\frac{3}{5}x)^{-5/2}$

3. Find the derivative of the following with respect t_0, x

a) $e^x \sin x$ b) $(\ln x)^2$

c) $\frac{x^2 - x + 1}{x^2 + x + 1}$

4. Integrate the following.

a) $\int_0^\pi (1+\sin x)dx$

b) $\int(\sin x + \frac{2}{x} - \frac{2}{x^3})dx$

c) $\int_0^t A\sin\omega t\, dt$ (where A and ω are constants)

1.1 Introduction :

The nature around us is vast, mysterious and diverse. The inquisitive and imaginative human mind has responded to this in making a systematic attempt to understand the natural phenomena in as much detail and depth as possible. This curiosity to learn has resulted in unravelling the secrets of nature, exploring the various laws governing them and suggesting various mathematical models. Careful experimentations have led to pick up the most correct mathematical models with some speculations and conjectures. On the other hand theoretical advances have also suggested looking for some experiments. [For example (i) α - scattering experiment has inspired Ernest Rutherford to suggest a Nuclear model of atom and Niels Bohr to propose a Quantum theory of Hydrogen atom.

(ii). electron - positron annihilation has thrown light on the existence of elementary particles beyond our knowledge of electron, proton and neutron. (iii). The concept of antiparticles came up as a prediction from Paul Dirac's Relativistic Quantum theory which was later on confirmed experimentally by the discovery of positron by Carl Anderson].

Physics is a basic discipline of **Natural Sciences.** A precise definition of Physics is neither possible nor necessary. However, we can broadly say **Physics is a study of basic laws of nature and its manifestations in different natural phenomena in terms of few concepts and laws.** No one can claim to have framed any rule or law of physics. One can only guess or discover the rules or laws that are operating in nature. The laws or rules once discovered or guessed may fail to explain some new phenomena that may be observed in future. For example Newton's laws of motion which was very elegant in explaining various observations in mechanics in a convincing manner, failed to explain events occurring at high velocities. Thus new and refined physics laws/theories are sure to come up and there should always be an attempt for improvement and innovations. Physics has two basic and principal thrusts: **Unification and Reduction.** For example Statistical Mechanics explained Themodynamical processes in terms of the properties of the molecules constituting the bulk system. There have been continuous attempts to unify the basic forces of nature (Unification of electromagnetic,), weak, strong force and gravitation).

1.2 Scope of Physics :

Basically one can consider physics in terms of two sub-disciplines : **Classical Physics** (dealing with macroscopic phenomena), and **Quantum Physics** (dealing with microscopic phenomena). Classical Physics includes subjects like Mechanics, Electrodynamics, Optics, and Thermodynamics. In mechanics one considers motion of bodies (particle/rigid bodies), gravitation, and elasticity and so on. Electrodynamics deals with electric and magnetic phenomena associated with charged and magnetic bodies. Optics deals with phenomena involving light. Thermodynamics, in contrast with mechanics deals with bodies as a whole. It deals with system in macroscopic equilibrium and is concerned with temperature, internal energy, and entropy etc. of the system. It also considers heat engines. On the other hand Quantum theory deals with the microscopic domain of physics. It considers the constitution and structure of matter and the interaction of the constituents with electrons, photons and other probes like neutrons, α -particles etc. Of late Quantum field theory has been developed which probes various phonomena involving elementary particles.

Thus scope of physics is very vast. It covers very large range of magnitude of physical quantities like length, mass, time, energy etc. At one end it deals with phenomena at very small scale of length ($10^{-14}\,m$ or even less), time scale of 10^{-22} sec. and mass scale of $10^{-30}\,kg$ involving electrons, protons, neutrons and other elementary particles; at the other end it deals with astronomical phenomena or even the entire Universe whose extent is of the order of (length scale 10^{26}m, time scale $10^{18}\,sec.$ and mass scale $10^{56}kg$)

1.3 Physics, Technology and Society:

Physics and Technology are complementary to each other. For example heat (steam) engines, which brought about an industrial revolution and had a great impact on the human civilization is a contribution of our knowledge on thermodynamics. On the contrary wireless communication technology led to the discovery of basic laws of electricity, magnetism and electromagnetic waves. Our atomic energy power plants are the contribution of our knowledge on nuclear physics. Our understandings on semiconductors has greatly contributed to the development of silicon chips which triggered the computer revolution. A significant and useful contribution of physics is the development of renewable energy sources like solar energy, geothermal energy etc.

1.4 Mathematics - "A Tool for Physics"

Laws of physics are easily comprehended by expressing them in the form of Mathematical equations. Also the simple statement of a physical law does not explain all connected phenomena. For example the simple statement of Newton's law of gravitation cannot throw light on the laws of planetary motion. Mathematical techniques are to be suitably applied to the mathematical statement of physical laws to arrive at certain concrete conclusions. Thus "Mathematics is the language of Physics". But one has to bear in mind that Mathematics itself is not physics. The aim of a physicist is to explore and understand nature precisely and as such suggest or guess empirical/ theoretical rules that govern the phenomena in nature. Mathematics helps in explaining various features of nature by suitable mathematical analysis of the suggested physical laws. Thus mathematics acts as a vehicle with a physicist in driver's seat that takes us from the fundamental physical laws to the conclusions of the physical laws, which explains various features of physical phenomena.

1.5 Physical Quantities :

The real test of any physical law or theory is its quantitative agreement with observations and measurements of various aspects of physical phenomena. Any quantity or set of quantities, used for a quantitative description of physical phenomena are called physical quantities.

To define a physical quantity one must either specify a procedure for measuring the quantity or specify a way to calculate the quantity from some other quantities that can be measured. For example to define length and time one has to prescribe a set of procedures for measuring length and time. This does not speak about what length is or what time is ; but says how to measure them. On the otherhand speed of a moving body is defined as distance ($\equiv$ length) divided by time.

A definition in terms of a procedure for measuring a physical quantity is called an operational definition. **Physical quantities defined by operational definitions are called fundamental quantities.** For convenience mass, length, time, thermodynamic temperature electric current, luminous intensity, amount of matter are chosen as fundamental (The choice is arbitrary and conventional). The fundamental quantities are independent of each other. **Physical quantities which are defined in terms of fundamental quantities are called derived quantities.**

1.6 Measurements :

Conclusions are drawn in a Scientific study, basing on precise measurements of various aspects of physical phenomena.

Measurement of a physical quantity always involves comparison with a similar physical quantity having a suitably chosen (assigned) value and the measured value is a multiple or sub-multiple of the chosen value. If Q be the value of the physical quantity and q be the chosen value, then $Q = nq$, where n is a real number. For example when we say a cloth piece of two metres it means the cloth piece is two times as long as the bar whose length is defined to be one metre.

Thus to carry out accurate measurements one has to establish a system of reference standards and a system of units, to express the standards. **Unit of a physical quantity is a conveniently and suitably chosen value of that quantity with which quantitative comparison can be made and a standard is the physical embodiment of a unit;** provided that certain conditions are fulfilled.

The value of a unit of a fundamental physical quantity is determined with the help of a standard for that quantity. Since conclusions in a scientific study are to be communicated to others for their knowledge and scrutiny, the units for measurements be chosen in a suitable manner to convey a precise idea about measurements. For example if one expresses the length of a police stick as few metres, others easily comprehend the size of the stick. But if one expresses the distance of Bhubaneswar to Delhi in metres it is not immediately comprehended, so we express this distance ($\equiv$ length) in kilometres ($1\text{km} = 10^3\text{m}$) In this sense, unit is a multiple or sub-multiple of a reference standard.

Since derived quantities are expressed in terms of fundamental quantities, we only need to define standards for fundamental quantities. But while choosing a standard for measurement one has to see that the following characteristics are fulfilled.

(a) It should be easily and conveniently accessible to all so that results obtained are agreed upon internationally. (b) It should be easily and accurately reproducible, so that it can be easily used by all people every where. (c) It should be invariable e.g. defining distance between tip of the finger and elbow as unit of length is not invariable. (d) It should be permanent and should be capable of withstanding mechanical stress and strain and undergo no change on variations of atmospheric conditions. (e) It should be in such a form and shape that it can be easily compared with other similar physical quantity.

Also while choosing a unit, one should take care to see that (a) it quickly conveys an easy conception about the measurement (b) it is conventional and adopted by all (c) it is of the same kind e.g. volume unit should be used for volume measurement, force unit for force measurements etc. (d) it is convenient for calculation (at least for scientific studies).

1.7 World Standards :

A. Standards of length :

(i) Earth size as standard of length :

Paris academy of Science in the year 1791 suggested a length standard called "metre". This was then defined as equivalent to one-ten millionth of the distance from the equator to the north pole along the meridian through Paris. But such a standard although fulfils the conditions of permanence and invariability, is impracticable for use. One cannot compare this standard for measurement of length of another body. Hence this standard was not acceptable and a modified view was taken.

(ii) Metre bar as standard of length

The international committee on weights and measures adopted a chemically stable metre-bar of platinum (90%)- iridium (10%) alloy as standard of length. **The distance between two lines engraved on gold plugs near the two ends of the platinum-iridium bar maintained at 0°C is considered to be one metre.** This primary standard bar is preserved at the International Bureau of weights and measures at serveres near Paris. To make it accessible accurate copies of this primary standard are provided to various standardising laboratories throughtout the world. These standards are treated as secondary standards. However the use of the metre - bar as a world standard of length has few short-comings. (a) It is destructible by natural calamities. (b) It is not easily accessible. (c) It is not accurately reproducible. (d) The maximum accuracy obtainable with a standard metre is 1 part in 10^7. This can cause large errors in calculations of space science.

(iii) Atomic standard of length :

In October 14,1960, the International Scientific Community agreed to accept an atomic constant, namely wavelength of orange-red light emitted by Krypton-86 ($5d_5 \rightarrow 2p_{10}$) as a standard of length. According to this. "One metre is 1,650,763.73 times the wavelength of orange-red light of Krypton-86 ($5d_5 \rightarrow 2p_{10}$)."

Such a standard has few advantages : (a) the light emitted is sharply defined (b) it is imperishable (c) these atoms are available everywhere in pure form and emit identical light of same wave length. Hence it is easily and conveniently reproducible (d) it has an accuracy of 1 part in 10^9.

(iv) Speed of light as standard of length :

This is based on the distance travelled by light in vaccum. A metre is the distance travelled by light in vaccum in $\frac{1}{2,99,792,458}$ second.

This standard is also relatively more accurate and reliable than atomic standard of length.

B. Standard of Mass :

(i) Cylinder as Standard of Mass :

A cylinder of platinum - iridium alloy, preserved at International Bureau of weights and measures at Severes near Paris, is designated as one kilogram.

This standard is simple for comparison purposes. However it lacks in fulfilling absolutely the conditions like invariability and indestructibility. In this respect an atomic standard appears to be a better substitute. But it has not yet been possible to measure masses on atomic scale with as great precision as on a macroscopic scale. Therefore the standard kilogram continues to be used as the primary standard. Secondary standards are then prepared and provided to different standardising laboratories.

C. Standards of Time :

A suitable phenomenon that repeats itself at regular interval of time can be used as a measure of time.

(i) Mean-solar day as standard of time :

"One second is defined as $\frac{1}{86,400}$ of a mean solar day". The mean solar day is the average of the times taken for one complete rotation of earth around its own axis. This time is called "Sideral time".

(ii) Tropical - year as standard of time

In this case "One second is defined as $\frac{1}{31,556,925.9747}$ part of the tropical year 1900". This time is called as "Ephemeris time".

(iii) Atomic standard of time

In 1967, an atomic standard was adopted, basing on a definite periodic vibration of Cs-133 atom. "One second is defined as the time required for 9,192,631,770 cycles of the radiation corresponding to transition between the two hyperfine levels of the ground state of Cs-133 atom". The atomic standard of time has few advantages such as : (a) It is indestructible (b) It is easily available and reproducible (c) It is unaffected by variations in atmospheric conditions (d) Its accuracy is 1 part in 10^{11}.

(D) Standard of Current : (Ampere)

It is based on the force between two long parallel wires carrying same current. One ampere is that current which on flowing through two parallel conductors of infinite length and negligible cross-section and placed one metre apart in vaccum produces a mutual force of 2×10^{-7} Newton per metre of each wire.

(E) Standard of Temperature : (Kelvin)

The standard is defined on the basis of thermodynamical scale of temperature, in which zero point is at absolute zero and reference point is the triple point of water.

One Kelvin is the fraction $\frac{1}{273.16}$ of the triple point of water.

(F) Standard of Intensity of Light : (Candela)

One 'Candela' is defined as the luminous intensity (luminous flux per unit area) in the direction normal to the surface of $\frac{1}{6,00,000}$ m^2 of a black body at the temperature of solidification of platinum at normal atmospheric pressure.

(G) Standard of amount of substance :

One mole is the amount of substance having 6.02×10^{23} entities.

1.8 Systems of Units

A **coherent system of units** is a complete set of fundamental units from which all derived units are obtained by simple multiplication and/ or division of the fundamental units without introducing any additional numerical factors.

Several systems of units are in use of which we give below three commonly used ones.

I. C.G.S. System of Units :

The C.G.S system of units is based on **centimeter** ($=10^{-2}$m) as unit of length, **gram** ($=10^{-3}$ kg) as unit of mass and **second** as unit of time.

II. F P S System of Unit :

The F P S system of units is based on **foot** as unit of length, **pound** as unit of weight (force) and **second** as unit of time. In this system

(a) 1 ft = 12 inches

1 yard = 3ft = 0.9144 metre

$$1 \text{ inch} = \frac{1}{36} \text{ yard} = 2.54 \text{ cm}$$

1 mile = 1760 yard = 5280 ft

6080ft = 1 nautical mile

(b) 1 pound force is equivalent to 0.45359237 kg force and produces an acceleration of 32.174 ft/sec^2 on acting on a body of mass 0.45359237 kg. The unit of mass in FPS system is **slug,** such that 1.1b-force causes an acceleration of 1ft/sec^2 in a mass of 1 slug.

$\frac{1}{16}$ 1b = 1 ounce (oz)

1 dram = $\frac{1}{10}$ oz

2 1b = 1 quarter (qr)

4qr = 1 hundred weight (wt)

20 cwt = 2240 1b = 1 tonne

1 slug $\cong$ 14.4 kg

III. MKS system of Units :

The MKS system of units is based on **metre** as unit of length, **Kilogram** as unit of mass and **second** as unit of time.

However the above three system of units is sufficient only to discribe physical quantities in mechanics and fails to describe quantities involved in electricity, optics, and thermodynamics. Therefore CGS, FPS, and MKS are not coherent. The general conference on weights and measures took care of the deficiency by developing a new coherent system of units called **International system** (S.I.) of units.

IV. International system (S.I.) of Units :

The base units in this system are **metre** for length, **kilogram** for mass, **second** for time, **ampere** for current, **kelvin** for temperature, **candela** for luminous intensity, and mole for amount of matter, as illustrated in table 1.1

Table 1.1

Physical quantity	S.I. Units	
	Name	Symbol
Length	metre	m
Mass	kilogram	kg
Time	second	S
Electric current	ampere	A
Thermodynamic temperature	kelvin	K
Luminous intensity	candela	cd
Amount of matter	mole	mol

The S.I. units also include two supplementary units :

(i) Unit of angle (radian (rd))

One Radian is the angle subtended at the centre of a circle, by an arc whose length is equal to the radius of the circle.

(ii) Unit of solid angle (steradian (sd))

One steradian is the solid angle subtended at the centre of a sphere, by a surface element of the sphere whose magnitude is equal to the square of the radius of the sphere.

Advantages of S.I. Units :

The following are the advantages of S.I. units :

(i) Seven basic units and two supplementary units cover all branches of physics. The units for the derived quantities are obtained as product or quotient of the unit of one or more fundamental physical quantities.

(ii) It absorbs the rationalised mks system.

S.I. Prefixes :

In actual measurements one come across very large and very small physical quantities compared to the corresponding base units. It is therefore necessary to define some multiple units and submultiple units. In S.I. system these units are related to the base units by powers of 10. Depending on the power of

10, suitable prefixes are attached to the base units. These are given in table 1.2.

Table 1.2

Prefix	Symbol	Factor	Prefix	Symbol	Factor
deci	d	10^{-1}	deca	da	10^{1}
centi	c	10^{-2}	hecto	h	10^{2}
milli	m	10^{-3}	Kilo	K	10^{3}
micro	μ	10^{-6}	Mega	M	10^{6}
nano	n	10^{-9}	Giga	G	10^{9}
pico	p	10^{-12}	Tera	T	10^{12}
femto	f	10^{-15}	Peta	P	10^{15}
atto	a	10^{-18}	Exa	E	10^{18}
Zepto	z	10^{-21}	Zetta	Z	10^{21}
yocto	y	10^{-24}	Yotta	Y	10^{24}

Practical Units (on common use)

There are some practical units in use for measuring macroscopic lengths, microscopic length, time, angle, volume, mass etc. as described below.

(i) **Astronomical unit (A.U.)**

It is the mean distance of earth from the sun.

1 A.U. $= 1.496 \times 10^{11}$m

(ii) **Light year :**

It is the distance travelled by light in vaccum in one year.

1 light year

$=3\times10^{8}\times365.25\times24\times60\times 60$m

$= 9.467 \times 10^{15}$ m.

(iii) **Parsec :**

It is the distance at which an arc of 1 AU subtends an angle of one second.

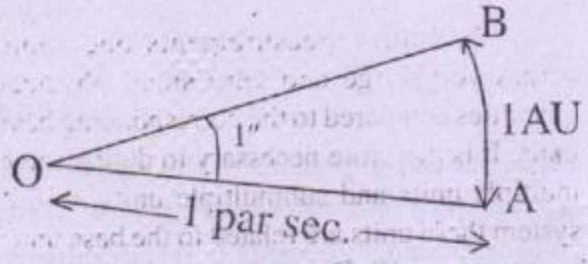


fig.1.1

(iv) **Angstrom :**

It is used in wave length measurement.

$1\ A^{0} = 10^{-10}$ m $= 10^{-8}$ cm.

(v) **Minute, hour, day and year :**

1 minute = 60 second, symbol used min

1 hour = 60 minute, symbol used hr(or h)

1 day = 24 hour, symbol used d

1 year = 365.25 days, symbol used yr.

(iv) **Degree, minute, second :**

1 degree $= \frac{\pi}{180}$ radian, symbol used °

e.g. $1^{0} = \frac{\pi}{180}$ rd

1 minute $= \left(\frac{1}{60}\right)$ deg $= \frac{\pi}{10800}$ rd,

symbol used "′"

e.g. $1' = \left(\frac{1}{60}\right)^{0}$

1 second $= \left(\frac{1}{3600}\right)^{0}$, symbol used is "″"

e.g. $1'' = \left(\frac{1}{60}\right)' = \left(\frac{1}{3600}\right)^{0}$

(vii) **Litre :**

It is a unit used in measurement of volume of liquids.

1 litre = 1000 cm^{3} = 10^{-3} m^{3} = 1 dm^{3}, symbol used is 'L' e.g. $1L = 10^{3}$ cm^{3}

(viii) **Tonne :**

It is a unit of mass used for expressing large amounts of mass.

1 Tonne = 10^{3} kg, symbol used is t.

Rules for using symbol for units :

(a) Small letters are ordinarily used for symbols of units. But if the unit is named after a person, then capital letter is used. But when the unit is written in full it begins with small letters e.g. it is correct to write 1 newton or 1 N but not 1 Newton or 1 n.

(b) Symbols are regarded as algebraic symbols (they are not followed by full stop, comma, dots, dashes etc.)

(c) The product of two or more units is written as follows :

Newton metre (or N-m)

(d) A derived unit, formed by division of two or more units is written as follows :

example m/s, or $\frac{m}{s}$ or ms^{-1}

m/s^2 or $\frac{m}{s^2}$ or ms^{-2}

$\frac{m}{LT^{-2}}$ or $mL^{-1}T^{-2}$

But it is wrong to write m/s/s, or m/L/ T^2

Rules for using S.I. prefixes :

(a) No space is left between the prefix and unit symbol. For example it is correct to write centimetre (but not centi-metre) and symbollically as cm.

(b) A unit symbol associated with a prefix is treated as a new symbol, which can be raised to any power.

e.g $1\ cm^3 = (0.01m)^3 = 10^{-6}\ m^3$

$1\mu s^{-1} = (10^{-6}s)^{-1} = 10^6 s^{-1}$

(c) Prefixes are not combined to form a compound prefix. For example 10^{-12} m is not written as μμm but simply as pm.

(d) Since kilogram (kg) already contains a prefix multiples or submultiples of this mass unit (kg) are expressed by associating prefix with gram. For example

$10^6\ kg = 10^9\ g = Gg$

$10^{-6}\ kg = 10^{-3}\ g = mg.$

1.9 Conversion among units :

While dealing with physical problems very often physical quantities are expressed (given) in different units. For example in a particular problem in mechanics mass might be given in kg, velocity in km/hr and time in second. Then it becomes necessary to bring them to **one** system of units and use the converted data for calculations. Thus it is imperative that one should know to convert from one unit system to another. This is achieved by following unit conversion method or direct method.

In **unit conversion method,** the given data is multiplied by a ratio equivalent to unit, called as **unity factor** or **conversion factor.** The conversion factors are derived from known identities.

For example

(i) 1 in = 0.0254 m = 2.54 cm

Here the unity factors are

$$1 = \frac{1\ in}{0.0254\ m} \quad ...(1.6.1)$$

$$1 = \frac{0.0254\ m}{1\ in} \quad ...(1.6.2)$$

$$1 = \frac{2.54\ cm}{1\ in} \quad ...(1.6.3)$$

$$1 = \frac{1\ in}{2.54\ cm} \quad ...(1.6.4)$$

Thus one can convert metre into inch by using (1.6.1), inch into metre by using (1.6.2), inch into centimetre by using (1.6.3) and centimetre into inch by using (1.6.4).

(ii) $1\ hr = 3600\ s$

$$\Rightarrow\quad 1 = \frac{3600\ s}{1\ hr} \quad ...(1.6.5)$$

$$\therefore\quad 1 = \frac{1\ hr}{3600\ s} \quad ...(1.6.6)$$

Thus to convent hour into second (1.6.5) is used and for converting second into hour (1.6.6) is used.

In **direct method** one substitutes the value of the unit in the desired unit directly.

For example :

(i) $18\ kg/L = \frac{18x10^3 g}{10^3 cm^3} = 18\ g/cm^3$

(ii) $36\ km/min^2 = \frac{36x10^3 m}{(60\ s)^2} = 10\ m/s^2$

(iii) $36\ km/hr = \frac{36x10^3 m}{(60\ x\ 60)s} = 10\ m/s$

Ex. 1.6.1 Express 18 km/hr in m/s

Soln.

$$18\ km/hr = \frac{18x10^3 m}{3600\ s} = 5 m/s$$

Ex. 1.6.2 How many neutrons are contained in 2 kg ? Given mass of neutron is 1.67×10^{-27} kg.

Soln.

$$n = \frac{2kg}{1.67x10^{-27} kg} = 1.198 \times 10^{27}$$

Ex. 1.6.3 How many electrons are there in 2g ? Given mass of an electron is 9.1×10^{-31} kg.

Soln.

$$n = \frac{2g}{9.1x10^{-31} kg} = \frac{2 \times 10^{-3} kg}{9.1x10^{-31} kg}$$

$$= 0.066 \times 10^{28}$$

$$n = 6.6 \times 10^{26}$$

Ex. 1.6.4 Convert an acceleration of 50 km/hr^2 into m/s^2.

Soln.

$$\frac{50km}{hr^2} = \frac{50x10^3 m}{(3600S)^2} = \frac{50x10^3 m}{1296x10^4 S^2}$$

$$= 3.85 \times 10^{-3}\ m/s^2$$

Ex. 1.6.5 How many nanograms are there in a mass of 2 exagram.

Soln.

$$2\ exagram = 2x10^{18}\ g = 2x10^{+27}x(10^{-9}g)$$

$$= 2 \times 10^{27}\ ng$$

Thus there are 2×10^{27} ng in 2 exagram.

1.10 Dimension and Dimensional Analysis

As discussed in sec. 1.3 the dirived physical quantities are expressed in terms of the fundamental (or. base) quantities through their defining relation.

e.g. speed = $\frac{\text{distance}}{\text{time}}$ = (length) (time)$^{-1}$

acceleration = $\frac{\text{velocity}}{\text{time}} = \frac{\text{length / time}}{\text{time}}$

= (length) (time)$^{-2}$

force = mass x accl^n = (mass) (length) (time)$^{-2}$

From the above it is observed that base quantities are raised to different exponents (or power) to represent a physical quantities. <u>The powers (or exponents) to which the base quantities must be raised to represent a physical quantity is called as its dimension or dimensional formula.</u>

For convenience the base quantities are represented by one letter symbol as given below

[mass] = M , [Luminous flux] = cd

[length] = L, [Thermodynamic temperature]=K

[Time] = T, [Amount of substance] = mol

[Electric current] = I

When the physical quantity is written on l.h.s and its dimensional formula on r.h.s, this constitutes a dimensional equation. Thus dimensional equation is an expression that indicates the relation of the derived physical quantity with the fundamental quantities.

e.g $[\text{force}] = [F] = M L T^{-2}$

$[\text{accl}^n] = [a] = L T^{-2}$ etc.

constitute dimensional equations.

Classification of physical quantities on the basis of dimension :

Physical quantities can be classified on the basis of dimensions as given below.

(i) **Dimensional variable :** Physical quantities which possess some finite dimension are called dimensional variables. e.g. force, velocity, work etc. are dimensional variables.

(ii) **Dimensionless variable :** Physical quantities which donot possess any dimension are called as dimensionless variable. e.g. specific gravity, strain, efficiency etc. are dimensionless variables.

(iii) **Dimensional constant :** Quantities which have constant magnitude but possess dimension are called dimensional constant. For example (a) speed of light (c) has a constant magnitude of 3×10^8 m/s. and has dimension LT^{-1}. (b) Universal gravitational constant (G) has a magnitude 6.67×10^{-11} $Nm^2 kg^{-2}$ and has dimension $M^{-1}L^3T^{-2}$.

(iv) **Dimensionless constant :** Constants which possess no dimension are called as dimensionless constants. e.g. 1,2,3,4π, e etc.

Dimension of derived physical quantities

In order to write the dimensional formula of derived quantities the following procedure may be adopted.

(a) Recall the definition (defining relation) of the physical quantity and write down the relation in the form of an equation with physical quantity on l.h.s and defining relation on r.h.s.

(b) Write the dimension of each of the quantity on r.h.s.

(c) Collect the dimensions of similar base quantities and write together.

Examples

(i) **Velocity (ϑ) :**

(a) $\vartheta = \dfrac{\text{displacement}}{\text{time}}$

(b) $[\vartheta] = \dfrac{L}{T}$

(c) $[\vartheta] = L T^{-1}$

(ii) **Acceleration (a)**

(a) $a = \dfrac{\Delta\vartheta}{\Delta t}$

(b) $[a] = \dfrac{LT^{-1}}{T}$

(c) $[a] = LT^{-2}$

(iii) **Momentum (p)**

(a) $p = m\vartheta$

(b) $[p] = M (LT^{-1})$

(c) $[p] = MLT^{-1}$

(iv) **Force (F)**

(a) $F = \dfrac{\Delta P}{\Delta t}$

(b) $[F] = \dfrac{MLT^{-1}}{T}$

(c) $[F] = MLT^{-2}$

(v) **Kinetic energy (T)**

(a) $T = \dfrac{1}{2} m\vartheta^2$

(b) $[T] = M.(LT^{-1})^2$

(c) $[T] = ML^2T^{-2}$

(vi) **Specific heat (s)**

(a) $Q = ms\theta$

(b) $[S] = \frac{[Q]}{[m][\theta]} = \frac{ML^2T^{-2}}{M.K.}$

(c) $[S] = L^2 T^{-2} K^{-1}$

(vii) **Electric field strength (E)**

(a) $E = \frac{F}{q}$

(b) $[E] = \frac{MLT^{-2}}{IT}$

(c) $[E] = MLT^{-3}I^{-1}$

Uses of Dimensional analysis.

Dimensional analysis can be used (i) to convert from one system of units to another system (ii) to check the correctness of a formula (iii) to derive a relation between various physical quantities.

The above uses are based on the principle of homogeneity of dimension, which states "the dimensional formula of every term on the two sides of a correct physical equation must be same".

(i) Conversion from one system to another system of units :

Suppose a physical quantity 'X' has dimensional formula.

$$[x] = M^aL^bT^cI^dK^e (cd)^f (mol)^g$$

Let the number associated with it in No 1 system of unit be n_1 and that in No 2 System of unit be n_2 then

$$n_1\ M_1^a\ L_1^b\ T_1^c\ I_1^d\ K_1^e\ (cd_1)^f\ (mol_1)^g$$

$$= n_2\ M_2^a\ L_2^b\ T_2^c\ I_2^d\ K_2^f\ (cd_2)^f\ (mol_2)^g$$

where subscripts 1 and 2 correspond to base units in No 1 system and No. 2 system respectively.

Then

$$n_2 = n_1 \left(\frac{M_1}{M_2}\right)^a \left(\frac{L_1}{L_2}\right)^b \left(\frac{T_1}{T_2}\right)^c \left(\frac{I_1}{I_2}\right)^d$$

$$\left(\frac{K_1}{K_2}\right)^e \left(\frac{cd_1}{cd_2}\right)^f \left(\frac{mol_1}{mol_2}\right)^g$$

Illustration : (i) conversion of 5N in S.I. system to dynes in C.G.S. system. We have

$$[F] = MLT^{-2}$$

so

$$n_2 = 5 \left(\frac{M_1}{M_2}\right)^1 \left(\frac{L_1}{L_2}\right)^1 \left(\frac{T_1}{T_2}\right)^{-2}$$

$$= 5 \left(\frac{1kg}{1g}\right)^1 x \left(\frac{1m}{1cm}\right)^1 x \left(\frac{1s}{1s}\right)^{-2}$$

$$= 5.\ 10^3.\ 10^2 = 5 \times 10^5$$

So $5N = 5 \times 10^5$ dyne

(ii) Conversion of 6 joules in S.I units to ergs in C.G.S system. We have

$$[E] = ML^2 T^{-2}$$

$$n_2 = n_1 \left(\frac{M_1}{M_2}\right)^1 \left(\frac{L_1}{L_2}\right)^2 \left(\frac{T_1}{T_2}\right)^{-2}$$

$$= 6 \times \left(\frac{1kg}{1g}\right)^1 \left(\frac{1m}{1cm}\right)^2 \left(\frac{1s}{1s}\right)^{-2}$$

$$n_2 = 6 \times 10^3 \times 10^4 = 6 \times 10^7$$

Hence 6 joul = 6×10^7 erg

[Note : The above proceedure only permits for conversion of fundamental values]

(ii) **Checking of correctness of relation :**

To check correctness of a given relation, dimension of each and every term on eitherside of an equation are calculated. If the dimensions of each term on either side are same then the formula is correct.

Illustratin : (i) check the correctness of the formula $S = ut + \frac{1}{2} at^2$

Now $[s] = L$

$[ut] = LT^{-1}\, T = L$

$[at^2] = LT^{-2}\, T^2 = L$

so formula is correct

(2) Time period $T = 2\pi \frac{\ell}{g}$

Now $[T] = T$

$$[\ell/g] = \frac{L}{LT^{-2}} = T^2$$

Since dimension of l.h.s. $\neq$ dimension of r.h.s

so the relation is not correct.

However the correctness of the formula can be tested up to the indeterminacy of the multiplying dimensionless constant if any. If a dimensional constant is present one should have prior knowledge about its dimension, so that the relation can be checked. For example consider

(3) the relation $T = 2\pi\sqrt{\frac{\ell}{g}}$

we find $[T] = T$

$$\left[\sqrt{\frac{\ell}{g}}\right] = \left[\frac{L}{LT^{-2}}\right]^{1/2} = T$$

Hence dimension of l.h.s. = dimension of r.h.s. and we are sure that $T \alpha \sqrt{\frac{\ell}{g}}$. But the above procedure does not throw any light on the constant 2π.

and

(4) the relation $F = G\,\frac{m_1 m_2}{r^2}$. In this case one can say whether the relation is correct or not with a prior knowledge about the dimesion of the universal gravitational constant.

(iii) **Derivation of relation between physical quantities.**

Relation of a physical quantity with others can be derived provided we know the factors on which the physical qantity depends.

Illustration

(a) **Centripetal force :**

When a body moves on a circular path with uniform speed, it experiences a force which could depend on the speed (ϑ), radius (r) of the circle and mass (m) of the body.

Let $F \;\alpha\; m^a$

$\alpha\; \vartheta^b$

$\alpha\; r^c$

Then by law of combination of variables

$$F = K\, m^a\, \vartheta^b\, r^c \qquad ...(1.7.1)$$

When 'K' is a dimensionless constant. Now dimension of l.h.s. = $[F] = MLT^{-2}$ and dimension of r.h.s. = $M^a (LT^{-1})^b (L)^c$

$= M^a\, L^{b+c}\, T^{-b}$

Equating the dimesions of l.h.s. and r.h.s. we obtain

$$a = 1$$
$$b + c = 1$$
$$-b = -2 \qquad ...(1.7.2)$$

Solution of (1.7.2) gives

$$a=1,\ b=2,\ c=-1 \qquad ...(1.7.3)$$

Hence equation (1.7.1) takes the form

$$F = K\,\frac{m\vartheta^2}{r} \qquad ...(1.7.3)$$

As said earlier the dimensional analysis cannot provide information about the dimensionless constant K. To find this value equation (1.7.3) shall have to be tallied with experiment. Experiment shows that K = 1. Hence

$$F = \frac{m\vartheta^2}{r} \qquad ...(1.7.4)$$

(b) **Time period of a simple pendulum :**

When a pendulum oscillates its time period could depend on the length of the suspension thread, and acceleration due to gravity. Therefore let

$$T \alpha \ell^a$$

$$T \alpha g^b$$

$$\Rightarrow T = K \ell^a g^b \qquad ...(1.7.5)$$

Now dimension of l.h.s. = [T] = T

and dimension of r.h.s = $L^a (LT^{-2})^b = L^{a+b} T^{-2b}$

Equating the dimensions of l.h.s. and r.h.s we obtain

$$a+b=0$$

$$-2b = 0 \qquad ...(1.7.6)$$

solution of eqn. (1.7.6) gives

$$b = -\frac{1}{2}, a = \frac{1}{2} \qquad ...(1.7.7)$$

Hence equation (1.7.5) takes the form

$$T = K\sqrt{\frac{\ell}{g}} \qquad ...(1.7.8)$$

Experiment suggests that $K = 2\pi$, so we write

$$T = 2\pi\sqrt{\frac{\ell}{g}} \qquad ...(1.7.9)$$

Limitations of Dimensional Anasysis

Dimensional analysis has few limitations.

(i) It gives no information regarding constant of proportionality present in a physical equation.

(ii) It cannot be used to derive formula involving trigonometric, exponential and logarithmic functions.

(iii) It **cannot** be used to derive experissions for a physical qantity if it depends on more than three quantities.

(iv) It **connot** be used to derive a formula if it involves a dimensional constant unless one has prior knowledge about the dimension of the constant.

(v) It cannot distinguish between two physical quantities having same dimension. e.g. work and torque.

(vi) It cannot reveal the inner mechanism of a phenomenon.

(vii) The method works only if the dependence is of the product type. For example one cannot derive the relation $S = ut + \frac{1}{2} at^2$ by dimensional method.

Ex. 1.7.1 Find the dimensional formula of the following (a) Pressure (b) Frequency (c) Power (d) Impulse (e) Angle.

Soln.

(a) Pressure $P = \frac{F}{A}$

$$\Rightarrow [P] = \frac{[F]}{[A]} = \frac{MLT^{-2}}{L^2} = ML^{-1}T^{-2}$$

(b) Frequencies $\nu = \frac{1}{T}$

$$\Rightarrow [\nu] = T^{-1}$$

(c) Power $P = \frac{\Delta W}{\Delta t}$

$$\Rightarrow [P] = \frac{ML^2T^{-2}}{T} = ML^2T^{-3}$$

(d) Impulse $\Delta P = m\,\Delta\vartheta$

$$\Rightarrow [\Delta P] = M.\,LT^{-1}$$

(e) Angle $\theta = \frac{\ell}{r}$

$$\Rightarrow [\theta] = \frac{L}{L} = L^0$$

$$\therefore [\theta] = M^0 L^0 T^0.$$

Ex. 1.7.2 Obtain the dimensions of the following quantities (a) Gravitational constant (b) planck's constant (c) gas constant.

Soln.

(a) $F = G\,\dfrac{m_1 m_1}{r^2}$

$$\Rightarrow [G] = \frac{[F][r^2]}{[m_1][m_2]} = \frac{MLT^{-2}.L^2}{M.M.}$$

$$\Rightarrow [G] = M^{-1} L^3 T^{-2}$$

(b) $E = h\nu$

$$\Rightarrow [h] = \frac{[E]}{[\nu]} = \frac{ML^2T^{-2}}{T^{-1}} = ML^2 T^{-1}$$

(c) $PV = RT$

$$\Rightarrow [R] = \frac{[P][V]}{[T]} = \frac{(MLT^{-2}/L^2)(L^3)}{K}$$

$$\Rightarrow [R] = ML^2 T^{-2} K^{-1}$$

Ex. 1.7.3 Which of the following quantities are dimensionally similar (a) force (b) pressure (c) work (d) torque (e) potential energy.

Soln.

(a) Force $[F] = MLT^{-2}$

(b) Pressure $[P] = \dfrac{[F]}{[A]} = ML^{-1} T^{-2}$

(c) Work $W = FS = MLT^{-2}. L = ML^2 T^{-2}$

(d) Torque $\vec{\tau} = \vec{r} \times \vec{F}$

$$[\tau] = L.\, MLT^{-2} = ML^2 T^{-2}$$

(e) Potential energy $V = mgh$

$$\Rightarrow [V] = M.\, LT^{-2}\, L = ML^2 T^{-2}$$

Thus inspection shows that work, torque and potential energy have similar dimensional formula.

Ex.1.7.4 The value of gravitational constant in C.G.S. system is 6.67×10^{-8} dyn. $cm^2\, g^{-2}$. What will be its value in S.I. units.

Soln.

$$G = 6.67 \times 10^{-8} \text{ dyne } cm^2.\, g^{-2}$$

$$[G] = M^{-1} L^3 T^{-2}$$

$$n_2 = n_1 \left(\frac{M_1}{M_2}\right)^{-1} \left(\frac{L_1}{L_2}\right)^3 \left(\frac{T_1}{T_2}\right)^{-2}$$

$$= 6.67 \times 10^{-8} \times \left(\frac{1g}{1kg}\right)^{-1} \left(\frac{1cm}{1m}\right)^3 \left(\frac{1s}{1s}\right)^{-2}$$

$$= 6.67 \times 10^{-8} \times 10^3 \times 10^{-6}$$

$$= 6.67 \times 10^{-11}$$

Hence in S.I. units $G = 6.67 \times 10^{-11}$ N.$m^2\, kg^{-2}$

Ex. 1.7.5 Convert an atmospheric pressure of 76 cm of mercury into S.I. units.

Soln.

$$P = h.\rho.g = (76 \times 13.6 \times 980) \text{ dyne/cm}^2$$

$$[P] = ML^{-1} T^{-2}$$

$$n_2 = n_1 \left(\frac{M_1}{M_2}\right)^1 \left(\frac{L_1}{L_2}\right)^{-1} \left(\frac{T_1}{T_2}\right)^{-2}$$

$$= 76 \times 13.6 \times 980 \times \left(\frac{1g}{1kg}\right)^1 \times \left(\frac{1cm}{1m}\right)^{-1} \left(\frac{1s}{1s}\right)^{-2}$$

$$= 76 \times 13.6 \times 980 \times 10^{-3} \times 10^2$$

$$n_2 = 1.0129 \times 10^5$$

$\therefore$ 1 atmospheric pressure $= 1.0129 \times 10^5\, Nm^{-2}$

Ex. 1.7.6 Acceleration due to gravity on the surface of earth is given by

$$g = G\,\frac{M}{R^2}$$

Where 'G' is the gravitational constant, M is the mass at earth and R is radius of earth. Using method of dimensional analysis check the relation.

Soln.

$$g = G\,\frac{M}{R^2}$$

Dimension of l.h.s. $[g] = LT^{-2}$

Dimension of r.h.s $G\,\dfrac{M}{R^2} = M^{-1}L^3T^{-2}.\dfrac{M}{L^2}$

$$= LT^{-2}$$

Thus dimension of l.h.s = dimension of r.h.s Hence relation is correct.

1.11 Modes of Expression of measurement :

(A) Exponentiation of 10 (powers of 10)

In physics we come across very large and very small measurements. For example diameter of the sun is 139 000 000 000 cm, while that of hydrogen atom is 0.0000000106 cm. But in this form it is cumbersome as well as incomprehensible. Therefore for the sake of convenience one expresses them as power of ten.

e.g $139000000000\text{ cm} = 1.39 \times 10^{11}\text{ cm}$

$0.0000000106 = 1.06 \times 10^{-8}\text{ cm}$

$682 = 6.82 \times 10^2$

$0.012 = 1.2 \times 10^{-2}$

The guide-lines for expressing a number in powers of 10 and use them in algebraic operations is as given below :

(i) A number in a decimal form is expressed as a number between 1 and 10, multiplied by a power of 10.

i.e. $Q = M \times 10^n$, with $1<M<10$

(ii) When the numbers are added or subtracted they are all written in terms of same power of 10.

e.g. $5 \times 10^3 + 2 \times 10^2 + 4 \times 10^{-2}$

$= 500000 \times 10^{-2} + 20000 \times 10^{-2} + 4\times10^{-2}$

$= 520004 \times 10^{-2} = 5200.04$

(iii) When the numbers are multiplied their exponents are simply added up and other factors are multiplied. Then finally the product is put in powers of 10.

e.g $(5\times10^m) \times (6\times10^n) = 30\times10^{m+n}$
$=3\times10^{m+n+1}$

$(5.2\times10^m) \times (6.3\times10^n) = 32.76\times10^{m+n}$

$= 3.276 \times 10^{m+n+1}$

(iv) When one number is divided by another, the exponent of the denominator is substracted from that of the numerator and other factors are calculated as usual. Then finally the result is given in exponent form.

e.g. $\dfrac{5\times10^m}{6\times10^n} = 0.833 \times 10^{m-n} = 8.33\times10^{m-n-1}$

(B) Rounding off a number :

Very often, depending upon the scope and necessity of a measurement, it is felt reasonable to round off a number after few digits beyond the decimal point. The rounding off is made following the rules as given below :

(i) If a digit <5 is dropped then the digit preceeding it is left unchanged.

e.g. $5.443 \rightarrow 5.44$

(ii) If a digit >5 is dropped, then the digit preceding it is raised by 1.

e.g. $5.446 \rightarrow 5.45$

(iii) If a digit = 5 is dropped, then an odd digit preceding it is raised by 1, while an even digit preceding it is left unchanged.

e.g. $5.435 \rightarrow 5.44$

$5.445 \rightarrow 5.44$

1.12 Significant figures :

Accuracy of a measurement depends on the least count (L.C.) of the instrument used for measurement. For example consider the measurement of the length of a stick by a metre scale (L.C. = 0.1cm), a slide callipers (L.C. = 0.01 cm) and a travelling microscope (L.C. = 0.001 cm). The metre scale finds its length to be 8.5 cm, the slide - callipers finds it to be 8.49 cm and the travelling microscope finds it to be 8.486 cm. This shows that (i) In the first measurement the digit '8' lying to the left of the decimal point is reliably known, but the digit 5, occuring at first decimal place is not reliable. The uncertainty, in this case, lies between $\frac{1}{10}$ to $\frac{1}{100}$. The observer, through eye estimation, rounds off at the first decimal place. (ii) In the second measurement the digit 8, lying to the left of decimal point and digit '4' at the first decimal place are reliably known, but the digit 9 occuring at the second decimal place is not reliable. The uncertainty in this case lies between $\frac{1}{100}$ to $\frac{1}{1000}$. Hence we round off at the second decimal place. (iii) Similarly in the third mesurement the digit '6' occuring at third decimal place is uncertain and we round off it at the third decimal place.

From the above discussion we see that the meaningful digits needed to be retained in a measurement are the digits which are reliably known plus the last digit that is uncertain. These figures are called significant figures. Thus **"Significant figures of a numerical value of measurement are the digits which are reliably known plus the last digit that is uncertain."** In the above example the number 8.5 has **two** significant figures, 8.49 has **three** significant numbers, and 8.486 has four significant figures.

Now suppose we express the first measurement in metres, then it is expressed as 0.085 m. Since the measuring apparatus is same, so the level of uncertainty, hence the number of significant figures should remain unchanged, i.e. 0.085 has two significant figures. This suggests that the zeroes occuring after the decimal point have no significance. But this does not hold good for the number of zeroes lying at the end of a numerical value. For example if one reports the length as 9.0 cm and another reports as 9 cm, then there is a difference. In the first measurement uncertainty occurs at the digit '0' at first decimal place, whereas in the second measurement the uncertainty occurs at the digit 9. Hence in first case number of significant figures is **two**, while in the second case the number of significant figures is **one**. This suggests that zeroes which appear after the decimal point are significant provided there is non-zero digit to the left of the decimal point.

Also zeroes occuring at the end without a decimal point are not significant.

e.g. $126000 = 1.26 \times 10^5$ has three significant figures.

The above discussion lead to the following conclusions

Conclusions :

i) All non-zero digits are significant

ii) The number of significant figures in a numerical value is equal to the number of digits counted from the first non-zero digit on the left to the last digit on the right.

iii) All zeroes occuring between two non-zero digits are significant e.g. 12.006 has five significant figures.

v) All zeroes lying in between the decimal point and the first non-zero digit on its right are not significant, if there is no non-zero digit to the left of the decimal point. e.g. 0.00126 has **three** significant figures but 1.00126 has **six** significant figures.

v) All zeroes appearing to the right of a decimal point with non-zero digits on the left of decimal point are significant.

e.g. 12.00 has **four** significant figures

vi) When there is no decimal point final zeroes are not signifiant e.g. 146000 has **three** significant figures.

vii) The last digit in the significant figures of a number is its uncertain digit.

Determination of significant figures :

Conclusions 1 to 7 can be used to determine the number of significant figures.

However one can also ascertain the number of significant figures by expressing the number in powers of 10 as illustrated below.

Number	Power of 10 form	No. of significant figures
0.387	3.87×10^{-1}	3
0.026	2.6×10^{-2}	2
1.00126	1.00126×10^{0}	6
0.00126	1.26×10^{-3}	3
6370000	6.37×10^{6}	3

Significant figures in algebraic operation :

Very often compounding of measurments becomes necessary. But all measurements are not made with same accuracy. Lack of accuracy in any one shall affect the accuracy of the final result, which is obtained after few algebraic operations.

For the sake of illustration consider the compounding of two numbers 5.4 and 13.72. In the number $n_1 = 5.4$ the uncertainty lies between 0.01 to 0.1. This means the range of values could be $5.5 > n_1 > 5.3$. In the number $n_2 = 13.72$, the uncertaintly lies between 0.001 to 0.01, so the range of values could be $13.73 > n_2 > 13.71$. So

(i) When we **add** them

$13.71 + 5.3 = 19.01$, $13.71 + 5.5 = 19.21$

$13.73 + 5.3 = 19.03$, $13.73 + 5.5 = 19.23$

$13.715 + 5.35 = 19.065$ etc.

i.e. $19.23 > n_1 + n_2 > 19.01$

This shows that the uncertainty starts at the first decimal place. Therefore we need to round off it at this digit (i.e. 1st decimal place). So we put

$13.72 + 5.4 = 19.12 = 19.1$

(ii) When we **subtract**

$13.71 - 5.3 = 8.41$, $13.71 - 5.5 = 8.21$

$13.73 - 5.3 = 8.43$, $13.73 - 5.5 = 8.23$

$13.715 - 5.35 = 8.365$ etc.

i.e. $8.43 > n_2 - n_1 > 8.21$

Here also uncertainty starts at the first decimal place. So we need to round off it at first decimal place.

i.e. $13.72 - 5.4 = 8.32 = 8.36$

(iii) When we **multiply**

$13.71 \times 5.3 = 72.663$, $13.71 \times 5.5 = 75.405$

$13.73 \times 5.3 = 72.769$, $13.73 \times 5.5 = 75.515$ etc.

i.e. $75.515 > n_1 n_2 > 72.663$

This shows that the uncertainty starts at the first digit before the decimal point. Hence final result should be rounded off at this first digit before decimal point.

i.e. $13.72 \times 5.4 = 74.088 = 74$

(iv) When we **divide**

$$\frac{13.71}{5.3} = 2.5867925; \frac{13.71}{5.5} = 2.4927273$$

$$\frac{13.73}{5.3} = 2.590566; \frac{13.73}{5.5} = 2.4963636$$

i.e. $2.590566 > \frac{n_2}{n_1} > 2.4927273$

This shows that the uncertainty starts at the first decimal place. Hence final result should be rounded off at first decimal place.

i.e. $\frac{13.72}{5.4} = 2.5407407 = 2.5$

The above analysis/exercise reveals that :

(i) When **numbers are added or subtracted** the final result shall have significant figures at those places where the constituent number having least significant figures had.

OR

The number of digits beyond the decimal point shall be equal to the smallest number of decimal places in the constituents.

(ii) When **product or quotient** of numbers are made the number of significant figures in the final result is equal to that of the constituent having the lowest number of significant figures.

Illustration

(i) $25.72 + 13.5 = 39.22 = 39.2$

(ii) $18.79 - 13.1 = 5.69 = 5.7$

(iii) $25.022 \times 10.31 = 257.97682 = 258.0$

(iv) $26.423 \div 9.072 = 2.9125882 = 2.912$

Ex.1.9.1 Determine the significant figures of the following measurements.

(i) 5.89 cm (ii) 38.25 kg (iii) 3.0089 g (iv) 0.0059 cm (v) 2.39×10^5 kg (vi) 3700 (vii) 16.0 (viii) 16.000.

Soln.

(i) 3 (ii) 4 (iii) 5 (iv) 2 (v) 3 (vi) 2 (vii) 3 (viii) 5.

Ex. 1.9.2 Obtain the following results with due regard to significant figures.

(i) 3.897+15.2 (ii) 19.388+0.003

(iii) 27.152+3.08 (iv) 15.2867-9.38

(v) 3.952-0.098 (vi) 12.98x3.2

(vii) 38.005x0.008 (viii) 9.58 ÷ 3.2

(ix) 7.925 ÷ 0.059

Soln.

(i) $3.897 + 15.2 = 19.097 = 19.1$

(ii) $19.388 + 0.003 = 19.391 = 19.391$

(iii) $27.152 + 3.08 = 30.232 = 30.23$

(iv) $15.2867 - 9.38 = 5.9067 = 5.91$

(v) $3.952 - 0.098 = 3.854 = 3.854$

(vi) $12.98 \times 3.2 = 41.536 = 42$

(vii) $38.005 \times 0.008 = 0.30404 = 0.3$

(viii) $9.58 \div 3.2 = 2.99375 = 3.0$

(ix) $7.925 \div 0.059 = 134.32203 = 130$

Ex. 1.9.3 A cylinder has length 8.9cm and radius 2.543 cm. Express its curved surface area and volume up to appropriate significant figures.

Soln.

$$S = 2\pi r \ell = 2 \times \pi \times 2.543 \times 8.9 \text{ cm}^2$$

$$= 142.20545 \text{ cm}^2 = 140 \text{ cm}^2$$

$$V = \pi r^2 \ell = \pi \times (2.543)^2 \times 8.9 \text{ cm}^3$$

$$= 180.81423 \text{ cm}^3 = 180 \text{ cm}^3$$

Ex. 1.9.4 A body has volume 14.56 cm^3 and mass 20.589 g. Calculate its density up to appropriate significant figure.

Soln.

$$\text{density } \rho = \frac{m}{V} = \frac{20.589 \text{ g}}{14.56 \text{ cm}^3}$$

$$\Rightarrow \rho = 1.4146747 \text{ cm}^3 = 1.415 \text{ cm}^3.$$

1.13 Order of magnitude

Sometimes order of magnitude of physical quantities are required than their exact values. For that a physical quantity is expressed as powers of 10, as

$$Q = M \times 10^n, \text{ with } 1<M<10$$

If $\frac{10}{M} < M$, then the order of magnitude is 10^{n+1}.

On the otherhand if $\frac{10}{M} > M$, then the order of magnitude is 10^n.

Ex. 1.10.1 $400 = 4 \times 10^2$

$$\frac{10}{4} = 2.5 < 4$$

$\Rightarrow$ order of magnitude 10^3

Ex. 1.10.2 $200 = 2 \times 10^2$

$$\frac{10}{2} = 5 > 2$$

$\Rightarrow$ order of magnitude 10^2

Ex. 1.10.3

(a) Charge of electron = 1.6×10^{-19} C.

$$\frac{10}{1.6} = 6.25 > 1.6$$

$\Rightarrow$ order of magnitude is 10^{-19} C

(b) Mass of earth = 5.983×10^{24} kg

$$\frac{10}{5.983} = 1.6 < 5.983$$

$\Rightarrow$ order of magnitude is 10^{25} kg

(c) radius of earth = 6.378×10^6 m

$$\frac{10}{6.378} = 1.5 < 6.378$$

$\Rightarrow$ order of magnitude is 10^7 m.

1.14 Errors in Measurement :

Measurement which is key to the study of science in general and physics in particular has two very important aspects accuracy and consistency (precision). The degree of difference between the measured or observed value (x_0) of a physical quantity and its true value (x) is a measure of the occuracy or error, and the degree of difference between the observed value (x_c) and mean of the observed values ($\bar{x}_0$) is a measure of the precision (consistency). For the sake of illustration consider the data in table 1.3, prepared by three observers A,B and C, while measuring mass of a body whose true mass is 2.30g (determined otherwise).

It is observed that the data in column A is accurate as well as consistent (precise), the data in column B is in accurate but consistent, and the data in column C is neither accurate nor precise.

It is a common experience that inspite of honest and sincere efforts some amount of inaccuracy and inconsistency creeps in and this leads to erratic results. Therefore it is necessary to locate various types of errors and the causes of such errors, so that one can find ways and means to minimise the error.

Types of errors :

Errors arising out of different reasons fall broadly into the following categories.

1. **Systematic error :** Errors which come into existence by virtue of definite rule are called systematic error. The following errors come under systematic error.

(a) **Instrumental error :** The error arising out of defect in the measuring instrument (like zero error, limits due to least count, constant error due to faulty graduation etc). This error can be ascertained.

(b) **Errors due to external cause :** This error arises due to variations in external conditions. For example while measuring

Table 1.3

A				B				C			
x_0	$\bar{x}_0$	$x-x_0$	$\bar{x}_0-x_0$	x_0	$\bar{x}_0$	$x-x_0$	$\bar{x}_0-x_0$	x_0	$\bar{x}_0$	$x-x_0$	$\bar{x}_0-x_0$
2.30		0	0	2.60		0.3	0.1	1.68		0.62	0.99
2.29	2.30	0.01	0.01	2.62	2.61	0.32	-0.1	2.67	2.67	-0.37	0
2.31		-0.01	-0.01	2.61		0.31	0	3.25		-0.95	-0.58

velocity of sound, the variations in temperature, affect the results. Corrections for such errors is possible.

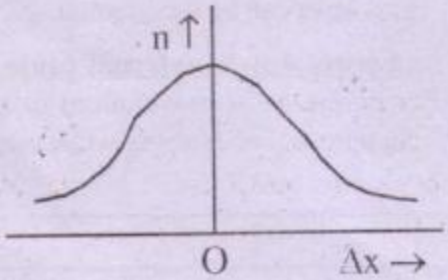
(c) **Error due to imperfection :** This error arises due to imperfection in experimental arrangement. For example in a heat experiment, in spite of all cares some radiation loss takes place. Similarly in a simple pendulum experiment air resistance cannot be avoided. However corrections for such errors are possible.

Thus when systematic error exists, the corrected value is

$$x_c = x_0 \pm a$$

Where x_0 is the observed value and 'a' is the correction due to a systematic error.

2. **Random error :** Error, which takes place in a random manner and cannot be associated with a systematic cause are called random errors. For example in the determination of focal length of concave mirror by u- ϑ method one records the positions of object pin and image pin sometimes avoiding parallax and sometimes not avoiding parallax. As a result the observed focal length is sometimes larger and sometimes smaller than the actual value. When the random error Δx is plotted against the number of times (n) this error is committed, the curve takes a Gaussian shape.



In such a distribution, the arithmatic mean of a number of observations is supposed to be most accurate. If $x_{10}, x_{20}, \ldots x_{n0}$ be values of various observations and $x_{1c}, x_{2c}, \ldots x_{nc}$, are the values after correction for systematic error, then the most occurate value $\bar{x}_0$ is given by

$$\bar{x}_0 = \frac{x_{1c} + x_{2c} + \ldots + x_{nc}}{n} = \frac{1}{n} \sum_{i=1}^{n} x_{ic} \quad \ldots(1.11.1)$$

Then we define

(a) Absolute error $= \Delta x_i = x_{ic} - \bar{x}_0$ ...(1.11.2)

(b) Mean absolute error $= \overline{\Delta x} = \frac{1}{n} \sum_{i=1}^{n} \Delta x_i$

...(1.11.3)

and the measured value can lie in between $\bar{x}_0 + \overline{\Delta x}$ and $\bar{x}_0 - \overline{\Delta x}$.

Sometimes it is also necessary to find

(c) Relative error (fractional error) defined as

$$\delta x_r = \frac{\overline{\Delta x}}{\bar{x}_0} \quad \ldots(1.11.4)$$

(d) Percentage of error defined as

$$\delta x_p = \frac{\overline{\Delta x}}{\bar{x}_0} \times 100 \quad \ldots(1.11.5)$$

and

(e) Standard deviation (standard error) defined by

$$\sigma = \sqrt{\frac{1}{n} \sum_{i=1}^{n} (\Delta x_i)^2} \quad \ldots(1.11.6)$$

Combination of errors :

In an actual esperiment result is obtained after taking measurements of different types of physical quantities, for example in the measurement of density one measures mass and volume separately and then calculates density from the observed velues of mass and volume. Since in each measurement some error is lileky to occur, so final result will depend on how the errors combine. For their purpose, we consider below various algebraic operations separately.

1. Error in a sum

Suppose Q is a physical quantity obtained from the sum of two physical quantities X and Y. Let the true values of X and Y be x and y respectively and x_o, y_o be the observed values. Then

$$x_o = x \pm \Delta x$$

$$y_o = y \pm \Delta y$$

Where Δx, and Δy are absolute errors in measurement of X and Y respectively. Then observed value of Q is given by

$$Z_o = x_o + y_o = (x \pm \Delta x) + (y \pm \Delta y) \quad ...(1.11.7)$$

since $z_o = z \pm \Delta z$,

so $z \pm \Delta z = (x+y) \pm \Delta x \pm \Delta y$

or $\pm \Delta z = \pm \Delta x \pm \Delta y \quad ...(1.11.8)$

Therefore greatest obsolute error in the sum is

$$\Delta z = \Delta x + \Delta y \quad ...(1.11.9)$$

This shows that

$$z + \Delta z > z_o > z - \Delta z$$

$$(x+y)+(\Delta x + \Delta y) > x_o + y_o > (x+y)-(\Delta x + \Delta y) \quad ...(1.11.10)$$

Equations (1.11.9) & (1.11.10) show that the greatest error in a sum is equal to the sum of individual obsolute errors.

2. Error in a difference

Similarly of Q is obtained by taking difference of two quantities X and Y, then

$$z_o = x_o - y_o = (x \pm \Delta x) - (y \pm \Delta y)$$

$$\Rightarrow \quad z_o = z \pm \Delta z = (x-y) \pm \Delta x \mp \Delta y$$

$$\Rightarrow \quad \pm \Delta z = \Delta x \mp \Delta y \quad ...(1.11.11)$$

Therefore the greatest obsolute error is

$$\Delta z = \Delta x + \Delta y \quad ...(1.11.12)$$

Equation (1.11.12) shows that

$$(x_o - y_o) + (\Delta x + \Delta y) > (x_o - y_o) > (x_o - y_o) - (\Delta x + \Delta y) \quad ...(1.11.13)$$

Equation (1.11.13) and (1.11.12) show that the greatest absolute error in a difference is the sum of individual absolute error.

3. Error in product

Suppose a physical quantity 'Q' is obtained by taking product of two measurable physical equantities X and Y. Then

$$z = xy$$

$$z_o = x_o y_o$$

$$x_o = x \pm \Delta x$$

$$y_o = y \pm \Delta y$$

$$z_o = x_o \cdot y_o = (x \pm \Delta x)(y \pm \Delta y)$$

$$= xy\left(1 \pm \frac{\Delta x}{x}\right)\left(1 \pm \frac{\Delta y}{y}\right)$$

$$\Rightarrow \quad z \pm \Delta z = z\left(1 \pm \frac{\Delta x}{x} \pm \frac{\Delta y}{y} \pm \frac{\Delta x . \Delta y}{xy}\right)$$

Retaining up to 1st order smallness in r.h.s

$$z \pm \Delta z = z\left(1 \pm \frac{\Delta x}{x} \pm \frac{\Delta y}{y}\right)$$

$$\Rightarrow \quad 1 \pm \frac{\Delta z}{z} = 1 \pm \frac{\Delta x}{x} \pm \frac{\Delta y}{y}$$

$$\Rightarrow \quad \pm \frac{\Delta z}{z} = \pm \frac{\Delta x}{x} \pm \frac{\Delta y}{y} \quad ...(1.11.14)$$

Therefore the maximum fractional (relative) error is

$$\left.\frac{\Delta z}{z}\right|_{max} = \frac{\Delta x}{x} + \frac{\Delta y}{y} \quad ...(1.11.15)$$

and percentage error is $\frac{\Delta z}{z} \times 100 = \left(\frac{\Delta x}{x} + \frac{\Delta y}{y}\right) \times 100$

Equation (1.11.15) shows that the maximum fractional error in a product is the sum of individual fractional errors. This maximum fractional error can also obtained by taking logarithmic differentiation as given below

$$z = xy$$

$$\log z = \log x + \log y$$

Differentiating

$$\frac{dz}{z} = \frac{dx}{x} + \frac{dy}{y}$$

4. Errors in a quotient :

If a physical quantity 'Q' is obtained as the quotient of two quantities X and Y then

$$z = \frac{x}{y}$$

$$z_o = \frac{x_o}{y_o}$$

$$x_o = x \pm \Delta x$$

$$y_o = y \pm \Delta y$$

$$z_o = z \pm \Delta z$$

$$\Rightarrow \quad z_o = z \pm \Delta z = \frac{x_o}{y_o} = \frac{x \pm \Delta x}{y \pm \Delta y}$$

$$\Rightarrow \quad \pm \Delta z = \frac{x\left(1 \pm \frac{\Delta x}{x}\right)}{y\left(1 \pm \frac{\Delta y}{y}\right)} = \frac{x}{y}\left(1 \pm \frac{\Delta x}{x}\right)\left(1 \pm \frac{\Delta y}{y}\right)^{-1}$$

$$z \pm \Delta z = z\left(1 \pm \frac{\Delta x}{x}\right)\left(1 \mp \frac{\Delta y}{y}\right)$$

$$= z\left(1 \pm \frac{\Delta x}{x} \mp \frac{\Delta y}{y}\right)$$

$$\Rightarrow \quad 1 \pm \frac{\Delta z}{z} = 1 \pm \frac{\Delta x}{x} \mp \frac{\Delta y}{y}$$

$$\Rightarrow \quad \pm \frac{\Delta z}{z} = \pm \frac{\Delta x}{x} \mp \frac{\Delta y}{y} \quad ...(1.11.16)$$

Therefore the greatest fractional error shall be

$$\left.\frac{\Delta z}{z}\right|_{max} = \frac{\Delta x}{x} + \frac{\Delta y}{y} \quad ...(1.11.16)$$

and percentage error is $\frac{\Delta z}{z} \times 100 = \left(\frac{\Delta x}{x} + \frac{\Delta y}{y}\right) \times 100$

Equation (1.11.16) is also obtained by logarithmic differentiation.

$$z = \frac{x}{y}$$

$$\log z = \log x - \log y$$

Differentiating

$$\frac{dz}{z} = \frac{dx}{x} - \frac{dy}{y}$$

so $$\left.\frac{dz}{z}\right|_{max} = \frac{dx}{x} + \frac{dy}{y}$$

5. Error in a combined calculation

Suppose $\quad z = A^p B^q C^{-r}$

$$A_o = a \pm \Delta a$$

$$B_o = b \pm \Delta b$$

$$C_o = c \pm \Delta c$$

$$Z_o = z \pm \Delta z$$

Then

$$z_o = z \pm \Delta z = A_o^p \, B_o^q \, C_o^{-r} = (a \pm \Delta a)^p (b \pm \Delta b)^q (c \pm \Delta c)^{-r}$$

$$\Rightarrow \quad z \pm \Delta z = a^p b^q c^{-r}\left(1 \pm \frac{\Delta a}{a}\right)^p \left(1 \pm \frac{\Delta b}{b}\right)^q \left(1 \pm \frac{\Delta c}{c}\right)^{-r}$$

$$= a^p b^q c^{-r}\left(1 \pm p\frac{\Delta a}{a}\right)\left(1 \pm q\frac{\Delta b}{b}\right)\left(1 \mp r\frac{\Delta c}{c}\right)$$

$$\Rightarrow \quad z \pm \Delta z = z\left(1 \pm p\frac{\Delta a}{a} \pm q\frac{\Delta b}{b} \mp r\frac{\Delta c}{c}\right)$$

(Retaining up to 1st order in r.h.s)

$$\Rightarrow \quad 1 \pm \frac{\Delta z}{z} = 1 \pm p\frac{\Delta a}{a} \pm q\frac{\Delta b}{b} \mp r\frac{\Delta c}{c}$$

$$\Rightarrow \quad 1 \pm \frac{\Delta z}{z} = \pm p\frac{\Delta a}{a} \pm q\frac{\Delta b}{b} \mp r\frac{\Delta c}{c} \quad ...(1.11.17)$$

This implies the greatest fractional error is

$$\left.\frac{\Delta z}{z}\right|_{max} = p\frac{\Delta a}{a} + q\frac{\Delta b}{b} + r\frac{\Delta c}{c} \quad ...(1.11.18)$$

and precentage error is $\frac{\Delta z}{z} \times 100$

$$= \left(p\frac{\Delta a}{a} + q\frac{\Delta b}{b} + r\frac{\Delta c}{c}\right) \times 100$$

This idea is used in estimating the permissible errors in actual physics experiments done in the laboratory as we discuss below :

Permissible error

In an actual experiment in physics done with extreme care and sincerity, errors arise due to the limitations of the apparatus. The maximum fractional error introduced by the limits of the apparatus used is called maximum permissible error.

For example consider the determination of the volume of a cylinder whose radius is measured by a screw gauge of L.C. = 0.001 cm and whose length is measured by a slide callipers of L.C. = 0.01 cm.

Now

$$V = \pi r^2 \ell$$

$$\Rightarrow \quad \frac{dV}{V} = 2\frac{dr}{r} + \frac{d\ell}{\ell}$$

$$\left.\frac{dV}{V}\right|_{max} = \frac{2}{\bar{r}_0}\Delta r + \frac{1}{\bar{\ell}_0}\Delta\ell$$

Where $\bar{r}_0$ and $\bar{\ell}_0$ are the mean observed values, supposed to be most accurate. Now $\Delta r = 0.001$ cm and $\Delta\ell = 0.01$ cm.

Then $\left.\frac{dV}{V}\right|_{max} = \left(\frac{2}{\bar{r}_0} \times 0.001 + \frac{1}{\bar{\ell}_0} \times 0.01\right)$

The r.h.s. gives a measure of maximum permissible error. The correction factor is

$$dV = \left(\frac{2}{\bar{r}_0} \times 0.001 + \frac{1}{\bar{\ell}_0} \times 0.01\right) \times \bar{V}_0$$

So the corrected value of volume shall be

$$V_c = \bar{V}_0 \pm dV$$

Thus while calculating the maximum permissible error, the obsolute error in a measureemnt is made equal to the L.C. of the instrument measuring that quantity.

Ex.1.11.1 Following observations were taken with a slide callipers while measuring the length of a cylinder, 3.29 cm, 3.28 cm, 3.29 cm, 3.31 cm, 3.28 cm, 3.27 cm, 3.29 cm, 3.30 cm. Find (a) most accurate length of the cylinder (b) Absolute error of the first and last observation (c) relative error (d) percentage error.

Soln.

(a) most accurate length = $\bar{\ell}$

$$= \frac{3.29 + 3.28 + 3.29 + 3.31 + 3.28 + 3.27 + 3.29 + 3.30}{8}$$

$$\Rightarrow \quad \bar{\ell} = 3.28875 \text{ cm} = 3.29 \text{ cm}$$

(b) Absolute error of first observation

$= 3.29 - 3.29 = 0.00$ cm

Absoulte error of last observation

$= 3.30 - 3.29 = 0.01$ cm

(c) Mean absolute error $\overline{\Delta \ell}$

$$= \frac{0+0.01+0+0.02+0.01+0.02+0+0.01}{8}$$

$$\overline{\Delta \ell} = \frac{0.07}{8} = 0.0088 = 0.01 \text{ cm}$$

$$\text{Relative error} = \frac{\overline{\Delta \ell}}{\ell} = \frac{0.01}{3.29} = 0.003039$$

$$= 0.003$$

(d) $\text{Percentage error} = \frac{\overline{\Delta \ell}}{\ell} \times 100 = 0.3\%$

Ex. 1.11.2 Find the sum of the following quantities (5.897 ± 0.3) cm and (17.87 ± 0.6) cm.

Soln.

$$(5.897 \pm 0.3) + (17.87 \pm 0.6)$$
$$= 23.767 \pm 0.9 = 23.77 \pm 0.9$$

Ex. 1.11.3 Calculate the density of a body whose mass $m = (32.98 \pm 0.9)$ g and volume $V = (5.79 \pm 0.3)$ cm^3.

Soln.

$$\rho = \frac{m}{V} = \frac{32.98}{5.79} = 5.696 = 5.70$$

$$\frac{d\rho}{\rho} = \frac{dm}{m} + \frac{dV}{V} = \frac{0.9}{32.98} + \frac{0.3}{5.79}$$

$$= 0.079$$

$$\therefore d\rho = 0.079 \times 5.7 \text{ g cm}^{-3} = 0.45 \text{ g cm}^{-3}$$

$$\text{density} = (5.70 \pm 0.45) \text{ g cm}^{-3}$$

Ex. 1.11.4 Find the volume of the glass slab whose measurements are (7.85 ± 0.3) cm, (2.98 ± 0.3) cm, and (5.387 ± 0.02) cm.

Soln.

$$V = 7.85 \times 2.98 \times 5.387 = 126$$

$$\frac{\Delta V}{V} = \frac{\Delta a}{a} + \frac{\Delta b}{b} + \frac{\Delta c}{c}$$

$$= \frac{0.3}{7.85} + \frac{0.1}{2.98} + \frac{0.02}{5.387} = 0.075$$

$$\Delta V = 0.075 \times 126 \text{ cm}^3 = 9.45 \text{ cm}^3$$

$$V = (126 \pm 9.45) \text{ cm}^3.$$

Ex. 1.11.5 A cylinder has length (12.87 ± 0.2) cm and has radius (3.85 ± 0.5) cm. What is the volume of the cylinder.

Soln.

$$V = \pi r^2 \ell = \pi \times (3.85)^2 \times (12.87)$$

$$V = 599 \text{ cm}^3.$$

$$\frac{\Delta V}{V} = 2 \frac{\Delta r}{r} + \frac{\Delta \ell}{\ell}$$

$$= 2 \times \frac{0.5}{3.85} + \frac{0.2}{12.87}$$

$$= 0.26 + 0.016 = 0.276$$

$$\therefore \Delta V = 0.276 \times 599 = 165.324$$

$$V = (599 \pm 165)$$

Summary

1. Physics is a study of basic laws of nature and its manifestations in different natural phenomena in terms of few concepts and laws.

2. Physics has two basic thrusts : Unification and Reduction.

3. Physics and Technology are complementary to each other. Some times technology gives rise to new physis where as other times physis induces a new technology. Both have direct impact on society.

4. The four basic forces of nature are : (a) electromagnetic force (b) weak force (c) strong force and (d) gravitation.

5. Sheldon Glashow, Abdus salam and Steven weinbergs (1979) have unified successfully weak and electromagretic force. The predictions of this unified theory was later on verified exprimentally by Carlo Rabia, and Semon Vander Meer.

6. Conclusions are drawn, basing on precise measurements on various aspects of physical phonomena.

7. Measurement of a physical quantity always involves comparison with a similar physical quantity, having a suitably chosen (1 assigned) value and the measured value is a multiple of the chosen value. e.g Q=nq, where q is chosen value, n is a real number, Q is the value of the physical quantity to be measured.

8. Unit of a physical quantity is a conveniently and suitably chosen value of that quantity with which comparisions can be made and a standard is the physical embodiment of unit.

9. Standards must fulfill the following.

(a) It should be easily and conveniently accesible to all.

(b) It should be easily and accurately reproduable

(c) It should be invariable

(d) It should be permanent

10. A unit is chosen so that :

(a) It easily and quickly conveys an easy conception about the measurement.

(b) It is conventional and adopted by all

(c) It is of the same kind

(d) It is convenient for calculation.

11. There are (i) length standards (1 meter) (ii) mass standards (1 kilogram) (iii) time standards (1sec) (iv) Current standards (1ampere) (v) Temperature (1kelvin) standard (vi) standard for intensity of light (1 candella) and (vii) standard for amount of substance (1mole)

12. The international system of units (SI) based on above seven base units is at present internationally accepted system of unit.

13. Derived quantities are expressed in term of derived units which are combinations of two or more of the seven fundamental units.

14. Small letters are ordinary used for symbols of units. But if the unit is named after a person, then capital letter is used. Symbals are not followed by full stop, dot, comma dashes etc.

15. Product of two or more units is written with-dashes.

16. No space is left between the prefix and unit symbol.

17. For the sake of convenience one expresses the magnitude of measurement as powers of ten.

e.g $682 = 6.82 \times 10^2$, $0.012 = 1.2 \times 10^{-2}$

i.e $Q = m \times 10^n$, with $1<m<10$

18. When the numbers are added or subtracted they are all written in terms of same power of 10.

e.g $5 \times 10^3 + 2 \times 10^2 + 4 \times 10^{-2}$

$= 500000 \times 10^{-2} + 2000 \times 10^{-2} + 4 \times 10^{-2}$

$= 520004 \times 10^{-2} = 5200.04$

19. When the numbers are multiplied their exponents are simply added up and other factors are multiplied.

e.g $(5 \times 10^m) \times (6 \times 10^n) = 30 \times 10^{m+n}$

$= 3.0 \times 10^{m+n+1}$

20. Significant figures of a numerical value of measurement are the digits which are reliably known plus the last digit that is uncentain.

(a) All non-zero digits are sgnificant

(b) The number of significant figunes in a numerical value is equal to the number of digits counted from the first non-zero digit on the left to the last digit on the right.

(c) All zeroes occuring between two non-zero digits are significants e.g. 12.000 has five significant figures.

(d) All zeroes lying in-between the decimal point and the first non-zero digits on its right are not significant, if there is no non-zero digit to the left of the decimal point. e.g. 0.00126 has three significant figures, but 1.00126 has six significant figures.

(e) All uses appearing to the right of a decimal point with non-zero digit on the left of the decimal point are significant. e.g. 12.00 has four significant figures.

(f) When there is no decimal point final zeros are not significant.

(g) The last digit is the significant figures of a number is its uncertain digit.

21. (a) When numbers are added or subtracted the final result shall have significant figures at those places where the constituent number having least significant figures had.

(b) When product or quotient of numbers are made the numbers of significant figures in the final result is equal to that of the constutuent having the lowest number of significant figure.

22. In physical measurements errors could occur due to instruments used, external cause, imperfection in arrangement of the expenment. These errors are called systematic error.

There could be also error due to the observers fault in a random manner.

23. The greatest error in a sum is equal to the sum of individual absulute errors.

$\Rightarrow \quad z \pm z = (x+4) \pm \Delta x \pm \Delta y$

$\Rightarrow \quad \pm z = \pm \Delta x \pm \Delta y$

Greatest absolute error $\Delta z = \Delta x + \Delta y$

This implics $z + \Delta z > z_0 > z - \Delta z$

$\Rightarrow (x+y) + (\Delta x + \Delta y) > x0 + y0 > (x+y) - (\Delta x + \Delta y)$

24. Errors in a difference : The greatest absolute errors in a difference is the sum of individual absolute error.

25. Error in a product : The maximum fraction error in a product is the sum of individual fractional errors.

26. The powers (or exponents) to which the base quantities must be raised to represent a physical quantity is called as its dimemsion or dimensimal formula.

27. The base quantities are (a) mass (M) (b) length (L) (c) time (T) (d) electric cument (I).

28. When the physical quantity is written on l.h.s and its dimensimal formula on r.h.s. this constitutes a dimensimal equation.

e.g. $[F] = MLT^{-2}$

$[a] = LT^{-2}$

MODEL QUESTIONS

A. Multiple Choice Type Questions:

1. Unit of illumination is
   (i) Lumen (ii) Candela
   (iii) Lux (iv) Photon

2. Watt is the unit of
   (i) Work (ii) Power
   (iii) Charge (iv) Pressure

3. Which of the following has energy units ?
   (i) Rate of doing work
   (ii) Watt
   (iii) Potential gradient
   (iv) Product of pressure and volume

4. Energy is expressed in the same unit as
   (i) Work (ii) Pressure
   (iii) Force (iv) Momentum

5. Which of the following is not a unit of time
   (i) Leap year (ii) Lunar month
   (iii) Light year (iv) micro-second

6. Density of wood is 0.5 g cm^{-3} in C.G.S. system of units. The corresponding value in SI units is
   (i) 500 kg m^{-3} (ii) 5 kg m^{-3}
   (iii) 0.5 kg m^{-3} (iv) 5000 kg m^{-3}.

7. Dimensions of velocity gredient are same as those of
   (i) time period
   (ii) frequency
   (iii) angular acceleration
   (iv) acceleration

8. Out of the following pairs, which pair does not have identical dimensions
   (i) angular momentum and planck's constant
   (ii) moment of inertia and moment of force
   (iii) work and torque
   (iv) impulse and momentum.

9. The dimensional formula of Young's modulus is
   (i) $ML^{2}T^{-2}$ (ii) MLT^{-3}
   (iii) $ML^{-1}T^{-2}$ (iv) none of these

10. The dimensional formula of rigidity modulus is
    (i) $ML^{-1}T^{-1}$ (ii) $ML^{-1}T^{-2}$
    (iii) $ML^{-1}T^{-3}$ (iv) $ML^{-2}T^{-2}$

11. The de-Broglie wavelength associated with a particle of mass m and energy E is $h/\sqrt{2mE}$. The dimensional formula of planck's constant is
    (i) $M^{2}L^{2}T^{-2}$ (ii) $ML^{2}T^{-1}$
    (iii) $ML^{2}T^{-2}$ (iv) MLT^{-2}

12. Choose the correct dimension for surface tension
    (i) MLT^{-2} (ii) $ML^{-1}T^{-2}$
    (iii) MT^{-2} (iv) $ML^{2}T^{-2}$

13. The dimensional formula of coefficient of viscocity is
    (i) $ML^{-1}T$ (ii) MLT
    (iii) $ML^{-1}T^{-1}$ (iv) $M^{2}L^{-1}T^{-2}$

14. What is the dimensional formula of 'a' in the equation $\left(P+\frac{a}{V^2}\right)(V-b) = RT$, Where letters have usual meanings ?
    (i) $ML^{5}T^{-2}$ (ii) $M^{-1}L^{5}T^{3}$
    (iii) $M^{0}L^{3}T^{-2}$ (iv) $M^{6}L^{7}T^{4}$

15. Which of the following statements is dimensionally correct
    - (i) Pressure is force per unit length
    - (ii) Pressure is force per unit volume
    - (iii) Pressure is the energy per unit area
    - (iv) Pressure is the energy per unit volume

16. When 400.1 is added to 20.54, we get
    - (i) 420 (ii) 420.65
    - (iii) 420.1 (iv) 420.6

17. The number of significant figures in 20.96 and 0.003 125 is
    - (i) 4,4 (ii) 3.,7
    - (iii) 4,7 (iv) 2,7

18. The number of significant figures in 40.00 is
    - (i) 4 (ii) 1
    - (iii) 3 (iv) 2

19. In a measurement maximum error in mass is 3%, and maximum error in volume is 2%. Then the maximum error in density is
    - (i) 1% (ii) 5%
    - (iii) 7% (iv) 9%

20. In measurement of length (L) and time (T) 0.1% and 2% errors are committed. What is the maximum percentage of error in LT^2
    - (i) 4.1% (ii) 3%
    - (iii) 2.1% (iv) 5%

21. The error in the measurement of radius of a sphere is 2%. The percentage error in the calculation of volume is
    - (i) 2% (ii) 4%
    - (iii) 6% (iv) 8%

22. What is the area of a circle (with due regards for significant figures) if the radius of the circle is 2.12 m^2
    - (i) 4.452m^2 (ii) 26.452m^2
    - (iii) 14.125257m^2 (iv) 14.1m^2

23. A time interval of 3.1 s is recorded by a stop-watch whose least count is 0.1 s. The limits between which the interval lies is
    - (i) (3.1 ± 0.1)s
    - (ii) (3.100 ± 0.1)s
    - (iii) (3.100 ± 0.001)s
    - (iv) (3.1 ± 0.2)s

24. The radius of a ball is (4.7 ± 0.2)cm. The percentage error in the volume of the ball is
    - (i) $3 \times \frac{0.2 \times 100}{4.7}$
    - (ii) $\frac{1}{3} \times \frac{0.2 \times 100}{4.7}$
    - (iii) $3 - \frac{0.2 \times 100}{4.7}$
    - (iv) $\frac{0.2}{4.7} \times 100$

B. Very Short Answer Type Questions :

1. Are there more microsconds in a second than there are seconds in a year ?
2. In the measurement of distance between earth and sun which unit you will chose light year or fermi ?
3. An equation is dimensionally correct, does it mean equation is necessarily correct ?
4. What are significant figures in 0.00985 ?
5. What do you mean by absolute error ?
6. How do the errors combine, when two quantities are added or subtracted.

7. How do the errors combine in multiplication or division ?
8. Add 27.53 + 0.032 to proper signific... figures.
9. Write the dimension of pressure.
10. Write the dimension of angular momentum.
11. Write the dimension of torque.
12. Give the unit of surface tension.
13. Give the unit of torque.
14. Give the dimension of Young's modulus
15. In what units wavelength of visible light is measured.
16. What does parsec measure ?
17. What does fermi measure ?
18. Write the dimension of planck's constant.
19. Express dyne/cm^2 in N/m^2.

C. Short Answer Type Questions :

1. What do you means by a unit ?
2. What do you mean by a standard ?
3. Determine dimensions of R from the gas equation PV = RT.
4. State the principle of homogeneity in dimensional analysis.
5. State the limitations of dimensional analysis.
6. What is maximum permissible error in an experiment.
7. Mention various types of systematic errors.
8. Show by dimensional analysis that $T = 2\pi\sqrt{\frac{\ell}{g}}$ is correct.
9. A stone weighs (10.0 ± 0.1) kg in air and (5.0 ± 0.1) kg in water. Calculate the maximum percentage error in the measurement of specific gravity.
10. The internal diameter of a cylinder is (3.89 ± 0.01) cm and its external diameter is (4.23 ± 0.01) cm. Calculate the thickness of the wall of the cylinder.
11. What are the basic requirements of a standard ?
12. What is the advantage of using wavelength as standard of length ?
13. What do you mean by a coherent system of units ?
14. Under what conditions zeroes are significant ?
15. Does a change in the units of a measurement have any effect on its significant figures ?
16. What information do we get from the significant figure of measurement ?
17. What are the limitations of dimensinal analysis ?

D. Numerical Problems :

1. Express 0.53 A^0 in meters
2. Using force (F), length (L) and time (T) as fundamental dimension, write down the dimension of mass.
3. The value of coefficient of viscosity is 12 in C.G.S. units. What is the value in SI unit ?
4. Velocity of sound in a gas depend on its pressure and density. Obtain the relation among velocity, pressure and density.
5. Surface tension of water in C.G.S units is 72 dyne/cm. What is its value in S.I. units ?
6. Using force (F), Length (L) and time (T) as fundamental dimension, find the dimension of surface tension, young's modulus.

7. Find the number of significant figures in the following (i) 0.009 m (ii) 0.3280 N (iii) 0.0009098 (iv) 9.28×10^{-23} kg.

8. Each side of a cube is 7.203 m. Find its total surface area and volume to proper significant figures.

9. In an experiment time period of a simple pendulum are measured as 2.63 s, 2.56 s, 2.71 s, 2.80 s. Find average absolute error and the percentage error.

10. Find the maximum error in the value of Y given by $Y = \frac{4MgL}{\pi D^2 \ell}$, where M = (1000 ± 0.05) g

$L = (200 \pm 0.1)$, $D = (0.075 \pm 0.001)$cm

and $\ell = (0.325 \pm 0.001)$cm

11. A force (5.87 ± 0.3) N is applied on an area $(2.85 \pm 0.6)\ m^2$. What is pressure on the area ?

E. Long Answer Type :

1. The frequency of transverse vibration of string depends on its tension T, length ℓ, and mass per unit length m. Derive the expression for its frequency by - dimensional analysis.

2. State the rules which govern the determination of significant figures in algebraic operations. Give one example for each.

3. How do errors combine together in algebraic operations. Discuss with examples.

4. Discuss the various types of errors likely to be introduced in a measurement.

F. Fill in the blank type :

1. Unit of illumination is.............

2. Dimension of velocity gradient is same as that of ....

3. The dimensional formula of coefficient of viscosity is............

4. Plank's constant has the dimensions.......

5. Energy and work have .... dimensions.

6. The error in the measurement of radius of a sphere is 2%. The percentage error in the calculation of volume is.........

7. The number of significant figures in 40.00 is ....

8. The radius of a ball is (4.7 ± 0.2) cm. The percentage error in the volume of the ball is............

G. True - False - Type

1. A pressure of 10^6 dyne/cm^2 is equivalent to $10^5\ N/m^2$.

2. The universal gas constant R, occuring in the equation of state PV = nRT for n moles of an ideal gas is a dimensional consistant.

3. Linear momentum and moment of a force have identical dimensions.

4. Pressure and Young's modulus have idential dimension.

5. Density has the dimensions ML^{-3}.

ANSWERS

A. MULTIPLE CHOICE TYPE :

1. (ii), 2. (ii), 3. (iv), 4. (i), 5. (ii), 6. (i), 7. (ii), 8. (ii), 9. (iii), 10. (ii), 11. (ii), 12. (iii), 13. (iii), 14. (i), 15. (iv), 16. (iv), 17. (i), 18. (i), 19. (ii), 20. (i), 21. (iii) 22. (iv), 23. (i), 24. (i)

D. NUMERICAL PROBLEMS :

1. 5.3×10^{-11} m
2. $FL^{-1}T^{2}$
3. 1.2
4. $\vartheta = \sqrt{\frac{P}{\rho}}$
5. 0.072 N/m
6. FL^{-1}; FL^{-2}
7. (i) 1 (ii) 4 (iii) 4 (iv) 3
8. 311.3 cm^2, 373.7cm^3
9. 0.08 s ; 2.99%
10. 3.03%
11. (2.06 ± 0.51) N/m^2

F. (1) candella (2) frequency (3) $ML^{-1}T^{-1}$ (4) $ML^{2}T^{-1}$ (5) Idntical (6) 6% (7) 4 (8) $3\times\frac{0.2\times100}{4.7}$

G. (1) True (2) True (3) False (4) (True) (5) True

2

Scalars and Vectors

The role of physical quantities is to describe various physical phenomena. For example when we study the density of a body, we need to know about the magnitude of its mass and magnitude of its volume. But when we study the effect of push or pull on a body we need to know how much the body is accelerated and in which direction the body is accelerated due to the push or pull. The first part i.e. amount of acceleration requires a knowledge about the magnitude of force applied and mass of the body but the second part requires a knowledge about the direction of force (Push or Pull) applied. This shows that mass, volume and density belong to one category and acceleration, force belong to another category.

Thus on the basis of requirements for complete specification, physical quantities can be broadly divided into two categories : (i) **Scalars** and (ii) **Vectors.**

2.1 Scalars :

Physical quantities, which are completely described by a number (with units specified and possibly with an algerbraic sign as for electric charge), are called **Scalar** quantities.

Mass, length, time, temperature, volume, density, energy, work, charge, potential, pressure etc. are few examples of scalar quantities. Calculations with scalars involve ordinary rules of algebra.

For Example

$5\ \text{kg} + 3\ \text{kg} = 8\ \text{kg}$

$10\ \text{cm}^3 - 7\ \text{cm}^3 = 3\ \text{cm}^3$

$5\ \text{m} \times 4\ \text{m} = 20\ \text{m}^2$ etc.

2.2 Vectors :

Physical quantities which require specification of magnitude (a number and unit) as well as direction for their complete description are called **Vectors.**

Displacement, velocity, acceleration, force, field intensity, momentum, torque, magnetic moment etc. are few examples of vectors. For example, we express velocity of a body as 10 m/s along a definite direction.

2.3 Representation of a vector :

(i) Symbolic representation :

A vector quantity is represented by a letter with an arrow head like $\vec{A}$, $\vec{a}$, $\vec{B}$, $\vec{C}$ etc. or by a bold-faced letter like **A, B, a, C** etc. However the former form is convenent while writing by hand.

The Magnitude of a vector is denoted by the same letter (not in bold face) without arrow head or as a modulus.

i.e. magnitude of $\vec{A} = A$ or $|\vec{A}|$

The magnitude is always a non-negative real quantity.

(ii) Geometrical (graphical) representation

A vector is represented geometrically by a straight line with an arrow head in the direction of the vector, such that the length of the straight line is proportional to the magnitude of the vector. The initial point (tail end) can be chosen at any point. The head (terminal point) is automatically fixed by the scale we choose to represent the vector.

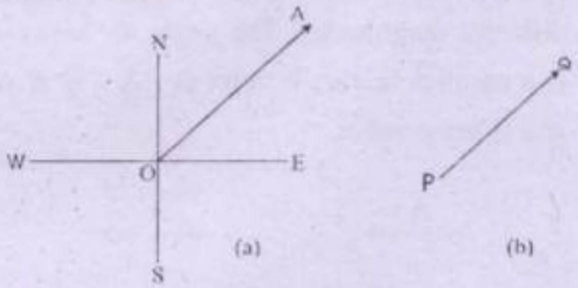


fig. 2.1

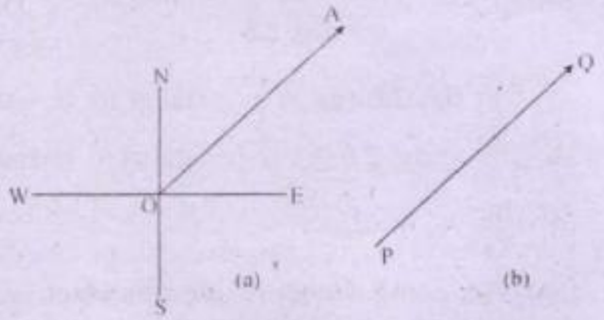


fig 2.2

For example a velocity vector of magnitude 5 m/s and directed along N-E direction, is represented by

$$\vec{\vartheta} = \overrightarrow{OA} \equiv \overrightarrow{PQ}$$

In fig 2.1 the scale is chosen such that 1 cm = 2 m/s while in fig (2.2) 1 cm = 1 m/s. The initial point could be O or P and accordingly the terminal point (head) is at A or Q, respectively. As shown in the figures, 2.1 and (2.2) the length of the straight line representing the vector depends on the scale chosen.

2.4 Important terms

(i) *Unit Vector :* A unit vector of any vector (say $\vec{A}$) is a unit-less vector having unit magnitude but lying in the direction of the vector. Its only purpose is to describe the direction.

It is denoted by giving a cap, or a hat or a carat over the letter representing the vector

e.g. $$\vec{A} = A\hat{A} = |\vec{A}|\hat{A}$$

or $$\hat{A} = \frac{\vec{A}}{A} = \frac{\vec{A}}{|\vec{A}|} \qquad ...(2.4.1)$$

Sometimes one uses fixed notations for fixed directions e.g. $\hat{i}, \hat{j}, \hat{k}$ or $\hat{e}_1, \hat{e}_2, \hat{e}_3$ are chosen as unit vector along X, Y and Z - directions respectively.

(ii) *Null Vector :* It is a vector having zero magnitude and arbitrary direction.

It is represented by a point. When a null vector is added to or substracted from a vector, the resultant vector is same as the given vector. The dot product of a null vector with a vector is always zero. The cross-product of a vector with a null vector gives rise to a null vector.

i.e. $$\vec{A} + \vec{O} = \vec{A}, \ \vec{A} - \vec{O} = \vec{A}$$

$$\vec{A} \times \vec{O} = \vec{O}, \ \vec{A} \cdot \vec{O} = O$$

(iii) *Equal Vectors :* Two vectors (representing same physical quantity) are said to be equal vectors if they have same magnitude and point in the same direction. That is

$$\vec{F}_1 = \vec{F}_2$$

if $|\vec{F}_1| = |\vec{F}_2|$ and $\hat{F}_1 = \hat{F}_2$

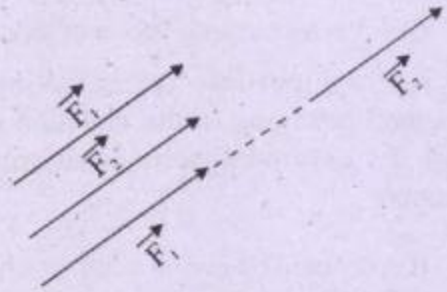


fig. 2.3

Two equal vectors are represented by (i) two parallel lines of equal length pointing in same direction or (ii) by two straight lines of equal length having same direction and same line of action as shown in fig. 2.3.

(iv) *Negative Vector :* A vector is said to be a negative vector of a similar kind of another vector if it has same magnitude as that of the vector but oppositely directed. That is

$$\vec{F}_1 = -\vec{F}_2$$

if $\quad \left|\vec{F}_1\right| = \left|\vec{F}_2\right|$ and $\hat{F}_1 = -\hat{F}_2$

A negative vector is represented by (i) a straight line of equal magnitude and antiparallel to the given vector, or (ii) by a straight line of equal length having same line of action but oppositely directed as shown in fig. 2.4

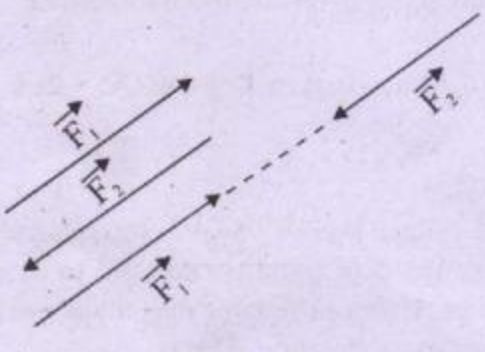


fig. 2.4

(v) *Collinear Vectors :* Vectors having a common line of action are called collinear vectors

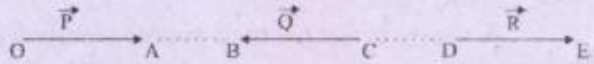


fig. 2.5

In fig. 2.5, $\vec{P}$, $\vec{Q}$ and $\vec{R}$ vectors are collinear as they have the same line of action

(vi) *Parallel Vectors :* Vectors having same direction (without common line of action) are called parallel vectors. They may have same or different magnitudes. The angle 'θ' between two parallel vectors is zero. i.e. $\hat{A} = \hat{B}$ if $\vec{A}$ and $\vec{B}$ are parallel.

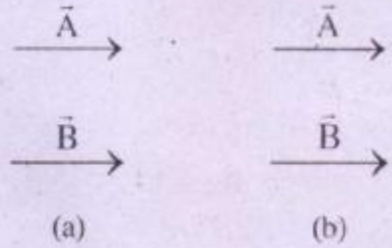


fig. 2.6

In fig. 2.6 (a) $\vec{A}$ is parallel to $\vec{B}$ but $\left|\vec{A}\right| \neq \left|\vec{B}\right|$. In fig. 2.6 (b) $\vec{A}$ is parallel to $\vec{B}$ and $\left|\vec{A}\right| = \left|\vec{B}\right|$.

(vii) *Antiparallel Vectors :* Vectors which are oppositely directed (with no common line of action) are called as antiparallel vectors. They may have same or different magnitude. The angle between them is 180.

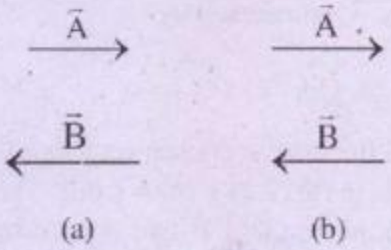


fig. 2.7

In fig. 2.6 (a) $\left|\vec{A}\right| \neq \left|\vec{B}\right|$, but in fig. 2.7 (b) $\left|\vec{A}\right| = \left|\vec{B}\right|$. In both cases the vectors are oppositely directed with no common line of action.

(viii) *Coplanar Vectors :* A number of vectors which lie on one plane are said to be coplanar.

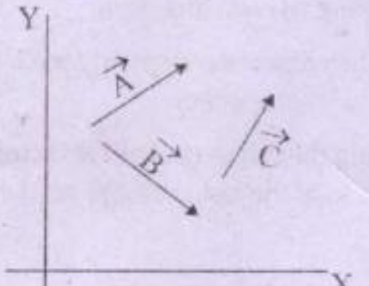


fig. 2.8

In fig. (2.8) $\vec{A}$, $\vec{B}$ and $\vec{C}$ lie on XY - plane so they are coplanar.

(ix) *Concurrent Vector :* A number of vectors which actually meet or appear to meet at a point are said to be concurrent. In fig. 2.9 (a) vectors $\vec{A}$, $\vec{B}$, $\vec{C}$ and $\vec{D}$ actually meet at 'O', while in 2.9 (b) $\vec{A}$, $\vec{B}$, $\vec{C}$

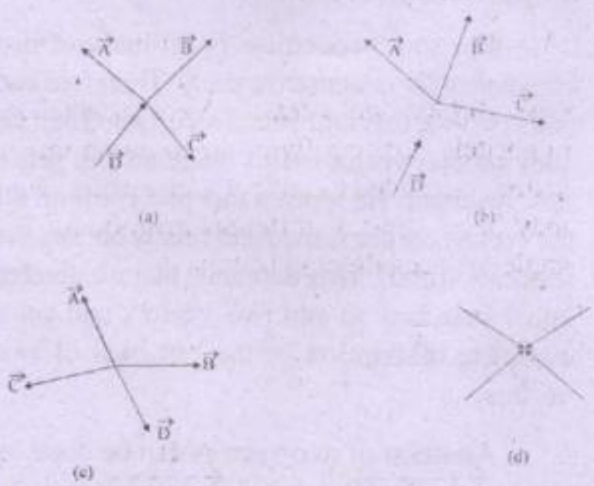


fig. 2.9

and $\vec{D}$ appear to meet at 0. Concurrent vectors which have a common initial point are called co-initial vector, and which have a common terminal point are called co-terminus.

(x) *Localised Vectors :* Vector whose initial point (tail) is fixed is called a localised fixed vector.

For example position vector at every point is a fixed vector as it always starts from a chosen origin (refrence point).

(ix) *Non-Localised Vector :* Vector whose initial point is not fixed is called a non-localised vector.

For example momentum, force, impulse etc. are non-localised vectors.

2.5 Multiplication of a Vector with a Scalar :

When a vector is multiplied by a scalar 'm' the result is another vector having magnitude m times that of the vector and directed along the original vector.

Thus if a vector $\vec{A}$ is multiplied by a scalar m, then the new vector $\vec{B}$ is given by

$$\vec{B} = m\vec{A}$$

$$\Rightarrow \quad B\,\hat{B} = m\,A\,\hat{A}$$

(i) If $m > 0$ then $\hat{B} = \hat{A}$, $B = mA$, $\vec{A}$ and $m\vec{A}$ are parallel.

(ii) If $m < 0$ then $\hat{B} = -\hat{A}$, $B = mA$, $\vec{A}$ and $m\vec{A}$ are parallel.

(iii) If $m = 0$, then $\vec{B} = 0$, a null vector.

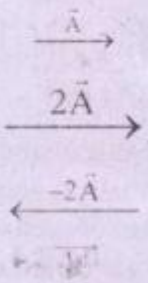


fig. 2.10

Fig 2.10, demonstrates the above facts.

2.6 Addition of Vectors :

Vectors representing similar physical quantities can be added together. The vector addition is essentially different from scalar addition. For example (i) if forces $\vec{F}_1$ and $\vec{F}_2$ act on a body 'O', as shown in fig. 2.11 (a), then the body neither moves in the direction of force $\vec{F}_1$ nor of $\vec{F}_2$; but along a different direction $\overrightarrow{OC}$. This indicates that $\vec{F}_1 + \vec{F}_2 = \vec{F}$ acts along $\overrightarrow{OC}$.

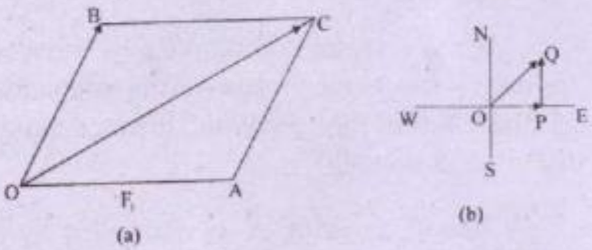


fig. 2.11

(ii) If a body starling from o suffers two successive displacements $\vec{S}_1 = \overrightarrow{OP}$ $\left(\left|\overrightarrow{OP}\right| = 4\text{cm}\right)$ along east direction and another $\vec{S}_2 = \overrightarrow{PQ}$ $\left(\left|\overrightarrow{PQ}\right| = 3\text{cm}\right)$ along North direction then the net displacement is $\overrightarrow{OQ}$, with $\left|\overrightarrow{OQ}\right| = 5$ cm. This indicates that $\left|\overrightarrow{OQ}\right| \neq \left|\overrightarrow{OP}\right| + \left|\overrightarrow{PQ}\right|$ (in this case at least). [See fig. 2.11 (b)]

The above discussion shows that vector addition is essentially a geometrical operation and when a number of vectors are added it gives rise to a new vector called **resultant**. One has to follow the following procedure for finding the **resultant** vector geometrically.

(i) Choose a suitable scale for vectors.

(ii) Choose any arbitrary origin (starting point). Keep the tail of the first vector at the origin and orient it in its direction.

(iii) Then keep the tail of the second vector at the head of the first vector and align it along its own direction.

(iv) Then repeat the step (iii) for all the vectors one after another.

(v) Join the origin (tail of 1st vector) with the head of the last vector to get the resultant.

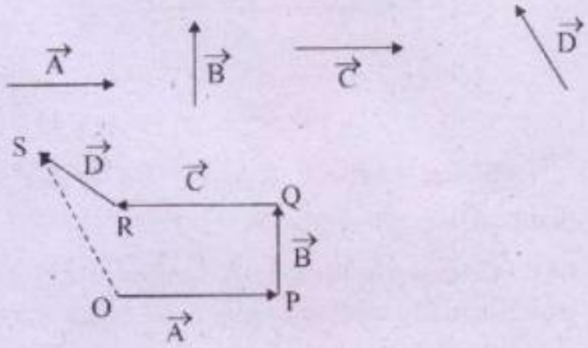


fig. 2.12

Fig 2.12 illustrates the above process of addition by geometrical method.

But this procedure is tedius and not convenient for quantitative study. Therefore one adds two vectors and gets a resultant. Then he adds another vector to this resultant and gets a new resultant. He repeats this process until all the vectors are consumed and finally obtains the resultant (final). This demands that we need to know first how to add two vectors and get a complete discription of the resultant of two vectors.

Addition of two vectors can be done by two methods (a) triangle method and (b) parallelogram method.

(a) *Triangle method of vector addition :*

If $\vec{A}$ and $\vec{B}$ be two vectors to be added, then a diagram is drawn with tail of $\vec{B}$ lying at the head of $\vec{A}$. The vector joining tail of $\vec{A}$ with head of $\vec{B}$, represents the **resultant** vector.

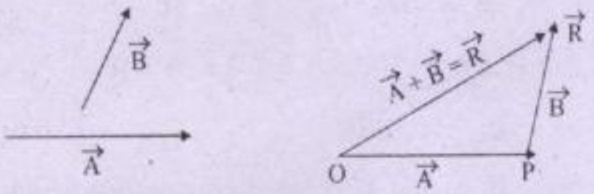


fig. 2.13

In fig. 2.13, $\overrightarrow{OP}$ represents $\vec{A}$, $\overrightarrow{PR}$ represents $\vec{B}$ and $\overrightarrow{OR}$ represents the resultant vectors. But O, P, and R are vertices of the triangle OPR. The above construction leads to the triangle law, stated as :

If two vectors ($\vec{A}$ & $\vec{B}$) are completely represented by two sides of a triangle taken in same order, then their resultant is represented completely by the third side taken in opposite order

i.e. $$\overrightarrow{OP}+\overrightarrow{PR}=\overrightarrow{OR} \quad ...(2.6.1)$$

<u>Magnitude and direction of $\vec{R}$</u>

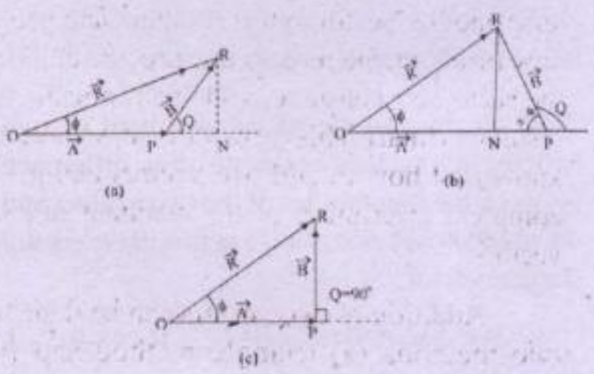


fig. 2.14

In fig. 2.14 we have drawn three different diagrams, where the angle between $\vec{A}$ and $\vec{B}$ could be acute ($\theta < 90$) obtuse ($\theta > 90$) and a right angle ($\theta = 90$).

Considering fig. 2.14 (a) we find

$$\overline{OR}^2 = \overline{ON}^2 + \overline{NR}^2 = (\overline{OP}+\overline{PN})^2 + \overline{NR}^2$$

$$= \overline{OP}^2 + \overline{PN}^2 + 2\overline{OP}.\overline{PN} + \overline{NR}^2$$

$$= \overline{OP}^2 + (\overline{PN}^2 + \overline{NR}^2) + 2\overline{OP}.\overline{PN}$$

$$= \overline{OP}^2 + \overline{PR}^2 + 2\overline{OP}.\overline{PN}$$

Since $\overline{PN} = \overline{PR}\cos\theta$, $\overline{NR} = \overline{PR}\sin\theta$,

so

$$\overline{OR}^2 = \overline{OP}^2 + \overline{PR}^2 + 2\,\overline{OP}.\overline{PR}\cos\theta$$

$$\Rightarrow \quad R^2 = A^2 + B^2 + 2AB\cos\theta$$

$$\Rightarrow \quad R = \sqrt{A^2 + B^2 + 2AB\cos\theta} \quad ...(2.6.2)$$

Equation (2.6.2) gives the magnitude of the resultant $\vec{R}$.

$$\text{Now } \tan\phi = \frac{\overline{NR}}{\overline{ON}} = \frac{\overline{NR}}{\overline{OP}+\overline{PN}} = \frac{\overline{PR}\sin\theta}{\overline{OP}+\overline{PR}\cos\theta}$$

$$\Rightarrow \quad \tan\phi = \frac{B\sin\theta}{A + B\cos\theta} \quad ...(2.6.3)$$

Equation (2.6.3) gives the direction of resultant $\vec{R}$, through the angle ϕ, subtended by $\vec{R}$ with $\vec{A}$. Sometimes it is convenient to express (2.6.2) and (2.6.3) as

$$R = \sqrt{A^2 + B^2 + 2A\cos(\hat{A},\hat{B})} \quad ...(2.6.4$$

$$\tan(\hat{R},\hat{A}) = \frac{B\sin(\hat{A},\hat{B})}{A + B\cos(\hat{A},\hat{B})} \quad ...(2.6.5)$$

We can also find

$$\tan(\hat{R},\hat{B}) = \frac{A\sin(\hat{A},\hat{B})}{B + A\cos(\hat{A},\hat{B})} \tan\phi' \quad ...(2.6.5)(a)$$

considering fig 2.14 (b) we find

$$\overline{OR}^2 = \overline{ON}^2 + \overline{NR}^2 = (\overline{OP} - \overline{NP})^2 + \overline{NR}^2$$

$$= \overline{OP}^2 + \overline{PR}^2 - 2\,\overline{OP}.\overline{NP}$$

$$= \overline{OP}^2 + \overline{PR}^2 - 2\,\overline{OP}.\overline{PR}\cos(\pi - \theta)$$

$$= \overline{OP}^2 + \overline{PR}^2 + 2\,\overline{OP}.\overline{PR}\cos\theta$$

$$\Rightarrow \quad R^2 = A^2 + B^2 + 2AB\cos\theta$$

or $\quad R = \sqrt{A^2 + B^2 + 2AB\cos\theta}$

and

$$\tan\phi = \frac{\overline{NR}}{\overline{ON}} = \frac{\overline{NR}}{\overline{OP} - \overline{PN}} = \frac{\overline{PR}\sin(\pi - \theta)}{\overline{OP} - \overline{PR}\cos(\pi - \theta)}$$

$$= \frac{\overline{PR}\sin\theta}{\overline{OP} - \overline{PR}\cos\theta} = \frac{B\sin\theta}{A + B\cos\theta}$$

$$\tan\phi = \frac{B\sin\theta}{A + B\cos\theta}$$

Thus we obtain same expressions as obtained while considering fig. 2.14(a).

similarly considering fig. 2.14 (c), we find

$$R = \sqrt{A^2 + B^2 + 2AB\cos\theta}, \ \theta = 90$$

$$\tan\phi = \frac{B\sin\theta}{A + B\cos\theta}, \ \theta = 90$$

In any case we find that equations (2.6.2) or (2.6.4) gives the mangnitude of resultant $\vec{R} = \vec{A} + \vec{B}$ and equations (2.6.3) or (2.6.5) gives the direction of resultant vector $\vec{R}$.

Special cases

(i) If $\vec{A}$ and $\vec{B}$ are parallel i.e. $\theta = 0$, then

$$R^2 = A^2 + B^2 + 2AB = (A + B)^2$$

$$\Rightarrow R = A + B.$$

and $\quad \tan\phi = \dfrac{B\sin\theta}{A + B\cos\theta} = 0.$

$$\Rightarrow \phi = 0.$$

But $R = R_{max}$, if $\cos\theta = 1$ i.e. $\theta = 0$

so $\quad R_{max} = A+B \qquad ...(2.6.6)$

(ii) If $\vec{A}$ and $\vec{B}$ are mutually perpendicular i.e. $\theta = \dfrac{\pi}{2}$

then $\quad R = \sqrt{A^2 + B^2}$

$$\tan\phi = \frac{B}{A}.$$

(iii) If $\vec{A}$ and $\vec{B}$ are antiparallel i.e. $\theta = 180$, then $R^2 = A^2 + B^2 - 2AB = (A - B)^2$

$\Rightarrow \quad R = A - B = +ve$ difference between the two.

$$\tan\phi = \frac{B\sin\pi}{A + B\cos\pi} = 0$$

$$\Rightarrow \quad \phi = 0 \text{ or } \pi$$

If $|\vec{A}| > |\vec{B}|$ then $R = A - B$, $\phi = 0$

If $|\vec{A}| < |\vec{B}|$ then $R = B - A$, $\phi = \pi$.

Thus the magnitude of the resultant of two antiparallel vectors is the positive difference between the magnitudes of the two vectors and the resultant lies along the direction of the greater (larger) vector.

We also note that $R = R_{min}$ if $\cos\theta = -1$ i.e. $\theta = \pi$.

$$\text{So } R = R_{min} = A - B = |A - B| \qquad ...(2.6.7)$$

Thus we find

$$A - B < |\vec{A} + \vec{B}| < A + B \qquad ...(2.6.8)$$

NOTE :

Since $\vec{A}+\vec{B}=\vec{R}$

$\Rightarrow \quad \vec{A}+\vec{B}-\vec{R}=0$

$$\Rightarrow \quad \overrightarrow{OP}+\overrightarrow{PR}+\overrightarrow{RO}=0 \qquad ...(2.6.9)$$

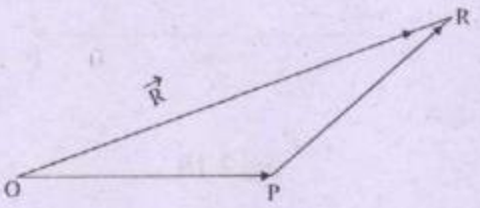


fig. 2.15

Equation (2.6.9) leads to an alternative statement of triangle law, stated as :

If three vectors acting at a point can be completely represented by the three sides of a triangle taken in same order, then their vector sum is zero.

(b) *Parallelogrom law of Vectors :*

If $\vec{A}$ and $\vec{B}$ be two vectors to be added, then the vectors are translated parallel to themselves until their tails meet. Then a parallelogram is constructed with these two sides as the adjacent sides. The diagonal of the parallelogram through the point of coincidence of tails, gives the resultant of the two vectors.

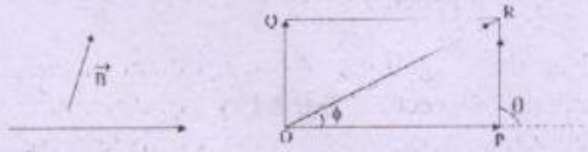


fig. 2.16

In fig. 2.16 $\overrightarrow{OP}$ represents vector $\vec{A}$, $\overrightarrow{OQ}$ represents vector $\vec{B}$ and OR represents the resultant vector $\vec{R}=\vec{A}+\vec{B}$. The above construction leads to the parallelogram law of vector addition stated as :

If two vectors are completely represented by the two adjacent sides of a parallelogram, then their resultant is represented by the diagonal passing through the point of intersection of the two vectors.

Magnitude and direction of $\vec{R}$ *:*

Since $\overrightarrow{OQ}=\overrightarrow{PR}$

so $\quad \overrightarrow{OP}+\overrightarrow{OQ}=\overrightarrow{OP}+\overrightarrow{PR}=\overrightarrow{OR}$

$$...(2.6.10)$$

Since equation (2.6.10) is same as (2.6.1) so one can go through the same steps as in triangle method and arrive at

$$R=\sqrt{A^2+B^2+2AB\cos\theta} \qquad ...(2.6.11)$$

$$\tan\phi = \frac{B\sin\theta}{A+B\cos\theta} \qquad ...(2.6.12)$$

Polygon Law of Vector addition :

If a number of vectors are completely represented by the sides of an open polygon, taken in same order, then their resultant is represented completely by the closing side taken in opposite order.

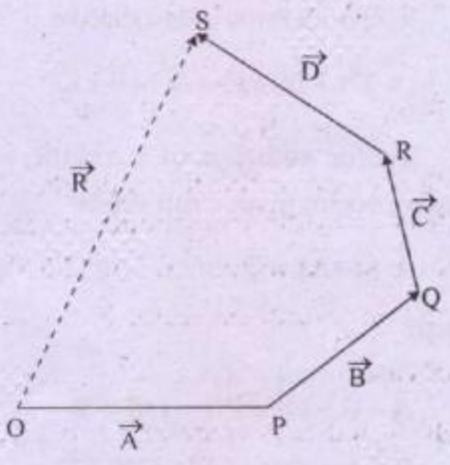


fig. 2.17

In fig. 2.17 $\overrightarrow{OP}=\vec{A}$, $\overrightarrow{PQ}=\vec{B}$, $\overrightarrow{QR}=\vec{C}$, $\overrightarrow{RS}=\vec{D}$. OP, PQ, QR, RS, are the sides of the open polygon OPQRS. The closing side $\overrightarrow{OS}$ represents the resultant.

We also note that

$$\overrightarrow{OP} + \overrightarrow{PQ} + \overrightarrow{QR} + \overrightarrow{RS} = \overrightarrow{OS}$$

$$\Rightarrow \overrightarrow{OP} + \overrightarrow{PQ} + \overrightarrow{QR} + \overrightarrow{RS} + \overrightarrow{SO} = \overrightarrow{OS} + \overrightarrow{SO} = 0 \quad ...(2.6.17)$$

This leads to an alternative statement of polygon law as :

If a number of vectors, acting at a point are completely represented by the sides of a polygon, taken in same order then their vector sum is zero.

Properties of vector addition :

The following properties are satisfied in vector addition.

(i) Vector addition is commutative

i.e. $\vec{A} + \vec{B} = \vec{B} + \vec{A}$

(ii) Vector addition a distributive

i.e. $m(\vec{A} + \vec{B}) = m\vec{A} + m\vec{B}$

(iii) Vector addition is associative

i.e. $(\vec{A} + \vec{B}) + \vec{C} = \vec{A} + (\vec{B} + \vec{C})$

(iv) Vector addition of a vector with its negative vector gives a rull vector

$$\vec{A} + (-\vec{A}) = \vec{O}$$

Proof:

(i) $\vec{A} + \vec{B} = \overrightarrow{OP} + \overrightarrow{PR} = \overrightarrow{OR} = \vec{R}$

$\vec{B} + \vec{A} = \overrightarrow{OQ} + \overrightarrow{QR} = \overrightarrow{OR} = \vec{R}$

$\Rightarrow \vec{A} + \vec{B} = \vec{B} + \vec{A}$

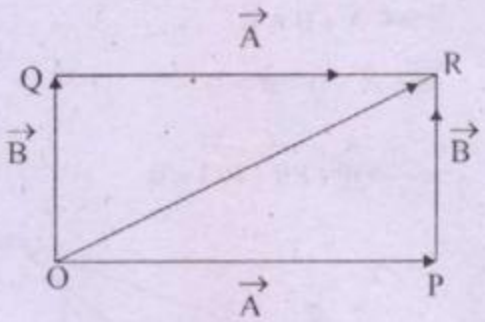


fig. 2.18

(ii) $\overrightarrow{OP} = \vec{A},\ \overrightarrow{OP'} = m\vec{A}$

$\overrightarrow{OQ} = \vec{B},\ \overrightarrow{OQ'} = m\vec{B}$

$\vec{A} + \vec{B} = \vec{R}$

$\Rightarrow \overrightarrow{OP} + \overrightarrow{OQ} = \overrightarrow{OR}$

$m(\vec{A} + \vec{B}) = m\vec{R} = \vec{R}'$

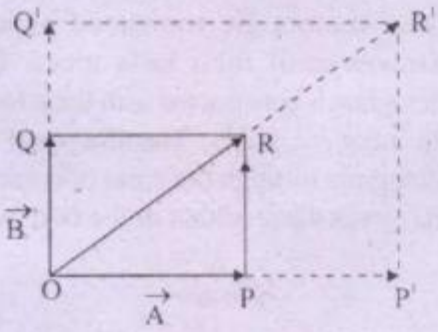


fig. 2.19

$$\Rightarrow |\vec{R}'| = |m(\vec{A} + \vec{B})| = m|\vec{R}|$$

and $\hat{R}' = \hat{R}$ i.e. direction $\vec{R}$ and $\vec{R}'$ are same.

Now $m\vec{A} + m\vec{B} = \vec{R}''$

$$\Rightarrow |\vec{R}''| = \sqrt{m^2A^2 + m^2B^2 + 2m^2AB\cos\theta}$$

$$= m\sqrt{A^2 + B^2 + 2AB\cos\theta} = m|\vec{R}|$$

i.e. $\left|\vec{R}''\right| = m\left|\vec{R}\right|$

Hence $\left|\vec{R}'\right| = m\left|\vec{R}\right| = \left|\vec{R}''\right|$

Now $\tan\phi'' = \dfrac{mB\sin\theta}{mA + mB\cos\theta} = \tan\phi$

$$\Rightarrow \phi'' = \phi$$

This means $\hat{R}'' = \hat{R} = \hat{R}'$

Thus we find that

$$m(\vec{A}+\vec{B}) = m\vec{A} + m\vec{B}$$

(iii)

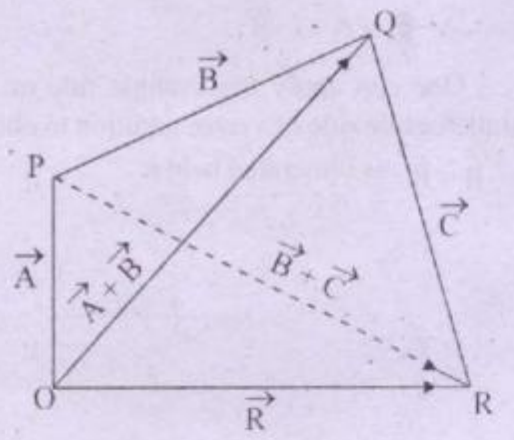


fig. 2.20

Now

$$(\vec{A}+\vec{B})+\vec{C} = (\overrightarrow{OP}+\overrightarrow{PQ})+\overrightarrow{QR}$$

$$= \overrightarrow{OQ}+\overrightarrow{QR} = \overrightarrow{OR}$$

and

$$\vec{A}+(\vec{B}+\vec{C}) = \overrightarrow{OP}+(\overrightarrow{PQ}+\overrightarrow{QR})$$

$$= \overrightarrow{OP}+\overrightarrow{PR} = \overrightarrow{OR}$$

Then $(\vec{A}+\vec{B})+\vec{C} = \vec{A}+(\vec{B}+\vec{C})$

<u>NOTE :</u> (i) The negative vector of the resultant of a number of vectors acting at a point is called **equilibriant.**

(ii) If a physical quantity specified by a magnitude and direction does not obey rules of vector addition then it cannot be called vector. For example - electric current.

Ex. 2.6.1 Two velocities, each 5 m/s. are inclined to each other at 60^0. Find the resultant.

Soln. Given $\left|\vec{\vartheta}_1\right| = \left|\vec{\vartheta}_2\right| = 5\,m/s$

$$\theta = 60^0$$

$$\therefore\ R = \sqrt{\vartheta_1^2 + \vartheta_2^2 + 2\vartheta_1\vartheta_2\cos 60}\ \text{m/s}$$

$$= \sqrt{5^2 + 5^2 + 2.5^2\frac{1}{2}}\ \text{m/s}$$

$$R = \sqrt{75}\ m/s = 5\sqrt{3}\ m/s = 8.66\ m/s$$

$$\tan\phi = \frac{\vartheta_2\sin 60}{\vartheta_1 + \vartheta_2\cos 60} = \frac{5\frac{\sqrt{3}}{2}}{5+5.\frac{1}{2}}$$

$$\tan\phi = 0.577$$

$$\Rightarrow \phi = 30^0$$

The resultant has magnitude 8.66 m/s and makes 30^0 with one of the velocities.

Ex. 2.6.2 Two forces equal in magnitude, have magnitude of their resultant equal to either. Find the angle between them and the direction of the resultant.

Soln. Given $\left|\vec{F}_1\right| = \left|\vec{F}_2\right| = \left|\vec{R}\right| = F$

Now $R = \sqrt{F_1^2 + F_2^2 + 2F_1F_2\cos\theta}$

$$\Rightarrow F = \sqrt{2F^2 + 2F^2\cos\theta}$$

$$F^2 = 2F^2(1+\cos\theta)$$

$$\Rightarrow 1+\cos\theta = \frac{1}{2}$$

$$\cos\theta = \frac{1}{2}$$

$$\theta = 120$$

The angle between vectors is 120^0

$$\tan\phi = \frac{F_2 \sin\theta}{F_1 + F_2\cos\theta} = \frac{F\sin 120}{F + F\cos 120}$$

$$= \frac{\sqrt{\frac{3}{2}}}{1-\frac{1}{2}} = \frac{\sqrt{\frac{3}{2}}}{\frac{1}{2}}$$

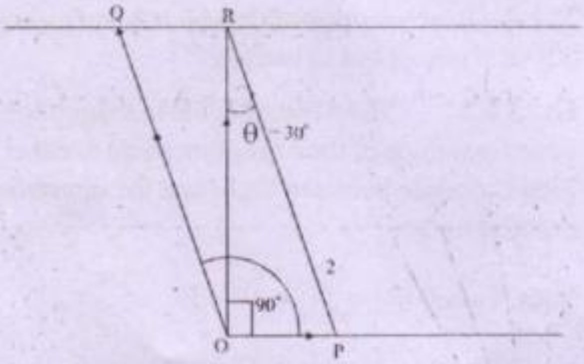
$$\Rightarrow \tan\phi = \sqrt{3}$$

$$\Rightarrow \phi = 60^\circ$$

Thus the resultant makes angle 60^0 with either of the force components.

Ex. 2.6.3 The stream of Mahanadi is flowing with a velocity of 1km/hr. What should be the direction of a swimmer to cross the river straight, if he can sweam at the rate 2 km/hour in still water ?

Soln.



$\overrightarrow{OP} = \vec{\vartheta}_R$ = velocity of stream

$\overrightarrow{OQ} = \vec{\vartheta}_M$ = velocity of man (swimmer)

$\overrightarrow{OR}$ = Resultant

$$\left|\overrightarrow{OR}\right| = \sqrt{2^2 - 1^2} = \sqrt{3} \text{ km/hr}$$

$$\tan\theta = \frac{OP}{OR} = \frac{1}{\sqrt{3}}$$

$$\Rightarrow \theta = 30^\circ$$

$\therefore$ Angle between stream and swimmer is 120^0

2.7 Subtraction of two vectors :

Subtraction of one vector (say $\vec{B}$) from another vector (say $\vec{A}$) is equivalent to adding vecotrially the negative of the vector ($\vec{B}$) to be subtracted.

i.e. $\vec{A} - \vec{B} = \vec{A} + (-\vec{B})$

One can apply the triangle rule or the parallelogram rule of vector addition to obtain $\vec{A} - \vec{B} = \vec{R}$, as illustrated below.

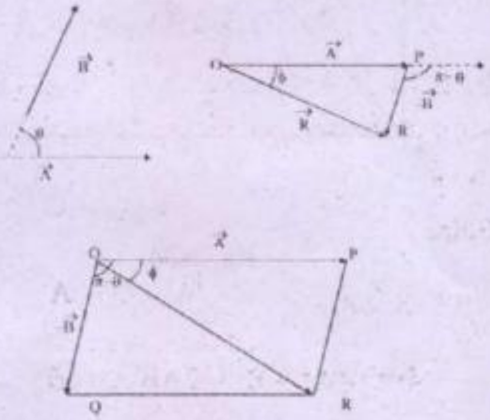


fig. 2.21

In this case if angle between $\vec{A}$ and $\vec{B}$ be θ, then angle between $\vec{A}$ and $-\vec{B}$ shall be $\pi - \theta$.

so

$$R = \sqrt{A^2 + B^2 - 2AB\cos\theta} \quad ...(2.7.1)$$

$$\tan\phi = \frac{B\sin\theta}{A - B\cos\theta} \quad ...(2.7.2)$$

or $\tan(\hat{R},\hat{A}) = \dfrac{B\sin(\hat{A},\hat{B})}{A - B\cos(\hat{A},\hat{B})}$ ...(2.7.3)

NOTE : If $\vec{A}$ and $\vec{B}$ are represented by the two adjacent sides of a parallelogram, then one diagonal passing through the point of intersection of $\vec{A}$ and $\vec{B}$ represents $\vec{R} = \vec{A} + \vec{B}$; and the other diagonal represents $\vec{R}' = \vec{A} - \vec{B}$.

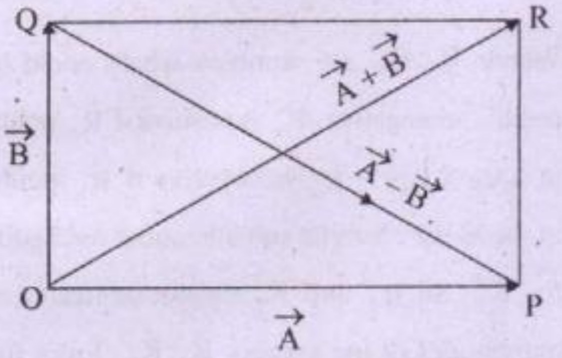


fig. 2.22

Ex 2.7.1 Two vecotors $\vec{A}$ and $\vec{B}$ have same magnitude and inclined at angle 60^0. Find $\vec{A} - \vec{B}$.

Soln.

$$\vec{R} = \vec{A} - \vec{B}; \qquad |\vec{A}| = |\vec{B}| = A$$

$$R = \sqrt{A^2 + B^2 - 2AB\cos 60}$$

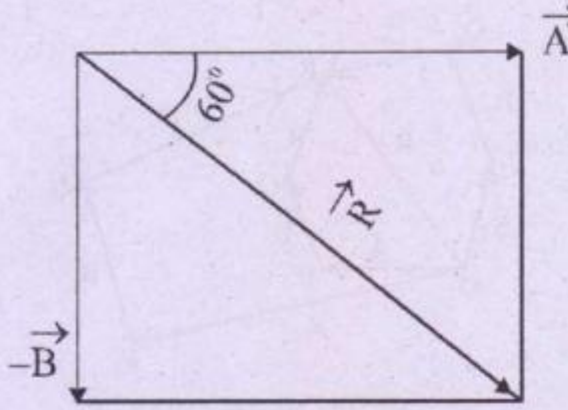
$$= \sqrt{2A^2 - 2A^2\frac{1}{2}} = A$$

$$R = A$$

$$\tan\phi = \frac{B\sin 60}{A - B\cos 60} = \frac{A\sqrt{3/2}}{A - A\frac{1}{2}}$$

$$= \frac{A\sqrt{3/2}}{A/2} = \sqrt{3}$$

$$\Rightarrow \phi = 60^0$$



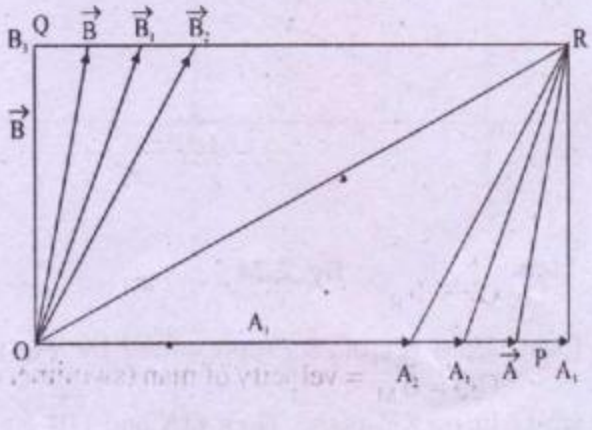
2.8 Resolution of vector :

We have seen in sec. 2.6 that $\vec{A} + \vec{B} = \vec{R}_1$; $\vec{A} + \vec{B} + \vec{C} = \vec{R}_2$; $\vec{A} + \vec{B} + \vec{C} + \vec{D} = \vec{R}_3$ etc. That is addition of two or more vectors gives rise to a single vector called resultant.

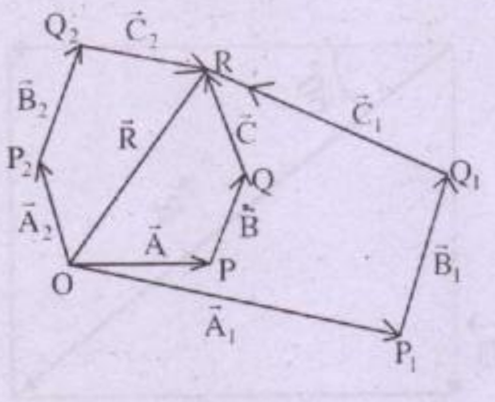
Conversely one can have $\vec{R}_1 = \vec{A} + \vec{B}$, $\vec{R}_2 = \vec{A} + \vec{B} + \vec{C}$; $\vec{R}_3 = \vec{A} + \vec{B} + \vec{C} + \vec{D}$, etc. That is we can split up a single vector into two or more (components) vectors. But this converse process may or may not coincide with the direct process. For example as shown in fig. 2.23.

$$\vec{A} + \vec{B} = \vec{R} = \vec{A}_1 + \vec{B}_1 = \vec{A}_2 + \vec{B}_2 = \vec{A}_3 + \vec{B}_3 \text{ etc.}$$

$$\vec{A} + \vec{B} + \vec{C} = \vec{R} = \vec{A}_1 + \vec{B}_1 + C_1 = \vec{A}_2 + \vec{B}_2 + \vec{C}_2 \text{etc.}$$



(a)



(b)

fig. 2.23

The process of splitting a single vector into a number of component vectors is called "resolution of vector"

The useful choice of components are the mutually perpendicular components. These mutally perpendicular components are called rectangular components.

2.9 (a) Rectangular components a vector in a plane

A vector lying on a plane possesses two mutually perpendicular components.

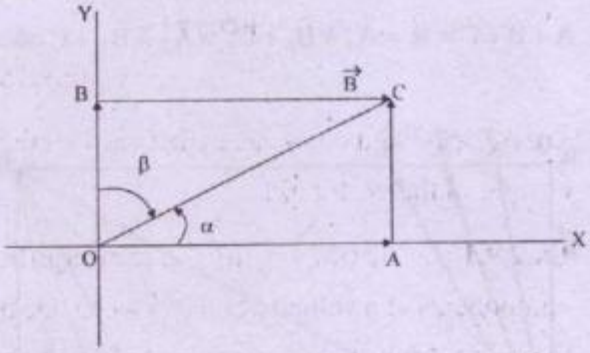


fig. 2.24

Consider a vector $\vec{R}$, represented by $\overrightarrow{OC}$; which lie on XY-plane. Then $\overrightarrow{OA}$ and $\overrightarrow{OB}$ are its two rectangular components, since

$\overrightarrow{OA}+\overrightarrow{OB}=\overrightarrow{OA}+\overrightarrow{AC}=\overrightarrow{OC}$ (By triangle law)

Let us put $\overrightarrow{OA}=\overrightarrow{R}_x;\ \overrightarrow{OB}=\overrightarrow{AC}=\overrightarrow{R}_y$

Then $\overrightarrow{R}=\overrightarrow{R}_x+\overrightarrow{R}_y$ ...(2.9.1)

If we choose $\hat{i},\hat{j}$ as unit vectors along +Ve X-dir^n and +Ve Y-dir^n, respectively then we write

$$\vec{R}_x = R_x\hat{i},\ \vec{R}_y = R_y\hat{j} \qquad ...(2.9.2)$$

Where R_x, R_y are numbers which could be positive or negative. R_x is positive if $\vec{R}_x$ points in +Ve X-dir^n; R_x is negative if $\vec{R}_x$ points in -ve X-dir^n. Similar considerations hold good for R_y. So R_x and R_y cannot be treated as magnitudes of the vectors $\vec{R}_x, \vec{R}_y$. From fig. 2.24 it is easy to see that

$$R_x = R\cos\alpha = R\cos\ (\hat{i},\vec{R}) \qquad ...(2.9.3)$$

$$R_y = R\sin\alpha = R\cos\beta = R\cos\ (\hat{j},\vec{R}) \qquad ...(2.9.4)$$

This gives

$$R^2 = R_x^2 + R_y^2 = (\cos^2\alpha + \cos^2\beta)$$

$$\Rightarrow\ R = \sqrt{R_x^2 + R_y^2} \qquad ...(2.9.5)$$

$$\cos^2\alpha + \cos^2\beta = 1 \qquad ...(2.9.6)$$

and

$$\tan\alpha = \tan(\hat{i},\vec{R}) = \frac{R_y}{R_x} \qquad ...(2.9.7)$$

$$\tan\beta = \tan\left(\hat{j},\hat{R}\right) = \frac{R_x}{R_y}$$

Thus when a vector $\vec{R}$ is written in its component forms

$$\vec{R} = R_x\hat{i} + R_y\hat{j} = R\cos(\hat{i},\vec{R})\hat{i}$$

$$+R\cos(\hat{j},\vec{R})\hat{j} \quad ...(2.9.8)$$

giving

$$\hat{R} = \frac{\vec{R}}{R} = \frac{R_x\hat{i} + R_y\hat{j}}{\sqrt{R_x^2 + R_y^2}}$$

$$= \cos(\hat{i},\vec{R})\hat{i} + \cos(\hat{j},\vec{R})\hat{j} \quad ...(2.9.9)$$

$\cos\alpha$, $\cos\beta$ are called direction - cosines of the vector $\vec{R}$, which is defined as the cosine of the angle subtended by the vector with the coordinate axes.

(b) Rectangular Components of a Vector in 3-dimension

A vector $\vec{R}$ in 3-dimensional space can be resolved into three mutually perpendicular components

$$\vec{R} = \vec{R}_x + \vec{R}_y + \vec{R}_z \quad ...(2.9.10)$$

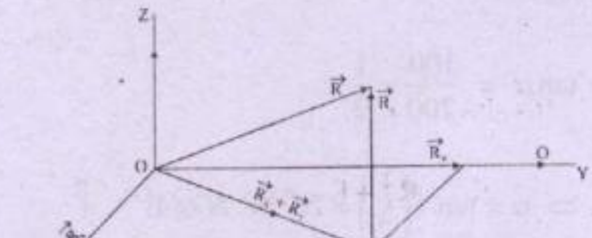


fig. 2.25

Choosing $\hat{i}, \hat{j}, \hat{k}$ as unit vectors along +Ve X, Y, and Z – dir^n respectively we write

$$\vec{R} = R_x\hat{i} + R_y\hat{j} + R_z\hat{k} \quad ...(2.9.11)$$

Here again R_x, R_y, R_z could be positive or negative depending on whether $\vec{R}_x, \vec{R}_y, \vec{R}_z$ are along +Ve or – Ve dir^ns of their respective axes.

If α, β and γ be the angles subtended by the vector $\vec{R}$ with X, Y, and Z - axes respectively, Then

$$R_x = R\cos\alpha = R\cos(\hat{i},\vec{R})$$

$$R_y = R\cos\beta = R\cos(\hat{j},\vec{R})$$

$$R_z = R\cos\gamma = R\cos(\hat{k},\vec{R}) \quad ...(2.9.12)$$

Giving

$$R^2 = R_x^2 + R_y^2 + R_z^2$$

$$= R^2(\cos^2\alpha + \cos^2\beta + \cos^2\gamma)$$

$$\Rightarrow R = \sqrt{R_x^2 + R_y^2 + R_z^2}$$

...(2.9.13)

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

...(2.9.14)

and

$$\hat{R} = \frac{\vec{R}}{R} = \frac{R_x\hat{i} + R_y\hat{j} + R_z\hat{k}}{\sqrt{R_x^2 + R_y^2 + R_z^2}}$$

$$= \cos\alpha\,\hat{i} + \cos\beta\,\hat{j} + \cos\gamma\,\hat{k} \quad ...(2.9.15)$$

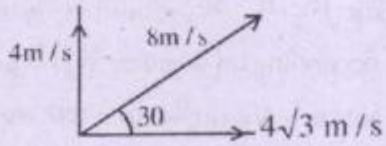
$\cos\alpha$, $\cos\beta$ and $\cos\gamma$ are called the direction cosines of the vector $\vec{R}$.

Ex.2.9.1 Find the rectangular components of a velocity of 8 m/s when one of the components makes an angle of 30^0 with the resultant.

Soln.

$$R_x = R\cos 30, \; R_y = R\sin 30$$

$$\Rightarrow R_x = R\sqrt{3/2} \; R_y = R/2$$



Hence one component is $4\sqrt{3}$ m/s. and another 4 m/s.

2.10 Addition of Vectors by Component method :

Let $\quad \vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$

$$\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$$

$$\vec{C} = C_x\hat{i} + C_y\hat{j} + C_z\hat{k}$$

Then $\vec{A} + \vec{B} + \vec{C} = \vec{R}$

$$\Rightarrow (A_x + B_x + C_x)\hat{i} + (A_y + B_y + C_y)\hat{d}$$

$$+(A_z + B_z + C_z)\hat{k}$$

$$= R_x\hat{i} + R_y\hat{j} + R_z\hat{k} \qquad ...(2.10.1)$$

This gives

$$R_x = A_x + B_x + C_x$$

$$R_y = A_y + B_y + C_y$$

$$R_z = A_z + B_z + C_z \qquad ...(2.10.2)$$

and

$$R = \sqrt{R_x^2 + R_y^2 + R_z^2}$$

$$= (A_x + B_x + C_x)^2 + (A_y + B_y + C_y)^2$$

$$+(A_z + B_z + C_z)^2 \qquad ...(2.10.3)$$

Thus the components add up in vector addition.

Ex. 2.10.1 Four forces act at a point. The first is 200N acting due north, the second of 100N acting due south, the third of 500N acting due east and the fourth of 300N due west. What is the magnitude and direction of resultant force ?

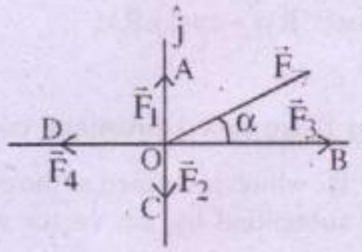
Soln.

Given $\vec{F}_1 = 200$N due north.

$\vec{F}_2 = 100$N due south

$\vec{F}_3 = 500$N due east

$\vec{F}_4 = 300$N due west



So $\quad \vec{F}_1 = 200\hat{j}; \quad \vec{F}_2 = -100\hat{j}; \quad \vec{F}_3 = 500\hat{i};$

$\vec{F}_4 = -300\hat{i}$

Hence resultant force $\vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \vec{F}_4$

$$\Rightarrow \vec{F} = (500 - 300)\hat{i} + (200 - 100)\hat{j} = 200\hat{i} + 100\hat{j}$$

$$|\vec{F}| = \sqrt{(200)^2 + (100)^2} = 100\sqrt{5}\ \text{N} = 223.6\text{N}$$

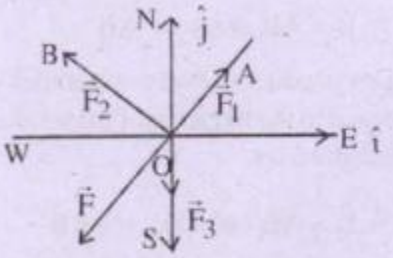
$$\tan\alpha = \frac{100}{200} = \frac{1}{2}$$

$$\Rightarrow \alpha = \tan^{-1}\left(\frac{1}{2}\right) = 26^\circ 34' \text{ N of E}$$

Thus the resultant has magnitude 223.6 N and is directed $26^\circ 34'$ North of East.

Ex.2.10.2 Find the resultant of the following forces acting at a point, (i) $100\sqrt{2}$ dyne along north-east (ii) $980\sqrt{2}$ dyne along north-west (iii) 2 g wt. along south.

Soln.



Given $\vec{F}_1 = 100\sqrt{2}$ dyne N - E

$\vec{F}_2 = 980\sqrt{2}$ dyne N - W

$\vec{F}_3 = 2\text{gwt} = 980\text{x}2$ dyne South

Resolving along East and North dir^n

$$\vec{F}_1 = F_{1x}\hat{i} + F_{1y}\hat{j}$$

$$= (100\sqrt{2}.\cos 45)\hat{i} + (100\sqrt{2}\ \sin 45)\hat{j}$$

$$\Rightarrow \vec{F}_1 = 100\hat{i} + 100\hat{j}$$

$$\vec{F}_2 = F_{2x}\hat{i} + F_{2y}\hat{j}$$

$$= (980\sqrt{2}.\cos 135^\circ)\hat{i} + (980\sqrt{2}\ \sin 135^\circ)\hat{j}$$

$$\Rightarrow \vec{F}_2 = -980\hat{i} + 980\hat{j}$$

$$\vec{F}_3 = -1960\hat{j}$$

So resultant force $\vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3$

$$\Rightarrow \vec{F} = (100 - 980)\hat{i} + (100 + 980 - 1960)\hat{j}$$

$$\vec{F} = -880\hat{i} - 880\hat{j}$$

Thus the resultant force lies in the 3rd quadrant

$$|\vec{F}| = \sqrt{(880)^2 + (880)^2} = 880\sqrt{2} \text{ dyne}$$

This is directed along S-W.

2.11 Product of two vectors :

Product of two vectors, which may be representing similar or different physical quantities gives rise to a new physical quantity. The product of two vectors is of two types : (a) Scalar product or dot product, which is represented by putting a dot between two vectors like $\vec{A}.\vec{B}$ (b) Vector product or cross product, which is represented by putting a cross between two vectors like $\vec{A} \times \vec{B}$.

(a) Scalar product of two vectors :

The Scalar product of two vectors is defined to be the product of their magnitudes and cosine of the smaller angle between them.

i.e. $\vec{A}.\vec{B} = AB\cos\theta = AB\cos(\vec{A},\vec{B})$ ...(2.11.1)

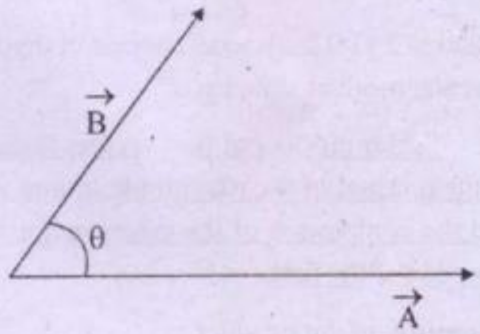


fig. 2.26

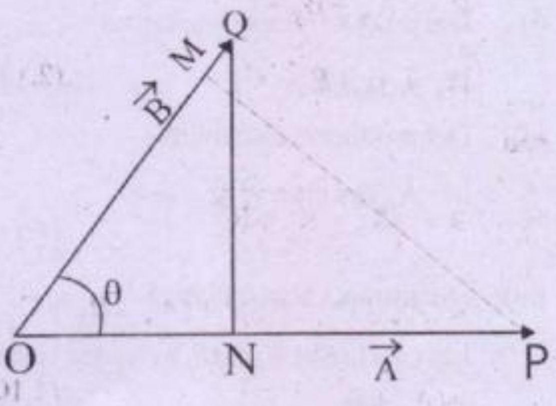


fig. 2.27

The ℓ.h.s. of equation 2.11.1 is read as $\vec{A}$ dot $\vec{B}$. One can re-write eqn. 2.11.1 as

$$\vec{A}.\vec{B} = A(B\cos\theta) = B(A\cos\theta) \quad ...(2.11.2)$$

It is easy to see from fig. 2.27, that

$$\vec{B} = \vec{OQ} = \vec{ON} + \vec{NQ}$$
$$\vec{A} = \vec{OP} = \vec{OM} + \vec{MP}$$

and

$\overline{ON} = B\cos\theta$ = component of $\vec{B}$ along $\vec{A}$

$\overline{OM} = A\cos\theta$ = component of $\vec{A}$ along $\vec{B}$

So

$$\vec{A}.\vec{B} = A \times (\text{component of } \vec{B} \text{ along } \vec{A})$$
$$= B \times (\text{component of } \vec{A} \text{ along } \vec{B}) \quad ...(2.11.3)$$

Equation 2.11.3 leads to an alternative definition of scalar product stated as

<u>Scalar product of two vectors is defined as the product of the magnitude of one vector and the component of the other vector in the direction of the first.</u>

Properties of dot product :

Dot product of two vectors satisfy the following properties.

(i) Dot product is communtative

i.e. $\vec{A}.\vec{B} = \vec{B}.\vec{A}$ ....(2.11.4)

(ii) Dot product is distributive

i.e. $\vec{A}.(\vec{B}+\vec{C}) = \vec{A}.\vec{B} + \vec{A}.\vec{C}$ ...(2.11.5)

(iii) Dot product is associative

i.e. $(m\vec{a}).(n\vec{b}) = mn(\vec{a}.\vec{b}) = (n\vec{a}).(m\vec{b})$ ...(2.11.6)

(iv) Dot product of two mutually perpendicular vectors is zero

i.e. $\vec{A}.\vec{B} = AB\cos 90 = 0$

So $\hat{i}.\hat{j} = \hat{j}.\hat{k} = \hat{k}.\hat{i} = 0$ ...(2.11.7)

(v) Dot product of two parallel vectors is equal to the product of their magnitudes.

$$\vec{A}.\vec{B} = AB\cos\theta = AB$$

(vi) Dot product of two antiparallel vectors is equal to the negative of the product of their magnitudes.

$$\vec{A}.\vec{B} = AB\cos 180 = -AB$$

(viii) Dot product of two equal vectors is equal to square of the magnitude of the vector.

$$\vec{A}.\vec{A} = A^2 \quad ...(2.11.8)$$

$$\Rightarrow \hat{i}.\hat{i} = \hat{j}.\hat{j} = \hat{k}.\hat{k} = 1 \quad ...(2.11.9)$$

<u>**Dot product in terms of component of vectors**</u>

Let $\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$

$$\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$$

Then

$$\vec{A}.\vec{B} = (A_x\hat{i} + A_y\hat{j} + A_z\hat{k}).(B_x\hat{i} + B_y\hat{j} + B_z\hat{k})$$

Using distributive property on r.h.s

$$\vec{A}.\vec{B} = A_xB_x\hat{i}.\hat{i} + A_xB_y\hat{i}.\hat{j}$$
$$+A_xB_z\hat{i}.\hat{k} + A_yB_x\hat{j}.\hat{i}$$
$$+A_yB_y\hat{j}.\hat{j} + A_yB_z\hat{j}.\hat{k} + A_zB_x\hat{k}.\hat{i}$$
$$+A_zB_y\hat{k}.\hat{j} + A_zB_z\hat{k}.\hat{k}$$

Using eqns. (2.11.7) and (2.11.9) on r.h.s

$$\vec{A}.\vec{B} = A_xB_x + A_yB_y + A_zB_z \quad ...(2.11.10)$$

Eqn. 2.11.10 shows that dot product of two vectors is equal to the sum of the products of their like rectangular components.

Since $A=\sqrt{A_x^2+A_y^2+A_z^2}$, $B=\sqrt{B_x^2+B_y^2+B_z^2}$

and

$$\vec{A}.\vec{B}=AB\cos\theta=A_xB_x+A_yB_y+A_zB_z$$

So

$$\cos\theta=\cos(\vec{A}.\vec{B})=\frac{\vec{A}.\vec{B}}{AB}$$

$$=\frac{A_xB_x+A_yB_y+A_zB_z}{\sqrt{A_x^2+A_y^2+A_z^2}\sqrt{B_x^2+B_y^2+B_z^2}}$$

...(2.11.11)

Component of a vector along any direction :

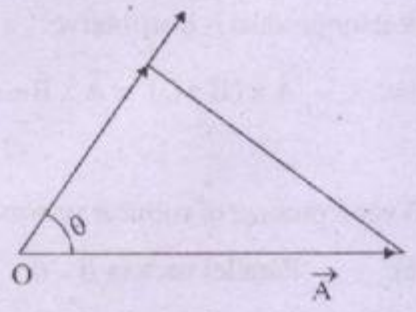


fig. 2.28

Suppose we have a vector $\bar{A}$ and we have a direction defined by unit vector $\hat{\ell}$. Then

$$\bar{A}.\hat{\ell}=A\cos\theta=\text{component of } \bar{A} \text{ along } \hat{\ell}$$

This shows that

$$A_\ell=\bar{A}.\hat{\ell}=A\cos(\vec{A},\hat{\ell}) \quad ...(2.11.12)$$

Eqn. (2.11.12) shows that component of a vector along any arbitrary direction can be obtained by taking the dot product of the vector with the unit vector along the desired direction.

Ex. 2.11.1 Find the angle between the two vector

$$\bar{A}=2\hat{i}+3\hat{j}+4\hat{k},\ \bar{B}=\hat{i}-2\hat{i}+3\hat{k}$$

Soln.

$$A=\sqrt{2^2+3^2+4^2}=\sqrt{29}$$

$$B=\sqrt{1^2+(-2)^2+3^2}=\sqrt{14}$$

$$\cos\theta=\frac{A_xB_x+A_yB_y+A_2B_2}{AB}$$

$$=\frac{2.1+3.(-2)+4.3}{\sqrt{29}\ \sqrt{14}}=\frac{8}{\sqrt{29}\ \sqrt{14}}$$

$$\Rightarrow\ \cos\theta\ =\ 0.397$$

$$\Rightarrow\ \theta=66.61^\circ$$

Ex. 2.11.2 Are the two vectors represented by $2\hat{i}+4\hat{j}+2\hat{k}$ and $\hat{i}+2\hat{j}+5\hat{k}$ perpendicular to each other ?

Soln.

$$\bar{A}=2\hat{i}+4\hat{j}+2\hat{k},\ \bar{B}=\hat{i}+2\hat{j}+5\hat{k}$$

$$\bar{A}.\vec{B}=2.1+4.2+2.(-5)=0=AB\cos\theta$$

Since $A\neq O$, $B\neq O$, $\cos\theta\ =\ 0$

The two vectors are perpendicular to each other

(b) **Vector product :**

Vector product of two vectors ($\vec{A}$ and $\vec{B}$) is a new vector $\vec{n}$, whose magnitude is given by the product of the magnitudes of the vectors and sine of the smaller angle between them and which is perpendicular to the plane containing the two vectors. The positive direction of the product vector $\vec{n}$; is given by right-hand screw rule or right-hand thumb rule.

i.e. $\bar{A}x\bar{B}\ =\ \bar{n}\ =\ AB\ \sin\theta\ \hat{n}$

...(2.11.13)

with $0^\circ\ \leq\ \theta\ \leq\ 180^\circ$

and $\bar{n}$ is perpendicular to $\vec{A}$ as well as $\vec{B}$, and also to the plane containing $\bar{A}$ and $\bar{B}$

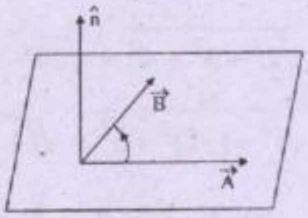


fig. 2.29

(i) *Right-hand screw rule :*

If a right-handed screw be held perpendicular to the plane containing the two vectors ($\vec{A}$, $\vec{B}$) and be rotated from the first vector towards the second vector through the smaller angle, then the direction of motion of the tip of the screw gives the positive direction of the vector $\vec{n} = \vec{A} x \vec{B}$.

For example in the vector product $\vec{A} x \vec{B}$, $\vec{A}$ is the first vector and $\vec{B}$ is the second vector. The right-handed screw is rotated

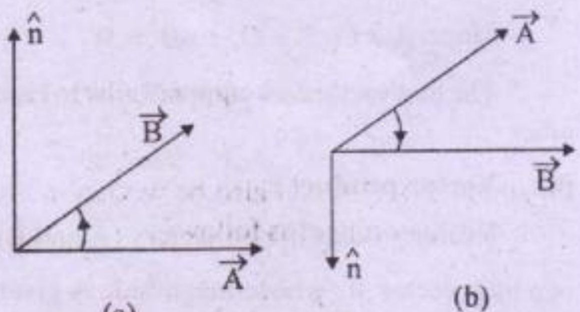


fig. 2.30

from $\vec{A}$ towards $\vec{B}$ through the smaller angle as shown in fig. 2.30. The screw tip moves either towards the reader or away from the reader along the perpendicular to the plane of $\vec{A}$ and $\vec{B}$.

(ii) *Right-hand thumb rule :*

If the finger of the right hand with thumb erect are closed such that the fingers curl in the direction from first vector towards the second vector, then the thumb gives the positive direction of the (vector) product vector.

Properties of Vector Product : The following properties are satisfied by vector product.

(i) Vector product is not commutative

i.e. $\vec{A} \times \vec{B} \neq \vec{B} \times \vec{A}$

But $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$

(since direction is reversed) ...(2.11.14)

(ii) Vector product is associative

i.e.

$$m\vec{A} \times \vec{B} = \vec{A} \times m\vec{B} = m(\vec{A} \times \vec{B}) \quad ...(2.11.15)$$

(iii) Vector product is distributive

i.e. $$\vec{A} \times (\vec{B}+\vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C} \quad ...(2.11.16)$$

(iv) Vector product of colinear vectors is zero

a) Parallel vectors $\theta = 0$

$$\vec{A} x \vec{B} = AB \sin\theta \hat{n} = \vec{O} = \text{null vector}$$

b) anti-parallel vectors $\theta = \pi$

$$\vec{A} x \vec{B} = AB \sin 180^{\circ} \hat{n} = \vec{O} = \text{null vector}$$

(v) Vector product of equal vectors is zero

i.e. $\vec{A} x \vec{A} = \vec{O}$

$$\Rightarrow \hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0 \quad ...(2.11.17)$$

(vi) Magnitude of vector product of two mutually perpendicular vectors is equal to the product of their, Magnitudes.

$$|\vec{A} x \vec{B}| = AB \sin 90 = AB$$

$$\Rightarrow \hat{i} \times \hat{j} = \hat{k},\ \hat{j} x \hat{k} = \hat{i},\ \hat{k} x \hat{i} = \hat{j} \quad ...(2.11.18)$$

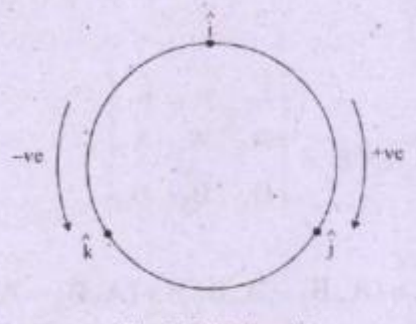


Fig : 2.32 (Cycle rule)

Vector product in terms of components.

Let $\bar{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$,

$\bar{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$

Then

$$\bar{A}x\bar{B} = (A_x\hat{i} + A_y\hat{j} + A_z\hat{k})$$

$$x\ (B_x\hat{i} + B_y\hat{j} + B_z\hat{k})$$

Using distributive property on r.h.s

$$\bar{A}x\bar{B} = A_xB_x\hat{i}\ x\ \hat{i} + A_xB_y\hat{i}\ x\ \hat{j} + A_xB_z\hat{i}\ x\ \hat{k})$$

$$+ A_yB_x\hat{j}\ x\ \hat{i} + A_yB_y\hat{j}\ x\ \hat{j} + A_yB_z\hat{j}\ x\ \hat{k}$$

$$+ A_zB_x\hat{k}\ x\ \hat{i} + A_zB_y\hat{k}\ x\ \hat{j} + A_zB_z\hat{k}\ x\ \hat{k}$$

Using eqns (2.11.17) and (2.11.18) on r.h.s we obtain

$$\bar{A}x\bar{B} = A_xB_y\hat{k} - A_xB_z\hat{j} - A_yB_x\hat{k} + A_yB_z\hat{i}$$

$$+ A_zB_x\hat{j} - A_zB_y\hat{i}$$

$$\Rightarrow \quad \bar{A}x\bar{B} = (A_yB_z - A_zB_y)\hat{i}$$

$$+(A_zB_x - A_xB_z)\hat{j} + (A_xB_y - A_yB_x)\hat{k} \quad ...(2.11.19)$$

Eqn. (2.11.19) implies that

$$\left(\bar{A}x\bar{B}\right)_x = A_yB_z - A_zB_y$$

$$\left(\bar{A}x\bar{B}\right)_y = A_zB_x - A_xB_z$$

$$\left(\bar{A}x\bar{B}\right)_z = A_xB_y - A_yB_x \quad ...(2.11.20)$$

and

$$\left|\bar{A}x\bar{B}\right| = \sqrt{(A_yB_z - A_zB_y)^2 + (A_zB_x - A_xB_z)^2 + (A_xB_y - A_yB_x)^2} \quad ...(2.11.21)$$

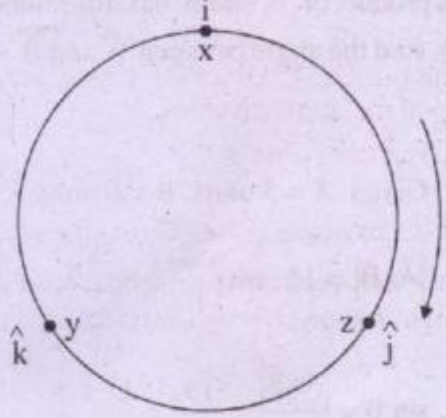


Fig : 2.33 (Component rule)

Equation (2.11.19) can also be written in the form of a determinant as follows :

$$\widehat{\left|\bar{A}x\bar{B}\right|} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \quad ...(2.11.22)$$

Since $\left|\bar{A}x\bar{B}\right| = AB\sin\theta$

$$\Rightarrow \quad \sin\theta = \frac{\left|\bar{A}x\bar{B}\right|}{AB} \quad ...(2.11.23)$$

If $\bar{A}$ and $\bar{B}$ are given in component form, then $\left|\bar{A}x\bar{B}\right|$ is obtained by eqn. (2.11.21) and put on r.h.s. of (2.11.23), so that the angle 'θ' between $\vec{A}$ and $\vec{B}$ can be obtained. Also

$$\vec{A}\times\vec{B} = \frac{\vec{A}\times\vec{B}}{|\vec{A}\times\vec{B}|} = \hat{n} \qquad ...(2.11.24)$$

defines a vector which is perpendicular to both $\vec{A}$ and $\vec{B}$ and the plane containing $\vec{A}$ and $\vec{B}$.

Ex. 2.11.3 The vector $\vec{A}$ has magnitude of 5 units and $\vec{B}$ has magnitude of 6 unit. If the cross product of $\vec{A}$ and $\vec{B}$ has magnitude of 15 units, find the angle between $\vec{A}$ and $\vec{B}$.

Soln.

Given $A = 5$ units, $B = 6$ units

$|\vec{A}\times\vec{B}| = 15$ unit

$$\sin\theta = \frac{|\vec{A}\times\vec{B}|}{AB} = \frac{15}{5\times 6} = \frac{1}{2}$$

$\therefore\ \theta = 30^\circ$ or 150°

Ex. 2.11.4 Given $\vec{A} = 2\hat{i}$, $\vec{B} = 3\hat{i} + 2\hat{j}$

Find (a) $\vec{A}\times\vec{B}$ and (b) $\vec{B}\times\vec{A}$

Soln.

(a) $\vec{A}\times\vec{B} = 2\hat{i}\times(3\hat{i}+2\hat{j}) = 4\hat{i}\times\hat{j} = 4\hat{k}$

(b) $\vec{B}\times\vec{A} = -(\vec{A}\times\vec{B}) = -4\hat{k}$

Ex. 2.11.5 Given $\vec{A} = 2\hat{i} + 3\hat{j} + 4\hat{k}$

and $\vec{B} = \hat{i} - 2\hat{j} + 3\hat{k}$

obtain $\vec{A}\times\vec{B}$

Soln.

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$= (A_yB_z - A_zB_y)\hat{i} + (A_zB_x - A_xB_z)\hat{j} + (A_xB_y - A_yB_x)\hat{k}$$

$$= (3.3 - 4.(-2))\hat{i} + (4.1 - 2.3)\hat{j} + (2(-2) - 3.1)\hat{k}$$

$\vec{A}\times\vec{B} = 17\hat{i} - 2\hat{j} - 7\hat{k}$

$|\vec{A}\times\vec{B}| = \sqrt{17^2 + (-2)^2 + (-7)^2} = 18.49$ units

2.12 Lami's theorem

When three vectors add to zero, the ratio of the magnitude of each vector to the sine of the angle between the other two vectors is a constant.

i.e. if $\vec{P} + \vec{Q} + \vec{R} = 0$

then $$\frac{P}{\sin(\vec{Q},\vec{R})} = \frac{Q}{\sin(\vec{P},\vec{R})} = \frac{R}{\sin(\vec{P},\vec{Q})} \qquad ...(2.12.1)$$

Proof

Given $\vec{P} + \vec{Q} + \vec{R} = 0$

$\Rightarrow\ \vec{P} + \vec{Q} = -\vec{R}$

Taking cross-product on both sides with $\vec{P}$ from the left

$\vec{P}\times(\vec{P}+\vec{Q}) = -\vec{P}\times\vec{R}$

$\Rightarrow\ \vec{P}\times\vec{Q} = \vec{R}\times\vec{P} \qquad ...(2.12.2)$

Again taking cross-product with Q from the left

$$\vec{Q} \times \vec{P} = -\vec{Q} \times \vec{R}$$

$$\Rightarrow \quad \vec{P} \times \vec{Q} = \vec{Q} \times \vec{R} \qquad ...(2.12.3)$$

From eqns (2.12.2) and (2.12.3)

$$\vec{P} \times \vec{Q} = \vec{Q} \times \vec{R} = \vec{R} \times \vec{P} \qquad ...(2.12.4)$$

$$\Rightarrow \quad |\vec{P} \times \vec{Q}| = |\vec{Q} \times R| = |\vec{R} \times \vec{P}|$$

Giving

$$PQ \sin(\vec{P}, \vec{Q}) = QR \sin(\vec{Q}, \vec{R}) = RP \sin(\vec{R}, \vec{P})$$

Dividing throughout by PQR

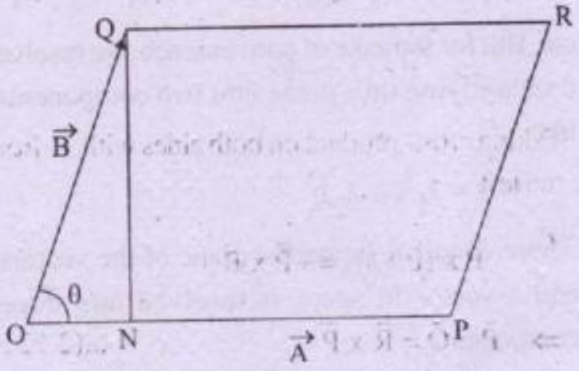
$$\frac{\sin(\vec{P}, \vec{Q})}{R} = \frac{\sin(\vec{Q}, \vec{R})}{P} = \frac{\sin(\vec{R}, \vec{P})}{Q}$$

Inverting the relation

$$\frac{P}{\sin(\vec{Q}, \vec{R})} = \frac{Q}{\sin(\vec{R}, \vec{P})} = \frac{R}{\sin(\vec{P}, \vec{Q})} \qquad ...(2.12.5)$$

Thus Lami's theorem is proved.

2.13 (a) Area of a parallelogram :



Consider a parallelogram OPRQ with $\overrightarrow{OP} = \vec{A}$ and $\overrightarrow{OQ} = \vec{B}$ as two adjacent sides.

Area of the parallelogram

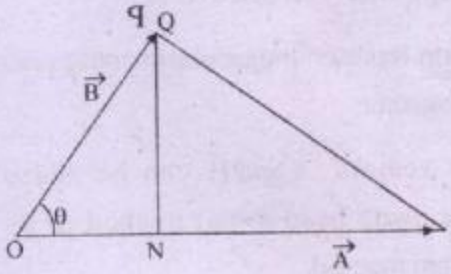
$$= \overline{OP}.\overline{ON} = A(B\sin\theta) = AB\sin\theta$$

$$= |\vec{A} \times \vec{B}| \qquad ...(2.13.1)$$

Eqn. (2.13.1) shows that "the magnitude of vector product is equal to the area of the parallelogram with the vectors as adjacent sides".

(b) Area of a triangle :

consider the triangle OPQ, with



$\overrightarrow{OP} = \vec{A}$ and $\overrightarrow{OQ} = \vec{B}$

as two sides of the triangle.

Area of the triangle $= \frac{1}{2}\overline{OP}.\overline{QN} = \frac{1}{2}A(B\sin\theta)$

$$= \frac{1}{2}AB\sin\theta = \frac{1}{2}|\vec{A} \times \vec{B}| \qquad ...(2.13.2)$$

Eqn. (2.13.2) whows that "Area of a triangle is equal to half of the magnitude of cross-product of two vectors representing two sides of the triangle".

Summary

1. Physical quantities which are completely described by a number (with units specified and possibly with an algebraic sign as for electric cherge) are called scalar quantities.

e.g. mass, length, time, temperature etc.

2. Physical quantities which require specification of magnitude (a number and unit) as well as direction for their complete discription are called vectors.

3. If $\vec{A}$ is a Vector then $\vec{B} = k\vec{A}$ is a vector. If k is a positive number then $\vec{B}$ has same direction as $\vec{A}$, but if k is a negative number then $\vec{B}$ is oppositely directed to $\vec{A}$.

4. Vectors representing similar quantities can be added together.

5. Two vectors $\vec{A}$ and $\vec{B}$ can be added graphically using head-to-tail method or by parallelogram method.

6. If $\vec{R} = \vec{A} + \vec{B}$ then $|\vec{R}| = \sqrt{A^2 + B^2 + 2AB}\cos\theta$, where θ is the smaller angle between $\vec{A}$ and $\vec{B}$ and the angle subtended by resultant $\vec{R}$ with $\vec{A}$ and $\vec{B}$ are given respectively as

$$\tan(\hat{R}, \hat{A}) = \tan\varphi = \frac{B\sin\theta}{A + B\cos\theta} = \frac{B\sin(\hat{A}, \hat{B})}{A + B\cos(\hat{A}, \hat{B})}$$

$$\tan(\hat{R}, \hat{B}) = \tan\varphi' = \frac{A\sin\theta}{B + A\cos\theta} = \frac{B\sin(\hat{A}, \hat{B})}{B + A\cos(\hat{A}, \hat{B})}$$

7. Vector addition is commutative :

$$\vec{A} + \vec{B} = \vec{B} + \vec{A}$$

It also obeys associative law :

$$\vec{A} + (\vec{B} + \vec{C}) = (\vec{A} + \vec{B}) + \vec{C}$$

It also obeys distributive law :

$$m(\vec{A} + \vec{B}) = m\vec{A} + m\vec{B}$$

8. A null vector or zero vector ($\vec{O}$) is a vector with zero maguitude and arbitrary direction. It has the proporties.

$$\vec{A} + \vec{O} = \vec{A} \qquad \alpha\vec{O} = \vec{O}$$

$$\vec{A} - \vec{O} = \vec{A} \qquad \vec{O}.\vec{A} = o, \vec{O} \times \vec{A} = \vec{O}$$

9. Subtraction of two vectors in defined as :

$$\vec{A} - (\vec{B}) = \vec{A} + (-\vec{B})$$

10. $\vec{A} + (-\vec{A}) = o$, $(-\vec{A})$ is called the negative vector of .

11. Two vectors are said to be colinear if their lines of action are one and same.

12. Vectors are said to be coplaner if they lie in same plane.

13. A unit vector of any vector (say $\vec{A}$) is a unit less vector having unit magnitude but lying in the same direction of the vectors. Its only purpose is to describe the direction.

$$\hat{A} = \frac{\vec{A}}{A} = \frac{\vec{A}}{|\vec{A}|}$$

14. A vector can be resolved into a number of components.

e.g

$$\vec{R} = \vec{A}_1 + \vec{B}_1 + \vec{C}_1 = \vec{A}_2 + \vec{B}_2 = \vec{A}_3 + \vec{B}_3 + \vec{C}_3 + \vec{D}_3$$

etc. But for the sake of convenience, we resolve a vector lying on a plane into two components like

$$\hat{A} = \lambda_1\vec{a} + \lambda_2\vec{b}$$

where $\vec{a}$ and $\vec{b}$ lie on the plane of the vectors and a vector in space is resolved into three components.

$$\hat{A} = \lambda_1\vec{a}_1 + \lambda_2\vec{a}_2 + \lambda_3\vec{a}_3$$

The component vectors are chosen to be perpendicular to each other. e.g

(i) $\vec{A} = \hat{i}A_x + \hat{j}A_y$ (When vector $\hat{A}$ lies on a plane

(ii) $\vec{A} = \hat{i}A_x + \hat{j}A_y + \hat{k}A_z$ (When vector $\vec{A}$ lies in 3 dim space)

In case (I) $|\vec{A}| = \sqrt{A_x^2 + A_y^2}$

case (II) $|A| = \sqrt{A_x^2 + A_y^2 + A_z^2}$

15. When two or more vectors are added their components add up.

e.g $\vec{A} + \vec{B} + \vec{C} = \hat{i}(A_x + B_x + C_x)$
$+ \hat{j}(A_y + B_y + C_y) + \hat{k}(A_z + B_z + C_z)$

16. Dot product or scalar product of two vectors is given as :

$\vec{A}.\vec{B} = AB\cos\theta$ where 'θ' is the smaller angle between the vectors. Also when expressed in terms of rectangular components.

$$\vec{A}.\vec{B} = A_xB_x + A_yB_y + A_zC_z$$

(a) Dot product is commutative i.e. $\vec{A}.\vec{B} = \vec{B}.\vec{A}$

(b) Dot product is distributive i.e.

$$\vec{A}.(\vec{B}+\vec{C}) = \vec{A}.\vec{B} + \vec{A}.\vec{C}$$

(c) Dot product is associative i.e.

$$(m\vec{A}).(n\vec{B}) = mn\,\vec{A}.\vec{B} = (n\vec{A}).(m\vec{B})$$

17. Vector product or cross product of two vectors is given as :

$\vec{A} \times \vec{B} = AB\sin\theta\hat{n}$, where $\hat{n}$ is the unit vector normal to the plane containing $\vec{A}$ and $\vec{B}$ and its positive direction is given by right hand screw rule or right hand thumb rule. In terms of components.

$$\vec{A} \times \vec{B} = \hat{i}(A_yB_z - A_zB_y) + \hat{j}(A_zB_x - A_xB_z) + \hat{k}(A_xB_y - A_yB_x)$$

(a) Cross product is not commutative i.e.

$$\vec{A} \times \vec{B} \neq \vec{B} \times \vec{A}$$

(b) Cross product is associative i.e.

$$(m\vec{A}) \times \vec{B} = \vec{A} \times m\vec{B} = m(\vec{A} \times \vec{B})$$

(c) Cross product is distributive i.e.

$$\vec{A} \times (\vec{B}+\vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$$

(d) $\vec{A} \times \vec{A} = 0$

MODEL QUESTIONS

A. Multiple Choice Type Questions :

1. Name the quantity which is a scalar
   (i) velocity (ii) momentum
   (iii) kineticenergy (iv) acceleration

2. Name the quantity which is a vector
   (i) work (ii) speed
   (iii) pressure (iv) displacement

3. Two forces $\vec{F}_1$ and $\vec{F}_2$ act simultaneously on a particle. If the forces are at right angles to each other their resultant is
   (i) $F_1 + F_2$ (ii) $\vec{F}_1 - \vec{F}_2$
   (iii) $F_1^2 + F_2^2$ (iv) $\sqrt{F_1^2 + F_2^2}$

4. Two forces of equal magnitude act, simultaneously, on a particle. If the magnitude of their resultant is equal to either of them, angle between them is
   (i) 30^0 (ii) 60^0
   (iii) 90^0 (iv) 120^0

5. A vector is not changed if
   (i) it is rotated through an arbitrary angle
   (ii) it is multiplied by an arbitrary scalar
   (iii) it is cross multiplied by a unit vector
   (iv) it is translated parallel to itself

6. Which of the following sets given below may represent the magnitudes of three vectors adding to zero ?
   (i) 2,4,8 (ii) 4,8,10
   (iii) 1,2,1 (iv) 0.5,1,2

7. The resultant of $\vec{A}$ and $\vec{B}$ makes an angle α with $\vec{A}$ and β with $\vec{B}$. Then
   (i) $\alpha < \beta$
   (ii) $\alpha < \beta$ if $A < B$
   (iii) $\alpha < \beta$ if $A > B$
   (iv) $\alpha < \beta$ if $A = B$

8. The component of a vector is
   (i) always less than its magnitude
   (ii) always greater than its magnitude
   (iii) always equal to its magnitude
   (iv) none of the above

9. A vector $\vec{A}$ points vertically upward and $\vec{B}$ points towards north. The vector product $\vec{A} \times \vec{B}$ is
   (i) along west
   (ii) along east
   (iii) zero
   (iv) vertically down-ward

10. If $\vec{R} = \vec{A} + \vec{B}$. Then
    (i) $|\vec{R}|$ is always greater than $|\vec{A}|$
    (ii) It is possible to have $|\vec{R}| < |\vec{A}|$ and $|\vec{R}| < |\vec{B}|$
    (iii) $|\vec{R}|$ is always equal to A+B
    (iv) $|\vec{R}|$ is never equal to A+B

11. If $\vec{A} = \vec{B} + \vec{C}$ and A=5, B= 4, C=3 units the angle between $\vec{A}$ and $\vec{C}$ is
    (i) $\cos^{-1}(3/5)$ (ii) $\cos^{-1}(4/5)$
    (iii) $\pi/2$ (iv) $\sin^{-1}(3/4)$

12. Given $\vec{A} = 2\hat{i} + 5\hat{k}$, $\vec{B} = 3\hat{j} + 4\hat{k}$. Then $\vec{A}.\vec{B}$ is equal to

    (i) 20 (ii) 23

    (iii) $5\sqrt{33}$ (iv) 26

13. If $|\vec{A} + \vec{B}| = |\vec{A} - \vec{B}|$, then angle between $\vec{A}$ and $\vec{B}$ is

    (i) 0^0 (ii) 60^0

    (iii) 90^0 (iv) 180^0

14. If vector $\vec{a} = 2\hat{i} + 4\hat{j} - 2\hat{k}$, $\vec{b} = 3\hat{i} - 2\hat{j} + x\hat{k}$ are perpendicular to each other then the value of x should be

    (i) 2 (ii) -2

    (iii) 1 (iv) -1

15. A force of 120N and a force of 20N acting simultaneously at a point may produce a resultant force of

    (i) 80 N (ii) 140 N

    (iii) 160 N (iv) none of the above

16. If $\hat{a}_1$ and $\hat{a}_2$ are non-collinear unit vectors and if $|\hat{a}_1 + \hat{a}_2| = \sqrt{3}$, then the value of $(2\hat{a}_1 - 5\hat{a}_2).(3\hat{a}_1 + \hat{a}_2)$ is

    (i) $\frac{41}{2}$ (ii) $\frac{11}{2}$

17. If $\vec{A} + \vec{B}$ is perpendicular to $\vec{A} - \vec{B}$, then $\vec{A}$ and $\vec{B}$ are

    (i) equal to each other

    (ii) equal in magnitude

    (iii) not equal in magnitude

    (iv) not predictable

18. The minimum number of unequal forces whose vector sum can be zero is

    (i) 2 (ii) 5

    (iii) 3 (iv) 4

19. The component of $9\hat{i} + 17\hat{k}$ along z-axis is :

    (i) zero (ii) 17

    (iii) 9 (iv) 26

20. The magnitude of vector product of two vectors is $\sqrt{3}$ times their scalar product. The angle between the vectors is

    (i) $\pi/2$ (ii) $\pi/6$

    (iii) $\pi/3$ (iv) $\pi/4$

21. The flight of a bird can be an example of

    (i) polygon law of vectors

    (ii) dot product of vectors

    (iii) cross product of vectors

    (iv) composition of vectors

22. If $|\vec{F}_1 + \vec{F}_2| = |\vec{F}_1 - \vec{F}_2|$, then

    (i) $\vec{F}_1$ is parallel to $\vec{F}_2$

    (ii) $\vec{F}_1 = \vec{F}_2$

    (iii) $|\vec{F}_1| = |\vec{F}_2|$

    (iv) $\vec{F}_1$ is perpendicular to $\vec{F}_2$

23. The resultant of $2\hat{i}$ and $3\hat{j}$ is

    (i) $2\hat{j}$ (ii) $3\hat{j}$

    (iii) $3\hat{j} - 2\hat{i}$ (iv) $2\hat{i} + 3\hat{j}$

24. Given $\vec{P} = 3\hat{i} + 4\hat{j} - 2\hat{k}$, $\vec{Q} = 4\hat{i} - 3\hat{j} + 2\hat{k}$ unit vector in the direction of $\vec{P} + \vec{Q}$ is
   (i) $7\hat{i} + \hat{j}$
   (ii) $\frac{1}{29}\left(2\hat{i} - 14\hat{j} - 25\hat{k}\right)$
   (iii) $\frac{1}{\sqrt{50}}\left(7\hat{i} + \hat{j}\right)$
   (iv) $2\hat{i} - 14\hat{j} - 25\hat{k}$

25. Given $\vec{A} = 2\hat{i} + 3\hat{j}$, $\vec{B} = 3\hat{i} - 2\hat{k}$ unit vector in the direction of $\vec{A} \times \vec{B}$ is
   (i) $4\hat{j}$
   (ii) $\frac{2\hat{i} + \hat{j}}{5}$
   (iii) $\frac{1}{\sqrt{133}}\left(-6\hat{i} + 4\hat{j} - 9\hat{k}\right)$
   (iv) $-6\hat{i} + 4\hat{j} - 9\hat{k}$

B. Very Short Answer Type Questions :

1. Is a vector necessarily changed if it is rotated through an angle ?
2. Is it possible to add two vectors of unequal magnitudes and get zero
3. Is it possible to add three vectors of equal magnitudes and get zero ?
4. Can you add three unit vectors to get a unit vector ?
5. If two unit vectors are along coordinate axes and third unit vector is arbitrary, will the three add to give a unit vector ?
6. Can a vector have zero component along a line and still have nonzero magnitude ?
7. Can the vector sum of two unit vector be a unit vector ?
8. Is the vector sum of $\hat{i}$ and $\hat{j}$ a unit vector ?
9. The dot product of two vector is zero. What is the angle between them ?
10. What are the maximum and minimum value of $\vec{A}.\vec{B}$.
11. If two vectors, mutually perpendicular, are increased twice each, how will their resultant vary ?
12. What is the condition that will show that two given non-zero vectors are either parallel or antiparallel ?
13. Will three vectors not lying on one plane ever add up to give a null vector ?
14. Write the expression for unit vector in the direction vector $\vec{A}$.
15. What is the angle between $\vec{A}$ and $m\vec{A}$, where m is a constant.
16. Find the magnitude of $\vec{R}$ given by $\vec{R} = 3\hat{i} + \sqrt{11}\,\hat{j} + 4\hat{k}$.
17. What will be the component of a vector along a line making an angle α with the vector ?

C. Short Answer Type Questions :

1. State triangle law of vectors.
2. State parallelogram law of vectors.
3. If two vectors are represented by the adjacent sides of a parallelogram, then what do the diagonals of the parallelogram represent ?
4. Which of the following has the maximum inclination with X - axis and what is its value ?

   (a) $3\hat{i} + 4\hat{j}$ (b) $5\hat{i} + 3\hat{j}$ (c) $10\hat{i} + 12\hat{j}$
5. If $\vec{A} + \vec{B} + \vec{C} = 0$, show that $\vec{A} \times \vec{B} = \vec{B} \times \vec{C}$

6. The magnitude of the resultant of $\vec{a}$ and $\vec{b}$ is equal to that of $\vec{a}$ and is perpendicular to $\vec{a}$. Show that $|\vec{b}| = \sqrt{2}.a$.

7. How can you explain the flying of a bird on the basis of law of vector addition ?

8. Explain walking of a man by the concept of resolution of vectors.

9. State polygon law of vectors.

10. If $\vec{P}+\vec{Q}=\vec{R}$ and $P+Q=R$, what is the angle $\vec{P}$ and $\vec{Q}$?

11. Find the angle between $\vec{A} = 2\hat{i}+4\hat{j}+4\hat{k}$ and $\vec{B} = 4\hat{i}+2\hat{j}-4\hat{k}$

12. A triangle is formed by vectors $\vec{a}, \vec{b}$ and $\vec{c} = \vec{a}+\vec{b}$.
Prove $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

D. Unsolved numericals

1. A vector $\vec{A}$ makes an angle of 20^0 and $\vec{B}$ makes an angle of 110^0 with x-axis. If A=3m, B=4m, find the resultant.

2. Given $\vec{a} = 4\hat{i}+3\hat{j}$, $\vec{b} = 3\hat{i}+4\hat{j}$ find the magnitudes of (i) $\vec{a}$ (ii) $\vec{b}$ (iii) $\vec{a}+\vec{b}$ (iv) $\vec{a}-\vec{b}$.

3. Given $\vec{a} = 2\hat{i}+3\hat{j}+4\hat{k}$, $\vec{b} = 3\hat{i}+4\hat{j}+5\hat{k}$ find the angle between $\vec{a}$ and $\vec{b}$.

4. If $\vec{A}, \vec{B}, \vec{C}$ are mutually perpendicular, show that $\vec{C} \times (\vec{A} \times \vec{B}) = 0$.

5. The mid-points of the sides of a quadrilateral are connected. Prove that the resulting quadrilateral is a perallelogram.

6. If AB be the diameter of a circle and C is any point on the circumference of the circle. Prove by vector method that angle ACB is right angle.

7. The diagonals of a parallelogram are represented by $\vec{R}_1 = 3\hat{i}-2\hat{j}+7\hat{k}$, $\vec{R}_2 = 5\hat{i}+6\hat{j}-3\hat{k}$, find the area of the parallelogram.

8. A particle gets displaced through $\hat{i}+2\hat{j}-4\hat{k}$ due to a force $2\hat{i}-3\hat{j}-6\hat{k}$. The displacement and force are measured in S.I. units. Find the work done.

9. Show that $\vec{A} \times \vec{B}$ is perpendicular to both $\vec{A}$ and $\vec{B}$.

10. Show that $\vec{A}.\vec{B} = \frac{1}{4}\left[|\vec{A}+\vec{B}|^2 - |\vec{A}-\vec{B}|^2\right]$

E. Long Answer Type Questions :

1. State Parallelogram law of vectors and derive an expression for the resultant of two vectors $\vec{P}$ and $\vec{Q}$.

2. State triangle law of vectors, and derive an expression for the resultant of the vectors $\vec{A}$ and $\vec{B}$.

3. What do you mean by resolution of vectors ? Explain the resolution of vectors into rectangular components.

4. Explain scalar product and vector product of two vectors.

F. Fill in the Blanks Type

1. Time is a .... quantity
2. Acceleration is a .... quantity
3. $\vec{F}.\vec{r}$ is a ....and $\vec{r} \times \vec{F}$ is a .... quantity
4. The maximum value of magnitude of $\vec{A}-\vec{B}$ is ....
5. The maximum value of magnitude of $\vec{A}+\vec{B}$ is ....

6. If $\vec{A}+\vec{B}$ is perpendicular to $\vec{A}-\vec{B}$, then $\vec{A}$ and $\vec{B}$ are ....

7. The minimum number of unequal forces whose vector sum can be zero is ....

8. If $\left|\vec{F}_1+\vec{F}_2\right|=\left|\vec{F}_1+\vec{F}_2\right|$, then F_1 is .... to $\vec{F}_2$.

G. **True - False Type**

1. The vector $\left(\frac{1}{\sqrt{2}}\right)\hat{i}+\left(\frac{1}{\sqrt{2}}\right)\hat{j}$ is a unit vector.

2. The vector $\hat{i}+\hat{j}$ is a unit vector.

3. The vectors $2\hat{i}+3\hat{j}$ and $4\hat{i}+6\hat{j}$ have the same direction.

4. A vector can have many sets of components.

5. Relations among vectors are variant with respect to translation or rotation of the coordinates.

6. A force of 120 N and a force of 20 N acting simultaneously at a point may produce a resultant force of 140 N.

7. The magitude of vector product of two vectors is $\sqrt{3}$ times their scalar product.

The angle between the vectors is $\pi/3$.

ANSWERS

A. MULTIPLE CHOICE TYPE :

1. (iii) 2. (iv) 3. (iv) 4. (iv) 5. (iv) 6. (iii) 7. (iii) 8. (iv) 9 (i) 10. (ii) 11. (i) 12. (i)

13. (iii) 14. (iv) 15. (ii) 16. (iii) 17. (ii) 18. (iii) 19. (ii) 20. (iv) 21. (iv) 22. (iv) 23. (iv) 24. (iii) 25. (iii)

D. NUMERICAL PROBLEMS :

1. 5m, makes 73.13^0 with X-axis,
2. (i) 5, (ii) 5, (iii) $7(\hat{i}+\hat{j})$ (iv) $\hat{i}-\hat{j}$
3. 3.69^0
7. 63.37
8. 20 J

F. (1) scalar (2) vector (3) scalar, vector (4) A + B (5) A + B (6) equal in magnitude (7) 3 (8) perpendicular.

G. (1) True (2) False (3) True (4) True (5) False (6) True (7) True.

3

Kinematics

Rest and motion are two commonly experienced phenomena. But they need some careful attention. For example a person inside a moving train feels that he is at rest with respect to his fellow passengers and persons standing on the platform are in motion. On the contrary a person on the platform feels that persons standing on the platform are at rest and the passengers inside the train are in motion. Thus the concept of rest and motion is a combined property of the object under study and the observer. It is always a relative concept and never an absolute one.

A body is said to be at rest (with respect to an observer) if its position does not change with respect to the observer. On the otherhand a body is said to be in motion (with respect to an observer) if its position changes with respect to the observer from time to time.

The branch of physics that deals with bodies at rest and in motion is called **mechanics.** Mechanics is divided into **statics** and **dynamics. Statics** is the study of objects at rest and dynamics is the study of objects in motion. Dynamics is further subdivided into **kinematics** and **kinetics.** Kinematics is concerned with the study of different characteristics associated with motion of the object(s) without looking for its cause. Kinetics deals with the study of motion alongwith the cause of motion.

In this chapter we shall discuss various aspects of kinematics, keeping in mind students, requirement, inadequacies and misconceptions. Hence we describe below concepts like frame of reference, position vector, types of motion, distance, displacement, speed, velocity, acceleration and equations involving them.

3.0 MATHEMATICAL NOTE:

We discuss here some mathematical preliminaries which shall be useful while studying the chapters to follow.

A. Variables and Functions:

Symbols such as x, t etc. which can stand for any set of numbers (physical quantities) are called variables.

If for each value of x there corresponds one or more values of a real variable ω, then we say that ω is a function of x, and write $\omega = f(x)$. The variable x is called as independent variable and ω is called a dependent variable. The value of the function at x=a is often written as f(a). Thus if $f(x) = x^2$, then $f(2) = 2^2 = 4$.

B. Single and Multiple valued functions:

If one value of $\omega = f(x)$ corresponds to each value of x, then $\omega = f(x)$ is a single valued function of x or one calls f(x) as single-valued.

If more than one value of $\omega = f(x)$ corresponds to each value of x, then $\omega = f(x)$ is a multiple valued function of x.

For example if $f(x) = x^2$, then for each value of x, f(x) has a single value. On the other hand if $f(x) = x^{1/2}$, then for each value of x, f(x) has two values of x (if x = 4, then $f(4) = 4^{1/2} = \pm 2$).

C. Limits:

Let f(x) be defined and single-valued, in the neighbourhood of x = a, with the possible exception of x = a, then we say that 'l' is the limit of f(x) as x approaches a (i.e. $x \to a$) and we write

$$\lim_{x \to a} . f(x) = l \qquad 3.0.1$$

If for any positive number ε (however small) we can find some positive number $\delta(\varepsilon)$, such that

$$| f(x) - l | < \varepsilon, \text{ whenever } 0 < | x - a | < \delta \qquad 3.0.2$$

D. Theorems on limits:

If $\lim_{x \to a} . f(x) = A$

$\lim_{x \to a} . g(x) = B$

Then,

1. $\lim_{x \to a} \{ f(x) \pm g(x) \} = \lim_{x \to a} f(x) \pm \lim_{x \to a} . g(x) = A \pm B \qquad 3.0.3$

2. $\lim_{x \to a} \{f(x). g(x)\} = \{\lim_{x \to a} f(x)\}.\{\lim_{x \to a} g(x)\} = A.B \qquad 3.0.4$

3. $\lim_{x \to a} \{f(x)/g(x)\} = \{\lim_{x \to a} f(x)\}/\{\lim_{x \to a} g(x)\} = A/B, \text{ if } B \neq 0 \qquad 3.0.5$

E. Infinity:

i) $\lim_{x \to a} f(x) = l$, if for any $\varepsilon > 0$, we can find $M > 0$ such that $|f(x) - l| < \varepsilon$ whenever $|x| > M$

ii) $\lim_{x \to a} f(x) = \infty$, if for any $N > 0$, we can find $\delta > 0$, such that $| f(x) | > N$

F. Continuity:

A function f(x) is said to be continuous at x=a if,

$$\lim_{x \to a} f(x) = f(a) \qquad 3.0.6$$

Equation 3.0.6 implies

i) $\lim_{x \to a} f(x) = l$ exists

ii) f(a) exists

iii) $l = f(a)$

G. Theorems on continuity:

i) If f(x) and g(x) are continuous at x=a, then $\{f(x) + g(x)\}$, $\{f(x) - g(x)\}$ and $\{f(x).g(x)\}$ are also continuous at x=a. Also $\{f(x)/g(x)\}$ is continuous at x=a, if $g(x) \neq 0$.

ii) If f(x) is continuous at x=a, g(y) is continuous at y=b, and if b=f(a), then g{f(x)} [called as a function of a function or composite function at x=a] is continuous at x=a.

H. Derivative (Differential Coefficient):

Let f be a function of a variable x over an interval (a, b). Let a variable quantity y be given by the rule y=f(x). If x changes to $x+\delta x$, then f(x) changes to $f(x+\delta x)$, where δx is a small increment in x (we shall continue to call δx an increment even when it is negative). The corresponding increment δy in y is given by

$$\delta y = f(x+\delta x) - f(x)$$

Then, $$\frac{\delta y}{\delta x} = \frac{f(x+\delta x) - f(x)}{\delta x} \qquad 3.0.7$$

In the above $\delta y/\delta x$ is the average rate of change of y. The instantaneous rate of change of y at the value of x is then given by the limit of ($\delta y/\delta x$) as $\delta x \to 0$, provided the limit exists. Therefore one defines: A function f : (a,b): $\to$ R is said to be derivable or differentiable at x ε (a, b) if

$$\lim_{\delta x\to 0} \frac{\delta y}{\delta x} = \lim_{\delta x\to 0} \frac{f(x+\delta x)-f(x)}{\delta x}$$

exists 3.0.8

This limit is the derivative (differential coefficient) of '**f**' with respect to (w.r.t) x. The derivative of '**f**' is denoted by f' or dy/dx

i.e. $f'(x) = dy/dx$

$$= \lim_{\delta x\to 0} \frac{f(x+\delta x)-f(x)}{\delta x} \qquad 3.0.9$$

One can also write

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \qquad 3.0.10$$

and

$$\frac{dy}{dx}\Big|_{x=c} = \lim_{h\to 0} \frac{f(c+h)-f(c)}{h} \qquad 3.0.11$$

Sometimes one defines:

Right hand derivative as

$$f'(c+) = \lim_{h\to 0} \frac{f(c+h)-f(c)}{h}, \quad h>0$$

3.0.12

and Left hand limit as

$$f'(c-) = \lim_{h\to 0} \frac{f(c-h)-f(c)}{-h}, \quad h>0 \qquad 3.0.12$$

It is easily seen that the (unique) derivative of f(x) exists at x = c if and only if $f'(c+)$ and $f'(c-)$ both exist and are equal and is denoted by eqn. (3.0.11).

If function f has a derivative at every point of the interval (a, b), then it is said to be differentiable on (a, b). The process of finding the derivative of a function is known as differentiation.

I. Some results of Differentiation.

1. **If y = c, where c is an isolated constant, then(dy/dx) =(dc/dx) = 0**
2. **If y = x^n, then (dy/dx) = n x^{n-1}**
3. **If y = u^n, and u = u(x), then (dy/dx) = n u^{n-1} (du/dx)**
4. **If y = u(x) ± v(x) ± ω(x) ± — — — — — — —, then (dy/dx) = du/dx ± dv/dx ± dω/dx ± — — — — —**
5. **If y = u(x)v(x), then dy/dx = v(du/dx) + u(dv/dx)**
6. **If y = u(x)/v(x), then dy/dx = {v(du/dx) – u(dv/dx)}/v^2**
7. **(i) (d/dx) sinx = cosx**
   (ii) (d/dx) cosx = -sinx
   (iii) (d/dx)tanx = $\sec^2 x$
   (iv) (d/dx) cotx = -$\text{cosec}^2 x$
   (v) (d/dx) secx = secx. tanx
   (vi) (d/dx) cosecx = -cosecx. cotx
8. **(i) If y = $\log_e x$, then dy/dx = 1/x,**
   (ii) If y = $\log_a x$, then dy/dx = (1/x) $\log_a e$
   (iii) If y = $\log_e u(x)$, then dy/dx = (1/u) (du/dx)
   (iv) If y = $\log_a u(x)$, then dy/dx = (1/u) (du/dx) $\log_a e$
   (v) If y = e^x, then dy/dx = e^x
   (vi) If y = e^u, then dy/dx = e^u (du/dx)
   (vii) If y = a^x, then dy/dx = $a^x \log_e a$

J. Some important applications of differentiation:

(i) Tangent to a curve at a point: If we have a curve y vs. x, i.e. y = f(x) then the tangent to the curve at any point P(a,f(a)) is (i) the line through P with slope $f'(a)$, if $f'(a)$ exists. (ii) the line x = a if

$$\lim_{x\to a} \left|\frac{f(x)-f(a)}{x-a}\right| = \infty$$

The tangent line at $P(x_0, f(x_0))$ does not exist if neither (i) nor (ii) holds. If one holds, then clearly the equation to the required tangent line is

$$y - f(a) = f'(a)(x - a) = \{ (dy/dx)_{x=a} \} . (x - a)$$

The value of $(dy/dx)_{x=a}$ will indicate the nature of the curve.

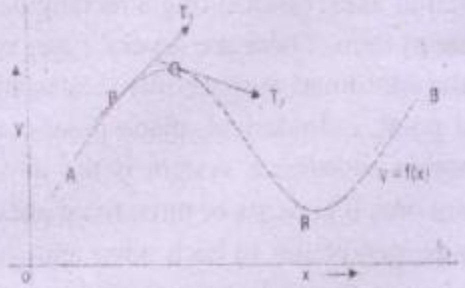


Fig. 3.0.1

2. If second derivative $f''(x) > 0$, then the curve is concave upwards w.r.t X-axis (portion QRB). On the other hand if $f''(x) < 0$, then the curve is convex upwards w.r.t X-axis (portion APQ). The point (Q) where the tangent changes its direction of rotation i.e. where $f'(x)$ attains an extreme value is called a point of inflexion. **A point (Q) on a curve is said to be a point of inflexion if the curve is concave on one side and convex on the other side of the point (Q) w.r.t X-axis.**

K. Integration, Antiderivative (Integral):

Integration is a reverse process of differentiation. If g(x) is the derivative of f(x), then f(x) is said to be an antiderivative (or integral) of g(x). For example cosx is derivative of sinx, and sinx is antiderivative of cosx. This fact is symbolically written as $\int \cos x \, dx = \sin x$. The function to be integrated (here cosx) is called the integrand, dx denotes the integration variable.

But some more thought is to be added to the above.

Suppose $g(x) = df/dx$

Then $g(x)$ is also equal to $(d/dx)(f(x) + c)$, where c is any arbitrary constant. Therefore it is always correct to write

$$\int g(x)\, dx = f(x) + c \qquad 3.0.14$$

The r.h.s $= f(x) + c$ is called the indefinite integral of g(x). In physics this arbitrary constant is obtained by imposing various physical conditions/boundary conditions (which we shall see in the following chapters).

L. Some integration formulae:

(i) $\int dx = x + c$

(ii) $\int x^n \, dx = x^{n+1}/n+1 + c, n \neq -1$

(iii) $\int (1/x)\, dx = \log_e x + c$

(iv) $\int \cos x \, dx = \sin x + c$

(v) +"sinx dx = -cosx + c

(vi) $\int \sec^2 x \, dx = \tan x + c$

(vii) $\int \mathrm{cosec}^2 x \, dx = -\cot x + c$

(viii) secx . tanx dx = secx + c

(ix) $\int \mathrm{cosec}\, x \,.\, \cot x \, dx = -\mathrm{cosec}\, x + c$

(x) $\int (\sin ax)\, dx = -((\cos ax)/a) + c$

(xi) $\int (\cos ax)\, dx = ((\sin ax)/a) + c$

(xii) $\int \tan x \, dx = -\log_e \cos x + c$

(xiii) $\int \cot x \, dx = \log_e \sin x + c$

(xiv) $\int e^x \, dx = e^x + c$

(xv) $\int a^x \, dx = (a^x / \log_e a) + c$

(xvi) $\int \cosh x \, dx = \sinh x + c$

(xvii) $\int \sinh x \, dx = \cosh x + c$

(xviii) $\int \operatorname{sech}^2 x = \tanh x + c$

(xix) $\int \operatorname{cosech}^2 x \, dx = -\coth x + c$

(xx) $\int \operatorname{sech} x \cdot \tanh x \, dx = -\operatorname{sech} x + c$

(xxi) $\int \operatorname{cosech} x \cdot \coth x \, dx = -\operatorname{cosech} x + c$

M. Definite Integration:

If, $\int g(x)\, dx = f(x) + c$,

Then, $\int_a^b g(x)dx = f(x)\Big|_a^b = f(b) - f(a)$; where a is the lower limit and b is the upper limit of the variable x.

It is to be noted that integration is also a substitute for area under a curve. For example consider the curve F(x) vs. x as shown below

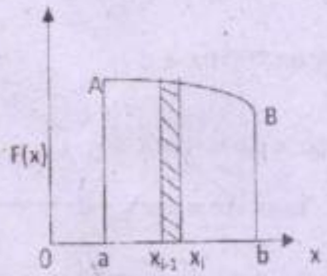


Fig. 3.0.2

Area of the shaded strip is approximately $\Delta A_i = F(x_i)(x_i - x_{i-1}) = F(x_i)\,\Delta x$; where Δx is the width of the shaded strip. When we sum the area of such strips, covering the area AabB under the curve then we obtain the area under the curve AB i.e. area ABba.

$$\text{Thus } A = \sum_{i=1}^{N} \Delta A_i = \sum_{i=1}^{N} \mathbf{F(x_i)\,\Delta x}$$

The limit of this sum as $N \to \infty$ is known as the integral of F(x) over x from a to b.

$$\text{i.e. } A = \int_a^b F(x)\, dx$$

3.1 Frame of reference :

Study of the position of a body at different times is of primary importance in dynamics. In order to locate the position of a body we need a frame of reference, consisting of a conveniently chosen reference point and a set of axes (called coordinate axes) passing through the reference point. A simple and convenient way to fix up coordinate axes is to choose three mutually perpendicular axes, constituting a rectangular coordinate system. There are several types of rectangular coordinate systems like : cartesian, spherical polar, cylinderical, plane polar etc. The cartesian coordinate system is the most convenient one. It consists of three fixed axes, mutually perpendicular to each other and are named as x-y-z axes. The coordinates (x,y,z) specify the position of a body with respect to the frame of reference. If we add a clock to this frame to measure time, then one is capable of measuring positional coordinates at different times; and the reference frame is then called **space-time frame.**

If cordinates x,y,z of a body remains unchanged as time progresses, we say that the body is at rest with respect to the frame. If all or any of (x,y,z) changes as time progresses then we say that the body is in motion with respect to the frame.

The choice of frame of reference can be made at one's convenience. For example if a passsenger, in a moving train, wishes to study the motion of a body in his own compartment, he may choose his frame of reference within the compartment.

There are two types of frames of reference (i) inertial and (ii) non-inertial.

(i) *Inertial frame of reference :* A frame of reference in which Newton's laws of motion are valid is called inertial frame.

(ii) *Non-inertial frame of reference :* A frame of reference in which Newton's laws of motion are not valid is called non-inertial frame.

3.2 Position and positon vector :

The position of a particled is fixed with respect to a reference point and a set of reference axes (i.e. a frame of reference).

The vector joining the reference point (origin) and the particle is called **position vector** ($\bar{r}$) of the particle and it specifics the position of the particle.

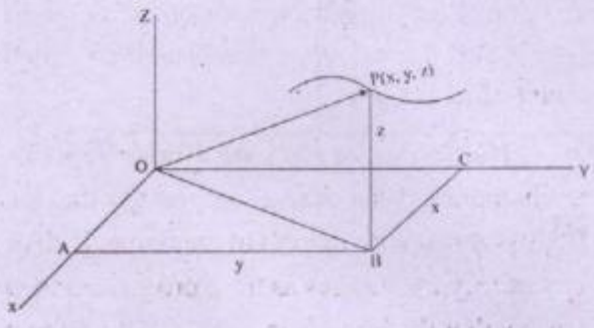


fig. 3.1

In fig. 3.1 $\overrightarrow{OP} = \bar{r}$, is the position vector, specifying the positon p of the particle at any time t. The coordinates of the point p are (x,y,z) and the position vector is given as

$$\overrightarrow{OP} = \bar{r} = x\hat{i} + y\hat{j} + z\hat{k} \qquad ...(3.2.1)$$

with $|\bar{r}| = r = \sqrt{x^2 + y^2 + z^2}$ ...(3.2.2)

Ex. 3.2.1 The position of a body is given by coordinates (2,3,-2). Give its position vector.

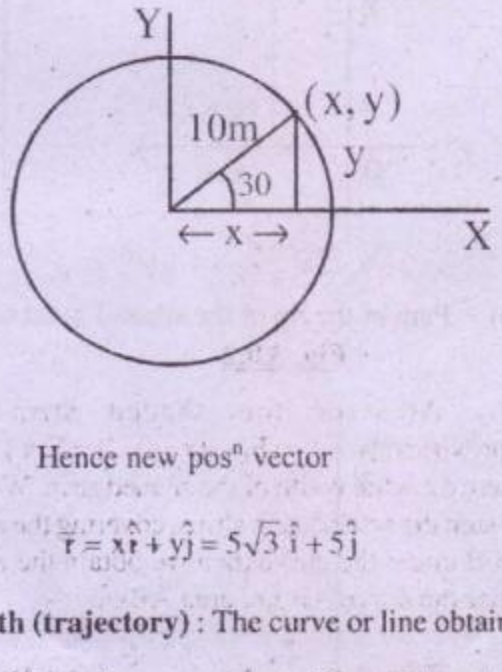
Soln. Position vector $\bar{r} = 2\hat{i} + 3\hat{j} - 2\hat{k}$.

Ex. 3.2.2 A body is moving on a circular path of radius 10m in xy-plane. If he turns through an angle 30^0 what is his new position vector w.r. to the centre of the circle.

Soln.

$$\text{Now } x = 10\cos 30 = 10\frac{\sqrt{3}}{2} = 5\sqrt{3}\text{ m}$$

$$y = 10\sin 30 = 10.\frac{1}{2} = 5\text{ m}$$



Hence new pos^n vector

$$\bar{r} = x\hat{i} + y\hat{j} = 5\sqrt{3}\,\hat{i} + 5\hat{j}$$

Path (trajectory) : The curve or line obtained by joining the successive positions of the particle during its motion is called its path.

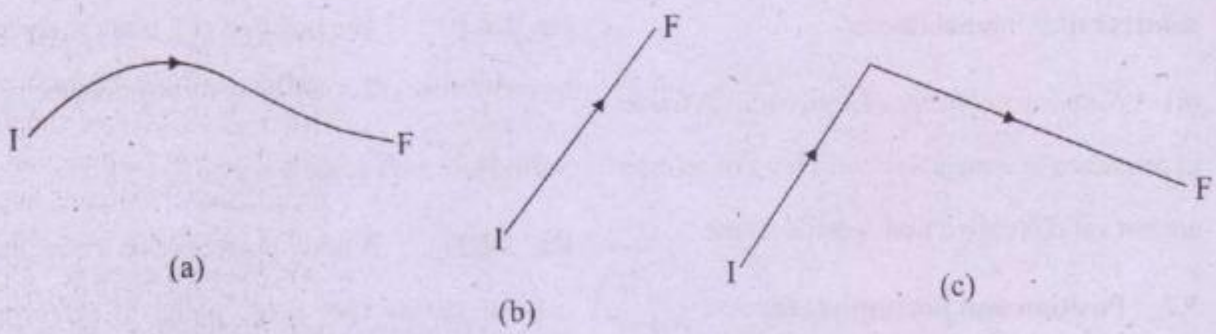


fig. 3.2

Fig. 3.2 shows different paths followed by three different bodies. It is important to note that the choice of reference point does not change the nature of path. For the sake of illustration we consider the following examples.

(a) Path of a batsman as he takes a run by the shortest route.

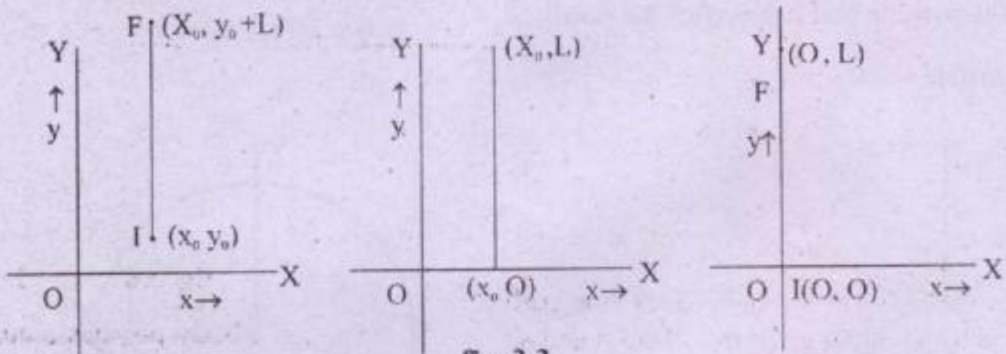


fig. 3.3

(b) Path of the tip of the second-hand of a clock

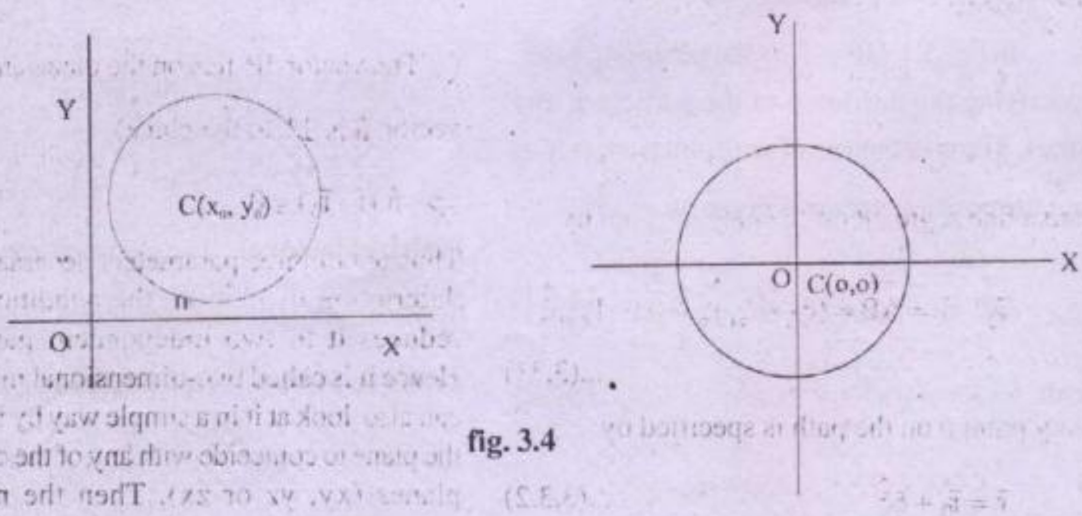


fig. 3.4

3.3 Types of motion :

Depending on the path pursued by a body during its motion, the motion can be classified as : (i) One-dimensional (ii) two-dimensional and (iii) three dimensional motion.

(i) **One - dimensional motion :** If a body during its motion, moves along a stright line, then the motion is called rectilinear or one-dimensional motion.

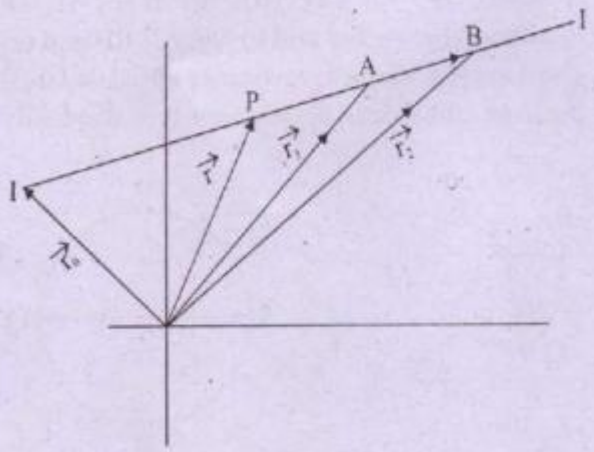


fig. 3.5

In fig. 3.5 II′ is the straight line path followed by a body. I is the initial point and A, B are two points on the path. Then IB = α.IA , where α is a real positive number. We can put

$\overrightarrow{IB} = \xi_2\hat{\xi}, \overrightarrow{IA} = \xi_1\hat{\xi}$ and $\xi_2 = \alpha\xi_1$, then

$$\bar{r}_2 = \bar{r}_0 + \xi_2\hat{\xi}$$

$$\bar{r}_1 = \bar{r}_0 + \xi_1\hat{\xi}$$

and a line segment on the path is given by

$$\bar{r}_2 - \bar{r}_1 = \overrightarrow{AB} = (\xi_2 - \xi_1)\hat{\xi} = (\alpha - 1)\xi_1\hat{\xi} \quad ...(3.31)$$

Any point p on the path is specified by

$$\bar{r} = \bar{r}_0 + \xi\hat{\xi} \quad ...(3.3.2)$$

Thus a single parameter 'ξ' is required to describe the motion of the particle. Hence it is called one-dimensional motion. Another simple way to look at the problem is to imagine a coordinate axis to coincide with the straight line. Then only that coordinate shall vary. So it shall be reckoned as one-dimensional motion.

(ii) **Two-dimensional motion :** If a body during its motion pusues a curved path, suct that the path is confined to a fixed plane, then the motion is called two-dimensional motion.

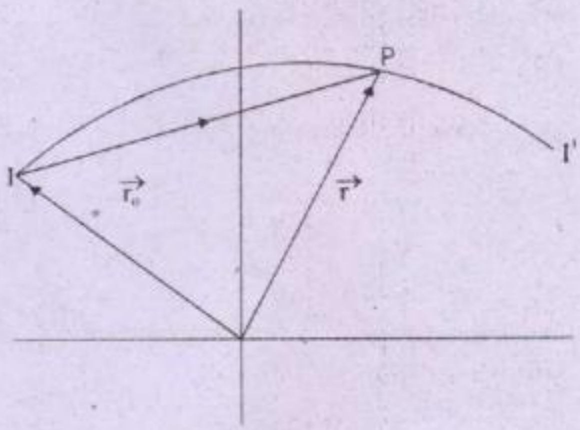


fig. 3.6

Suppose a body pursues a curved path II′ on a plane. Let $\hat{n}$ be perpendicular to this plane. Then

$$\hat{n}.\overrightarrow{IP} = 0$$

($\because$ The vector $\overrightarrow{IP}$ lies on the plane and the unit vector $\hat{n}$ is $\perp^r$ to the plane)

$$\Rightarrow \hat{n}.(\bar{r} - \bar{r}_0) = 0 \quad ...(3.3.3)$$

Thus out of three parameters necessary for the description of motion, the condition (3.3.3) reduces it to two independent parameters. Hence it is called two-dimensional motion. We can also look at it in a simple way by imagining the plane to coinceide with any of the coordinate planes (xy, yz or zx). Then the motion is described by two coordinates only.

(iii) **Three - dimensional motion :** If a body during its motion pursues a curved path, not being restricted to a plane, then the motion is called three-dimensional motion.

In this case three-coordinates are necessary to describe the motion.

3.4 Distance and displacement :

Distance is the length of the path travelled during a time interval. It is always measured from the initial position to the final position along the path. For example consider different paths shown is fig. 3.7.

travelled is 2L. Distance is a scalar quiantity and has the dimension of length. It is measured in cm or m.

Displacement is the straight line joining the initial position to the final position and is directed from the initial position to the final position. It is a vector quantity and has the dimension of length. It is also measured in cm or m.

In fig. 3.7 (a), (b), (c) $\vec{IF} = \vec{S}$ is the displacement vector and in fig. 3.7 (d) and (e) displacement vector is zero since initial and final position coincide in these cases. It is also easily

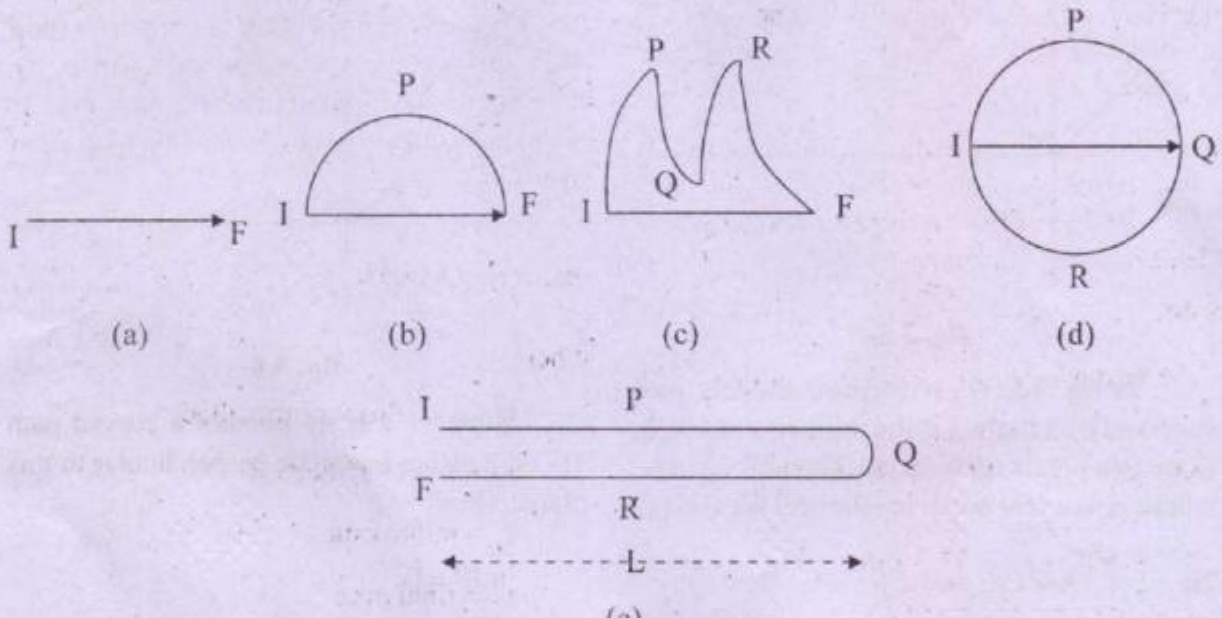


fig. 3.7

In fig. 3.7 (a) path followed is IF and distance (D) travelled is equal to $\overline{IF}$. In fig. 3.7(b) path followed is IPF and distance travelled is $\pi.\overline{IF}/2$. In fig. 3.7 (c) path followed is IPQRF and distance travelled is equal to sum of lengths of the curves IP, PQ, QR and RF. In fig. 3.7 (d) the path followed is IPQRI and distance travelled is $\pi.\overline{IQ}$. In fig. 3.7 (e) path followed is IPQRF and distance

seen that in fig. 3.7(a) $|\vec{S}|$=D but in fig. 3.7 (b), (c), (d) and (e) $|\vec{S}| \neq D$. **Hence except for unidirectional motion magnitude of displacement is not equal to distance travelled:**

The displacement vector can be expressed in terms of position vector.

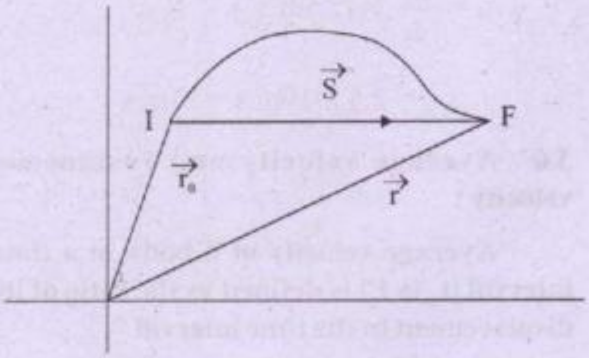


fig. 3.8

$$\vec{S} = \vec{r}(t) - \vec{r}_0 \qquad ...(3.4.1)$$

It is also worth mentioning that displacement is defined over a time interval, whereas position-vector is defined at an instant of time.

Ex. 3.4.1 A man moves along straight paths from P (2,3,2)m to Q (4,7,6)m and then to R (-2,1,3)m. Find the distance travelled and displacement of the man.

Soln.

Distance travelled $D = \overline{PQ} + \overline{QR}$

$$\overline{PQ} = \sqrt{(4-2)^2 + (7-3)^2 + (6-2)^2}$$

$$= \sqrt{2^2 + 4^2 + 4^2} = 6\text{m}$$

$$\overline{QR} = \sqrt{(-2-4)^2 + (1-7)^2 + (3-6)^2}$$

$$= \sqrt{6^2 + 6^2 + 3^2} = 9\text{m}$$

$\therefore$ D = 6m+9m = 15m.

The displacement

$$\vec{S} = \overrightarrow{PR} = (-2-2)\hat{i} + (1-3)\hat{j} + (3-2)\hat{k}$$

$$\Rightarrow \quad \vec{S} = -4\hat{i} - 2\hat{j} + \hat{k}$$

$$|S| = \sqrt{4^2 + 2^2 + 1^2} = \sqrt{21}$$

Ex.3.4.2 A body walks on a semicircular path of radius 50 m. If he starts at one end and reaches at the other end. Find the distance covered and the displacement of the boy.

Soln.

Given r = 50 m

Length of the semicircular path = $\pi r = 50\pi$ m

Distance travelled D = 50π m = 157m

Displacement $|\vec{S}| = 2r = 100\text{m}$

3.5 Average speed and Instantaneous speed :

The average speed of a body in a time interval is defined as the distance travelled by the body divided by the time interval (or ratio of the total distance travelled to the total time taken)

$$\text{i.e. average speed} = \langle v \rangle = \frac{D}{t_2 - t_1} \qquad ...(3.5.1)$$

Where,

D = total distance travelled

$t_2 - t_1$ = time interval

t_1 = initial time

t_2 = final time

Thus the average speed gives the overall rapidity with which a body covers a distance. For example if a person covers a distance of 40 km on road in 1hr, one naively says that the person has covered the distance at a speed of 40 km/hr. But actually this is not the speed at all instants during his course of travel. To know the speed at an instant we need to define instantaneous speed.

Instantaneous speed at any instant of time, is defined as the speed possessed by a body at that particular instant.

Let ΔD be the distance travelled in time interval Δt. Then average speed over this interval is $\langle \upsilon \rangle = \frac{\Delta D}{\Delta t}$. Now if the time interval is made vanishingly small then it shall correspond to speed at time t. So we write

$$\upsilon = \lim_{\Delta t \to 0} \frac{\Delta D}{\Delta t} = \frac{dD}{dt} \qquad ...(3.5.2)$$

Instantaneous speed is generally called speed.

A body is said to be moving with uniform speed if its speed is constant althroughout the motion. In that case $\langle \upsilon \rangle = \upsilon$. Speed is a scalar quantity and has the dimension LT^{-1}. It is expressed in cms^{-1} or ms^{-1}.

Ex. 3.5.1 A train covers a distance of 500km in 10hour. Calculate its average speed.

Soln.

$$\upsilon_{av} = \langle \upsilon \rangle = \frac{\text{Distance travelled}}{\text{Time taken}}$$

$$\Rightarrow \langle \upsilon \rangle = \frac{500\text{km}}{10\text{hr}} = 50\text{km/hr}$$

Ex.3.5.2 The distance travelled by a particle in time t is given by $D=(2.5\ m/s^2).t^2$. Find (a) the average speed of the particle during time 0 to 5.0 s; and (b) the instantaneous speed at t = 5.0s.

Soln.

$$D_{t=5s} = (2.5.\text{x}25)\text{m} = 62.5\text{m}$$

$$t = 5\text{s}.$$

$$\therefore \langle \upsilon \rangle = \frac{62.5}{5}\text{m/s} = 12.5\text{m/s}$$

Instantaneous speed

$$\upsilon = \frac{dD}{dt} = (2.5\text{m/s}^2) \times 2t$$

$$\therefore \upsilon_{t=5s} = 2.5 \times 10\text{m/s} = 25\text{m/s}.$$

3.6 Average velocity and Instaneous velocity :

Average velocity of a body in a time interval (t_1 to t_2) is defined as the ratio of its displacement to the time interval

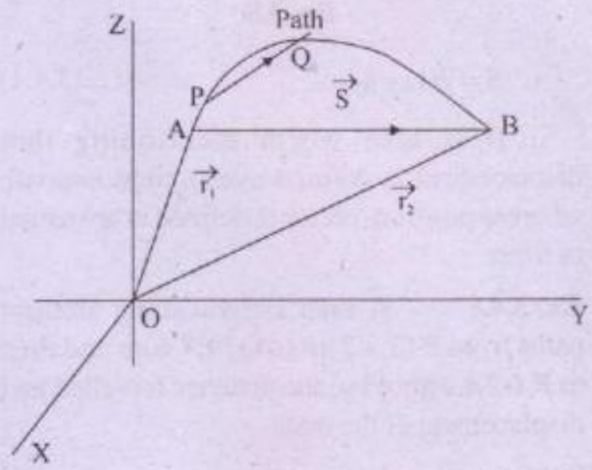


fig.3.9

Thus if the body is at A at time t_1 and at B at time t_2 (see fig. 3.9), then average velocity is given as

$$\langle \vec{\upsilon} \rangle = \frac{\vec{AB}}{t_2 - t_1} = \frac{\vec{S}}{t_2 - t_1} = \frac{\vec{r}_2 - \vec{r}_1}{t_2 - t_1} \qquad ...(3.6.1)$$

If time interval $t_2 - t_1 = \Delta t$, and $\vec{r}_2 - \vec{r}_1 = \vec{\Delta r}$ then

$$\langle \vec{\upsilon} \rangle = \frac{\vec{\Delta r}}{\Delta t} \qquad ...(3.6.2)$$

But actually when we travel an appreciable distance along a curved path the direction of motion could change and also speed of motion could change. Even if when we travel along a straight road the speed of motion could change, so we need to define **instantaneous velocity.**

Instantaneous velocity of a body is its velocity at a particular instant or at a particular point on its path.

Suppose $\Delta\vec{S} = \Delta\vec{r}$ is the displacement in the time interval $\Delta t \left(t - \frac{\Delta t}{2} \text{ to } t + \frac{\Delta t}{2} \right)$. Then average velocity over this interval of time is

$$\langle \vec{\upsilon} \rangle = \frac{\Delta\vec{r}}{\Delta t} = \frac{\Delta\vec{S}}{\Delta t}$$

Now if the time interval is made vanishingly small, then the velocity will correspond to time t. So **instantaneous velocity** is given as

$$\vec{\upsilon} = \lim_{\Delta t \to 0} \frac{\Delta\vec{r}}{\Delta t} = \frac{d\vec{r}}{dt} = \frac{d\vec{s}}{dt} \qquad ...(3.6.3)$$

and $\vec{g}$ is along the line of displacement $\Delta\vec{r}(=\Delta\vec{s})$ Eqn. (3.6.3) leads to define; **instantaneous velocity is the time rate of displacement;** or time rate of change of position vector. Instantaneous velocity is generally called velocity. Velocity, whether average or instantaneous, is a vector quantity and has the dimension of LT^{-1}. It is expressed in cms^{-1} or ms^{-1}. When a body moves with constant velocity, it is said to be moving with uniform velocity; and in that case $\langle \vec{\upsilon} \rangle = \vec{\upsilon}$.

It is worth noting that **unless the motion is rectilinear, magnitude of average velocity is not equal to the average speed.** For example consider the motion, shown in fig. 3.10

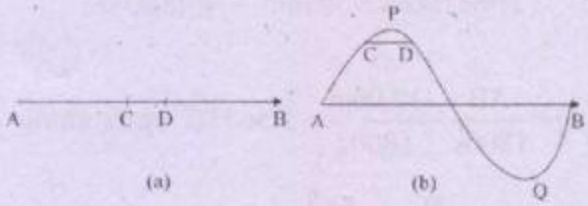


fig. 3.10

In fig. 3.10 (a) $\langle \vec{\upsilon} \rangle = \frac{\vec{AB}}{t}$, giving $|\langle \vec{\upsilon} \rangle| = \frac{\overline{AB}}{t}$ and $\langle \upsilon \rangle = \frac{\overline{AB}}{t}$. So $|\langle \vec{\upsilon} \rangle| = \langle \upsilon \rangle$. Bt in fig. 3.10 (b) $|\langle \vec{\upsilon} \rangle| = \frac{\overline{AB}}{t}$, $\langle \upsilon \rangle = \frac{APQB}{t}$, so $|\langle \vec{\upsilon} \rangle| \neq \langle \upsilon \rangle$.

However the magnitude of instantaneous velocity is equal to instantaneous speed.

Because $|\upsilon| = \left| \lim_{\Delta t \to 0} \frac{\Delta\vec{r}}{\Delta t} \right| = \lim_{\Delta t \to 0} \frac{\text{Chord CD}}{\Delta t}$

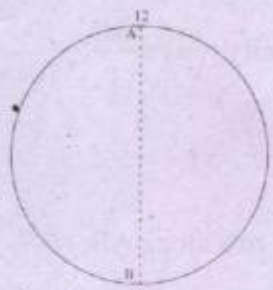
and $\upsilon = \lim_{\Delta t \to 0} \frac{\text{Arc CD}}{\Delta t}$. Since for small Δt, arc CD = Chord CD, so $|\vec{\upsilon}| = \upsilon$.

Ex. 3.6.1 A table clock has its minute hand 5.0 cm long. Find the average velocity of the tip of the minute hand (a) between 6.0 am to 6.30 am and (b) between 6.00 am to 6.30 p.m. also find any speed in each case.

Soln.

(a) Between 6.00 am to 6.30 am

to displacement = $\vec{AB}$



Time taken = 30 min = 1800 sec.

$$\left\langle \vec{\upsilon} \right\rangle = \frac{\vec{AB}}{1800s} = \frac{10.0cm}{1800s} = 5.56\times10^{-3}\text{ cm/s along}$$

$$\vec{AB}\ \langle\upsilon\rangle = \frac{\pi r}{18005} = \frac{\pi\times5}{1800}\text{cm/s} = 8.73\times10^{-3}\text{ cm/s}$$

(b) Between 6.00 am to 6.30 pm displacement = $\vec{AB}$ time taken = 12hr 30 min = 750 x 60s.

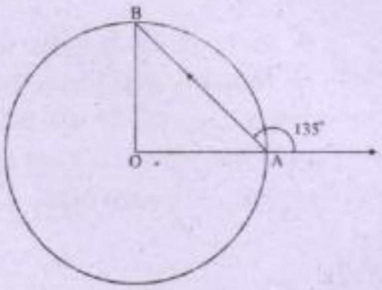
$$\therefore \left\langle \vec{\upsilon} \right\rangle = \frac{\overline{AB}}{750\times60} = \frac{10}{45000}\text{ cm/s along }\vec{AB}$$

$$\Rightarrow \left\langle \vec{\upsilon} \right\rangle = 2.22\times10^{-4}\text{ cm/s along }\vec{AB}.$$

$$\langle\upsilon\rangle = \frac{25\pi r}{45000S} = 8.73\times10^{-3}\text{ cm/s}$$

Ex.3.6.2 A body moves in a circle of radius 100 m. If it describes 90^0 at the centre what is its displacement ? If he takes 10 minute, to cover this, what is his average velocity ?

Soln.



Initial position is A. Final position is B Displacement $\vec{S} = \vec{AB} = 100\sqrt{2}$ m along i.e. $100\sqrt{2}$ m at 135^0 with the initial radius OA.

Average velocity $\left\langle \vec{\upsilon} \right\rangle = \frac{\overline{AB}}{t} = \frac{100\sqrt{2}}{10\times60s}$ **along** $\vec{AB}$

$$\Rightarrow \left\langle \vec{\upsilon} \right\rangle = \frac{\sqrt{2}}{6}\text{ m/s along }\vec{AB}.$$

i.e. $\frac{1}{3\sqrt{2}}$ m/s at 135^0 with the initial radius.

3.7 Average acceleration and Instantaneous acceleration

If the velocity of a body changes from time to time, it is said to be accelerated.

Average acceleration of a body is defined as the ratio of the change in velocity to the time interval over which the change occured.

$$\text{i.e. } \langle\vec{a}\rangle = \frac{\vec{\upsilon}_2 - \vec{\upsilon}_1}{t_2 - t_1} = \frac{\vec{\Delta\upsilon}}{\Delta t} \qquad ...(3.7.1)$$

Where,

$\vec{\upsilon}_1$ = velocity of the body at time $t_1 = \upsilon_1\hat{\xi}_1$

$\vec{\upsilon}_2$ =velocity of the body at time $t_2 = \upsilon_2\hat{\xi}_2$

$\Delta\vec{\upsilon}$ = change in velocity

Δt = time interval.

Since velocity is a vector quantity, a change in its magnitude or direction or both will cause a change in velocity, hence an accelerated motion However if a body executes a **rectilinear** as well as **unidirectional motion,** then

$$\left\langle \vec{a} \right\rangle = \frac{\upsilon_2 - \upsilon_1}{t_2 - t_1}\xi \qquad ...(3.7.2)$$

Instantaneous acceleration of a body at time t is defined as the time rate of change of velocity

$$\text{i.e. } \vec{a} = \lim_{\Delta t \to 0} \frac{\vec{\Delta\upsilon}}{\Delta t} = \frac{d\vec{\upsilon}(t)}{dt} \qquad ...(3.7.3)$$

In general $\vec{\upsilon}(t) = \upsilon(t)\,\hat{\xi}(t)$, so

$$\vec{a} = \frac{d\upsilon(t)}{dt}\hat{\xi}(t) + \upsilon(t)\frac{d\hat{\xi}(t)}{dt} = a(t)\hat{a}(t) \qquad ...(3.7.4)$$

Thus in general acceleration could be due to change in speed and/or direction of velocity. But if a body performs a **unidirectional** motion

then $\frac{d\hat{\xi}}{dt}=0$ and $\bar{a}=\frac{d\upsilon(t)}{dt}\hat{\xi}$ ...(3.7.5)

Which means in unidirectional motion, acceleration is caused due to change in its speed only.

A body which moves with a constant acceleration is said to be uniformly accelerated. So in case of uniform acceleration

$\langle\bar{a}\rangle = \bar{a}$ = A constant vector

It implies that in case of uniform acceleration, the magnitude as well as directin of acceleration remains constant. However uniformly accelerated motion need not be a rectilinear motion. For example in fig. 3.11. we find that even though $\vec{a}$ is constant the velocity vector changes its direction from time to time.

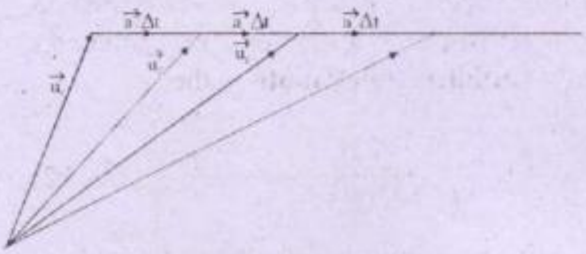


fig. 3.11

Thus acceleration is a vector quantity and has the dimension of LT^{-2}. Its magnitude is expressed in $cm\,s^{-2}$ or $m\,s^{-2}$.

Ex. 3.7.1 A particle moving along the positive x-axis has a speed of 20 m/s at t=2s. After 3 seconds its speed becomes 40 m/s. Compute the average acceleration of the particle during this time interval.

Soln.

Average acceleration $\langle\bar{a}\rangle = \frac{\vec{\upsilon}_2-\vec{\upsilon}_1}{t_2-t_1}$

$\vec{\upsilon}_1=20\frac{m}{s}.\hat{i},\ \vec{\upsilon}_2=40\frac{m}{s}\hat{i}$

$t_1 = 2s,\ t_2 = 3s$

$\therefore \langle\bar{a}\rangle = \frac{40-20}{3-2}\ ms^{-2}\hat{i} = 20m/s^2$ in the dir^n of position x-axis.

Ex. 3.7.2 A particle moving along positive x-axis, has its position given by

$x = At + Bt^2$

find its instantaneous velocity and acceleration

Soln.

Given $x = At + Bt^2$

$\upsilon = \frac{dx}{dt} = A+2Bt$

$\therefore\ \vec{\upsilon}=(A+2Bt)\ \hat{i}$

$a = \frac{d\upsilon}{dt} = 2B$

$\therefore\ \bar{a} = 2B\hat{i}$

Ex. 3.7.3 The length of the second hand of a watch is 1 cm. What is the change in velocity of the tip of the second hand of the watch in 15s.

Soln.

Initial velocity $\vec{\upsilon}_1 = \upsilon\hat{i}$ (along $\vec{AB}$)

final velocity $\vec{\upsilon}_2 = -\upsilon\hat{j}$ (along $\vec{BC}$)

$\vec{\Delta\upsilon} = \vec{\upsilon}_2 - \vec{\upsilon}_1 = \upsilon(-\hat{j}-\hat{i})$

$\therefore \left|\vec{\Delta\upsilon}\right| = \upsilon\sqrt{(1+1)} = \upsilon\sqrt{2}$

It is directed along $\vec{BO}$, i.e. it makes 45^0 with the final velocity.

Now $\upsilon = \frac{2\pi r}{60s} = \frac{2\pi}{60}.1\ \frac{cm}{s} = \frac{\pi}{30}\frac{cm}{s}$

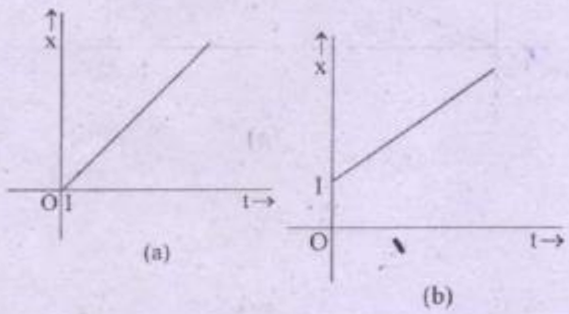
$\therefore \left|\vec{\Delta\upsilon}\right| = \frac{\pi}{30}\sqrt{2}$ cm/s.

3.8 Graphs in Kinematics :

(A) Position - time graph :

Position - time graph can be plotted for rectilinear motion only. Some examples are shown below :

(i) Uniform motion along a straight line



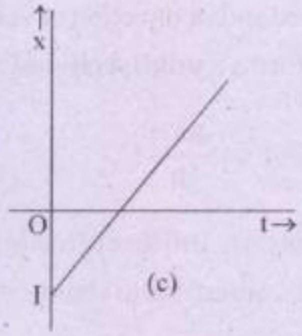
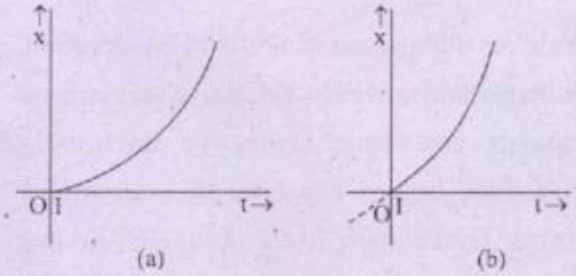


fig. 3.12

(I denotes the initial position)

(ii) Uniformly accelerated motion along a straight line.



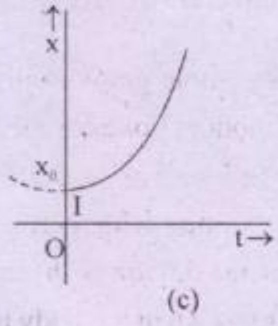


fig. 3.13

(iii) Ball thrown upwards

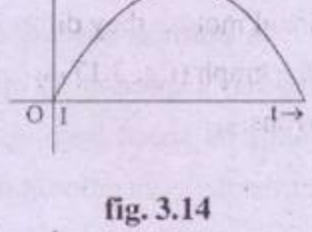


fig. 3.14

(iv) A batsman taking two runs.

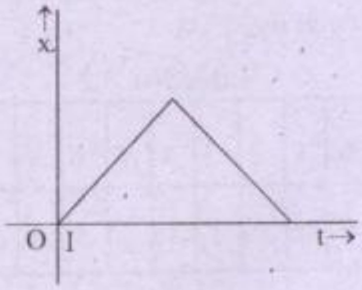


fig. 3.15

(B) Distance - time graph :

Distance - time graph can be plotted for curvilinear as well as rectilinear motions.

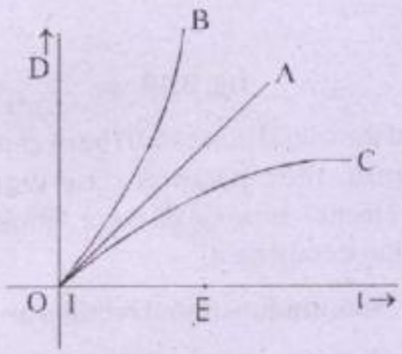


fig. 3.16

The distance-time graph shall depend on the nature of motion. For example in fig. 3.16 OA represents motion with uniform speed, OB represents motion with increasing speed, and OC represents motion with decreasing speed. OE represents when the body is at rest.

It is to be noted that in case of unidirectional motion position time graph is equivalent to distance-time graph (Compare fig. 3.12, 3.13 with fig. 3.16). But in non-unidirectional motion they differ for example consider the graph (fig. 3.17) when a batsman takes two runs.

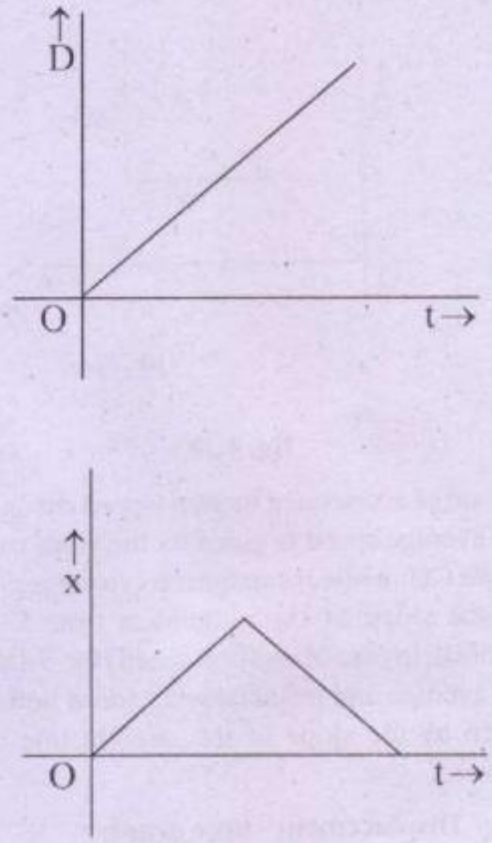
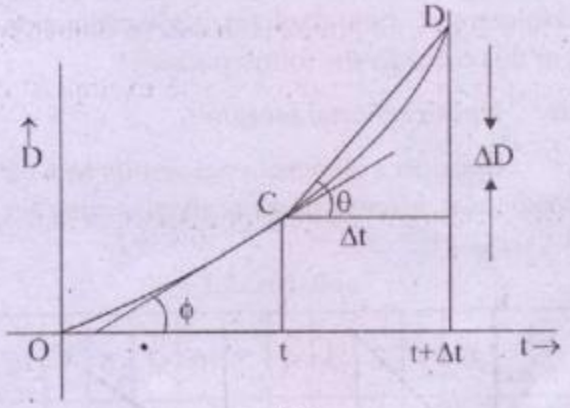


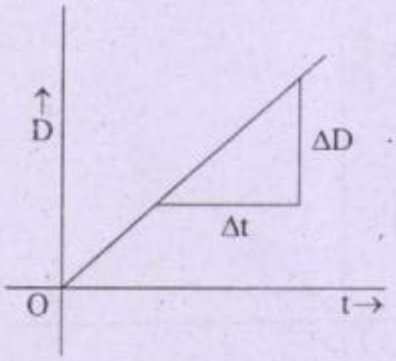
fig. 3.17 (a)

(Non-Unidirectional motion)

From distance-time graph one can find average speed and instantaneous speed.



(a)



(b)

fig. 3.18

In case of accelerated motion (speed changing) the average speed is given by the slope of the chord CD; while instantaneous speed is given by the slope of the tangent at time t (fig. 3.18(a)). In case of uniform speed (fig. 3.18(b)) the average and instantaneous speed both are given by the slope of the straight line (D-t curve).

(C) Displacement - time graph :

Displacement time graph can be plotted for rectilinear motion only, by attributing the negative sign (arising due to direction) to the negative magnitude. For curvilinear motion displacement-time graph cannot be plotted.

We now make a comparison between position-time graph. distance - time graph and displacement - time graph for rectilinear motion. For this consider the following cases.

(i) Unidirectional motion :

Consider a unidirectional motion with the positions at different times as given in table No. 3.1

Table No. 3.1

t in sec	0	1	2	3	4	5	6	7	8	9	10
x in cm	.02	.05	.09	.14	.20	.27	.37	.48	.60	.73	.85

Then the distance and dlisplcement are as given in table No. 3.2 and the corresponding graphs are shown in fig. 3.19.

Table No. 3.2

t in sec	0	1	2	3	4	5	6	7	8	9	10
D/s in cm	0	.03	.07	.12	.18	.25	.35	.46	.58	.71	.83

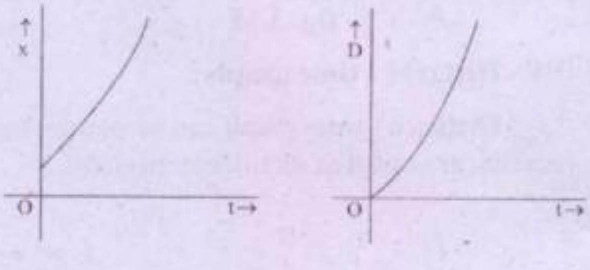


fig. 3.19

But had the initial point (t=0) been choosen as the origin, then position-time graph and displacement - time or distance - time graph would have coincided.

(ii) Non-unidirectional rectilinear motion

Consider a non-unidirectional motion with positions at different times as given in table No. 3.3

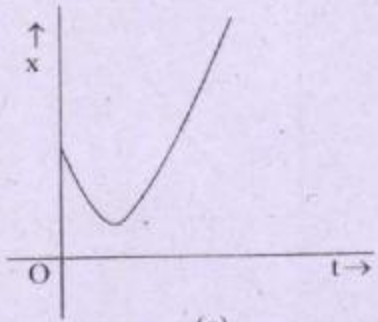
Table No. 3.3

t in sec	0	1	2	3	4	5	6	7	8	9	10
x in cm	.08	.05	.04	.05	.08	.13	.19	.28	.39	.51	.64

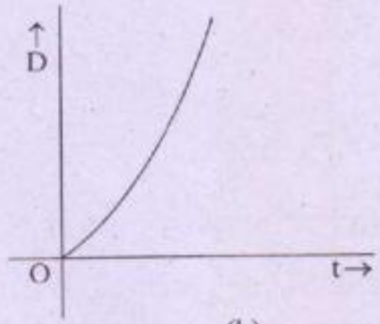
Then the distance and displacement are as given in table No. 3.4. The corresponding graphs are shown in fig. 3.20

Table No. 3.4

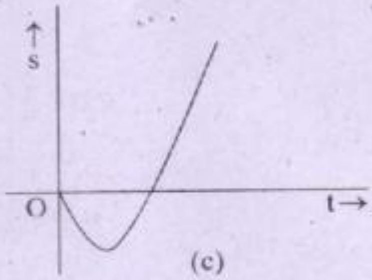
t in sec	0	1	2	3	4	5	6	7	8	9	10
D in cm	0	.03	.04	.05	.08	.13	.19	.28	.39	.51	.64
S in cm	0	-.03	-.04	-.03	0	.05	.11	.20	.31	.43	.56



(a)
(Position - time graph)



(b)
(Distance - time graph)



(c)
(Displacement - time graph)

fig. 3.20

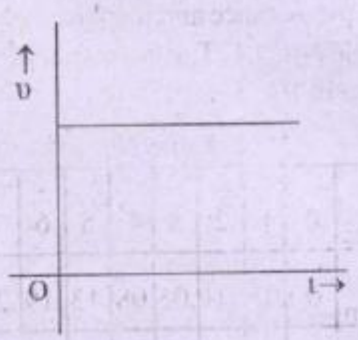
(Graphs only show the nature)

But if the initial point is chosen as the origin, then the displacement-time graph and position-time graph shall coincide.

(D) Speed-time graph :

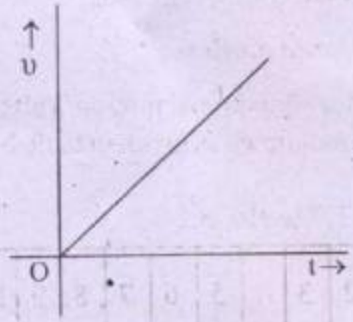
Speed-time graph can be plotted for rectilinear as well as curvilinear motion. For the sake of illustration we discuss below few examples.

(i)



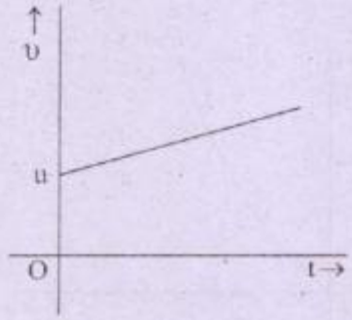
(uniform speed)

(ii)



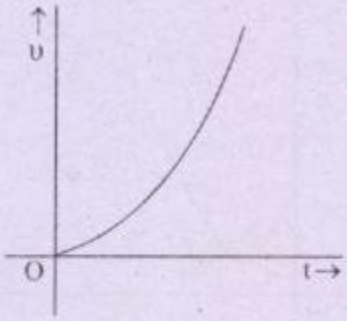
(Speed increasing uniformly from zero initial speed)

(iii)



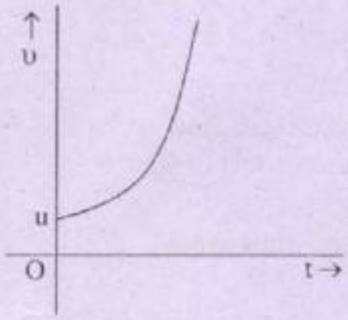
(Speed increasing uniformly from non-zero initial speed)

(iv)



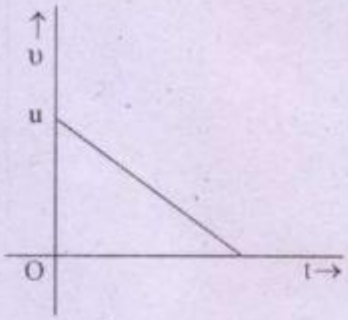
(Speed increasing from zero initial speed)

(v)



(Speed increasing from non-zero initial speed)

(vi)



(Speed decreasing uniformly)

(vii) Motion with non-uniform rate of decrease of speed.

(vii)

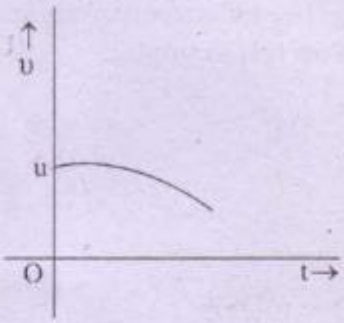


fig. 3.21

Speed - time graph for other specific cases can be drawn from the knowledge of the actual motion. For example consider a ball thrown upwards with a speed u which comes down after reaching highest point. The graph shall be as show in fig. 3.22

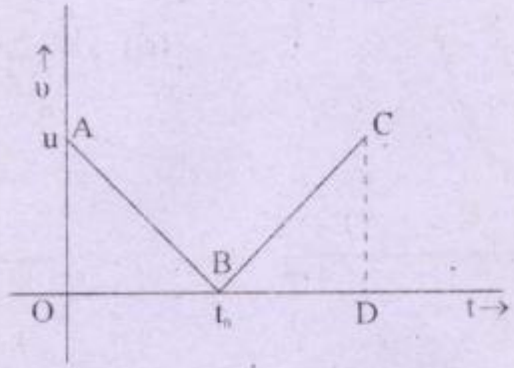


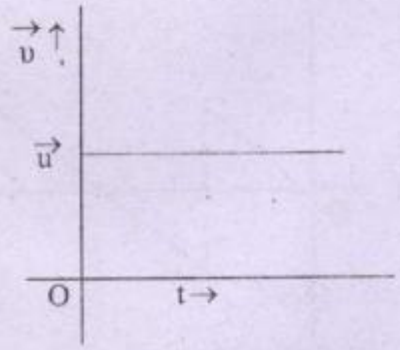
fig. 3.22

Since $D=\int \vartheta dt$. One can find the distance travelled from the area under the curve (ϑ -t). For example in fig. 3.22 total distance travelled is equal to the area of $\Delta OAB + \Delta BCD$.

(E) Velocity - time graph :

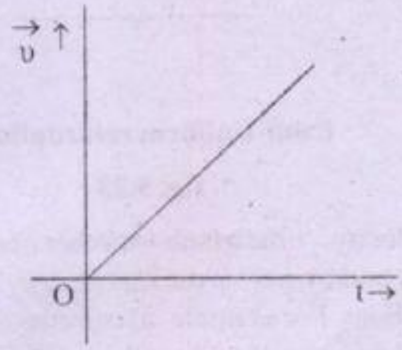
In general one cannot plot velocity time graph as velocity is a vector quantity. However in case of rectilinear motion, by attributing the negative direction to the regative magnitude, we can plot velocity-time graph. We discuss below few examples of rectilinear motion.

(i)



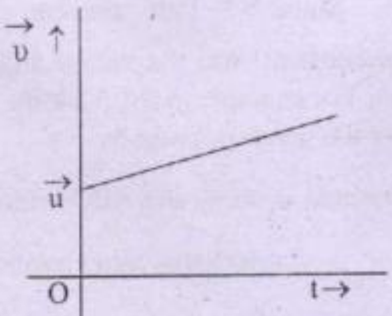
(Constant velocity)

(ii)



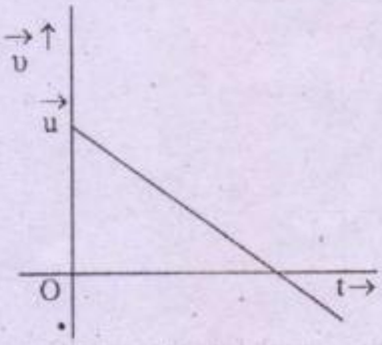
(Uniform accl^n with zero initial velocity)

(iii)



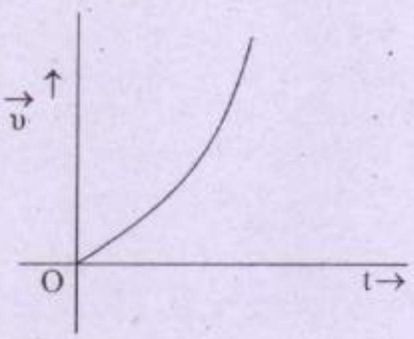
(Uniforma accl^n with non-zero initial velocity)

(iv)



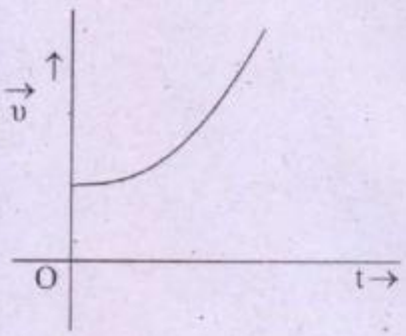
(Uniform retardation)

(v)



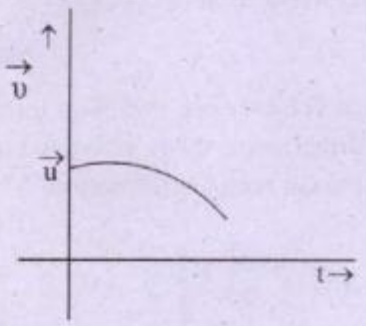
(Non-uniform accl^n with zero initial velocity)

(vi)



(Non-uniform accl^n with non- zero initial velocity)

(vii)



(Non-Uniform retardation)

fig. 3.23

Velocity - time graph for other specific cases can be drawn from the knowledge of the actual motions. For example (a) consider the case of a ball thrown upwards with a velocity $\vec{u}$ which comes down after reaching the highest point. The graph shall be as shown in fig. 3.24.

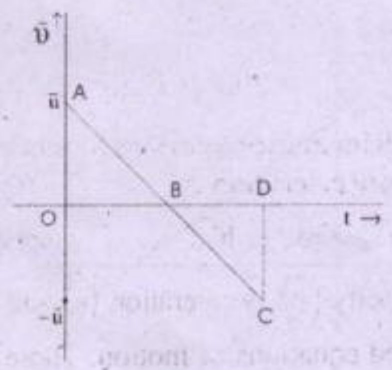


fig. 3.24

(b) Elastic collision of a ping-pong ball on the floor, falling from a height.

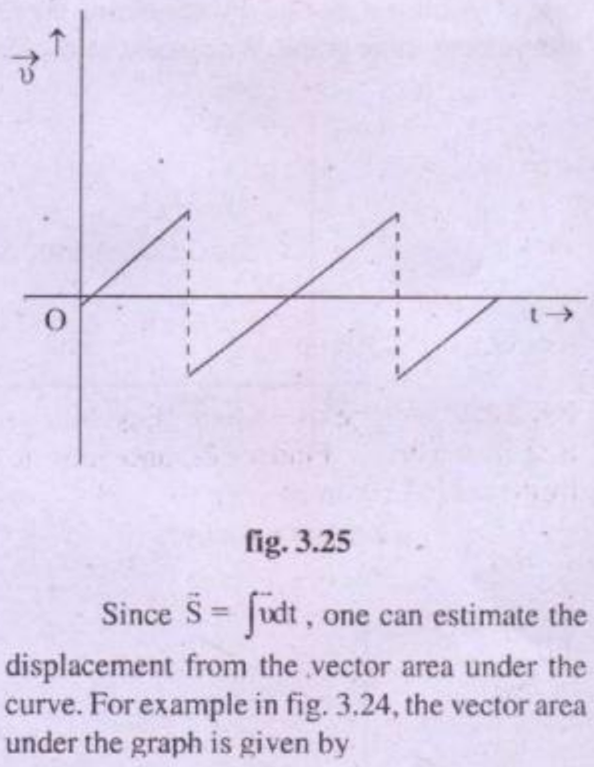


fig. 3.25

Since $\vec{S} = \int \vec{u}dt$, one can estimate the displacement from the vector area under the curve. For example in fig. 3.24, the vector area under the graph is given by

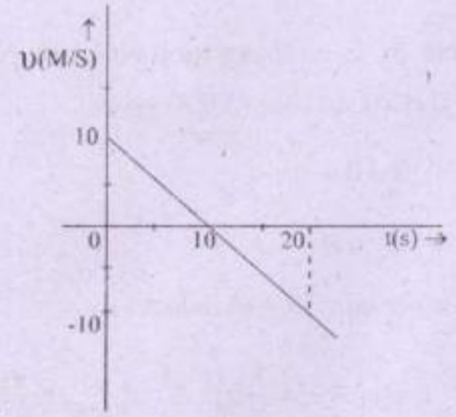
vector area = vector area ABO+vector area BCD

= clock wise area + anticlock wise area

= 0

Hence the displacement is zero.

Ex. 3.81 The velocity time plot for a particle moving on a straight line is as shown in the figure below. What type of motion it is ?

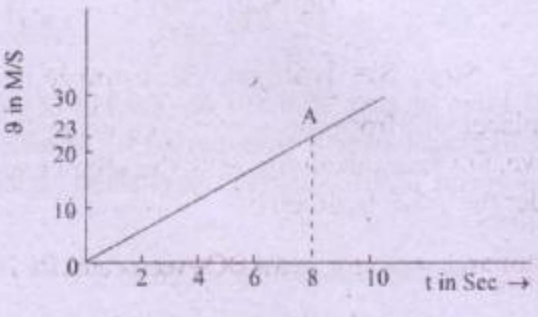


What is its average speed between 0 to 10 s and 10s to 20s.

Soln.

(i) It is a motion with deceleration (retardation). The direction of velocity changes after 10s. (ii) Average speed between 0 to 10 sec. $\langle \upsilon \rangle = \frac{10+0}{2} = 5\text{m/s}$ and average speed between 10s to 20s is $\langle \upsilon \rangle = \frac{0+10}{2} = 5\text{m/s}$.

Ex. 3.8.2 The speed-time graph for a car is as shown below. Find the distance travelled by the car in 8 seconds.



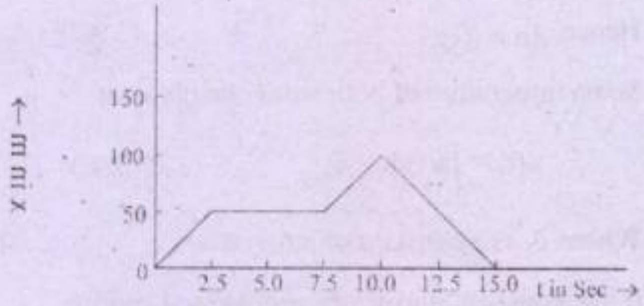
Soln.

distance travelled = area under the graph

$= \text{Area OAB}$

$= \frac{1}{2} \times 8 \times 23\text{m} = 92\text{m}$

Ex. 3.8.3. The figure below shows the graph of x-coordinates of a particle moving along x-axis as a function of time. Find (a) average velocity during 0 to 10s (b) instantaneous velocity at 2,5,8 and 12s.



Soln.

(a) Average velocity between 0 to 10s is

$\langle \vec{\upsilon} \rangle = \frac{100}{10}\text{m/s} = 10\text{m/s along x - dir}^n$

(b) Instantaneous velocity at 2s

$= \frac{50}{2.5} = 20\text{m/s along x - dir}^n$

Inst. vel. at 5 s = 0

Inst. vel. at 8s $= \frac{50}{2.5}$

$= 20\text{m/s along x - dir}^n$

Inst. vel. at 12s $= \frac{100}{5}$

$= 20\text{m/s along -ve x - dir}^n$

3.9 Kinematic equations of motion with uniform acceleration

Equations which relates displacement $(\vec{s})$, velocity $(\vec{\upsilon})$, acceleration $(\vec{a})$ and time (t) are called equations of motion. These can be deduced by analytic method in general and by graphical method for rectilinear motion only.

(i) **Analytic (calculus) method**

By definition

$$\vec{a}(t) = \frac{d\vec{\upsilon}(t)}{dt} \qquad ...(3.9.1)$$

Hence $d\vec{\upsilon} = \vec{a}\, dt$ ...(3.9.2)

So on integration of both sides one obtains

$$\vec{\upsilon}(t) = \int \vec{a}(t)dt + \vec{c}_1 \qquad ...(3.9.3)$$

Where $\vec{c}_1$ is a constant of integration.

If we consider uniformly accelerated motion then $\vec{a}$ =constant, and equation (3.9.3) reduces to

$$\vec{\upsilon}(t) = \vec{a} \int dt + \vec{c}_1 \text{, giving}$$

$$\vec{\upsilon}(t) = \vec{a}\, t + \vec{c}_1 \qquad ...(3.9.4)$$

Now if the body possesses velocity $\vec{u}$ at initial time (t=0) (i.e. $\vec{u}$ is initial velocity)

then from (3.9.4) we have

$$\vec{u} = \vec{a}.o + \vec{c}_1$$

$$\Rightarrow \vec{c}_1 = \vec{u} \qquad ...(3.9.5)$$

Using value of $\vec{c}_1$ in (3.9.4) we get

$$\vec{\upsilon}(t) = \vec{u} + \vec{a}\, t \qquad ...(3.9.6)$$

Equation (3.9.6) gives the velocity at time t.

Further by definition

$$\vec{\upsilon}(t) = \frac{d\vec{s}(t)}{dt} = \frac{d\vec{r}(t)}{dt}$$

$$\Rightarrow d\vec{s}(t) = \vec{\upsilon}(t)\, dt \qquad ...(3.9.7)$$

Using (3.9.6) on r.h.s. of (3.9.7)

$$d\vec{s}(t) = (\vec{u} + \vec{a}t)dt$$

Integrating both sides

$$\vec{s}(t) = \vec{u}\int dt + \vec{a}\int t\, dt + \vec{c}_2$$

$$\Rightarrow \vec{s}(t) = \vec{u}t + \frac{1}{2}\vec{a}\, t^2 + \vec{c}_2 \qquad ...(3.9.8)$$

Where $\vec{c}_2$ is an integration constant. Now at t=0 $\vec{s}(t)$ =0, so eqn. (3.9.8) gives

$$0 = 0 + 0 + \vec{c}_2$$

$$\Rightarrow \vec{c}_2 = 0 \qquad ...(3.9.9)$$

Therefore eqn. (3.9.8) reduces to

$$\vec{s}(t) = \vec{u}t + \frac{1}{2}\vec{a}\, t^2 \qquad ...(3.9.10)$$

Eqn. (3.9.9) gives displacement over time period t.

Again taking $\vec{\upsilon} = \frac{d\vec{r}}{dt}$

$$d\vec{r} = \vec{\upsilon}dt = (\vec{u} + \vec{a}\, t)dt$$

We obtain on integration of both sides

$$\vec{r}(t) = \vec{u}\, t + \frac{1}{2}\vec{a}\, t^2 + \vec{r}_0 \qquad ...(3.9.10)(a)$$

Where $\vec{r}_0$ is the position vector of the body at the initial time (t=0). This can be re-written as

$$\vec{r} - \vec{r}_0 = \vec{u}\, t + \frac{1}{2}\vec{a}\, t^2 \qquad ...(3.9.11)$$

The l.h.s. of eqn. (3.9.10) & (3.9.11) is the displacement over a time-period t., whereas l.h.s. of eqn. (3.9.10)(a) gives the position vector at time t.

We can also further see that

$$\vec{\upsilon}.d\vec{\upsilon} = (\vec{u} + \vec{a}\, t).\vec{a}\, dt$$

$$\Rightarrow d\left(\frac{1}{2}\vec{\upsilon}.\vec{\upsilon}\right) = \vec{u}.\vec{a}\, dt + a^2 t\, dt$$

Integrating both sides

$$\frac{1}{2}\upsilon^2 = \vec{u}.\vec{a}\int dt + a^2\int t\,dt + c_3$$

$$\Rightarrow \frac{1}{2}\upsilon^2 = \vec{u}.\vec{a}\,t + \frac{1}{2}a^2t^2 + c_3 \qquad ...(3.9.12)$$

Where c_3 is a constant of integration. Since at $t=0$, $\vec{\upsilon} = \vec{u}$, so

$$\frac{1}{2}u^2 = 0 + 0 + c_3$$

$$\Rightarrow \quad c_3 = \frac{1}{2}u^2 \qquad ...(3.9.13)$$

Using the value of c_3 on r.h.s. of (3.9.12) we obtain

$$\frac{1}{2}\upsilon^2 = \vec{u}.\vec{a}\,t + \frac{1}{2}a^2t^2 + \frac{1}{2}u^2$$

$$\Rightarrow \upsilon^2 - u^2 = 2\,\vec{a}.\left(\vec{u}\,t + \frac{1}{2}\vec{a}\,t^2\right)$$

Using eqn. (3.9.10) on r.h.s. we get

$$\upsilon^2 - u^2 = 2\,\vec{a}.\vec{s}$$

$$\Rightarrow \upsilon^2 = u^2 + 2\,\vec{a}.\vec{s} \qquad ...(3.9.14)$$

Eqn. (3.9.14) gives the velocity after suffering a displacement $\vec{s}$.

We can also find the displacement suffered in the the n^{th} second. This is given by

$$\overrightarrow{\Delta S}_n = \vec{S}_n - \vec{S}_{n-1} = \left(\vec{u}\,n + \frac{1}{2}\,\vec{a}\,n^2\right)$$

$$-\left(\vec{u}(n-1) + \frac{1}{2}\vec{a}(n-1)^2\right)$$

$$= \vec{u}\left[n-(n-1)\right] + \frac{1}{2}\vec{a}\left[n^2-(n-1)^2\right]$$

$$\overrightarrow{\Delta S}_n = \vec{u} + \frac{1}{2}\vec{a}\,(2n-1) \qquad ...(3.9.15)$$

Thus the various equations of motion for a uniformly accelerated motion are

$$\vec{\upsilon} = \vec{u} + \vec{a}\,t \qquad (a)$$

$$\vec{r}(t) - \vec{r}_0 = \vec{S}(t) = \vec{u}t + \frac{1}{2}\vec{a}\,t^2 \qquad (b)$$

$$\upsilon^2 = u^2 + 2\,\vec{a}.\vec{s} \qquad (c)$$

$$\overrightarrow{\Delta S}_n = \vec{u} + \frac{1}{2}\vec{a}\,(2n-1) \qquad (d)$$

...(3.9.16)

Equation of motion for uniformly accelerated rectilinear motion

When a body performs a uniformly accelerated rectilinear motion, we have

$$\vec{u} = u\hat{\xi}$$

$$\vec{a} = a\,\hat{\xi} \qquad ...(3.9.17)$$

Where (i) a is positive if the acceleration is in the direction $\vec{u}$ and is negative if it is in opposite direction to $\vec{u}$ and (ii) $\hat{\xi}$ is the initial direction of motion. So using eqn. (3.9.17) in the eqns (3.9.16), we obtain

$$\vec{\upsilon}(t) = (u+at)\hat{\xi}^{o} \equiv \hat{\xi} \qquad ...(3.9.18)$$

$$\vec{S}(t) = \left(ut + \frac{1}{2}a\,t^2\right)\hat{\xi} \equiv S(t)\hat{\xi} \qquad ...(3.9.19)$$

$$\upsilon^2 = u^2 + 2\,as \qquad ...(3.9.20)$$

$$\overrightarrow{\Delta S}_n = \left[u + \frac{1}{2}a(2n-1)\right]\hat{\xi} \equiv \Delta S_n\hat{\xi} \qquad ...(3.9.21)$$

From eqns (3.9.18) to (3.9.21) we obtain

$$\upsilon(t) = u + at \quad ...(3.9.22)$$

$$S(t) = ut + \frac{1}{2} a t^2 \quad ...(3.9.23)$$

$$\upsilon^2 = u^2 + 2 a s \quad ...(3.9.24)$$

$$\Delta S_n = u + \frac{1}{2} a (2n - 1) \quad ...(3.9.25)$$

The above eqns. show that when 'a' is positive υ, S, and ΔS_n are all positive. But when 'a' is negative υ could be positive or negative. When υ is negative, the negative sign. should be associated with the unit vector $\hat{\xi}$, (i.e. $\vec{\upsilon}$ is oppositely directed to $\vec{u}$) and $|\upsilon|$ shall give the speed at time t. Similarly when 'a' is negative, S and ΔS_n could be positive or negative. When their values are negative, it is implied that they are directed opposite to $\vec{u}$ (the initial direction of motion). <u>But $|S|$ and $|\Delta S_n|$ donot give distance travelled when 'a' is negative.</u>

(ii) **Graphical method :**

Equations of motion for uniformly accelerated (/retarded) rectilinear motion can be obtained from the velocity- time graph. Since there could be unidirectional and non-unidirectional rectilinear motion, so the possible graphs shall be as shown in fig. 3.26.

Since $\vec{a} = \frac{d\vec{\upsilon}}{dt}$, so the slope of the tangent to the υ~t curve at any point gives the acceleration. As in uniformly accelerated or retarded rectilinear motion, the υ~t graph is a straight line, so the slope of υ~t straight line (AB) shall give the acceleration. Considering fig. 3.26 (a,b & c)

$$\text{Slope} = a = \frac{\upsilon - u}{t - 0}$$

$$\Rightarrow \quad \upsilon - u = at$$

$$\text{or} \quad \upsilon = u + at \quad ...(3.9.26)$$

Further $S = \int_0^t \upsilon dt$. So the area under the υ~t curve, shall give the displacement over a time period t. Considering fig. 3.26 (a) and (b) we find

S = Area of the trapezium ABNO

$$= \frac{1}{2} (AO + BN).ON$$

$$S = \frac{1}{2}(u + \vartheta).t \quad ...(3.9.27)$$

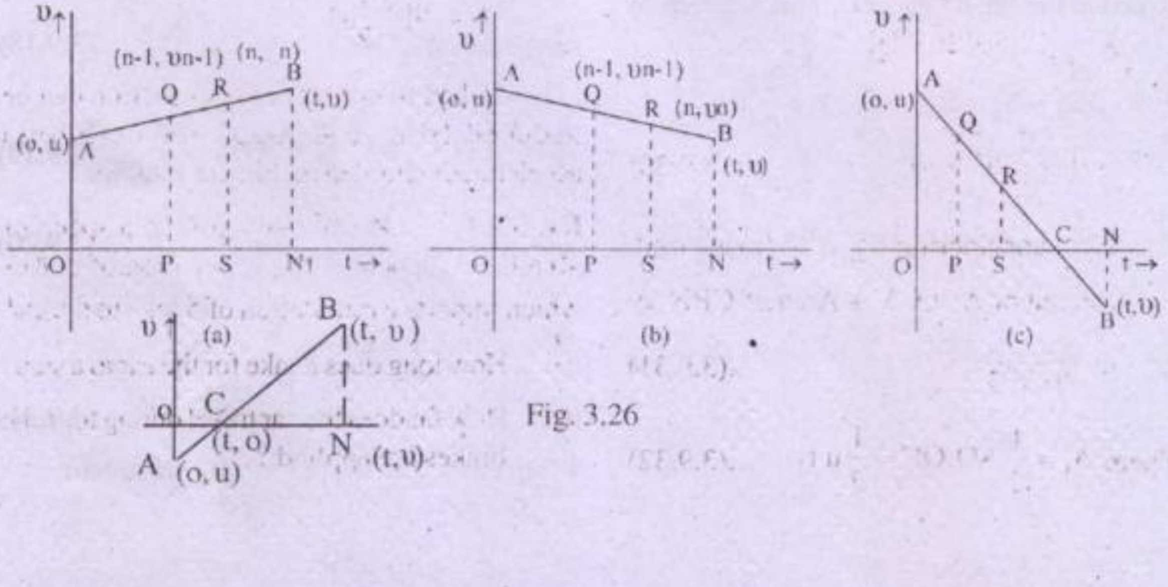


Fig. 3.26

Using (3.2.26) on r.h.s

$$S = \frac{1}{2}(2u+at)t$$

$$\Rightarrow \; S = ut + \frac{1}{2}at^2 \qquad ...(3.9.28)$$

Substituting for $t = \frac{\upsilon - u}{a}$, (which follows from eqn. (3.9.26) in (3.9.27) we obtain

$$S = \frac{1}{2}(u+\upsilon)\left(\frac{\upsilon - u}{a}\right) = \frac{\upsilon^2 - u^2}{2a}$$

$$\Rightarrow \; \upsilon^2 = u^2 + 2as \qquad ...(3.9.29)$$

The displacement suffered in the n^{th} second is given by $\Delta S_n = \int_{n-1}^{n} \upsilon dt$. So

ΔS_n = Area of the trapezium PQRS

$$= \frac{1}{2}(PQ+RS).PS$$

$$= \frac{1}{2}(\upsilon_{n-1} + \upsilon_n).1$$

$$= \frac{1}{2}\left[\{u + a(n-1)\} + \{u + an\}\right]1$$

$$= \frac{1}{2}\left[2u + a(2n-1)\right]$$

$$\therefore \; \Delta S_n = u + \frac{1}{2}a(2n-1) \qquad ...(3.9.30)$$

Now considering fig. 3.26 (c) we find

S=area of ACO Δ + Area of CBN Δ

$$= \Delta_1 + \Delta_2 \qquad ...(3.9.31)$$

$$\text{Where } \Delta_1 = \frac{1}{2}AO.OC = \frac{1}{2}u\,t_1 \qquad ...(3.9.32)$$

$$\Delta_2 = \frac{1}{2}BN.CN = \frac{1}{2}\upsilon(t - t_1) \quad ...(3.9.33)$$

Considering the slope of straight line AB

$$\frac{u-0}{0-t_1} = \frac{\upsilon - 0}{t - t_1}$$

$$\Rightarrow u(t - t_1) = -\upsilon t_1$$

$$\text{giving } t_1 = \frac{ut}{u - \upsilon} \qquad ...(3.9.34)$$

$$\text{and } t - t_1 = -\frac{\upsilon t}{u - \upsilon} \qquad ...(3.9.35)$$

Using eqns. (3.9.34) and (3.9.35) in (3.9.32) and (3.9.33) we obtain

$$\Delta_1 = \frac{1}{2}\frac{u^2 t}{u - \upsilon} \qquad ...(3.9.36)$$

$$\Delta_2 = -\frac{1}{2}\frac{\upsilon^2 t}{u - \upsilon} \qquad ...(3.9.37)$$

Using (3.9.36) and (3.9.37) in (3.9.31) we obtain

$$S = \frac{1}{2}\frac{t}{u - \upsilon}.(u^2 - \upsilon^2) = \frac{1}{2}(u + \upsilon).t \quad ...(3.9.38)$$

Therefore this shall give

$$S = ut + \frac{1}{2}at^2$$

Thus in general eqns of motion can be deduced from $\upsilon \sim t$ graph for uniformly accelerated/ retarded rectilinear motion.

Ex. 3.9.1 A car moving with a speed of 40 m/s is stopped by the application of brakes which imparts a retardation of 5 ms^{-2} to the car.

(i) How long does it take for the car to a stop?

(ii) How far does the car travel during the time brakes are applied ?

Soln.

Given initial velocity 'u' = 40 m/s

final velocity $\vartheta = 0$

acceleration a = -5 m/s²

(i) $\therefore \upsilon = 0 = u + at$

$$\Rightarrow t = \frac{-u}{a} = -\frac{40 m/s}{-5m/s^2} = 8\ s$$

(ii) $\upsilon^2 = 0 = u^2 + 2\ a\ s$

$$\Rightarrow S = -\frac{u^2}{2a} = -\frac{(40)^2 m^2/s^2}{-2 \times 5m/s^2} = 160m$$

Ex. 3.9.2 Starting from rest a body accelerates at 4 m/s². How far will it travel during the 4th second ?

Soln.

Given a = 4 m/s²

u = 0

$$\Delta S_n = u + \frac{1}{2}a(2n-1)$$

$$= 0 + \frac{1}{2} x4x(2 \times 4 - 1)$$

$$\Delta S_n = 14m.$$

Ex. 3.9.3 A ball is thrown straight upward with an initial speed of 12 m/s. Find (a) the displacement (b) velocity after 1s.

Soln.

Given u=12 m/s

$$a = -g\ m/s^2 = -9.8\ m/s^2$$

(b) $\therefore \upsilon = u + at = 12m/s - 9.8\frac{m}{5^2} \times 1s$

$= 12$ m/s $- 9.8$ m/s $= 2.2$ m/s

$\therefore \upsilon = 2.2m/s$

(a) $S = ut + \frac{1}{2}at^2 = 12x1 - \frac{1}{2}x9.8x1^2$

$S = 12 - 4.9 = 7.1$ m

Ex. 3.9.4 A body describes 10 m in 3rd second, 28 m in 6th sec. of its unidirectional uniformly accelerated motion. How far will it go in 7th second ?

Soln.

We have

$$\Delta S_n = u + \frac{1}{2}a(2n-1)$$

Given $\Delta S_3 = 10 = u + \frac{1}{2}a(2x3-1)$

$$\Delta S_6 = 28 = u + \frac{1}{2}a(2x6-1)$$

$$\Rightarrow u + \frac{5}{2}a = 10$$

$$u + \frac{11}{2}a = 28$$

Substracting

3a = 18

a = 6m/s²

Hence $u = 10 - \frac{5}{2}a = 10 - \frac{5}{2}x6 = -5m/s$

Now $\Delta S_7 = u + \frac{1}{2}x6x(13)$

$$\Delta S_7 = 34m$$

Ex. 3.9.5 A body describes half of the distance with speed υ_1 and the second half with speed υ_2. Calculate the average speed.

Soln.

Let total distance be = D.

$\therefore$ Time taken to cover 1st half $t_1 = \dfrac{D/2}{\upsilon_1}$

Time taken to cover 2nd half $t_2 = \dfrac{D/2}{\upsilon_2}$

average speed $\langle \upsilon \rangle = \dfrac{\text{Total distance}}{\text{Total time}}$

$$\Rightarrow \langle \upsilon \rangle = \frac{D}{\frac{D/2}{\upsilon_1} + \frac{D/2}{\upsilon_2}} = \frac{2\upsilon_1\upsilon_2}{\upsilon_1+\upsilon_2}$$

Ex. 3.9.6 A person travelling on straight line unidirectionally moves with a uniform velocity $\vec{\upsilon}_1$ for some time and with uniform velocity $\vec{\upsilon}_2$ for the next equal time. calculate the average velocity.

Soln.

Displacement in 1st period t is $\vec{s}_1 = \vec{\upsilon}_1 t$

Displacement in 2nd period t is $\vec{s}_2 = \vec{\upsilon}_2 t$

Net displacement $\vec{S} = \vec{S}_1 + \vec{S}_2 = \left(\vec{\upsilon}_1 + \vec{\upsilon}_2\right) t$

Net time = 2t

$\therefore$ average velocity $\langle \vec{\upsilon} \rangle = \dfrac{\left(\vec{\upsilon}_1 + \vec{\upsilon}_2\right) t}{2t} = \dfrac{\vec{\upsilon}_1 + \vec{\upsilon}_2}{2}$

Ex. 3.9.7 A car accelerates from rest at a constant rate α for some time after which it decelerates at a constant rate β to come to rest. If t is total time elapsed, then calculate the maximum velocity acquired by car.

Soln.

Given u = 0

Let it be accelerated for time t_1

Then max. velocity reached $\vartheta_1 = \alpha t_1$ (1)

Then it comes to rest within time $(t-t_1)$

$\therefore \; 0 = \upsilon_1 - \beta(t-t_1)$ (2)

Using (1) in (2)

$$\alpha t_1 - \beta(t-t_1) = 0$$

$$\Rightarrow \; (\alpha+\beta)t_1 = \beta t$$

$$\Rightarrow \; t_1 = \frac{\beta}{\alpha+\beta} t \quad (3)$$

So max. velocity $\upsilon_1 = \alpha\, t_1 = \dfrac{\alpha\beta}{\alpha+\beta} t$ (4)

Ex. 3.9.8 A man can swim in still water at a speed of 3km/hr. He wants to cross a river that flows at 2km/hr. and reach the point directly opposite his starting point. (a) In which direction should he try to swim. (b) How much time will he take to cross the river, if the river is 500m wide ?

Soln.

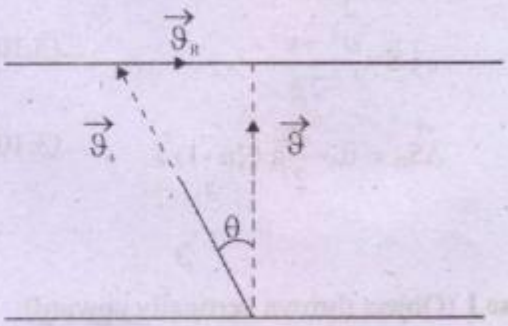


Fig : A

Let velocity of river be $\vec{\vartheta}_R$

and that of swimmer be $\vec{\vartheta}_S$

Then $\vec{\vartheta}_S + \vec{\vartheta}_R = \vec{\vartheta}$ υ

and $\vec{\vartheta}$ is $\perp^r$ to the

length of the river (see fig. A)

$$\upsilon_S = 3km/hr \quad \upsilon_R = 2km/hr$$

$$\sin\theta = \frac{\upsilon_R}{\upsilon_S} = \frac{2}{3}$$

$$\upsilon = \sqrt{\upsilon_S^2 - \upsilon_R^2} = \sqrt{3^2 - 2^2} = \sqrt{5} \quad \text{km/hr}$$

$$\therefore t = \frac{500m}{\sqrt{5}\ km/hr} = \frac{500m}{\sqrt{5}x10^3 m/hr}$$

$$= \sqrt{5}x10^{-1} hr = \frac{1}{2\sqrt{5}} hr$$

3.10 Motion under gravity :

Consider a body moving under gravity. Then acceleratin of the body is equal to 9-8 m/s^2 in the vertically downward direction

i.e. $\bar{a} = -g\,\hat{j}$ ...(3.10.1)

Where $\hat{j}$ is the unit vector along the vertically upward direction. (i.e. +ve y-dir^n): Then equations of motion (3.9.16) reduce to

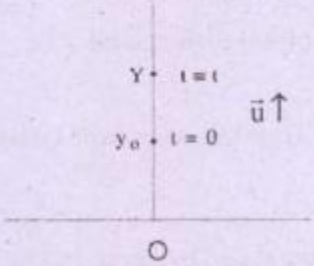
$$\vec{\upsilon} = \bar{u} - gt\,\hat{j} \qquad ...(3.10.2)$$

$$\bar{S} = ut\frac{1}{2}gt^2\hat{j} \qquad ...(3.10.3)$$

$$\hat{j}.\bar{S} = \frac{\upsilon^2 - u^2}{-2g} \qquad ...(3.10.4)$$

$$\overrightarrow{\Delta S_n} = \bar{u} - \frac{1}{2}g\,(2n-1)\,\hat{j} \qquad ..(3.10.5)$$

Case I (Object thrown vertically upward)



Then initial velocity $\bar{u} = u\hat{j}$ and equations 3.10.2,3,4 reduce to

$$\vec{\upsilon} = (u - gt)\,\hat{j} \qquad ...(3.10.6)$$

$$\bar{S} = \left(ut\frac{1}{2}gt^2\right)\hat{j} \qquad ...(3.10.7)$$

$$\hat{j}.\bar{S} = \frac{\upsilon^2 - u^2}{-2g} \qquad ...(3.10.8)$$

$$\overrightarrow{\Delta S_n} = \left[u - \frac{1}{2}g(2n-1)\right]\hat{j} \qquad ..(3.10.9)$$

If u-gt >0, then it will imply that the object is moving vertically upward; with velocity $|u-gt|$. If u-gt <0, then the object is moving vertically downward with velocity $|u-gt|$. Similarly if $ut - \frac{1}{2}gt^2 > 0$, then displacement is vertically upward w.r. to the starting point, and if $ut - \frac{1}{2}gt^2 < 0$, then the displacement is vertically downward w.r.to the starting point.

Equations 3.10.6,7,8 &9 can be re-written as

$$\upsilon = u - gt \qquad ...(3.10.10)$$

$$y - y_o = S = ut - \frac{1}{2}gt^2 \qquad ...(3.10.11)$$

$$y - y_o = S = \frac{\upsilon^2 - u^2}{-2g} \qquad ...(3.10.12)$$

$$\Delta y_n = u - \frac{1}{2}g(2n-1) \qquad ...(3.10.13)$$

In equations 3.10.10,11,12&13, if r.h.s. is +ve it shall mean direction is upwards and if r.h.s is $-$ve , then direction is downwards, 'y_0' is the vertical position of the point of throw.

The **maximum displacement** in the upward dirction is obtained by putting $\vartheta = 0$, so that we obtain

$$S_{max} = \frac{0 - u^2}{-2g} = \frac{u^2}{2g} \qquad ...(3.10.11)$$

This gives the maximum height (y_{max}) reached by putting

$$y_{max} = S_{max} + y_o = y_o + \frac{u^2}{2g}$$

...(3.10.12)

The time of upward flight is given by

$$t_u = \frac{\upsilon - u}{-g} = \frac{0 - u}{-g} = \frac{u}{g} \qquad ...(3.10.13)$$

Using (3.10.11) we find

$$t_u = \frac{u}{g} = \sqrt{\frac{2S_{max}}{g}} \qquad ...(3.10.14)$$

We define time of flight t_f as the time for which, the object remains in space. During this time the net displacement $\vec{S} = -y_o\hat{j}$ therefore

$$-y_o = ut_f - \frac{1}{2}gt_f^2$$

$$\Rightarrow \frac{1}{2}gt_f^2 - ut_f - y_o = 0$$

This gives $t_f = \dfrac{u \pm \sqrt{u^2 + 2gy_o}}{g}$

Since $t_f > t_u$ i.e. $t_f > \dfrac{u}{g}$, so

$$t_f = \frac{u + \sqrt{u^2 + 2gy_o}}{g} \qquad ...(3.10.15)$$

From eqns (3.10.15) and (3.10.14) we find the time for downward flight as

$$t_d = t_f - t_u = \frac{\sqrt{u^2 + 2gy_o}}{g} \qquad ...(3.10.16)$$

Case II (Object falling freely from a height)

In this case $\vec{u} = 0$, so

$$\vec{\upsilon} = -gt\,\hat{j} \qquad ...(3.10.17)$$

$$\vec{S} = -\frac{1}{2}gt^2\,\hat{j} \qquad ...(3.10.18)$$

$$\hat{j}.\vec{S} = \frac{\upsilon^2}{-2g} \qquad ...(3.10.19)$$

The '$-$ve' sign on r.h.s of equation 3.10.17,18 & 19 indicates that the velocity displacement are all in downward direction. The above equations can also be written as

$$\upsilon = -gt \qquad ...(3.10.20)$$

$$y - y_o = S = -\frac{1}{2}gt^2 \qquad ...(3.10.21)$$

$$y - y_o = S = \frac{\upsilon^2}{-2g} \qquad ...(3.10.22)$$

and the '$-$ve' sign on r.h.s shall mean the motion is down-wards.

The time of flight is obtained by putting y=o in eqn. (3.10.21), giving

$$t_f = \sqrt{\frac{2y_o}{g}} \qquad ...(3.10.23)$$

Ex. 3.10.1 A stone is dropped from a baloon at a height 10m while moving upwards with a speed of 5m/s. Calculate maximum height reached and the time of flight of the stone from the instant of its release.

Soln.

Given u = 5 m/s

$y_o = 10$ m

$$y_{max} = y_o + \frac{u^2}{2g} = 10 + \frac{(5)^2}{2x9.8} = 11.28m$$

$$t_f = \frac{u + \sqrt{u^2 + 2gy_o}}{g}$$

$$= t_f = \frac{5 + \sqrt{5^2 + 2x9.8x10}}{9.8} \text{ s.}$$

$$t_f = 1.15 \text{ sec}$$

3.11 Relative velocity :

We have discussed in the begining of this chapter that all motions are relative. When we define $\vec{\upsilon} = \frac{d\vec{r}}{dt}$, the position vector $\vec{r}$ is measured with respect to some reference point o. Suppose we have two bodies A and B whose positions are specified by position vectors $\vec{r}_{AO}$ and $\vec{r}_{BO}$ (see fig. 3.27). Then position

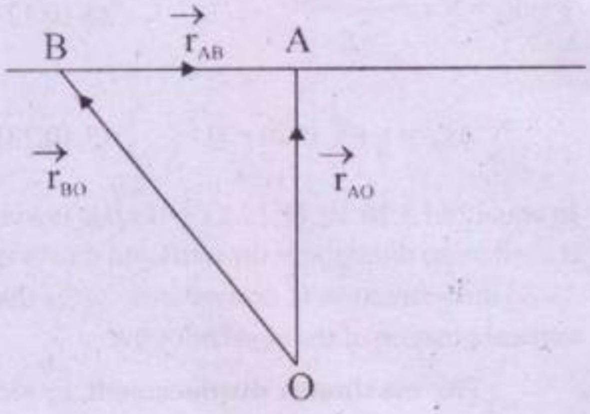


fig. 3.27

of A with respect to B is given by the position vector $\vec{r}_{AB} = \vec{r}_{AO} - \vec{r}_{BO}$ ...(3.11.1)

$$\text{Then } \frac{d}{dt}(\vec{r}_{AB}) = \frac{d}{dt}(\vec{r}_{AO}) - \frac{d}{dt}(\vec{r}_{BO}) \qquad ...(3.11.2)$$

The l.h.s. of eqn. (3.11.2) gives the time rate of displacement of A w.r. to B. while 1st and 2nd term on r.h.s. give respectively the time rate of displacement of A and B w.r. to the reference point O. So by definition eqn. (3.11.2) gives

$$\vec{\upsilon}_{AB} = \vec{\upsilon}_{AO} - \vec{\upsilon}_{BO} \qquad ...(3.11.3)$$

In eqn. (3.11.3) $\vec{\upsilon}_{AB}$ represents the velocity of A with respect to B and is called the **relative velocity** of A with respect to B. $\vec{\upsilon}_{AO}$ and $\vec{\upsilon}_{BO}$ represent respectively the velocity of A and B w.r.to the reference point O. This implies that relative velocity of A with respect to B is equal to the velocity of A minus the velocity of B, both being measured with respect to same reference point O.

From (3.11.3) we find that

$$\vec{\upsilon}_{AO} = \vec{\upsilon}_{AB} + \vec{\upsilon}_{BO} \qquad ...(3.11.4)$$

One can continue this process and write

$$\vec{\upsilon}_{AO} = \vec{\upsilon}_{AB} + \vec{\upsilon}_{BC} + \vec{\upsilon}_{CD} + \vec{\upsilon}_{DO} \qquad ...(3.11.5)$$

We also notice that

$$\vec{\upsilon}_{AB} = \vec{\upsilon}_{AO} - \vec{\upsilon}_{BO} = -(\vec{\upsilon}_{BO} - \vec{\upsilon}_{AO}) = -\vec{\upsilon}_{BA} \quad ...(3.11.6)$$

This implies that relative velocity of A w.r.to B is equal to the negative of relative velocity of B w.r.to A.

Since $\vec{\upsilon}_{AB} = \vec{\upsilon}_{AO} - \vec{\upsilon}_{BO}$; we can find its magnitude and direction as follows :

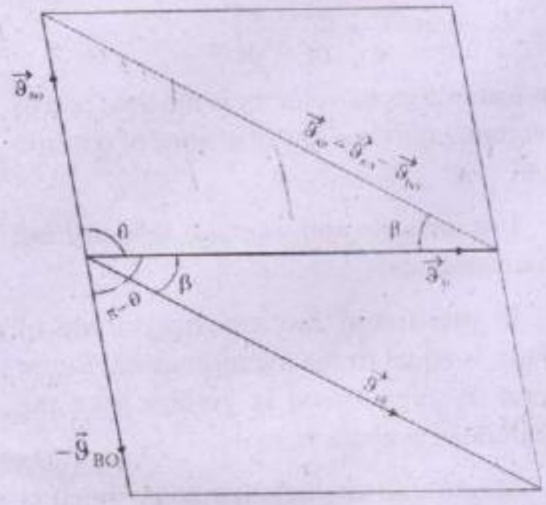


Fig. 3.28

$$\left|\vec{\upsilon}_{AB}\right| = \sqrt{\upsilon_{AO}^2 + \upsilon_{BO}^2 - 2\upsilon_{AO}.\upsilon_{BO}\text{Cos}\theta} \quad ...(3.11.7)$$

$$\tan\beta = \frac{\upsilon_{BO}\sin(\pi-\theta)}{\upsilon_{AO} + \upsilon_{BO}\cos(\pi-\theta)} \quad ...(3.11.8)$$

Eqn. (3.11.7) gives the magnitude and eqn. (3.11.8) gives the direction of relative velocity of A w.r. to B.

Special cases :

(i) If $\vec{\upsilon}_{AO}$ and $\vec{\upsilon}_{BO}$ are in same direction then $\theta = 0$, and we have

$$\left|\vec{\upsilon}_{AB}\right| = \left|\vec{\upsilon}_{AO}\right| \sim \left|\vec{\upsilon}_{BO}\right|$$

$$\tan\beta = 0 \text{, implying } \beta = 0 \text{ or } \pi$$

If $\upsilon_{AO} > \upsilon_{BO}$, then $\beta = 0$, and if $\upsilon_{AO} < \upsilon_{BO}$ then $\beta = \pi$.

(ii) $\vec{\upsilon}_{AO}$ and $\vec{\upsilon}_{BO}$ are oppositely directed then $\theta = \pi$ and we have

$$\left|\vec{\upsilon}_{AB}\right| = \upsilon_{AO} + \upsilon_{BO}$$

$$\tan\beta = 0 \text{ implying } \beta = 0$$

3.12 Relative acceleration :

The time rate of change of velocity of A with respect to B is called the relative acceleration of A with respect to B.

$$\text{i.e } \bar{a}_{AB} = \frac{d\vec{\upsilon}_{AB}}{dt} = \frac{d}{dt}\left(\vec{\upsilon}_{AO}\right) - \frac{d}{dt}\left(\vec{\upsilon}_{BO}\right)$$

$$\Rightarrow \bar{a}_{AB} = \bar{a}_{AO} - \bar{a}_{BO} \quad ...(3.12.1)$$

If velocity of B w.r. to O is constant i.e. $\vec{\upsilon}_{BO}$ = constant, then $\bar{a}_{BO} = 0$, and we have

$$\bar{a}_{AB} = \bar{a}_{AO} \quad ...(3.12.2)$$

Eqn. (3.12.2) implies that observers B and O measure the same value of acceleration of A.

Summary

1. A body is said to be at rest (with repect to an observer) if its position does not change with respect to the observer.

2. A body is said to be in motion (w.r. to an observer) if its position changes with respect to the observer from time to time.

3. The position of a particle is fixed with respect to (w.r.to) a reference point and a set of reference axes. (i.e. a frame of reference).

4. The vector joining the reference point (origin) and the particle is called position vector($\vec{r}$) of the particle and it specifics the position of the particle. In general

5. The curve or line obtained by joining suceessive positions of the particle during a motion is called its path (/ trajectory).

6. Distance (/ path length) is the length of the path travelled during a time interval and it is always measured along the path.

7. Displecement is the straight line joining the initial position to the final position and is directed from the initial position to the final position. It is a vector quantity.

8. Except for unidirectional motion magnitude of displacement is not equal to distance travelled.

9. Displacement vector ($\vec{s}$) can be expressed in terms of position vector as :

$$\vec{s} = \vec{r}(t) - \vec{r}_0$$

10. Average speed of a body in a time enterval is defined as the distance travelled by the body divided by the time interval.

$$\langle \upsilon \rangle = \frac{D}{t_2 - t_1}$$

11. Instantaneous speed at any instant of time, is defined as the speed possessed by the body at the particular instant.

$$\upsilon = \lim_{\Delta t \to 0} \frac{\Delta D}{\Delta t} = \frac{dD}{dt}$$

12. Average velocity of a body in a time enterval (t_1, t_0, t_2) is defined as the ratio of its displacement to the time internal.

$$\langle \upsilon \rangle = \frac{\vec{s}}{t_2 - -t_1} = \frac{\vec{r}(t_2) - \vec{r}(t_1)}{t_2 - t_1}$$

13. Instantaneous velocity of a body is its velocity at a particular instant or at a particular point on its path.

$$\vec{\upsilon} = \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t} = \frac{d\vec{r}}{dt} = \frac{d\vec{s}}{dt}$$

Thus instantaneous velocity is the time rate of displacement or time rate of change of position vector.

14. Instanneous and average velocity are vector quantities.

15. In rectilinear motion, magnitude of velouity is equal to the average speed. But in general average speed is greater than the magnitude of average velocity.

16. Magnitude of instantancous speed is always equal to the magnitude of instantameous velocity.

17. Average acceleration of a body is defined as the ratio of the change in velocity to the time interval over which the change occurred.

$$\langle \vec{a} \rangle = \frac{\vec{\upsilon}_2 - \vec{\upsilon}_1}{t_2 - t_1} = \frac{\Delta \vec{\upsilon}}{\Delta t}$$

Since vilocity is a vector quantity a change in its magnitude or direction or both will cause a change in velocity, hence an accelerated motion.

18. In case of rectilinear as well as unidirectioned motion.

$$\langle \vec{a} \rangle = \frac{\upsilon_2 - \upsilon_1}{t_2 - t_1} \hat{\xi}$$

19. Instantaneous accoleration of a body at time t is defined as the time rate of change of velocity.

$$\vec{a} = \lim_{\Delta t \to 0} \frac{\Delta \vec{\vartheta}}{\Delta t} = \frac{d\vec{\vartheta}}{dt}$$

20. In general $\vec{\upsilon}(t) = \upsilon(t)\,\hat{\xi}(t)$, giving

$$\vec{a} = \frac{d\vec{\upsilon}}{dt} = \frac{d\,\upsilon\,(t)}{dt}\hat{\xi}t + \upsilon(t)\frac{d\hat{\xi}(t)}{dt} = a(t).\,\hat{a}(t)$$

In case of unidirectional motion

$$\frac{d\hat{\xi}(t)}{dt} = 0, (\because \hat{\xi}(t) = \hat{\xi})$$

Hence $\vec{a} = \dfrac{d\,\upsilon\,(t)}{dt}\hat{\xi}t$

21. Position time graph can be plotted for rectilinear motion only.

22. Distance-time graph can be plotted for rectilinear as well as curilinear mation.

23. Displacement-time graph can be plotted for rectilinear motion only, by altributing the negative sign (arising due to direction) to the negative magnitude.

24. Speed -time graph can be plolted for rectilinear as well as curvilinear mation.

25. Velocity-time graph can be plotted for rectilinear motion only, by attributing the negative sign (arising due to direction) to the negative magnitude.

The area under velocity-time graph between times t_1 and t_2 is equal to the displacement.

26. For uniformly accelerated motion

(i) $\vec{\upsilon}(t) = \vec{u} + \vec{a}t$

(ii) $\vec{s}(t) = \vec{u}t + \frac{1}{2}\vec{a}t^2$

(iii) $\upsilon^2 = u^2 + 2\vec{a}.\vec{s}$

(iv) $\Delta\vec{s}_n = \vec{u} + \frac{1}{2}\vec{a}(2n-1)$

For uniformly accelerated rectilinear motion

(i) $\upsilon\,(t) = u + at$

(ii) $s(t) = ut + \frac{1}{2}at^2$

(iii) $\upsilon^2 = u^2 + 2as$

(iv) $\Delta s_n = u + \frac{1}{2}a(2n-1)$

MODEL QUESTIONS

A. Multiple Choice Type Questions :

1. A motor car is moving due north at a speed of 50 km/hr. It makes a 90^0 left turn without changing the speed. The change in the velocity of the car is about

(i) 50 km/hr towards west

(ii) 70 km/hr towards south west

(iii) 70 km/hr towards north-west

2. The fig. 3.Q.1 shows the displacement time graph of a particle moving along x-axis.

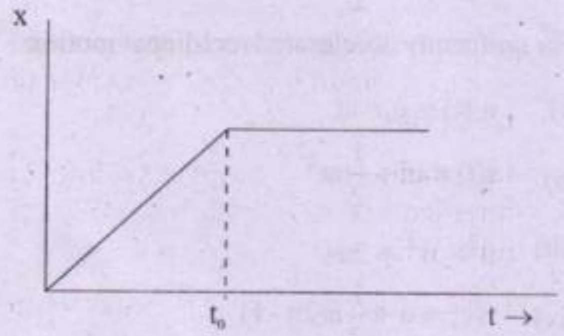


fig. 3.Q.1

(i) The particle is continuously going in positive x-direction.

(ii) The particle is at rest

(iii) The velocity increases upto time t_0 and then becomes constant.

(iv) The particle moves at a constant velocity upto a time t_0 and then stops.

3. A stone is released from an elevator going up with an acceleration $\bar{a}$. This acceleration of the stone after the release is

(i) a upward

(ii) (g-a) upward

(iii) (g-a) downward

(iv) g downward

4. An iron ball and a wooden ball of same radius are released from a height h in vaccum. The time taken by both of them to reach the ground are

(i) roughly equal

(ii) exactly equal

(iii) unequal

(iv) equal only at equator

5. The displacement of a particle moving along x-axis is given by

$x = a_0 + a_1 t + a_2 t^2$

Where 't' is time. The acceleration of the particle is

(i) a_0 (ii) a_1

(iii) a_2 (iv) $2a_2$

6. A stone released with zero velocity from the top of a tower, reaches the ground in 4-seconds. The height of the tower is about

(i) 20 m (ii) 40 m

(iii) 80 m (iv) 160 m

7. A rocket fired vertically upwards with constant acceleration has its fuel exhausted in 10 s. What is the maximum height reached by the rocket if its velocity at the end of 10th second is 600 m/s (Assume $g=10m/s^2$)

(i) 600x5m (ii) 600x25m

(iii) 6000x35m (iv) 600x40m

8. A body falling from a vertical height of 10 m pierced through a distance of 1m in sand. The amount of retardation in sand is

(i) g (ii) 10 g

(iii) 100 g (iv) 1000 g

9. A stone is thrown upward with an initial velocity of 30m/s. The time taken to reach the maximum height is

   (i) 0.326 s (ii) 3.26 s

   (iii) 30.6 s (iv) 3.06 s

10. The distance travelled by a body falling from rest in the rirst, second and third seconds are in the ratio :

    (i) 1:2:3 (ii) 1:3:5

    (iii) 1:4:9 (iv) None of the above

11. The displacement time graph of a particle moving with uniform velocity is

    (i) a point (ii) a parabala

    (iii) a circle (iv) a straight line

12. The velocity of a particle moving along a straight line is given by : $\upsilon = 10 + 3t^2$. The change in velocity in the interval from t=2s to t=5s is

    (i) 63 m/s (ii) 60 m/s

    (iii) 5 m/s (iv) 0 m/s

13. A particle moves along a straight line with a speed (i) υ (ii) 2υ. The ratio of the shortest distances in which the particle can be stopped is

    (i) 3:1 (ii) 2:1

    (iii) 1:16 (iv) 1:4

14. Two bodies are moving in two mutually perpendicular directions with speed ϑ. The magnitude of their relative velocity is

    (i) $\upsilon\sqrt{2}$ (ii) $2\sqrt{\upsilon}$

    (iii) $\frac{\upsilon}{\sqrt{2}}$ (iv) 2υ

B. Very short answer type questions :

1. Under what conditions the average velocity of a body is equal to its instantaneous velocity ?

2. A train is moving forward with a velocity of 20 km/hr. A man inside a boggie is walking backward with a velocity 10 km/hr. What is the velocity of the man with respect to platform ?

3. If the displacement-time graph is parallel to the time axis, what is the velocity of the body ?

4. Is it possible that displacement is zero but distance travelled is not zero ?

5. Can a body have constant velocity but varying speed ?

6. Can a body have constant speed, but varying velocity ?

7. For a body moving unidirectionaly, the distance travelled is directly proportional to the equare of the time. What type of motion is it ?

8. In a unidirectional motion, the distance travelled is directly proportional to the time. What type of motion is it ?

9. Is the acceleration of a car greater when the accelerator is pushed to the floor or when the brake pedal is pushed hard ?

10. What is the quantity which is measured by the area occupied below the velocity - time graph ?

11. What type of position time graph do we get for a body at rest ?

12. What does the slope of velocity-time graph represent ?

13. Can an object have an Eastward velocity while - experiencing a Westward accleration ?

14. Can a body have zero velocity and still be accelerated ?

15. What is the displacement of a particle moving is a circular path of radius R, in half the period of revolution ?

C. Short Answer Type :

1. A body is thrown upwards with a velocity of 19.6m/s. Find the height the body will reach.

2. A body starting from rest moves with constant velocity for 4 seconds and then comes to rest. Draw its displacement - time graph.

3. A ball thrown vertically upwards from ground comes back to the thrower in 4 second. How high did the ball rise ? (Take $g=10\ m/s^2$).

4. Draw the displacement - time graph of a body which starting from rest moves with uniform velocity for two seconds and then remains at rest for 3 seconds.

5. Distinguish between distance and displacement giving one example for each.

6. Distinguish between speed and velocity giving one example for each.

7. A body 'A' travels with velocity $\bar{u}$ due east and another body B moves with velocity $\vec{\upsilon}$ due west. What is the relative velocity of B w.r. to A ?

8. Two bodies A and B are moving with velocities $\bar{u}$ and $\vec{\upsilon}$ at angle θ. Give the expression for relative velocity of B w.r. to A.

9. Three particles A,B and C move in such a way that their distance from a fixed point, at any instant is given by

$$x_A = 5t + 6$$

$$x_B = 7t^2 + 9t + 10$$

$$x_C = 4t^3 + 5t^2 + 6t$$

Which of them moves with uniform acceleration ?

10. Why do rain drops appear to be falling at an angle to a man moving at uniform speed ?

D. Unsolved Numericals :

1. A railway train 100m long passes over a bridge 400m long at the rate of 72 km/hr. How long will the train take to cross the bridge ?

2. A person travelling at 43.2 km/hr applies the brake giving a deceleration of 6.0 m/s^2 to his scooter. How far will it travel before stopping ?

3. A bullet travelling with a velocity of 16 m/s penetrates a tree trunk and comes to rest in 0.4m. Find the time taken during the retardation.

4. A stone is dropped from the top of a cliff and is found to travel 14.7m in the last second before it reaches the ground. Find the height of the cliff. ($g=9.8\ m/s^2$)

5. A car travelling at 60 km/hr slows down at a uniform rate of $2 m/s^2$. How far will it travel before its speed is $\frac{1}{4}$ th of its original value ?

6. A uniformly accelerated body describes a displacement of 3 m during 5th second and 50 m during 8th second of its motion. Compute initial velocity and acceleration of the body.

7. A car moving with uniform acceleration covers the distance between two points 60m apart in 6s. Its speed as it passes the second point is 15 m/s. (a) What is its speed at the first point ? (b) What is its acceleration ?

8. A stone is thrown vertically upwards with an initial velocity of 14 m/s. Find the maximum height reached and the time of desecent.

9. A stone falls from a height of 19.6m. With what velocity does it strike the ground ?

10. A stone is dropped from the top of a tower 400m high. At the same time another stone is projected vertically upward with a velocity of 100 m/s. Find when and where they meet.

11. A body travels 100m in first two seconds and 104m in the next 4 second. How far will it move in the next 1.25 second if the acceleration is uniform.

12. A stone is dropped from a rising baloon at a height of 76 m above the ground and reaches the ground in 6s. What was the velocity of the baloon, when the stone was dropped.

13. A stone falls from a baloon that is ascending at the rate of 12m/s. Calculate the velocity and displacement of the stone from the point of release after 10s.

14. A man walking towards east with a velocity of 3km/hr encounters rain falling vertically, with a velocity of $3\sqrt{3}$ km/hr. At what angle should he hold the umbrella in order to protect himself from the rain ?

15. To a person going westward with a speed of 6km/hr, rain appears to fall vertically downward with a speed of 8km/hr. Find the actual speed and direction of rain.

16. To a man walking due east at the rate of 2km/hr, rain appears to fall vertically. When he increases his speed to 4km/hr, it appears to meet him at an angle of 4s. Find speed and direction of rain.

E. Long Answer type Questions :

1. Using calculus method derive the expression for the displacement in the nth second of a body moving with uniform acceleration along a straight line.

2. Derive the relation $s = ut + \frac{1}{2}at^2$, where the symbols have their usual meanings.

3. Obtain velocity time and position time relation for the uniform and uniformly accelerated motion. Explain the results with the help of position - time and velocity - time graph for both cases.

4. Using calculus method prove that

   $\upsilon^2 = u^2 + 2as$

   Where the symbols have their usual meanings.

5. Discuss velocity - time graph. Derive relation $s = ut + \frac{1}{2}at^2$ from this graph.

6. Discuss about position - time graph, distance time - graph. Find the distinctive features.

7. Discuss about speed - time graph and velocity - time graph. Give the distinctive features.

F. Fill in the Blank Type

1. Four persons K,L,M,N are initially at the four corners of a square of side d. Each person now moves with a uniform speed ϑ in such a way that K always moves directly towards L, L directly towards M, M directly towards N and N directly towards K. The four persons will meet at a time......

2. A man leaves his house and goes on an automobile trip returning to his house in a time Δt, after he left it. His average velocity for the trip is ....

3. A stone is released from an elevator going up with an acceleration $\vec{a}$. This acceleration of the stone after the release is ....

4. A stone released from the top of a tower with zero velocity, reaches the ground in 4-seconds. The height of the tower is ....

5. A train is moving with velocity 30 km/h and a car is moving with a velocity of 40 km/h at right angles to the direction of motion of the car. To a passanger in the train, the car appears to be moving with a velocity of ....

6. A body falling from a vertical height of 10m pierced through a distance of 1m in sand. The amount of retardation in sand is.........

G. True-False Type

1. A body can have east ward velocity while experiencing a west-ward acceleration.

2. A body is moving with uniform velocity in one frame A, then there is another frame B in which it is accelerating.

3. A lorry and a car moving with same kinetic energy are brought to rest by the application of brakes which provide equal retarding forces. Both come to rest in equal distance.

4. A car moving due east takes a turn and moves due north, the speed remaining unchanged. The acceleration of the car is zero.

5. A train is moving with a speed of 60 km/hr and a car is moving by its side in the same direction with a speed of 20 km/hr. The speed of the car relative to train is 80 km/hr.

6. Two cars A and B are moving in the same direction with equal speeds. A passenger in the car A finds that the car B is at rest.

ANSWERS

A. MULTIPLE CHOICE TYPE :

1. (ii), 2. (iv), 3. (iv), 4. (ii), 5. (iv), 6. (iii), 7. (iii), 8. (ii), 9. (iv), 10. (ii), 11. (iv), 12. (i), 13. (iv), 14. (i).

D. UNSOLVED NUMERICALS :

(1) 25 s

(2) 12m

(3) 0.05s

(4) 19.6m

(5) 65.1 m

(6) -67.5m/s, 15.67 ms^{-2}

(7) 5m/s; 1.67 m/s^2

(8) 10m, 1.428 s

(9) 19.6 m/s

(10) 4 s after dropped, 78.4m below the top.

(11) 6.25 m

(12) 16.73 m/s

(13) 86m/s moving downward, - 370m i.e. 370 m below the point of release of stone.

(14) 30^0 east of north

(15) 10 km/hr, $36^0 52'$ east of vertical

(16) 2.828 km/hr, 45^0 west of vertical.

F. (1) d/ g (2) zero (3) g downwards (4) 78.4m (5) 50 km/h (6) 10g.

G. (1) True (2) True (3) True (4) False (5) False (6) True

4

Laws of Motion

In chapter - 3 we have discussed about the motion of a particle, without enquiring about the cause of motion. In the present chapter we shall study the motion alongwith its cause.

4.1 Force :

In order to create a feeling about the cause of motion we consider a four-wheeler push trolly resting on a smooth and level surface and go through the following activities.

Activity	Observation
1. No push is given	Trolly remains at rest
2. (a) A push ($\vec{F}$) is given	Trolly starts moving in the direction of push.
(b) Push($\vec{F}$) is continued for some time t	Trolly moves in the same direction with increasing speed.
(c) Push is stopped	Trolly keeps on moving with the acquired velocity.
(d) Some luggage is kept on the trolly and steps 2(a) and (b) are repeated with the same push ($\vec{F}$)	Trolly now attains less speed within the same time interval 't', implying less acceleration.
3. A larger push is given and steps 2(a) and (b) are repeated.	Trolly attains larger speed within the same time t.
4. A person is asked to sit on the trolly and give push.	Trolly does not move
5. Two persons are asked to push the trolly from outside equally in opposite Direction.	Trolly does not move

The above activities and corresponding observations lead to the following conclusions.

(i) Unless a net push is applied from outside a body continues in its own state of motion i.e. the state of motion is not disturbed. This property is called inertia.

(ii) Larger the push (F), larger is the acceleration (a) i.e. $a \alpha F$.

(iii) Larger the mass, smaller is the acceleration i.e. $a \alpha \frac{1}{m}$.

Also if the above activities are repeated with a pull, the same conclusions can be arrived at. One could also observe that a strong wind can set the trolly into motion.

Thus the above discussion clearly indicates that the push or pull, applied by an external agency is the cause of motion, and it is in general called **force.** Thus **force is defined as the push or pull, applied by an external agency (animate or inanimate) which produces or tends to produce changes in the**

state of motion of a body. It is also quite apparent that **force is an interaction between two objects.** Force is always exerted by an object (say A) on another object (say B).

A. Contact and non-contact force

The above examples might lead one to think that force can only be exerted by contact. But this is not so. Attraction of earth on bodies near its surface, attraction of a magnet on a piece of iron are few examples of force, exerted without contact.

Generally, forces between two bodies in contact act along the common normal to the surface of contact. For example the table pushes the book kept on it upwards (away from it) and the book pushes the table downward (again away from it). However the force between two bodies in contact may have a component parallel to the surface of contact. This component accounts for friction.

B. Types of forces

We come across several types of forces arising out of different types of interaction between bodies. For example in the interaction between two static charges each charge experiences an electrostatic force, a piece of iron experiences a magnetic force due to its interaction with a magnet, all material particles experience gravitational pull due to their interaction with earth. Similarly we have other types of forces like : Viscous force, nuclear force, elastic force, etc. But all the forces we experience can be grouped together into four types of forces called **Basic forces.**

C. Basic forces

All the known forces encountered in nature, are not independent. They can be described in terms of four basic forces : (i) strong (ii) electromagnetic (iii) weak and (iv) gravitational. The range (i.e. the distance within which the effect can be felt) and relative strength are given in table 4.1.

Table No. 4.1

Basic force	Range	Relative strength
Strong	10^{-15}m	1
Electromagnetic (e.m.)	∞	10^{-2}
Weak	$< 10^{-15}$m	10^{-5}
Gravitational	∞	10^{-39}

Strong force operates at subnuclear level and is responsible for the binding of the proton and neutrons inside the atomic nucleus. Electromagnetic force operates on electrically charged particles. This force is responsible for formation of molecules, solids, liquids etc. Forces like elastic force, viscous force, and all contact forces are due to e.m. force. The weak force governs the radioactive decay and other slow decays of the subnuclear particles. Gravitational force is an attractive force that operates between any two material particles.

Even the above four basic forces are seen to be not independent. Physicists like S.Weinberg, A. Salam, Glashow etc. have conclusively shown that weak and e.m. force are different manifestations of a single force called "electro-weak force". Grand unified theories have been put forward to unify strong, e.m. and weak forces.

However we shall now attempt to investigate the relation between force and motion without considering what type of force it is.

4.2 Momentum

As illustrated in sec 4.1, greater force is required to cause a greater change in velocity in a certain time interval; and this change also depends on the mass of the body. So Newton considered that mass should also be included in the definition of force by defining a physical quantity momentum ($\vec{p}$) as mass times velocity.

i.e. $$\vec{p} = m\vec{\upsilon} \qquad ...(4.2.1)$$

Thus momentum is the amount of motion contained in a body and change of state of motion is the change of momentum. For example consider a car and a scooter moving with same velocity $\vec{\vartheta}$. It is seen that it is easier to stop the scooter than the car. Because car possesses more motion than the scooter $\left(m_c\left|\vec{\upsilon}\right|>m_s\left|\vec{\upsilon}\right| \text{as } m_c>m_s\right)$. Also it is easier to stop a slow moving scooter than a fast moving scooter $\left(m_s\left|\vec{\upsilon}_1\right|>m_s\left|\vec{\upsilon}_2\right| \text{as} \left|\vec{\upsilon}_1\right|>\left|\vec{\upsilon}_2\right|\right)$.

As defined in 4.2.1, momentum is a vector quantity. It has the dimension of MLT^{-1}. It is expressed in $kg.ms^{-1}$ (in S.I. units) and $g.cm.s^{-1}$ (in C.G.S. units).

4.3 Newton's Laws of Motion

Basing on experimental observations Sir Isaac Newton presented three laws of motion in his famous book "Philosophiae Naturalis Principia Mathematica" (the mathematical principle of natural philosophy). These three laws are commonly known as Newton's first law, second law and third law of motion. These laws along with their implications are discussed below.

A. Newton's first law of motion

It states that **"everybody continues in its state of rest or of uniform motion in a straight line (i.e. moves with a constant velocity) until and unless it is compelled by a net external force to change that state".**

OR

If the vector sum of all the external forces acting on a body is zero, then and only then the body remains unaccelerated (i.e. remains at rest or moves with constant velocity).

i.e. $\vec{a}=0$ if and only if $\sum \vec{F}^e=0$ where $\vec{a}$ is the acceleration and $\sum \vec{F}^e$ is the vector sum of all the external forces.

Implications of first law of motion

(1) Inertia

The first law of motion can be divided into two parts :

(a) continuance in the state of rest

(b) continuance with constant velocity.

when no net external force acts on a body. The first part implies static equilibrium and second part implies dynamic equilibrium. This behaviour is a natural property of the material bodies and is known as **inertia** of a body. So we define **"the property of a material body by virtue of which it continues in its state of rest or of uniform motion along a straight line until and unless it is acted upon by a net force is called inertia".** Inertia is of two types (i) enertia of rest and (ii) inertia of motion.

(a) *Inertia of rest* :**It is the property of a material body by virtue of which it continues in its state of rest until and unless a net external force acts on it.**

The following examples shall illustrate the statement.

(i) Place a card board on a glass and place a coin on the card board. When we give a sudden jerk to the card board it will be observed that the card board flies away but the coin falls inside the glass.

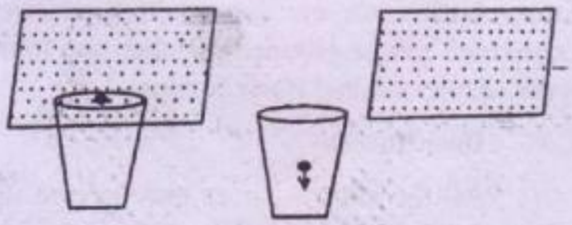


fig 4.1

This happens because while the card board is forced to change its state of rest, the coin continues in its state of rest and remained in its position. Hence it fell into the glass.

(ii) A passenger sitting in a bus at rest gets a backward jerk, when the bus suddenly starts.

Because when the bus starts the lower portion of the body, which is in contact with the bus, shares the motion, while the upper portion of the boy still continues in its state of rest. Hence the passenger leans backward.

(iii) Similarly a horse rider gets a backward jerk when the horse suddenly starts running.

(iv) The dust particles get separated from a blanket if the blanket is beaten with a stick or given a sudden jerk.

When we beat the blanket, the clothes are set into motion, while the dust particles still continue in its state of rest. Hence they get separated.

(v) A bullet fired into a window pane makes a clean hole through it while a stone breaks the whole of it.

The bullet has a larger speed than stone, hence its time of contact with the glass pane is very small. Motion of the bullet could be communicated to a small portion of the glass pane, hence a neat hole. On the otherhand the stone has larger time of contact because of its low speed. As a result the motion is communicated to a larger portion of the glass pane, thus causing damage to a larger portion of glass pane.

(vi) The leaves or fruits of a tree fall when we shake a tree.

Because the leaves or fruits are still in the state of rest while other portions of the tree share the motion because of shaking.

(b) **Inertia of motion : It is the property of a material body by virtue of which it continues in its state of uniform motion along a straight time until and unless it is impressed upon by a net external force.**

The following examples shall illustrate the property.

(i) A passenger in a bus leans forward when the bus suddenly stops.

Because when the bus stops, the lower portion of body in contact with the bus, comes to rest whereas the upper portion of the body continues in its state of motion. So the upper portion leans forward.

(ii) A man jumping from a running train falls with his head towards the engine. That is why the man coming down from a moving train is advised to run in the direction of motion of the train; to keep himself stable.

Because while jumping from a running train, the foot suddenly comes to rest, but the head continues in its state of motion. Hence the man falls with his head towards the engine.

(iv) While taking a long jump an athlete has to run a long distance so that inertia of motion may help him to have a longer jump.

(v) The mud sticking to a cycle, scooter, or any vehicle - tyre, flies off tangentially. That is why mudguards are provided on the front and rear wheels.

(vi) A stone tied to one end of a string and rotating in a horizontal circle flies off tangentially when the string breaks.

(vii) The person sitting in a bus gets an outward jerk if the bus takes a sharp turn.

(viii) The spark coming out of a grinding stone appears to be tangential to the same.

2. Definition of force

Newton's first law of motion contains a qualitative definition of force. According to first law. "A force is an agent capable of producing a change in the state of rest or of uniform motion".

3. Definition of inertial frame of reference

According to first law a body is unaccelerated ($\vec{a} = 0$) if and only if $\vec{F}_{ext} = 0$. Thus it implies that whenever $\vec{a} = 0$, we can conclude that $\vec{F}_{ext} = 0$ or viaversa. However, as said earlier in sec 3.1, the concept of rest, motion or acceleration are meaningful only when a frame of reference is specified. The acceleration measured in different frames of reference differ (sec 3.12).

For example consider a cabin with a ball, hanging from its roof. Let the cabin fall down. The cabin and all bodies fixed in the cabin (say the ball C and observer A) are accelerated w.r.to the earth and the acceleration is about 9.8 m/s^2 in the downward direction as measured by an observer B, on earth surface.

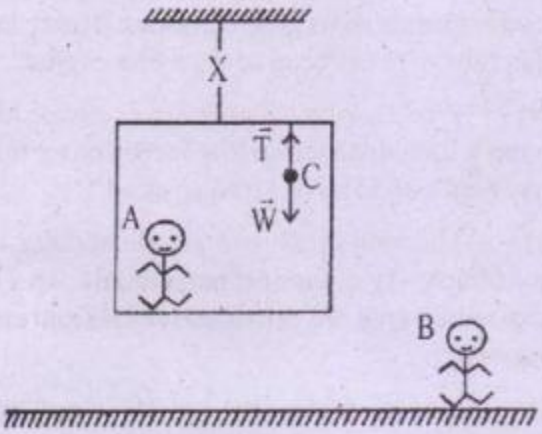


fig. 4.2

Now consider the ball within the cabin. The forces acting on the ball are (i) gravitation force $\vec{W}$ $(= m\vec{g})$ (ii) tension force $\vec{T}$ of the rope supporting the ball. The net force acting on the ball is $\vec{W} + \vec{T} = (W - T)(-\hat{j})$.

The observer 'A' measures the acceleration of the ball as zero. So if 'A' uses first law then $W - T = 0$ or W=T. $(\because \vec{a} = 0)$

On the contrary the observer B measures the acceleration of the ball as g=9.8 m/s^2 (i.e.$|\vec{a}| \neq 0$). Hence when observer B applies first law he finds $W \neq T$.

Thus we find that when A and B measure acceleration and apply first law, they donot agree. Therefore either of them must be wrong i.e. one of the frame is a bad frame or not suitable for applying first law of motion. This implies one is allowed to use only those frames for measuring acceleration in which Newton's first law of motion is valid. Such frames in which Newton's law of motion is valid are called **inertial frames.**

In the present case 'A' is in an accelerated frame (not an inertial frame) hence A is not allowed to apply Newton's first law.

But earth is taken to be (approximately) an inertial frame, hence man B is allowed to apply Newton's first law of motion.

In this sense Newton's first law by implication defines an inertial frame.

Inertial frame other than earth

Suppose S is an inertial frame and S' is a frame moving uniformly w.r. to S-frame. Let a particle P have acceleration $\vec{a}_{ps}$ w.r.to S-frame and $\vec{a}_{ps'}$ w.r.to S'-frame. So as shown in sec 3.12

$$\vec{a}_{ps} = \vec{a}_{ps'} + \vec{a}_{s's}$$

But $\quad \vec{a}_{s's} = 0$, so $\vec{a}_{ps} = \vec{a}_{ps'}$

Now S is an inertial frame implies

$$\vec{a}_{ps} = 0 \text{ if and only if } \vec{F} = 0$$

so $\quad \vec{a}_{ps'} = 0$ if and only if $\vec{F} = 0$

i.e. $\quad$ S' is also an inertial frame.

Thus any frame moving with uniform velocity with respect to an inertial frame is inertial.

Limitations of first law of motion

The first law of motion suffers from the following limitations.

(i) It does not attempt to distinguish

between the situations where the forcs are completely absent and where the vector sum of the forces is zero.

(ii) It gives no qantitative idea about force.

B. Newton's Second Law of Motion

It states that the time rate of change of momentum of a body is directly proportional to the net external force impressed on it and takes place in the direction of net external force.

i.e. $\frac{d\vec{p}}{dt} \alpha \vec{F}$

$$\Rightarrow \vec{F} = K\frac{d\vec{p}}{dt} = K\frac{d}{dt}(m\vec{\upsilon}) \qquad ...(4.3.1)$$

Where 'K' is a constant of proportionality - whose value depends on the choice of the unit of force. The unit of force is so chosen that K=1. Then equation 4.3.1 reduces to

$$\vec{F} = \frac{d\vec{p}}{dt} = \frac{d}{dt}(m\vec{\upsilon}) \qquad ...(4.3.2)$$

If mass 'm' of a body is considered to be constant (which is true at low velocities) then equation (4.3.2) reduces to

$$\vec{F} = m\frac{d\vec{\upsilon}}{dt} = m\vec{a} \qquad ...(4.3.3)$$

Implications of second law of motion

Equations 4.3.2 and 4.3.3 provide us a handle to measure force and inertia of a body. The second law of motion implies that a force $\vec{F}$ acting on a body of mass m, induces in it an acceleration of $\vec{F}$ /m w.r. to the inertial frame, and this is a law of nature. If the force ceases to act at some instant, the acceleration becomes zero at the same instant. In equation 4.3.3 $\vec{a}$ and $\vec{F}$ are measured at the same instant of time.

1. Measurement of force

Equation 4.3.2 and 4.3.3 gives that

$$\vec{F} = \frac{d\vec{p}}{dt} = m\vec{a}$$

Hence the dimension of force is given by

$$[F] = MLT^{-2} \qquad ...(4.3.4)$$

In order to measure force a unit of force is to be defined through the equation 4.3.3 or 4.3.2. There are two types of units used in measuring force (a) absolute unit and (b) gravitational unit.

(a) Absolute unit

In equation 4.3.3 if we consider a body of mass 1 unit (m=1) and if the body is accelerated through 1 unit (a=1), then

F = 1 unit. 1 unit = 1 unit.

This implies that an absolute unit of force is the amount of force which induces an acceleration of 1 unit in a body of mass 1 unit.

(i) C.G.S. absolute unit

Absolute unit of force in C.G.S. system is 1 dyne and

$$1 \text{ dyne } 1 \text{ g.cm.s}^{-2} \qquad ...(4.3.5)$$

i.e. A dyne is the amount of force which produces an acceleration of $1 cm/s^2$ in a body of mass 1 g.

(ii) S.I. absolute unit

Absolute unit of force in S.I. system is 1 newton and

$$1 \text{ N} = 1 \text{ kg. m.s}^{-2} \qquad ...(4.3.6)$$

i.e. A newton is the amount of force that produces an acceleration of $1 m/s^2$ in a body of mass 1 kg.

Relation between newton and dyne

$$1 \text{ N} = 1\text{kg. m.s}^{-2} = 10^3 \text{ g. } 10^2 \text{ cm.s}^{-2}$$

$$1 \text{ N} = 10^5 \text{ g. cm.s}^{-2} = 10^5 \text{ dyne} \qquad ...(4.3.7)$$

(b) Gravitational unit

In gravitational units the fundamental unit of force is defined in terms of the pull of the earth upon an arbitrarily but suitably chosen mass at a conveniently chosen place. Since the pull F of earth on a body of mass m is given by

$$F = mg \qquad ...(4.3.8)$$

Where 'g' is acceleration due to gravity, so equation 4.3.8 may be taken as defining relation for gravitational unit.

Thus "Gravitational unit of force is the amount of force which produces an acceleration equal to the acceleration due to gravity at a suitably chosen place, in a body of unit mass".

OR

Gravitational unit of force is the force with which a unit mass at a suitably chosen place is attracted towards the earth.

The suitably chosen place is where $g = 980.665\ cm/s^2$ or $9.80665 m/s^2$.

(i) **C.G.S. gravitational unit**

In C.G.S. gravitational system the unit of force is called 1 gram - force or 1 gram-weight.

One gram-force is the amount of force that produces an acceleration of $980.665\ cm/s^2$ in a body of mass 1 gm.

Thus 1 g-force = 1 g x $980.665\ cm/s^2$ =980.665 dyne.

i.e. 1 gf = 980.665 dyne 980 dyne.

(ii) **S.I. gravitational unit**

In S.I. gravitational system the unit of force is called 1 kilogram-force (1kgf) or 1 kilogram weight (1kg.wt).

One kilogram-force is the amount of force that produces an acceleration of 9.80665 m/s^2 in a body of mass 1kg.

so 1kgf = 1kg x 9.80665 m/s^2=9.80665 N

i.e. 1 kgf = 9.80665 N 9.8 N.

2. **Measurement of inertia (Inertial mass)**

From equation 4.3.3 we find that a=f/m. This implies if m is small a is large and vice versa. So 'm' decides how fast the inertia or the state of motion is disturbed when a fixed quantity of force is applied. Hence 'mass' is a measure of inertia of a body. That is why 'm' in equation 4.3.3 is often called inertial mass.

The inertial mass has the following characteristics :

(i) Inertial mass is an additive quantity.

(ii) It is proportional to the quantity of matter contained in the body.

(iii) It does not depend on shape and size of the body.

(iv) It is independent of the state of the body.

(v) It is not affected by presence of other bodies near it.

(vi) It varies with velocity as

$$m = \frac{m_o}{\sqrt{1 - \frac{\upsilon^2}{C^2}}}$$

Where 'm_o' is the rest mass, C is the speed of light.

Ex. 4.3.1 A body of mass 10 kg traverses in successive seconds distances of 5,6 and 7 m respectively. What force is acting on it ?

Soln.

Given distances travelled in succesive seconds is 5m, 6m, and 7m. Hence

$$S_n - S_{n-1} = \frac{1}{2} a(2n + 1) = 5m$$

$$S_{n+1} - S_n = \frac{1}{2} a(2n + 3) = 6m$$

$$S_{n+2} - S_{n+1} = \frac{1}{2} a(2n + 5) = 7m$$

$$\Rightarrow a = 1 m/s^2$$

Hence force acting $F = ma = 10kg \times 1 m/s^2 = 10N$.

Ex. 4.3.2 A motor car running at the rate of 7m/s can be stopped by its brakes in 5m. Prove that the total resistance to the car's motion, when the brakes are on, is one half of the weight of the car.

Soln.

Given $u = 7m/s,\ \upsilon = 0,\ S = 5m$

Hence $a = \dfrac{\upsilon^2 - u^2}{2S} = \dfrac{-49}{2\times 5} = -4.9m/s^2$

Hence resistance $F = ma = m \times 4.9\ kgm/s^2$
Weight of the car $= mg = 9.8\ kgm/s^2$

Hence resistance $F = \dfrac{1}{2}\ mg = \dfrac{1}{2}$ wt of the car.

Ex. 4.3.3 A force of 15 N is applied to a mass m. The mass moves in a straight line with its speed increasing by 10m/s every 2 S. Find the mass m.

Soln.

As given $\upsilon = u + at$

$\upsilon + 10 = u + a\,(t+2)$

$\Rightarrow\quad 2a = 10m/s^2$

$\Rightarrow\quad a = 5m/s^2$

Hence $m = \dfrac{F}{a} = \dfrac{15N}{5m/s^2} = 3kg$

Ex. 4.3.4 A single force 10N acts on a mass m. The mass starts from rest and travels in a straight line a distance of 18m in 6 sec. Find the mass.

Soln.

Given force $F = 10N$

initial velocity $u = 0$

distance travelled $S = 18m$

time taken $t = 6\ S$

$S = ut + \dfrac{1}{2}\ at^2 = \dfrac{1}{2}\ at^2$

$a = \dfrac{2S}{t^2} = \dfrac{2\times 18}{6\times 6} = 1m/s^2$

Hence $m = \dfrac{F}{a} = \dfrac{10N}{1m/s^2} = 10\ kg$

C. Newton's Third law of motion

It states that "To every action there is an equal and opposite reaction."

This means that if a body A exerts a force $\vec{F}$ on another body B, then B exerts a force $(-\vec{F})$ on A.

i.e. if $\vec{F}_{A\to B} = \vec{F}$

then $\vec{F}_{B\to A} = -\vec{F}$

Although these two forces are equal in magnitude and their lines of action coincide, they donot cancel each other. Because they act on two different bodies. For example if we consider a book kept on a table top, then the book exerts a force $\vec{W} = \vec{F}_{B\to A}$ on the table top, while the table top offers a reaction force $\vec{R} = \vec{F}_{A\to B}$ on the book, such that $\vec{W} = -\vec{R}$

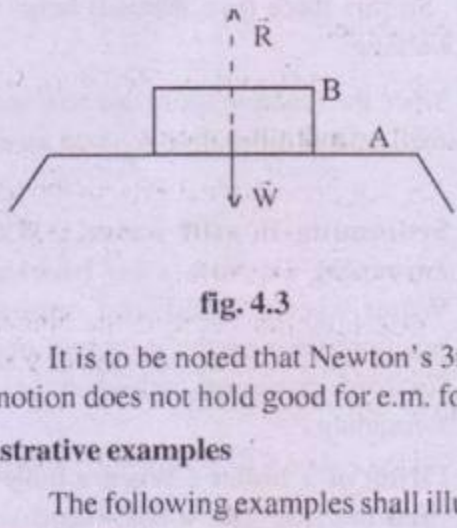


fig. 4.3

It is to be noted that Newton's 3rd law of motion does not hold good for e.m. forces.

Illustrative examples

The following examples shall illustrate Newton's third law of motion.

(i) **Walking on ground :** While walking we press the ground with a force with our feet in a backward direction (as shown in fig. 4.4). The reaction froce offered by ground acts on the man. So the forces acting

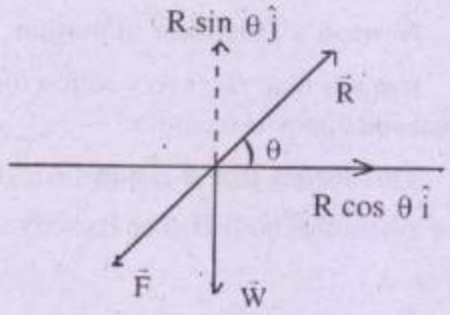


fig. 4.4

on the man are (i) $\vec{R}$ and (ii) $\vec{W}$, the weight of man, due to gravity. Resolving along horizontal direction ($\hat{i}$) and vertical direction ($\hat{j}$).

We have

$$\vec{R} = R\cos\theta\,\hat{i} + R\sin\theta\,\hat{j}$$

Since there is no vertical motion so

$$R\sin\theta\,\hat{j} = -\vec{W}$$

and unbalanced force acting on the man is $R\cos\theta\,\hat{i}$. So this force (unbalanced) helps the man in walking.

Since the reaction force due to a sandy road is small, so it is difficult to walk on a sandy road.'

(ii) **Swimming in still water :** While swimming we press water backward. Water exerts equal and opposite reaction force on the swimmer in the forward direction, which helps in swimming

(iii) **Firing of a bullet :** When a bullet is fired the rifle gets a (backward jerk) recoil in the backward direction due to reaction of the bullet.

(iv) **Rebound of rubber ball :** When a rubber ball is struck against a wall it gets rebound due to reaction from the wall.

(v) **Flying of jets :** Jet plane rejects burnt gases at high speed. As a result of this the jet plane gets a jerk in the forward direction.

(vi) **Jumping of a man from a boat :** When a man jumps from a boat on to the shore, the boat gets a recoil in the backward direction.

(vii) **Rotatory lawn sprinkler :** The design is made as shown in fig. 4.5. When water ejects out the reaction forces form two couples which rotate the sprinkler.

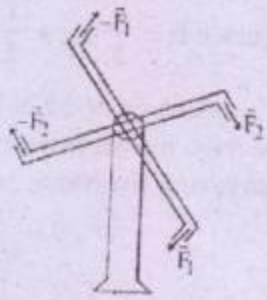


fig. 4.5

(viii) Lifting of water from a well by means of a rope passing over a pulley.

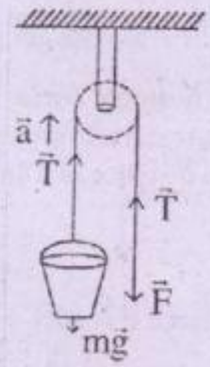


Fig. 4.6

When we pull the rope with a force $\vec{F}$, a tension $\vec{T}$ acts in upward direction implying $\vec{F} = -\vec{T}$, and $\vec{T} = |\vec{F}|$.

If $\vec{T} + m\vec{g} = m\vec{a}$

then $\vec{T} = m(-\vec{g} + \vec{a})$

$\vec{T}\,\hat{j} = m(+g\,\hat{j} + a\,\hat{j})$

$=m\,(a+g)\,\hat{j}$

$\Rightarrow T = m(a+g) = F$

4.3.D. Application of Newton's laws of motion

Newton's 2nd law refers to a point particle and relates the forces acting on the particle with its acceleration and mass. Therefore before writing the equation one should very clearly and carefully understand which particles are to be considered.

In actual practice we deal with extended objects, which are collections of large number of particles. However Newton's 2nd law can be applied to an extended body or a system provided each part of this body or system has same acceleration (in magnitude and direction). The following **algorithm** (procedure) may be followed.

Step I Draw a neat digram showing all the salient features of the problem.

Step II A single particle, or a combination of particles, or an extended object or a combination of extended objects, which have identical acceleration should be included in one system.

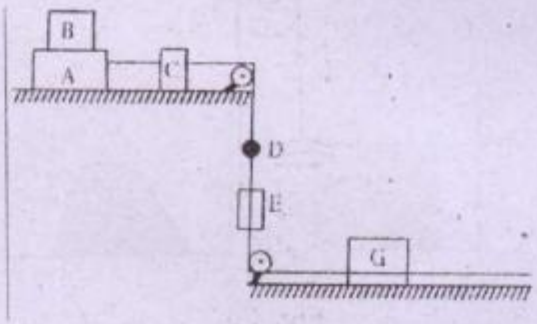


fig. 4.7

In the example shown in fig. 4.7 A,B,C and G move with same acceleration $\vec{a}$. So we can consider them individually or some combinations of them e.g A,B,C,G or A+B, A+C, A+G, B+G etc. and A+B+C, A+B+G, C+A+G, A+B+C+G etc. as constituting a system for application of 2nd law. Similarly D and E move with same acceleration $\vec{b}$ (and different from that of A,B,C,G in direction) hence we consider D,E or D+E as forming a system. But we **should not** take D+G, E+G, A+E etc. as forming a system.

Step III. After selecting the system, one has to list the forces acting on the system due to all the objects other than the system.

For example consider a body standing on a floor with a load on his head. So various forces operating are

(i) Load pressing the boy down ward

(ii) Boy pushes the load upward

(iii) Boy presses the floor downward

(iv) Floor pushes the boy upward

(v) Earth attracts the load downward

(vi) Earth attracts the boy downward

(vii) Load attracts earth upward

(viii) Boy attracts earth upward

The relevant forces to be taken into consideration depend on the system we select.

Suppose we consider the state of the motion of the boy. Then the forces relevant are

(i) Force exerted by earth on the body $= \vec{W} = m\vec{g}$, acting downward.

(ii) Force $\vec{N}$ exerted by load on head acting downward.

(iii) Reaction force $\vec{R}$ due to floor acting upward.

If, on the otherhand we consider the load, the relevant forces are

(i) Force due to earth $= \vec{W}'$, acting downward.

(ii) Force $\vec{N}$ due to boy acting upward.

Step - IV Represent the system by a point and draw vectors representing the various forces acting on the system with the point as the common origin.

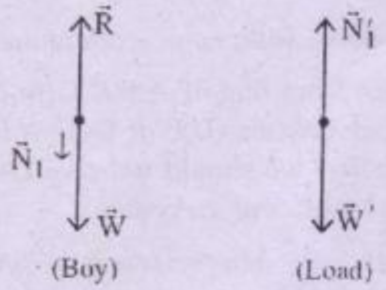


fig. 4.8

Step - V Choose axes and write equations

4.4 Impulse :

Impulse of a force $\vec{F}$, acting for a time interval dt is defined as the product of force $\vec{F}$ and time interval dt.

i.e. $d\vec{j} = \vec{F}\ dt$ ...(4.4.1)

If the force $\vec{F}$ acts for a finite time interval $(t_1 \to t_2)$, then impulse of the force is given as

$$\vec{j} = \int_{t_1}^{t_2} \vec{F}\ dt \quad ...(4.4.2)$$

Impulse is a vector quantity and has the dimension of MLT^{-1} (same as that of momentum).

Impulse-Momentum therem :

According to Newton's 2nd law of motion, if a force $\vec{F}$ acts on a body of mass m then

$$\vec{F} = \frac{d\vec{p}}{dt} = \frac{d}{dt}(m\vec{v})$$

$$\Rightarrow d\vec{j} = \vec{F}.dt = d\vec{p} \quad ...(4.4.3)$$

If we consider a finite time enterval we obtain from (4.4.3).

$$\vec{j} = \int_{t_1}^{t_2} \vec{F}.dt = \int_{t_1}^{t_2} d\vec{p} = \vec{P}_2 - \vec{P}_1 = \overrightarrow{\Delta p} \quad ..(4.4.4)$$

Equations 4.4.3 and 4.4.4 show that the impulse of the force is equal to the change in linear momentum of the body. This statement is some-times known as **impulse - momentum theorem** for a body.

Equation (4.4.4) also yields

$$J_x = \int_{t_1}^{t_2} F_x dt = P_{2x} - P_{1x} = \Delta P_x$$

$$J_y = \int_{t_1}^{t_2} F_y dt = P_{2y} - P_{1y} = \Delta P_y$$

$$J_z = \int_{t_1}^{t_2} F_z dt = P_{2z} - P_{1z} = \Delta P_z \quad ...(4.4.5)$$

The impulse of any force component or any force whose direction is constant (e.g. one-diemensional motion) can be computed from the area under the F ~ t curve.

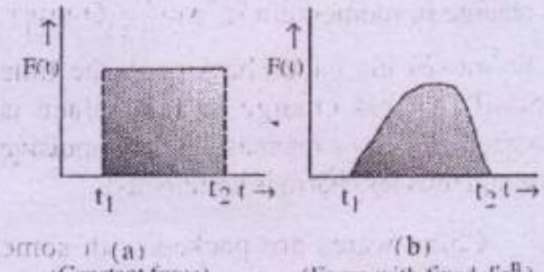


fig. 4.8(a)

However, when a force acts for short time interval, causing appreciable change in linear momentum, we call it an impulsive force.

So impulsive force is a large force acting for a short time interval.

Illustrative example

Since impulse $\overrightarrow{\Delta J} = \vec{F}\Delta t = \overrightarrow{\Delta P}$, so $|\vec{F}| = \frac{|\overrightarrow{\Delta P}|}{\Delta t}$. This implies that the amount of impulsive force shall depend on the time interval over which the change in momentum has taken place. For the sake of illultration. We consider the following examples.

(i) A man falling from a height on a cemented floor gets hurt to a greater extent than when falling on a heap of sand or a cushion.

Because when the man falls on the cemented floor he suddenly comes to rest. So the charge in momentum $|\vec{O} - \vec{P}| = m\upsilon$ takes place in a vary short time interval. Hence the impulsive force is more and it causes more harm. On the otherhand when the man falls on a heap of sand the change in momentum $m\upsilon$, takes place in a larger time interval. Hence the impulsive force is less and less harm is caused.

(ii) A cricket player moves his hands backward while taking a catch.

Because when the player takes the catch the change in momentum is $m\upsilon = (|\vec{O} - m\vec{\upsilon}|)$. As he moves his hands backward, the time interval for this change to take place is increased, causing a decrease in the impulsive force and thus less harm is anticipated.

(iii) China wares are packed with some flexible materials (like straw) in between them. In case of agitation or jerking the flexible material provides sufficient time for change in momentum and thus reduces the impulsive force. As a result less damage is caused.

(iv) Shock absorbers provided in the motor - vehicles reduce the impulsive force arising out of jerks, by prolonging the time due to its spring system.

Ex. 4.4.1 A cricket ball of mass 100 g moving with velocity of 20 m/s is brought to rest by a player in 0.05s. Find the impulse of the force and the average force applied by the player.

Soln.

$$\text{Impulse } J = |m(\upsilon - u)| = 100 \text{ g x } 20 \text{ m/s}$$

$$= 0.1\text{kg x } 20 \text{ m/s} = 2 \text{ Ns.}$$

$$\text{Average force} = F = \frac{J}{\Delta t} = \frac{2\text{Ns}}{0.05\text{s}} = 40\text{N}$$

Ex. 4.4.2 Two bodies of equal masses are placed side by side. A constant force F acts on first while an impulse J is given to the other, along the line of action of force. Show that they will meet after a time $\frac{2J}{F}$.

Soln.

For first body $a = \frac{F}{m}$

Hence distance travelled in time t, is

$$S = ut + \frac{1}{2}at^2 = \frac{1}{2}at^2 = \frac{F}{2m}t^2$$

For second body impulse given is $J = m(\vartheta - u)$

$$\Rightarrow J = m(\upsilon - o) = m\upsilon$$

Initial velocity of 2nd body is $\upsilon = \frac{J}{m}$.

So distance travelled in 't' seconds is

$$S' = \upsilon.t = \frac{J}{m}t$$

Hence two bodies shall meet if $S = S'$

i.e. $\frac{F}{2m}t^2 = \frac{J}{m}t$

$$\Rightarrow t = \frac{2J}{F}$$

4.5 Static and translational equilibrium :

A body in general can be at rest, in translational motion or in rotational motion.

A body maintained in a state of rest is said to be in **static equilibrium.** A body maintained in a state of uniform translational motion is said to be in **translational equilibrium.** In both these cases linear acceleration of the body is zero. Since according to Newton's 2nd law of motion $\sum_i \vec{F}_i = m\vec{a}$, so for static or translational equilibrium.

$$\sum_i \vec{F}_i = 0 \qquad ...(4.5.1)$$

i.e. vector sum of all the external forces $\left(\sum_i \vec{F}_i\right)$ acting on the body need be zero. This is known as **first condition of equilibrium.** It can be stated as :

A body/system is said to be in static or translational equilibrium, if the net external force acting on the body/system is zero.

When a single force acts, the body is not in equilibrium, and the single force is called **unbalanced force.**

There should be a minimum of two forces acting on the body so that the body can be in equilibrium, and this can happen if and only if

$$\vec{F}_1 + \vec{F}_2 = 0$$

$$\Rightarrow \vec{F}_1 = -\vec{F}_2$$

i.e. the two forces are equal in magnitude and opposite in direction, and their lines of action coincide.

If more than two forces, acting on a body, keep it in translational equilibrium, then

$$\vec{F}_1 + \vec{F}_2 + \vec{F}_3 + ... = 0$$

which implies that the forces can be represented by the sides of a closed polygon taken in one order.

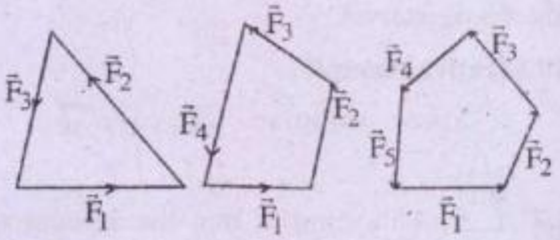


fig. 4.9

Condition 4.5.1 can also be given in component from. Since

$$\sum_i \vec{F}_i = 0$$

$$\Rightarrow \quad \hat{i}.\sum_i \vec{F}_i = \sum_i \vec{F}_{ix} = 0$$

$$\hat{j}.\sum_i \vec{F}_i = \sum_i \vec{F}_{iy} = 0$$

$$\hat{K}.\sum_i \vec{F}_i = \sum_i \vec{F}_{iz} = 0$$

...(4.5.2)

Note : A detailed discussion on equilibrium shall be mode in a later chapter.

4.6 Some typical and common problems

We discuss below few typical but common problem encountered in our day to day life.

(i) **Tension force :**

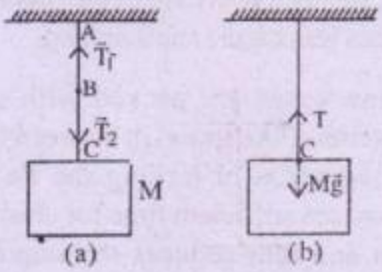


Fig. 4.10

(a) Suppose a mass M is hanging from the ceiling (see fig. 4.10) with the help of a massless string. The cross - section at B, divides the string into two parts - lower part and upper part. The two parts are in physical contact at B. The lower part exerts a force $\vec{T}_2$ on the upper part and the upper part exerts a force $\vec{T}_1$ on the lower part. According to 3rd law of motion, the two forces will have equal magnitude i.e. $|\vec{T}_1| = |\vec{T}_2| = T$. This common magnitude is called as the tension in the string at B.

At the lower and C, the block and the string are in physical contact, so considering the equilibrium of the block

$$\vec{T} + M\vec{g} = 0$$

$$T = -M\vec{g} \text{ and } |\vec{T}| = Mg \qquad ...(4.6.1)$$

At the upper end A, the ceiling and the string are in contact. So if $\vec{R}_c$ be the reaction force of the ceiling, then $\vec{R}_c = -\vec{T}_c$ (by 3rd law). Since the string is in equilibrium

so $\quad \vec{R}_c + M\vec{g} = 0$

$$\Rightarrow \quad \vec{R}_c = -M\vec{g} \qquad ...(4.6.2)$$

Therefore $\quad \vec{R}_c = -M\vec{g} = -\vec{T}_c$

$$\Rightarrow \quad \vec{T}_c = M\vec{g} \qquad ...(4.6.3)$$

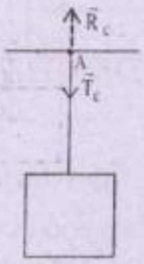


fig. 4.10 (c)

Thus from (4.6.3) and (4.6.1) it follows that

$$\Rightarrow \quad \vec{T} = -M\vec{g} = -\vec{T}_c = \vec{R}_c \qquad ...(4.6.4)$$

$\Rightarrow \quad T = Mg =$ Reaction force of the ceiling

(b) Suppose the block is hanging from a ceiling by a massive string.

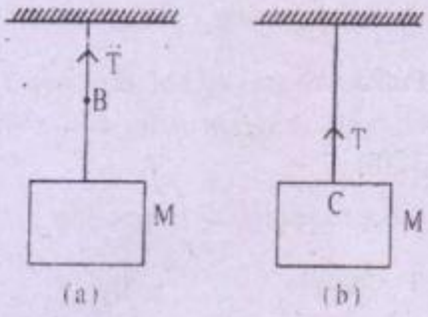


fig. 4.11

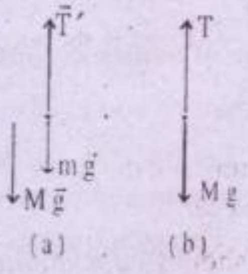


fig 4.12

Let m be mass of the string below B. Now choose the string below B and the block as our system. Forces acting on this system are

(i) $M\vec{g}$, acting downward due to block

(ii) $\vec{T}'$, acting upward due to upper portion of the string.

(iii) $m\vec{g}$, acting downward, due to weight of the string below B. The **free-body diagram** is as shown in fig. 4.12(a).

Since the lower part has no acceleration, so

$$\vec{T}' + M\vec{g} + m\vec{g} = 0$$

$$\Rightarrow \quad \left[\vec{T}' - (M+m)g\right]\hat{j} = 0$$

$$\Rightarrow \quad \vec{T}' = (M+m)g \qquad ...(4.6.6)$$

Considering the lower end c (see fig 4.11.(b)), let us choose the block as our system. The forces acting on this are

(i) Pull of the string, $\vec{T}$ upward

(ii) Pull due to gravity $M\vec{g}$, downward. The free body diagram is as shown in fig. 4.12(b).

Since the point 'c' is at rest so

$$\vec{T} + M\vec{g} = 0$$

$$\Rightarrow \quad (T - Mg)\hat{j} = 0$$

$$\Rightarrow \quad T = Mg \qquad ...(4.6.7)$$

Thus it is observed that when the string is massive, the tension is not uniform throughout the string.

(ii) **Motion of bodies in contact :**

Consider two bodies of masses m_1 and m_2 in contact. Let force $\vec{F} = F\hat{i}$ be applied on m_1. Let the two masses move with a common acceleration $\vec{a}$. Forces acting on

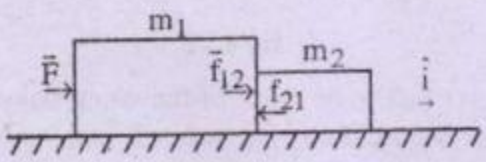


fig. 4.13

m_1 are $\vec{F}$ (external force) and $\vec{f}_{21}$ (reaction force due to m_2 on m_1).

i.e. $\vec{F}_1 = \vec{F} + \vec{f}_{21} \qquad ...(4.6.8)$

Force acting on m_2 is $\vec{f}_{12}$ (force exerted by m_1 on m_2)

i.e. $\vec{F}_2 = \vec{f}_{12} \qquad ...(4.6.9)$

Now since the two bodies are in contact always so

$$\vec{f}_{12} = -\vec{f}_{21} = f\hat{i} \qquad ...(4.6.10)$$

and $\vec{F} = F\hat{i} \qquad ...(4.6.11)$

Writing equations of motion for m_1 and m_2 separately we have

$$\vec{F}_1 = \vec{F} + \vec{f}_{21} = m_1\vec{a} \qquad ...(4.6.12)$$

and $\vec{F}_2 = \vec{f}_{12} = m_2\vec{a} \qquad ...(4.6.13)$

Using equations (4.6.10) and (4.6.11) in 4.6.12) and (4.6.13) we obtain

$$(F - f)\hat{i} = m_1\vec{a}$$

$$\Rightarrow \quad F - f = m_1 a \qquad ...(4.6.13)$$

and $\vec{f}_{12} = f\hat{i} = m_2\vec{a}$

$$\Rightarrow \quad f = m_2 a \qquad ...(4.6.14)$$

Adding (4.6.13) and (4.6.14) we obtain

$$F = (m_1 + m_2)\, a \qquad ...(4.6.15)$$

Thus $\vec{a} = \dfrac{\vec{F}}{m_1 + m_2} \qquad ...(4.6.16)$

$$\vec{f} = m_2\vec{a} = \frac{m_2}{m_1 + m_2}\vec{F} \qquad ...(4.6.17)$$

(iii) **Motion of bodies connected by string**

(a) *Two bodies*

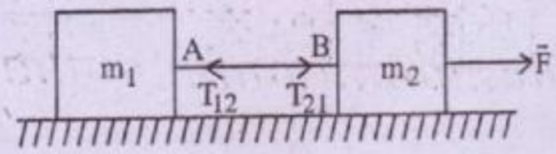


fig. 4.14

Consider two bodies of mass m_1 & m_2 connected by a string AB and pulled by a force $\vec{F}$ (see fig. 4.14). Forces acting on m_2 are (i) pull $\vec{F}$ and (ii) tension $\vec{T}_{12}$ (through the string). Forces acting on mass m_1 is tension $\vec{T}_{21}$ (acting through the string). So writing equations of motion for m_1 and m_2 separately we have

$$\vec{F}_2 = \vec{F} + \vec{T}_{12} = m_2\vec{a} \qquad ...(4.6.18)$$

$$\vec{F}_1 = \vec{T}_{21} = m_1\vec{a} \qquad ...(4.6.19)$$

Taking $\vec{T}_{21} = T\hat{i}$ and since every point of the string is in equilibrium so that $\vec{T}_{21} = -\vec{T}_{12}$, we have

$$\vec{T}_{21} = -\vec{T}_{12} = T\hat{i} \qquad ...(4.6.20)$$

Using (4.6.20) in (4.6.18) and (4.6.19) we obtain

$$(F - T)\hat{i} = m_2\vec{a} \qquad ...(4.6.21)$$

$$T\hat{i} = m_1\vec{a} \qquad ...(4.6.22)$$

Adding (4.6.21) and (4.6.22) we obtain

$$F\hat{i} = (m_1 + m_2)\vec{a}$$

$$\Rightarrow \quad \vec{a} = \frac{\vec{F}}{m_1 + m_2} \qquad ...(4.6.23)$$

and $$\vec{T} = \frac{m_1}{m_1 + m_2}\vec{F} \qquad ...(4.6.24)$$

(b) **Three bodies**

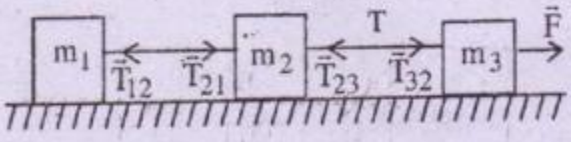


fig. 4.15

Proceeding in a similar manner we find

$$\vec{F}_1 = \vec{T}_{21} = m_1\vec{a} \qquad (a)$$

$$\vec{F}_2 = \vec{T}_{12} + \vec{T}_{32} = m_2\vec{a} \qquad (b)$$

$$\vec{F}_3 = \vec{F} + \vec{T}_{23} = m_3\vec{a} \qquad ...(4.6.25)\,(c)$$

But $$\vec{T}_{21} = -\vec{T}_{12} = T'\hat{i} \qquad (a)$$

$$\vec{T}_{32} = -\vec{T}_{23} = T\hat{i} \qquad (b)$$

$$\vec{F} = F\hat{i} \qquad ...(4.6.26)\,(c)$$

Using (4.6.26) in (4.6.25) we obtain

$$T'\hat{i} = m_1\vec{a} \qquad (a)$$

$$(T - T')\hat{i} = m_2\vec{a} \qquad (b)$$

$$(F - T)\hat{i} = m_3\vec{a} \qquad ...(4.6.27)(c)$$

Adding equations 4.6.27 (a), (b) and (c) we have

$$F\hat{i} = (m_1 + m_2 + m_3)\vec{a}$$

$$\Rightarrow \quad \vec{a} = \frac{\vec{F}}{m_1 + m_2 + m_3} \qquad ...(4.6.28)$$

and $$\vec{T}' = \frac{m_1}{m_1 + m_2 + m_3}\vec{F} \qquad ...(4.6.29)$$

...(4.6.30)

(c) **n-bodies connected by string**

Proceeding as above we shall have

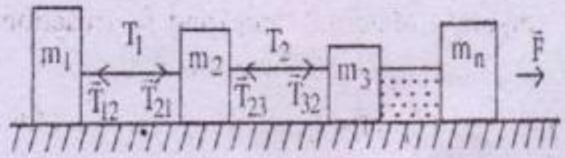


fig. 4.16

$$\vec{F}_1 = \vec{T}_{21} = m_1\vec{a}$$

$$\vec{F}_2 = \vec{T}_{32} + \vec{T}_{12} = m_2\vec{a}$$

$$\vec{F}_3 = \vec{T}_{43} + \vec{T}_{23} = m_3\vec{a}$$

$$\vec{F}_{n-1} = \vec{T}_{n,n-1} + \vec{T}_{n-2,n-1} = m_{n-1}\vec{a}$$

$$\vec{F}_n = \vec{F} + \vec{T}_{n-1,n} = m_n\vec{a}$$

...(4.6.31)

Adding we have

$$\vec{F} = (m_1 + m_2 + ... + m_n)\vec{a}$$

$$\Rightarrow \vec{a} = \frac{1}{m_1+m_2+......+m_n}\vec{F} = \frac{1}{\sum_{i=1}^{n} m_i}\vec{F}$$

$$\vec{T}_1 = \vec{T}_{21} = \frac{m_1}{\sum_{i=1}^{n} m_i}\vec{F}$$

$$\vec{T}_2 = \vec{T}_{32} = \frac{m_1 + m_2}{\sum_{i=1}^{n} m_i}\vec{F}$$

$$\vec{T}_3 = \vec{T}_{43} = \frac{m_1 + m_2 + m_3}{\sum_{i=1}^{n} m_i}\vec{F}$$

$$\vec{T}_{n-1} = \vec{T}_{n,n-1} = \frac{m_1 + m_2 + ... + m_{n-1}}{\sum_{i=1}^{n} m_i}\vec{F}$$

...(4.6.32)

(iv) **Horse and Cart Problem**

Consider the problem of horse pulling a cart (as shown in fig.4.17) along the horizontal direction ($\hat{i}$)

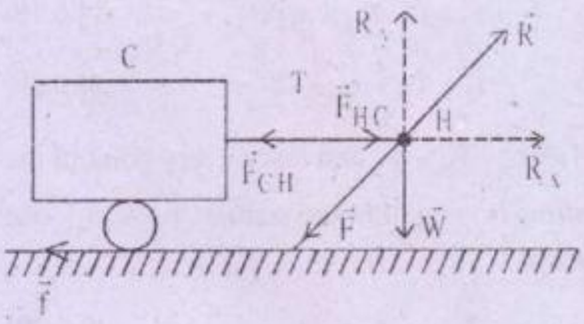


fig. 4.17

The horse exerts a force $\vec{F}$ on the ground. The reaction force $\vec{R}$ due to ground acts on the horse. The horizontal component $\vec{R}_x$ enables the horse to move forward. The vertically upward component $\vec{R}_y$ blances the weight of the horse. As the horse moves in the forward direction, a tension force $\vec{T} = \vec{F}_{HC}$ acts along the string. The cart also exerts a force $\vec{F}_{CH} (= -\vec{F}_{HC})$ on the horse. So proceeding as per sec 4.4 we obtain for,

(a) *Motion of horse :*

Net force acting on horse is given as

$$\vec{F}_H = \vec{R} + \vec{F}_{CH} + \vec{W}$$

$$\Rightarrow \vec{F}_H = (\vec{R}_x + \vec{R}_y) + \vec{F}_{CH} + \vec{W}$$

$$= (\vec{R}_x + \vec{R}_y) - \vec{T} + \vec{W}$$

Hence by 2nd law of motion

$$\vec{F}_H = (\vec{R}_x + \vec{R}_y) - \vec{T} + \vec{W} = m_2\vec{a}$$

...(4.6.33)

Where m_2 is mass of horse and $\vec{a}$ is acceleration of the horse. If $\vec{a} = a\hat{i}$, for the system, then

$$R_x - T = m_2 a \quad ...(4.6.34)$$

and $$R_y + W = 0 \quad ...(4.6.35)$$

(b) *Motion of cart*

Net force acting on the cart is $\vec{F}_c$, given as

$$\vec{F}_c = \vec{F}_{HC} + \vec{f} = m_1\vec{a} \quad ...(4.6.36)$$

Where m_1 is mass of cart and $\vec{f}$ is the force of friction between wheel of cart and the ground. Equation (4.6.36) leads to

$$F_{HC} - f = m_1 a$$

$$\Rightarrow \quad T - f = m_1 a \quad ...(4.6.37)$$

Adding (4.6.34) and (4.6.37) we obtain

$$R_x - f = (m_1 + m_2)a \quad ...(4.6.38)$$

Equation (4.6.34) shows that only when $R_x > T$ the horse moves forward with an acceleration. Similarly equation (4.6.37) shows that only when $T > f$, the cart is accelerated. Equation (4.6.38) shows that when $R_x > f$ the system of cart and horse is accelerated. Thus the seemingly paradox that horse pulls the cart and the cart pulls the horse vanishes, when we carefully take into account all the forces acting on the system and its constitutents.

(v) **Comparision of the effect of push & pull**

(a) Consider a block of mass m being subjected to a push ($\vec{F}$) along a direction making an angle , with the direction of motion. The forces acting on the block are

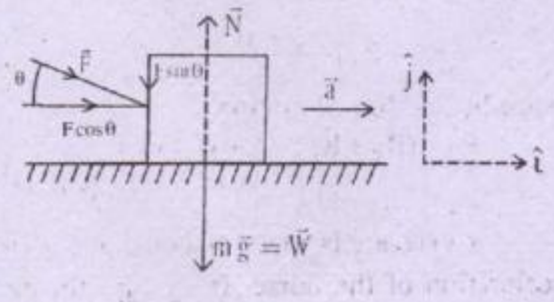


fig. 4.18

(i) Push $\vec{F}$ (ii) Normal reaction $\vec{N}$ and

(iii) Weight $\vec{W}(= m\vec{g})$. So by 2nd law of motion.

$$\vec{F} + \vec{W} + \vec{N} = m\vec{a} \quad ...(4.6.39)$$

Resolving along the direction of motion ($\hat{i}$) and perpendicular to it

$$F\cos\theta\,\hat{i} + (N - F\sin\theta - mg)\,\hat{j} = ma\hat{i}$$

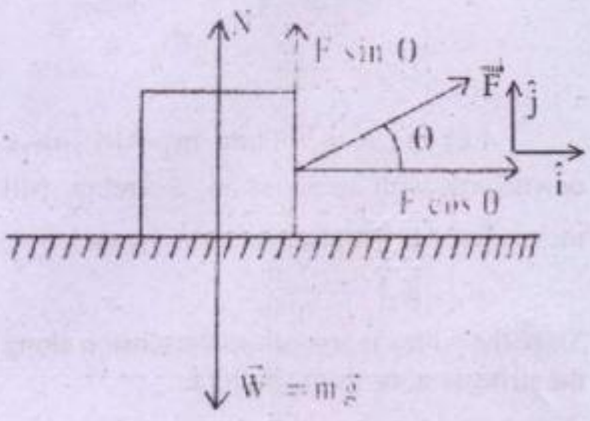
$$\Rightarrow \quad F\cos\theta = ma \quad ...(4.6.40)$$

and $$N = F\sin\theta + mg \quad ...(4.6.41)$$

This shows that $N \neq mg$ (which is the case when the block is at rest), but $N > mg$. This implies as if the weight of the body has been increased. Hence it is **difficult to push.**

(b) On the otherhand of the block is pulled by the same force $\vec{F}$ making an angle θ (as shown in fig. 4.19) with the

fig. 4.19



direction of motion, then

$$\vec{F} + \vec{W} + \vec{N} = m\vec{a}$$

giving

$$F\cos\theta\,\hat{i} + (F\sin\theta - W + N)\,\hat{j} = ma\hat{i}$$

$$\Rightarrow \quad F\cos\theta = ma \quad ...(4.6.42)$$

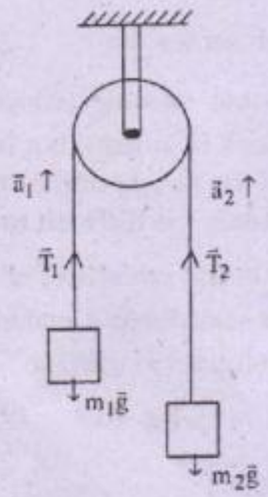
and $$N = W - F\sin\theta = mg - f\sin\theta \quad ...(4.6.43)$$

This shows that that but $N<mg$. This in turn implies as if the weight of the body has been reduced. Hence it is **easier to pull.**

(vi) Atwood machine

In Atwood machine a massless (light) and in extensible strong connecting masses m_1 and m_2 passes over a smooth pulley.

fig. 4.20



Let $m_2 > m_1$. Then m_2 will move downwards with acceleration $\vec{a}_2$ and m_1 will move upwards with acceleration $\vec{a}_1$, such that

$$|\vec{a}_1| = |\vec{a}_2| = a$$

Since the pulley is smooth so the tension along the string is same throughout i.e.

$$|\vec{T}_1| = |\vec{T}_2| = T$$

Now considering mass m_1 the forces acting on m_1 are $\vec{T}_1$ and $m_1\vec{g}$, so

$$\vec{T}_1 + m_1\vec{g} = m_1\vec{a}_1$$

$$\Rightarrow \quad T - m_1 g = m_1 a \qquad ...(4.6.44)$$

Considering mass m_2, the forces acting are $\vec{T}_2$ and $m_2\vec{g}$, so

$$\vec{T}_2 + m_2\vec{g} = m_2\vec{a}_2$$

$$\Rightarrow \quad m_2 g - T = m_2 a \qquad ...(4.6.45)$$

From equations 4.6.44 and 4.6.45 we obtain on adding

$$(m_2 - m_1)g = (m_2 + m_1)a$$

$$\Rightarrow \quad a = \frac{m_2 - m_1}{m_1 - m_2} g \qquad ...(4.6.64)$$

The tension $T = m_1(g+a) = \frac{2m_1m_2}{m_1 + m_2} g$ ...(4.6.47)

Force acting on the pulley by the string is

$$F_s = |\vec{T}_1| + |\vec{T}_2| = 2T = \frac{4m_1m_2}{m_1 + m_2} g \qquad ...(4.6.48)$$

(vii) Apparent weight of a body

It is a common experience that when one moves up in an elevetor he feels heavier, whereas when one moves down in an elevator he feels lighter. This can be explained as follows :

Consider an elevator capable of moving upward and downward. Let a spring scale be placed on its floor to record weight of a body placed on it.

The forces acting on the body are

(i) $m\vec{g} = mg(-\hat{j})$ acting vertically downward

(ii) $\vec{R} = R$, reaction force acting vertically upward.

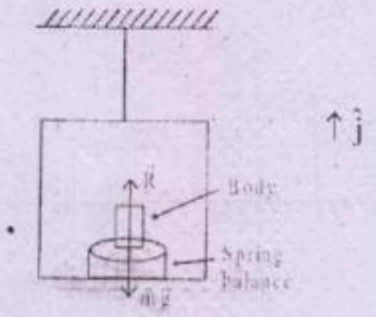


fig. 4.21

(a) *Elevator at rest :* When the elevator is at rest

$$\vec{R} + m\vec{g} = 0$$

$$\Rightarrow \quad R = mg \qquad ...(4.6.49)$$

(b) **Elevator moving up with an acceleration :**

When the elevator is moving up with acceleration $\vec{a}_{LE} = a\hat{j}$ w.r.to earth the spring scale and the body are also moving up with acceleration $\vec{a}_{LE}$. Therefore

$$\vec{R} + m\vec{g} = m\vec{a}_{LE}$$

$$\Rightarrow \quad \vec{R} = m(\vec{a}_{LE} - \vec{g}) = m(a+g)\hat{j}$$

$$\Rightarrow \quad R = m\,(a + g) \qquad ...(4.6.50)$$

The apparent weight of the body, which is equal to the reaction of the spring balance is greater than the true weight mg. Thus when the elevator moves up the weight of the body appears to increase.

(c) *Elevator moving down :*

In this case $\vec{a}_{LE} = -a\hat{j}$, so

$$\vec{R} + m\vec{g} = m\vec{a}_{LE}$$

gives

$$\vec{R} = m(g - a)\hat{j}$$

$$\Rightarrow \quad R = m(g - a) \qquad ...(4.6.51)$$

This shows that the apparent weight, when the elevator moves down, is less than the true weight.

(d) *Elevator falling freely :*

In this case $\vec{a}_{LE} = -g\hat{j}$

$$\therefore \vec{R} = m(\vec{a}_{LE} - \vec{g}) = m(-g\hat{j} + g\hat{j}) = 0$$

$$\Rightarrow \quad R = 0 \qquad ...(4.6.52)$$

i.e. the apparent weight is zero. Thus a freely falling body appears weightless.

(e) Elevator moving down with $|\vec{a}_{LE}| > |\vec{g}|$.

Then $\vec{R} = m(\vec{a}_{LE} - \vec{g})$

$$= m(-a\hat{j} + g\hat{j})$$

$$= m(a - g)(-\hat{j}) \qquad ...(4.6.53)$$

i.e. the body shall move upwards until it touches the ceiling.

(viii) **Pseudo Forces**

Sometimes it is required to solve a physical problem in a noninertial frame of reference.

Suppose S' - frame moves with a constant acceleration a_o w.r.to S-frame. Then the acceleration of a particle 'p' measured w.r.to S'-frame is $\vec{a}_{ps'} = \vec{a}$, and w.r.to s-frame shall be $\vec{a}_{ps}$. Now $\vec{a}_{s's} = \vec{a}_o$. So

$$\vec{a}_{ps'} = \vec{a}_{ps} + \vec{a}_{ss'} = \vec{a}_{ps} - \vec{a}_{s's}$$

$$\Rightarrow \vec{a} = \vec{a}_{ps} - \vec{a}_o$$

Multiplying both sides by 'm'

$$m\vec{a} = m\vec{a}_{ps} - m\vec{a}_o \qquad ...(4.6.54)$$

Since 's' is an inertial frame so $m\vec{a}_{ps}$ is equal to the sum of all the forces $\vec{F}$ acting on the particle. Hence

$$m\vec{a} = \vec{F} - m\vec{a}_o \qquad ...(4.6.55)$$

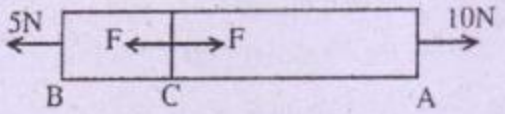
or

$$\vec{a} = \frac{\vec{F} - m\vec{a}_o}{m} \qquad ...(4.6.56)$$

Eqn. (4.6.56) shows that an extra term $m\vec{a}_o$ has been added to sum of all the forces. This term is called <u>pseudoforce</u>

Ex. 4.6.1 · Two forces 10N and 5N are applied at the two ends A and B of a uniform rod of length 3m. Determine the tension acting on a cross-section 2 m from A.

Soln.



Let 'm' kg be mass of rod

mass per meter $= \frac{m}{3}$

mass of portion AC $m_1 = \frac{2}{3}m$

mass of portion BC $m_2 = \frac{m}{3}$.

Acceleration of rod 'a' $= \frac{(10-5)N}{m} = \frac{5}{m} m/s^2$

Eqn. of motion of AC is

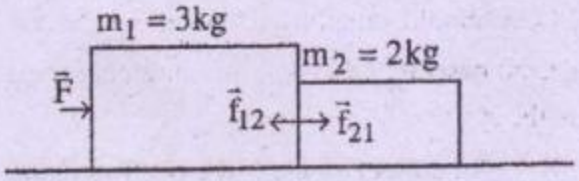
$$F_1 - F = m_1.a$$

$$\Rightarrow \quad 10 - F = \frac{2}{3}ma = \frac{2}{3}.5N = \frac{10}{3}N$$

$$\therefore \quad F = \left(10 - \frac{10}{3}\right)N = \frac{20}{3}N. \cong 6.67N$$

Ex.4.6.2 Two blocks of masses 3kg and 2kg are placed in contact with each other on a frictionless table. Find the force on the common cross-section of contact if a force of 5N is applied on. (i) bigger block (ii) smaller block.

Soln.



Using eqn. (4.6.17) (i) when force is applied on 3kg block

$$\vec{f}_{12} = \vec{f} = m_2\vec{a} = \frac{m_2}{m_1 + m_2}\vec{F}$$

$$\Rightarrow \quad \left|\vec{f}\right| = \frac{2}{3+2} \times 5N = 2N$$

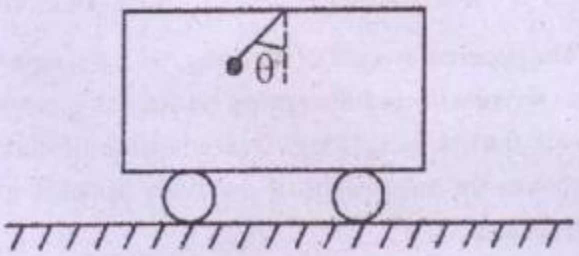
(ii) When force is applied on smaller block

$$\vec{f}' = m_1\vec{a} = \frac{m_1}{m_1 + m_2}\vec{F}$$

$$\Rightarrow \quad \left|\vec{f}'\right| = \frac{3}{3+2} \times 5N = 3N$$

Ex. 4.6.3 A pendulum is hanging from the ceiling of a car having an acceleration , w.r.to the road. Find the angle made by the string with the vertical.

Soln.



The car is moving with accl $\vec{a}_0$

It is an accelerated frame, hence non-inertial.

Therefore in this frame

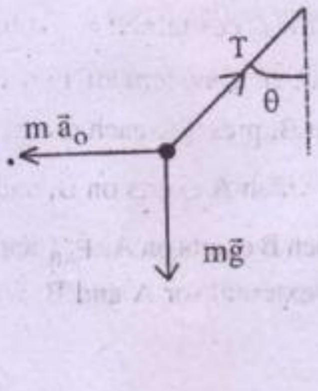
$$\vec{a} = \frac{\vec{F} - m\vec{a}_0}{m}$$

The forces acting on the bob are

(i) wt $m\vec{g}$ acting vertically downwards

(ii) Tension $\vec{T}$ along the strong

(iii) Pseudo force $m\vec{a}_o$



Now resolving along horizontal and vertical dir^n.

$$mg = T\cos\theta$$
$$ma = T\sin\theta - ma_o$$

Since the pendulum is at rest w.r.to the car, so a= o, and we have

$$T\sin\theta = ma_o$$
$$T\cos\theta = mg$$

This gives $\tan\theta = \frac{a_o}{g}$

or $\theta = \tan^{-1}\left(\frac{a_o}{g}\right)$.

4.7 Newton's 2nd law is the real law of motion :

It can be shown that Newton's 1st law is contained in 2nd law and 3rd law is contained in 1st law, thus implying that Newton's 2nd law is the real law or key to laws of motion.

(i) 1st law is contained in 2nd law :

According to 2nd law of motion

$$\vec{F} = \frac{d\vec{p}}{dt} = m\vec{a}$$

Hence if no external force acts on the system (i.e. for an isolated system) the acceleration $\vec{a} = 0$ (since $m \neq 0$). Zero acceleration means no change in velocity. This in turn implies that if a body is initially at rest then it will continue to be at rest, and if a body is in motion it will keep on moving with same velocity (with magnitude and direction intact). This is the content of Newton's 1st law of motion.

Thus we conclude that 1st law is contained in 2nd law.

(ii) 3rd law is contained in 1st law

Consider a system of two identical bodies A and B, pressing each other. Let $\vec{F}_{AB}$ be the force which A exerts on B; and $\vec{F}_{BA}$ be the force which B exerts on A. $\vec{F}_{AB}$ and $\vec{F}_{BA}$ are respectively external for A and B, when they are considered separately. But when A and B are considered as constituents of the system consisting of A and B, the forces $\vec{F}_{AB}$ and $\vec{F}_{BA}$ are to be considered as internal forces.

Now before pressing each other let their c.o.m be G. When they press

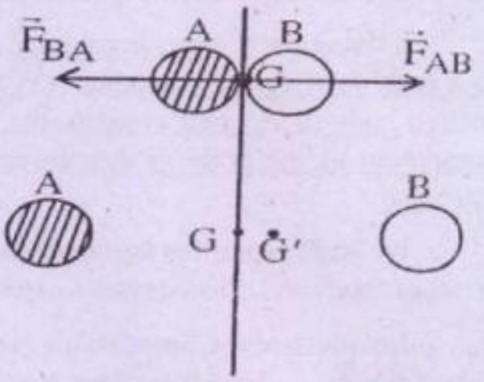


fig. 4.22

each other the acceleration induced are

$$\vec{a}_A = \frac{\vec{F}_{BA}}{m}, \vec{a}_B = \frac{\vec{F}_{AB}}{m}$$

If the two forces are unequal in magnitude (i.e. $|\vec{F}_{BA}| \neq \vec{F}_{AB}$) then $|\vec{a}_A| \neq |\vec{a}_B|$ as a result of which A and B will be displaced through unequal distances from G. Hence there shall be a shifting of C.O.M. But this is contrary to 1st law. Because according to 1st law so long as no external force acts on the system the C.O.M should remain unchanged and in the present case $\vec{F}_{AB}$ and $\vec{F}_{BA}$ are internal forces for the system.

On the otherhand if the two forces are equal in magnitude i.e. $|\vec{F}_{BA}| = |\vec{F}_{AB}|$ then $|\vec{a}_A| = |\vec{a}_B|$.and A & B will be displaced through equal distances from G; causing no change in the position of C.O.M. This is in accordance with.1st law.

Thus the above discussion shows that action and reaction forces are equal in magnitude and opposite in direction. This is the content of 3rd law of motion. Hence we conclude that 3rd law follows from 1st law.

Thus we see that 2nd law is the real law or the cream law.

4.8 Conservation of linear momentum :

It states that if net external force acting on a body or a system vanishes (i.e. for an isolated body or isolated system), the linear momentum of the body or system remains conserved.

We shall prove this law separately for an isolated body and isolated system as follows :

(i) *Isolated body :* Consider an isolated body of mass 'm'. The external force acting on the body is zero. Hence applying Newton's 2nd law of motion

$$\vec{F} = \frac{d\vec{p}}{dt} = 0$$

$\Rightarrow \quad \vec{p} =$ constant in time.

Thus law of conservation of linear momentum is proved for an isolated body.

Since $\vec{p} = p_x\hat{i} + p_y\hat{j} + p_z\hat{k}$ and $\hat{i}, \hat{j}, \hat{k}$ are constant unit vectors, so when $\vec{F} = 0$,

$p_x =$ constant

$p_y =$ constant

$p_z =$ constant

(ii) *Isolated system :* Consider a system of particles of masses $m_1, m_2, m_3 .... m_n$ and moving respectively with velocities $\vec{\upsilon}_1, \vec{\upsilon}_2, \vec{\upsilon}_3 .... \vec{\upsilon}_n$ at time t. Then the linear momentum $\vec{P}_s$ of the system (of particles) is given by $\quad \vec{P}_s = \vec{P}_1 + \vec{P}_2 + \vec{P}_3 + ... + \vec{P}_n$

$$= m_1\vec{\upsilon}_1 + m_2\vec{\upsilon}_2 + m_3\vec{\upsilon}_3 + ... + m_n\vec{\upsilon}_n$$

$$= \vec{P}_{cm} \qquad ...(4.8.1)$$

Therefore

$$\frac{d\vec{p}_s}{dt} = \frac{d\vec{p}_1}{dt} + \frac{d\vec{p}_2}{dt} + ... + \frac{d\vec{p}_n}{dt}$$

$$= \vec{f}_1 + \vec{f}_2 + ... + \vec{f}_n$$

$$= \sum_{i=1}^{n} \vec{f}_i = \vec{f}_{net} \qquad ...(4.8.2)$$

Where $\vec{f}_1, \vec{f}_2 .... \vec{f}_n$ are the external forces acting respectively on particles of masses $m_1, m_2 .... m_n$ and $\vec{f}_{net}$ is the net external force acting on the system. So if $\vec{f}_{net} = 0$, then

$$\frac{d\vec{p}_s}{dt} = \frac{d\vec{p}_{cm}}{dt} = 0$$

$\Rightarrow \quad \vec{P}_s = \vec{P}_{cm} =$ constant in time.

Thus law of conservation of linear momentum is proved for a system of particles.

Applications

(i) *Recoil of a gun :* A gun and a bullet form an isolated system. On firing the gun, the bullet moves forward while the gun recoils back. This can be explained as follows.

Let mass of bullet be m and mass of gun be M. The velocity of bullet and gun before firing is zero. Let the velocities of bullet and gun be $\vec{\upsilon}$ and $\vec{V}$ respectively after firing. So before firing, momentum of the system = M.0+m.0=0 and after firing, momentum of the system = $M\vec{V} + m\vec{\upsilon}$. Since the system is isolated, so linear momentum has to be conserved. Hence

$$M\vec{V} + m\vec{\upsilon} = 0$$

$$\Rightarrow \quad \vec{V} = -\frac{m}{M}\vec{\upsilon} \qquad ...(4.8.3)$$

Equation 4.8.2 indicates that the direction of motion of gun is opposite to that of the bullet.

Further if m<<M, then $|\vec{V}| << |\vec{\upsilon}|$ i.e. a heavier gun recoils mildly.

(ii) **Rocket propulsion :**

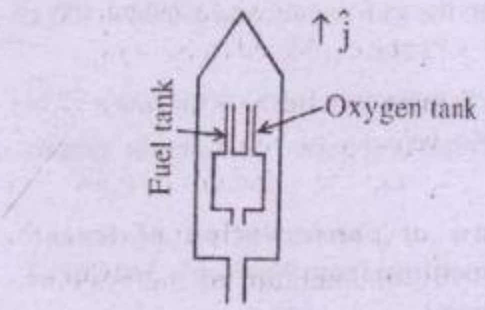
In a rocket, fuel and oxygen are burnt in an ignition chamber and hot gases escape through a rear opening. As a result the rocket moves forward.

Let the mass of the rocket with fuel at any time t, be M. Let its upward velocity at this instant be $\vec{\upsilon}_{RE}$. Let ΔM mass of gas be ejected out in time Δt (small). Let υ_{gE} be velocity of gas ejected out. Then velocity of gas w.r.to rocket shall be

$$\vec{\upsilon}_{gR} = \vec{\upsilon}_{gE} - \vec{\upsilon}_{RE} = \upsilon_{gR}(-\hat{j}) \qquad ...(4.8.4)$$

When the gas is ejected out let the change in velocity of the rocket be $\Delta\vec{\upsilon}_{RE}$. Then change in momentum of the system of rocket and gas shall be

fig. 4.23



$$\Delta\vec{p} = (M-\Delta M)\left(\vec{\upsilon}_{RE} + \Delta\vec{\upsilon}_{RE}\right) + \Delta M\vec{\upsilon}_{gE} - M\vec{\upsilon}_{RE}$$

= Impulse imparted to the rocket

$$= M.\Delta\vec{\upsilon}_{RE} - \Delta M.\vec{\upsilon}_{RE} + \Delta M.\vec{\upsilon}_{gE} - \Delta M.\Delta\vec{\upsilon}_{RE}$$

Neglecting the 2nd order small term $\Delta M.\Delta\vec{\upsilon}_{RE}$ we obtain

$$\Delta\vec{p} = M\Delta\vec{\upsilon}_{RE} + \Delta M(\vec{\upsilon}_{gE} - \vec{\upsilon}_{RE})$$

$$\Rightarrow \quad \Delta\vec{p} = M.\Delta\vec{\upsilon}_{RE} + \Delta M.\vec{\upsilon}_{gR} \qquad ...(4.8.5)$$

$$\text{So} \quad \frac{\Delta\vec{p}}{\Delta t} M.\frac{\Delta\vec{\upsilon}_{RE}}{\Delta t} + \vec{\upsilon}_{gR}.\frac{\Delta M}{\Delta t} \qquad ...(4.8.6)$$

= Ext. Force acting on the rocket

The only external force acting on the rocket is the gravitational pull due to earth. Hence

$$M\vec{g} = \frac{\Delta\vec{p}}{\Delta t}$$

$$\Rightarrow \quad -Mg\hat{j} = M\frac{\Delta\upsilon_{RE}}{\Delta t}\hat{j} - \frac{\Delta M}{\Delta t}\upsilon_{gR}\hat{j}$$

$$\Rightarrow \quad -Mg = M\frac{\Delta\upsilon_{RE}}{\Delta t}\hat{j} - \frac{\Delta M}{\Delta t}.\upsilon_{gR}\hat{j}$$

$$\Rightarrow \quad M\frac{\Delta\upsilon_{RE}}{\Delta t} = \frac{\Delta M}{\Delta t}.\upsilon_{gR} - Mg \qquad ...(4.8.7)$$

This gives

$$a_{RE} = \frac{\Delta\upsilon_{RE}}{\Delta t} = \frac{\upsilon_{gR}}{M}.\frac{\Delta M}{\Delta t} - g \qquad ...(4.8.8)$$

Equation 4.8.8 gives the acceleration of the rocket. As the rocket moves higher and higher 'g' decreases, hence increases, and this continues until all the fuel is exhausted.

From (4.8.8) we find that

$$d\upsilon_{RE} = \upsilon_{gR}\frac{dM}{M} - gdt$$

Integrating

$$\int d\upsilon_{RE} = \upsilon_{gR}\int_{M_0}^{M}\frac{dM}{M} - g\int dt$$

$$\Rightarrow \quad \upsilon - \upsilon_o = \upsilon_{gr}(\log_e M - \log_e M_o) - gt$$

$$\Rightarrow \quad \upsilon - \upsilon_o = \upsilon_{gR} \log_e \left(\frac{M}{M_o} \right) - gt \quad ...(4.8.9)$$

If the initial velocity of the rocket be zero, then $(\upsilon_o = 0)$ and

$$\upsilon = \upsilon_{gR} \log_e \left(\frac{M}{M_o} \right) - gt \quad ...(4.8.10)$$

However after the rocket escapes earth's attraction g=0 and then equation (4.8.10) reduces to

$$\upsilon = \upsilon_{gR} \log_e \left(\frac{M}{M_o} \right) \quad ...(4.8.11)$$

The burnt out speed 'υ'_b of the rocket is the speed acquired by the rocket when whole of the fuel gets burnt; and is the maximum speed of the rocket. So

$$\upsilon_b = \upsilon_{gR} \log_e \left(\frac{M_c}{M_o} \right) \quad ...(4.8.12)$$

Where 'M_c' is mass of the empty container.

(iii) **Explosion of a bomb :** Momentum of the bomb before emplosion is zero. Since there is no external force acting, so after explosion the fragments fly in such directions that the vector sum of their momenta is zero. The momenta of the fragment can be represented by the sides of a closed polygon.

(iv) When a man jumps from a boat, the boat is pushed backward. This can be explained as follows. Before jumping momentum of the system of man and boat is zero. Since no external force acts on the system so after jumping the momenta or the system should be zero. Therefore

$$m\vec{\upsilon} + M\vec{V} = 0 \quad ...(4.8.13)$$

$$\Rightarrow \quad \vec{V} = -\frac{m}{M}\vec{\upsilon} \quad ...(4.8.14)$$

Where m is mass of the man and $\vec{\upsilon}$ is velocity of man; M is mass of the boat and $\vec{V}$ is velocity of boat. Equation (4.8.14) shows that the boat is pushed backward. If the boat is heavy, then the pushing shall be less.

Ex. 4.8.1 A gun of mass 10 kg fires a bullet of 20 g with a velocity of 300 m/s. With what velocity does the gun recoil ?

Soln.

$$0 = m\vec{\upsilon} + m\vec{V}$$

$$\vec{V} = -\frac{m}{M}\vec{\upsilon},$$

Recoil velocity

$$\vec{V} = \frac{m}{M}\upsilon = \frac{20g}{10 \times 10^3 g} \times 300 m/s$$

$$V = 0.6 m/s$$

Ex. 4.8.2 A machine gun fires 30 g bullet at the rate of 6 bullet per second with a speed of 400 m/s. What force in newton is needed to keep the gun in position.

Soln.

Mass of each bullet m=30 g = 0.03 kg. Momentum of each bullet = m υ =0.03 x 400 kg m/s. Momentum carried by bullets per second = 0.03 x 400 x 6 kg m/s^2. So momentum imparted to the gun per second = 0.03 x 400 x 6 kg. m/s^2 =72 N.

Hence reaction force on the gun = 72 N

So 72 N force is required to keep the gun in position.

4.9 Law of conservation of linear momentum from Newton's 3rd law of motion :

Consider two bodies A and B with masses m_1 and m_2 respectivety. Let A have velocity $\vec{u}_1$ and B have velocity $\vec{u}_2$ before they collide. Let their velocities after collision be $\vec{\upsilon}_1$

and $\vec{\upsilon}$ respectively. Let A exert a force $\vec{F}_{AB}$ on B and B exert force $\vec{F}_{BA}$ on A. Then as per 3rd law.

$$\vec{F}_{AB} = -\vec{F}_{BA} \qquad ...(4.9.1)$$

But according to 2nd law

$$\vec{F}_{BA} = \frac{m_1\vec{\upsilon}_1 - m_1\vec{u}_1}{\Delta t} \qquad ...(4.9.2)$$

and $$\vec{F}_{AB} = \frac{m_2\vec{\upsilon}_2 - m_2\vec{u}_2}{\Delta t} \qquad ...(4.9.3)$$

So using (4.9.2) and (4.9.3) in (4.9.1)

$$\frac{m_2\vec{\upsilon}_2 - m_2\vec{u}_2}{\Delta t} = -\frac{m_1\vec{\upsilon}_1 - m_1\vec{u}_1}{\Delta t}$$

$$\Rightarrow \quad m_2\vec{\upsilon}_2 - m_2\vec{u}_2 = -m_1\vec{\upsilon}_1 - m_1\vec{u}_1$$

$$\Rightarrow \quad m_1\vec{\upsilon}_1 + m_2\vec{\upsilon}_2 = m_1\vec{u}_1 + m_2\vec{u}_2 \quad ...(4.9.4)$$

Equation (4.9.4) shows that Initial momentum of the system = final momentum of the system i.e. law of conservation of momentum is proved.

4.10 Newton's 3rd law from law of conservation of momentum

Consider two bodies A and B of masses m_1 and m_2 respectively. Let the two bodies possess momenta $\vec{P}_{1i} = m_1\vec{u}_1$ and $\vec{P}_{2i} = m_2\vec{u}_2$ before collision. Let them acquire momenta $\vec{P}_{1f} = m_1\vec{\upsilon}_1$ and $\vec{P}_{2f} = m_2\vec{\upsilon}_2$ after collision. Let the time during which collision takes place be Δt. Then according to law of conservation of momenta

$$\vec{P}_{1i} + \vec{P}_{2i} = \vec{P}_{1f} + \vec{P}_{2f} \qquad ...(4.10.1)$$

$$\Rightarrow \quad \vec{P}_{1f} - \vec{P}_{1i} = \vec{P}_{2i} - \vec{P}_{2f} = -(\vec{P}_{2f} - \vec{P}_{2i}) \qquad ...(4.10.2)$$

Therefore

$$\frac{\vec{P}_{1f} - \vec{P}_{1i}}{\Delta t} = -\frac{\vec{P}_{2f} - \vec{P}_{2i}}{\Delta t}$$

$$\Rightarrow \quad \frac{\Delta\vec{P}_1}{\Delta t} = -\frac{\Delta\vec{P}_2}{\Delta 2}$$

$$\Rightarrow \quad \vec{F}_{BA} = -\vec{F}_{AB} \qquad ...(4.10.3)$$

Equation (4.10.3) is the content of Newton's 3rd law.

4.11 Work, Power and Energy

In the previous sections we have discussed how dynamics of a particle can be studied on the basis of Newton's laws of motion. This method is suitable when the force acting on the body is constnat. Because we obtain a constant acceleration $\vec{a} = \vec{F}/m$ through Newton's 2nd law and then through kinematic equations we obtain speed, velocity, position etc. But when the force is not constant, the acceleration is not constant, and it becomes difficult to have knowledge about speed, velocity, position etc. Therefore it is necessary to develop a new approach to deal with motion under a variable force. Hence in the following sections we shall show how dynamics can also be studied on the basis of another vital concept in physics, called energy, which is linked with two more physical quantities known as work and power.

4.12 Work

The meaning of the term work is quite different from the view point of a common man and a physicist.

From a physicist's view point. "Work is said to be done if a force acts on a body, the body suffers displacement and the force has a component in the direction of displacement".

Quantitatively we define

$$dW = \vec{F}.\vec{dr} = \vec{F}.\vec{ds} = (F\cos\theta)ds$$

$$= F(ds\ \cos\theta) \qquad ...(4.12.1)$$

as the work done by a force $\vec{F}$ on the particle which undergoes a small displacement $\vec{ds}(=\vec{dr})$. For finite displacement the total work done is defined as

$$W = \int_A^B \vec{F}.\vec{ds} \qquad ...(4.12.2)$$

Where the integration is to be performed along the path followed.

4.12.(A) Calculation of work (analytic method)

(i) *Constant force :* If the force is constant then equation (4.12.2) reduces to

$$W = \vec{F}.\int_A^B \vec{dr} = \vec{F}.\vec{S} = (F\cos\theta)S = F(S\cos\theta)$$

...(4.12.3)

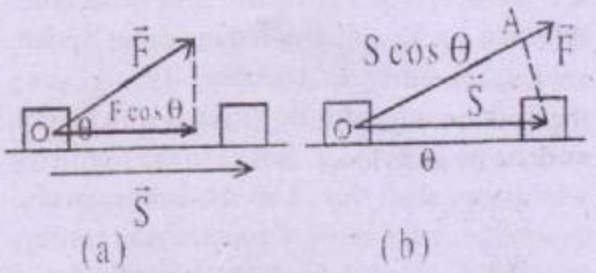


fig. 4.24

Equation (4.12.3) shows that "Work done by a constant force is the product of displacement and component of force in the direction of displacement". Alternatively (i) Work done by a constant force is defined as the product of force and component of displacement in the direction of force (ii) "Work done by a constant force is defined as the dot (scalar) product of force and displacement vectors".

$$w = \vec{F}.\vec{S} \qquad (4.12.3a)$$

Equation (4.12.3) reveals that in case of constant force, work done depends on the positions of A and B and not on the path followed. For example consider the work done by gravitational force on a body of mass m as it moves from A to B.

The work done W=

$$-mg\hat{j}.\vec{AB} = -mg(y_2 - y_1)mgh$$

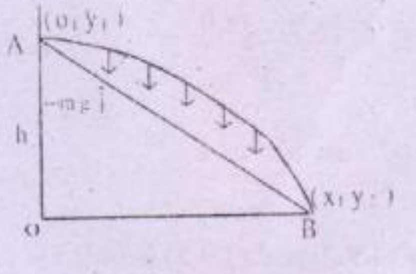


fig. 4.24 (c)

Special Cases :

(i) *Positive Work :* When $'0' < \theta < 90^0$ $\cos\theta > 0$ and force has a component in the direction of displacement. Hence work done by the force is positive.

(b) Further when $\theta = 0$, $\cos\theta = +1$ and W=FS. That is when the body is displaced in the direction of force, work done by the force on the body is positive and maximum.

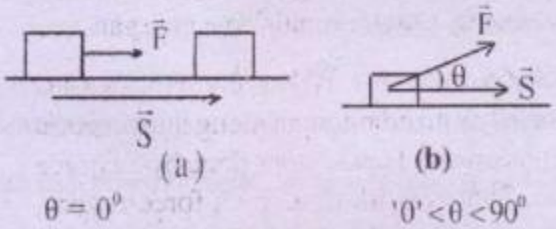


fig. 4.25

For example (1) when a horse pulls a cart, horse does positive work on the cart (2) when a body falls under gravity, workdone by the force of gravity on the body is positive. (3) When a body is lifted the work done by the lifting force is positive (4) when a wire is stretched, the work done by stretching force is positive. (5) In the compression of a gas work done by compressive force is positive.

(ii) *Negative Work :* When $180^0 \geq \theta > 90$, $\cos\theta < 0$ and the force has a component in the opposite direction of displacement. Hence work done by the force is negative. Further when $\theta = 180^0$, $\cos\theta = -1$, and $W = -FS$. That is when the body is displaced in a direction opposite to the direction of force, work done by the body against the force is negative and maximum.

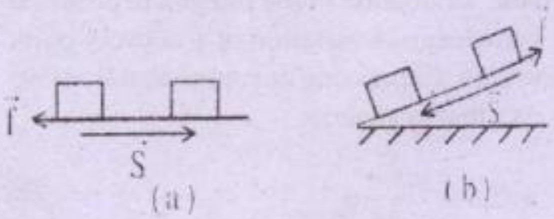


fig. 4.26

For example (1) when a body slides over a rough inclined plane work is done by the body against the force of friction. So we say work done by the force of friction $(\vec{f})$ is negative. (2) When a body is pulled over a rough horizontal sufrace work is done by the body against the force of friction $(\vec{f})$. So work done by the force of friction is negative. (3) When a body is lifted up work is done against gravitational force. So work done by gravitational force is negative.

(iii) *Zero Work :* When $\theta = 90$, $\cos\theta = 0$ and force has no component along the direction of displacement. Hence work done by the force is zero. That is work done by a force is zero, when displacement is at right angles to displacement. For example (1) When a body rotates in a circular path in a horizontal plane, work done by centripetal force is zero. (2) A coolie carrying a luggage on his head from one end of a bus-stop to another end does no work.

Also when $\vec{S} = 0$, W=0, For example (1) A coolie standing with a load on his head does no work (2) A person pushing a rigid wall does no work on the wall.

From the above discussion it is quite apparent that while speaking about work done, one has to mention the body which does work and the body on which work is done. For example while lifting a load the work done by the lifting force on the load is positive. But the load does work against the force of gravity, or we say gravity performs negative work on the body.

(ii) *Several constant forces :*

If a number of constant forces $\vec{F}_1, \vec{F}_2, \vec{F}_3$ .... etc. act on a body and the body undergoes displacement $\vec{S}$, then

$$W = \vec{F}_1.\vec{S} + \vec{F}_2.\vec{S} + \vec{F}_3.\vec{S} + ...$$

$$\Rightarrow W = \left(\vec{F}_1 + \vec{F}_2 + \vec{F}_3 + ...\right)\vec{S} = \vec{F}.\vec{S} \quad ...(4.12.4)$$

Where $\vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + ...$ is the resultant force. Equation 4.12.4 shows that work done by the resultant force is equal to the sum of the work done by individual forces.

(iii) *Variable force :*

When the force acting on a body vary in magnitude, direction or both, the work done given by

$$W = \int_A^B \vec{F}.\vec{dr} = \int_A^B F\cos\theta\ dr \quad ...(4.12.5)$$

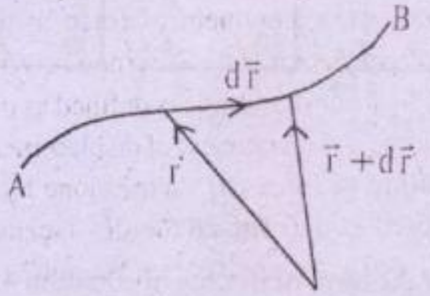


fig. 4.27

Where the integration is to be carried out along the path followed. The evaluation of the integral is possible provided we know how magnitude of force and angle θ vary from point to point along the path.

If a number of variable forces $\vec{F}_1, \vec{F}_2, \vec{F}_3$ ... etc. are acting then

$$W = \int_A^B \vec{F}.\vec{dr} = \int_A^B (\vec{F}_1 + \vec{F}_2 + ...).\vec{dr}$$

$$= \int_A^B \vec{F}_1.\vec{dr} + \int_A^B (\vec{F}_2.\vec{dr} + ...) \qquad (4.12.6)$$

Equation 4.12.6 shows that the total work done by the resultant force is equal to the sum of work done by the individual forces.

F.12. (B) *Graphical Calculation of work :*

In few specific cases work can be calculated by graphical method.

(i) *Constant force :* When the force is constant $W = \vec{F}.\vec{S} = F_x(x_2 - x_1) + F_y(y_2 - y_1) + F_z(z_2 - z_1)$ ...(4.12.7)

Since $\vec{F}$ is constant, so F_x, F_y, F_z are also constant as $\hat{i}, \hat{j}, \vec{k}$ are constant unit vectors. So if we plot $F_x \sim x, F_y \sim y$, and $F_z \sim z$ graph, the graphs shall be straight line as shown in fig. 4.28.

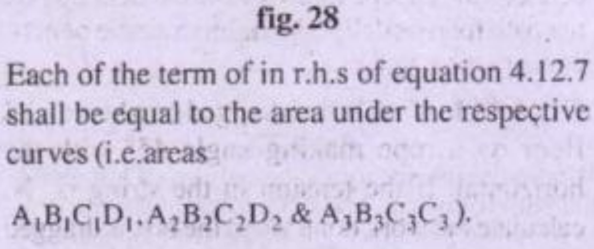


fig. 28

Each of the term of in r.h.s of equation 4.12.7 shall be equal to the area under the respective curves (i.e.areas

$A_1B_1C_1D_1, A_2B_2C_2D_2$ & $A_3B_3C_3C_3$).

If the motion takes place along the direction of force, (say x-dir") then only the 1st term of 4.12.7 shall give the work and is equal to area $A_1B_1C_1D_1$.

If the motion takes place in a plane (say xy-plane) then the 1st two terms shall give the work and is equal to the sum of areas of $A_1B_1C_1D_1$ and $A_2B_2C_2D_2$.

(ii) *Variable force :*

As said earlier when the force is variable, evaluation of the integral depends on the knowledge of variation of $\vec{F}$ at every point of the path. Expressing in component from we observe that in general.

$$W = \int_A^B F_x(x,y,z)dx + \int_A^B F_y(x,y,z)dy + \int_A^B F_z(x,y,z)dz$$

But these integrals cannot be evaluated by graphical method. Only when $\vec{F} = F(x)\hat{i}$ and the motion also takes place along the direction of force, the integral can be evaluated by graphical method. In this case

$$W = \int_{x_1}^{x_2} F(x)dx = Area ABCD$$

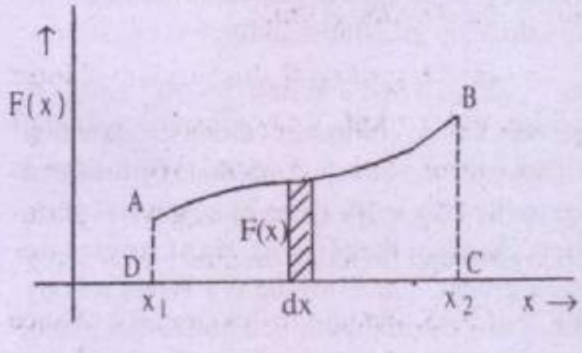


fig. 4.29

4.12 (c) Dimension and Unit of work :

Dimension : Dimension of work is computed from the defining equation 4.12.1 or 4.12.2.

$$[W] = [F]\,[S] = MLT^{-2}.\,L = ML^2T^{-2} \quad ...(4.12.8)$$

(ii) ***Unit :*** From equation 4.12.2 it follows that if $\theta = 0$, S=1 unit and F=1 unit, then W=1 unit, Since force can be expressed in absolute unit as well as gravitational unit so there shall be two types of units for work.

(i) ***Absolute unit***

(a) S.I. system : In this case the unit of force is 1N, unit of displacement is 1m and unit of work done is therefore 1 Nm, called **1 joule** (1 J).

i.e. $1\,J = 1\,N.m = 1\,kg.\,m^2S^{-2}$ ...(4.12.9)

"Work done is said to be 1 joule, if a force of 1N, displaces a body through 1m along the direction of force".

(b) ***C.G.S. Unit*** **:** In this system unit of force is 1 dyne, unit of displacement as 1 cm, unit of work done is therefore 1 dyn.cm called **1 erg**.

i.e. $1\,erg = 1\,dyn.\,cm = 1g.cm^2S^{-2}$ ...(4.12.10)

"Work done is said to be 1 erg if a force of 1 dyne displaces a body through 1cm, along the direction of force."

(ii) ***Gravitational unit :***

(a) S.I. system : In this case unit of force is **1 kg. wt.** or **1 kgf**, unit of displacement is **1 m**, so unit of work is **1kg.m**. "Work done is said to be 1 kg.m if a force of 1 kg wt displaces a body through 1m along the direction of force".

(b) ***C.G.S. system :*** In this case unit of force is 1g wt or 1 gf. unit of displacement is 1cm, therefore unit of work is **1 g.**cm.

"Work done is said ot be 1g.cm if a force of 1g.wt. displaces a body through 1cm along the direction of force".

Relation among the units

1 joule = 1 N.m = 10^7 dyne.cm = 10^7 erg 1 kg. m=9.8 N.m = 9.8 J = 9.8 x 10^7 erg 1 g.cm=980 dyne. cm= 980 erg.

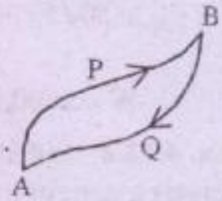
4.12 (d) Work done by Conservative force and non-conservative force :

If a force $\vec{F}$ is conservative then work done by the force around a closed path is zero.

i.e. $$\oint \vec{F}.\vec{ds} = 0 \quad ...(4.12.11)$$

for a conservative force $\vec{F}$.

Eqn. 4.12.11 implies



$$\oint_{APB} \vec{F}.\vec{ds} + \oint_{BQA} \vec{F}.\vec{ds} = 0$$

$\Rightarrow$

$$\int_{APB} \vec{F}.\vec{ds} = -\int_{AQB} \vec{F}.\vec{ds} = \int_{AQB} \vec{F}.\vec{ds}$$

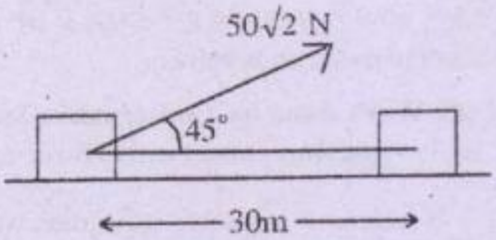
$$\Rightarrow \quad W_{APB} = W_{AQB} \quad ...(4.12.12)$$

Eqn. 4.12.12 implies that the work done on a particle by a conservative force is independent of the path followed; but depends on the end points only. Such a force can depend only on the position of a particle; it does not depend on velocity of the particle or the time explicitly. Gravitational force, spring force, columb force etc. are few examples of conservative force.

On the otherhand if $\oint \vec{F}.\vec{ds} \neq 0$ i.e. $W_{APB} \neq W_{AQB}$, then the force is called non-conservative force and work done depends on the path followed. Friction is an example of non-conservative force.

Ex. 4.12.1 A box is dragged on a horizontal floor by a rope making angle 45^0 with the horizontal. If the tension in the string is N, calculate the work done when the box is dragged through 30 m.

Soln.



$$W = F\ S\cos\theta$$

$$= 50\sqrt{2} \times 30 \times \cos 45$$

$$= 50\sqrt{2} \times 30 \times \frac{1}{\sqrt{2}} \text{Joul}$$

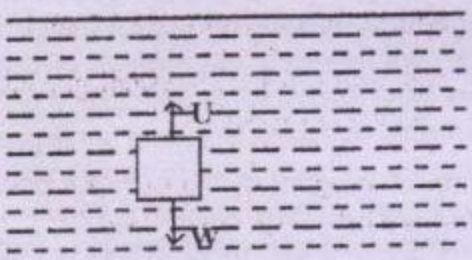
$$W = 1500 \text{ J.}$$

Ex. 4.12.2 Calculate the work done in raising a stone of mass 10 kg of specific gravity 2 immersed in water from a depth of 5m to one metre below the surface.

Soln.

Forces acting on the stone

(i) Weight W= 10 x 9.8 = 98N



(ii) Upward thrust U= V σ g

$$= \frac{10}{\rho} x\sigma x g$$

$$= \frac{10}{(\rho/\sigma)} xg = \frac{10}{2} xg$$

$$= 5 \text{ x } 9.8 = 49 \text{ N}$$

Net force acting = 98 - 49 = 49N down ward
Workdone W= FS = 49 x 4 = 196 J.

4.12.3 A body id subjected to a constant force $\vec{F}$ given by $\vec{F} = (2\hat{i} - 3\hat{j} + 5\hat{k})$ N. Find the work done by this force in displacing the body through a distance of am along z-axis.

Soln.

Work done by the constant force is

$$W = \vec{F}.\vec{S}$$

$$= (2\hat{i} - 3\hat{j} + 5\hat{k}).(4\hat{k}) \qquad [\because \vec{S} = 4\hat{k}]$$

$$= 20 \text{ J}$$

4.12.4 A particle moves along x-axis from x=0 to x=2m under the influence of a force F = $(3x^2 - 2x)$N. Calculate the work done.

Soln.

Work done

$$W = \int_0^2 F dx \text{ [As the force varies with 'x']}$$

$$= \int_0^2 (3x^2 - 2x)\, dx$$

$$= 3\int_0^2 x^2 dx - 2\int_0^2 x dx$$

$$= 3\left[\frac{x^3}{3}\right]_0^2 - 2\left[\frac{x^2}{2}\right]_0^2$$

$$\text{or } w = 3\left[\frac{2^3}{3}\right] - 2\left[\frac{2^2}{2}\right]$$

$$\text{or } w = 3.\frac{8}{3} - 2 \times 2$$

$$w = 4 \text{ J}$$

4.13 Power :

While defining work, the time during which work has been done, is not taken into consideration. But same amount of work may be done in different times by different agencies. For example, a pump may lift 1000 litre of water in 15 mins., another may take 30 min. and some other may take 1 hr. But in all these cases work done is same. In many such instances, time factor is also important. So it is necessary to know the rate at which work is done by the agent. **The rate at which work is done by the agent is called the *Power of the Agent.***

If an agent performs work $\Delta \vec{w}$ in a time interval $\Delta \vec{t}$, then average power is defined as

$$\langle P \rangle = P_{av} = \frac{\Delta w}{\Delta t} \qquad ...(4.13.1)$$

If the rate at which work is done is not constant, (i.e. different amount of work is done in equal time intervals) then we define instantaneous power P as

$$P = \lim_{\Delta t \to 0} \frac{\Delta w}{\Delta t} = \frac{dw}{dt} \qquad ...(4.13.2)$$

Thus instantaneous power is defined as the rate at which work is being done or the time rate of doing work.

Since $\Delta w = \vec{F}.\vec{\Delta s}$

So $\frac{\Delta w}{\Delta t} = \vec{F}.\frac{\vec{\Delta s}}{\Delta t}$

$$\Rightarrow \quad \lim_{\Delta t \to 0} \frac{\Delta w}{\Delta t} = \lim_{\Delta t \to 0} \vec{F}.\frac{\vec{\Delta s}}{\Delta t} = \vec{F}.\vec{v} \quad ...(4.13.3)$$

Equation 4.13.3 shows that power delivered to a particle by a force at any instant (or instantaneous power) is equal to the dot product of instantaneous velocity and the force acting on the body at that instant.

Power is a scalar quantity.

4.13(a) Dimension and unit of power :

I.Dimension :

Eqn. 4.13.1 or 4.13.2 gives

$$[P] = [W]/[T] = ML^2T^{-2}T^{-1} = ML^2T^{-3} \qquad ...(4.13.4)$$

II. Unit of Power :

Power is measured in absolute as well as gravitational unit.

i) Absolute Unit

a) S.I. System :- In S.I. system, absolute unit of power is 1 watt = 1 J/S.

"Power of an agent is said to be 1 watt if it performs 1 joule of work in 1 sec."

b) C.G.S. System :- In C.G.S. system absolute unit of power is 1 erg/s.

"Power of an agent is said to be 1 erg/s, if it performs 1 erg of work in 1 sec."

A commonly used unit of power is Horse power (hp). It is the amount of work done by a normal healthy horse in one second and is equal to 746 watts or 550 ft. lb/s.

1 hp = 746 watt.

(ii) Gravitational Unit :

(a) S.I. system : In S.I. system, gravitational unit of power is 1 kg. m/s.

"Power of an agent is said to be 1 kg.m/s if it performs work of 1 kg. m in 1 sec."

(b) C.G.S. systm : In C.G.S system, gravitational unit of power is 1g. cm/s.

"Power of an agent is said to be 1g. cm/s if it performs work of 1g. cm in 1 sec."

Ex.4.13.1 A car weighing 500 kg is running on a level road with a uniform speed of 72 km/hr. If the frictional resistance is 0.002 kg wt per kg, find the power of the engine in watt.

Soln.

Frictional resistance F = 0.002 x 500 kgwt
= 0.002 x 500 x 9.8 N.

Power $P = \vec{F}.\vec{v} = Fv$

$= 0.002 \times 500 \times 9.8 \times \frac{72 \times 10^3}{3600}$ J/s

$P = 196$ watt

Ex. 4.13.2 Calculate the power of a motor in watt which is capable of raising 200 kg of water in five minutes from a well of 120 m deep.

Soln.

Work done $W = mgh = 200 \times 9.8 \times 120$ J

Time taken $t = 5$ min $= 300$ s.

$$\text{Power } P = \frac{W}{t} = \frac{200 \times 9.8 \times 120}{300}$$

$P = 784$ watt.

Ex. 4.13.3 : A boy pulls 2 kg box horizontally along a straight line at a constant speed of 0.5 m/s by exerting a force of 5N at angle 30^0 with the horizontal. Calculate the average power of the boy.

Soln.

$$P = \vec{F}.\vec{\upsilon} = F\upsilon\cos\theta$$

$$= 5\text{N} \times 0.5\text{m/s} \times \cos 30^0$$

$$= 5 \times 0.5\text{m/s} \times \cos 30^0$$

$$= 5 \times 0.5 \times \frac{\sqrt{3}}{2} = 2.5 \times 866 = 2.1550$$

$P \simeq 2.16$ watt.

4.13.4

A particle moves with a velocity $\vec{V} = \left(6\hat{i} - 4\hat{j} + 3\hat{k}\right) m5^{-1}$ under the influence of a force $\vec{F} = \left(20\hat{i} + 15\hat{j} - 5\hat{k}\right)$N. Find the power.

Soln - Power $P = \vec{F}.\vec{V}$

$$= \left(20\hat{i} + 15\hat{j} - 5\hat{k}\right).\left(6\hat{i} - 4\hat{j} + 3\hat{k}\right)$$

$$= 45 \text{ w}$$

4.14: Energy

Energy of a body is defined as its capacity for doing work. Greater the energy contained in a body, greater the work it can perform. For example, (i) a speedy bullet is able to penetrate a block of wood (ii) A nail is driven into wood, when it is struck with hammer at its head (iii) water falling from a height rotates the turbine of a generator.

Thus energy of a body or agent is measured in terms of the work done by the body or agent. Hence energy is a scalar quantity. It has the same dimension and unit as that of work.

Energy can exist in various forms such as mechanical energy, heat energy, light energy, sound energy, electrical energy, etc. But in the present section we shall consider mechanical energy only.

Mechanical energy is of two types (i) Kinetic energy (ii) Potential energy.

4.14 (A) Kinetic energy :

Energy possessed by a body by virtue of its motion is called Kinetic energy.

OR

Kinetic energy of an object is a measure of the work it can do by virtue of its motion.

For example (i) Kinetic energy of running water is used to run the water mills (ii) Kinetic energy of a hammer is used to drive a nail into the wall, etc.

Expression for K.E. :

Kinetic energy of a body of mass m moving with velocity $\vec{\vartheta}$ can be found in two ways : (I) By calculating the amount of work done against external force before it comes to rest (II) By calculating the amount of work necessary to impart a velocity $\vec{\vartheta}$ to a body of mass m and initially at rest.

I.

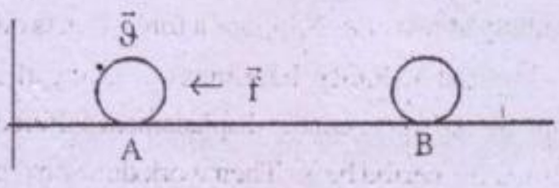


fig. 4.31

Consider a body of mass m, moving on a rough road, which provides a frictional force $\vec{f}$. Suppose the body has velocity $\vec{\upsilon}$ while at A and comes to rest at B (i.e. $\vec{\upsilon} = 0$ at B). In travelling from A to B the body does work against the force of friction ($\vec{f}$) and utilises its energy possessed.

Now work done by the body for a small displacement $\vec{ds}$ in a time interval dt is given by

$$dW = \vec{f}.\vec{ds} = -fds = -\left|\frac{d\vec{p}}{dt}\right||d\vec{s}|$$

$$= -\left|\frac{d}{dt}\left(m\vec{\upsilon}\right)\right||d\vec{s}| = -m\left|d\vec{\upsilon}\right|.\left|\frac{d\vec{s}}{dt}\right|$$

$$\Rightarrow \quad dW = -m\left|d\vec{\upsilon}\right|.\left|\vec{\upsilon}\right| = -m\upsilon d\upsilon \qquad ...(4.14.1)$$

(where we have assumed mass to be constant)

Hence net work done

$$W = -m\int_{\upsilon}^{0}\upsilon d\upsilon = m\frac{1}{2}\upsilon^2$$

Since K.E. is equal to the net work done against friction, so K.E. of body of mass m, moving with a speed υ is $\frac{1}{2}m\upsilon^2$.

$$\text{i.e. } E_k = \frac{1}{2}m\upsilon^2 \qquad ...(4.14.3)$$

It is to be noted, however, that the expression 4.14.3 is valid in non-relativistic limit, where mass is considered to be constant.

II. Consider a body of mass 'm' which is initially at rest at A. Suppose a force $\vec{F}$ acts on it until its velocity becomes $\vec{\upsilon}$ along the direction of force. Let the displacement suffered during the period be $\vec{S}$. Then work done on the body by force $\vec{F}$ is

$$W = \int_A^B \vec{F}.\vec{ds} = \int_A^B \frac{d}{dt}(m\vec{\upsilon}).\vec{ds}$$

$$= \int_A^B d(m\vec{\upsilon}).\frac{d\vec{s}}{dt}$$

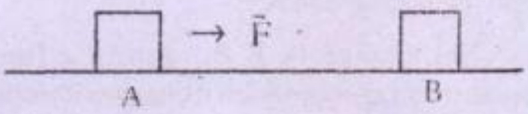


fig. 4.32

$$W = \int_0^{\upsilon} d(m\vec{\upsilon}).\vec{\upsilon} = m\int_0^{\upsilon}\vec{\upsilon}.d\upsilon$$

(where we have assumed mass to be constant)

$$= m\int_0^{\upsilon}\frac{1}{2}d(\vec{\upsilon}.\vec{\upsilon})$$

$$\Rightarrow \quad W = \frac{1}{2}m\upsilon^2 \qquad ...(4.14.4)$$

This work done on the body is stored as K.E.

So K.E. $E_k = \frac{1}{2}m\upsilon^2$ as expressed in 4.14.3.

Work Energy Theorem :

It states that work done on a body or by a body is equal to the change in kinetic energy of the body.

Consider the work dW done during a small displacement $d\vec{s}$, under the action of a force $\vec{F}$. Then

$$dW = \vec{F}.d\vec{s} = \frac{d}{dt}(m\vec{\upsilon}).\vec{ds}$$

$$= d(m\vec{\upsilon})\frac{.d\vec{s}}{dt} = d(m\vec{\upsilon}).\vec{\upsilon}$$

$$= m\vec{\upsilon}.d\vec{\upsilon} = md\left(\frac{1}{2}\upsilon^2\right)$$

(where mass is assumed to be constant)

Thus $dW = \vec{F}.\vec{ds} = dE_k$ ...(14.4.5)

When work is done on the body dW is +ve and therefore dE_k is +ve, implying increase in K.E. On the otherhand when work is done by the body dW is '- ve' and therefore dE_k is '- ve' implying decrease in K.E.

Net change in K.E., during a finite displacement during which it changes its initial velocity $\vec{u}$ to final velocity $\vec{\upsilon}$ is given by

$$W = \int \vec{F}.\vec{ds} = \int dE_k = \Delta E_k = \frac{1}{2} m \int_u^{\vartheta} d(\vartheta^2)$$

$$\Rightarrow W = \Delta E_k = \frac{1}{2} m(\vartheta^2 - u^2) = E_{kf} - E_{ki} \quad ...(4.14.6)$$

Where $E_{kf} = \frac{1}{2} m\upsilon^2$ = final K.E.

$E_{ki} = \frac{1}{2} mu^2$ = initial K.E.

Eqn. 4.14.6 is the mathematical statement of Work-energy theorem.

It is to be noted that when we consider a system of particles the change in K.E. of the system is equal to the work done on the system by external and internal forces.

Dependence of K.E. on frame of reference :

The Kinetic energy even in the non-relativistic limit depends on frame of reference. For example the Kinetic energy of a passanger in a moving train is zero w.r. to a fellow passanger. But w.r. to a person on the ground it is not zero.

Note :

Relation between kinetic energy and momentum.

$$E_k = \frac{1}{2} mv^2 = \frac{m^2 v^2}{2m}$$

or $E_k = \frac{p^2}{2m}$ where p=mv is the momentum of the body

momentum $p = \sqrt{2mE_k}$

Ex. 4.17.1

The kinetic energy of a body is 20 J. What will be its kinetic energy if its momentum is doubled ?

Soln. Let K_1 is the K.E. when momentum is P_1.

Then $K_1 = \frac{P_1^2}{2m} = 20$ J

K_2 is the K.E. When memotum is $P_2 = 2P_1$.

$$\therefore K_2 = \frac{P_2^2}{2m} = \frac{(2P_1)^2}{2m} = 4\frac{P_1^2}{2m} = 4 \times 20\,J = 80J$$

4.14 (B) Potential energy function :

We have shown in sec 4.12 (d) that in case of conservative force

$$\oint \vec{F}.\vec{ds} = 0 \quad ...(4.14.7)$$

However from calculus we know that if U = U (x,y,z) and dU is an exact differential then

$$\oint dU\,(x,y,z) = 0 \quad ...(4.14.8)$$

So comparing 4.14.7 and 4.14.8 we put

$$\vec{F}.\vec{ds} = -dU \quad ...(4.14.9)$$

Where the '-ve' sign is chosen for future convenience. Eqn. (4.14.9) gives

$$\int_A^B \vec{F}.\vec{ds} = W_{AB} = -\int_A^B dU = -(U_B - U_A)$$

$$\Rightarrow U_B - U_A = -\int_A^B \vec{F}.\vec{ds} \quad ...(4.14.10)$$

The functions $U_A(x_1, y_1, z_1)$ and $U_B(x_2, y_2, z_2)$ are called potential energy functions at A and B respectively. The change in potential energy function of a particle in going from A to B along an arbitrary path in a conservative force field is equal to the negative of the work done by the conservative force.

As discussed above, a potential energy function can be associated with a conservative force only. It cannot be associated with non-conservative force.

Potential energy of a particle at a point :

If we choose A as a reference point and measure P.E. relative to the potential energy at A, then we can arbitrarily put $U_A = 0$ and write

$$U_B = -\int_A^B \vec{F}.\vec{ds} \qquad ...(4.14.11)$$

Usually the reference point is chosen where the force is zero. For example in case of gravitation the reference point is chosen at the surface of earth and in case of spring force it is chosen at mean position of rest.

Eqn. 4.14.11 defines potential at a point. That is **the potential energy of a particle at any point in a conservative force field is equal to the negative of the work done on it by the conservative force when the particle moves from a point at zero potential energy to that point.**

It means work done is stored as potential energy which the body can recover and use it to come back to the reference point (the zero potential energy point). For example a brick taken to a roof acquires potential energy by way of work done against gravity and thus possesses more ability to do work than it has while on ground. Similarly a compressed spring possesses potential energy by way of work done against spring force and thus has the ability to come cank to its zero potential energy state.

It also follows from the above discussion that depending on the conservative forces the potential energy is acquired in different manner. In case when work is done against gravity, position of the body is changed only; but in case of spring force, the configuration is changed. So we sometimes define **"Potential energy of a particle is the energy possessed by a body by virtue of its position or configuration".**

Relation between P.E. function and conservative force :

The components of conservative force $\vec{F}$ are related with potential energy function as given below

$$F_x = -\frac{dU}{dx}; F_y = -\frac{dU}{dy}; F_z = -\frac{dU}{dz} \qquad ...(4.14.12)$$

Gravitational Potential energy :

Energy possessed by a body by virtue of its position in a gravitational field is called its gravitational potential energy.

Consider a block of mass m kept near the surface of the earth. Suppose it is raised through a height 'h'. Consider the (earth + block) as the system. The gravitational force between earth and block is internal for the system and is conservative. Hence we can define P.E. corresponding to this force. As the earth is very heavy compared to the block, so acceleration of earth is nearly zero. Hence earth can be safely taken as an inertial frame. The work done by the gravitational force due to block ($= \vec{F}_{BE}$) is zero in this frame. The gravitational force on the block due to the earth is $m\vec{g} = mg(-\hat{j}) = \vec{F}_{EB}$

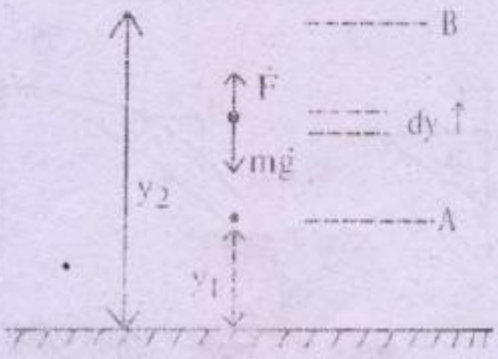


fig. 4.33

So in order to displace the body upwards, work is to be done against the gravitational pull of earth, by applying a force $\vec{F} = -\vec{F}_{BE} = mg\hat{j}$. Work done on the body by the force $\vec{F}$ in displacing the body through dy is given by

$$dW = \vec{F}.\vec{dy} = mg\hat{j}.\hat{j}dy = mgdy \quad ...(4.14.13)$$

Therefore total work done in displacing the body from A to B is given by

$$W = mg\int_{y_1}^{y_2} dy = mg(y_2 - y_1) \quad ...(4.14.14)$$

If the body is raised from ground level (earth's surface) to the point, then $y_1 = 0$ and $y_2 = h$ and eqn. 4.14.14 reduces to

$$W = mgh \quad ...(4.14.15)$$

This work done against the gravitational force of earth is stored as the P.E. of the block and the P.E. of the (earth + block) system increases by mgh. If the block descends then the P.E. of the (earth + block) system decreases by mgh.

Since the earth remains almost fixed, we recognise the P.E. of the (earth + block) system as P.E. of the block only. So we say gravitational P.E. of the block at a height 'h' above the ground is mgh.

Conservative nature of gravitational force :

We show below that the work done in raising a body from the ground level to a height 'h' is independent of the path followed to justify that gravitational force is a conservative force.

Consider a body of mass m. Let it be taken from A to B along the arbitrary path APB.

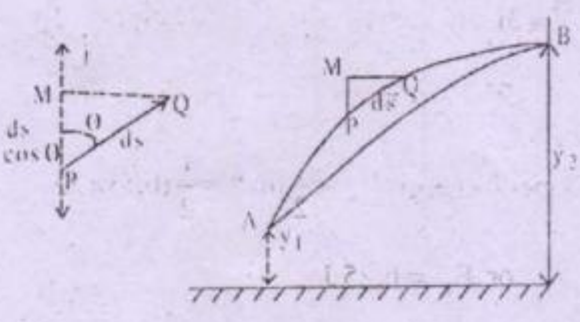


fig. 4.34

Let the body be at P at any instant of time. The force acting on the body at that instant is $\vec{F}_g = m\vec{g} = -mg\hat{j}$, due to gravity. Therefore the lifting force required is $\vec{F} = -\vec{F}_g = +mg\hat{j}$. If the body is displaced through $\vec{ds}$ along the path then work done by the lifting force is

$$dW = \vec{F}.\vec{ds} = mg\hat{j}.\vec{ds} = mg \; ds \cos\theta$$

But $PM = ds \cos\theta = dy$

$$\text{So } dW = mg\,dy \quad ...(4.14.16)$$

Therefore net work done in going from A to B along the path APB is

$$W = mg\int_{y_1}^{y_2} dy = mg(y_2 - y_1) \quad ...(14.14.17)$$

This eqn. 4.14.17 is same as eqn. 4.14.14 and it shows that work done does not depend on the path followed but on the end points only. Hence the gravitational force is conservative.

Potential energy of a spring :

Consider a massless, perfectly elastic spring. Let its natural length be 'ℓ'. Let one of its ends be fixed firmly to a wall and the other end be attached to a block of mass 'm'.

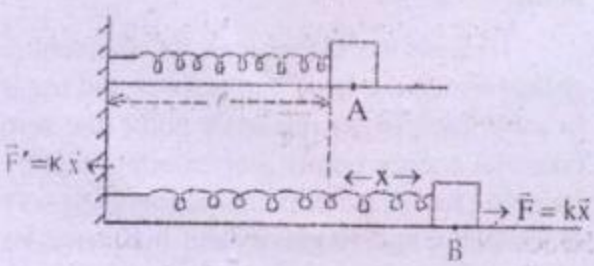


fig. 4.35

When the spring is unstretched let the position of the block be A. The block is slowly pulled on a smooth horizontal surface to extend the spring by a distance x. When it is elongated the tension in the spring is Kx, where K is spring constat. It pulls the wall towards right and the block towards left, by forces of magnitude Kx. The wall also pulls the spring by a force $\vec{F}'$, with $|\vec{F}'| = Kx$. So the forces acting on the spring are (i) Kx towards right by the block and (ii) Kx towards left by the wall.

The work done on the spring by these two forces is

$$F'.0 + \int \vec{F}.\vec{dx} = \int Kx dx = \frac{1}{2}Kx^2 \quad ...(4.14.18)$$

($\because$ the force by the wall does no work as the point of application is fixed)

Thus the total external work done on the spring is $\frac{1}{2}Kx^2$, when the spring is elongated by an amount x from the natural length. This is stored as P.E. of the spring.

When the spring is compressed, the amount of work done shall be also $\frac{1}{2}Kx^2$.

Ex. 4.14.1 A force of 10 N acts on a particle of mass 10 g. for 10 s. Calculate the velocity of the particle and the K.E. produced.

Soln.

Force acting F = 10 N

Mass of the body m = 10 g. = 0.01 kg.

$$accl^n = \frac{F}{m} = \frac{10N}{0.01kg} = 1000 m/s^2$$

Hence $\upsilon = u + at = 0 + 1000\frac{m}{s^2} \times 10.s$

$\upsilon = 10^4 m/s$

$$K.E. = \frac{1}{2}m\upsilon^2 = \frac{1}{2} \times 0.01 \times 10^8 j$$

K.E. = 5×10^5J

Ex. 4.14.2 Compare the K.E. of a football player of mass 70 kg running at 3 m/s with the potential energy of a spectator of mass 35 kg sitting on a gallery 2m above the ground level. ($g = 9.8\ m/s^2$)

Soln.

$$\text{K.E. of football player} = \frac{1}{2}m\upsilon^2$$

$$= \frac{1}{2} \times 70kg \times (3m/s)^2 = 315J$$

P.E. of the spectator = mgh = $35kg \times 9.8\frac{m}{s^2} \times 2$

$= 686$ J

$$\therefore \frac{K.E}{P.E} = \frac{315}{686} = \frac{1}{2.18}$$

$\therefore$ K.E. : P.E. = 1 : 2.18

Ex. 4.14.3 A spring of force constant 100N/m stretches 0.12 m in excess of its equilibrium length. Calculate the P.E. stored in the spring.

Soln.

Given K = 100 N/m

x = 0.12 m

P.E. $U = \frac{1}{2}KX^2 = \frac{1}{2} \times 100 \times (0.12)^2$

$\Rightarrow$ U = 0.72 J

Ex : 4.14.4 A body of mass 500g has a velocity $(3\hat{i} - 4\hat{j})ms^{-1}$. Find its kinetic energy.

Soln- Mass M = 500g = 0.5 kg

$$\vec{V} = 3\hat{i} - 4\hat{j} \Rightarrow V^2 = \vec{V}.\vec{V} = (3\hat{i} - 4\hat{j}).(3\hat{i} - 4\hat{j})$$

or $V^2 = 25\ m^2s^{-2}$

$$\therefore \text{kinetic energy } E_k = \frac{1}{2}mv^2 = \frac{1}{2}(0.5) \times 25$$

or $E_k = 6.25$ J

4.15 Conservation of energy :
(Robert Meyer 1842)

It states that, **"Energy can neither be created nor be destroyed; it can only be transformed from one form to another. The sum total of energy in the universe remains constant."**

The following examples shall illustrate the statement.

(i) When electric current flows through a wire, it gets heated up. It indicates that electrical energy is converted to heat energy in exact equivalance.

(ii) In photo-electricity, incidence of light on certain photo-sensitive metals, results in ejection of electrons. Here light energy is converted to electrical energy.

(iii) In hydro-electric power plants, P.E. of water is converted to K.E. of water, and this in turn rotates the armature of a dynamo, which produces electric current. Thus mechanical energy is converted to electrical energy.

(iv) When heat is supplied to one of the junctions of a thermo couple, electric current starts flowing. Thus heat energy is converted to electrical energy.

(v) When electirc current passes through the armature of a fan, the armature rotates. Thus electrical energy is converted to mechanical energy.

4.16 Conservation of Mechanical energy :

The total mechanical energy of a particle moving in a conservative force field remains constant in time or conserved.

The above statement can be proved from laws of motion.

I. The mechanical energy of a particle in a conservative force field is the sum of its K.E. and P.E.

i.e. $$E = E_k + U = \frac{1}{2} m\upsilon^2 + U(x,y,z) \qquad ...(4.16.1)$$

Therefore

$$\frac{dE}{dt} = m\upsilon\frac{d\upsilon}{dt} + \frac{dU(x,y,z)}{dt} \qquad ...(4.16.2)$$

Since $U = U(x,y,z)$

So $$dU = \frac{dU}{dx}.dx + \frac{dU}{dy}.dy + \frac{dU}{dz}.dz$$

$$\Rightarrow \frac{dU}{dt} = \frac{dU}{dx}.\frac{dx}{dt} + \frac{dU}{dy}.\frac{dy}{dt} + \frac{dU}{dz}.\frac{dz}{dt}$$

$$= \frac{dU}{dx}.\upsilon_x + \frac{dU}{dy}.\upsilon_y + \frac{dU}{dz}.\upsilon_z$$

Using eqn. 4.14.5 on r.h.s.

$$\frac{dU(x,y,z)}{dt} = -F_x\upsilon_x - F_y\upsilon_y - F_z\upsilon_z = -\vec{F}.\vec{\upsilon} \qquad ...(4.16.3)$$

Since by Newton's 2nd Law $\vec{F} = \frac{d}{dt}(m\vec{\upsilon})$

So $$\frac{dU(x,y,z)}{dt} = -m\frac{d\vec{\upsilon}}{dt}.\vec{\upsilon} = -m\upsilon\frac{d\upsilon}{dt} \qquad ...(4.16.4)$$

Using eqn. 4.16.4 in (4.16.2) we obtain

$$\frac{dE}{dt} = m\upsilon\frac{d\upsilon}{dt} - m\upsilon\frac{d\upsilon}{dt} = 0$$

$$\Rightarrow E = \text{constant} \qquad ...(4.16.5)$$

Equation (4.16.5) shows that mechanical energy is constant in time.

The laws of conservation of mechanical energy reveals that during motion K.E. changes to P.E. and vice versa, but net energy remains constant.

II. The above conclusion can also be drawn in this alternative manner.

From work-energy-Theorem we have (see 4.14.5)

$$\Delta E_k = \vec{F}.d\vec{s}$$

and from definition of potential energy function (see eqn. 4.14.9)

$$\vec{F}.d\vec{s} = -\Delta U$$

So we have

$$\Delta E_k = \vec{F}.d\vec{s} = -\Delta U$$

Hence on integration

$$\int_1^2 dE_k = -\int_1^2 dU$$

$$\Rightarrow \quad E_{k2} - E_{k1} = -U_2 + U_1$$

$$\Rightarrow \quad E_{k1} + U_1 = E_{k2} + U_2 \qquad ...(4.16.6)$$

Eqn. (4.16.6) shows that the sum

$$E_k + U = \text{constant}$$

Illustrative Examples :

1. Conservation of mechanical energy by a freely falling body :-

Consider a body of mass 'm' falling freely from a point A at height 'h' above the ground. While falling it crosses an arbitrary point B at a height y above the ground and reaches the point C on the ground.

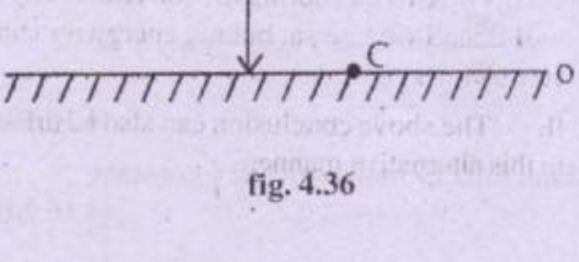


fig. 4.36

Let its velocity at the arbitrary points 'B' be $\vec{\upsilon}$. Then its total energy at this instant is

$$E = \frac{1}{2}m\upsilon^2 + mgy \qquad ...(4.16.7)$$

This gives

$$\frac{dE}{dt} = m\upsilon\frac{d\upsilon}{dt} + mg\frac{dy}{dt}$$

$$= m\frac{d\vec{\upsilon}}{dt}.\vec{\upsilon} + mg(-\upsilon)$$

where we have put $\frac{dy}{dt} = -\upsilon$, since as y decreases υ increases. We re-write the above as

$$\frac{dE}{dt} = \frac{d}{dt}(m\vec{\upsilon}).\vec{\upsilon} - mg\upsilon$$

$$= \vec{F}.\vec{\upsilon} - mg\upsilon = F\upsilon - mg\upsilon$$

But the force acting on the body is force due to gravity (=mg). Hence

$$\frac{dE}{dt} = mg\upsilon - mg\upsilon = 0 \qquad ...(4.16.8)$$

This shows that E = constant. That is the total energy is conserved.

Value of E

At the initial point A, $\upsilon_A = 0$ so total energy is

$$E = E_k + U = 0 + mgh = mgh \qquad ...(4.16.9)$$

At B,

$$\upsilon^2 = 2g(h - y)$$

$$\therefore \quad E = E_k + U = \frac{1}{2}m\upsilon^2 + mgy$$

$$= \frac{1}{2}m.\ 2g(h - y) + mgy$$

$$\Rightarrow \quad E = mgh$$

At C, $\upsilon_c^2 = 2gh$

$$E = E_k + U = \frac{1}{2}m\upsilon_c^2 + 0 = \frac{1}{2}m.2gh$$

$\Rightarrow$ E= mgh

Thus everywhere the body possesses a constant energy mgh.

The inter-convertibility of P.E. and K.E. can be known from the graphs P.E. ~ y and K.E. ~ y as shown below. Both K.E. and P.E. varies linearly with y; with K.E. increasing and P.E. decreasing as the body falls.

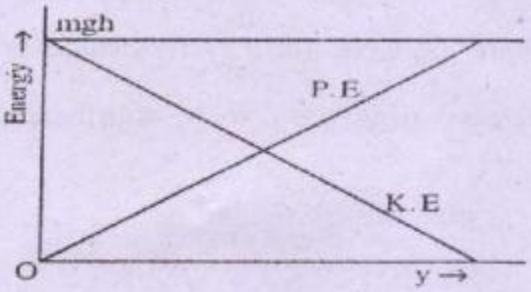


fig. 4.37

2. Conservation of energy by Simple pendulum :

Consider a simple pendulum, oscillating in a vertical plane by an inelastic thread of length ℓ.

The equation of motion for simple pendulum is given by

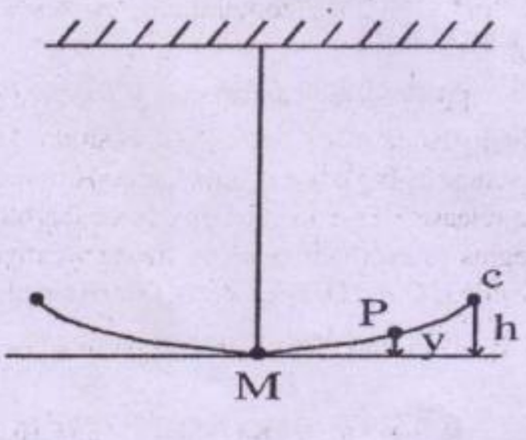


fig. 4.38

$$-mg\sin\theta = ma = m\ell\frac{d^2\theta}{dt^2} \quad ...(4.16.10)$$

Which for small θ reduces to

$$\frac{d^2\theta}{dt^2} = -\frac{g}{\ell} = -\omega^2\theta \quad ...(4.16.11)$$

This admits solution

$$\theta = A\sin\omega t$$

Suppose the simple pendulum has velocity ϑ at any arbitrary position P, which is at height y, w.r. to the mean position M. Then

$$y = \ell(1-\cos\theta) - \ell\frac{\theta^2}{2} \quad ...(4.16.13)$$

and velocity is given as

$$\upsilon = l\theta = A\omega l\cos\omega t = \omega l\sqrt{A^2 - \theta^2} \quad ...(4.16.14)$$

The energy of the simple pendulum at this arbitrary position P is given by

$$E = E_k + U = \frac{1}{2}m\upsilon^2 + mgy \quad ...(4.16.15)$$

where P.E. 'U' is measured w.r.to the reference point M, the mean position. Differentiating (4.16.15) w.r.to time we have

$$\frac{dE}{dt} = m\upsilon\frac{d\upsilon}{dt} + mg\frac{dy}{dt}$$

$$= m\upsilon\, a + mg\,\dot{y} \quad ...(4.16.16)$$

From (4.16.13) $\dot{y} = \ell\theta\,\dot{\theta}$ ...(4.16.17)

and from (4.16.10) $a = \ell\frac{d^2\theta}{dt^2} = -g\theta$ ...(4.16.18)

Using eqns. 4.16.14, 17 & 4.16.18 in eqn. 4.16.16 we obtain

$$\frac{dE}{dt} = m\ell\theta(-g\theta) + mg\,\ell\theta\,\dot{\theta} = 0 \quad ...(4.16.19)$$

Eqn. 4.16.19 implies that E = constant ...(4.16.20)

Value of E

At any point P, $E = \frac{1}{2}m\upsilon^2 + mgy$

Using eqn. (4.16.14), (4.16.13) and (4.16.12) on r.h.s we have

$$E = \frac{1}{2}mA^2\omega^2\ell^2\cos^2\omega t + mg\ell\frac{\theta^2}{2}$$

$$= \frac{1}{2}mA^2\omega^2\ell^2\cos^2\omega t + \frac{1}{2}mg\ell A^2\sin^2\omega t$$

$$= \frac{1}{2}mA^2\omega^2\ell^2\cos^2\omega t + \frac{1}{2}m\ell^2\omega^2A^2\sin^2\omega t$$

$$E = \frac{1}{2}mA^2\omega^2\ell^2 \qquad ...(4.16.21)$$

Since when $\theta = A$, $y = h$, (maximum height) so

$$h = \ell\frac{A^2}{2} \qquad ...(4.16.22)$$

Using (4.16.22) in (4.16.21) we obtain

$$E = \frac{1}{2}m\frac{2h}{\ell}.\frac{g}{\ell}.\ell^2 = mgh \qquad ...(4.16.23)$$

At the mean position P.E. = 0

So $E = \frac{1}{2}m\,\upsilon_m^2 = \frac{1}{2}m(\omega\ell A)^2 = mgh$

At the extreme position 'C', we have $\upsilon = 0$, so K.E. = 0, and

$E = 0 + mgh = mgh$

Thus the total energy is mgh everywhere.

We further see that

$$K.E. = E_k = \frac{1}{2}m\upsilon^2 = \frac{1}{2}m\omega^2\ell^2(A^2 - \theta^2)$$

$$= \frac{1}{2}m\omega^2\ell^2(A^2 - \frac{2y}{\ell})$$

$$= mgh - m\omega^2\ell y = mgh - mgy$$

$$E_k = mg(h - y)$$

and

$$P.E. = U = mgy$$

So their variation with y is as shown below.

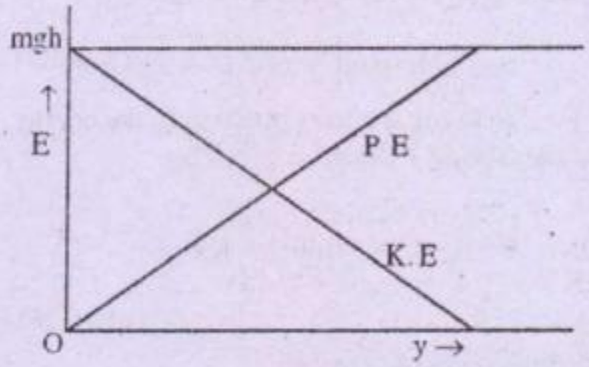


fig. 4.39

3. Conservation of energy by loaded spring:

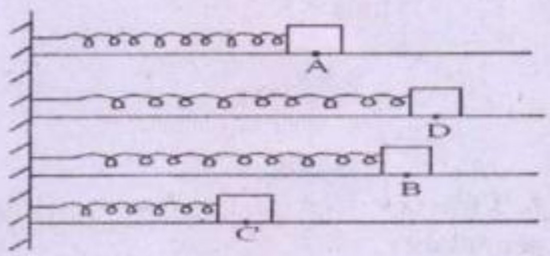


fig. 4.40

Consider a spring of spring constant k whose one end is rigidly fixed to a wall and the other end is attached to a block of mass m, which can move on a horizontal plane (as shown in fig. 4.40)

Let A be the equilibrium position of the block. Let the block be pulled to position D at a distance 'a' from the equilibrium position and be released. Due to restoring force the block begins to oscillate between the two extreme positions C and D. The force acting on the block is $\vec{F}_e = -k\vec{x}$ and the equation of motion is

$$m\frac{d^2x}{dt^2} = ma = -Kx \qquad ...(4.16.24)$$

This admits a solution

$$x = a\sin\omega t \qquad ...(4.16.25)$$

where $\omega = \sqrt{\dfrac{K}{m}}$ ...(4.16.26)

This gives velocity at any time 't' as

$$\upsilon = a\omega\cos\omega t = \omega\sqrt{a^2 - x^2} \quad ...(4.16.27)$$

So at any arbitrary position B, the energy of the (spring + block) is given by

$$E = E_k + U = \frac{1}{2}m\upsilon^2 + \frac{1}{2}Kx^2 \quad ...(4.16.28)$$

Differentiating w.r.to time

$$\frac{dE}{dt} = m\upsilon\frac{d\upsilon}{dt} + Kxx$$

$$= m\upsilon a + Kx\upsilon$$

$$= m\upsilon\left(\frac{Kx}{x}\right) + Kx\upsilon$$

$$\Rightarrow \quad \frac{dE}{dt} = 0 \quad ...(4.16.29)$$

This shows that total energy

$$E = \text{constant} \quad ...(4.16.30)$$

Value of E

At any position B,

$$E = \frac{1}{2}m\upsilon^2 + \frac{1}{2}Kx^2$$

Using eqn. (4.16.27) on r.h.s

$$E = \frac{1}{2}m\omega^2(a^2 - x^2) + \frac{1}{2}Kx^2$$

$$= \frac{1}{2}Ka^2 - \frac{1}{2}Kx^2 + \frac{1}{2}Kx^2$$

$$\Rightarrow \quad E = \frac{1}{2}Ka^2 \quad ...(4.16.31)$$

At the extreme position D, $\upsilon = 0$ so $E_k = 0$, and

$$E = 0 + \frac{1}{2}Ka^2 = \frac{1}{2}Ka^2$$

At the equilibrium position A, $\upsilon = \upsilon_A$ given by $\upsilon_A = \omega a$, and P.E. = 0

So $\quad E = \frac{1}{2}m\upsilon_A^2 + 0 = \frac{1}{2}m\omega^2 a^2$

$$\Rightarrow \quad E = \frac{1}{2}Ka^2$$

Thus we find that the total energy of a loaded spring remains constant and has the value $\frac{1}{2}Ka^2$.

The variation of K.E. and P.E. is as shown 4.41 in fig.

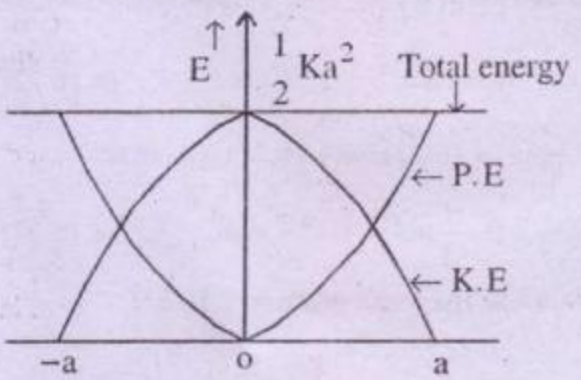
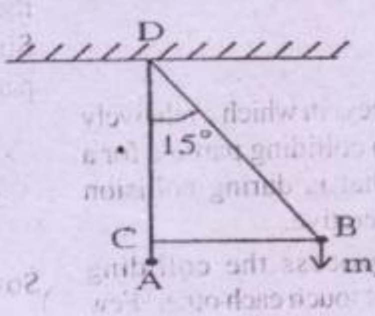


fig. 4.41

Ex. 4.16.1 The bob of a pendulum has its rest point 1m below the support. The bob is pulled aside until the string makes an angle of 15° with the vertical. Upon releasing, with what speed does the pendulum swing past the rest point ?



Soln.

At A $E=\frac{1}{2}m\upsilon^2$

At B $E = mgh = mg.\ AC$

$= mg(\ell - \ell\cos 15^0)$

$\therefore \frac{1}{2}m\upsilon^2 = mgl(1\text{-}\cos 15^0)$

$\Rightarrow \quad \upsilon^2 = 2gl(1\text{-}\cos 15^0) = 2g.(1\text{-}\cos 15^0)$

$\upsilon = \sqrt{2g(1-\cos 15^0)} = \sqrt{2\times 9.8(1-\cos 15)}$

$\Rightarrow \quad \upsilon = 0.817 \text{m/s}$

Ex.4.16.2 A 2 kg block is dropped from a height of 40 cm on to a spring of force constant 1960 N/m. Find the maximum distance the spring will be compressed.

Soln.

Velocity after fall is

$$\upsilon = \sqrt{2gh} = \sqrt{2\times 9.8\times 0.4\text{m/s}}$$

K.E. of the block $= \frac{1}{2}m\upsilon^2$

P.E. of the spring $= \frac{1}{2}kx^2$

From conservation of energy

$$\frac{1}{2}m\upsilon^2 = \frac{1}{2}kx^2$$

$$\Rightarrow \quad x = \sqrt{\frac{m}{k}\upsilon^2} = \sqrt{\frac{2}{1960}\times 2\times 9.8\times 0.4\text{m/s}}$$

$x = 0.089$ m.

4.17 COLLISION :

Collision is a process in which a relatively large force acts on each colliding particle for a short time interval. That is during collision impulsive forces are operative.

In a collision process the colliding particles may or may not touch each other. Few common examples in which the colliding particles touch each other are :

i) Collision between bat and a base ball.

ii) Collision between a scooter and a car.

iii) Collision between a truck and a tree etc.

In such cases both the colliding particles get deformed and then restore their original shape and size either fully or partially.

But in processes like

i) Scattering of α -particles by a nucleus

ii) Deflection of path of a comet when it approaches sun.

the colliding particles donot touch each other. These cases are called collision because impulsive forces operate.

Conservation of linear momentum in collision :

During collision the colliding particles exert large forces on each other. At any instant let $\vec{F}_{21}$ be force of particle 2 on particle 1; and $\vec{F}_{12}$ be force of particle 1 on 2. By Newton's 3rd law

$$\vec{F}_{12} = -\vec{F}_{21} \qquad ...(4.17.1)$$

The change in linear momentum of particle 1 is given by

$$\Delta\vec{p}_1 = \int_{t_1}^{t_2}\vec{F}_{21}dt = \langle\vec{F}_{21}\rangle.\Delta t \qquad ...(4.17.2)$$

where Δt is the time of collision and $\langle\vec{F}_{21}\rangle$ is the average force during the collision time. Similarly the change in linear momentum of particle 2 is given by

$$\Delta\vec{p}_2 = \int_{t_1}^{t_2}\vec{F}_{12}dt = \langle\vec{F}_{12}\rangle.\Delta t \qquad ...(4.17.3)$$

So change in momentum of the system is

$$\vec{\Delta p}_1 + \Delta p_2 = \left(\langle \vec{F}_{21} \rangle + \langle \vec{F}_{12} \rangle\right)\Delta t = 0$$

...(4.17.4)

$(\because \vec{F}_{12} = -\vec{F}_{21} \Rightarrow \langle \vec{F}_{12} \rangle_{\Delta t} = -\langle \vec{F}_{21} \rangle_{\Delta t})$

If we consider the two colliding particles as constituting an isolated system, then momentum of the system before collision is

$$\vec{p} = \vec{p}_1 + \vec{p}_2$$

and momentum of the system after collision is

$$\vec{p}' = \left(\vec{p}_1 + \vec{\Delta p}_1\right) + \left(\vec{p}_2 + \vec{\Delta p}_2\right)$$

$$= (\vec{p}_1 + \vec{p}_2) + \left(\vec{\Delta p}_1 + \vec{\Delta p}_2\right)$$

$$\Rightarrow \vec{p}' = \vec{p}_1 + \vec{p}_2 = \vec{P} \qquad ...(4.17.5)$$

Eqn. 4.17.5 implies that linear momentum is conserved in the collision.

In the above we have taken into consideration only the impulsive forces and neglected other external forces. The effect of other external forces are infact comparatively negligible. For example in the collision between bat and ball, the gravitational force that is also present has very little effect on the process.

Elastic and in-elastic collision

In order to understand what is elastic collision, we consider the collision between two bodies which come in physical contact during collision. At the instant they come in contact, they get deformed and as long as they are in contact, they move with a common velocity. The deformed bodies push each other and the velocities of the two bodies change. If they regain their original shape and size and the energy used to cause deformation is restored, then the collision is *perfectly elastic.*

If the original shape and size is not fully restored, then the collision is called *inelastic collision.*

But in actual collision this restoration is not complete. We study this through a factor, called coefficient of restitution (e).

Coefficient of restitution

Let us divide the time of collision 'Δt' into two parts (i) time of deformation Δt_1, and (ii) time of restoration, Δt_2 with $\Delta t = \Delta t_1 + \Delta t_2$.

Let the two bodies have mass m_1 and m_2 and their initial velocities be $\vec{u}_1$ and $\vec{u}_2$ respectively. Let the common velocity at the time of deformation be $\vec{u}$ and velocities after restoration be $\vec{\upsilon}_1$ and $\vec{\upsilon}_2$ respectively. Let the force acting on particle 1 at the time of deformation be $\vec{F}_{21}$ and at the time of restoration be $\vec{F}_{21}$. Similarly the force acting on particle 2 at the time of deformation be $\vec{F}'_{12}$ and at the time of restoration be $\vec{F}'_{12}$.

The coefficient of restitution (e) is defined as

$$e = \frac{\left|\int \vec{F}'_{21} dt\right|}{\left|\int \vec{F}_{21} dt\right|} = \frac{\left|m_1(\vec{\upsilon}_1 - \vec{u})\right|}{\left|m_1(\vec{u} - \vec{u}_1)\right|} = \frac{\left|\vec{\upsilon}_1 - \vec{u}\right|}{\left|\vec{u} - \vec{u}_1\right|}$$

...(4.17.6)

Similarly considering particle 2

$$e = \frac{\left|\int \vec{F}'_{12} dt\right|}{\left|\int \vec{F}_{12} dt\right|} = \frac{\left|m_2(\vec{\upsilon}_2 - \vec{u})\right|}{\left|m_2(\vec{u} - \vec{u}_2)\right|} = \frac{\left|\vec{\upsilon}_2 - \vec{u}\right|}{\left|\vec{u} - \vec{u}_2\right|}$$

...(4.17.7)

Since momentum is conserved through out the collision process so

$$m_1\vec{u}_1 + m_2\vec{u}_2 = (m_1 + m_2)\vec{u} = m_1\vec{\upsilon}_1 + m_2\vec{\upsilon}_2$$

...(4.17.8)

Using (4.17.8) in (4.17.6) and (4.17.7) we obtain

$$e = \frac{\left|\vec{\upsilon}_1 - \vec{\upsilon}_2\right|}{\left|\vec{u}_2 - \vec{u}_1\right|} = \frac{\left|\vec{\upsilon}_2 - \vec{\upsilon}_1\right|}{\left|\vec{u}_1 - \vec{u}_2\right|} \quad ...(4.17.9)$$

Thus coefficient of restitution is the ratio of the magnitude of relative velocity after collision to the magnitude of relative velocity before collision.

A collision is said to be perfectly elastic if original shape and size is fully restored i.e. $\left|\vec{F}_{12}\right| = \left|\vec{F}'_{12}\right|; \left|\vec{F}_{21}\right| = \left|\vec{F}'_{21}\right|$. This implies that in elastic collision e = 1, hence

$$\left|\vec{\upsilon}_1 - \vec{\upsilon}_2\right| = \left|\vec{u}_1 - \vec{u}_2\right|$$

$$\Rightarrow \quad \upsilon_1^2 + \upsilon_2^2 - 2\vec{\upsilon}_1.\vec{\upsilon}_2 = u_1^2 + u_2^2 - 2\vec{u}_1.\vec{u}_2$$

$$\Rightarrow \quad \frac{1}{2}m_1m_2\upsilon_1^2 + \frac{1}{2}m_1m_2\upsilon_2^2 - m_1m_2\vec{\upsilon}_1.\vec{\upsilon}_2$$

$$= \frac{1}{2}m_1m_2u_1^2 + \frac{1}{2}m_1m_2u_2^2 - m_1m_2\vec{u}_1.\vec{u}_2 \quad ...(4.17.10)$$

From conservation of momentum

$$m_1\vec{\upsilon}_1 + m_2\vec{\upsilon}_2 = m_1\vec{u}_1 + m_2\vec{u}_2$$

This gives

$$\frac{1}{2}m_1^2\upsilon_1^2 + \frac{1}{2}m_2^2\upsilon_2^2 + m_1m_2\vec{\upsilon}_1.\vec{\upsilon}_2$$

$$= \frac{1}{2}m_1^2u_1^2 + \frac{1}{2}m_2^2u_2^2 + m_1m_2\vec{u}_1.\vec{u}_2 \quad ...(4.17.11)$$

Adding eqns (4.17.10) and (4.17.11) we obtain

$$\frac{1}{2}m_1\upsilon_1^2(m_1+m_2) + \frac{1}{2}m_2\upsilon_2^2(m_1+m_2)$$

$$= \frac{1}{2}m_1u_1^2(m_1+m_2) + \frac{1}{2}m_2u_2^2(m_1+m_2)$$

$$\Rightarrow \quad \frac{1}{2}m_1\upsilon_1^2 + \frac{1}{2}m_2\upsilon_2^2 = \frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 \quad ...(4.17.12)$$

Eqn. (4.17.12) implies that Kinetic energy is conserved in perfectly elastic collision. **So a collision in which linear momentum, total energy and kinetic energy are conserved is called elastic collision.**

(ii) If the original shape and size is partially restored, then $\left|\vec{F}'_{12}\right| < \left|\vec{F}_{12}\right|$ and $\left|\vec{F}'_{21}\right| < \left|\vec{F}_{21}\right|$. This gives e < 1

i.e. $\left|\vec{\upsilon}_1 - \vec{\upsilon}_2\right| < \left|\vec{u}_1 - \vec{u}_2\right|$

In this case K.E. is not conserved.

(iii) If the deformation is permanent then $\left|\vec{F}'_{12}\right| = 0 = \left|\vec{F}_{21}\right|$ gives e = 0

Thus the coefficient of restitution lies between the two extreme value 0 & 1 i.e.

$0 \le e \le 1.$

<u>Elastic collision in one-dimension :</u>

Consider two bodies of masses m_1 and m_2 moving along a straight line with initial velocities $\vec{u}_1$ and $\vec{u}_2$ respectively; just before collision. Let their velocities be $\vec{\upsilon}_1$ and $\vec{\upsilon}_2$ after collision. If the collision is elastic, then linear momentum and kinetic energy are both conserved.

i.e. $m_1\vec{u}_1 + m_2\vec{u}_2 = m_1\vec{\upsilon}_1 + m_2\vec{\upsilon}_2$ ...(4.17.13)

and $\frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 = \frac{1}{2}m_1\upsilon_1^2 + \frac{1}{2}m_2\upsilon_2^2$ ...(4.17.14)

Eqn. 4.17.13 gives

$m_1(\vec{u}_1 - \vec{\upsilon}_1) = m_2(\vec{\upsilon}_2 - \vec{u}_2)$ ...(4.17.15)

and eqn. (4.17.14) gives

...(4.17.16)

As the motion is along a straight line we put

$$\vec{u}_1 = u_1\hat{i}, \vec{u}_2 = u_2\hat{i}, \vec{\upsilon}_1 = \upsilon_1\hat{i}, \vec{\upsilon}_2 = \upsilon_2\hat{i} \quad ...(4.17.17)$$

where u_2 is positive if m_2 moves in the direction of m_1 and negative of it moves in a direction opposite to m_1. The values of υ_1 and υ_2 are obtained (after solution) in terms of u_1 and u_2.

Using (4.17.17) in (4.17.15) we have

$$m_1(u_1-\upsilon_1)=m_2(\upsilon_2-u_2) \quad ...(4.17.18)$$

Using (4.17.18) in (4.17.16) we obtain

$$m_1(u_1-\upsilon_1)(u_1+\upsilon_1)=m_2(\upsilon_2-u_2)(\upsilon_2+u_2)$$

$$\Rightarrow \quad u_1+\upsilon_1=\upsilon_2+u_2 \quad ...(4.17.19)$$

Solving (4.17.18) and (4.17.19) we obtain

$$\upsilon_1=\frac{m_1-m_2}{m_1+m_2}u_1+\frac{2m_2}{m_1+m_2}u_2 \quad ...(4.17.20)$$

$$\upsilon_2=\frac{2m_1}{m_1+m_2}u_1+\frac{m_2-m_1}{m_1+m_2}u_2 \quad ...(4.17.21)$$

Eqns. (4.17.20) and (4.17.21) give the final velocities υ_1 and υ_2. If their values are positive, then we conclude that they move in $\hat{i}$ -direction; if negative we say that they move in $(-\hat{i})$-direction.

Special Cases

A: Colliding bodies moving in same direction

A.1 $(m_1 >> m_2)$:

In this case eqn. (4.17.20) and (4.17.21) reduce to

$$\upsilon_1=u_1; \upsilon_2=2u_1-u_2 \quad ...(4.17.22)$$

This shows that the heavier body moves undisturbed and the light body moves with a velocity $(2u_1-u_2)\hat{i}$.

A.2 $m_1 >> m_2$, and m_1 at rest :

In this case eqns. (4.17.20) and (4.17.21) reduce to

$$\upsilon_1=0, \upsilon_2=-u_2 \quad ...(4.17.23)$$

This shows that when a light body collides with a heavy body at rest the heavy body remains at rest but the light body returns back with same speed.

A.3 (m_2 at rest) i.e. $u_2 = 0$:

Then $\upsilon_1=\frac{m_1-m_2}{m_1+m_2}u_1$;

$$\upsilon_2=\frac{2m_1}{m_1+m_2}u_1 \quad ...(4.17.24)$$

This shows that when a body collides with another body of unequal mass, at rest, the body is slowed down and the collided body gains a velocity in the direction of first body.

The change in K.E. of the 1st body is

$$\Delta E_{k1}=\frac{1}{2}m_1\upsilon_1^2-\frac{1}{2}m_1u_1^2$$

$$=\frac{1}{2}m_1u_1^2\left[\frac{(m_1-m_2)^2}{(m_1+m_2)^2}-1\right]$$

$$\Rightarrow \quad \Delta E_{k1}=-\left(\frac{1}{2}m_1u_1^2\right)\frac{4m_1m_2}{(m_1+m_2)^2} \quad ...(4.17.25)$$

The change in K.E. of 2nd body is

$$\Delta E_{k2}=\frac{1}{2}m_2\upsilon_2^2=\left(\frac{1}{2}m_1u_1^2\right)\frac{4m_1m_2}{(m_1+m_2)^2} \quad ...(4.17.26)$$

Eqns. (4.17.25) and (4.17.26) show that the loss in K.E. by 1st body is equal to the gain in K.E. of the 2nd body. The fractional loss of K.E. by 1st body is given by

$$\frac{\Delta E_{k1}}{E_{k1}} = \frac{4m_1m_2}{(m_1+m_2)^2} = 4\left(\frac{m_1}{m_2}\right)\frac{1}{\left(1+\frac{m_1}{m_2}\right)^2} \quad ...(4.17.27)$$

Eqn. (4.17.27) shows that if $\left(\frac{m_1}{m_2}\right)$ is zero (i.e. a light body collides with a heavy body at rest) than the fractional loss is zero. Also if $\frac{m_1}{m_2} \to \infty$, the fractional loss is zero (i.e. when a heavy body collides with a very light body at rest). Fig4.42 shows the variation of fractional loss of K.E. with m_1/m_2. We also note that the maximum fractional energy decrease occur when $m_1 = m_2$.

This fact has been made use of in slowing down fast moving neutrons by passing the neutrons through moderators.

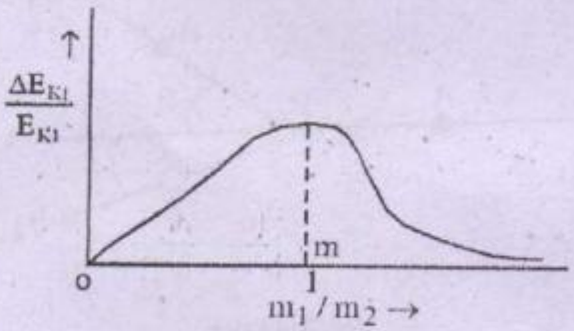


fig. 4.42

A.4 (m_1>> m_2 and m_2 at rest)

In that case eqns. (4.17.20) and (4.17.21) reduce to

$$\upsilon_1 = u_1; \upsilon_2 = 2u_1 \quad ...(4.17.28)$$

This shows that when a heavy body collides with a light body at rest, the heavy body moves undisturbed and the light body moves approximately at twice the velocity of the heavy body.

A. 5 ($m_1 = m_2$)

In this case we obtain from 4.17.20 and 4.17.21

$$\upsilon_1 = u_2; \upsilon_2 = u_1 \quad ...(4.17.29)$$

This shows that when two bodies are of equal mass they exchange their velocities.

A. 6 ($m_1 = m_2$ and m_2 at rest)

In this case we obtain

$$\upsilon_1 = 0; \upsilon_2 = u_1 \quad ...(4.17.30)$$

That is the body colliding with an identical body at rest comes to rest, and the body initially at rest moves with the velocity of the colliding body.

B. Colliding bodies moving in opposite direction :

In this case eqns. (4.17.20) and (4.17.21) reduce to

$$\upsilon_1 = \frac{m_1-m_2}{m_1+m_2}u_1 - \frac{2m_2}{m_1+m_2}u_2 \quad ...(4.17.31)$$

$$\upsilon_2 = \frac{2m_1}{m_1+m_2}u_1 - \frac{m_2-m_1}{m_1+m_2}u_2 \quad ...(4.17.32).$$

by replacing u_2 by $-u_2$. Then for

B. 1 ($m_1 >> m_2$)

$$\upsilon_1 = u_1; \; \upsilon_2 = 2u_1 + u_2 \quad ...(4.16.33)$$

This shows that the heavy body moves undisturbed after collision; and light body moves with velocity $(2u_1 + u_2)\hat{i}$.

B. 2 ($m_1 = m_2$)

$$\upsilon_1 = u_2; \; \upsilon_2 = u_1 \quad ...(4.17.34)$$

This shows that when bodies of equal masses moving in opposite direction collide, they exchange their velocities.

Perfectly inelastic collision in one-dimension

In perfect inelastic collision $e = 0$, which implies $\upsilon_1 = \upsilon_2$. That is the two bodies move together with a single velocity $\upsilon = \upsilon_1 = \upsilon_2$. So by conservation of linear momentum

$$m_1\vec{u}_1 + m_2\vec{u}_2 = (m_1 + m_2)\vec{\upsilon}$$

$$\Rightarrow \quad \vec{\upsilon} = \frac{m_1\vec{u}_1 + m_2 u_2}{m_1 + m_2} \qquad ...(4.17.35)$$

The K.E. before collision is

$$E_{ki} = \frac{1}{2}m_1 u_1^2 + \frac{1}{2}m_2 u_2^2 \qquad ...(4.17.35)$$

and that after collision is $\frac{1}{2}(m_1 + m_2)\upsilon^2$.

Using eqn. (4.17.35) the loss in K.E. is

$$\Delta E_k = \frac{1}{2}m_1 u_1^2 + \frac{1}{2}m_2 u_2^2 - \frac{1}{2}(m_1 + m_2)\upsilon^2$$

$$= \frac{1}{2}m_1 u_1^2 + \frac{1}{2}m_2 u_2^2 - \frac{1}{2}(m_1 + m_2)\left(\frac{m_1\vec{u}_1 + m_2\vec{u}_2}{m_1 + m_2}\right)^2$$

$$\Delta E_k = \frac{m_1 m_2 (\vec{u}_1 - \vec{u}_2)^2}{2(m_1 + m_2)} \qquad ...(4.17.37)$$

Eqn. (4.17.37) shows that $\Delta E_k > 0$.

If $m_1 >> m_2$, then $\Delta E_k \approx 0$. This means when a heavy body makes an inelastic collision with a light body the change in K.E. is almost negligible.

On the otherhand when $m_2 >> m_1$ and m_2 is at rest ($u_2 = 0$), the final velocity

$$\vec{\upsilon} = \frac{m_1\vec{u}_1 + m_2\vec{u}_2}{m_1 + m_2} \approx 0$$

and final K.E. is

$$E_{kf} = \frac{1}{2}(m_1 + m_2)\upsilon^2 = 0$$

For example (i) when a meteorite hits the earth it remains embeded within it (ii) when a bullet is fired into a heap of sand it remains embeded within it.

Elastic Collision in two-dimension :

In an elastic collision in 2-dimension, there shall be four unknowns - the two components of final velocity of each colliding particle. But the conservation of momentum and K.E. shall provide only three equations - two from the conservation of components of momentum and one from the conservation of K.E. Therefore another additional information such as one recoil angle is necessary for complete solution of a collision in two-dimension.

For example consider two bodies A and B of mass m_1, m_2 respectively kept on X-axis and colliding. Initially B is at rest and A moves towards B with a velocity $\vec{u}_1$.

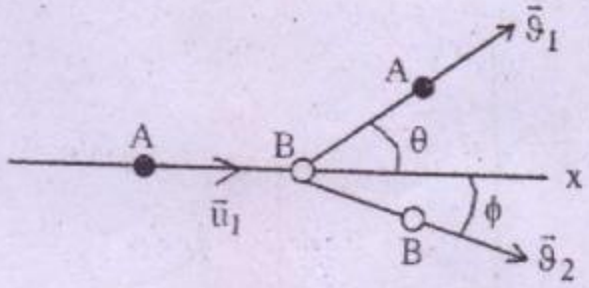


fig. 4.42 (a)

If the force during collision is not along the initial velocity, then the collision is not head-on and the objects are deflected along different directions. Let A move with velocity $\vec{\upsilon}_1$, making angle θ with X-axis and B move with velocity $\vec{\upsilon}_2$, making angle ϕ with X-axis. We assume that $\vec{\upsilon}_1$ and $\vec{\upsilon}_2$ lie on a plane say XY-plne. Then by applying law of conservation of momentum for X & Y components separately, we obtain

$$m_1 u_1 = m_1\upsilon_1\cos\theta + m_2\upsilon_2\cos\phi \qquad ...(4.17.35)$$

$$0=m_1\upsilon_1\sin\theta-m_2\upsilon_2\sin\phi \quad ...(4.17.36)$$

Using principle of conservation of K.E.

$$\frac{1}{2}m_1u_1^2=\frac{1}{2}m_1\upsilon_1^2+\frac{1}{2}m_2\upsilon_2^2 \quad ...(4.17.37)$$

Thus the three eqns. 4.17.35, 36, & 37 are not sufficient to know $\vec{\upsilon},\vec{\upsilon}_2,\theta$ or $\vec{\upsilon}_1,\vec{\upsilon}_2,\theta$ and ϕ. Hence additional information is needed from the experiment.

Ex.4.17.1 Two balls of masses 50 g and 30 g and having velocities 50cm/s and 30cm/s respectively, collide head on. Find the veolocities after collision.

Soln.

$$\upsilon_1=\frac{m_1-m_2}{m_1+m_2}u_1+\frac{2m_2}{m_1+m_2}u_2$$

$$\upsilon_2=\frac{2m_1}{m_1+m_2}u_1+\frac{m_2-m_1}{m_1+m_2}u_2$$

So $$\upsilon_1=\left(\frac{50-30}{50+30}\times50+\frac{2\times30}{50+30}\times30\right)cm/s$$

$$=(12.5+22.5)cm/s=35cm/s$$

$$\upsilon_2=\left(\frac{2\times50}{80}\times50+\frac{-20}{80}\times30\right)cm/s$$

$$=(62.5-7.5)cm/s=55cm/s$$

Thus υ_1=35cm/s, υ_2=55cm/s.

Ex.4.17.2 A 2kg ball 'A' moving with a velocity of 17m/s strikes a 20kg ball 'B' which is moving westward with a velocity 12 m/s. Find the velocity of each after collision.

Soln.

Given $u_A=17cm/s$ eastward

$u_B=12cm/s$ westward

Now

$$\upsilon_A=\frac{m_A-m_B}{m_A+m_B}u_A-\frac{2m_B}{m_A+m_B}u_B$$

$$\upsilon_B=\frac{2m_A}{m_A+m_B}u_A-\frac{m_B-m_A}{m_A+m_B}u_B$$

$$\Rightarrow \quad \upsilon_A=\left(\frac{2-20}{2+20}\times17-\frac{2\times20}{2+20}\times12\right)m/s$$

$$=\left(\frac{-18}{22}\times17-\frac{40}{22}\times12\right)m/s$$

$$\Rightarrow \quad \upsilon_A=-35.73m/s$$

$$\upsilon_B=\left(\frac{2\times2}{22}\times17-\frac{18}{22}\times12\right)=-6.73m/s$$

'-ve' sign indicates both moves in westward direction.

Ex.4.17.3 A ball is dropped from a height 'h' on to a fixed horizontal plane. Find the height to which it rebounds after collision if 'e' is the coefficient of restitution of the collision.

Soln.

The ball falls from height 'h'

The velocity before collision is $u_1=\sqrt{2gh}$ downward

Let υ_1 be the velocity just after collision.

The velocity of the plane before collision $u_2=0$ and velocity of the plane after collision $\vartheta_2=0$

Now

$$e=\frac{|\vec{\upsilon}_2-\vec{\upsilon}_1|}{|\bar{u}_2-\bar{u}_1|}=\frac{\upsilon_1}{u_1}$$

$$\Rightarrow \quad \upsilon_1=eu_1=e\sqrt{2gh}$$

Let the height to which it shall rise be h'.

Then $\upsilon_1^2=2gh'$

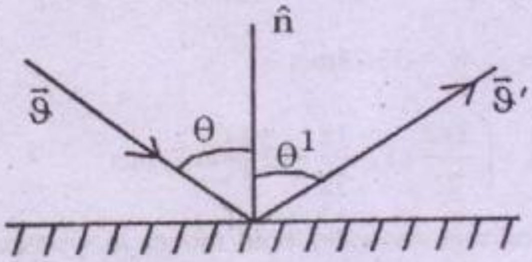
$\Rightarrow \quad e^2.2gh = 2gh'$

$\Rightarrow \quad h' = e^2h$

Ex.4.17.4 A ball of mass m hits a floor with a speed ϑ making an angle of incidence θ with the normal. The coefficient of restitution is e. Find the speed of the reflected ball and angle of reflection of the ball.

Soln.

During collision the floor exerts a force along the normal and no force parallel to the floor.



Therefore

$(\vec{P}_{11})^{\text{before collision}} = (\vec{P}_{11})_{\text{after collision}}$

$\Rightarrow \quad m\upsilon \sin\theta = m\upsilon' \sin\theta'$

$\Rightarrow \quad \sin\theta = \upsilon' \sin\theta' \qquad ...(1)$

The velocity component of the ball normal to the floor

before collision =

$\upsilon\cos\theta(\text{down ward}) = \upsilon \cos\theta\ (-\hat{j})$

after collision =

$\upsilon'\cos\theta'(\text{up ward}) = \upsilon'\cos\theta'(\hat{j})$

Hence

$$e = \frac{\upsilon'\cos\theta'}{\upsilon\cos\theta} \qquad ...(2)$$

$\Rightarrow \quad \upsilon'\cos\theta' = e\upsilon \cos\theta \qquad ...(3)$

From (1) & (3)

$\upsilon'^2 = \upsilon^2\sin^2\theta + e^2\upsilon^2\cos^2\theta$

$\Rightarrow \quad \upsilon' = J\sqrt{\sin^2\theta + e^2\cos^2\theta} \qquad ...(4)$

From (1) & (3) also

$$\tan\theta' = \frac{1}{e}\tan\theta \qquad ...(5)$$

Eqn. (4) gives the velocity and eqn. (5) gives the angle of reflection.

In case of elastic collision e = 1, so $\upsilon' = \upsilon$ and $\theta' = \theta$.

4.18: FRICTION

When the engine of a running car is cut-off, the car stops after travelling some distance, indicating that a retarding (opposing) force exists between the surface of the wheel of the car and road. We also observe that when an almirah is pushed by a single person it does not move; but when pushed by few it starts moving. This also shows that there exists an opposing force between surface of contact of almirah and floor. These opposing forces are called **"force of friction."**

4.18(a) Force of friction as component of contact force :

When two bodies are kept in contact, electromagnetic forces act between the charged particles at the surfaces of the bodies. As a result each body exerts a contact force on the other. These forces are equal in magnitude and opposite in direction. The component $\vec{N}$ perpendicular to the contact surface is called normal contact force or simply normal force and the parallel component $\vec{f}$ is called **friction.**

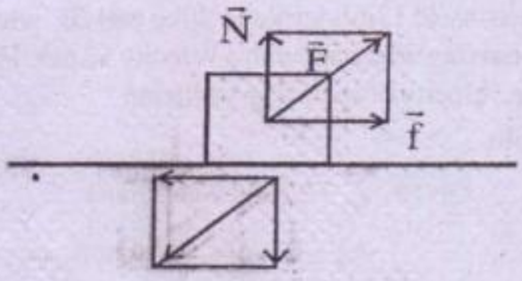


fig. 4.43

Friction can operate between a given pair of solids, between a solid and fluid or between a pair of fluids. Frictional force exerted by fluids is called viscous force. Frictional force between two unlubricated (dry) solid surfaces is called dry friction.

Dry friction is of three types (i) Static friction (ii) Kinetic friction and (iii) rolling friction.

4.18(b) STATIC FRICTION :

The force of friction that arises between two surfaces in contact when there is no relative motion between the surfaces is called **Static Friction.**

Consider a block placed on a horizontal table top. One end of the string AB is attached to the block and the other end to a pan. The string passes over a pulley as shown in fig. 4.44

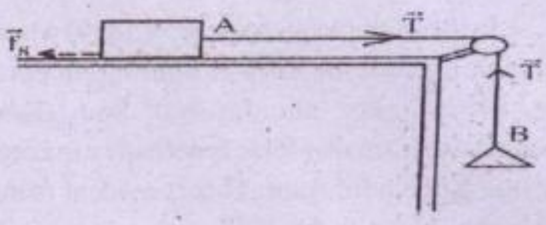


fig. 4.44

A horizontal force is applied on the block through the tension of the string by placing weights on the pan. It is observed that :

i) When small weight is placed on the pan, the block does not move.

ii) The block continues in the state of rest until weight on the pan reaches a certain value W_L. At this value the block just begins to slide, on gentle tapping.

iii) The block is accelerated when more load is put on the pan.

The above observations reveal that as long as the weight on the pan 'W' is less than W_L, the block is at rest. The free-body diagram for the block (fig. 4.45) gives

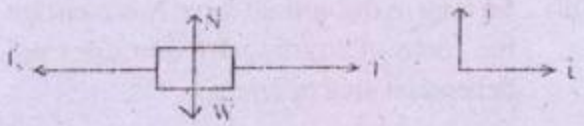


fig. 4.45

$$\vec{F}_{net} = \vec{f}_s + \vec{T} + \vec{N} + \vec{W} = 0$$

$$\Rightarrow \quad -\vec{f}_s\hat{i} + \vec{T}\hat{i} + \vec{N}\hat{j} - \vec{W}\hat{j} = 0$$

$$\Rightarrow \quad f_s = T$$

and $N = W$ = Weight of the block

This analysis shows that the block continues to be at rest so long as $\vec{f}_s = -\vec{T}$. But T increases as load is added to the pan. Hence it implies that force of static friction f_s increases as applied force increases. Thus 'f_s' is self-adjusting. But this self-adjustment continues only to a limit. When $T_L = W_L$, the block begins to slide and the value of force of static friction at this instant is the maximum value of f_s.

Laws of Static friction :

The following laws follow from several observations.

(i) The force of static friction is self-adjusting and increases upto a limiting (maximum) value called *Force of limiting friction* (f_L).

(ii) The force of limiting friction is directly proportional to the normal force.

i.e. $f_s\big|_{max} = f_L = \mu_s N$ ...(4.18.1)

where, 'μ_s' is called *coefficient of static friction*. Its value depends on the nature of the material and roughness of the two surfaces in contact. *Thus coefficient of static friction for a given pair of surfaces* *may be defined as the ratio of the force of limiting friction to the normal force acting on the contact surface.*

(iii) So long as the normal force N is constant the force of limiting friction does not depend on area of contact.

(iv) The actual force of static friction may be smaller than $\mu_s N$ and its value depends on other forces acting on the body.

i.e. $f_s \le f_s|_{max} (= \mu_s N)$.

Direction of f_s:

The direction of f_s on a body is such that the total force acting on it keeps it at rest w.r.to the body in contact.

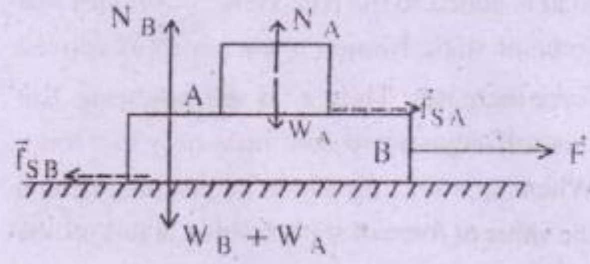


fig. 4.46

Suppose B is pulled by a force F towards right and F being small B does not move. The block B is at rest w.r.to the ground. So resultant force on B is zero. Hence for static equilibrium of B we should have

$\Rightarrow \quad \vec{f}_{sB} = -\vec{F}$ ...(4.18.2)

and $\vec{W}_A + \vec{W}_B = -\vec{N}_B$ ...(4.18.3)

Similarly for static equilibrium of A, we should have

$$\vec{F}_{net}^{A} = \vec{W}_A + \vec{N}_A = 0$$

As no net external force acts on A, so $f_{sA} = 0$ i.e. force of static friction on A due to lower block B is zero.

If F is increased so that B slides towards right over the surface, then block A also moves to the right w.r.to the ground. Hence a net force must also be acting on the upper block, so it is the frictional force which acts on the upper block and it must be towards right.

Note that it is the friction on the upper block which accelerates it towards right contrary to the misconception that friction always opposes motion. Also a vehicle accelerates on road because of friction. It is not possible to drive on a frictionless road.

4.18 (c) KINETIC FRICTION :

When two bodies in contact move with respect to each other, rubbing the surfaces in contact, the friction between them is called Kinetic friction.

The direction of frictional force is such that it tries to destroy the relative motion (i.e. it tends to oppose the relative motion). Hence it acts in a direction opposite to relative motion.

In the example given in sec 4.18 (b) when the force exceeds the force of limiting friction, the block gets accelerated and then comparatively smaller force is necessary to keep the block in equilibrium. This is evident from the graph shown in fig. 4.47.

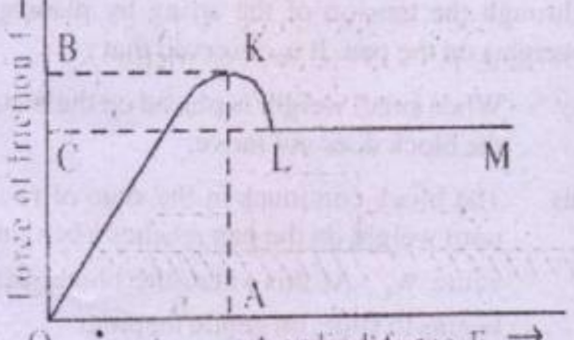


fig. 4.47

For an applied force OA (F_s) the maximum of static friction is OB. When force $F > F_s$, the body starts sliding and the force of friction then slightly decreases, and then remains constant throughout. Further increase in F will produce an acceleration in the body.

Laws of Kinetic friction :

(i) The magnitude of kinetic friction is proportional to the normal force acting between the two bodies.

i.e. $f_k = \mu_k N$ ...(4.18.4)

where f_k is force of kinetic friction, μ_k is coefficient of Kinetic friction and N is normal force. **The coefficient of Kinetic friction between two surfaces is defined as the force of Kinetic friction between two given surfaces per unit normal force.**

(ii) So long as the normal force N remains constant, the force of Kinetic friction (hence μ_k) does not depend on the area of contact.

(iii) The force of Kinetic friction (hence μ_k) is almost independent of the relative speed of the sliding bodies (provided $\upsilon < 10 m/s$)

Direction of 'f_k' :

The direction of Kinetic friction on a body 'A' slipping against another body B is opposite to the velocity of A w.r.to B.

Ex (i)

Force of friction on A $\vec{f}_A$ due to B is towards left and the force of friction $\vec{f}_B$ on B due to A, is towards right.

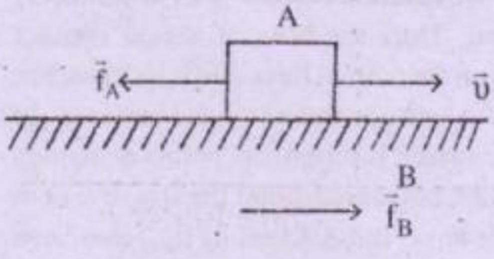


fig. 4.48

(ii)

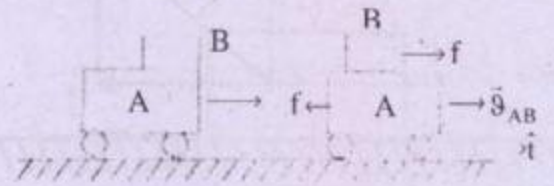


fig. 4.49

Consider a small box placed on a long box with wheels and moving on a horizontal road. The small box slips on the long box to fall from the rear end. Here

$$\vec{\upsilon}_{BR} = \upsilon_{BR}\,\hat{i}$$

$$\vec{\upsilon}_{AR} = \upsilon_{AR}\,\hat{i}$$

where 'R' stands for road. If $\upsilon_{AR} > \upsilon_{BR}$, then $\upsilon_{AR} - \upsilon_{BR} = u > 0$. Then $\vec{\upsilon}_{BA} = \vec{\upsilon}_{BR} - \vec{\upsilon}_{AR}$ $= -\left(\vec{\upsilon}_{AR} - \vec{\upsilon}_{BR}\right) = -u\hat{i}$. That is the velocity of box B w.r.to box A is towards left. Hence the force of friction on box B due to A will be towards right. On the otherhand $\vec{\upsilon}_{AB}$ is towards right, hence force of friction on 'A' due to 'B' will be towards left.

Relative values of μ_s and μ_k

For two given surface , so long as normal force 'N', is constant

Angle of friction :

The angle made by the resultant of limiting friction '$\vec{f}_L$' and normal force '$\vec{N}$', with the normal force '$\vec{N}$', is called angle of friction θ.

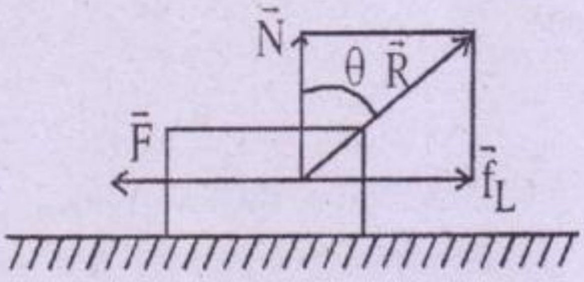


fig. 4.50

From fig. 4.50 it follows that

$$\tan\theta = \frac{f_L}{N} = \mu_s \qquad ...(4.18.5)$$

This implies that coefficient of static friction is equal to the tangent of the angle of friction.

Inclined plane and angle of repose :

The angle of repose (for a given inclined plane and contact surface of another body kept on the inclined plane), is defined as the angle which the inclined plane makes with the horizontal so that the body just begins to slide.

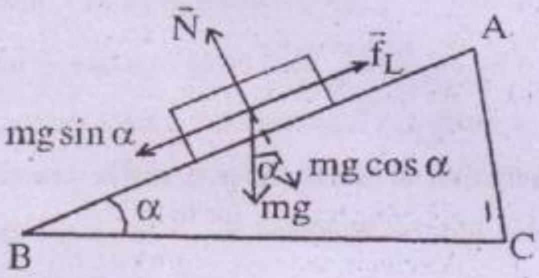


fig. 4.51

Consider the inclined plane ABC making angle α with the horizontal base BC.

Let the mass of the block on the inclined plane be=m. Then the forces acting on this are :

i) Wt. of the body = $m\vec{g}$, acting vertically down ward.

ii) Normal force $\vec{N}$ acting perpendicular to the contact surface. i.e. the inclined plane.

iii) The force of friction $\vec{f}_L$ acting parallel to the inclined plane and upwards. (see fig. 4.51)

When the block just begins to slide

$$|\vec{f}_L| = |\mu_s \vec{N}| \qquad ...(4.18.6)$$

For equilibrium

$$\vec{f}_L + m\vec{g} + \vec{N} = 0 \qquad ...(4.18.7)$$

Resolving along and to the inclined plane and equating for equilibrium.

$$f_L = mg\sin\alpha \qquad ...(4.18.8)$$

$$N = mg\cos\alpha \qquad ...(4.18.9)$$

This gives

$$\tan\alpha = \frac{f_L}{N} = \mu_s \qquad ...(4.18.10)$$

Equation 4.18.10 shows that the coefficient of static friction is equal to the tangent of the angle of repose. Comparing equations (4.18.5) and (4.18.10) we find that

angle of friction = angle of repose.

Causes of sliding friction :

The force of friction is due to the atomic and molecular forces of attraction between molecules on two surfaces in contact.

A surface which appears to be smooth to naked eye is actually rough microscopically. When two sufraces are in contact the actual area of contact is much smaller than the area of the surfaces. The distance between the molecules of the two bodies at the points of contact is small, therefore molecular force of attraction starts operating at these points. When the normal force is increased, the actual area of contact is increased. Thus the area of actual contact depends on the normal force and is independent of the size and roughness of the surfaces. In case the surface is rough, the points of contact are smaller, but at each point the size of area in contact is large, thus increasing the actual area of contact when the surface is rough. On the otherhand when load on the body is increased, the irregularities or projections get crushed and as a result the number of points of contact increases (but the size of area of contact at each point decreases). However on the whole total area of contact increases.

This molecular force of attraction at the points of contact oppose the relative motion between two surfaces.

Therefore the force of friction between any two surfaces is in fact the force required to break the bonds of attraction that exist between the two surfaces at the points of actual contact.

Since force of friction is directly propertional to actual area of contact and actual area of contact is directly propertional to the normal force, so

force of friction α normal force.

4.18.(d) Rolling Friction :

When a body rolls over another surface the points of contact of both sufraces keep on changing. But incase of sliding the points of contact of at least one of the surfaces is constant.

Motion of a wheel, a ball, or a cylinder over a level surface are examples of rolling. It is observed that whenever such rolling bodies are left to themselves they gradually slow down and come to rest. This slowing down is due to an opposing force, called **rolling friction.**

Thus the force of friction that arises between two surface in contact when one body rolls over the other is called **rolling friction.**

Causes of rolling friction :

The rolling friction arises due to these two effects.

i) As a body rolls, it creates a depression below it and an elevation in front of it. As a result the body feels as if it climbs up an inclined plane. This causes on opposing force.

ii) Due to the weight of the body the lower portion of the body gets slightly flattened, thus increasing the actual area of contact. This causes an increase in force of friction.

Relative values of μ_s, μ_k and μ_r **:**

In case of rolling points of contact is very much reduces, hence rolling friction is very much less. It is observed that

$\mu_s > \mu_k > \mu_r$

That is why it is easier to roll a barrel than slide it.

Advantages and disadvantages of friction :

Friction is sometimes advantageous and sometimes disadvantageous, depending on the situations.

Advantages :

i) We can hold a pen and write, due to friction between finger tip and pen surface.

ii) A nail can be fixed in a wall, due to friction between wall and nail surface.

iii) We are able to walk on road due to friction.

iv) We are able to stop a moving vehicle by applying brakes, due to friction.

v) A vehicle can take a turn on a level road due to friction.

Disadvantages :

i) It causes wear and tear in the moving parts of the machineries.

ii) Energy has to be spent against friction.

iii) It slows down motion of moving objects.

Thus friction has both advantages and disadvantages, hence it is a necessary evil.

Methods of reduction of friction :

Friction between two surfaces can be reduced by employing few methods as given below.

i) By polishig the surfaces

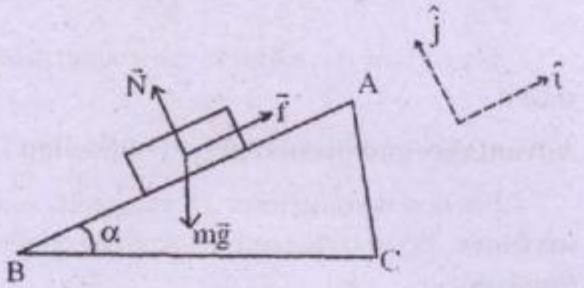
ii) By using lubricants

iii) By using ball. bearings

iv) By streamlining (properly shaping) the body. For example pin pointed shape has less friction.

Motion on Inclined plane :

Case (a) : *Rough inclined plane*



Consider rough inclined plane AB of length ℓ. Let a block of mass m rest on the inclined plane. The forces acting on the block are

i) Weight of the body $m\vec{g}$, acting vertically downwards.

(ii) Normal force $\vec{N}$ acting perpendicular to the surface of contact i.e. inclined plane AB.

(iii) The force of friction $\vec{f}$

Hence net force acting on the body is

$$\vec{F} = m\vec{g} + \vec{N} + \vec{f} \qquad ...(4.18.11)$$

Taking the inclined plane direction $\overrightarrow{BA}$ as $\hat{i}-\text{dir}^n$ and perpendicular to it as $\hat{j}-\text{dir}^n$ we re-write eqn (4.18) as

$$\vec{F} = (f - mg\sin\alpha)\hat{i} + (N - mg\cos\alpha)\hat{j} \qquad ...(4.18.12)$$

Now as the inclined plane is raised gradually the angle 'α' hence mg sinα increases and due to self-adjusting property the force of friction 'f' also increases, until it reaches the limiting value f_L. Suppose the angle of inclination corresponding to force of limiting friction is α_0. Then if the inclined plane is just further raised, then the block starts sliding so we have

$$0 = (f_L - mg\sin\alpha_0)\hat{i} + (N - mg\cos\alpha_0)\hat{j} \qquad ...(4.18.13)$$

This gives

$$f_L = mg\sin\alpha_0 \qquad ...(4.18.14)$$

$$N = mg\cos\alpha_0 \qquad ...(4.18.15)$$

and $$\tan\alpha_0 = \frac{f_L}{N} = \mu_s \qquad ...(4.18.16)$$

When the inclined plane is raised beyond α_0, i.e. for $\alpha > \alpha_0$ the block slides down the inclined plane. Under these conditions

$$\sin\alpha > \sin\alpha_0$$

$$\Rightarrow \quad mg\sin\alpha > mg\sin\alpha_0$$

$$\Rightarrow \quad mg\sin\alpha - f_L > mg\sin\alpha_0 - f_L$$

i.e $$mg\sin\alpha - f_L > 0$$

Further when the block slides down the inclined plane, the force of friction ('f_K') is slightly decreased (i.e. $f_K < f_L$). Therefore now net force acting on the body is

$$\vec{F} = (f_k - mg\sin\alpha)\hat{i} + (N - mg\cos\alpha)\hat{j} \qquad ...(4.18.17)$$

with $\alpha > \alpha_0$. As the block sticks to the inclined plane so

$$N - mg\cos\alpha = 0$$

$$\Rightarrow \quad N = mg\cos\alpha \qquad ...(4.18.18)$$

and

$$\vec{F} = (f_k - mg\sin\alpha)\hat{i} \qquad ...(4.18.19)$$

But $f_k = \mu_k N = \mu_k mg\cos\alpha$

So eqn. (4.18.19) reduces to

$$\vec{F} = (\mu_k\cos\alpha - \sin\alpha)mg\hat{i} \qquad ...(4.18.20)$$

Eqn. (4.18.20) gives

$$\vec{a} = g(\mu_k\cos\alpha - \sin\alpha)\hat{i} \qquad ...(4.18.21)$$

$$\Rightarrow \quad a = |g(\mu_k\cos\alpha - \sin\alpha)| \qquad ...(4.18.22)$$

Eqn (4.18.22) gives the acceleration of the block.

Using kinematic equations the velocity on reaching the bottom is given as

$$\vartheta^2 = u^2 + 2as = 0 + 2a\ell$$

$$\Rightarrow \quad \vartheta = \sqrt{2a\ell} = \sqrt{2g\ell(\mu_k \cos\alpha - \sin\alpha)}$$

The time taken is given by

$$S = ut + \frac{1}{2}at^2 = \frac{1}{2}at^2$$

$$\Rightarrow \quad t = \sqrt{\frac{2s}{a}} = \sqrt{\frac{2\ell}{g(\mu_k \cos\alpha - \sin\alpha)}}$$

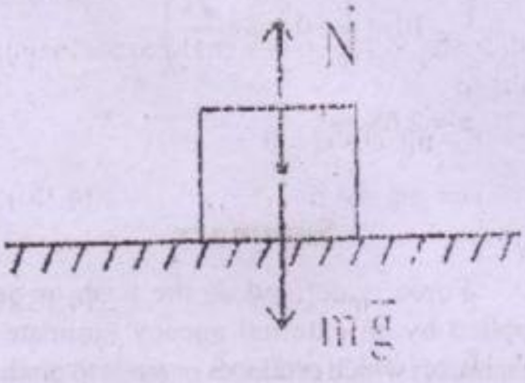
Case (b) *Smooth Plane*

If the plane is smooth, $\mu_k = 0$ and

$$\vartheta = \sqrt{2g\ell\sin\alpha}, \; a = g\sin\alpha \qquad ...(4.18.24)$$

$$t = \sqrt{\frac{2\ell}{g\sin\alpha}} \qquad ...(4.18.25)$$

Ex.4.18.1 Calculate the horizontal force required to move a body weighing 200 kg on a rough horizontal surface having coefficient of friction 0.4.



Soln.

Given m = 200 kg

=> N = mg = 200 x 9.8 N

$\mu_s = 0.4$

$f_L = \mu_s N = 0.4 \times 200 \times 9.8N$

Hence force required is 784 N.

Ex.4.18.2 A force of 4 kg wt is just sufficient to pull a block of 4 kg wt over a flat surface. What is the angle of friction.

Soln.

Given $F = f_L = 4$ kg wt = 4 x 9.8 N

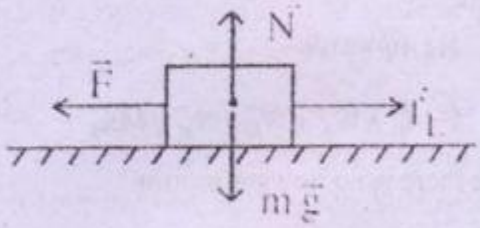
N = mg = 4 kg wt = 4 x 9.8 N

$$\Rightarrow \quad \mu_s = \tan\theta = \frac{f_L}{N} = 1$$

$$\Rightarrow \quad \theta = 45^0$$

Ex. 4.18.3 A horizontal force of 90 N is applied on a block of 15 kg which rests on a horizontal surface. If the coefficient of friction be 0.2 find the acceleration produced in the block (g = 10 m/s²)

Soln.



Given m = 15 kg

N = mg = 15 x 10 = 150 N

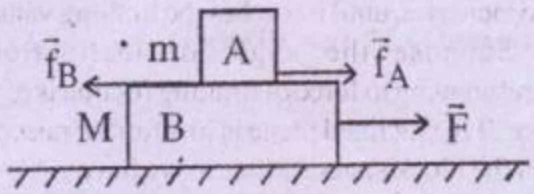
$f_L = \mu N = 0.2 \times 150 = 30$ N

Net force $F_{net} = F - f_L = (90 - 30)N = 60N$

Hence $a = \frac{f_{net}}{m} = \frac{60N}{15kg} = 4m/s^2$

Ex.4.18.4 The coefficient of static friction between two blocks A and B is μ. and B is kept on the smooth table. What maximum horizontal force F can be applied to the block B of mass M so that the blocks move together.

Soln.



Max. force applied = F

The only horizontal force on 'A' is the force of friction $\vec{f}_A$. Hence it should act towards right. Hence force of friction $\vec{f}_B$ on B should be towards left. Now $|\vec{f}_A| = \mu|\vec{N}_A| = \mu mg$

Considering motion of A

$$f_A = m\,a_A$$

$$=> \quad \mu\, mg = m\,a_A$$

$$=> \quad a_A = \mu g$$

Now considering the motion of B, the forces acting on B, are

(i) $\vec{f}_B$ towards left (ii) towards right

(iii) $\vec{W}_A$ downwards, (iv) $\vec{W}_B$ downwards

(iv) $\vec{N}_B$ upwards.

So $\quad \vec{F} + \vec{f}_B + \vec{W}_A + \vec{W}_B + \vec{N}_B = M\vec{a}_B$

Since there is no upward motion

$$\vec{W}_A + \vec{W}_B + \vec{N}_B = 0$$

and $\quad F - f_B = M\,a_B$

But $\quad f_B = f_A = f = \mu mg$

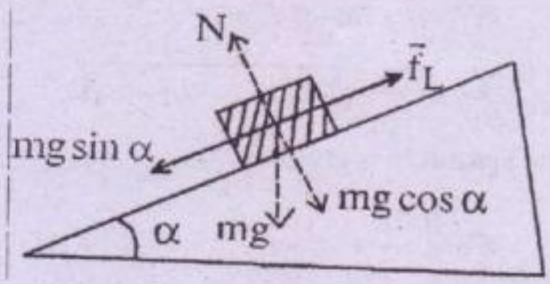
and $\quad a_A = a_B = a$

So $\quad F = \mu mg + M\,a = \mu mg + M\mu g$

$=> \quad F = \mu g\,(m+M)$

Ex.4.18.5 A 10 kg block slides without acceleration down a rough inclined plane making an angle of 15^0 with the horizontal. Find the acceleration when angle of inclination is increased to 30^0 ($g = 10$ m/s^2).

Soln.



Given angle of repose = 15^0

$$\mu_s = \tan 15^0 = 0.268$$

$$N = mg\cos\alpha$$

$$f_L = \mu_s N = \mu_s\, mg\cos\alpha$$

Eqn of motion of block is

$$mg\sin\alpha - f_L = m\,a$$

$$=> \quad mg\sin\alpha - \mu_s\, mg\cos\alpha = ma$$

$$=> \quad a = g\,(\sin\alpha - \mu_s\cos\alpha)$$

$$= g\,(\sin 30^0 - \mu_s.\cos 30)$$

$$= 10\times\left(\frac{1}{2} - 0.268\times\frac{\sqrt{3}}{2}\right)$$

$$a = 2.68 \text{ m/s}^2.$$

Summary

1. Force is defined as the push or pull, applied by an external agency (animate or inanimate) which produces or tends to produce changes in the state of motion of a body.

Force is an interaction between two objects. Force is always exerted by an object (say A) on another object (say B).

2. There are four basic forces (i) strong (ii) elecbomaguatic (e.m) (iii) weak and (iv) gravitational.

3.

Basic force	Range	Strength (Relative)
Strong	10^{-15}m	1
e.m	∞	10^{-2}
weak	$< 10^{-15}$m	10^{-5}
gravitational	∞	10^{-39}

4. Momentum ($\vec{p}$) of a body is the product of its mass (m) and velocity ($\vec{\upsilon}$) : $\vec{P} = m\,\vec{\upsilon}$

5. Newton's 1st law of motion : Every body continues in its state of rest or of uniform motion in a straight line (i.e moves with a constant velocity) until and unless it is compelled by a net external force to change that state.

OR

If the vector sum of all the external forces acting on a body is zero, then and only then the body remains unaccelerated (i.e remains at rest or moves with constant velocity).

6. Newton's 2nd law of motion :

The time rate of change of momentum of a body is directly proportional to the net external force impressed on it and takes place in the direction of net external force.

i.e. $\vec{F} = k\dfrac{d\vec{p}}{dt} = km\vec{a}$, where k is a constant of proportionality. We set value of k=1 in S.I units. In this case mass 'm' is considered constant (which is true at low velocities).

8. Unit of force is S.I units is $1N = 1kg.ms^{-2}$

9. (a) 2nd law is consistent with first law $(\vec{F} = 0 \Rightarrow \vec{a} = 0)$.

(b) It is a vector equation.

(c) It is applicable to a particle and a system of particles.

(d) Second law is a local law : i.e $\vec{a}$ at an instant does not depend on the history of motion.

10. Newton's 3rd law states : To every action there is an equal and opposite reaction.

Action and reaction forces are simultaneous force. Any of the two can be called action and the other reaction. e.g. $\vec{F}_{A\to B} = -\vec{F}_{B\to A}$.

If one is action the other is reaction.

11. Impulse is the product of force and time which equals change in momentum. i.e. $\vec{F}dt = d\vec{p}$.

12. Law of conservation of linear momentum : The total linear momentum of an isolated body or system is conserved.

13. Work is defined as the scalar product of force and displacement $dw = \vec{F}.d\vec{s}$. It is a scalar quantity.

14. Unit of work is S.I. units is Joule. (1 Joule = 1N-m)

15. If work done by a force around a closed path is zero, then the force is said to be conservative. i.e. $\oint \vec{F}.d\vec{s} = 0$ for a conserative force.

16. The rate at which work is done by the agent is called the power of the agent. It is a scalar quantity.

17. Energy of a body is defined as its capacity for doing work.

18. Mechanical energy is of two types (i) potential and (ii) kenetic energy.

19. A body of mass m moving with velocity $\vec{\upsilon}$, posseses K.E. given as $E_k = \frac{1}{2}m\upsilon^2 = \frac{P^2}{2m}$

20. P.E. of a body at height h is V= mgh.

21. P.E. of a spring with spring constant k is.

$$V = \frac{1}{2}kx^2$$

22. Law of conservation of energy : Energy can neither be created nor be destroyed, it can only be transformed from one form to another. The sum total energy in the universe remains constant.

23. Coefficient of restitution :

$$e = \left|\frac{\vec{\upsilon}_1 - \vec{\upsilon}_2}{\vec{u}_2 - \vec{u}_1}\right| = \frac{|\vec{\upsilon}_2 - \vec{\upsilon}_1|}{|\vec{u}_1 - \vec{u}_2|}$$

24. In elastic collision linear momention and K.E are conserved.

18. Friction :

Frictional force opposes any relative motion (whether impending or actual) between two surfaces in contact. Static friction f_s opposes impending relative motion where as kinetic friction f_k opposes actual relative motion. They are independent of area of contact and satisfy the following laws.

$$f_s \le f_s^{max} (= \mu_s N)$$

$$f_k = \mu_k N$$

where μ_s and μ_k are coefficients static and kinetic frictron.

It is found experimentally that $\mu_k < \mu_s$.

MODEL QUESTIONS

A. MULTIPLE CHOICE

1. Definition of a force comes from Newton's
   - (a) first law (ii) second law
   - (c) third law (iv)non of the above

2. If a constant force is applied to a mass, it moves with a uniform
   - (a) velocity (b) acceleration
   - (c) angular velocity (d) momentum

3. There are three Newton's laws of motion. We can derive
   - (a) Second and third law from first law
   - (b) third and first law from second law
   - (c) first and second law from third law
   - (d) All the laws are independent of each other

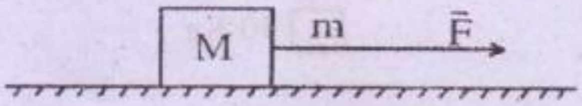
4. The linear momentum p of a particle varies with time as $p = a + bt^2$; where a and b are constats. The net force acting on the particle is
   - (a) proportional to t
   - (b) proportional to t^2
   - (c) zero
   - (d) constant

5. A boy sitting in a moving train is facing the engine. He tosses a coin up. The coin falls behind him. The train is moving
   - (a) forward with uniform speed
   - (b) backward with uniform speed
   - (c) forward with acceleration
   - (d) foward with deceleration

6. Newton's second law of motion gives the measure of
   - (a) force (b) energy
   - (c) momentum (d) acceleration

7. A man of weight W is standing on a lift which is moving upward with an acceleration a. The apparent weight of the man is
   - (a) W (1+a/g) (b) W
   - (c) W (1- a/g) (d) $W (1- a^2/g^2)$

8. A block of mass M is pulled along a horizontal frictionless surface by a rope of mass m. Force P is applied at one end of the rope. The force which the rope exerts on the block is



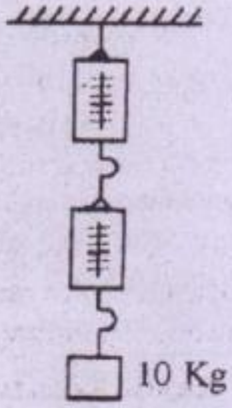
   - (a) $\frac{P}{M-m}$ (b) $\frac{P}{M(M+m)}$
   - (c) PM / (M+m) (d) Pm (M - m)

9. A body of weight w is suspended from the ceiling of a room through a chain of weight W. The ceiling pulls the chain by a force
   - (a) w (b) W
   - (c) W + w (d) (W + w) / 2

10. When a horse pulls a cart, the force that helps the horse to move forward is the force exerted by
    - (a) the cart on the horse
    - (b) the ground on the horse
    - (c) the ground on the cart
    - (d) the horse on the ground.

11. A car accelerates on a horizontal road due to the force exerted by
    (a) the engine of the car
    (b) the driver of the car
    (c) the earth
    (d) the road.

12. A block of mass 10 kg is suspended through two light spring balances as shown



    (a) Both scales will read 10 kg
    (b) Both scales will read 5 kg
    (c) The upper scale will read 10 kg and the lower zero
    (d) The readings may be anything but their sum shall be zero.

13. A block of mass m is placed on a smooth inclined plane of inclination θ with the horizontal. The force exerted by the plane on the block has a magnitude
    (a) mg (b) $mg/\cos\theta$
    (c) $mg\cos\theta$ (d) $mg\tan\theta$

14. A uniform rope of length L resting on a frictionless horizontal surface is pulled at one end by force F. The tension in the rope at a distance from the end where the force is applied is
    (a) F (b) l F
    (c) $(1 - l/L)F$ (d) $(1 + l/L)F$

15. A body of mass 2kg moving on horizontal surface with an initial velocity of 4m/s comes to rest after 2 s. If one wants to keep this body moving on the same surface with a velocity 4 m/s, the force required is
    (a) 8 N (b) 4 N
    (c) zero newton (d) 2 N

16. Two bodies of masses m and 4m are moving with equal Kinetic energies. The ratio of their linear momentum is
    (a) 1 : 4 (b) 4 : 1
    (c) 1 : 2 (d) 1 : 1

17. When mass and speed are doubled the K.E. increases
    (a) 2 times (b) 4 times
    (c) 8 times (d) 16 times

18. A vertically projected body has its energy
    (a) maximum at highest point
    (b) minimum at highest point
    (c) same at all points
    (d) none of the above

19. A body 'A' and another body 'B' have the same K.E. of translation. Then momentum
    (a) A is larger than B
    (b) B is larger than A
    (c) A and B are same
    (d) Data is inadequate.

20. A ball is thrown up vertically in air. As it rises there is an increase in its
    (a) Potential energy
    (b) Kinetic energy
    (c) Acceleration
    (d) none of these

21. The bob of an oscillating pendulum comes to rest after sometime due to
    (a) its mass (b) gravity
    (c) friction of air (d) tension in the string.

22. The K.E. of a body is doubled. Its momentum will
    (a) remain unchanged
    (b) be doubled
    (c) becomes 1/2 times
    (d) becomes $\sqrt{2}$ times

23. The following four particles have same momentum which one has maximum K.E. ?
    (a) electron (b) proton
    (c) deutron (d) α – particle

24. A bullet in motion hits and gets embeded in a solid block resting on a frictionless table. What is conserved ?
    (a) momentum alone
    (b) K.E. alone
    (c) momentum and K.E.
    (d) neither momentum nor K.E.

25. A ball falls from a height of 10m on a horizontal fixed plane. After rebounding twice, it attains a height of 2.5 m. The coefficient of restitution is
    (a) 2 (b) 1
    (c) $\frac{1}{2}$ (d) 1/4

26. A 150 gm mass has a velocity $\left(2\hat{i}+6\hat{j}\right)$ m/s at a certain instant. Its K.E. is
    (a) 3 joule (b) 8 joule
    (c) 10 joule (d) 11 joule

27. Angle of repose is equal to
    (a) coefficient of friction
    (b) angle of friction
    (c) angle at which external force is applied
    (d) angle at which frictional force acts.

28. The tyres are made of rubber and not of iron because
    (a) friction between concrete and iron is higher than the friction between rubber and concrete
    (b) rubber is cheaper than iron
    (c) rubber can be given circular shape and not iron
    (d) iron tyres will produce sound.

29. Aeroplanes and jets are streamlined to reduce
    (a) dynamic friction
    (b) sliding friction
    (c) rolling friction
    (d) fluid friction

30. Which of the following should be decreased to transport a massive cylinder by rolling on a flat surface ?
    (a) coefficient of Kinetic friction
    (b) limiting firction
    (c) normal reaction
    (d) weight

B. Very Short Answer Type Questions :

1. When a bullet is fired the gun is pushed back. State the law that explains this phenomenon.
2. Distinguish between inertial mass and weight.
3. What is weight of a freely falling body during its free fall ?
4. What property of body represents its inertia ?
5. How does momentum of an isolated system vary in time ?
6. What is the relation between kg.wt and newton ?
7. What is the working principle of rocket?

8. A man sitting in a train with face towards engine tosses up a coin. The coin falls behind. Describe the motion of the train.

9. Name the quantity which measures the motion of a body.

10. State the law of inertia.

11. According to Newton's 3rd law, every force is accompained by an equal and opposite force. How can a movement ever take place ?

12. A person sitting in a train moving with constant velocity along a straight line throws a ball vertically upward. Will the ball return to thrower's hand ?

13. Two bodies of mass M and m are allowed to fall down from same height. If air resistance for each be same, then will both the bodies reach the earth simultaneously.

14. A soda-water bottle is falling freely. Will the bubbles of the gas rise in water of the bottle ?

15. There is some water in a beaker placed on the pan of a spring balance. If we dip our finger in this water without touching bottom of beaker then what would be the effect on the reading of the balance?

16. For which physical quantity Kilo-watt hour is used as a unit?

17. If momentum of a body is doubled, how its K.E. will change ?

18. By which factor the momentum be changed so that K.E. is doubled?

19. What is K.E. of 1g body having velocity 1cm/s.

20. In elastic collision what are conserved ?

21. State gravitational unit of work.

22. Convert 1 joule into erg.

23. A bus and car, moving with same K.E. are brought to rest by application of brakes which provides equal retarding force. Which of them will come to rest faster ?

24. Define power.

25. Give dimension of power.

26. Give SI unit of power.

27. How much work is done when a labourer carrying books on his head moves on a level road from one place to another.

28. Does a man rowing a boat upstream that remains at rest w.r.to the shore do work ?

29. Define horse power.

30. How much work is done when a car moves with a uniform speed on a level road?

31. In a tug of war what is the work done ?

32. When a constant force is applied to a body moving wiht constant acceleration, is power of the force constant ? If not, how would force have to vary with speed for the power to be constant ?

GROUP - C

1. State the law of conservation of momentum.

2. What will be the direction in which the bob of a simple pendulum, at rest suspended inside a train, moves when the train starts speeding up ?

3. Why does a cricket player lower his hands to catch a ball?

4. Why is it difficult to walk on sand ?

5. When a bullet is fired from a gun, what is the origin of the force that accelerates the bullet ?

6. If a man in an elevator drops his briefcase but it does not fall to the floor, what can he conclude about the elevators motion ?

7. A force of 7 N acts on a body for 1 minute. What is the change in momentum produced in the body.

8. State the law of Conservation of energy.

9. A body is dragged 50m horizontally on a frictionless surface by applying a force of 60N at angle of 60^0 with ground. Find the work done.

10. A man raises a stone of 40kg through a distance of 1m in 4 seconds. While a child raises a stone of 10 kg through 4m in 1s. Which of them has greater power ?

11. Under the action of a force F a body of mass m kg moves such that its position x is a function of time t given by $x = \frac{1}{3}t^3$, where x is in meter and 't' in second. Find the work done by the force in first second.

12. Two bodies of masses m_1 and m_2 have equal K.E. show that their momentum p_1 and p_2 are in the ratio $\sqrt{\frac{m_1}{m_2}}$

13. A man ran up a stair case 9m height in 20 secs. If the man has a power of 270 W what is his weight ?

14. A very heavy body moving with a velocity of 10 m/s makes a head on collision with a very light body at rest. What are the velocities of two bodies after collision ?

15. A light body moving in a straight line with a velocity of 710 m/s, makes a head on collision with a very heavy body at rest. What are the velocities after collision ?

16. A light body and heavy body have same K.E. which one will have a greater momentum ?

17. A light body and a heavy body have same momentum which one will have greater K.E.

18. Will water at the foot of a waterfall be at different temperature from that at the top? If yes, explain.

19.. A meteorite burns in the atmosphere before it reaches earth's surface. What happens to its momentum ?

20. When a ball is thrown up its velocity decreases and then increases. Is the principle of conservation of momentum violated ?

21. Can a body have energy without having momentum ? explain.

22. Can a body have momentum without having energy ? Explain.

23. When two protons brought towards each other what happens to P.E.

24. When a proton and an electron are brought nearer what happens to P.E. ?

25. Explain how friction is a necessary evil.

26. What is the minimum stopping distance of a car of mass m moving with speed ϑ along a level road if the coefficient of static friction between tyres and road is μ.

27. What are the methods for reducing friction?

28. Give two examples where friction is artificially increased.

29. State the laws of friction.

30. Why does lubrication reduce friction?

31. Why is it easier to roll a barrel than push it along a road ?

32. What are the causes of sliding friction ?

GROUP - D

1. A force of 3000 dyne acts on a mass of 400g for 10 sec. Find the velocity and momentum of body after 10s.

2. A force $\vec{F} = (6\hat{i} - 8\hat{j})N$ acts on a mass of 2kg. Find the acceleration . What is its magnitude ?

3. A constant force acts on a body of mass 1kg for 5 s; and then ceases to work. In the next 5 seconds the body describes 100m. Calculate the force that acted on the body.

4. A 1000 kg car is travelling at 36 km/hr. By the application of brakes it is brought to rest in 10m. Find the average force.

5. What force is required to change the velocity from 20 cm/s to 1000 cm/s in 20s. of a body of mass 500g.

6. A mass of 5 kg descending vertically draws up a mass of 3kg by means of a light string passing over a pulley. At the end of 4 s, the string breaks. How much higher the 3kg mass would go ?

7. Two metal blocks A and B of mass 2kg and 1kg respectively are in contact on a frictionless table. A constant horizontal force of 3N is applied to block A. Calculate the force of contact between the two blocks. If the same force is applied to block B, then what will be the force of contact ?

8. A glass marble whose mass is 100g falls from a height of 40m and rebounds to a height of 10m. Find the impulse and average force between the marble and the floor if the time of contact is 0.1 S.

9. A stone of mass 5kg falls from a height 40m and buries itself 2m deep in sand. Find the average resistance offered by sand and the time it takes to penetrate.

10. A machine gun of mass 10kg fires 25g bullets at the rate of 5 bullets per second with a speed of 500 m/s. What force in newton must be applied to the gun to hold it in position.

11. A hammer weighing 2kg and moving with a speed of 5m/s strikes the head of a nail and drives it 10cm into a wall. Neglecting the mass of the nail, calculate

(i) acceleration during impact

(ii) interval during the impact

(iii) impulse.

12. A 5kg shell is fired with a nozzle velocity of 700 m/s from a gun of barrel length 2.4m long. Calculate the average value of

(i) acceleration (ii) the time, (iii) the force (iv) impulse (v) momentum

13. The mass of a cyclist together with the bike is 90 kg. Calculate the increase in K.E. if the speed increases from 6km/h to 12 km / h.

14. A block of mass 2.0 kg moving at a speed of 10 m/s accelerates at 3.0 m/s^2 for 5.0 s. Compute its final kinetic energy.

15. A box is pushed through 4.0 m across a floor offering 100N. resistance. How much work is done by the resisting force?

16. A block of mass 5.0 kg slides down on incline of inclination 30^0 and length 10m. Find the work done by the force of gravity.

17. A particle moves from a point $\vec{r}_1 = 2m\hat{i} + 3m\hat{j}$ to another point $\vec{r}_2 = 3m\hat{i} + 2m\hat{j}$, during which a force $\vec{F} = 5N\hat{i} + 5N\hat{j}$ acts on it. Compute the work done by the force during the displacement.

18. A 12 hp motor has to be operated 8h/day. How much will it cost at the rate of 50 paise per KWh in 10 days ?

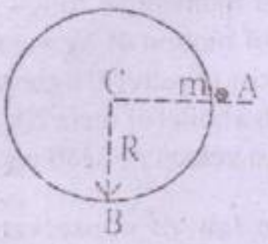
19. A 1hp motor operates for 1 hour per day. Calculate energy spent in 30 days.

20. If a man speeds up by 1 m/s, his K.E is doubled. Find his original speed.

21. If the momentum of a body is increased by 30% what will be the percentage increase in K.E. ?

22. A man of mass m falls through a height h and lands on a spring of spring constant k. Find the compression x in the spring. Assume that $x << h$.

23. A body of mass m slides down a curved frictionless path (that is quarter of a circle of radius R) from point A to point B. What is its velocity at B ?



24. A student holds a book of mass 1kg between his hands. Each hand exerts horizontally a force of 49 N. Find the coefficient of static friction between the hand and the book.

25. The coefficient of friction between a block of wood of mass 20kg and table surface is 0.25. What force is needed to give it an acceleration of $0.2\ m/s^2$.

26. A pick-up truck of 3000 kg going at 25 m/s has its breakes locked and starts skidding with the engine shut off. If the coefficient of friction is 0.2, how far will the truck travel before stopping ?

27. A particle is dropped on a surface from a height of 9m. If the coefficient of restitution is 0.5, what is the total distance covered by the particle before coming to rest.

28. A block of mass 'm' moving at a speed ϑ collides with another block of mass 2m at rest. The lighter block comes to rest after collision. What is the coefficient of restitution?

29. A ball is dropped from a height 'h' on a horizontal surface. It bounces up and down and finally comes to rest in time t. Show that

$$t = t_0\left(\frac{1+e}{1-e}\right)$$

where t_0 is the time taken for the first fall and e is the coefficient of restitution.

30. A plastic ball is dropped from a height of 1 m and rebounds several times from the floor. If 1.3 s elapse from the moment it is dropped to the second impact with the floor, what is the coefficient of restitution?

31. The velocity of a body of mass 0.1 kg changes from $V_0 = (3\hat{i} + 4\hat{j})ms^{-1}$ to $\vec{V} = (7\hat{i} - 8\hat{j})ms^{-1}$. What is the change in its kinetic energy ?

32. A 5 kg object is raised vertically upward with a uniform speed to a height of 2m from the ground. Find the change in its potantial energy.

33. An elevator weighing 500 kg is to be lifted at a constant speed of $0.2\ ms^{-1}$. What would be the minimum horse power of the motor?

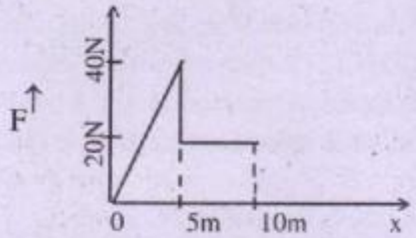
34. A force of 20 N is applied on a body of mass 5kg which is initially at rest. Find its kinetic energy at the end of 5s.

35. A block of mass 2kg slides down the surface of a frictionless bowl of diameter 1m, from the edge. Find the velocity of the block at the lowest point.

36. A block is gently placed at the top of an inclined plane 640 cm long. Find the time taken by the block to side down to the bottom of the plane. [Given that angle of inclination of the plane is 30°, $\mu_k = 0.2$ and $g = 9.8\ ms^{-1}$].

37. A particle moves from a position $(3\hat{i} + 2\hat{j} - 6\hat{k})$ to a position $(14\hat{i} + 13\hat{j} + 9\hat{k})$, due to a constant force $(4\hat{i} + 2\hat{j} + 3\hat{k})$. Find the work done by the force if all are in SI units.

38. Find the work done by a force as it pushes the body from x=0 to x=10m as shown in the figure.



39. Two bodies having kinetic energies in the ration of 4:1 are moving with equal linear momentum. Find the ratio of their masses.

40. A body of mass 5kg moving with a velocity of $10ms^{-1}$ collides with another body of mass 20 kg at rest and comes to rest. Find the velocity of the second body.

41. A metal ball of mass 2kg moving with a speed of 36 km/hr has a head-on collision with a stationary ball of mass 3kg. If after collision, both the balls move together, find the loss in kinetic energy due to collision.

42. A body of mass 3kg is under a force which causes a displacement in it given as $s = \frac{t^2}{3}$ (m). Find the work done by the force in 2s.

43. A child is swinging a swing. Maximum and minimum heights of the swing from ground are 2m and 0.75m respectively. Find the maximum velocity of the swing. $(g = 10ms^{-1})$

GROUP - E

Long Answer Type Qustions :

1. State Newton's 2nd law of motion and show how it leads to (i) force = mass x acceleration. (ii) show how Newton's 1st law follows from 2nd law.

2. How does the law of conservation of linear momentum follow from the third law of motion of Newton ? What is the velocity of recoil of a gun weighting 15kg when a bullet of mass 20g is fired from it with a velocity of $250\ ms^{-1}$.

3. State law of conservation of linear momentum. Use it to explain rocket propulsion.

4. What is kinetic energy of a body ? Derive an expression for the same.

5. Derive an expression for gravitational potential energy and prove that it is independent of the path followed.

6. State law of conservation of energy. Show that a freely falling body conserves energy.

7. State the principle of conservation of linear momentum. Show that when a body collides head on with a stationary body of same mass, the first body comes to rest and the second body moves with a velocity equal to that of the first body before collision.

8. Explain elastic and inelastic collision. Obtain equations governing elastic collision in 1dimension and solve for final velocities of two colliding particles considering one particle to be at rest before collision.

9. Define angle of repose and angle of friction. Establish relation between them.

10. State the laws of limiting friction ? How would you verify them.

F. Fill in the Blank Type

1. Definition of force comes from Newton's .... law.

2. The linear momentum p of a particle varies with time as $p = a + bt^2$, where a and b are constants. The net force acting on the particle is ....

3. A man of weight w is standing on a lift which is moving upward with an acceleration a. The apparent weight of the man is............

4. A body of weight w^1 is suspended from the ceiliing of a room through a chain of weight W. The ceiling pulls the chain by a force ....

5. Newton's second law of motion gives the measurement of ....

6. The working of a rocket is based on ....

7. The distance travelled by a body before coming to rest, if its initial velocity is 36 km/hr and coefficient of friction between the body and surface is 0.2, is ....

G. True - False Type

1. Newton stated that only an external force can change the state of motion of the centre of mass of body. Now the internal force of the brakes can stop a running car. It proves that Newton's law is wrong.

2. A rocket moves forward by pushing the surrounding air backward.

3. Since action and reaction are always equal and opposite, they cancel each other.

4. A bullet is fired from a rifle. If the rifle recoils freely, the kinetic energy of the rifle is less than that of the bullet.

5. When a person walks on a rough surface the frictional force exerted by the surface on the person is opposite to the direction of motion.

ANSWERS

I Multiple Choice Type Questions :

1. (a)	2. (b)	3. (b)	4. (a)	5. (c)	6. (a)	7. (a)	8. (c)
9. (c)	10. (b)	11. (d)	12. (a)	13. (c)	14. (c)	15. (b)	16.(c)
17. (c)	18. (c)	19. (d)	20. (a)	21. (c)	22. (d)	23. (a)	24.(a)
25. (c)	26. (a)	27. (b)	28. (a)	29. (d)	30. (a)		

GROUP - D

1. 75 cm/s, 3×10^4 g.cm/s
2. $\vec{a}=(3\hat{i}-4\hat{j})ms^{-2}$, a = $5ms^{-2}$
3. 4N.
4. 5000 N
6. 4.9 m
7. (i) i N (ii) 2 N
8. 4.2 Ns, 42 N
9. 980 N, 0.143 s
10. 62.5 N
11. 125 m s^{-2}, 0.04 s, 10 Ns
12. (i) $1.021\times10^5 ms^{-2}$ (ii) 6.857×10^{-3}s
(iii) 5.1×10^5N. (iv) 3500 Ns
(v) 5300 N/m
13. 375 J
14. 625 J
15. 400 J
16. 245 J
17. zero
18. Rs. 358.08
19. 22.38 KWh
20. $\frac{1}{\sqrt{2}-1}ms^{-1}$
21. 69 %
22. $x=(2mgh/k)^{1/2}$
23. $\sqrt{2gR}$
24. 0.1
25. 53 N 26. 159.44 m
27. 15 m
28. 0.5 30. 0.94 31. 4.4 J
32. 98 J 33. 13.14 HP 34. 1000 J
35. 4.43 ms^{-1} 36. 20 s 37. 111 J
38. 200 J 39. 1:4 40. 2.5 ms^{-1}
41. 60 J 42. $\frac{8}{3}J$ 43. 4.95 ms^{-1}

[Hints : $h_1=e^2h$, $h_2=e^2h_1=e^4h$,

$h_3=e^2h_2=e^6h$ ... and so on

$$\therefore H=h+2h_1+2h_2+2h_3+...$$
$$=h+2e^2h+2e^4h+2e^6h+...$$
$$=h+2he^2\left(1+e^2+e^4+e^6...\right)$$
$$=h+2he^2\frac{1}{1-e^2}=\frac{1+e^2}{1-e^2}h=15m]$$

F. (1) First (2) 2bt (3) W (1+a/g) (4) W+W' (5) force (6) law of concervation of momentum (7) 25.5 m.

G. (1) False (sliding friction between wheels and the road stops the car, which is an external force) (2) False (3) False (4) True (5) False.

5

Circular Motion

5.1 Circular Motion

When a body (treated as a point particle) moves in such a way that its distance from a fixed point always remains constant, then the body is said to be executing circular motion.

The fixed point is called centre of the circular path and the fixed distance is called the radius of the circle.

Circular motion is an example of motion in 2-dimensions. We come across such motion very often e.g. the tip of the second-hand of a clock moves along a circular path, electrons move around the nucleus in an atom almost in circular path.

5.2 Rotational Motion

In case of an extended body if the particles constituting the body, moves along circles with centres on a straight line, called axis of rotation, then the body is said to be executing rotational motion.

Thus circular motion is the rotational motion of a point particle.

5.3 Rotational Kinematics

For study of rotational kinematics one needs to be acquainted with the following angular variables.

(i) Path :

The curve traced by the tip of the radius vector $\vec{OP}$ (straight line joining the fixed point 'O' and the particle 'P') which is either a complete circles as in fig. 5.1 (a) or part of a circle as in fig 5.1 (b), is called the path.

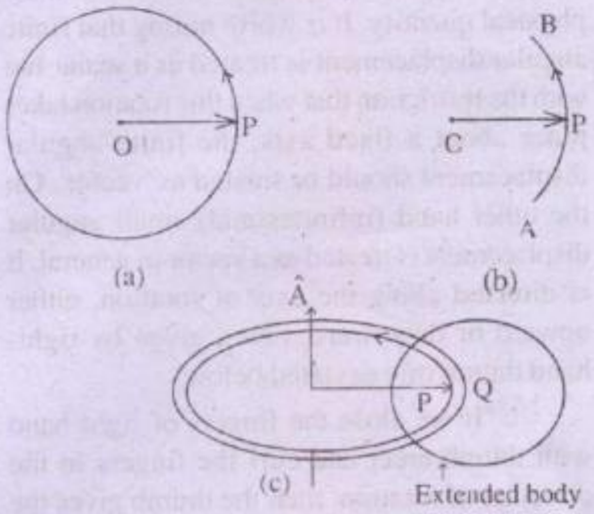


Fig. 5.1

In case of an extended body each particle constituting the body describes a circular path about the axis of rotation, with centres on the axis of rotation.

(ii) Angular Displacement :

The angle described by the radius vector $\vec{OP}$ during a time interval is called angular displacement.

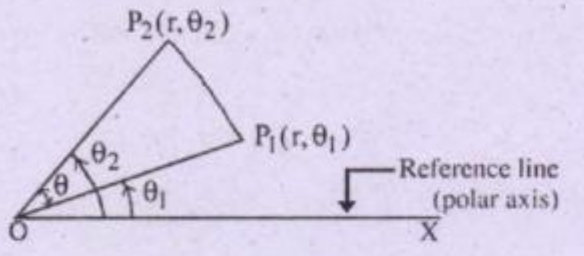


Fig. 5.2

Suppose a particle is at P_1 at time t_1 and at P_2 at a later time t_2, then

angular displacement $(\theta) = \angle P_1OP_2$

Since angles are always measured w.r. to a reference line, called as Polar-axis, so in fig. 5.2, we have $\theta = \theta_2 - \theta_1$. The angles θ_1 and θ_2 specify the positions P_1 and P_2 respectively w.r. to the polar axis (OX).

Angular displacement is a dimensionless physical quantity. It is worth noting that finite angular displacement is treated as a scalar but with the restriction that when this rotation takes place about a fixed axis, the finite angular displacement should be treated as 'vector'. On the other hand (infinitesimal) small angular displacement is treated as a vector in general. It is directed along the axis of rotation, either upward or downward, being given by right-hand thumb rule as stated below.

"If we close the fingers of right hand with thumb erect and curl the fingers in the direction of rotation, then the thumb gives the direction of angular displacement. Thus

$$\overrightarrow{\Delta\theta} = \Delta\theta\hat{A}$$

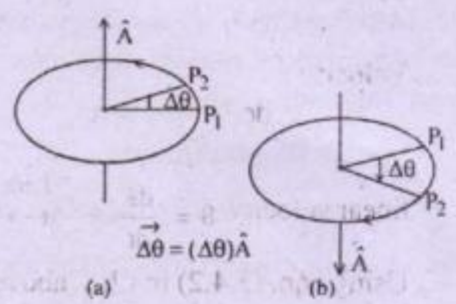


Fig. 5.3

Angular displacement is measured in (i) degrees (ii) radians or in (iii) revolutions; and they are related as

1 rev. = 2 π rad. = 360 deg.

(iii) Angular Velocity (ω) :

Angular velocity is defined as the time rate of change of angular displacement.

However, we need to distinguish between average angular velocity and instantaneous angular velocity.

(a) Average angular velocity :

Suppose as shown in fig. 5.2, θ_1 and θ_2 specify the position of the particle at times t_1 and t_2 respectively. If the angular displacement $\theta = \theta_2 - \theta_1$ and time interval $t = t_2 - t_1$ are finite, then the average angular velocity is defined as

$$\langle\omega\rangle = \frac{\theta}{t} = \frac{\theta_2 - \theta_1}{t_2 - t_1} \qquad ...(5.3.1)$$

Thus except in case of rotation about fixed axis, average angular velocity is treated as a scalar. Hence one sometimes gets inclined to call it angular speed. But usually we donot take note of this distinction.

(b) Instantaneous angular velocity :

If the time interval is small so that the angular displacement is also infinitesimally small, then instantaneous angular velocity is given as

$$\vec{\omega} = \lim_{\Delta t \to 0} \frac{\overrightarrow{\Delta\theta}}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t}\hat{A}$$

$$\Rightarrow \quad \vec{\omega} = \frac{d\theta}{dt}\hat{A} \qquad ...(5.3.2)$$

Instantaneous angular velocity is a vector quantity and it is in the direction of angular displacement. Its dimension is given as

$$[\omega] = \frac{[\theta]}{[t]} = T^{-1}$$

and is measured in (i) deg/sec, (ii) rad/sec or (iii) rev./sec.

(iv) Angular Acceleration (α)

The time rate of change of angular velocity is called angular acceleration.

Here again we can have average angular acceleration and instantaneous angular acceleration.

(a) Average Angular Acceleration $\langle\vec{\alpha}\rangle$

Suppose a body executing rotational motion possesses angular velocity $\vec{\omega}_1$ at time t_1 and $\vec{\omega}_2$ at time t_2. Then average angular acceleration is given as

$$\langle\vec{\alpha}\rangle = \frac{\vec{\omega}_2 - \vec{\omega}_1}{t_2 - t_1} \quad ...(5.3.3)$$

(b) Instantaneous Angular Acceleration $\langle\vec{\alpha}\rangle$

If the time interval is so small that the change in angular velocity is also very small, then one defines

$$\vec{\alpha} = \lim_{\Delta t \to 0} \frac{\Delta\vec{\omega}}{\Delta t} = \frac{d\vec{\omega}}{dt} = \frac{d\omega}{dt}\hat{A} \quad ...(5.3.4)$$

Angular acceleration has dimension T^{-2} and is measured in (i) deg/sec^2 (ii) rad / sec^2 or (iii) rev./sec^2.

5.4 Relation Between Linear and Angular Motion

(i) Displacement

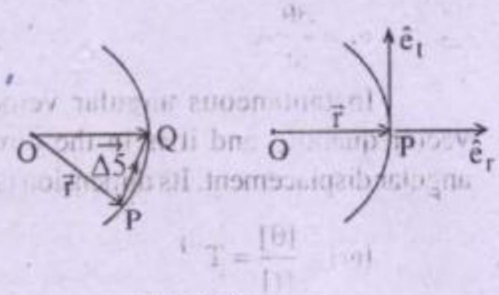


Fig. 5.4

Consider a body executing circular motion about a fixed point 'O' and moving from P to Q.

Then

$$\text{linear displacement } \vec{PQ} = \vec{\Delta S} = \Delta S.\hat{e}_t$$

$$\text{angular displacement } \vec{\Delta\theta} = \Delta\theta.\hat{A}$$

Where $\hat{A}$ is the unit vector, perpendicular to the plane of the circle and passes through the point 'o', and obeys the right hand thumb rule.

Now, $\lim_{\Delta t\to 0}$ Arc PQ $\rightarrow$ Chord. PQ = ΔS

But arc $PQ = r.\Delta\theta$

Therefore

$$\lim_{\Delta t\to 0} r.\Delta\theta = \Delta S$$

i.e. for infinitesimal angular and linear displacements, occuring over infinitesimal time interval.

$$\Delta S = r.\Delta\theta \quad ...(5.4.1)$$

Eqn. (5.4.1) is the scalar form of relation between linear displacement (ΔS) and angular displacement ($\Delta\theta$).

Now we note that $\hat{A} \times \hat{e}_r = \hat{e}_t$, which gives

$$\Delta\theta\hat{A} \times r\hat{e}_r = r\Delta\theta\hat{e}_t$$

i.e. $$\vec{\Delta\theta} \times \vec{r} = \Delta S\hat{e}_t = \vec{\Delta S} \quad ...(5.4.2)$$

(ii) Velocity :

As defined earlier

$$\text{linear velocity } \vec{\vartheta} = \frac{d\vec{s}}{dt} = \lim_{\Delta t\to 0}\frac{\vec{\Delta S}}{\Delta t}$$

Using eqn. (5,4.2) in r.h.s. above

$$\vec{\vartheta} = \lim_{\Delta t \to 0} \frac{\vec{\Delta\theta} \times \vec{r}}{\Delta t} = \left(\lim_{\Delta t \to 0} \frac{\vec{\Delta\theta}}{\Delta t} \right) \times \vec{r}$$

$$\Rightarrow \vec{\vartheta} = \vec{\omega} \times \vec{r} \quad ...(5.4.3)$$

Using $\omega = \omega\hat{A}$, $\vec{r} = r\hat{e}_r$, $\hat{A} \times \hat{e}_r = \hat{e}_t$ in eqn. (5.4.3) one obtains

$$\vec{\vartheta} = r\omega\,\hat{e}_t \quad ...(5.4.4)$$

This gives

$$\left| \vec{\vartheta} \right| = r\omega \quad ...(5.4.5)$$

Eqn. (5.4.3) and (5.4.4) are vector relations while eqn. (5.4.5) is the scalar relation.

(iii) **Acceleration :**

By definition ,

linear acceleration $\vec{a} = \frac{d\vec{\vartheta}}{dt} = \frac{d}{dt}(\vec{\omega} \times \vec{r})$

$$\Rightarrow \vec{a} = \frac{d\vec{\omega}}{dt} \times \vec{r} + \vec{\omega} \times \frac{d\vec{r}}{dt}$$

$$\Rightarrow \vec{a} = \vec{\alpha} \times \vec{r} + \vec{\omega} \times \vec{\vartheta} \quad ...(5.4.6)$$

Since

$\vec{\alpha} = \alpha\hat{A}$, $\vec{\omega} = \omega\hat{A}$, $\vec{r} = r\hat{e}_r$, $\vec{\vartheta} = r\omega\,\hat{e}_t = r\omega(\hat{A} \times \hat{e}_r)$

so

$$\vec{a} = (\alpha\hat{A} \times r\hat{e}_r) + \omega\hat{A} \times r\omega(\hat{A} \times \hat{e}_r)$$

$$= r\alpha\hat{e}_t + r\omega^2\left[(\hat{A}.\,\hat{e}_r)\hat{A} - (\hat{A}.\hat{A})\hat{e}_r\right]$$

$$\vec{a} = r\alpha\hat{e}_t - r\omega^2\hat{e}_r \quad ...(5.4.7)$$

$$\text{Thus } \vec{a} = \vec{a}_t + \vec{a}_r \quad ...(5.4.8)$$

where $\vec{a}_t = r\alpha\hat{e}_t$ is the tangential acceleration, and $\vec{a}_r = -r\omega^2\hat{e}_r$ is the radial acceleration (See fig. 5.5)

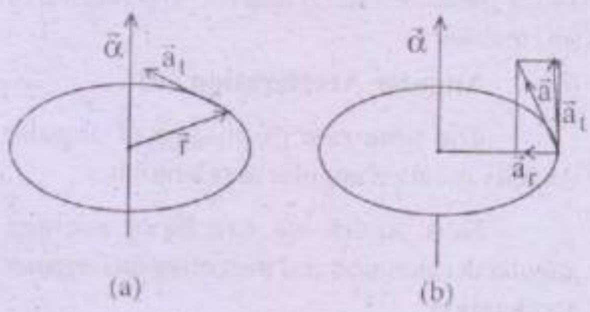


fig. 5.5

5.5 Equations of Rotational Motion :

The equations of rotational motion follow from the defining equations

$$\vec{\alpha} = \frac{d\vec{\omega}}{dt} = \frac{d\omega}{dt}\hat{A}$$

and

$$\vec{\omega} = \frac{d\vec{\theta}}{dt} = \frac{d\theta}{dt}\hat{A}$$

Where $\hat{A}$ is the unit vector along the axis of rotation obeying right hand thumb rule.

Since $\vec{\alpha} = \alpha\hat{A}$, and $\vec{\omega} = \omega\hat{A}$ so

$$\alpha = \frac{d\omega}{dt} \quad ...(5.5.1)$$

and

$$\omega = \frac{d\theta}{dt} \quad ...(5.5.2)$$

Eqn. (5.5.1) gives

$$d\omega = \alpha\,dt \quad ...(5.5.3)$$

Integrating both sides of eqn 5.5.3 within proper limits one obtains

$$\int_{\omega_0}^{\omega} d\omega = \int_0^t \alpha dt \quad ...(5.5.4)$$

where, 'ω_0' is the angular velocity of the body at the initial time t = 0, and ω is the angular velocity at time t. If the body is uniformly accelerated (ie α = constant), then eqn. (5.5.4) reduces to

$$\int_{\omega_0}^{\omega} d\omega = \alpha \int_0^t dt$$

giving

$$\omega - \omega_o = \alpha t$$

or $$\omega = \omega_o + \alpha t \quad ...(5.5.5)$$

Similarly considering eqn. (5.5.2) one obtains

$$d\theta = \omega\, dt \quad ...(5.5.6)$$

Using eqn. (5.5.5) on r.h.s. of eqn (5.5.6)

$$d\theta = (\omega_o + \alpha t)\, dt \quad ...[5.5.6(a)]$$

Integrating both sides within proper limits

$$\int_{\theta_o}^{\theta} d\theta = \int_o^t (\omega_o + \alpha t) dt$$

One obtains

$$\theta - \theta_o = \omega_o t + \frac{1}{2}\alpha t^2 \quad ...(5.5.7)$$

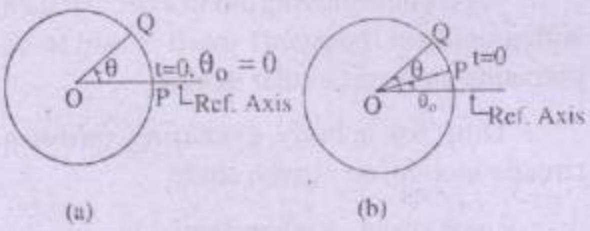


Fig 5.6

The l.h.s. of eqn (5.5.7) gives the angular displacement over a time interval t and θ_0 refers to the angular position of the body at time t = 0, with respect to ref. axis. If we choose the initial position vector $\vec{OP}$ as the ref. axis, then $\theta_0 = 0$ and eqn. (5.5.7) reduces to

$$\theta = \omega_0 t + \frac{1}{2}\alpha t^2 \quad ...(5.5.8)$$

In any case r.h.s. of eqns. (5.5.7) & (5.5.8) gives the angular displacement over a time interval t.

Combining eqns. (5.5.1) and (5.5.2) we have

$$\omega \frac{d\omega}{dt} = \frac{d\theta}{dt}\alpha \quad ...(5.5.9)$$

$$\Rightarrow \frac{d}{dt}\left(\frac{1}{2}\omega^2\right) = \frac{d}{dt}(\alpha\theta) \quad (\because \alpha \text{ is constant})$$

Integrating both sides w.r. to time

$$\int_{\omega_o}^{\omega} \frac{d}{dt}\left(\frac{1}{2}\omega^2\right) dt = \int_{\theta_0}^{\theta} \frac{d}{dt}(\alpha\theta)\, dt$$

One obtains

$$\frac{1}{2}(\omega^2 - \omega_0^2) = \alpha(\theta - \theta_0)$$

giving

$$\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0) \quad ...(5.5.10)$$

It is to be noted that $(\theta - \theta_0)$ occuring in r.h.s. of eqn (5.5.10) is the angular displacement over the time interval t. If the initial position vector $(\vec{OP})$ is chosen as ref. axis so that $\theta_0 = 0$, then eqn. (5.5.10) assumes the form

$$\omega^2 = \omega_0^2 + 2\alpha\,\theta \quad ...(5.5.11)$$

The angular displacement suffered by a body in the n_{th} second is obtained on integration of eqn. 5.5.6.(a)

i.e. $$\int_{\theta_{n-1}}^{\theta_n} d\theta = \omega_0 \int_{n-1}^{n} dt + \alpha \int_{n-1}^{n} t\,dt$$

This gives

$$\theta_n - \theta_{n-1} = \theta = \omega_o + \frac{1}{2}\alpha(2n-1) \quad ...(5.5.12)$$

Thus eqns (5.5.5) to (5.5.12) are the required equations of motion for a body rotating with uniform (constant) angular acceleration.

Ex. 5.5.1 A particle travels in a circle of radius 20 cm at a speed that uniformly increases. If the speed changes form 5 m/s to 6m/s in 2.0 s, find the angular acceleration.

Soln.

Given $\vartheta_1 = 5$ m/s, $\vartheta_2 = 6$m/s; r = 20cm

= 0.2m, t= 2.0 s.

Hence $\omega_1 = \frac{\vartheta_1}{r} = 25$ rad/s; $\omega_2 = \frac{\vartheta_2}{r} = 30$rad/s

$$\therefore \alpha = \frac{\omega_2 - \omega_1}{t} = \frac{(30-25)\text{rad/s}}{2.0\text{s}}$$

$= 2.5$ rad/s^2

Ex. 5.5.2 A wheel has its speed increased from 120 to 240 rev/min in 20 s. (a) what is the angular acceleration, (b) how many revolutions of the wheel are required.

Soln.

Given $\omega_1 = 120$ rev/min = 2 rev/s

$\omega_2 = 240$ rev/min = 4 rev/s

t = 20 s.

Hence (a)

$$\alpha = \frac{\omega_2 - \omega_1}{t} = \frac{(4-2)\text{rev/s}}{20\text{s}} = \frac{1}{10}\text{rev/s}^2$$

Now (b) $\theta = \omega_1 t + \frac{1}{2}\alpha t^2$

$$= 2\frac{\text{rev}}{\text{s}}20\text{s} + \frac{1}{2}\cdot\frac{1}{10}\frac{\text{rev}}{\text{s}^2}\cdot(20\text{s})^2$$

$\theta = 40$ rev + 20 rev = 60 rev.

Thus angular acceleration is $\frac{1}{10}$ rev/s^2, and 60 revolutions are required.

Ex. 5.5.3 A particle moves along a circle of radius 1.0 cm at a speed given by $\vartheta = 2.0$ t, where ϑ is in cm/s and t in seconds. (a) Find the tangential acceleration at t= 1 s. (b) Find the radial acceleration at t = 1s. (c) Find the magnitude of the acceleration.

Soln.

$$a_t = r\alpha = r\frac{d\omega}{dt} = \frac{d}{dt}(r\omega) = \frac{d\vartheta}{dt}$$

Therefore

$$a_t = 2\text{ cm/s}^2$$

Radial acceleration

$$a_r = r\omega^2 = \frac{\vartheta^2}{r} = \frac{[2.(1)]^2}{1}\text{cm/s}^2$$

$$\therefore a_r = 4\text{ cm/s}^2$$

$$a = \sqrt{a_t^2 + a_r^2} = \sqrt{2^2+4^2}\text{cm/s}^2 = \sqrt{20}\text{ cm/s}^2$$

5.6 Uniform Circular Motion :

A particle moving along a circular path with a uniform (constant) speed is said to be executing uniform circular motion.

Thus for a body executing uniform circular motion, on a given circle

linear speed, ϑ = constant

radius, r = constant

Hence, angular speed ω = constant ... (5.6.1)

However, although the speed remains constant, its direction constantly changes. Hence the velocity vector $\vec{\vartheta}$ is not constant. Therefore a uniform circular motion is an accelerated motion and an unbalanced force must exist which generates the acceleration.

We now proceed to find expression for the acceleration and the force generating it.

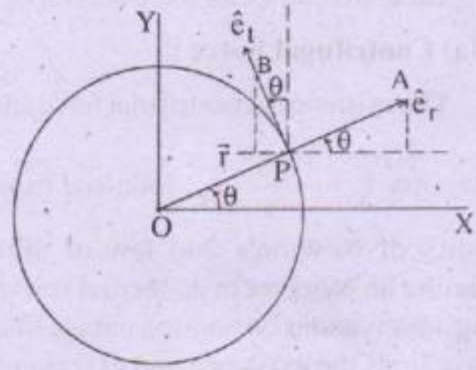


fig. 5.7

Consider a body of mass 'm' executing uniform circular motion along a circular path of radius r with linear speed ϑ. Let its centre 'o' be chosen as origin. Then $\overrightarrow{OP} = \vec{r}$ is the position vector. If $\hat{e}_r$ be unit vector along this radial direction, then

$$\vec{r} = r\hat{e}_r \qquad ...(5.6.2)$$

Let $\hat{e}_t$ be the unit vector along the tangential direction at P. From fig. 5.7 one easily sees that

$$\overrightarrow{PA} = (\overline{PA})\,\hat{e}_r = (\overline{PA}\cos\theta)\hat{i} + (\overline{PA}\sin\theta)\hat{j}$$

$$\overrightarrow{PB} = (\overline{PB})\,\hat{e}_t = (\overline{PB}\sin\theta)(-\hat{i}) + (\overline{PB}\cos\theta)\hat{j}$$

and these reduce to

$$\hat{e}_r = \text{Cos}\theta\;\hat{i} + \text{Sin}\theta\;\hat{j} \qquad ...(5.6.3)$$

$$\hat{e}_t = -\text{Sin}\theta\;\hat{i} + \text{Cos}\theta\hat{j} \qquad ...(5.6.4)$$

Since θ change with time (ie $\theta = \theta(t)$), so $\hat{e}_r$ and $\vec{e}_t$ are functions of time , On differentiation of eqns (5.6.3) &(5.6.4) one finds

$$\frac{d\hat{e}_r}{dt} = -\text{Sin}\theta\frac{d\theta}{dt}\;\hat{i} + \text{Cos}\theta\frac{d\theta}{dt}\;\hat{j}$$

$$\Rightarrow \frac{d\hat{e}_r}{dt} = \omega\,(-\sin\theta\;\hat{i} + \text{Cos}\;\hat{j}) = \omega\hat{e}_t \qquad ...(5.6.5)$$

and

$$\frac{d\hat{e}_t}{dt} = -\text{Cos}\theta\frac{d\theta}{dt}\;\hat{i} - \text{Sin}\theta\frac{d\theta}{dt}\;\hat{j}$$

$$\Rightarrow \frac{d\hat{e}_t}{dt} = -\omega(\cos\theta\;\hat{i} + \sin\theta\;\hat{j}) = -\omega\hat{e}_r \qquad .....(5.6.6)$$

Therefore for a body executing uniform circular motion, the linear velocity is given as

$$\vec{\vartheta} = \frac{d\vec{r}}{dt} = \frac{d}{dt}(r\hat{e}_r) = r\frac{d\hat{e}_r}{dt} = r\omega\hat{e}_t \qquad ... (5.6.7)$$

[where we have treated r = constant as per eqn. (5.6.1)]

Now the linear acceleration for the body is given as

$$\vec{a}_c = \frac{d\vec{\vartheta}}{dt} = \frac{d}{dt}(r\omega\hat{e}_t) = r\omega\frac{d\hat{e}_t}{dt}$$

[$\because$ r and ω are constant as per eqn (5.6.1)]

Using (5.6.6.) on r.h.s.

$$\vec{a}_c = -r\omega^2\hat{e}_r = -\frac{\vartheta^2}{r}\hat{e}_r \qquad ...(5.6.8)$$

Equation (5.6.8) shows that the acceleration of a body, executing uniform circular motion is always directed towards the centre and has a constant magnitude equal to $r\omega^2$ $(= \vartheta^2/r)$. This acceleration is called **centripetal acceleration.**

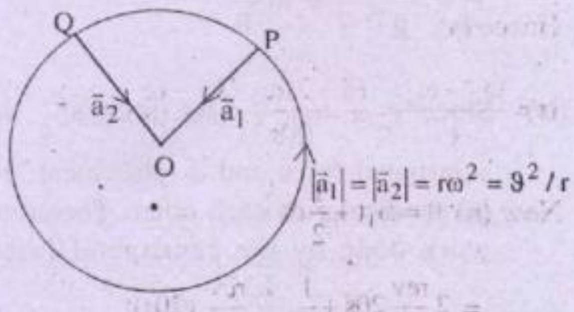


fig. 5.8

Eqn. (5.6.8) shows that centripetal acceleration a_c satisfies $a_c \alpha\ \vartheta^2$ and $a_c\ \alpha\ 1/r$.

Now according to Newton's 2nd law of motion the force acting on the body of mass 'm' executing uniform circular motion must be

$$\vec{f}_C = m\vec{a}_C = -\frac{m\vartheta^2}{r}\hat{e}_r . \qquad ...(5.6.9)$$

This force $\vec{f}_C$ is known as **centripetal force.** Thus **"Centripetal force is the force required to keep a body moving along a circular path with uniform speed".** But although a separate name is attached to it (as it is centre seeking), it is just like any other ordinary force that generates acceleration. Eqn. (5.6.9) shows that, $f_c\ \alpha\ m.\ f_c\ \alpha\ \vartheta^2$ and $f_c\ \alpha\ 1/r$ and is more sensitive to variations in speed ϑ.

We note the following features about centripetal force ($\vec{f}_C$).

(i) It is always directed towards the centre.

(ii) It depends on mass (m), speed (ϑ) and radius (r) as $f_c \alpha\ m,\ f_c \alpha\ \vartheta^2, f_c\ \alpha\ 1/r$.

(iii) In the Newton's equation of motion $\Sigma\vec{F} = m\vec{a}$, the sum of forces must include the real physical forces and the quantity $a = \vartheta^2/r$, must be included in r.h.s. For example in the planetary motion of earth around sun $F\left(=\frac{GMm}{R^2}\right) = m\frac{\vartheta^2}{R}$

(iv) Since $\vec{f}_C = -r\omega^2\,\hat{e}_r$ and $\vec{ds} = ds\,\hat{e}_t$, so centripetal force and displacement are perpendicular to each other. Therefore work done by the centripetal force $= \vec{f}_c \cdot \vec{ds} = 0$

(v) The body executing uniform circular motion is not in equilibrium as a net force equal to centripetal force acts on it.

5.6 (a) Contrifugal Force :

The existence of centripetal force and the expression $\vec{f}_C = -\frac{m\vartheta^2}{r}\,\hat{e}_r$, followed from the validity of Newton's 2nd law of motion. Therefore an observer in an inertial frame and taking observations on uniform circular motion always finds the existence and expression for centripetal force correct. But one has to investigate what would be the observations by an observer in a non-inertial frame of reference. For that we discuss as below:

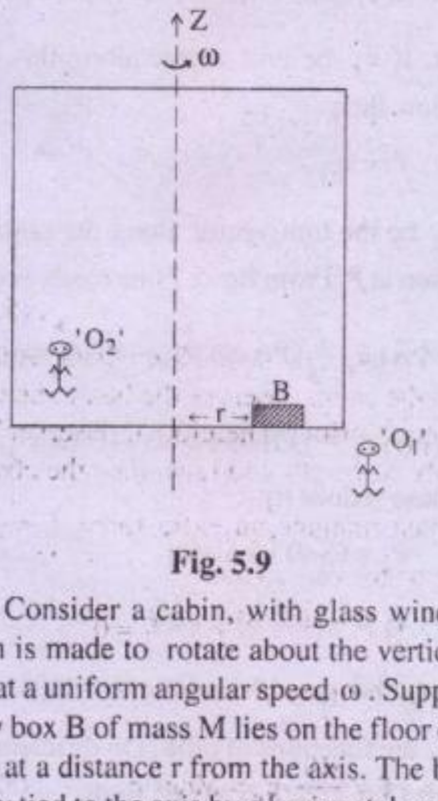


Fig. 5.9

Consider a cabin, with glass windows, which is made to rotate about the vertical Z-axis, at a uniform angular speed ω . Suppose a heavy box B of mass M lies on the floor of the cabin at a distance r from the axis. The box is rigidly tied to the axis by a horizontal rope.

An observer 'O_1' standing outside the cabin (ground inertial frame) observes that the box is rotating with an angular speed 'ω'. According to observer 'O_1', the forces acting on the box are :

i) Weight $M\vec{g}$, acting vertically downward.

ii) Normal reaction $\vec{N}$, by the floor of the cabin, acting vertically upward.

iii) Tension $\vec{T}$, acting along the rope acting towards the axis of rotation.

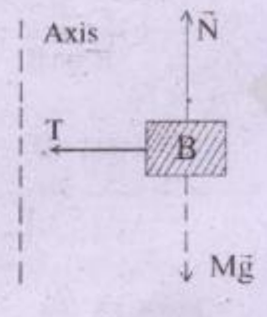


Fig. 5.10

such that

$$\Sigma\vec{F} = M\vec{g} + \vec{N} + \vec{T} = \vec{T} \qquad \dots (5.6.10)$$

Therefore the tension $\vec{T}$ must be providing the necessary centripetal force to keep the box moving in circular path with speed ϑ.

i.e. $$\vec{T} = -\frac{M\vartheta^2}{r}\hat{e}_r = -Mr\omega^2\,\hat{e}_r \qquad \dots (5.6.11)$$

On the other hand observer 'O_2' standing within the cabin observes the box to be at rest i.e. in equilibrium. Therefore if observer 'O_2' is to apply Newton's 2nd Law, then the observer 'O_2' must imagine an extra force $\vec{f}_i = -M\vec{a}_c$ acting on the box so that

$$\Sigma\vec{F} = M\vec{g} + \vec{N} + \vec{T} + \vec{f}_i = 0$$

$$\Rightarrow \vec{T} + \vec{f}_i = 0$$

i.e $$\vec{f}_i = -\vec{T} = \frac{M\vartheta^2}{r}\hat{e}_r = Mr\omega^2\,\hat{e}_r \qquad \dots (5.6.12)$$

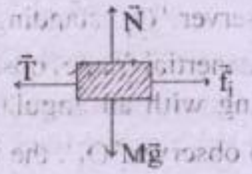


Fig. 5.11

Thus the observer 'O_2' (a non-inertial observer) should include this pseudo force if he wished to apply Newton's 2nd Law in his frame of reference. This psecudoforce is called **centrifugal force**. As an example consider a rotating merry-go-round on which a piece of marble is kept on a raised rim at the outer edge. If the marble is slightly pulled from the rim towards the axis and released; then it goes back to its previous position. It should also be noted that the centrifugal force is a sufficient pseudo force for an observer analysing a particle at rest on a uniformly rotating frame. But if we wish to analyse the motion of a particle that moves in the rotating frame, we may have to take account of other pseudo forces over and above the centrifugal force. Thus

"Centrifugal force is the pseudoforce that one has to take into account over and above other forces, for analysing the motion of a particle at rest in a uniformly rotating frame (applying Newton's Laws of motion)".

This force is called a fictitious or pseudo force (not a real force) as this does not arise due to the interaction of the body with the surrounding. Further one should also note that "the centrifugal force is not the reaction force of centripetal force". The reaction of the centripetal force acts on the agent, which exerts centripetal force, whereas centrifugal force acts on the body as viewed by the non-inertial observer in a uniformly rotating frame.

5.6(b) Examples of Uniform Circular Motion :

The effects of circular motion are most often felt in our day-to-day life. We discuss few of them.

(i) Centrifuge :

A centrifuge is a device to separate particles of different densities in a mixture.

It is essentially a bowl made to rotate with an angular speed ω, about an axis. Therefore the force on an element of mass m lying at distance r from the axis is

$$f_c = mr\omega^2 \quad \text{... (5.6.13)}$$

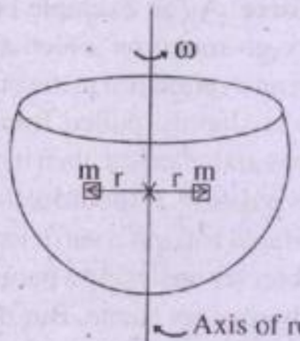


Fig. 5.12

Every element of mass 'm' in the layer at distance r must experience the force f_c. This implies that the acceleration experienced by an element of volume dv shall be

$$a_c = \frac{f_c}{\rho\, dv} \quad \text{... (5.6.14)}$$

Eqn. 5.6.14 implies that particles of larger density in the layer at distance r, shall have less (inward) acceleration, than particles of smaller density. Therefore particles of less density collect near the axis than particles of larger density, which are collected near the rim e.g. (a) In a mixture of iron dust and sand, iron dust is collected near the rim while sand is collected near the axis.

(b) When milk is rotated in a cylinder at high speed, cream is collected near the axis while creamless milk is collected near the edge.

(ii) Bending of a Cyclist :

Consider a cyclist negotiating a curve of radius 'r'. The cyclist tilts himself so that he is inclined at an angle θ to the vertical. Let 'G' be C.O.G. (Centre of Gravity) of the cyclist and cycle.

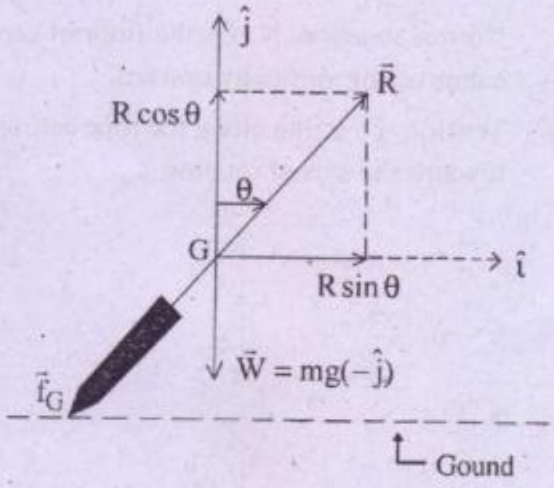


Fig 5.13

The forces acting on the cycle and cyclist are :

i) $\vec{W} = mg(-\hat{j})$, due to gravitational pull of the earth.

ii) $\vec{R}$ = Reaction force of the ground on the cycle and cyclist.

Therefore net force acting on the cycle and cyclist is

$$\vec{f}_{net} = \vec{R} + \vec{W}$$

$$= (R\sin\theta)\,\hat{i} + (R\cos\theta)\,\hat{j} - (mg)\,\hat{j}$$

$$\Rightarrow \vec{f}_{net} = (R\sin\theta)\,\hat{i} + (R\cos\theta - mg)\,\hat{j} \quad \text{...(5.6.15)}$$

Since there is no vertical component of motion so

$$R\cos\theta - mg = 0$$

$$\Rightarrow R\cos\theta = mg \quad \text{... (5.6.16)}$$

Hence eqn. (5.6.15) becomes

$$\vec{f}_{net} = R\sin\theta\,\hat{i} \quad \text{... (5.6.17)}$$

This force $\vec{f}_{net}$ given by eqn. (5.6.17) must provide the necessary centripetal force to keep the cyclist on the circular track of radius r. Therefore

$$f_{net} = R\sin\theta = \frac{m\vartheta^2}{r} \quad ...(5.6.18)$$

Eqn. (5.6.16) and (5.6.18) give

$$\tan\theta = \frac{\vartheta^2}{rg} \quad ...(5.6.19)$$

or

$$\theta = \tan^{-1}\left(\frac{\vartheta^2}{rg}\right) \quad ...(5.6.20)$$

Eqn. (5.6.19) or (5.6.20) gives the angle of inclination with the vertical, the cyclist has to make in order to negotiate a curve of radius r with speed ϑ. The eqns (5.6.19) and (5.6.20) show that the angle depends on

(a) Speed of cyclist : (greater the speed larger the angle and vice versa)

(b) radius of the curve : (smaller the radius of the curve, larger the angle and vice versa)

(iii) Motion of vehicle on a flat (horizontal) circular road :

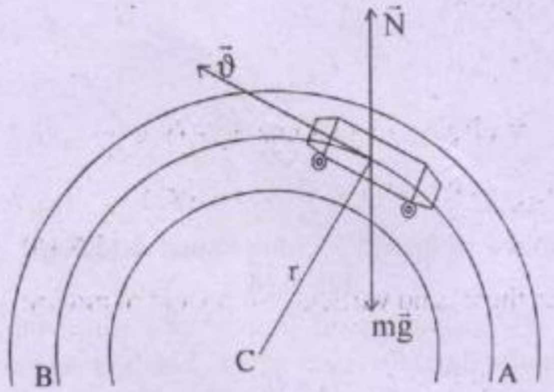


Fig. 5.14

Consider an automobile taking a turn on a horizontal road along the path AB (Fig. 5.14), the part of a circular track of radius r, with centre at C. The forces acting on the automobile are :

i) $\vec{W} = m\vec{g} = mg(-\hat{j})$ = Weight of the automobile.

ii) $\vec{N} = N\hat{j}$ = Normal reaction of the ground (vertically upwards as the road is horizontal)

iii) $\vec{f}_s$ = The force of friction.

The force of friction $\vec{f}_s$ is the only horizontal force that can act towards the centre. This arises due to the tendency of tyres to skid outward, so that force of friction is towards the centre. Therefore net force acting on the vehicle is

$$\vec{f}_{net} = N\hat{j} - mg\hat{j} + \vec{f}_s = (N - mg)\hat{j} + \vec{f}_s \quad ...(5.6.21)$$

Since there is no vertical component of motion so $N = mg$ ... (5.6.22)

and $$f_s = \frac{m\vartheta^2}{r} \quad ...(5.6.23)$$

Eqn. (5.6.23) shows that for a given mass m and given circular path of radius r, the force of friction is directly proportional to the square of the speed. It implies that if a vehicle is to negotiate with a large speed then the force of friction should be large. However there is a limit to force of friction i.e.

$f_s \le \mu_s N$. Hence

$$\frac{m\vartheta^2_{max}}{r} = (f_s)_{max} = \mu_s N = \mu_s mg$$

(using 5.6.22)

$$\Rightarrow \vartheta^2_{max} = \mu_s rg \quad ...(5.6.24)$$

So for safe turning, the vehicle should negotiate with a safe speed ϑ_s, given below

$$\vartheta_s \le \sqrt{\mu_s rg} \quad ...(5.6.25)$$

(iv) Motion of an automobile on a banked road : (neglecting friction) :

Since friction is not always reliable at circular turns for helping vehicle taking sharp turns, so usually roads are banked near the turns for an average speed of the vehicles i.e. the outer edge of the road is raised w.r. to the inner edge.

Consider an automobile moving on a smooth banked road as shown in fig. 5.15

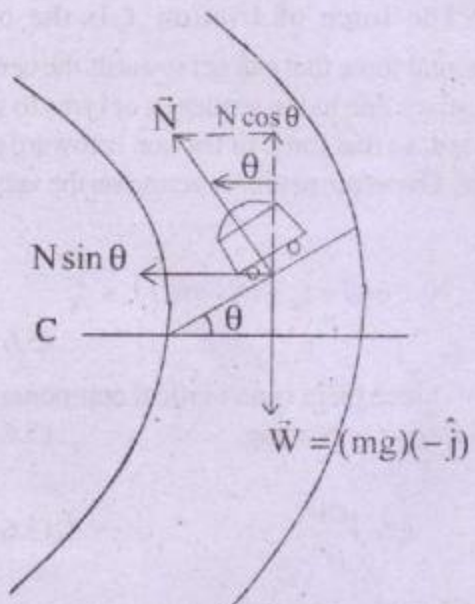


Fig. 5.15

The forces acting on the vehicle are :

i) $\vec{W} = -mg\,\hat{j}$ = Weight of the vehicle acting vertically down wards.

ii) $\vec{N}$ = Normal reaction of the road acting perpendicular to the surface of the road.

Therefore the net force acting on the vehicle is

$$\vec{f}_{net} = \vec{W} + \vec{N}$$

$$= -mg\,\hat{j} + N\,Cos\theta\,\hat{j} - N\,Sin\theta\,\hat{i}$$

$$\Rightarrow \vec{f}_{net} = -N\,Sin\theta\hat{i} + (N\,Cos\theta - mg)\hat{j} \quad ...(5.6.26)$$

Since there is no vertical component of motion so $N\,Cos\,\theta = mg$ ... (5.6.27)

and the horizontal component $N\sin\theta$, must provide the necessary centripetal force. So

$$N\sin\theta = m\frac{\vartheta^2}{r} \quad ... (5.6.28)$$

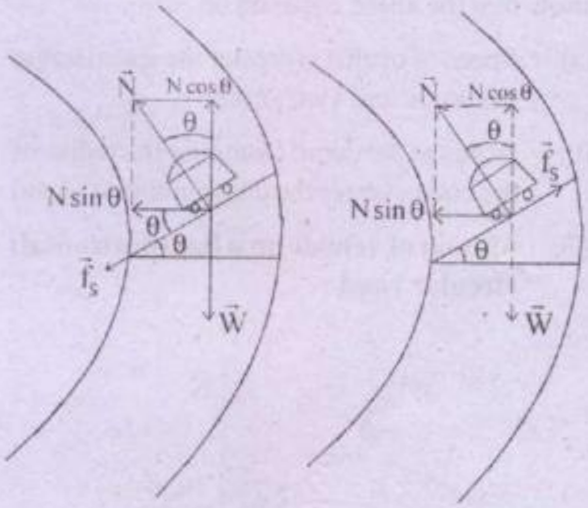
From eqns. (5.6.27) & (5.6.28) one finds

$$\tan\theta = \frac{\vartheta^2}{rg} \quad ... (5.6.29)$$

As for a given banked road θ, r and g are fixed so for safe journey, the vehicle should take a turn with safe-speed ϑ_s given by

$$\vartheta_s = \sqrt{rg\tan\theta} \quad ... (5.6.30)$$

(v) Motion of an automobile on a banked road (including friction) :



(a) (upward skidding) **(b) (downward skidding)**

Fig 5.16

Consider a road, banked near a circular turn of radius 'r' by an angle θ. Let the road be rough so that friction cannot be neglected. Suppose a vehicle of mass 'm' negotiates the banked road near the circular turn with a speed ϑ. Then the forces acting on the vehicle are :

i) $\vec{W} = mg(-\hat{j})$ = weight of the body acting vertically downward.

ii) $\vec{N}$ = Normal reaction of the road, acting perpendicular to the surface of the road

iii) $\vec{f}_s$ = force of friction, whose direction depends on the direction in which the vehicle tends to skid. [See fig. 5.16(a) & (b)]

Therefore the net force acting on the vehicle is

$$\vec{f}_{net} = \vec{W} + \vec{N} + \vec{f}_s \qquad ...(5.6.31)$$

Case (a):

Referring to fig. 5.16(a) (upward skidding)

$$\vec{f}_{net} = -mg\,\hat{j} + (N\,Cos\theta\,\hat{j} - N\,Sin\theta\,\hat{i}) + (-f_s\,Cos\theta\,\hat{i} - f_s\,Sin\theta\,\hat{j})$$

$$\vec{f}_{net} = -(N\,Sin\theta + f_s\,Cos\theta)\,\hat{i} + (N\,Cos\theta - f_s\,Sin\theta - mg)\,\hat{j} \quad ...(5.6.32)$$

Since there is no vertical component of motion and the vehicle is to negotiate the turn so

$$N\,Cos\theta - f_s\,Sin\theta = mg \qquad ...(5.6.33)$$

and

$$N\,Sin\theta + f_s\,Cos\theta = \frac{m\vartheta^2}{r} \qquad ...(5.6.34)$$

Eqns (5.6.33) & (5.6.34) give

$$\frac{\vartheta^2}{rg} = \frac{N\,Sin\theta + f_s\,Cos\theta}{N\,Cos\theta - f_s\,Sin\theta} \qquad ...(5.6.35)$$

For a banked road θ and r are fixed. AB for a given vehicle and road N is fixed. Therefore from eqn. (5.6.34) it follows that the maximum speed with which the vehicle can negotiate is given by

$$N\,Sin\theta + (f_s)_{max}\,Cos\theta = \frac{m\vartheta^2_{max}}{r} \quad ...(5.6.36)$$

But $(f_s)_{max} = \mu_s N$

Hence eqn (5.6.35) leads to

$$\frac{\vartheta^2_{max}}{rg} = \frac{N\sin\theta + \mu_s N\cos\theta}{N\cos\theta - \mu_s N\sin\theta}$$

$$\Rightarrow \frac{\vartheta^2_{max}}{rg} = \frac{Sin\theta + \mu_s Cos\theta}{Cos\theta - \mu_s Sin\theta} = \frac{\tan\theta + \mu_s}{1 - \mu_s\tan\theta} \quad ...(5.6.37)$$

It is worth noting that (i) if the road is banked for a specified optimum speed $\vartheta = \vartheta_o$, so that there is no skidding or slipping, then sidewise frictional force f_s is zero; and eqn. (5.6.35) reduces to $\frac{\vartheta_0^2}{rg} = \tan\theta$. (ii) when the vehicle negotiates with maximum velocity, maximum force of friction acts downward. This causes maximum strain on tyres.

Case (b)

Using $\tan\theta = \frac{\vartheta_o^2}{rg}$ in eqn. (5.6.37) one obtains

$$\frac{\vartheta^2}{rg} = \frac{N.\frac{\vartheta_o^2}{rg} + f_s}{N - f_s.\frac{\vartheta_o^2}{rg}}$$

giving

$$N\left(\frac{\vartheta^2 - \vartheta_0^2}{rg}\right) = f_s\left(1 + \frac{\vartheta^2\vartheta_0^2}{r^2g^2}\right) \qquad ...(5.6.38)$$

Putting $N = f_s / \mu_s$

$$\mu_s = \frac{(\vartheta - \vartheta_0)(\vartheta + \vartheta_0)}{rg\left(1 + \frac{\vartheta^2\vartheta_0^2}{r^2g^2}\right)} \qquad ...[5.6.38(a)]$$

Eqn. (5.6.38) implies that if $\vartheta_0 > \vartheta > 0$, then f_s is negative i.e. the force of friction will act upward, implying in turn that the vehicle while negotiating with a speed ϑ, less than the optimum speed it will tend to skid downward. Consideration of fig. 5.16 (b) leads to

$$\vec{f}_{net} = -mg\,\hat{j} + (N\cos\theta\,\hat{j} - N\sin\theta\,\hat{i}) + (f_s\sin\theta\,\hat{j} + f_s\cos\theta\,\hat{i})$$

As there is no vertical motion and the vehicle is to negotiate the circular turn so

$$N\,Cos\theta + f_s\,Sin\theta = mg \qquad ...(5.6.39)$$

and

$$N\ \text{Sin}\theta - f_s\ \text{Cos}\theta = \frac{m\vartheta^2}{r} \qquad \dots (5.6.40)$$

This gives

$$\frac{\vartheta^2}{rg} = \frac{\tan\theta - \mu_s}{1 + \mu_s \tan\theta} \qquad \dots [5.6.40\ (a)]$$

It is easy to note that (i) when 'f_s' is replaced by - f_s in (5.6.33) and (5.6.34) eqns (5.6.39) and (5.6.40) follow. (ii) when $\vartheta = 0$ (i.e. vehicle is at rest on a banked road) eqn. (5.6.38) shows that f_s is negative implying that the vehicle would tend to skid downward, so that force of friction acts upward. If the force of friction is not sufficient, then the vehicle will skid downward.

NOTES

(i) A curve in a railway track is also banked. The outer rail is slightly raised w.r. to the inner rail to create a suitable angle of banking. The necessary centripetal force is provided by the horizontal component of the reaction force.

(ii) In case of an aeroplane the pilot suitably bends the aeroplane so that the resultant of the vector sum of lifting force and weight of the aeroplane provides the necessary centripetal force.

(vi) Motion on a vertical circle :

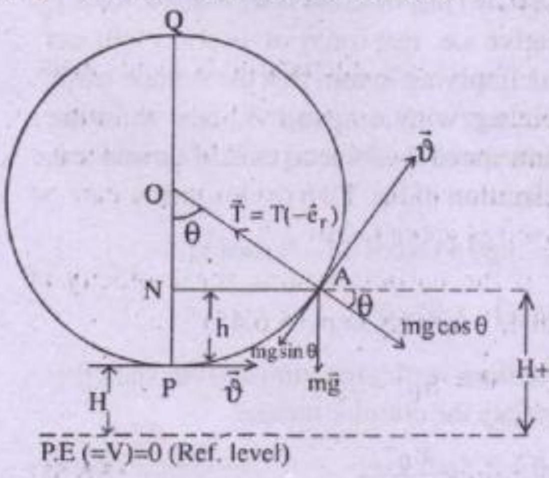


Fig. 5.17

Consider a small body of mass 'm' tied to the end of a string of length 'r' and revolving in a vertical circle about a fixed point 'O', the centre of the vertical circle.

Let the body have velocity $\vec{\vartheta}_0$ at the lowest point P, and velocity $\vec{\vartheta}$ at any point A (see fig. 5.17). The forces acting on the body at any instant are

i) $\vec{T} = T(-\hat{e}_r) =$ Tension along the string directed towards the centre.

ii) $\vec{W} = m\vec{g} =$ Weight of the body acting vertically downward.

Therefore the net force acting on the body is

$$\vec{f}_{net} = \vec{T} + m\vec{g} \qquad \dots (5.6.41)$$

The component of force acting towards the centre, which provides the necessary centripetal force 'f_c' is

$$f_c = (-\hat{e}_r).\ \vec{f}_{net}$$

$$= (-\hat{e}_r).(\vec{T} + m\vec{g}) = T - mg\ \text{Cos}\theta \quad \dots (5.6.42)$$

Hence at any instant

$$f_c = T - mg\ \text{Cos}\theta = \frac{m\vartheta^2}{r} \quad \dots (5.6.43)$$

The tangential component of force is given as

$$(f_{net})_t = \hat{e}_t.\vec{f}_{net} = \hat{e}_t.(\vec{T} + m\vec{g}) = -mg\sin\theta \qquad \dots (5.6.44)$$

Now applying the Law of Conservation of Energy

$$\frac{1}{2}m\vartheta_0^2 + mgH = \frac{1}{2}m\vartheta^2 + mg(H + h)$$

$$\Rightarrow \vartheta^2 = \vartheta_0^2 - 2gh \qquad \dots (5.6.45)$$

Further it follows from fig. 5.17.

$$\text{Cos}\theta = \overline{ON} / \overline{OA} = \frac{r-h}{r} \quad ...(5.6.46)$$

So using eqns 5.6.45 & 5.6.46 in (5.6.43) we obtain

$$T - mg.\frac{r-h}{r} = m\frac{\vartheta_0^2 - 2gh}{r}$$

Which on simplification gives

$$T = \frac{m}{r}[\vartheta_0^2 + g(r-3h)] \quad ...(5.6.47)$$

Using eqns. (5.6.42) to (5.6.47) one obtains

(i) At the lowest point 'P' (h = 0)

$\text{Cos}\,\theta = 1 \quad \Rightarrow \text{Sin}\,\theta = 0$

Hence

tangential accln.

$$a_t = (f_{net})_t / m = -g\,\text{Sin}\theta = 0 \quad ...(a)$$

centripetal (Radial) acln. 'a_r' =

$$\frac{f_c}{m} = \frac{\vartheta^2}{r} = \frac{\vartheta_0^2 - 2gh}{r} = \frac{\vartheta_0^2}{r} \quad ...(b)$$

and

$$T = T_1 = \frac{m}{r}(\vartheta_0^2 + rg) \quad ...(5.6.48)\,(c)$$

(ii) At the highest point Q (h=2r)

$\cos\theta = -1 \quad \Rightarrow \sin\theta = 0$

Hence

tangential accl$_n$ $a_t = 0$ ...(a)

centripetal (Radial) accl$_n$ $a_r = \frac{\vartheta_0^2}{r} - 4g$ ...(b)

and

$$T = T_2 = \frac{m}{r}(\vartheta_0^2 - 5rg) \quad ...[5.6.49(c)]$$

The above analysis shows that nature of motion in a vertical circle shall very much depend on the value of velocity at the lowest point (initial point). So we proceed as follows :

Case (A) :

For complete revolution, the string should not loosen at any instant. Since eqn (5.6.47) shows that T decreases as h increases, so the minimum tension would be occuring at the highest point 'Q' (h= 2r). So for complete revolution it is required that

$$T_{min} = T_2 \geq 0$$

i.e $\quad \vartheta_0^2 \geq 5rg$

$$\Rightarrow (\vartheta_0)_{min} = \sqrt{5rg} \quad ...(5.6.50)$$

Thus for complete revolution the minimum speed with which a particle must start at the lowest point is $\sqrt{5rg}$. Corresponding to this speed the tension at the highest point is zero and the speed at the highest point is

$$\vartheta_c = \sqrt{\vartheta_0^2 - 4gr} = \sqrt{rg} \quad ...(5.6.51)$$

This speed is sometimes called **critical speed.** Thus **critical speed**, for motion in a vertical circle, is the minimum speed of the particle at the highest point, below which the string becomes slack.

Case (B) :

If $\vartheta_0 < \sqrt{5rg}$, then the body cannot make complete revolution. So the body will either oscillate about the lowest point 'P' or will leave the circular path. These two cases can be analysed as given below :

If the particle attains zero velocity at height 'h_1', then by eqn. (5.6.45)

$$0 = \vartheta_0^2 - 2gh_1$$

$$\Rightarrow h_1 = \frac{\vartheta_0^2}{2g} \quad ...(5.6.52)$$

On the other hand if the tension of the string becomes zero at some height h_2, then by eqn. (5.6.47)

$$\frac{m}{r}[\vartheta_0^2 + g(r - 3h_2)] = 0$$

$$\Rightarrow h_2 = \frac{\vartheta_0^2 + rg}{3g} \quad \text{... (5.6.53)}$$

So **for Oscillation** to occur, velocity should vanish before tension vanishes (i.e. $h_1 < h_2$) implying

$$\frac{\vartheta_0^2}{2g} < \frac{\vartheta_0^2 + rg}{3g}$$

$$\Rightarrow \vartheta_0 < \sqrt{2rg} \quad \text{...(5.6.54)}$$

and **for Leaving the circular path,** the tension should vanish before velocity vanishes i.e. $h_2 < h_1$

$$\Rightarrow \frac{\vartheta_0^2 + rg}{3g} < \frac{\vartheta_0^2}{2g}$$

$$\Rightarrow \vartheta_0 > \sqrt{2rg} \quad \text{...(5.6.55)}$$

Thus one observes that (i) if $\sqrt{2rg} < \vartheta_0 < \sqrt{5rg}$, then the particle will leave the (vertical) circular path some-where (ii) if $\vartheta_0 < \sqrt{2rg}$, then the particle will oscillate in a vertical plane (iii) if $\vartheta_0 > \sqrt{5rg}$, the particle will make complete revolutions on a vertical circle.

Ex. 5.6.1. A stone weighing 500gm is tied to a string 50 cm long and whirled in a horizontal plane top of a frictionless table with a speed of 150 cm per second. Determine the tension of the string. Also find out the maximum speed with which the stone can be whirled if the string can bear at most load of 20 kg.

Soln.

Given m = 500 gm, r = 50 cm, ϑ= 150 cm/s.

Here Centripetal force = Tension along the string = m ϑ^2 / r

$$\text{Tension} = \frac{(500\text{gm})(150\text{cm/s})^2}{50\text{cm}}$$

$$= 22500 \times 10 = 2.25 \times 10^5 \text{ dynes}$$

$$= 2.25 \text{ N.} \quad \text{(Ans)}$$

Maximum speed is attained when the tension of the string becomes equal to the breaking forces of the string.

$$\frac{m\vartheta_{max}^2}{r} = 20 \times 9.8\text{N}$$

$$\vartheta_{max} = \sqrt{\frac{20 \times 9.8 \times 0.5}{0.5}} = 14 \text{ m/s} \quad \text{(Ans)}$$

Ex. 5.6.2. A small body of mass 280 gm. revolves in a circle on a horizontal frictionless surface attached by a cord 14 cm long to a pin fixed in the surface. If the body makes 2 revolutions per second, find the force exerted by it on the cord.

Soln.

Given m = 280 gm, r = 14 cm, ω = 2 rev/s.

Force exerted on the cord = Tension along the cord = m r ω^2

$$= 280 \text{ gm} \times 14 \text{ cm} \times (4\pi \text{ rad/s})^2$$

$$\doteq 6.19 \times 10^5 \text{ dyn} = 6.19 \text{ N} \quad \text{(Ans)}$$

Ex. 5.6.3 An unbanked curve has a radius of 13.07 m. What is the maximum speed at which a vehicle can make a turn if $\mu_s = 0.75$?

Soln.

Given r = 13.07 m, $\mu_s = 0.75$

Safe (maximum) speed = $\vartheta_{max} = \sqrt{\mu_s rg}$

$$= \sqrt{0.75 \times 13.07 \times 9.8}$$

$$\Rightarrow \vartheta_{max} = 9.8 \text{m/s} \quad \text{(Ans)}$$

Ex. 5.6.4 A maximum speed at which a scooter can negotiate a curve of radius 100 m.

is 48 km / hr. What is the banking angle if the road is frictionless ?

Soln.

$$\text{Given } \vartheta = 48\frac{\text{km}}{\text{hr}} = \frac{48 \times 10^3}{3600}\text{m/s} = \frac{40}{3}\text{m/s}$$

$$r = 100 \text{ m}$$

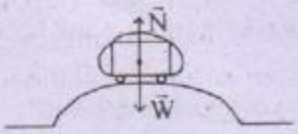
Banking angle

$$\theta = \tan^{-1}\left(\frac{\vartheta^2}{rg}\right) = \tan^{-1}\left(\frac{(40/3)^2}{100 \times 9.8}\right)$$

$$\Rightarrow \theta = 10.28 \text{ deg} = 10\ 16\ 48''$$

Ans

Ex. 5.6.5 There is a bridge in the form of an arc of radius 9.8.m. What should be the maximum speed of the car at the highest point of the bridge, so that it may not leave the ground.



Soln.

Given $r = 9.8$ m

$$N - mg = -m\vartheta^2/r \Rightarrow N = mg - m\vartheta^2/r \geq 0$$

$$\Rightarrow \quad rg \geq \vartheta^2$$

The maximum speed at the highest point = $\vartheta = \sqrt{rg}$

$$\Rightarrow \vartheta = \sqrt{9.8 \text{x} 9.8} = 9.8 \text{m/s} \qquad \text{(Ans)}$$

Ex. 5.6.6 An aeroplane loops the loop at a speed of 180 km/hr. What is the largest circular loop possible ?

Soln.

The aeroplane is looping at a speed of

$$\vartheta = 180\text{km/hr} = \frac{180\text{x}10^3}{60\text{x}60}\text{m/s} = 50\text{m/s}$$

This is the maximum speed at the highest point.

Hence $\therefore \vartheta^2_{\max} = rg$

Hence the radius of largest circular loop is

$$r = \vartheta^2_{\max}/g = 255\text{m}$$

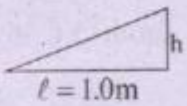
Ex. 5.6.7 The radius of curvature of a railway track at a place is 400m. and the distance between the rails is 1.0 m. What should be the elevation of the outer rail above the inner rail for a safe speed of 36 km/hr.

Soln.

$$\text{Given } \vartheta_s = 36\text{km/hr.} = \frac{36\text{x}10^3}{3600}\text{m/s} = 10\text{m/s}$$

$$r = 400 \text{ m}$$

$$\tan\theta = \frac{\vartheta_s^2}{rg}$$

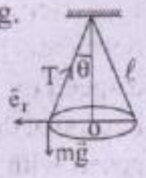


$$\Rightarrow \quad \frac{h}{\ell} = \frac{\vartheta_s^2}{rg}$$

$$h = \frac{\vartheta_s^2}{rg}.\ell = \frac{(10)^2}{400\text{x}9.8} \text{ x } 1$$

$$= 0.026 \text{ m} = 2.6 \text{ cm}$$

Ex. 5.6.8 A conical pendulum consists of small bob of mass m suspended from a rigid support by a string of length 'ℓ'. As the bob, revolves at constant speed 'ϑ' in a horizontal circle of radius r, the string sweeps a cone. Show that $\vartheta^2 = rg\tan\theta$. Find the time period of revolution, and tension of the string.



Soln.

The forces acting are

(i) $\vec{T}$ (ii) $m\vec{g}$

Hence

$$\vec{f}_{net} = \vec{T} + m\vec{g}$$

Centripetal force

$$= (-\hat{e}_r).\vec{f}_{net} = -\hat{e}_r.\vec{T} + (-\hat{e}_r).m\vec{g}$$

$$= -\hat{e}_r.\vec{T} = \frac{m\vartheta^2}{r}$$

$$\Rightarrow T\ Sin\theta = \frac{m\vartheta^2}{r} \quad ...(5.6.56)$$

Vertical component = $\hat{j}.\vec{f}_{net} = 0$

($\because$ There is no vertical component of motion)

$$\Rightarrow \hat{J}.\vec{T} + m\vec{g}.\hat{J} = 0$$

$$\Rightarrow T\cos\theta = -m\vec{g}.\hat{J} = mg$$

...(5.6.57)

From (5.6.56) and (5.6.57)

$$\tan\theta = \frac{\vartheta^2}{rg}$$

$$\Rightarrow \vartheta^2 = rg\tan\theta \quad ...(5.6.58)$$

Now

$$\tan\theta = \frac{r}{\sqrt{\ell^2 - r^2}},\ Sin\ \theta = \frac{r}{\ell}$$

Time period of revolution $T = \frac{2\pi r}{\vartheta}$

$$\Rightarrow \quad T^2 = \frac{4\pi r^2}{\vartheta^2} = \frac{4\pi^2 r^2}{rg\tan\theta}$$

$$T^2 = \frac{4\pi^2 r/g}{\tan\theta} = \frac{4\pi^2 r}{g(r/\sqrt{\ell^2 - r^2})}$$

$$T^2 = \frac{4\pi^2}{g}\sqrt{\ell^2 - r^2}$$

$$T = \frac{2\pi}{\sqrt{g}}(\ell^2 - r^2)^{\frac{1}{4}} = 2\pi\left(\frac{\ell^2 - r^2}{g^2}\right)^{\frac{1}{4}} \quad ...(5.6.59)$$

Using (5.6.58) in (5.6.56)

$$T\sin\theta = \frac{m}{r}.rg\tan\theta$$

$$\Rightarrow T = mg/\cos\theta = mg.\frac{\ell}{\sqrt{\ell^2 - r^2}} \quad ...(5.6.60)$$

Ex. 5.6.9. One end of massless spring of spring constant 100 N/m and natural length 0.5 m is fixed and the other end is connected to a particle of mass 0.5 kg lying on a frictionless horizontal table. The spring remains horizontal. If the mass is made to rotate at an angular velocity of 2 rad/s, find the elongation of the spring.

Soln.

The centripetal force is provided by the spring due to tension developed along the spring arising out of its extension. Therefore

$$\frac{m\vartheta^2}{r} = k\ell = m\omega^2 r = m\omega^2(\ell + \ell_0)$$

Where 'ℓ_0' is original length and 'ℓ' is extension.

$$\Rightarrow \quad \ell\ (k - m\ \omega^2) = m\ \omega^2\ \ell_0$$

$$\Rightarrow \quad \ell = \frac{m\omega^2\ell_0}{k - m\omega^2}$$

Given $\ell_0 = 0.5$m, k= 100N/m, $\omega = 2$ rad/s, m = 0.5 kg.

$$So \quad \ell = \frac{0.5x(2)^2 x0.5}{100 - 0.5x(2)^2}\ m$$

$$\Rightarrow \quad \ell = 0.01\ m = 1\ cm. \quad (Ans)$$

Ex. 5.6.10 A simple pendulum is constructed by attaching a bob of mass 'm' to a thread of length 'ℓ'. The simple pendulum oscillates in a vertical plane. If the speed of the bob is ϑ, when the thread makes angle θ with the vertical, then find the tension along the thread at this instant.

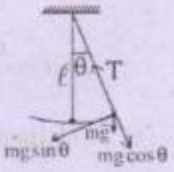
Soln. Forces acting (i) $\vec{T}$ (ii) $m\vec{g}$

$$\vec{f}_{net} = \vec{T} + m\vec{g}$$

Centripetal force

$= (-\hat{e}_r).\vec{f}_{net} = -\hat{e}_r.\vec{T} + (-\hat{e}_r).m\vec{g}$

$= -\hat{e}_r.\vec{T} = \frac{m\vartheta^2}{\ell}$



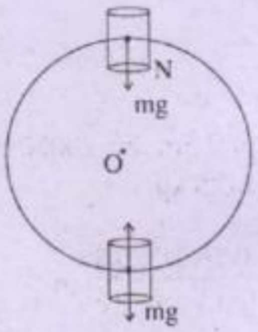
$T - mg\,\text{Cos}\theta = \frac{m\vartheta^2}{\ell}$

$\Rightarrow \quad T = m\left(g\,\text{Cos}\,\theta + \frac{\vartheta^2}{\ell}\right)$ (Ans)

Ex. 5.6.11. A cylinderical bucket filled with water is whirled around in a vertical circle of radius 'r'. What can be the minimum speed at the top of the path if water does not fall out from the bucket ? If it continues with this speed, what normal contact force the bucket exerts on water at the lowest point of the path ?

Soln.

Consider water in the bucket as a system. At the top of the circle, the forces acting on it are



i) $m\vec{g}$ = acting vertically downward $= -mg\hat{e}_r$

ii) $\vec{N}$ = acting vertically downward $= -N\hat{e}_r$

Therefore $(mg + N)\,(-\hat{e}_r) = \frac{m\vartheta^2}{r}(-\hat{e}_r)$

$$\Rightarrow \quad mg + N = \frac{m\vartheta^2}{r}$$

Since water sticks to the bucket so $N \geq 0$

$$\Rightarrow \quad N = \frac{m\vartheta^2}{r} - mg \geq 0$$

$$\Rightarrow \quad \vartheta^2 \geq rg$$

$\therefore\ \vartheta^2_{min} = rg \Rightarrow \vartheta_{min} = \sqrt{rg}$

At the lowest point forces acting are

i) $m\vec{g}$ = acting vertically downward $= mg\hat{e}_r$

ii) $\vec{N}'$ = acting inwards $= N'(-\hat{e}_r)$

Therefore $(N' - mg)(-\hat{e}_r) = \frac{m\vartheta^2}{r}(-\hat{e}_r)$

$$\Rightarrow \quad N' - mg = \frac{m\vartheta^2}{r}$$

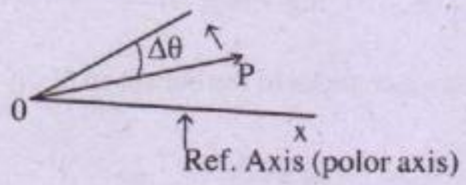
If $\vartheta = \vartheta_{min}$

Then $N' = mg + \frac{m\vartheta^2}{r} = 2mg$ (Ans)

Summary

1. Cercular motion - see sec. 5-1
2. Rotational motion. see.sec. 5-2
3. Angular displacement is the angle

described by the radius vector $\overline{op}$ during a time interval.



4. Angular velocity

$$\overline{\overline{\omega}} \equiv \langle\overline{\omega}\rangle = \frac{\theta_2 - \theta_1}{t_2 - t_1}$$

$$\overline{\omega} = \frac{d\theta}{dt}\hat{A}$$

where $\hat{A}$ is the unit vector perpendicular to the plane of rotation and its +ve direction is given by right-handed-screw rule.

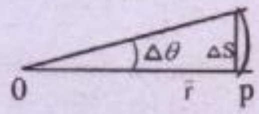
5. Angular acceleration

$$\langle\vec{\alpha}\rangle = \frac{\vec{\omega}_2 - \vec{\omega}_1}{t_2 - t_1}$$

$$\vec{\alpha} = \frac{d\bar{w}}{dt}$$

6. Linear displacement $\Delta s = r\Delta\theta$

$$\overline{\Delta s} = \overline{\Delta\theta} \times \bar{r}$$



7. Linear velocity $\bar{\upsilon} = d\bar{s} = \overline{\omega} \times \bar{r}$
8. $\theta = \omega_0 t + \frac{1}{2}\alpha t^2$

$$\omega = \omega_0 + \alpha t$$

$$\omega^2 = \omega_0^{\ 2} + 2\alpha\theta$$

MODEL QUESTIONS

A. Multiple Choice Type Questions:

1. When a body is executing uniform circular motion it is
   (i) in equilibrium
   (ii) not in equilibrium
   (iii) in unstable equilibrium
   (iv) none of the above

2. An object moves along a curved path. Which of the following quantities may remain constant during motion
   (i) speed
   (ii) velocity
   (iii) acceleration
   (iv) magnitude of acceleration

3. A particle moves in circular path of radius 'r'. In half the period of revolution its displacement and distance covered are
   (i) $2r, 2\pi r$
   (ii) $r\sqrt{2}, \pi r$
   (iii) $2r, \pi r$
   (iv) $r, \pi r$

4. A particle is moving along a circular path. Which of the following relations is / are correct among angular velocity ($\vec{\omega}$), linear velocity ($\vec{\vartheta}$), angular acceleration ($\vec{\alpha}$) and centripetal acceleration ($\vec{a}_c$)
   (i) $\vec{\omega}\perp\vec{\vartheta}$ (ii) $\vec{\omega}\perp\vec{\alpha}$
   (iii) $\vec{\omega}\perp\vec{a}_c$ (iv) $\vec{\vartheta}\perp\vec{a}_c$

5. In a circus, a motor cyclist rides in circular track of radius 'r' in the vertical plane. The minimum velocity at the highest point of the track will be
   (i) $\sqrt{2gr}$ (ii) $\sqrt{3g}$
   (iii) $\sqrt{5gr}$ (iv) $\sqrt{gr}$

6. For traffic moving at 60 km/hr, along a circular track of radius 100m, the correct angle of banking is
   (i) $(60)^2 / 0.1$
   (ii) $\tan^{-1}\left[\frac{(50/3)^2}{100\text{x}9.8}\right]$
   (iii) $\tan^{-1}\left[\frac{100\text{x}9.8}{(50/3)^2}\right]$
   (iv) $\tan^{-1}\sqrt{60\text{x}0.1\text{x}9.8}$

7. A point mass M is hanging by a string of length ℓ. The velocity ϑ_0 which must be imparted to it in order for it to just barely reach the top is
   (i) $\sqrt{3g\ell}$ (ii) $\sqrt{4g\ell}$
   (iii) $\sqrt{5g\ell}$ (iv) $\sqrt{6g\ell}$

8. A mass M is hanging by a rod of length ℓ. The velocity ϑ_0 which must be imparted to it in order for it to just barely reach the top is
   (i) $\sqrt{2g\ell}$ (ii) $\sqrt{3g\ell}$
   (iii) $\sqrt{4g\ell}$ (iv) $\sqrt{5g\ell}$

9. What tangential velocity be given to the bob of a simple pendulum of length 'ℓ' so that it can oscillate
   (i) $\sqrt{g\ell}$ (ii) $\sqrt{2g\ell}$
   (iii) $\sqrt{4g\ell}$ (iv) $\sqrt{5g\ell}$

10. If the bob of a simple pendulum is given a tangential velocity $\sqrt{2g\ell}$, the angle described by the string going from one extreme position to the other will be
    (i) $\pi/4$ (ii) $\pi/2$
    (iii) $3\pi/2$ (iv) π

11. A particle is executing uniform circular

motion in a circle of radius 'r'. Its angular acceleration is

(i) r (ii) ℓ/r

(iii) r^2 (iv) Zero

12. A car of mass m is taking a circular turn of radius r on a frictional level road with a speed ϑ. In order that the car does not skid

(i) $\frac{m\vartheta^2}{r} \geq \mu mg$ (ii) $\frac{m\vartheta^2}{r} \leq \mu mg$

(iii) $\frac{m\vartheta^2}{r} = \mu mg$ (iv) $\frac{\vartheta}{r} = \mu g$

13. If the bob of a simple pendulum of length 'ℓ' is given a tangential velocity $\sqrt{5g\ell}$ at the lowest point, the kinetic energy of the bob at the highest point shall be

(i) Zero (ii) mg ℓ

(iii) 1/2 mg ℓ (iv) 2 mg ℓ

14. When a particle is in uniform motion, it does not have
    (i) radial acceleration
    (ii) radial velocity and radial acceleration
    (iii) transverse acceleration
    (iv) radial and transverse acceleration

15. A bottle of soda lime grasped by the neck and swung briskly in a vertical circle. Near which portion of the bottle do the bubbles collect ?
    (i) near the bottom
    (ii) in the middle of bottle
    (iii) near the neck
    (iv) bubbles remain distributed unriformly throughout the volume of bottle.

B. Very Short Answer Type Questions:

1. State unit of angular velocity.
2. State unit of angular acceleration.
3. Give an example of motion where an accelerated body moves with uniform speed.
4. When will the magnitude of angular velocity and linear velocity be same ?
5. Calculate the angular velocity of minute hand of a clock.
6. If a bottle of soda water is grasped by the neck and swung briskly in a vertical circle, near which portion of bottle do the bubble collect ?
7. Which provides centripetal force for rotation of earth ?
8. Is a body executing uniform circular motion in equilibrium ?
9. Write the relation between linear displacement and angular displacement .
10. Write the relation between linear velocity and angular velocity.
11. Write the relation between linear acceleration and angular acceleration.
12. What is the direction of instantaneous velocity in case of particle executing uniform circular motion ?
13. Write the expression for centripetal force involving frequency of revolution.
14. Write the physical quantities which remain constant for a particle executing uniform circular motion.
15. What provides centripetal force to an electron in an atom ?
16. What provides centripetal force to a car taking a turn on a level road ?
17. In a circular motion, how much work is done by the centripetal force.
18. Give the ratio of the angular speed of minute hand and hour hand of a watch.
19. Is the K.E. of a body moving in a vertical circle same ?
20. When is the tension maximum in the thread of a simple pendulum ?
21. If both speed and radius are doubled, how will centripetal force change ?
22. If a stone, tied at the end of a string is whirled and the string breaks, how does the stone fly off ?

23. Why are railways banked near a turn ?
24. Is it necessary to express all angles in radian while using the equation $\omega = \omega_0 + \alpha t$?
25. A small coin is placed on a record rotating at $33\frac{1}{3}$ rev/min. The coin does not slip on the record. Where does it get the required centripetal force from ?

C. Short Answer Type Questions :

1. Calculate the angular speed of the minute hand of a clock. Compare it with that of the second hand .
2. A body moves with a speed of 6 m/s in a circle of radius 4m. Find the angular velocity.
3. The centripetal force acting on a body moving with uniform speed in circular path is 10 N. Calculate the centripetal force when the speed is doubled ?
4. A cyclist moves with a speed of 20 km/hr in a circular path of 100m radius. How much will he bend from the vertical while moving ?
5. What is the maximum speed with which an automobile can move safely round a curved track of radius 5 m and banked at 45^0 ?
6. A cyclist goes round a circular path of radius $20\sqrt{3}$ at speed of $24\sqrt{3}$ m/s. What is its inclination with vertical ?
7. The minute hand of a watch which points towards 3 is turned through 5π radian, where will it come ?
8. What are the factors on which angle of banking depends ?
9. A centrifuge rotor is accelerated from rest to 330 rps in 220 seconds. Calculate the angular acceleration.
10. Two particles of equal masses are revolving in circular path of radius 'r_1', and 'r_2' with same period calculate the ratio of their centripetal force.

(b) Conceptual :

11. Why does passenger of a car rounding a curve are thrown outward ?
12. If a small can filled with water is rapidly swung in a vertical circle the water does not fall why ?
13. Why did the earth bulge at the equator ?
14. If the centripetal force is taken as a force of action, can the centrifugal force be its reaction force ? Explain.
15. Work done by the centripetal force is zero. Explain.
16. In a banked frictionless road what provides the centripetal force for an automobile negotiating a curve.

D. Numerical Problems :

1. Express (a) 45^0 in radians (b) 15 rad in degrees.
2. Determine the angular speed of a 40 rpm turntable in rad/s.
3. How much distance is covered by the tip of a minute hand of length 5 cm when it moves from 3 to 6.
4. A car travels at a constant speed of 36 km/hr. If the radius of the wheel of the car be 35 cm calculate the angular speed of the wheels in rpm.
5. How much centripetal force is required to keep a 0.3 kg stone moving on a horizontal circle of radius 1.5 m, at a speed of 36 km/hr.
6. A motor cyclist goes round a circular race course at 150 km/hr. How far from the vertical must he lean inward to keep his balance if the track is 1.5 km long ?

7. A gramophone disc is set revolving in a horizontal plane at the rate $33\frac{1}{3}$ rev per min. It is found that a small coin placed on the disc will remain there if it is not more than 5 cm from the axis of rotation. Calculate the coefficient of friction between the coin and the disc.

8. A coin of mass 0.05 g is lying 10 cm from the centre of phonograph record revolving at 45 rpm (a) what is the linear speed of the coil, (b) what is its acceleration, (c) what is the magnitude of centripetal force on the coin, (d) what is the maximum value of coefficient of static friction between surfaces of coin and phonograph record.

9. What is the maximum speed with which an automobile can round an unbanked curve of radius 200m without skidding if the coefficient of friction between tyre and road is 0.3 ?

10. A particle moves along a circle of radius r met. at a constant speed of 10 m/s. Determine the change in velocity corresponding to a movement of 1/3rd way around the circle.

Hints : $\overrightarrow{\Delta\vartheta} = \bar{\vartheta}_2 - \bar{\vartheta}_1, |\bar{\vartheta}_1| = |\bar{\vartheta}_2| = \vartheta$

$$\left|\overrightarrow{\Delta\vartheta}\right| = 2\vartheta \sin\frac{\theta}{2}$$

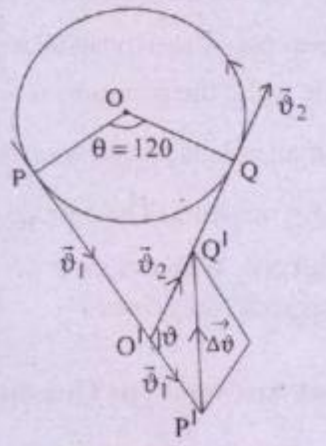
$$\overrightarrow{\Delta\vartheta}.\bar{\vartheta}_2 = \Delta\vartheta.\vartheta_2 \cos\beta = \vartheta_2^2 - \bar{\vartheta}_1.\bar{\vartheta}_2$$

$$\Rightarrow \cos\beta = \frac{\vartheta^2(1-\cos\theta)}{2\vartheta^2 \sin\frac{\theta}{2}} = \sin\frac{\theta}{2}$$

$$\Rightarrow 90 - \beta = \frac{\theta}{2}$$

$$\Rightarrow \beta = 90 - \frac{\theta}{2}$$

$$\Rightarrow \beta = 90 - \frac{\theta}{2} = 90^\circ - 60^\circ = 30$$



11. A park has a radius of 10m. If a vehicle goes round it at an average speed of 18 km/hr, what should be the proper angle of banking ?

12. In Bohr's H-atom, the electron is supposed to be revolving around a fixed proton in circular orbit. In the ground state the electron goes round the proton in a circle of radius 5.3×10^{-11} m. Find the speed of the electron in the ground state if mass of electron $= 9.1 \times 10^{-13}$ kg and charge of electron $= 1.6 \times 10^{-19}$ c.

13. A stone is fastened to one end of a string and is whirled in a vertical circle of radius R. Find the minimum speed the stone can have at the highest point of the circle.

14. A simple pendulum is suspended from the ceiling of a car taking a turn of radius 10m. at a speed of 36 km/hr. Find the angle made by the string of the pendulum with the vertical if this angle does not change during the turn.

15. The bob of a simple pendulum of mass 100 g has a speed of 1.4 m/s, when the string of length 1m makes an angle of 0.2 radian with the vertical. Find the tension along the string at this instant.

16. A turn of radius 20m is banked for vehicles going at a speed of 36 km/hr. If the coefficient of static friction between the road

and the tyre is 0.4 what are the possible speeds of a vehicle so that it neither slips down nor skids up ?
Take $g = 10 \ m/s^2$

17. A car goes on a horizontal circular road of radius R. The speed increases at a constant rate $\frac{d\vartheta}{dt} = a$. The coefficient of friction between the road and tyre is μ. Find the speed at which the car will skid.

18. The maximum force a road can exert on the tyre of a 1000 kg car is 5000 N. What is the maximum velocity at which the car can round a turn of radius 100 m ?

19. A level road makes 90^0 turn with a radius of curvature 80m. What is the maximum speed with which a car can negotiate this turn (a) when $\mu_s = 0.09$ (b) when $\mu_s = 0.25$?

20. An unbanked circular highway curve on level ground makes a turn of 90^0. The highway carries traffic at 60 km/hr; and the centripetal force on a vehicle is not to exceed $\frac{1}{10}$ of its weight. What is the minimum length of the curve in kms ?

E. Long Answer Type Questions :

1. Define angular displacement, angular velocity and angular acceleration. Show how they are related with linear displacement, velocity and acceleration.

2. Define angular displacement and angular velocity. Find the relation between angular velocity and linear velocity. Also show that in circular motion the linear velocity is along the tangential direction.

3. What is centripetal force ? Find expression for it for a body executing uniform circular motion.

4. Derive an expression for centripetal force acting on a body executing uniform circular motion. Distinguish between centripetal and centrifugal force.

5. What do you mean by banking of a road near a curve ? Find expression for the safe speed with which a vehicle can take a turn on a frictionless banked road.

F. Fill in the Blank Type

1. When a body is executing uniform circular motion it is ....

2. In a circus, a motor cyclist rides in a circular track of radius 'r' in the vertical circle. The minimum velocity at the highest point of the track will be...........

3. .... tangential velocity be given to the bob of a simple pendulum of length 'l' so that it can oscillate........

4. When a particle is in uniform motion it does not have ....

5. A bottle of soda lime grasped by the neck and swung briskly in a vertical circle. The bubbles collect near .... portion of the bottles.

G. True - False Type

1. It is not possible for a body to have a constant speed in an accelerated motion.

2. The passengers of a car rounding a curve are thrown inward.

3. The kinetic energy of a body moving along a horizontal circle remains same at every point but for a vertical circle it changes from point to point.

4. In circular motion work done by the contripetal force in zero.

5. If a body is moving on a curved path with a constant speed then its acceleration is perpendicular to the direction of motion.

6. If the speed of a body is constant, the body cannot have a path other than a circular or straight line path.

ANSWERS

A. Multiple Choice Questions :

(1) ii, (2) i & iv, (3) iii, (4) i, iii, & iv, (5) iv, (6) ii, (7) iii, (8) i, (9) ii, (10) iv, (11) iv, (12) ii, (13) iii, (14) iii, (15) iii

B. Very Short Type Questions :

1. rad/s, deg/s, rev/s
2. rad/s^2, deg/s^2, rev/s^2
3. Uniform circular motion
4. Radius of circular path is 1 unit.
5. $\frac{\pi}{180}$ rad/s
6. Neck
7. Grav. attraction of sun on earth.
8. No
9. $\Delta\vec{s} = \Delta\vec{\theta} \times \vec{r}$
10. $\vec{\vartheta} = \vec{\omega} \times \vec{r}$
11. $\vec{\alpha} = \vec{\alpha} \times \vec{r} + \vec{\omega} \times \vec{\vartheta}$
12. Along the tangent.
13. $f_c = 4\pi^2 m\upsilon^2 r$
14. Speed, K.E., angular velocity, angular momentum.
15. Attraction of nucleus
16. Frictional force between ground and wheel
17. Zero
18. 12
19. No
20. At its lowest (mean) position
21. Centripetal force will be increased two times.
22. Flies tangentially
23. To provide necessary centripetal force.
24. No
25. Force of friction.

C. Short Answer Type Questions :

1. minute hand = (π/180) rad/s, second hand = (π/30) rad/s
2. $\frac{3}{2}$ rad/s
3. 40 N
4. 1:08^0
5. 7 m/s
6. $\theta = 78.88^0$
7. 9
8. (i) r, (ii) ϑ, (iii) g, (iv) μ_s
9. 3π rad/s^2
10. $r_1 : r_2$

D. Numerical Problems :

1. (a) $\frac{\pi}{4}$ (b) 859. 43^0
2. 4.19 rad/s
3. $\frac{5\pi}{2}$ cm = 7.85 cm
4. 272. 84 rpm
5. 20 N
6. 36.58^0

7. 0.062

8. (a) 47.12 cm/s, (b) 222.07 cm/s^2, (c) 11.10 dyne, (d) 0.227

9. 24.25 m/s

10. $|\Delta\vartheta| = 10\sqrt{3}\,\mathrm{m/s}$ makes 30^0 with the final velocity.

Hints :

$$\Delta\vec{\vartheta} = \vec{\vartheta}_2 - \vec{\vartheta}_1,\ \left|\vec{\vartheta}_1\right| = \left|\vec{\vartheta}_2\right| = \vartheta,\ \left|\Delta\vec{\vartheta}_1\right| = 2\vartheta\sin\tfrac{\theta}{2}$$

$$\Delta\vec{\vartheta}.\vec{\vartheta}_2 = \Delta\vartheta.\vartheta_2\cos\beta = \vartheta_2^{\,2} - \vec{\vartheta}_1.\vec{\vartheta}_2$$

$$\Rightarrow\ \mathrm{Cos}\beta = \frac{\vartheta^2(1-\mathrm{Cosc}\theta)}{2\vartheta^2.\mathrm{Sin}\dfrac{\theta}{2}} = \mathrm{Sin}\frac{\theta}{2}$$

$$\Rightarrow\ 90-\beta = \tfrac{\theta}{2}$$

$$\Rightarrow\ \beta = 90-\tfrac{\theta}{2} = 90^\circ - 60^\circ = 30^\circ$$

11. 14.31 = 0.25 rad

12. 2.18×10^6 m/s

13. $\sqrt{Rg}$

14. 45^0

15. 1.16 N

16. 14.7 km/hr and 54 km/hr

17. $[(\mu^2 g^2 - a^2/R^2]^{\frac{1}{4}}$

Hints :

$$\vec{\vartheta} = \vartheta\hat{e}_t,\ \frac{d\vec{\vartheta}}{dt} = \frac{d\vec{\vartheta}}{dt}\hat{e}_t + \vartheta\frac{d\hat{e}_t}{dt} = a\hat{e}_t - \vartheta\omega\hat{e}_r$$

$$\left|m\frac{d\vec{\vartheta}}{dt}\right| = m\sqrt{a^2 + \left(\frac{\vartheta^2}{R}\right)^2} = (\mu_s mg)$$

$$\Rightarrow\ a^2 + \frac{\vartheta^4}{R^2} = (\mu_s g)^2$$

$$\vartheta^4 = (\mu_s^2 g^2 - a^2)R^2$$

$$\vartheta = [(\mu_s^2 g^2 - a^2)R^2]^{\frac{1}{4}}$$

18. 22.36 m/s

19. (a) 8.4 m/s,(b) 14 m/s

20. 0.445 km

F. (1) Not in equalibrium (2) $\sqrt{rg}$ (3) $\sqrt{2gl}$ (4) transverse acceleration (5) Near the neck.

G. (1) False (2) False (3) True (4) True (5) True (6) False.

6

Rotation

6.1 Centre of Mass : -

The Newton's 2nd Law of Motion

$$\frac{d\vec{p}}{dt} = \frac{d}{dt}(m\vec{\vartheta}) = (\vec{F}_{ext})_{net} \quad ...(6.1.1)$$

supplemented by $\vec{\vartheta} = \frac{d\vec{r}}{dt}$ is key to the description of the dynamics of a point mass 'm' subjected to an external force.

However, in actual practice we deal with bodies which can be considered to be a combination of number of point particles. But it is inconvenient to describe the motion of the constituent particles separately, as it will involve internal force, (arising out of the interaction among the constituent particles). Therefore it becomes necessary to develop a mechanism so that a single point mass (equal to the mass of the whole body), situated at a convenient point, carries the net momentum of the particles constituting the body. Such a point shall be recognised as "Centre of Mass" (C.O.M.)

We shall develop this concept in a gradual manner as follows :

6.2 Two-Particle System :

Consider a system of two particles of masses m_1, and m_2. Let their velocities be $\vec{\vartheta}_1$ and $\vec{\vartheta}_2$ respectively at any time 't'. Let $\vec{r}_1$ and $\vec{r}_2$ be the position vectors describing the two bodies of masses m_1 and m_2 respectively. Then net momentum $\vec{P}$ of the system is given by

$$\vec{P} = m_1\vec{\vartheta}_1 + m_2\vec{\vartheta}_2 = \vec{P}_1 + \vec{P}_2 \quad ...(6.2.1)$$

Since $\vec{\vartheta}_1 = \frac{d\vec{r}_1}{dt} = \dot{\vec{r}}_1$; $\vec{\vartheta}_2 = \frac{d\vec{r}_2}{dt} = \dot{\vec{r}}_2$

So $$\vec{P} = m_1\dot{\vec{r}}_1 + m_2\dot{\vec{r}}_2 \quad ...(6.2.2.)$$

Let external forces $\vec{f}_{1e}$ and $\vec{f}_{2e}$ act respectively on m_1 and m_2. Let $\vec{f}_{12}$ be the force of m_1 on m_2 and $\vec{f}_{21}$ be the force of m_2 on m_1. Then equations of motion for m_1 and m_2 are given as

$$\frac{d\vec{p}_1}{dt} = \frac{d}{dt}(m_1\vec{\vartheta}_1) = \vec{f}_{1e} + \vec{f}_{21} \quad ...(6.2.3)$$

$$\frac{d\vec{p}_2}{dt} = \frac{d}{dt}(m_2\vec{\vartheta}_2) = \vec{f}_{2e} + \vec{f}_{12} \quad ...(6.2.4)$$

Adding eqns. (6.2.3) & (6.2.4) we obtain

$$\frac{d}{dt}(\vec{p}_1 + \vec{p}_2) = \frac{d}{dt}(m_1\vec{\vartheta}_1 + m_2\vec{\vartheta}_2)$$

$$= \vec{f}_{1e} + \vec{f}_{2e} + (\vec{f}_{12} + \vec{f}_{21}) \qquad \ldots (6.2.5)$$

But by the law of action and reaction

$$\vec{f}_{12} + \vec{f}_{21} = 0$$

Hence eqn (6.2.5) reduces to

$$\frac{d}{dt}(\vec{p}_1 + \vec{p}_2) = \frac{d}{dt}(m_1\vec{\vartheta}_1 + m_2\vec{\vartheta}_2)$$

$$= \vec{f}_{1e} + \vec{f}_{2e} = (\vec{F}_{ext})_{net} \qquad \ldots (6.2.6)$$

Using eqn. (6.2.1) on L.H.S. of eqn. (6.2.6) we obtain

$$\frac{d\vec{p}}{dt} = (\vec{F}_{ext})_{net} \qquad \ldots (6.2.7)$$

Eqn. (6.2.7) is in a form similar to eqn. (6.1.1), the eqn. of motion for a point particle. So we define

$$\vec{P} = M\dot{\vec{R}} = M\vec{V}_{cm} \qquad \ldots (6.2.8)$$

where, $M = m_1 + m_2$ is the total mass of the system and $\dot{\vec{R}}$ is the velocity of this total mass, supposed to be concentrated at a point C, specified by the position vector $\vec{R}$.

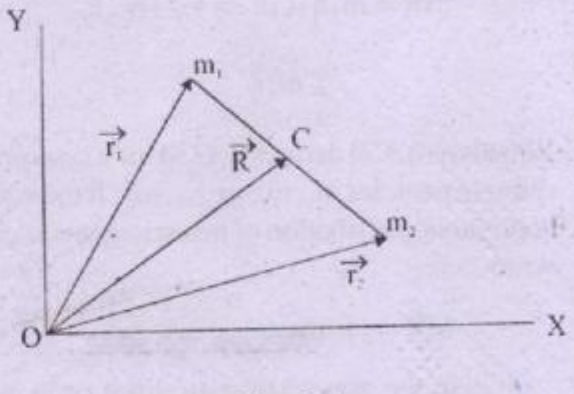


Fig. 6.1

Thus one can apply Newton's Laws of Motion to a mass M, situated at $\vec{R}$ and describe the dynamics of the system of two particles. Using eqn. (6.2.8) in eqn. (6.2.2) we obtain

$$M\dot{\vec{R}} = m_1\dot{\vec{r}}_1 + m_2\dot{\vec{r}}_2 \qquad \ldots (6.2.9)$$

Eqn. (6.2.9) can be considered as defining the position of C.O.M. Thus "Centre of Mass is a point where the entire mass is supposed to be concentrated, which moves in such a manner that the net momentum of the system during motion is qual to momentum of this mass point."

In the non-relativistic limit, where mass is considered to be constant or in any other type of motion where mass is considered to be constant, eqn. (6.2.9) reduces to

$$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2} \qquad \ldots (6.2.10)$$

It is instructive to note that in the C.O.M. coordinate system (where C.O.M. is considered to be at rest) the net momentum of the system is zero. Hence $\vec{p}_1 = -\vec{p}_2$. We further note that when net external force on the system is zero the momentum of C.O.M. is constant.

6.3. <u>Many-particle system :</u>

Consider a system of particles of masses m_1, m_2, m_3, ... m_n; moving with velocities $\vec{\vartheta}_1$, $\vec{\vartheta}_2$, ... $\vec{\vartheta}_n$ respectively at any time t. Let their positions at time t be specified by position vectors $\vec{r}_1, \vec{r}_2, \ldots \vec{r}_n$ respectively. Then total momentum $\vec{P}$ of the system is given as

$$\vec{P} = m_1\vec{\vartheta}_1 + m_2\vec{\vartheta}_2 + \cdots + m_n\vec{\vartheta}$$

$$= \vec{p}_1 + \vec{p}_2 + \cdots + \vec{p}_n$$

$$= m_1\dot{\vec{r}}_1 + m_2\dot{\vec{r}}_2 + \cdots + m_n\dot{\vec{r}}_n \qquad \ldots (6.3.1)$$

The equations of motion for the different particles are given as

$$\frac{d\vec{p}_1}{dt} = \frac{d}{dt}(m_1\vec{\vartheta}_1) = \vec{f}_{1e} + \vec{f}_{21} + \vec{f}_{31} + \cdots + \vec{f}_{n1} = \vec{f}_{1e} + \sum_{i \neq 1}^{n} \vec{f}_{i1}$$

$$\frac{d\vec{p}_2}{dt} = \frac{d}{dt}(m_2\vec{\vartheta}_2) = \vec{f}_{2e} + \vec{f}_{12} + \vec{f}_{32} + \cdots + \vec{f}_{n2} = \vec{f}_{2e} + \sum_{i \neq 2}^{n} \vec{f}_{i2}$$

$$\frac{d\vec{p}_n}{dt} = \frac{d}{dt}(m_n\vec{\vartheta}_n) = \vec{f}_{ne} + \vec{f}_{1n} + \vec{f}_{2n} + \cdots + \vec{f}_{n-1,n} = \vec{f}_{ne} + \sum_{i \neq n}^{n} \vec{f}_{in}$$

... (6.3.2)

Where $\vec{f}_{1e}, \vec{f}_{2e}, \ldots \vec{f}_{ne}$ are the external forces on masses $m_1, m_2, m_3, \ldots m_n$ respectively and $\sum_{i \neq j} \vec{f}_{ij}$ are the internal forces. Adding vertically eqn. (6.3.2) gives

$$\frac{d}{dt}(\vec{p}_1 + \vec{p}_2 + \cdots + \vec{p}_n)$$

$$= \frac{d}{dt}(m_1\vec{\vartheta}_1 + m_2\vec{\vartheta}_2 + \cdots + m_n\vec{\vartheta}_n)$$

$$= (\vec{f}_{1e} + \vec{f}_{2e} + \cdots + \vec{f}_{ne}) + (\vec{f}_{21} + \vec{f}_{31} + \cdots)$$

$$+ (\vec{f}_{12} + \vec{f}_{32} + \cdots) + \cdots\cdots$$

$$= \sum_i \vec{f}_{ie} + \sum_i \sum_{\substack{j \\ i \neq j}} \vec{f}_{ij}$$

...(6.3.3)

Since $\sum_i \sum_{\substack{j \\ i \neq j}} \vec{f}_{ij} = 0$ (by the law of action and reaction) so eqn. (6.3.3) reduces to

$$\frac{d\vec{P}}{dt} = \frac{d}{dt}(m_1\vec{\vartheta}_1 + m_2\vec{\vartheta}_2 + \ldots) = \sum_i \vec{f}_{ie} = (F_{ext})_{net} \quad \ldots(6.3.5)$$

Equation (6.3.5) is in a form similar to the equation of motion of a point mass. So we define

$$\vec{P} = M\dot{\vec{R}} = M\vec{V}_{CM} \quad \ldots (6.3.6)$$

where $M = m_1 + m_2 + m_3 + \cdots + m_n$ is the total mass of the system and $\dot{\vec{R}}$ is the velocity of this total mass, supposed to be concentrated at a point C, specified by the position vector $\vec{R}$.

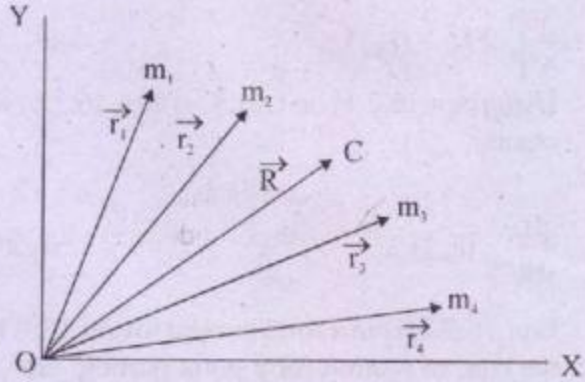


Fig. 6.2

Thus one can apply Newton's Laws of Motion to a mass M, situated at a point C, specified by the position vector $\vec{R}$, and describe the dynamics of the system of many particles. Using eqn. (6.3.6) in L.H.S. of eqn. (6.3.1) we obtain

$$M\dot{\vec{R}} = m_1\dot{\vec{r}}_1 + m_2\dot{\vec{r}}_2 + \ldots + m_n\dot{\vec{r}}_n$$

$$= \sum_i m_i\dot{\vec{r}}_i \quad \ldots(6.3.7)$$

Equation (6.3.7) defines C.O.M. of a system of discrete particles $m_1, m_2, m_3, \ldots m_n$. If there is a continuous distribution of masses, then one can write

$$M\dot{\vec{R}} = \int \dot{\vec{r}}\, dm \quad \ldots(6.3.8)$$

In the non-relativestic limit or in any other type of motion where mass is considered to be constant, equations (6.3.7) and (6.3.8) reduce respectively to

$$\vec{R} = \frac{\sum_i m_i \vec{r_i}}{\sum_i m_i} \qquad ...(6.3.9)$$

and

$$\vec{R} = \frac{I}{M}\int \vec{r}\, dm \qquad ...(6.3.10)$$

It follows from eqn. (6.3.4) that when net extrnal force on the system is zero, the momentum of C.O.M. remains constant.

6.4 Rigid body and rotation of rigid body:

A body is said to be rigid if the distance between any two particles, constituting the body does not change, under application of force.

When a rigid body rotates about an axis, each particle, constituting the rigid body moves in circular path about the axis, with centres on the axis; and all particles possess same angular velocity at any instant of time. They may possess different angular velocities at different times but all particles do possess same angular velocity at any instant (e.g. rotation of a fan).

6.5 Cause of rotation :

In order to investigate the causes of rotation we consider the following cases of rotation.

(i) Rotation of a fan :

We know that C.O.M. of a fan lies at its centre, and the axis of rotation passes through it. When the fan rotates the C.O.M. is at rest. This implies that net external force acting on the fan is zero. So we conclude that "a body can rotate even though net external force acting on it is zero."

(ii) Rotation of a rotating chair :

In this case one finds that unless a force is applied, the chair does not rotate on its own. This leads to the conclusion that "a force is necessary for rotation".

(iii) Rotation of a cycle wheel :

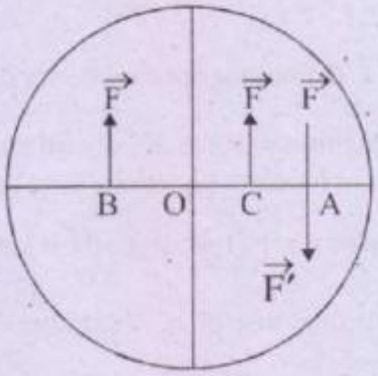


Fig. 6.3

When we apply a force $\vec{F}$ at A (See fig. 6.3), the wheel rotates anticlockwise. But when we apply the same force $\vec{F}$ at B, the wheel rotates clockwise.

We also observe that when same force $\vec{F}$ is applied at 'C' (nearer to axis of rotation passing through 'O'), the rotation is anticlockwise but less. One also observes that when a force $\vec{F'}$ $(= -\vec{F})$ is applied at A, the wheel rotates clockwise. We also observe that when a larger force is applied at any of the points, the rotation is larger, (preserving the sense of rotation).

From above discussion we conclude that:

i) Rotation depends upon force but in a special manner, such that even if the net force is zero, there is rotation.

ii) Rotation depends upon the point of application of the force w.r. to the axis of rotation.

iii) Rotation depends on the relative orientation of the force and position vector of the point of application of the force.

Taking into account the above conclusions one defines a quantity

$$\vec{\tau} = \vec{r} \times \vec{F} = rF\sin\theta\hat{n} \qquad ...(6.5.1)$$

where $\vec{r}$ defines the position vector of the point of application of force $\vec{F}$, w.r. to an origin on the axis of rotation. Equation (6.5.1) shows that $|\vec{\tau}|$ is large when $|\vec{r}|$ is large, $|\vec{F}|$ is large and $\theta = 90^0$. The direction of $\vec{\tau}$ shall also depend on the direction of $\vec{r}$ and $\vec{F}$. Therefore $\vec{\tau} = \vec{r} \times \vec{F}$, called as torque (or moment of a force) can be taken as a physical quantity, which could explain the phenomenon of rotation.

We now proceed to learn more about this torque.

6.6 Torque :

Torque of a force about a point is defined as

$$\vec{\tau} = \vec{r} \times \vec{F} = rF\sin\theta\,\hat{n} \qquad ...(6.6.1)$$

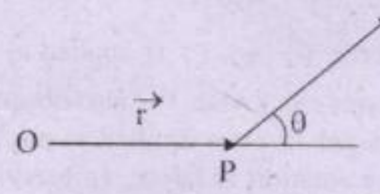


Fig. 6.4

The difinition (6.6.1) is, however, not suitable for the description of a real physical rotation.For example consider a plane lamina ABCD as shown in fig. 6.5. Let the plane be taken to be XY-plane, with AB along X- direction and AD along Y-diretion. Let the lamina be made to rotate about an axis (zoz') passing through its centre 'o' and perpendicular to its plane (i.e. along z-dir^n). If a force $\vec{F}$ is applied perpendicular to its plane (ie along z-dir^n) at P, the lamina does not rotate, although rotque about

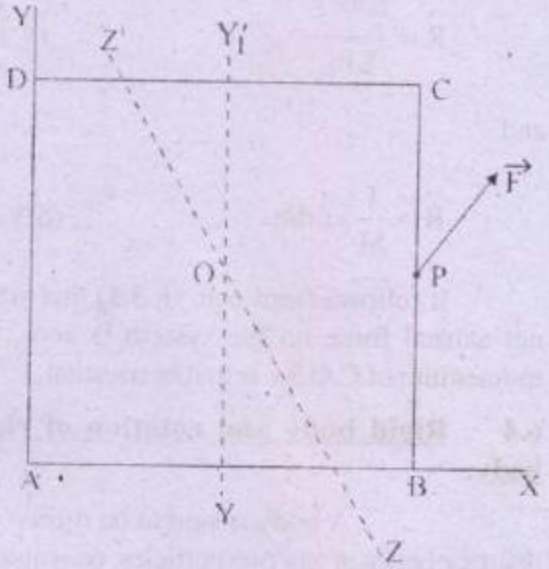


Fig. 6.5

the point O is finite. On the otherhand if the axis of rotation be parallel to Y-direction like $Y_1OY'_1$, but still passing through O, then the lamina rotates. This example illustrates that in order to describe rotation,one has to take into account the effect of axis of rotation. Therefore we define torque about a given axis of rotation (specified by aunit vector $\hat{A}$) as

$$\tau_A = \hat{A} \cdot \vec{\tau} = \hat{A}.(\vec{r} \times \vec{F}) \qquad ...(6.6.2)$$

where $\vec{r}$ is the position vector of the point of application of force $\vec{F}$, measured with respect to an origin lying on the axis of rotation. Equation (6.6.2) helps us in understanding the rotation of the plane lamina described above. The positive direction of $\hat{A}$ is given by right hand thumb rule.

The definition (6.6.2) necessitates to show that the torque of a force about an axis is independent of the choice of origin on the axis of rotation.

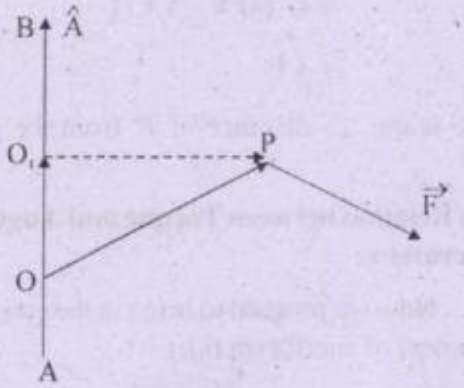


Fig. 6.6

Consider a force $\vec{F}$ acting at P. The torque of this force about the axis AB, with origin at 'O' is given (as per eqn. (6.6.2) by

$$\tau_A = \hat{A}.(\vec{op} \times \vec{F}) \qquad ...(6.6.3)$$

The torque of the same force $\vec{F}$ about the axis AB with origin at O_1, is given by

$$\tau'_A = \hat{A}.(\vec{o_1p} \times \vec{F}) \qquad ...(6.6.4)$$

Now $\vec{o_1p} = \vec{op} - \vec{oo_1} = \vec{op} - (\overline{oo_1})\hat{A} \quad ...(6.6.5)$

Using (6.6.5) in (6.6.4) we obtain

$$\tau'_A = \hat{A}.[(\vec{op} - (\overline{oo_1})\hat{A}) \times \vec{F}]$$

$$= \hat{A}.(\vec{op} \times \vec{F}) - (\overline{oo_1})\hat{A}.(\hat{A} \times \vec{F})$$

$$\Rightarrow \tau'_A = \hat{A}.(\vec{op} \times \vec{F}) = \tau_A \qquad ...(6.6.6)$$

$$[\because \hat{A}.(\hat{A} \times \vec{F}) = \vec{F}.(\hat{A} \times \hat{A}) = 0]$$

The above exercise shows that the definition (6.6.2) is independent of the choice of origin on the axis of rotation.

If there are more than one force acting on the body, then torque <u>about a point</u> is given as

$$\vec{\tau} = (\vec{r_1} \times \vec{F_1} + \vec{r_2} \times \vec{F_2} + ...) \qquad ...(6.6.7)$$

and about an axis $\hat{A}$ is given as

$$\vec{\tau_A} = \hat{A}.(\vec{r_1} \times \vec{F_1} + \vec{r_2} \times \vec{F_2} + ...) \qquad ...(6.6.8)$$

It may be noticed from (6.6.7) and (6.6.8) that even if $\vec{F_1} + \vec{F_2} = 0$, the torque may not vanish. This explains the fact that even if the net force acting on a body is zero, there could be rotation.

<u>Special Cases</u> :- When we consider torque about a given axis the following special cases may be noted.

i) $\vec{F_1}$ ll to axis of rotation

Then $\vec{F} = F\hat{A}$, yielding

$$\tau_A = \hat{A}.(\vec{r} \times \vec{F}) = \hat{A}.(\vec{r} \times F\hat{A}) = 0$$

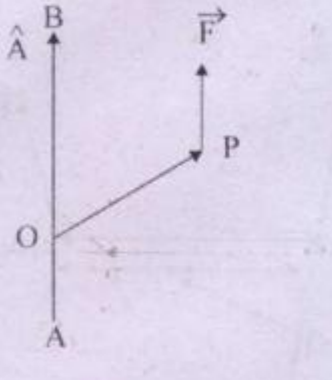


Fig. 6.7

ii) $\vec{F}$ <u>intersects the axis</u> :

When line of action of force $\vec{F}$ intersects the axis of rotation, we can choose the point of intersection as origin so that

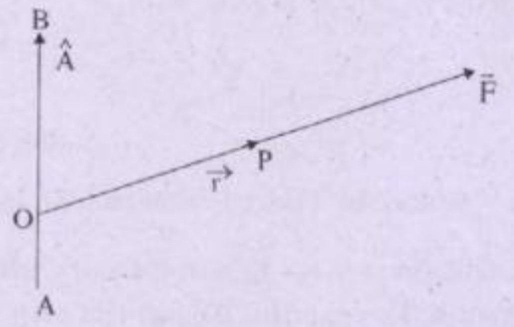


Fig. 6.8

$\vec{r} \times \vec{F} = 0$, yielding $\tau_A = \hat{A}.(\vec{r} \times \vec{F}) = 0$.

iii) $\vec{F}$ is $\perp^r$ to axis AB, without intersecting it.

Then

$\tau_A = \hat{A}.(\vec{r} \times \vec{F})$ = Force x perpendicular distance of pt. of application of force from axis.

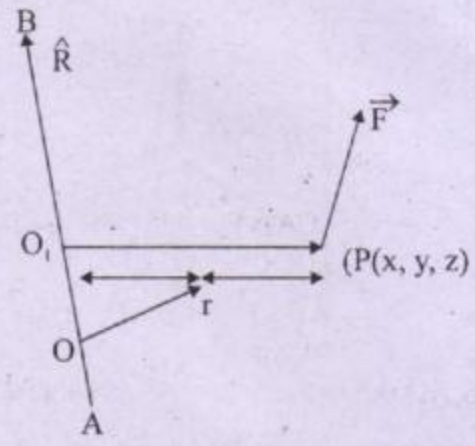


Fig. 6.9

To fix our ideas we proceed as follows:

Let $\hat{A} = \hat{k}$, and $\vec{F} = F\hat{j}$

Then $\tau_A = \hat{A}.(\vec{r} \times \vec{F})$

$$= \hat{k}.[(x\hat{i} + y\hat{j} + z\hat{k}) \times F\hat{j}]$$

$$= \hat{k}.[xF\hat{k} - zF\hat{i}]$$

$$= x\,F$$

But 'x' is the $\perp^r$ distance of 'P' from the axis AB.

6.6(a) Relation between Torque and Angular acceleration :

Now we proceed to bring in the concept of moment of enertia (m.o.I.)

Consider a rigid body A, rotating about an axis AB (as shown in fig 6.10)

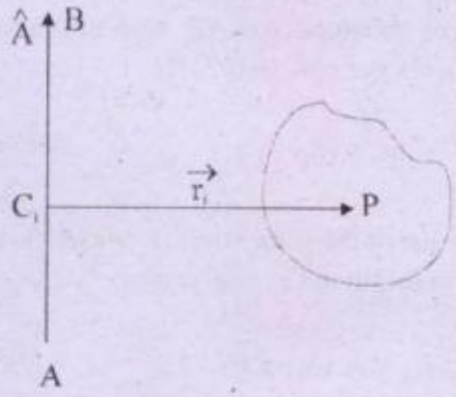


Fig 6.10

Let P be position of i_{th} particle and $\vec{F_i}$ be the force on this particle. As the body rotates the particle at 'P' moves along a circular path with centre at C_i, lying on the axis of rotation, which is perpendicular to the plane of the circle. Then

$\left|\vec{C_iP}\right| = |\vec{r_i}|$ shall be perpendicular distance of P from the axis of rotation.

Now from eqn. (5.4.7)

$$\vec{a} = r\alpha\hat{e}_t - r\omega^2\hat{e}_r$$

So in non-relationstic limit or when mass is taken to be constant, we can write

$$\vec{F_i} = m_i r_i \alpha \hat{e}_t - m_i r_i w^2 \hat{e}_r \quad ...(6.6.a)$$

Where $\overrightarrow{C_iP} = \vec{r}_i = r_i\,\hat{e}_r$ and $\hat{e}_t$ is the unit vector along the tangent to the circular path. Therefore torque of this force about the axis AB, shall be

$$\tau_{iA} = \hat{A}.(\vec{r}_i \times \vec{F}_i)$$

$$= \hat{A}.[\vec{r}_i \times (m_i r_i\,\alpha\,\hat{e}_t - m_i r_i w^2\,\hat{e}_r)]$$

$$= m_i r_i^2\,\alpha\,\hat{A}.(\hat{e}_r \times \hat{e}_t)$$

$$\Rightarrow \tau_{iA} = m_i r_i^2\,\alpha \qquad ...(6.6.10)$$

$$\left[\because \hat{A}.(\hat{e}_r \times \hat{e}_t) = 1\right]$$

Considering all the particles in the above manner we obtain

$$\tau_A = \sum_i \tau_{iA} = (\sum_i m_i r_i^2).\alpha$$

Defining $I_A = \sum_i m_i r_i^2$ ...(6.6.12)

We obtain

$$\tau_A = I_A \alpha \qquad ...(6.6.13)$$

where the subscript 'A' refers to the axis $\hat{A}$.

We can re-write (6.6.13) as

$$\hat{A}.\vec{\tau} = \tau_A = I_A(\hat{A}.\hat{A})\alpha = \hat{A}.I_A(\hat{A}\alpha) = \hat{A}.I_A\,\vec{\alpha}$$

i.e. $\hat{A}.\vec{\tau} = \hat{A}.I_A\,\vec{\alpha}$ ...(6.6.14)

So we are used to write

$$\vec{\tau} = I_A\,\vec{\alpha} \qquad ...(6.6.15)$$

The quantity I_A defined in (6.6.12) is called moment of inertia whose physical significance is hidden within eqn. (6.6.13). Eqn. (6.6.13) can be treated as Newton's equation of motion for rotation. It is easy to note that M.O.I. plays a role in rotatioon similar to mass in linear motion. We also further note that the eqn. (6.6.13) or (6.6.15) holds good in an inertial frame.

6.7 Couple :

Consider the torque due to two equal and opposite forces acting on a body.

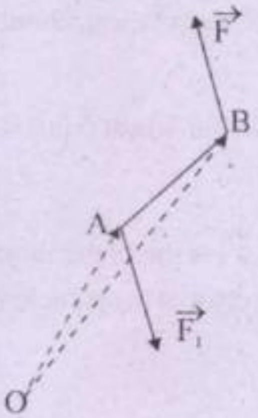


Fig. 6.11

Let force $\vec{F}_1$ act at A and $\vec{F}_2$ at B. Let $\vec{F}_1 = -\vec{F}_2 = -\vec{F}$. Then torque about any arbitrary point 'O' is given as

$$\vec{\tau}_0 = \vec{OA} \times \vec{F}_1 + \vec{OB} \times \vec{F}_2$$

$$= (\vec{OB} - \vec{OA}) \times \vec{F}_2 \qquad (\because \vec{F}_1 = -\vec{F}_2)$$

$$\vec{\tau}_0 = \vec{AB} \times \vec{F}_2 = \vec{AB} \times \vec{F} \qquad ...(6.7.1)$$

Eqn. (6.7.1) shows that when $\vec{AB}$ is parallel to $\vec{F}_2$ the torque about the point 'O' vanishes (i.e. when the lines of action of $\vec{F}_1$ and $\vec{F}_2$ coincide). Otherwise $\vec{\tau}_0$ does not vanish and its rotational effect about any axis $\hat{A}$, passing through 'O' is given by

$$\tau_A = \hat{A}.\vec{\tau}_0 = \hat{A}.(\vec{AB} \times \vec{F}) \qquad ...(6.7.2)$$

Thus "two equal and opposite forces, whose lines of action donot coincide constitute a couple", and its torque (moment of the couple) about any point 'O' is given by eqn. (6.7.1) and about any axis by eqn. (6.7.2)

6.8. Angular Momentum

Angular momentum of a particle undergoing rotation, about a point 'O' is defined as :

$$\bar{L} = \bar{r} x \bar{p} = (rp\sin\theta)\hat{n} = m\vartheta(r\sin\theta)\hat{n} = m\vartheta b\hat{n} \quad ...(6.8.1)$$

where, $\vec{p}$ $(=m\vec{\vartheta})$ is the linear momentum, $\vec{r}$ is the position vector of the particle with respect to the point 'O'.

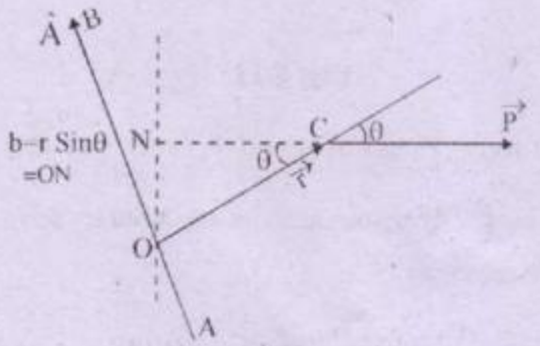


Fig. 6.12

The angular momentum of a system of particles about a point 'O' is given as

$$\vec{L} = \sum_i \vec{L}_i = \sum_i \vec{r}_i \times \vec{p}_i \quad ...(6.8.2)$$

The angular memontum about an axis (any line) is defined as the component of $\vec{r} \times \vec{p}$ along the axis.

i.e. $L_{AB} = L_A = \hat{A}.\vec{L} = \hat{A}.(\vec{r} \times \vec{p})$ ...(6.8.3)

where $\vec{r} \times \vec{p}$ is calculated about any point on the axis AB.

6.8. (a) Reltion between $\vec{L}$, I_A and $\vec{\omega}$

Consider a rigid body M rotating about an axis AB (denoted by $\hat{A}$). Let P be the position of the i_{th} particle in the rigid body. All the

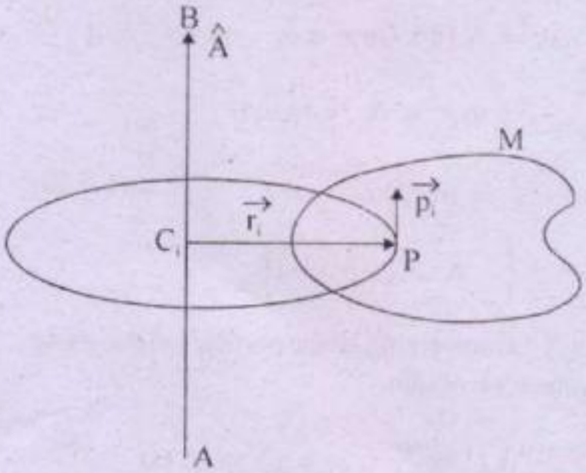


Fig. 6.13

particles in the rigid body move along circles with centres lying on the axis of rotation AB; and the axis is perpendicular to the plane of the circles. Let C_i be the centre of the circle, pursued by the i_{th} particle at P. The angular momentum of this particle about the axis AB is by (6.8.3) given as

$$(L_i)_{AB} = L_{iA} = \hat{A}.\vec{L}_i = \hat{A}.(\vec{r}_i \times \vec{p}_i)$$

$$= \hat{A}.(r_i\hat{e}_r \times m_i\vartheta_i\hat{e}_t)$$

$$= m_i r_i \vartheta_i \hat{A}.(\hat{e}_r \times \hat{e}_t)$$

$$\Rightarrow \quad L_{iA} = m_i r_i \vartheta_i = m_i r_i^2 \omega \quad ...(6.8.4)$$

$[\because \hat{A}.(\hat{e}_r \times \hat{e}_t) = 1, \text{and } \vartheta_i = r_i\omega]$

Considering all the particles constituting the rigid body we have

$$L_A = L_{AB} = \sum_i (L_i)_{AB} = \sum_i L_{iA} = (\sum_i m_i r_i^2)\omega$$

Using the difinition (6.6.12) on R.H.S. we have

$$L_{AB} = L_A = I_A \omega \qquad ...(6.8.6)$$

where the subscript 'A' refers to the axis $\hat{A}$. We can re-write (6.8.6) as

$$L_A = \hat{A}.\vec{L} = I_A(\hat{A}.\hat{A})\omega = \hat{A}.I_A(\omega\hat{A}) = \hat{A}.I_A\vec{\omega}$$

i.e. $$\hat{A}.\vec{L} = \hat{A}.I_A\vec{\omega} \qquad ...(6.8.7)$$

So we are used to write

$$\vec{L} = I_A\vec{\omega} \qquad ...(6.8.8)$$

6.8.(b) Relation between Torque and angular momentum :

By definition angular momentum $\vec{L}$ about a point 'o' is given as

$$\vec{L} = \vec{r} \times \vec{p}$$

Differentiating both sides w.r. to time 't'

$$\frac{d\vec{L}}{dt} = \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt}$$

$$= \vec{\vartheta} \times m\vec{\vartheta} + \vec{r} \times \vec{F}$$

$$\Rightarrow \quad \frac{d\vec{L}}{dt} = \vec{r} \times \vec{F} = \vec{\tau}$$

i.e. $$\frac{d\vec{L}}{dt} = \vec{\tau} \qquad ...(6.8.9)$$

where R.H.S. of eqn. (6.8.9) is the torque about the point 'o'. Similarly if we consider angular - momentum about a given axis $\hat{A}$, then

$$L_A = \hat{A}.\vec{L}$$

$$\frac{dL_A}{dt} = \hat{A}.\frac{d\vec{L}}{dt} = \hat{A}.\vec{\tau} = \tau_A \quad ...(6.8.10)$$

where r.h.s. of eqn (6.8.10) refers to torque about axis $\hat{A}$. Thus equation (6.7.9) and (6.8.10) leads to the statement "the time rate of change of angular momentum about a point or a given axis is equal to the corresponding torque (about the point or the axis as the case may be)". This statement is equivalent to Newton's 2nd Law for linear motion.

It is worth while to mention that eqns (6.8.9) and (6.8.10) hold good in general as $\vec{F} = \frac{d\vec{p}}{dt}$ holds good. But the relation $\vec{\tau} = I\vec{\alpha}$ or $\tau_A = I_A\,\alpha$ holds good in non-relativistic limit or where mass remains constant as the relation $\vec{F} = m\vec{a}$ holds.

6.8 (c) Law of Conservation of angular momentum

<u>It states that when net torque acting on a body (or system) vanishes, the angular momentum of the body (or system) remains constant in time.</u>

The above statement follows from eqn. (6.8.9) and (6.8.10). When $\vec{\tau} = 0$, $\frac{d\vec{L}}{dt} = 0$ implying $\vec{L}$ = constant. Also when $\tau_A = 0$, $\frac{dL_A}{dt} = 0$, implying L_A = constant.

Illustrative Examples

(i) When a man, standing on a turn table with dumb bell in each stretched hand, pulls down his hands, the angular speed of the turn table increases.

This phenomenon can be explained as follows : The turn table and man with dumb-bell form an isolated system. As no external torque acts on this system, Its angular momentum is to be conserved.

i.e. $L_i = L_f$

$\Rightarrow \; I_i\omega_i = I_f\omega_f$

$$\frac{\omega_f}{\omega_i} = \frac{I_i}{I_f}$$

Since $I_i > I_f$ so $\omega_f > \omega_i$.

(ii) Consider a stone tied to one end of a string and whirled around a finger. When the finger stops supplying centripetal force the string wraps around the finger with increasing angular speed.

This can be explained on the following line. As no external torque acts, so

$I_i = I_f$

$\Rightarrow \; I_i\,\omega_i = I_f\,\omega_f$

$$\frac{\omega_f}{\omega_i} = \frac{I_i}{I_f}$$

As $I_i > I_f$ so $\omega_f > \omega_i$.

(iii) A spring board diver, while jumping into a swimming pool, curls himself in such a way that his feet and head are driven close to each other. This results in a decrease in moment of inertia of his body about a central axis, and this enables him to rotate his body at a faster rate. Thus he is able to complete more number of revolutions before hitting the surface of water.

(iv) Use of auxillary propeller on helicopter :

When the main lift rotor of helicopter rotates, its angular momentum points in one direction. In order to conserve angular momentum, the body of the helicopter would tend to rotate in opposite direction to that of the rotor. In order to avoid this an auxillary propeller is provided at the tail. The axis of rotation of the propeller is in a position perpendicular to the longitudinal axis of the helicopter. It provides a torque that opposes the reaction torque of the main rotor. Hence rotation of the body of the helicopter is prevented.

6.9. Kinetic energy due to rotation :

Consider a rigid body rotating about an axis (AB) with an angular speed ω. The i_{th} constituent particle moves in a circle of radius

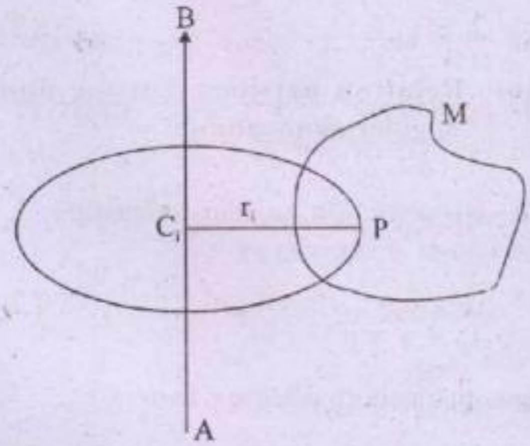


Fig. 6.14

r_i, with a linear speed $\vartheta_i = r_i\,\omega$. Therefore the K.E. of the ith particle is $K_i = \frac{1}{2}\,m_i\,\vartheta_i^2 = \frac{1}{2}\,m_i r_i^2 w^2$ (non-relativistic limit). Considering all such constituent particles the total kinetic energy of the rigid body is given as

$$K = \sum_i K_i = \sum_i \frac{1}{2} m_i r_i^2 \omega^2$$

$$\Rightarrow \quad K = 1/2(\sum_i m_i r_i^2)\omega^2 = 1/2\, I_A \omega^2 \quad ...(6.9.1)$$

Although it is called rotational kinetic energy, but it is not a new type of kinetic energy. It is just the sum of $1/2\ m_i \vartheta_i^2$ of all the particle as shown above.

6.9.(a) Work done by torque, Power dissipation in rotation and Work-energy relation in rotation :

Consider a rigid body of mass M executing rotation about an axis (AB) due to a torque τ_A, about an axis AB ($= \hat{A}$). Let a force $\vec{F}$ acting at a point P, specified by position vector $\vec{r}$, produce this torque. Then work done for an elementary angular displacement $\vec{d\theta}$, about the axis of rotation, causing a linear displacement $\vec{ds}$ of the point P, is given by

$$dW = \vec{F}.\vec{ds} = \vec{F}.(\vec{d\theta} \times \vec{r}) \qquad ...(6.9.2)$$

($\because \vec{ds} = \vec{d\theta} \times \vec{r}$ by eqn. 5.4.2)

Since $\vec{A}.(\vec{B} \times \vec{C}) = \vec{B}.(\vec{C} \times \vec{A})$, eqn (6.9.2) reduces to

$$dW = \vec{d\theta}.(\vec{r} \times \vec{F}) = \vec{d\theta}.\vec{\tau}$$

$$= d\theta\ \hat{A}.\vec{\tau} = d\theta.\ \tau_A$$

i.e. $$dw = \tau_A . d\theta \qquad ...(6.9.3)$$

The total work done by the torque τ_A of the force $\vec{F}$ when angular position of point P changes from θ_1 to θ_2 is given by

$$W = \int_{\theta_1}^{\theta_2} \tau_A d\theta \qquad ...(6.9.4)$$

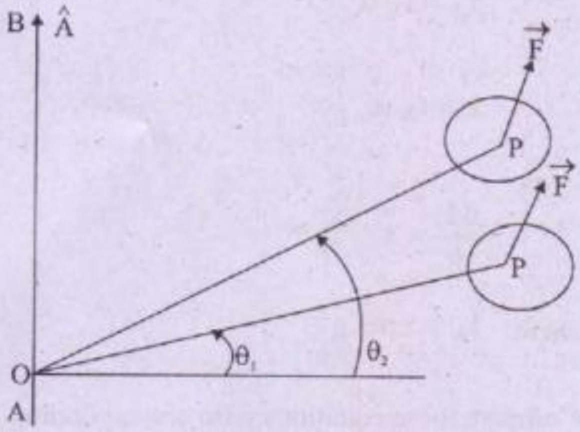


Fig. 6.15

Power 'P' delivered by the torque τ_A, during rotation, is the rate of work done (as defined earlier). Hence

$$P = \frac{dW}{dt} = \tau_A \frac{d\theta}{dt} = \tau_A \omega \qquad ...(6.9.5)$$

In the non-relativistic limit or where mass remains constant, the kinetic energy is given by eqn (6.9.1). It follows from (6.9.1)

$$\frac{dK}{dt} = \frac{d}{dt}(1/2 I_A \omega^2) = I_A \omega \frac{d\omega}{dt} = I_A \alpha . \omega$$

$$\Rightarrow \frac{dK}{dt} = \tau_A . \omega \qquad ...(6.9.6)$$

Eqns. (6.9.5) and (6.9.6) implies that the rate of increase of kinetic energy is equal to the rate of work done during a rotation (in non-relativistic limit). This is the work-energy relation in rotation.

6.10. <u>Moment of Inertia</u> :

In Sec. 6.6(a) we brought about the concept of moment of inertia and related it with torque, and angular acceleration. In sec. 6.8 we also established the relation of angular momentum ($\vec{L}$) with M.O.I. To be specific we have established

$$\vec{\tau}_A = I_A \vec{\alpha}$$

$$\vec{L} = I_A \vec{\omega}$$

$$\frac{d\vec{L}}{dt} = \vec{\tau}$$

where $I_A = \sum_i m_i . r_i^2$

[Compare these equations with corresponding equations for linear motion $\vec{F} = m.\vec{a}$, $\vec{p} = m.\vec{\vartheta}$, $\frac{d\vec{p}}{dt} = \vec{F}$]

These equations imply that "Moment of inertia of a body is a property by virtue of which the body tends to continue in its state of rest or of uniform rotation until and unless it is acted upon by a net external torque; and is equal to the sum of the products of masses and square of their perpendicular distances from the axis of rotation."

Thus we have

(i) For a system, consisting of discrete masses (see fig 6.16 (a)

$$I_A = \sum_i m_i r_i^2 \quad ...(6.10.1)$$

(ii) For a system, consisting of continuous mass distribution (see fig. 6.16 (b)

$$I_A = \int r^2 dm \quad ...(6.10.2)$$

where dm is an elementary mass of the body situated at a perpendicular distance 'r' from the axis of rotation and integration is carried over the whole body.

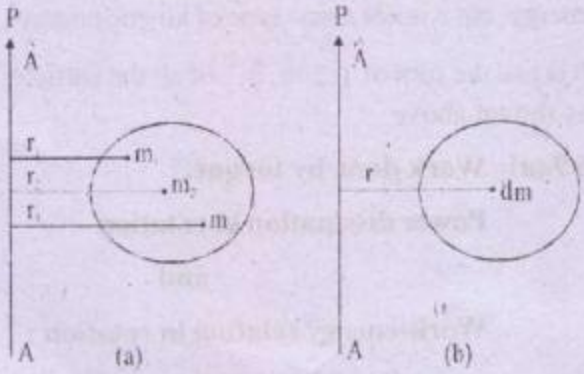


Fig. 6.16

We note that M.O.I. depends on (i) total mass (ii) the mass distribution and (iii) axis of rotation. As M.O.I. depends on the axis of rotation so it has no unique value.

6.10(a) Radius of Gyration

Radius of gyration of a body about an axis of roatation is defined as the distance (perpendicular distance) of a point from the axis of rotation, where if the entire mass of the body is supposed to be concentrated, then it would yield the same moment of inertia about the given axis as the body would have given about the axis, i.e.

$$I_A = MK_A^2 = \sum_i m_i r_i^2 \quad ...(6.10.3)$$

OR

$$I_A = MK_A^2 = \int r^2 dm \quad ...(6.10.4)$$

where K_A is radius of gyration, M is total mass. Eqn. (6.10.3) holds for discrete mass distribution and eqn. (6.10.4) holds for continuos mass distribution.

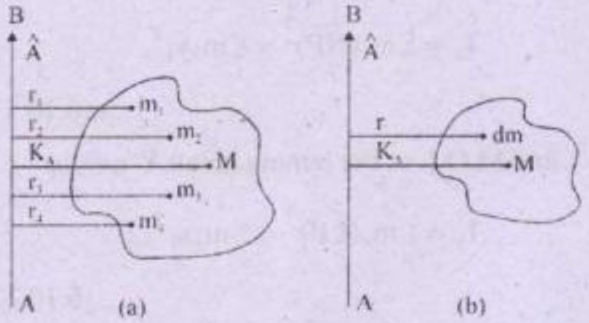


Fig. 6.17

6.10.(b) Theoroms on M.O.I.

i) Parallel axis Theorem :

It states that M.O.I. of a body about a given axis is equal to the sum of the M.O.I. of the body about a parallel axis through the centre of gravity and the product of the mass of the body and square of the distance between the axes.

If I_A be M.O.I. of the body of mass M about an axis AB ($= \hat{A}\,\overline{AB}$) and I_G be M.O.I. of the body about an axis $A'B'$ ($= \hat{A}\,\overline{A'B'}$), passing through C.O.G. 'G' and at a distance 'd' from AB, then

$$I_A = I_G + Md^2 \qquad ...(6.10.5)$$

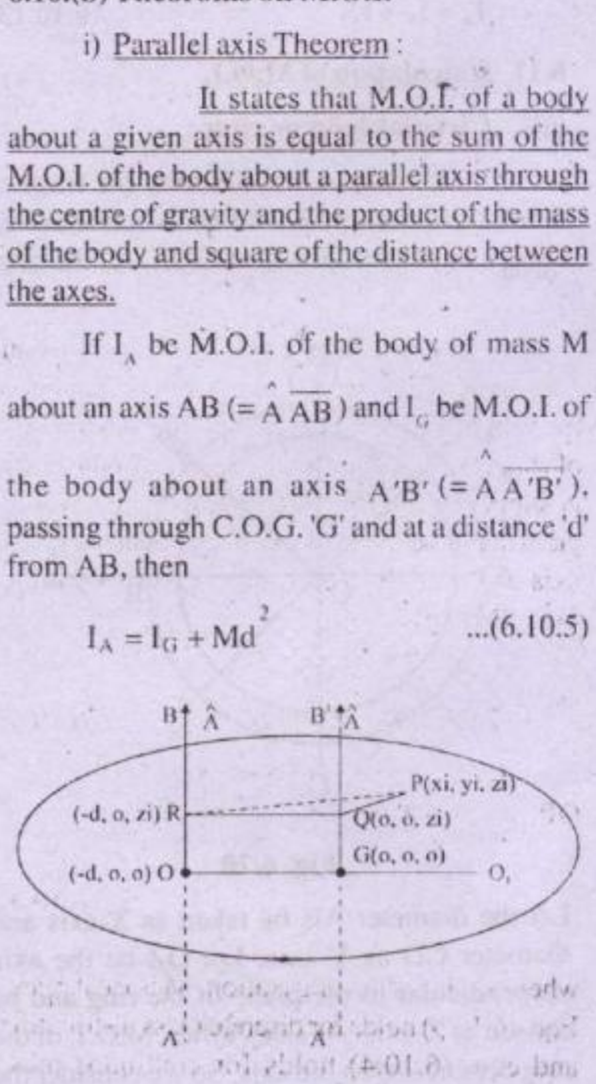


Fig. 6.18

Proof : Consider a rigid body of mass M. AB and $A'B'$ are two parallel axes separated by distance d, with $A'B'$ passing through C.O.G. ('G'). Let us choose G as our origin. Let P be any arbitary point (x_i, y_i, z_i), where the i_{th} particle is present. Let us drop a perpendicular GO on to AB, a perpendicular PQ on to $A'B'$ and another perpendicular PR on to AB. Let us choose GO' as X- axis and GB' as Z-axis. Then 'O' has coordinates (-d,o,o), 'Q' has coordinates (o,o,z_i) and R has coordinates $(-d,0,z_i)$.

So M.O.I. of the body about the axis AB shall be

$$I_A = \sum_i m_i (\overline{PR})^2$$

$$I_A = \sum_i m_i [(x_i + d)^2 + (y_i - o)^2 + (z_i - z_i)^2]$$

$$=> I_A = \sum_i m_i [(x_i^2 + y_i^2) + d^2 + 2x_i.d] \qquad ...(6.10.6)$$

The coordinates of C.O.G. are given by

$$M x = \sum m_i x_i$$

$$M y = \sum m_i y_i$$

$$M z = \sum m_i z_i$$

Since we have chosen C.O.G. as origin (i.e. x = 0=y=z) so $\sum m_i x_i = 0 = \sum m_i y_i = \sum m_i z_i$. Therefore eqn. (6.10.6) reduces to

$$I_A = \sum_i m_i (x_i^2 + y_i^2) + Md^2 \qquad ...(6.10.7)$$

Now M.O.I. of the body about the axis A'B' is given by

$$I_G = \sum_i m_i (\overline{PQ})^2$$

$$= \sum_i m_i [(x_i - o)^2 + (y_i - o)^2 + (z_i - z_i)^2]$$

$$=> \quad I_G = \sum_i m_i (x_i^2 + y_i^2) \qquad ...(6.10.8)$$

Using eqn. (6.10.8) in eqn. (6.10.7) we obtain

$$I_A = I_G + Md^2 \quad ...(6.10.9)$$

Thus parallel axis theorem is proved.

(ii) Perpendicular -axis Theorem

It states that the moment of inertia of a plane lamina about an axis perpendicular to the plane of the lamina, is equal to the sum of M.O.I. of thelamina about two mutually perpendicular axes lying on the plane of the lamina and intersetcting each other where the perpendicular axis passes through the body.

Proof :

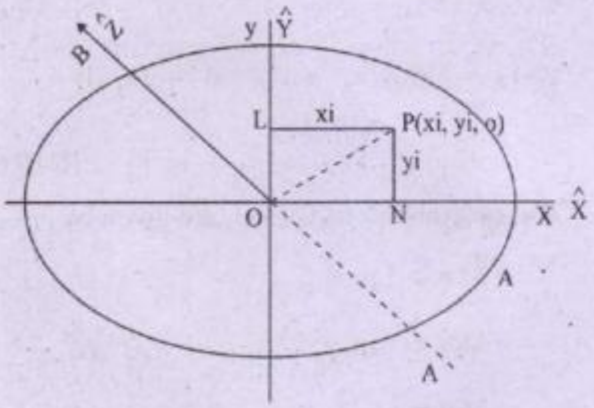


Fig. 6.19

Consider a plane lamina A. Let this plane be chosen as XY-plane. Let an axis AB be perpendicular to this plane and meet the plane of the lamina at O. Choose 'O' as origin and OB as Z-axis. Let OX and OY be chosen as X and Y -axis respectvely. Let P be any arbitrary point (x_i, y_i, o). Then $\overline{OP}$ shall be the perpendicular distance of P from the axis AB. Hence M.O.I. of the body about the axis AB, shall be

$$I_z = \sum_i m_i(\overline{op})^2 = \sum_i m_i[(x_i - o)^2 + (y_i - o)^2 + (o - o)^2]$$

$$\Rightarrow I_z = \sum_i m_i x_i^2 + \sum_i m_i y_i^2 \quad ...(6.10.10)$$

But M.O.I. of the lamina about X-axis is

$$I_x = \sum_i m_i(\overline{NP})^2 = \sum_i m_i y_i^2 \quad ...(6.10.11)$$

and M.O.I. of the lamina about Y-axis is

$$I_y = \sum_i m_i(\overline{LP})^2 = \sum_i m_i x_i^2 \quad ...(6.10.12)$$

Hence using eqns. (6.10.11) and (6.10.12) in (6.10.10) we obtain,

$$I_z = I_x + I_y \quad ...(6.10.13)$$

6.11 Calculation of M.O.I.

(a) Circular loop or ring :

Consider a circular loop (or ring) of radius r and mass m. Let 'O' be its centre and be chosen as origin. 'O' is also C.O.G. of the ring.

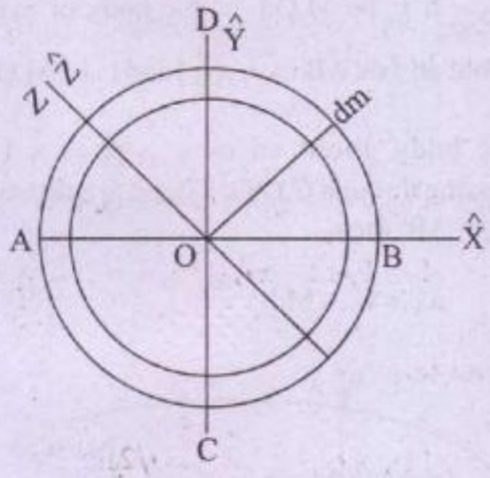


Fig. 6.20

Let the diameter AB be taken as X-axis and diameter CD as Y-axis. Let OZ be the axis perpendicular to the plane of the ring and be chosen as Z-axis. As said earlier, M.O.I. of the ring depends upon the axis. So we consider the following cases.

(i) Axis perpendicular to the plane of the ring and passing through C.O.G. (i.e. about OZ-axis)

Consider an element of the ring and let its mass be dm. Its perpendicular distance from axis oz is r. Hence its M.O.I. about the axis oz, shall be

$$dI_z = r^2 dm \qquad ...(6.11.1)$$

Therefore M.O.I. of the entire ring about oz-axis shall be

$$I_z = \int r^2 dm = r^2 \int dm = mr^2$$

i.e. $$I_z = mr^2 \qquad ...(6.11.2)$$

Then by definition (6.10.4), the radius of gyration

$$K_A = \sqrt{I_z/m} = r \qquad ...(6.11.3)$$

(ii) Axis coinciding with a diameter

Symmetry of the ring suggests that M.O.I. of the ring about diameters AB and CD should be equal.

i.e. $I_{AB} = I_{CD}$

Applying perpendicular axis theorem

$$I_z = I_x + I_Y = I_{AB} + I_{CD} = 2I_{AB} = 2I_{CD} \qquad ...(6.11.4)$$

Therefore

$$I_{AB} = I_{CD} = 1/2\, I_z = 1/2\, mr^2 \qquad ...(6.11.5)$$

and corresponding radius of gyration is given as

$$K_A = \sqrt{\frac{I}{m}} = \frac{r}{\sqrt{2}} \qquad ...(6.11.6)$$

(iii) Axis coinciding with a tangent in the plane of the ring

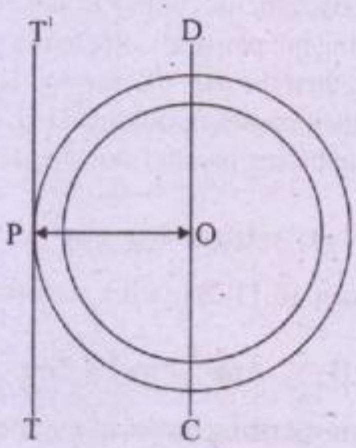


Fig. 6.21

The tangent TT' is parallel to a dieameter CD, which passes through C.O.G. of the ring. Therefore applying parallel axis theorem (eqn. 6.10.5)

$$(I_T)_{II} = I_{TT'} = I_{CD} + mr^2$$

Using eqn. (6.11.5) in r.h.s of above equation, we get

$$(I_T)_{II} = I_{TT'} = \frac{1}{2}mr^2 + mr^2 = \frac{3}{2}mr^2 \qquad ...(6.11.7)$$

The corresponding radius of gyration is

$$K_A = \sqrt{\frac{I}{m}} = \sqrt{\frac{3}{2}}.r \qquad ...(6.11.8)$$

(iv) Axis coinciding with a tangent perpendicular to the plane of the ring :

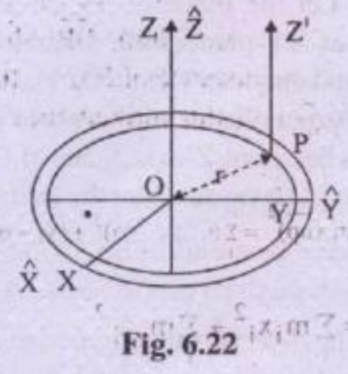


Fig. 6.22

If the axis coincides with PZ', which is tangent to the ring but perpendicular to the plane of the lamina, then the axis PZ' is parallel to the axis OZ; which passes through C.O.G. of the ring. Hence applying parallel axis therorem

$$(I_T)_\perp = I_{PZ'} = I_{OZ} + mr^2$$

Using eqn. (6.11.2) in r.h.s. we obtain

$$(I_T)_\perp = mr^2 + mr^2 = 2mr^2 \quad ...(6.11.9)$$

and corresponding radius of gyration as

$$K_A = \sqrt{\frac{I}{m}} = r\sqrt{2} \quad ...(6.11.10)$$

(b) M.O.I. of a circular disc :

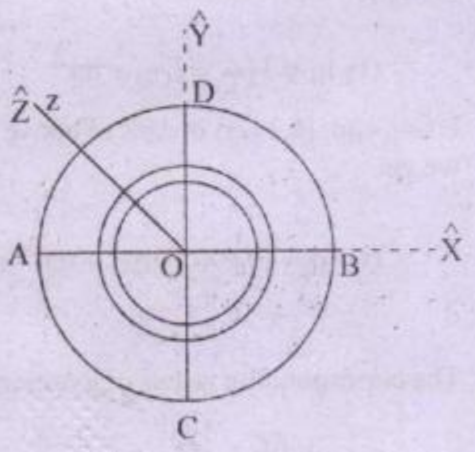


Fig. 6.23

Consider a circular disc of radius R and mass m. Let 'O' be its centre which is also its C.O.G. Let the plane of the circular disc be chosen as XY-plane, with diameter AB along x-axis and diameter CD along Y-axis. The axis OZ is perpendicular to the plane of the disc, and thus lies along Z-axis. Since M.O.I. depends on the axis, so we consider the following cases:

(i) Axis perpendicular to its plane and passing through its centre : (OZ - axis)

The disc can be imagined to be made up of several rings starting from the centre to the rim of the disc. Consider one such elementary ring of inner radius r and radial thickness dr (see fig. 6.23) Then

area of the elementary ring $dA = 2\pi r dr$

mass of the elementary ring $dm = (2\pi r dr)\sigma$

Where,

$$\sigma = \frac{m}{\pi R^2} = \text{mass per unit area of the disc}$$

Therefore M.O.I. of this elementary ring about the axis OZ ($\perp$' to the plane of the ring and passing through its centre) is given by eqn. 6.11.2 as

$$dI_Z = r^2\, dm = 2\pi\sigma\, r^3\, dr \quad ...(6.11.11)$$

Considering all such elementary rings starting from centre to the rim of the disc and summing such M.O.I'. s we obtain

$$I_Z = \sum dI_Z = \int dI_Z = 2\pi\sigma \int_0^R r^3 dr.$$

$$\Rightarrow I_Z = 2\pi\sigma \frac{R^4}{4} = \tfrac{1}{2}(\pi R^2).\sigma.R^2 = \tfrac{1}{2} mR^2$$

$$\Rightarrow I_Z = \tfrac{1}{2} mR^2 \quad ...(6.11.12)$$

The corresponding radius of gyration is given as

$$K_A = \sqrt{I/m} = \frac{R}{\sqrt{2}} \quad ...(6.11.13)$$

(ii) Axis coinciding with a diameter

Symmetry of the disc suggests that M.o.I. about diameters AB and CD should be equal

i.e. $I_{AB} = I_{CD}$

Therefore applying perpendicular axis theorem

$$I_Z = I_X + I_Y = I_{AB} + I_{CD} = 2I_{AB} = 2I_{CD}$$

This gives using eqn. 6.11.12

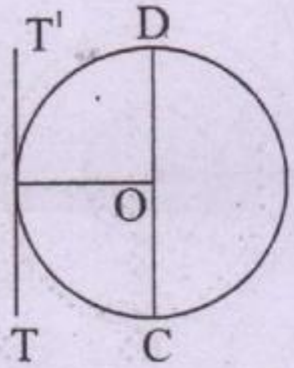
$$I_{AB} = I_{CD} = \frac{1}{2}I_z = \frac{1}{4}\,mR^2 \quad ...(6.11.13)$$

and corresponding radius of gyration

$$K = \sqrt{\frac{I}{m}} = \frac{R}{2} \quad ...(6.11.14)$$

(iii) Axis coinciding with a tangent, in the plane of the disc

The tangent TT ' is parallel to a diameter CD, which passes through the C.O.G. 'O' of the disc. Therefore applying parallel axis theorem (eqn. 6.10.5)



$$I_{TT'} = (I_T)_{||} = I_{CD} + mR^2$$

Using eqn 6.11.13 on r.h.s. we obtain

$$I_{TT'} = \frac{1}{4}mR^2 + mR^2 = \frac{5}{4}mR^2 \quad ...(6.11.15)$$

and corresponding radius of gyration as

$$K_A = \sqrt{\frac{I}{m}} = \frac{\sqrt{5}}{2}R \quad ...(6.11.16)$$

(iv) Axis coinciding with tangent, perpendicular to the plane of the disc.

The tangent PZ' is perpendicular to the plane of the disc, hence parallel to the axis OZ; which passes through the centre (C.O.G.)

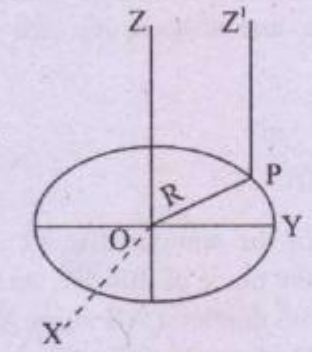


Fig. 6.24

Therefore applying parallel axis theorem

$$(I_T)_\perp = I_{pz'} = I_{oz} + mR^2$$

Using eqn (6.11.12) on r.h.s. we obtain

$$(I_T)_\perp = \frac{1}{2}mR^2 + mR^2 = \frac{3}{2}mR^2 \quad ...(6.11.16)$$

and corresponding radius of gyration as

$$K_A = \sqrt{\frac{I}{m}} = \sqrt{\frac{3}{2}}.R \quad ...(6.11.17)$$

(C) <u>M.O.I. of annular disc</u> :

An annular disc is a disc from which a concentric circular portion has been removed (fig.6.25)

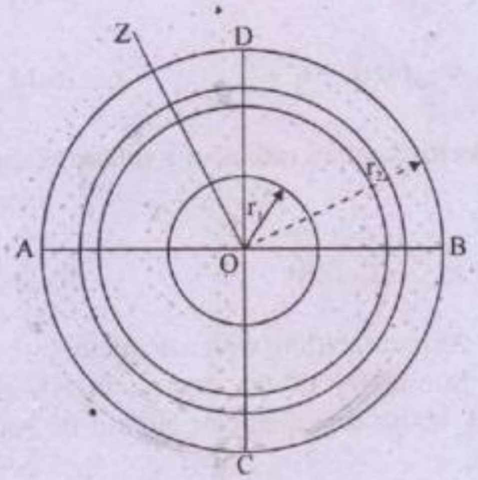


Fig. 6.25

It has inner radius r_1 and outer radius r_2. Let its mass be m, so that mass per unit area σ is given by

$$\sigma = \frac{m}{\pi(r_2^2 - r_1^2)} \qquad ...(6.11.18)$$

The centre of the annular disc 'O' is also its C.O.G. Let the plane of this disc be chosen as XY-plane with diameter AB along X-axis and diameter CD along Y-axis. The axis OZ is perpendicular to the plane of the disc and is along Z-axis. As M.O.I. depends on the axis, so we consider the following cases :

(i) Axis along OZ, (i.e. $\perp^r$ to the plane of disc & passing through C.O.G.)

Consider an elementary ring of radius r and radial thickenss dr. Its M.O.I. as done in eqn. (6.11.11) is given as

$$dI_z = 2\pi\sigma r^3 dr$$

Hence M.O.I. of the annular disc is given by

$$I_z = 2\pi\sigma\int_{r_1}^{r_2} r^3 dr = \frac{\pi\sigma}{2}(r_2^4 - r_1^4)$$

$$= \frac{\pi}{2}.\frac{m}{\pi(r_2^2 - r_1^2)}.(r_2^4 - r_1^4)$$

$$\Rightarrow I_z = \frac{1}{2}m(r_2^2 + r_1^2) \qquad ...(6.11.19)$$

and corresponding radius of gyration is given as

$$K_A = \sqrt{\frac{r_2^2 + r_1^2}{2}} \qquad ...(6.11.20)$$

(ii) Axis coinciding with a diameter :

Summetry of the disc suggetsts that M.O.I. about any diameter should be same. Hence

$$I_{AB} = I_{CD}$$

Applying perpendicular axis theorem

$$I_z = I_{AB} + I_{CD} = 2I_{AB} = 2I_{CD}$$

This gives

$$I_{AB} = I_{CD} = \frac{1}{2}I_z = \frac{1}{4}m(r_2^2 + r_1^2) \qquad ...(6.11.21)$$

and radius of gyration

$$K_A = \frac{1}{2}\sqrt{r_2^2 + r_1^2} \qquad ...(6.11.22)$$

(iii) Axis coinciding with a tangent, lying in the plane of the disc.

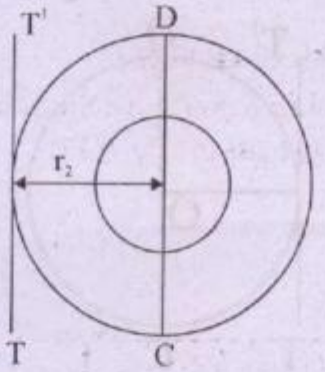


Fig. 6.26

Applying parallel axis theorem, we find

$$(I_{TT'}) = I_{CD} + mr_2^2$$

$$= \frac{1}{4}m(r_2^2 + r_1^2) + mr_2^2$$

$$(I_T)_{11} = \frac{1}{4}m(5r_2^2 + r_1^2) \qquad ...(6.11.23)$$

The corresponding radius of gyration is given by

$$K_A = \frac{1}{2}\sqrt{5r_2^2 + r_1^2} \qquad ...(6.11.24)$$

(iv) Axis coinciding with tangent, $\perp^r$ to the plane of the disc :

Then this axis shall be parallel to OZ-axis passing through C.O.G. and shall be at a distance r_2. Hence applying parallel-axis theorem we obtain

$$(I_T)_\perp = I_z + mr_2^2$$

$$= \frac{1}{2}m(r_2^2 + r_1^2) + mr_2^2$$

$$\Rightarrow (I_T)_\perp = \frac{1}{2}m(3r_2^2 + r_1^2) \quad ...(6.11.25)$$

and radius of gyration

$$K_A = \sqrt{\frac{3r_2^2 + r_1^2}{2}} \quad ...(6.11.26)$$

(D) <u>M.O.I. of a thin rod</u> :

Consider a thin and uniform rod AB of length ℓ and mass m. (fig. 6.27).

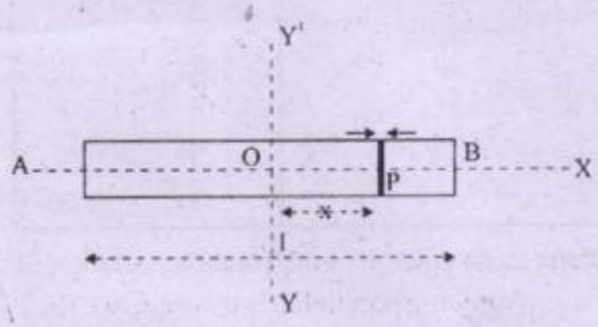


Fig. 6.27

Then mass per unit length shall be

$$\lambda = \frac{m}{\ell} \quad ...(6.11.27)$$

As M.O.I. of the rod shall depend upon the choice of the axis, so we consider the following cases.

(i) Axis $\perp^r$ to its length and passing through its mid point.

Axis YY' passes through mid point 'O' and is perpendicular to it length (chosen as X-axis) consider an element of thickness dx at P, at a distance x from 'O' chosen as origin. The M.O.I. of this element about axis YY'- shall be

$$dI_Y = x^2 dm = x^2(\lambda dx) = \lambda x^2 dx \quad ...(6.11.28)$$

Hence M.O.I. of the rod is obtained as

$$I_Y = \lambda \int_{-\ell/2}^{\ell/2} x^2 dx = \lambda \frac{l^3}{12} \quad ...(6.11.29)$$

Using eqn. (6.11.27), we obtain

$$I_Y = \frac{1}{12} ml^2 \quad ...(6.11.30)$$

Corresponding radius of gyration is given as

$$K_A = \frac{\ell}{2\sqrt{3}} \quad ...(6.11.31)$$

(ii) Axis $\perp^r$ to its length and lying at one end

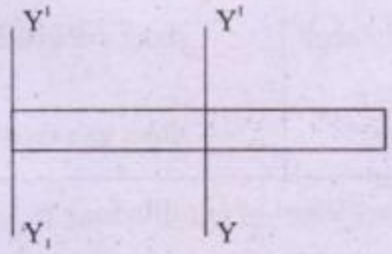


Fig. 6.28

Let $Y_1Y'_1$ be an axis, perpendicular to its length and lying at one end. This axis is parallel to axis YY' which passes through C.O.G. Hence applying parallel axis theorem M.O.I. about Y_1Y_1' axis be given as

$$I_{Y_1Y_1'} = \frac{1}{12}ml^2 + \frac{1}{4}ml^2 = \frac{1}{3}ml^2 \quad ...(6.11.32)$$

The corresponding radius of gyration is

$$K_A = \frac{\ell}{\sqrt{3}} \quad ...(6.11.33)$$

(E) Table 6.1 (M.O.I. of different bodies)

Body	Specification of axis	Expression for M.O.I.
Ring	(a) Passing thorugh its centre & $\perp^r$ to its plane	mr^2
	(b) about a diameter	$\frac{1}{2}mr^2$
Circular disc	(a) Passing through its centre & $\perp^r$ to its plane	$\frac{1}{2}mr^2$
	(b) about a diameter	$\frac{1}{4}mr^2$
Annular disc	(a) Passing through its centre & $\perp^r$ to its plane	$\frac{1}{2}m(r_2^2+r_1^2)$
	(b) about a diameter	$\frac{1}{4}m(r_2^2+r_1^2)$
Thin rod	Passing through its centre & $\perp^r$ to its length	$\frac{1}{12}ml^2$
Sphere (solid)	about a diameter	$\frac{2}{5}mr^2$
Spherical shell	about a diameter	$\frac{2}{3}mr^2$
Cylinder	about its axis of symmetry	$\frac{1}{2}mr^2$

6.12. Conditions of equilibrium :

From the study of linear motion we have seen that when $\sum_i \vec{F}_i = 0$ ie vector sum of all the forces acting on a body or a system vanishes, the momentum of the body remains constant. This implies that there is no change in the linear motion.

We have also noted that when $\sum_i (\tau_A)_i = 0 = \sum_i \hat{A}.(\vec{r}_i x \vec{F}_i)$ ie. sum of the torques due to all the forces about any axis is zero, the angular momentum of the body remains constant. If the axis $\hat{A}$ is to be any axis, then we should have = 0. This implies that there is no change in the rotational motion, if $\sum_i \vec{r}_i x \vec{F}_i = 0$.

Thus when

$$\sum_i \vec{F}_i = 0 \qquad ...(6.12.1)$$

and

$$\sum_i (\vec{r}_i x \vec{F}_i) = 0 \qquad ...(6.12.2)$$

are satisfied simultaneously, there is no change in linear motion as well as rotational motion (about any axis $\hat{A}$). Then the body is said to be in equilibrium. For a body at rest if the above conditions are satisfied, then the body is said to be in static equilibrium. On the otherhand for a body in motion, if the above conditions are satisfied, then the body is said to be in <u>dynamic equilibrium</u>.

Equilibrium can also be calssified basing on its response to slight disturbance at its equilibrium position as :

i) Stable equilibrium

ii) Unstable equilibrium

iii) Neutral equilibrium

(i) <u>Stable Equilibrium</u>

A body is said to be in stable equilibrium if it returns back to its original equilibrium position when slightly disturbed at its equilibrium position.

<u>Examples</u>

(a) A cubical box resting on a level floor on one of the faces. (b) A funnel resting on its mouth on a level surface. (c) A cone resting on its base on a horizontal surface. (d) A ball resting on the lowest point of bowl.

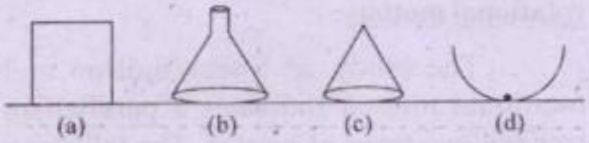


Fig.6.29

(ii) <u>Unstable Equilibrium</u> :

A body is said to be in unstable equilibrium, if it does not return back to its original equilibrium position, on being slightly disturbed at its equilibrium position.

<u>Examples</u> :

(a) A cone standing on its apex. (b) A funnel standing on its stem. (c) A pencil standing on its base. (d) A ball resting at the top of a bowl.

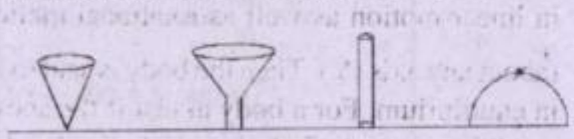


Fig. 6.30

(iii) <u>Neutral Equilibrium</u> :

A body is said to be in neutral equilibrium if it tends to remain at rest in its new position, on being slightly disturbed.

<u>Examples</u> :

(a) A cone resting on its curved surface on a level floor (b) A funnel lying on a level floor on its curved surface, (c) A cylinder lying on a level surface on its curved surface. (d) A ball lying on a level floor.

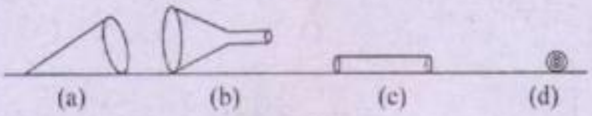


Fig. 6.31

6.13 <u>Centre of Gravity</u>

Consider a body of weight W. It is constituted of large number of particles. Hence the weight of the body given by

$$\vec{W} = \sum_i m_i \vec{g}_i = \sum_i \vec{W}_i \qquad ...(6.13.1)$$

must act through a point G, specified by $\vec{R}$. If the body is in stable equilibrium, then we must imagine a force $\vec{F}$, acting on the body at a suitable point G', specified by the position vector $\vec{R}'$, such that

$$\vec{W} + \vec{F} = 0 \qquad ...(6.13.2)$$

and

$$\vec{\tau}_{net} = 0 \qquad ...(6.13.3)$$

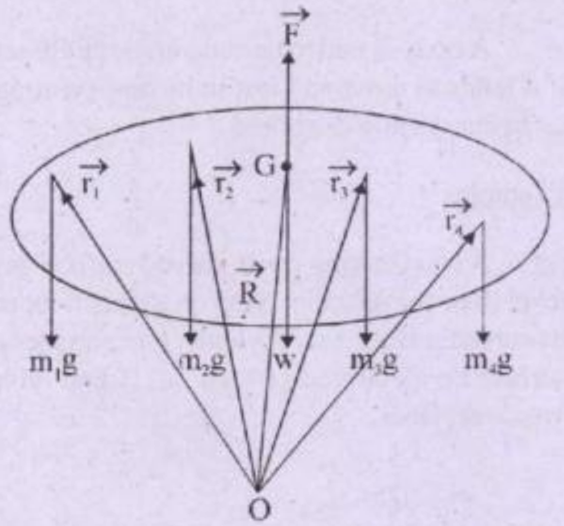


Fig. 6.32

Since $\vec{W} = -\vec{F}$ and $\vec{\tau}_{net} = 0$, so as dicussed in sec. 6.7, $\vec{W}$ and $\vec{F}$ must act along the same line and G,G' must coincide. Thus we find that there exists a point G where the entire weight of the body is supposed to act and this point is called as C.O.G.

From eqn. (6.13.3) we have

$$\sum_i \vec{r}_i x m_i \vec{g}_i + \vec{R} x \vec{F} = 0$$

$$\Rightarrow -\vec{R}x\vec{F} = \vec{R}x\vec{W} = \sum \vec{r}_i x m_i \vec{g}_i \quad ...(6.13.4)$$

where $\vec{R}$ is the position vector of C.O.G.

Since $\sum_i m_i \vec{g}_i = \sum m_i g_i(-\hat{k}) = W(-\hat{k})$

So $\quad \vec{R}xW(-\hat{k}) = \sum \vec{r}_i x m_i g_i(-\hat{k})$

$$\Rightarrow W\vec{R} = \sum_i m_i g_i \vec{r}_i$$

ie $$\vec{R} = \frac{\sum_i m_i g_i \vec{r}_i}{W} \quad ...(6.13.5)$$

This eqn. (6.13.5) defines the position of C.O.G. If the body is very extensive (like say a large portion of earth's surface) g_i may vary. If the body is not very extensive, so that 'g' is same throughhout, eqn. (6.13.5) reduces to

$$\vec{R} = \frac{\sum m_i \vec{r}_i}{M} \quad ...(6.13.6)$$

Eqn. (6.13.6) is identical with the position vector for C.O.M. (in the non-relativistic limit).

6.14 <u>Comparative study of linear and rotational motion</u> :

The study of linear motion and rotational motion indicates a parallelism between two types of motion. The following table summarises the analysis.

Concept	Linear motion	Rotational motion
Distance	Changes	remains constant
Displacement	$\vec{S}$	$\theta, \Delta\vec{S} = \Delta\vec{\theta} x \vec{r}$
Velocity	$\vec{\vartheta} = \frac{d\vec{s}}{dt}$	$\vec{\omega} = \frac{d\theta}{dt}\hat{A}, \vec{\vartheta} = \vec{\omega} x \vec{r}$
Acceleration	$\vec{a} = \frac{d\vec{\vartheta}}{dt}$	$\vec{\alpha} = \frac{d\vec{\omega}}{dt}\hat{A}, \vec{a} = \vec{\alpha} x \vec{r} + \vec{\omega} x \vec{\vartheta}$
Cause of motion	force $\vec{F}$	Torque $\vec{\tau} = \vec{r} x \vec{F}$
Resistance to motion	mass m	M.O.I. $I_A = \sum_i m_i r_i^2$

Quantity of motion	momentum $\vec{p} = m\vec{\vartheta}$	Ang. momentum $\vec{L} = \vec{r} \times \vec{p} = I_A\vec{\omega}$
Kinetic energy	$\frac{1}{2}m\vartheta^2$	$\frac{1}{2}I_A\omega^2$
Kinetic equation	$\vec{F} = \frac{d\vec{p}}{dt}$	$\vec{\tau} = \frac{d\vec{L}}{dt}$
Power	$P = \vec{F}.\vec{\vartheta}$	$P = \vec{\tau}.\vec{\omega}$
Impulse	$\int \vec{F}dt$	$\int \vec{\tau}dt$
Kinematic equation	$\vec{\vartheta} = \vec{u} + \vec{a}t$	$\vec{\omega} = \vec{\omega}_0 + \vec{\alpha}\, t$
	$\vec{s} = \vec{u}t + \frac{1}{2}\vec{a}t^2$	$\vec{\theta} = \vec{\omega}_0 t + \frac{1}{2}\vec{\alpha}\, t^2$
	$\vartheta^2 = u^2 + 2\,\vec{a}.\vec{s}$	$\omega^2 = \omega_0{}^2 + 2\vec{\alpha}.\vec{\theta}$

6.15 Rolling (combined rotation and translation)

Rolling motion of a rigid body is a combination of rotation and translation.

Suppose there exists a frame of reference B in which the motion of the rigid body appears to be a pure rotation about a fixed line $\hat{A}$. If the frame B is inertial then we can apply $\vec{\tau}_{ext} = I_A\vec{\alpha}$ where the quantities are measured in frame B. Then we can add the motion of B w.r. to the actual frame (Lab-frame) and determine the actual motion (rolling) w.r. to the lab-frame.

However if the frame B is non-inertial we cannot apply $\vec{\tau}_{ext} = I_A\vec{\alpha}$. If the frame B has an acceleration $\vec{a}$ with respect to an inertial frame 'C', we have to apply a pseudo force $-m\vec{a}$ to each particle and one has to now take into account the torque due to these pseudo forces so that $\vec{\tau}_{ext} = I_A\vec{\alpha}$ can be applied in the frame B, where $\vec{\tau}_{net} = \vec{\tau}_{ext} + \vec{\tau}_{ps}$. The torque $\vec{\tau}_{ps}$ due to pseudo forces is given by

$$\vec{\tau}_{ps} = \sum \vec{r}_i x(-m_i\vec{a}) = -\sum_i m_i \left(\frac{\sum m_i \vec{r}_i}{\sum_i m_i}\right) x\vec{a}$$

$$\Rightarrow \vec{\tau}_{ps} = -M\left(\frac{\sum_i m_i \vec{r}_i}{M}\right) x\, \vec{a} \qquad ...(6.16.1)$$

where $\vec{r}_i$ is the position vector of a constituent particle of mass m_i, measured w.r.to an origin on the axis of rotation. But it has been shown in sec. 6.13. (eqn. 6.13.6) that when the rigid body is not very very extensive, the position vector of C.O.G. is given by

$$\vec{R} = \left(\frac{\sum_i m_i \vec{r}_i}{M}\right)$$

So using this in eqn. (6.15.1) we obtain

$$\vec{\tau}_{ps} = -M\vec{R} \times \vec{a} \qquad ...(6.15.2)$$

Now suppose our axis of rotation (specified by $\hat{A}$) passes through C.O.G., then

$$(\vec{\tau}_{ps})_A = \hat{A}.\vec{\tau}_{ps} = \hat{A}.(-M\vec{R} \times \vec{a}) \qquad ...(6.15.3)$$

We have also shown in sec. 6.6. that the choice of the origin on the axis of rotation is arbitrary. Therefore without losing generality we can

Where I_G is the M.O.I. of the body about an axis $(\equiv \hat{A})_1$ passing through C.O.G. and α is the angular acceleration. Now the linear acceleration of the C.O.G. is given by eqn. (6.15.5) as

$$\vec{a}_G = \frac{\Sigma \vec{F}}{m} = (m\vec{g} + \vec{R} + \vec{f})/m$$

$$\Rightarrow \quad m\vec{a}_G = (mg\sin\theta - f)\hat{i} + (R - mg\cos\theta)\hat{j} \qquad ...(6.15.10)$$

where $\hat{i}$ is the unit vector down the inclined plane

$\hat{j}$ is the unit vector $\perp^r$ to the inclined plane.

Since there is no motion along the direction perpendicular to the inclined plane so

$$R = mg\cos\theta \qquad ...(6.15.11)$$

and

$$m\vec{a}_G = (mg\sin\theta - f)\hat{i} \qquad ...(6.15.12)$$

This implies equation of motion along the inclined plane is

$$ma_G = m\,g\,\sin\theta - f \qquad ...(6.15.13)$$

Using (6.15.9) in (6.15.13) we obtain

$$ma_G = m\,g\,\sin\theta - \frac{I_G\alpha}{r}$$

$$\Rightarrow \quad ma_G + \frac{I_G\alpha}{r} = mg\,\sin\theta$$

Using $\alpha = a_G/r$

$$a_G\left(m + \frac{I_G}{r^2}\right) = mg\,\sin\theta$$

giving

$$a_G = \frac{mg\,\sin\theta}{m + \frac{I_G}{r^2}} \qquad ...(6.15.14)$$

If K_A be the radius of gyration of the body about an axis passing through C.O.G. and perpendicular to the plane of motion, then $I_G = mK_A^2$. So

$$a_G = \frac{mg\,\sin\theta}{m + \frac{mK_A^2}{r^2}}$$

$$\Rightarrow \quad a_G = \frac{g\,\sin\theta}{1 + \frac{K_A^2}{r^2}} \qquad ...(6.15.15)$$

Equation (6.15.15) gives the acceleration of C.O.G. (hence of the body) down the inclined plane and it depends on the value K_A^2/r^2 of the body.

Using $I_G = mK_A^2$, $\alpha = a_G/r$ and eqn. (6.15.15) in (6.15.9) we obtain

$$f = \frac{[\text{illegible}]\; a_G}{r^2} = \frac{m\,K_A^2}{r^2}\left(\frac{g\sin\theta}{1 + \frac{K_A^2}{r^2}}\right) \qquad ...(6.15.16)$$

From eqns. (6.15.16) and (6.15.11) we obtain

$$\frac{f}{R} = \frac{m\,K_A^2}{r^2}\left(\frac{g\sin\theta}{1 + \frac{K_A^2}{r^2}}\right) \Big/ mg\cos\theta$$

$$\Rightarrow \quad \frac{f}{R} = \frac{K_A^2/r^2}{1 + \frac{K_A^2}{r^2}}\tan\theta \qquad ...(6.15.17)$$

Equation (6.15.16) and (6.15.11) show that as θ increases $|\vec{f}|$ increases and $|\vec{R}|$ decreases. Let for $\theta = \theta_{max}$, $|f|$ attains its maxmimum value. then $f = \mu R = f_{max}$ gives

$$\mu = \frac{K_A^2 / r^2}{1 + \frac{K_A^2}{r^2}} \tan\theta_{max} \qquad ...(6.15.18)$$

Now we are in a position to discuss the rolling of these cases separetely.

(a) Cylinder :

In this case the cylinder rolls on its curved surface. The axis of rotation passing through its C.O.G. is the symmarty axis. Hence in this case

$I_G = \frac{1}{2} m r^2$, $K_A^2 = \frac{r^2}{2}$

Therefore $a_G = \frac{2}{3} g \sin\theta$ (by eqn. 6.15.15). ...(6.15.19)

$$f = \frac{1}{3} m g \sin\theta \quad \text{(by eqn. 6.15.16)} \qquad ...(6.15.20)$$

$$\mu = \frac{1}{3} \tan\theta_{max}$$

$$\theta_{max} = \tan^{-1}(3\mu) \quad \text{(by eqn. 6.15.18)} \qquad ...(6.15.21)$$

(b) Sphere :

In this case the axis of rotation passing through its C.O.G. is one of its diameter. Hence in this case

$$I_G = \frac{2}{5} m r^2,\ K_A^2 = \frac{2}{5} r^2$$

Therefore

$$a_G = \frac{5}{7} g \sin\theta \qquad ...(6.15.22)$$

$$f = \frac{2}{7} mg \sin\theta \qquad ...(6.15.23)$$

$$\theta_{max} = \tan^{-1}(\frac{7}{2}\mu) \qquad ...(6.15.24)$$

(c) Circular disc

In this case the disc rolls in its rim. The axis of rotation passing through its C.O.G. is the axis perpendicular to its plane. Hence

$$I_G = \frac{1}{2} m r^2,\ K_A^2 = \frac{r^2}{2}$$

Therefore,

$$a_G = \frac{2}{3} g \sin\theta \qquad ...(6.15.25)$$

$$f = \frac{1}{3} mg \sin\theta \qquad ...(6.15.26)$$

$$\theta_{max} = \tan^{-1}(3\mu) \qquad ...(6.15.27)$$

(d) A ring

In this case the ring rolls on its rim. The axis of rotation passing through its C.O.G. is the axis perpendicular to its plane. Hence

$$I_G = m r^2,\ K_A^2 = r^2$$

Therefore

$$a_G = \frac{1}{2} g \sin\theta \qquad ...(6.15.28)$$

$$f = \frac{1}{2} mg \sin\theta \qquad ...(6.15.29)$$

$$\theta_{max} = \tan^{-1}(2\mu) \qquad ...(6.15.30)$$

Summary

1. A rigid body is one for which the distances between different particles do not change even though they move.

2. For a system of N particles having mass is $m_1, m_2 .... m_i ..... m_N$ placed at $\vec{r}_1, \vec{r}_2 .... \vec{r}_i .... \vec{r}_N$, the centre of gravity (also called centre of mass) is located at

$$\vec{R} = \frac{\sum_i m_i \vec{r}_i}{\sum_i m_i} = \frac{1}{m} = \sum_i m_i \vec{r}_i$$

3. Moment of inertia of an N particle system is

$$I = \sum_{i=1}^{N} m_i r_i^2$$

where $\bar{r}_i$ is the perperidicular distance of m_i from the axis of rotation.

4. Moment of intertia (I) depends on the mass of the system as well as the choice of axis of rotation.

5. Torgue of a force about a point is defined as :

$$\bar{\tau} = \vec{r} \times \vec{F}$$

6. For a rigid body rotating about a fixed axis.

$$\tau = I\alpha$$

7. Angular momention of an N particle system is

$$\vec{L} = \sum_i \vec{r}_i \times \vec{p}_i = \sum_i m_i \vec{r}_i \times \vec{v}_i$$

8. Torgue is related to angular momention

as $\quad \bar{\tau} = \frac{d\bar{L}}{dt}$

9. Kinetic energy of a rigid body due to rotation is

$$K = \frac{1}{2} I\omega^2$$

10. Law of conservation of angular momention states that if total external torque on a system is zero, its angular momentum remains constant.

11. A rigid body is said to be in equilibrium, if following two conditions are satisfied simultaneously.

a) The sum total of all the forces is zero.

$$\sum_i \bar{F}_i = 0$$

b) The sum total of all the torques relative to any point is zero

$$\sum_i \bar{\tau}_i = 0$$

12.

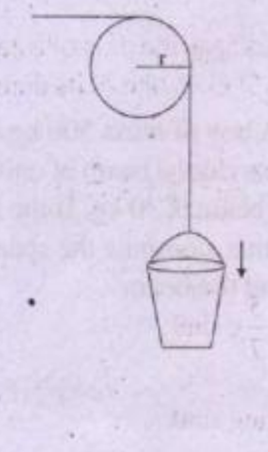
Physical quantity	Dimensional Formula	SI unit
Angular velocity (ω)	$[T^{-1}]$	rad s^{-1}
Angular momention ($\bar{L}$)	$[ML^2T^{-1}]$	JS
Torque ($\bar{\tau}$)	$[ML^2T^{-2}]$	NM
Moment of intertia (I)	$[ML^2]$	kg m^2
Angular acceleration (α)	$[T^{-2}]$	rad s^{-2}

Solved examples

Ex. 6.1 A bucket is being lowered down into a well through a rope passing over a fixed pulley of radius 10 cm. Assume that the rope does not slip on the pulley. Find the angular speed and acceleration at an instant when the bucket is going down at a speed of 30 cm/s and has an acceleration of 6 m/s^2.

Ans :

As the rope does not slip, the speed of the bucket is equal to the speed of the rim of the pulley.



$$\vartheta = r\omega$$

$$\Rightarrow \omega = \frac{\vartheta}{r} = \frac{30 cm/s}{10 cm}$$

$$\Rightarrow \omega = 3 rad/s$$

$$a_t = r\alpha$$

$$\Rightarrow \alpha = \frac{a_t}{r} = \frac{6\ m/s^2}{10\ cm} = \frac{6\ m/s^2}{0.1\ m} = 60\ rad/s^2$$

Ex. 6.2 There particles of masses 100 gm, 200 gm and 300 gm have, at a given instant positions specified by $2\hat{i}+3\hat{j}+4\hat{k}$; $\hat{i}$ - $2\hat{j}$ + $3\hat{k}$ and $3\hat{i}$ + $5\hat{j}$ - $4\hat{k}$ respectively. Find the position of C.O.M. at that instant.

Ans :

The position vector $\vec{R}$ of C.O.M. is given by

$$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3}{m_1 + m_2 + m_3}$$

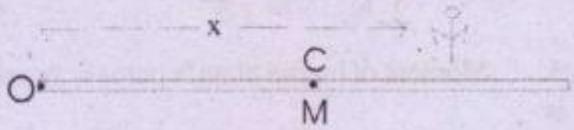
$$= \frac{100(2\hat{i}+3\hat{j}+4\hat{k}) + 200(\hat{i}-2\hat{j}+3\hat{k}) + 300(3\hat{i}+5\hat{j}-4\hat{k})}{100+200+300}$$

$$= \frac{1300\hat{i} + 1400\hat{j} - 200\hat{k}}{600}$$

$$= \frac{13\hat{i} + 14\hat{j} - 2\hat{k}}{6} cm$$

Ex. 6.3 A boy of mass 500 kg walks along a stationary horizontal beam of uniform density. The mass of beam is 30 kg. If the boy walks at a speed of 1 m/s, compute the speed of C.O.M. of the boy and the beam.

Soln.



C.O.M. of the beam alone is obtained as

$$X_c = \frac{\int_0^{\ell} \lambda x dx}{M} = \frac{\lambda \ell^2 / 2}{M} = \frac{\ell}{2}$$

If at any instant 'x' be position of the boy of mass m then position of new C.O.M. shall be

$$X = \frac{M\frac{\ell}{2} + mx}{m + M}$$

Therefore speed of C.O.M. shall be

$$\dot{X} = \frac{\frac{M}{2}\dot{\ell} + m\dot{x}}{m + M}$$

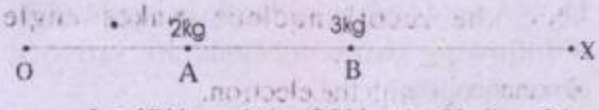
Since beam is stationary $\dot{\ell} = 0$

$$\therefore \dot{X} = \frac{m\dot{x}}{m+M} = \frac{50kg \times 1m/s}{(50+30)kg} = \frac{5}{8} m/s$$

Thus speed of C.O.M. = 0.625 m/s.

Ex. 6.4 A 2 kg body and a 3 kg body are moving along the X-axis. At a particular instant the 2 kg body is 1 m from the origin and has a velocity of 3 m/s, and the 3 kg body is 2m from the origin and has a velocity of -1m/s. find the position of the C.O.M. and also find the total momentum.

Soln.



Let 'O' be origin, OX be the X-dim. 2 kg body is at A and 3 kg body is at B. Let coordinates of A be (1,0) and that of B (2,0).

Coordinates of C.O.M.

$$X = \frac{2.1 + 3.2}{2+3}\text{m} = \frac{8}{5} = 1.6\text{m}.$$

$$Y = 0$$

Velocity of C.O.M.

$$V_{cm} = \frac{2.3 - 3.1}{5} = +\frac{3}{5}\ \frac{\text{m}}{\text{s}} = 0.6\text{m / s}$$

Total momentum = 3 kg. m/s.

Ex. 6.5 During radioactive decay of a nucleus, originally at rest, an electron of momentum 2 MeV/C, and a neutrino of momentum 1 MeV/C are emitted at right angles to each other. In what direction, does the nucleus recoil ? Calculate its momentum in MeV/C. If the mass of the residual nucleus is 3.9×10^{-22} g, what is K.E. in eV. ?

Soln.

$$\vec{P}_{Ni} = 0 = \vec{P}_N + \vec{p}_e + \vec{p}_v$$

$$\Rightarrow \quad \vec{P}_N = -(\vec{p}_e + \vec{p}_v)$$

As given $\vec{p}_e = 2\hat{i}\ \text{MeV/C}$

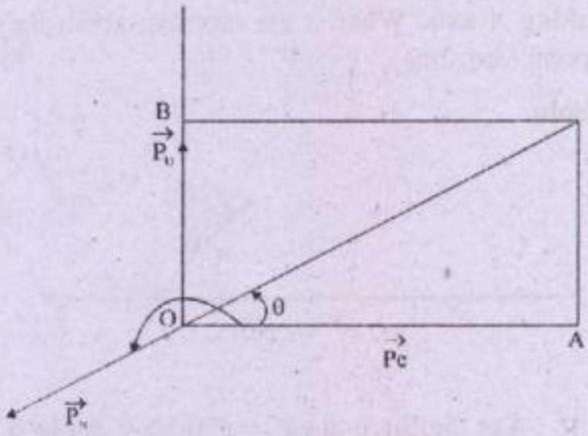
$$\vec{p}_v = \hat{j}\ \text{MeV/C}$$

$$\therefore \vec{P}_N = -(2\hat{i} + \hat{j})\text{MeV/C}$$

$$\tan(\hat{i}, \vec{p}_N) = \tan(\vec{p}_e, \vec{p}_N) = \frac{|\vec{p}_e \times \vec{p}_N|}{|\vec{p}_e \cdot \vec{p}_N|}$$

$$= \frac{|2\hat{i} \times (-2\hat{i} - \hat{j})|}{|2\hat{i} \cdot (-2\hat{i} - \hat{j})|} = \frac{2}{4} = \frac{1}{2}$$

Thus the recoil nucleus makes angle $\theta = \tan^{-1}(\frac{1}{2})$ with the electron.



$$|p_N| = \sqrt{5}\ \text{MeV / C}$$

$$\text{K.E. of residual nucleus} = \frac{p_N^2}{2m_N}$$

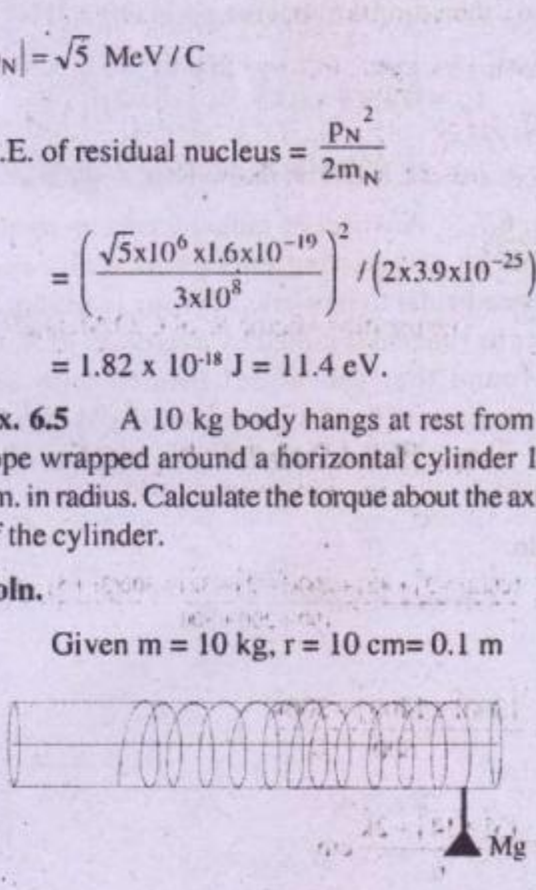
$$= \left(\frac{\sqrt{5} \times 10^6 \times 1.6 \times 10^{-19}}{3 \times 10^8}\right)^2 / \left(2 \times 3.9 \times 10^{-25}\right)$$

$$= 1.82 \times 10^{-18}\ \text{J} = 11.4\ \text{eV}.$$

Ex. 6.5 A 10 kg body hangs at rest from a rope wrapped around a horizontal cylinder 10 cm. in radius. Calculate the torque about the axis of the cylinder.

Soln.

Given m = 10 kg, r = 10 cm= 0.1 m



Torque about the axis $(\hat{i})$ is

$$\tau_a = \hat{i}.(\vec{r} \times \vec{f}) = \hat{i}.(r\hat{j} \times mg(-\hat{k}))$$

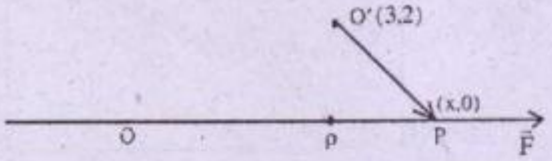
$$= -mgr = -10\ \text{kg} \times 9.8\ \text{m/s}^2 \times 0.1\ \text{m}$$

$$\tau_a = 9.8.\ \text{N.m}$$

(The -ve sign shall give the sense of rotation ie clockwise rotation)

Ex. 6.6 A force of magnitude 50 N acts along X-axis. What is the moment about the point (3m, 2m)

Soln.



Let the force $\vec{F} = F\hat{i} = (50\text{ N})\ \hat{i}$ act at p (x,o). then torque about the point O' (3,2) is

$$\vec{\tau}_{O'} = \overrightarrow{O'P} \times \vec{F} = [(x-3)\hat{i} + (0-2)\hat{j}] \times F\hat{i}$$

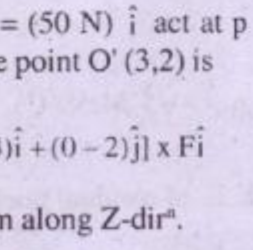
$$= 2F\,\hat{K} = 100\text{ N.m along Z-dir}^n.$$

Ex. 6.7 A wheel of radius 5 cm can rotate about an axis passing through its centre and perpendicular to its plane. A string is wrapped over its rim and is pulled by a force of 30 N. It is found that the wheel rotates with an acceleration of 3 rad/s^2. Calculate the M.O.I. of the wheel. Hence calculate the mass of the wheel.

Soln.

Given r = 5 cm

F = 3 N



$$\tau_a = I_a\alpha$$

$$\tau_a = \hat{a}.(\vec{r} \times \vec{F})$$

$$= \hat{j}.(-\hat{i}r \times F\hat{k})$$

$$\tau_a = rF = 0.05 \text{x} 3 = 0.15\text{N.m}$$

Now $I_a = \frac{\tau_a}{\alpha} = \frac{0.15\text{N.m}}{3\text{rad}/\text{s}^2} = 0.05\text{kg.m}^2$

$$I_a = \frac{1}{2}mr^2$$

$$\Rightarrow \quad m = \frac{2I_a}{r^2}$$

$$\Rightarrow \quad m = \frac{2 \times .05}{(0.05)^2} = \frac{2}{.05} = 40\text{kg}$$

Ex. 6.8 A wheel having M.O.I. 2 kg. m^2 about its axis rotates at 50 rpm about this axis. Find the torque that can stop the wheel in one minute.

Soln.

$$\text{Given } \omega_0 = 50\text{ rpm} = \frac{50 \text{x} 2\pi}{60}\text{ rad/s}$$

$$\omega = 0,\ t = 60\text{ sec.}$$

Angular accleration

$$\alpha = \frac{\omega - \omega_0}{t} = \frac{0 - \frac{5}{3}\pi\text{ rad/s}}{60\text{ sec}}$$

$$\Rightarrow \alpha = -\frac{5\pi}{180} = -\frac{\pi}{36}\text{ rad/s}^2$$

$$\tau_a = I_a\alpha = 2\text{kg.m}^2 \times \frac{\pi}{36}\text{rad/s}^2 = \frac{\pi}{18}\text{N.m}$$

Ex. 6.9 A force $\vec{F} = (\hat{i} - \hat{j} + 2\hat{k})$N acts on a body at a point P (0,a,6) and produces a torque of $10\hat{i} + 6\hat{j} - 2\hat{k}$ - about the origin. Find the value of a.

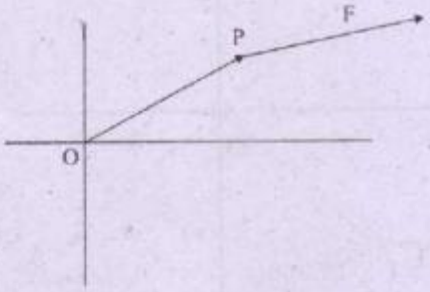
Soln.

$$\text{Given } \vec{F} = (\hat{i} - \hat{j} + 2\hat{k})\text{N}$$

$$\vec{\tau}_o = 10\hat{i} + 6\hat{j} - 2\hat{k}$$

$$\vec{OP} = \vec{r} = a\hat{j} + 6\hat{k}$$

$$\vec{\tau}_o = \vec{r} \times \vec{F}$$



$$= (a\hat{j} + 6\hat{k}) \times (\hat{i} - \hat{j} + 2\hat{k})$$

$$= -a\hat{k} + 2a\hat{i} + 6\hat{j} + 6\hat{i}$$

$$= (6 + 2a)\hat{i} + 6\hat{j} - a\hat{k}$$

$$= 10\hat{i} + 6\hat{j} - 2\hat{k}$$

We find $a = 2$.

Ex. 6.10 Three point masses A, B and C 30g, 20 g, and 10 g respectively are connected by rigid rods AB, BC, and CA of length 5 cm, 3 cm. and 4 cm. What is the M.O.I. of the system (a) about an axis through A and perpendicular to their plane and (b) about an axis coinciding with rod BC ?

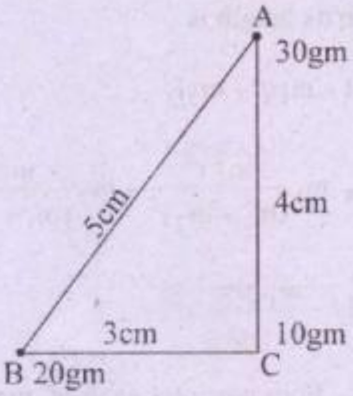
Soln.

Given AB = 5 cm

BC = 3 cm

CA = 4 cm

$\Rightarrow \angle ACB = 90^0$



(a) M.O.I. about an axis $\perp^r$ to plane ABC and passing through 'A' is

$$I = 20 \times AB^2 + 10 \times AC^2$$

$$= 20 \times 25 + 10 \times 16$$

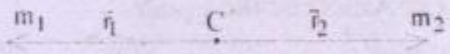
$$= 660\ g.\ cm^2.$$

(b) M.O.I. about axis BC is

$$I_{BC} = 30 \times 4^2 = 480 g.cm^2$$

Ex. 6.11 Two point masses m_1 and m_2 are joined by a weightless rod of length r. Calculate M.O.I. of the system about an axis passing through its C.O.M. and perpendicular to the rod.

Soln.



Choosing C.O.M. as origin

$$O = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 m_2}$$

$$\Rightarrow \quad m_1\vec{r}_1 = -m_2\vec{r}_2 = -m_2 r_2 \hat{i}$$

$$\Rightarrow \quad m_1 r_1 = m_2 r_2 \qquad ...(1)$$

Again $\Rightarrow r_1 + r_2 = r \qquad ...(2)$

From (1) & (2)

$$r_1 = \frac{m_2 r}{m_1 + m_2}, \quad r_2 = \frac{m_1 r}{m_1 + m_2} \qquad ...(3)$$

Now M.O.I. about an axis, passing through 'C' and $\perp^r$ to its length is

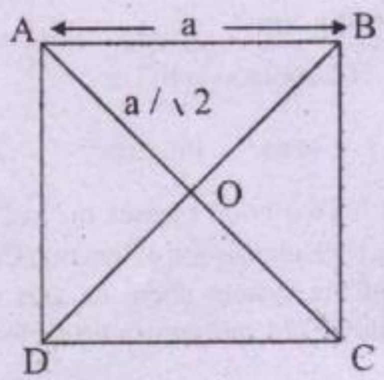
$$I = m_1 r_1^2 + m_2 r_2^2$$

$$= m_1 \frac{m_2^2 r^2}{(m_1 + m_2)^2} + m_2 \frac{m_1^2 r^2}{(m_1 + m_2)^2}$$

$$I = \frac{m_1 m_2}{m_1 + m_2} r^2 \qquad \text{(Ans)}$$

Ex. 6.12 Four particles each of mass m are kept at the four corners of a square of edge a. Find moment of inertia of the system about a line perpendicular to the plane of the square and passing through the centre of the square.

Soln.



Let ABCD be the square.

Each side = a

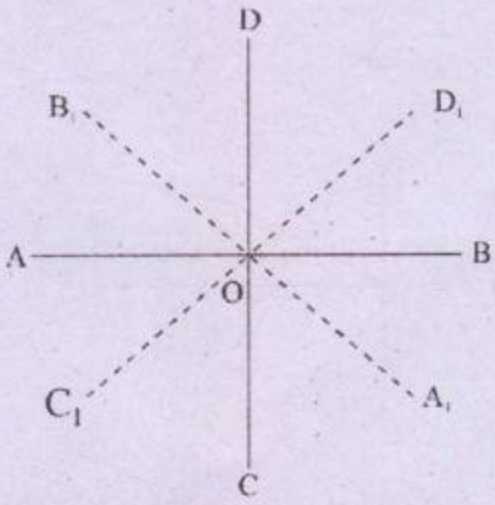
$\therefore$ OA = OB = OC = OD = $a/\sqrt{2}$

M.O.I. about an axis passing through 'O' and $\perp^r$ to its plane is

$$I_o = [m(a/\sqrt{2})2] \times 4 = 2\,ma^2$$

Ex. 6.13 Two uniform identical rods each of mass M and length ℓ are joined to form a cross as shown below. Find the M.O.I. of the cross about a bisector (shown by dotted line)

Soln.



Let AB and CD be two rods. A_1B_1 and C_1D_1 are the two bisectors mutually $\perp^r$ to each other and lying in the plane of the cross.

Now M.O.I. of each rod about an axis through 'O' and $\perp^r$ to their length is

$$I_o = \frac{1}{12} m\ell^2$$

Hence M.O.I. of two cross about an axis through 'O' and $\perp^r$ to its plane is

$$(I_o)_{net} = 2 \times \frac{1}{12} M\ell^2 = \frac{1}{6} M\ell^2$$

By perpendiculat axis theorem

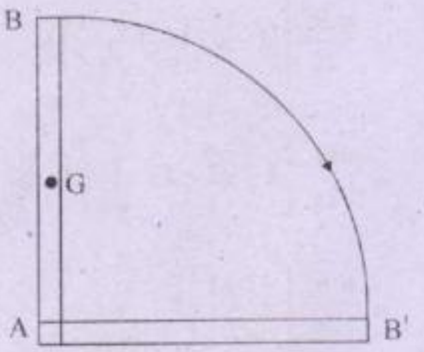
$$(I_o)_{net} = I_{A_1B_1} + I_{C_1D_1}$$

By symmetry $I_{A_1B_1} = I_{C_1D_1}$

$$\therefore (I_o)_{net} = 2I_{A_1B_1} = 2I_{C_1D_1}$$

$$\Rightarrow \; I_{A_1B_1} = I_{C_1D_1} = \frac{1}{2}(I_o)_{net} = \frac{1}{12} M\ell^2 \qquad \text{(Ans)}$$

Ex. 6.14 A thin uniform rod AB of mass m and length ℓ is hinged at one end to the level floor and stands vertically. If it is allowed to fall, with what angular velocity will it strike the floor.



Soln.

$$\text{Initial energy} = Mg.\frac{l}{2}$$

$$\text{Final energy} = \frac{1}{2}I_A\omega^2$$

$$I_A = \frac{1}{12}Ml^2 + M(l/2)^2 = \frac{1}{3}Ml^2$$

$$\therefore \frac{1}{2}.\frac{1}{3}Ml^2.\omega^2 = Mg\,\frac{l}{2}$$

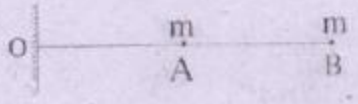
$$\Rightarrow \omega = \sqrt{\frac{3g}{l}} \qquad \text{(Ans)}$$

Ex. 6.15 Two particles of mass m each are attached to a light rod of length l; one at its mid point and the other at one end. The rod is fixed at the other end, and is made to rotate in a plane with angular speed w. Calculate the angular momentum of the particle at the end w.r.to the particle at the mid point and also w.r.to the fixed end.

Soln.

M.O.I. of mass 'm' at A w.r.to fixed end 'O' is

$$I_{AO} = m\left(\frac{l}{2}\right)^2 = \frac{1}{4}ml^2$$



M.O.I of mass 'm' at B w.r.to fixed end 'O' is

$$I_{BO} = ml^2$$

Angular momentum of mass at A w.r.to fixed end 'O' is

$$L_{AO} = I_{AO}\omega = \frac{1}{4}ml^2\omega \qquad ...(1)$$

Angular momentum of mass at B w.r.to fixed end 'O' is

$$L_{BO} = I_{BO}\omega = ml^2\omega \qquad ...(2)$$

Now angular momentum of mass at fixed end B w.r.to the mid point is

$$L_{BA} = I_{BA}\omega = m\left(\frac{l}{2}\right)^2.\omega$$

$$L_{BA} = \frac{1}{4}ml^2.\omega$$

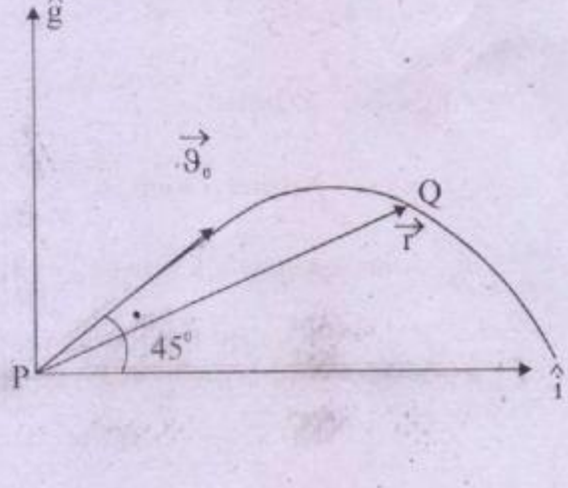
It is easy to note that we can also find

$$L_{BA} = m(\vartheta_B - \vartheta_A).\frac{l}{2} = m[(l\omega - \frac{l}{2}\omega)]\frac{l}{2}$$

$$= \frac{1}{4}ml^2\omega$$

Ex. 6.16 A particle is projected at time t= 0 from a point P with speed ϑ_o, at angle 45^0 to the horizontal. Find the magnitude and the direction of angular momentum of the particle about the point P at time t= ϑ_o/g.

Soln.



We have

$$\vartheta_x = \vartheta_{ox} = \vartheta_o \cos 45^0$$

$$\Rightarrow \quad \vartheta_x = \vartheta_{ox} = \frac{\vartheta_o}{\sqrt{2}} \qquad ...(1)$$

$$\vartheta_y = \vartheta_{oy} - gt = \vartheta_o \sin 45^0 - gt$$

$$\Rightarrow \quad \vartheta_y = \frac{\vartheta_o}{\sqrt{2}} - gt \qquad ...(2)$$

$$x = \vartheta_{ox}.t = \frac{\vartheta_o}{\sqrt{2}}.t$$

$$y = \vartheta_{oy}.t - \frac{1}{2}gt^2 = \frac{\vartheta_o}{\sqrt{2}}.t - \frac{1}{2}gt^2$$

At time 't' = $\frac{\vartheta_o}{g}$, the particle is at Q, with

$$\vartheta_x = \frac{\vartheta_o}{\sqrt{2}},$$

$$\vartheta_y = \frac{\vartheta_o}{\sqrt{2}} - g\frac{\vartheta_o}{g} = \vartheta_o\left(\frac{1}{\sqrt{2}} - 1\right)$$

$$x = \frac{\vartheta_o}{\sqrt{2}}.t = \frac{\vartheta_o^2}{g\sqrt{2}}$$

$$y = \frac{\vartheta_o}{\sqrt{2}}.\frac{\vartheta_o}{g} - \frac{1}{2}g.\frac{\vartheta_o^{\ 2}}{g^2}$$

$$y = \frac{\vartheta_o^2}{g}\left(\frac{1}{\sqrt{2}} - \frac{1}{2}\right) = \frac{\vartheta_o^2}{2g}\left(\sqrt{2} - 1\right)$$

Now $\vec{L}_p = \vec{r} x \vec{p} = \overrightarrow{PQ} x (m\vec{\vartheta})$

$$= (x\hat{i} + y\hat{j}) x (m\vartheta_x \hat{i} + m\vartheta_y \hat{j})$$

$$= mx\vartheta_y \hat{k} - my\vartheta_x \hat{k} = m(x\vartheta_y - y\vartheta_x)\hat{k}$$

$$\vec{L}_p = m\left(\frac{\vartheta_o^2}{g\sqrt{2}}.\frac{\vartheta_o(1-\sqrt{2})}{\sqrt{2}} - \frac{\vartheta_o^2}{2g}(\sqrt{2}-1).\frac{\vartheta_o}{\sqrt{2}}\right).\hat{k}$$

$$= \frac{m\vartheta_o^3}{2g}\left[\left(1-\sqrt{2}\right) - \frac{\sqrt{2}-1}{\sqrt{2}}\right]\hat{k}$$

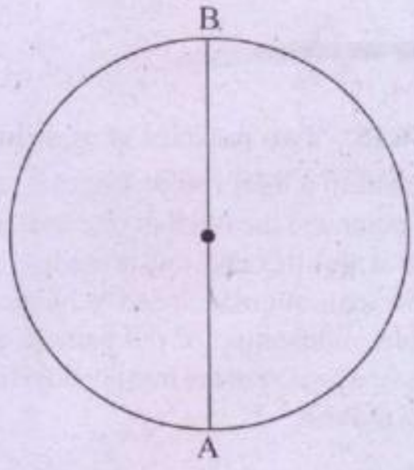
$$= \frac{m\vartheta_o^3}{2g}\left[1 - \sqrt{2} - 1 + \frac{1}{\sqrt{2}}\right]\hat{k}$$

$$= \frac{m\vartheta_o^3}{2g}\left[\frac{-2+1}{\sqrt{2}}\right]\hat{k}$$

$$\vec{L}_p = -\frac{m\vartheta_o^3}{2\sqrt{2}g}\hat{k}$$

Ex. 6.17 A uniform circular disc of mass 100g. and radius 4.0 cm is rotated about one of its diameter at an angular speed of 10 rad/s. Find the angular momentum about the axis of rotation and calculate the kinetic energy of the disc.

Soln.



M.O.I. of the disc about one of the diameter is

$$I_{AB} = \frac{1}{4}mr^2$$

$$= \frac{1}{4}x100x4^2$$

$$I_{AB} = 400g.cm^2$$

Hence

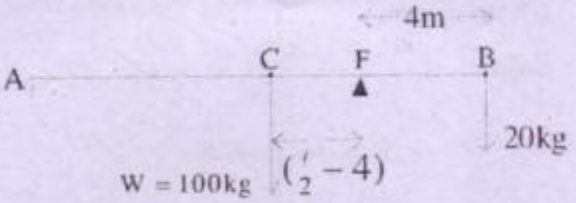
$$L_{AB} = I_{AB}\,\omega = 400 \times 10 = 4000 \text{ g.cm}^2/\text{s}$$

Kinetic Energy

$$E_k = \frac{1}{2} I_{AB}\,\omega^2 = \frac{1}{2} \times 400 \times 10^2 = 20000 \text{ erg}$$

Ex. 6.18 A rod weighs 100 kg. With a weight of 20 kg at one end it balances at a point 4 m. from the same end. Find the length of the rod.

Soln.



Let the rod be AB, 'C' its C.O.G., where its own wt 100 kg. acts. The load 20 kg acts at B. The rod balances at F. Therefore if 'ℓ' be length of the rod then for equilibrium

$$100 \times \left(\frac{\ell}{2} - 4\right) = 20 \times 4$$

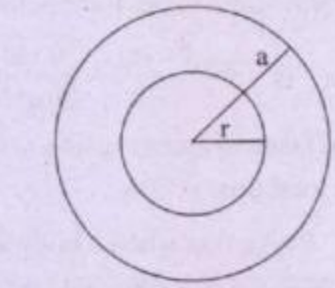
$$\Rightarrow \quad \frac{\ell}{2} - 4 = \frac{4}{5}$$

$$\Rightarrow \quad \frac{\ell}{2} = 4 + \frac{4}{5} = \frac{24}{5}$$

$$\ell = \frac{48}{5} = 9.6\text{m}$$

Ex. 6.19 The surface density (mass/area) 'σ' of a circular disc of radius 'a' depends on the distance from the centre as $\sigma(r) = A + Br$. Find its M.O.I. about the line perpendicular to the plane of the disc though its centre.

Soln.



Consider a ring of radius r and radial thickness dr.

The area of this ring = $2\pi r dr$

Mass of this ring = $2\pi\sigma(r)\, r\, dr$

$\Rightarrow dm = 2\pi(A + Br)\; r\, dr$

M.O.I of this ring about the axis = $r^2\, dm$

$= 2\pi(A + Br)\; r^3\, dr$

Hence M.O.I of the disc

$$= \int_0^a 2\pi(A + Br)\, r^3 dr$$

$$= 2\pi\left(A\frac{a^4}{4} + B\frac{a^5}{5}\right)$$

$$I = \frac{1}{2}\pi a^2\left(A + \frac{4}{5}Ba\right)a^2 \qquad \text{Ans.}$$

Ex. 6.20 A body rotating at 20 rad/s is acted upon by a constant torque providing it a deceleration of 2 rad/s². At what time will the body have kinetic energy same as the initial value, if the torque continues to act ?

Soln.

Given $\omega_0 = 20$ rad/s

$\alpha = 2$ rad/s^2

Initial K.E. = $\frac{1}{2} I\omega_0^2$.

So the body shall again acquire angular speed ω_0 after passing through zero angular speed situation. Now time taken for reaching $\omega = 0$, condition is $t = \frac{0-\omega_0}{-\alpha} = \frac{20\ rad/s}{2\ rad/s^2} = 10\ s$.

Hence time taken to again regain ω_0 is also 10s. Therefore total time is 20 s. (Ans.

Ex. 6.21 Prove that when a body starts from rest and rotates about a fixed axis with constant angular acceleration, the radial acceleration of a point in the body is directly proportional to its angular displacement.

Soln.

Given $\omega_0 = 0$

$\therefore \omega = \alpha t$

$$\theta = \frac{1}{2}\alpha t^2$$

$$\Rightarrow t = \sqrt{\frac{2\theta}{\alpha}}$$

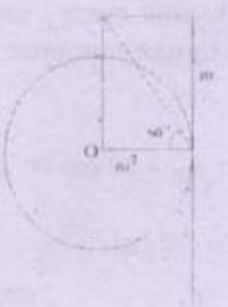
$$\therefore \omega = \alpha.\sqrt{2\theta/\alpha} = \sqrt{2\alpha\theta}$$

Now radial acceleration $a_r = r\omega^2 = 2r\alpha\theta$

$\Rightarrow a_r \propto \theta$

Ex. 6.22 In the above question through what angle has the body turned at the instant when the resultant acceleration of point makes an angle of 60^0 with the radial direction ?

Soln.



$$\frac{r\alpha}{r\omega^2} = \tan 60 = \sqrt{3}$$

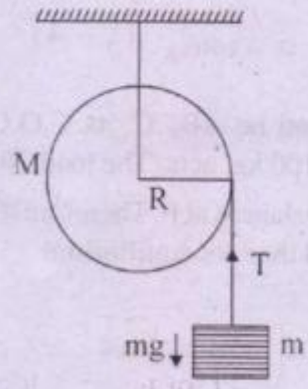
$$\alpha = \sqrt{3}\omega^2 = \sqrt{3}(2\alpha\theta)$$

$$\Rightarrow \theta = \frac{1}{2\sqrt{3}}\ rad$$

$$\theta = 0.289 rad$$ (Ans)

Ex. 6.23 A light thread with a body of mass m tied to its end is wound on a uniform solid cylinder of mass M and radius R. At a moment t = 0, the system is set in motion. Assuming the friction in the axle of the cylinder to be negligible, find the time dependence of (a) angular velocity of the cylinder (b) K.E. of the whole system.

Soln.



(a) at $t = 0$ $u = 0$

Now $mg - T = ma_t$ ...(1)

The toque that causes rotation is

$$\tau = T.R = I\alpha = \frac{1}{2}MR^2\alpha \quad ...(2)$$

$$\Rightarrow T = \frac{1}{2}MR\alpha \quad ...(3)$$

Using (3) in (1)

$$mg - \frac{1}{2}MR\alpha = ma_t = mR\alpha \quad (\because a_t = R\alpha)$$

$$\Rightarrow \; mg = \frac{1}{2}MR\alpha + mR\alpha$$

$$\Rightarrow \; 2mg = (M + 2m)R\alpha$$

$$\Rightarrow \; \alpha = \frac{2mg}{(M+2m)R} \qquad ...(4)$$

$$\omega = \alpha t = \frac{2mgt}{(M+2m)R} \qquad ...(5)$$

(b) K.E. of the whole system

$= \text{K.E. of the body} + \text{K.E. of the cylinder}$

$$= \frac{1}{2}m\vartheta^2 + \frac{1}{2}I\omega^2$$

$$= \frac{1}{2}mR^2\omega^2 + \frac{1}{2}.\frac{1}{2}MR^2.\omega^2$$

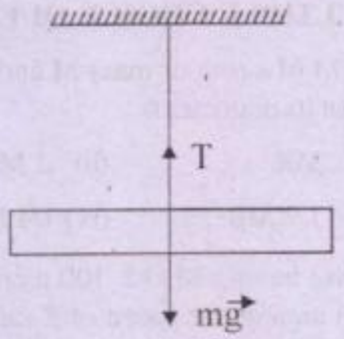
$$= \frac{1}{4}(2m + M)R^2.\omega^2$$

$$= \frac{1}{4}(2m + M)R^2.\frac{4m^2g^2t^2}{(2m+M)^2R^2}$$

$$\text{K.E.} = \frac{m^2g^2t^2}{2m+M} \qquad \text{(Ans)}$$

Ex. 6.24 M.O.I. of the reel of thread about its axis is MK^2. If the loose end of the thread is held in hand and the rest is allowed to uroll itself while falling down under the action of gravity, show that it falls down with an acceleration $gr^2/(r^2 + k^2)$.

Soln.



$$mg - T - ma_1 \qquad ...(1)$$

$$\text{Torque } \tau = T.r = I\alpha = mk^2\alpha$$

$$\Rightarrow \; T = \frac{mk^2\alpha}{r} \qquad ...(2)$$

Using (2) in (1)

$$mg - \frac{mk^2\alpha}{r} = ma_1 = mr\alpha$$

$$\Rightarrow \; g = \left(\frac{k^2}{r} + r\right)\alpha = \left(k^2 + r^2\right)\frac{\alpha}{r}$$

$$\Rightarrow \; \alpha = \frac{rg}{r^2 + k^2} = \text{angular accl}^n$$

Hence accl^n 'a_1' with which the reel falls is

$$a_1 = r\alpha = \frac{r^2g}{r^2+k^2} \qquad \text{(Ans)}$$

MODEL QUESTIONS

A. MULTIPLE CHOICE QUESTIONS

(1) M.O.I of a ring of mass M and radius R about its diameter is

(i) MR^2 (ii) $2\ MR^2$

(iii) $1/2\ MR^2$ (iv) $1/4\ MR^2$

(2) A disc having M.O.I. 100 $g.cm^2$ rotates with an angular speed of 2 rad/s. What will be its kinetic energy ?

(i) 100 erg. (ii) 200 erg.

(iii) 400 erg. (iv) 50 erg.

(3) A solid sphere, a thin ring and a cylinder of same mass and radius, held at rest are released simultaneously to roll down on an inclined plane. Which one will reach the bottom first ?

(i) sphere

(ii) ring

(iii) cylinder

(iv) All will reach simultaneously

(4) The centre of gravity and centre of mass of bodies ... coincide

(i) always

(ii) may

(iii) never

(iv) None of the above

(5) An electron of mass 9.1×10^{-31} kg moves in a circular orbit of radius 0.5×10^{-10} m at a speed of 2.2×10^6 m/s. The angular momenutm of electron is

(i) $1.0 \times 10^{-44}\ kg\ m^2\ s^{-1}$

(ii) $1.0 \times 10^{-34}\ kg\ m^2\ s^{-1}$

(iii) $2.0 \times 10^{-44}\ kg\ m^2 s^{-1}$

(iv) $2.0 \times 10^{-38}\ kg\ m^2 s^{-1}$

(6) M.O.I. is the ratio of

(i) mass and velocity

(ii) linear momentum and acceleration

(iii) linear momentum and angular velocity

(iv) angular momentum and angular velocity

(7) What will be the duration of day and night if the earth shrinks to half the present radius ?

(i) 24 hr (ii) 12 hr

(iii) 6 hr (iv) 3 hr

(8) The spokes used in a cycle wheel increase its

(i) M.O.I.

(ii) Centripetal acceleration

(iii) tangential acceleration

(iv) total acceleration

(9) A meter stick held vertically with one end on the floor is allowed to fall. The speed of the other end, when it hits the floor is nearly

(i) $3\ ms^{-1}$ (ii) $5.4\ ms^{-1}$

(iii) $7\ ms^{-1}$ (iv) $9\ ms^{-1}$

(10) A sphere of M.O.I. about its C.O.G. and mass m rolls from rest down an inclined plane without slipping. Its energy is

(i) $1/2\ I\omega^2$

(ii) $1/2\ m\vartheta^2$

(iii) $I\omega + m\vartheta$

(iv) $1/2\ I\omega^2 + 1/2\ m\vartheta^2$

(11) A particle of mass M tied to a string of length L is rotating along a circular path with constant speed ϑ. The torque on the particle is

(i) zero (ii) $M\vartheta L$

(iii) $M\vartheta / L$ (iv) $M\vartheta^2 / L$

(12) M.O.I. of a disc is minimum about an axis

(i) through the centre of the disc and parallel to the surface of the disc.

(ii) through the centre of the disc and perpendicular to the surface of the disc.

(iii) tangential and in the plane of the disc.

(iv) tangential and perpendicular to the plane of the disc.

(13) A ring of mass 'm' and radius 'r' is rolling on a smooth horizontal surface with speed ϑ. Its total kinetic energy is

(i) $1/2\ mr^2\ \omega^2$ (ii) $3/2\ m\vartheta^2$
(iii) $1/2\ m\vartheta^2$ (iv) $m\vartheta^2$

(14) A solid cylinder rolls down an inclined plane 1 in 10. Then its acceleration is nearly

(i) $0.1\ ms^{-2}$ (ii) $0.2\ ms^{-2}$
(iii) $0.5\ ms^{-2}$ (iv) $0.7\ ms^{-2}$

(15) The ratio of the magnitude of the angular speed of the hour hand of a watch to that of earth's rotation about its own axis is

(i) 1 : 1 (ii) 3 : 1
(iii) 1 : 2 (iv) 2 : 1

B. VERY SHORT ANSWER TYPE QUESTIONS :

(1) Write the dimension of angular momentum

(2) Which quantity in rotational motion plays the same role as mass in linear motion ?

(3) Write the unit of torque .

(4) State dimension of M.O.I.

(5) Write unit of M.O.I.

(6) If the radius of a circular disc is doubled how many times its M.O.I. about an axis passing through the centre and $\perp^r$ to the plane will increase ?

(7) Express angular momentum of $5gm\ cm^2/s$ in S.I. unit :

(8) If ice on polar caps of earth melts how will it affect the duration of a day ?

(9) Define C.O.M.

(10) What is the acceleration of C.O.M. of N-particle system when no external force acts ?

(11) Write expression for position vector of C.O.M. in terms of position vectors of a two body system.

(12) Define radius of gyration.

(13) Name the agent which produces rotation.

(14) Write down Newton's Law in rotation.

(15) Define angular momentum.

(16) Write the relation between torque and angular momentum.

(17) If angular momentum is conserved in a system, whose M.O.I. is decreased, will rotational K.E. be also conserved ?

(18) A body is rotating. Is it necessarily being acted on by an external torque ?

(19) A man standing on a turn-table raises his hands suddenly. What will happen ?

(20) There are two spheres of same radius and same mass. One is solid and the other is hollow. Which of them has a larger M.O.I. about a diameter.

(21) You are given two circular discs of equal masses and thickness, but made of different metals. Which one will have larger M.O.I. ?

C. SHORT ANSWER TYPE QUESTIONS :

(1) Under what conditions angular momentum is conserved ?

(2) What is the M.O.I. of a ring of mass 10 g and redius 10 cm, about an axis passing through its centre and perpendicular to its plane ?

(3) Define M.O.I. in terms of angular momentum and rotational K.E.

(4) Keeping the mass constant if the radius of earth is reduced to half of its present radius what will be the duration of the day ?

(5) A person sitting at the rim of a rotating platform starts moving towards the centre. What will happen ?

(6) If m, $\vec{\vartheta}$, $\vec{F}$ and $1/2\ m\vartheta^2$ stand for mass, linear velocity, force, and kinetic energy in case of linear motion. Write the corresponding quantities for rotational motion.

(7) Express angular momentum in terms of rectangular components.

(8) A thin circular ring of mass M and radius R is rotating about its axis ($\perp^r$ to its plane) in a horizontal plane; with angular speed ω. If two objects, each of mass 'm' are gently attached to the opposite ends of the diameter of the ring, calculate the new angular speed.

(9) Explain why spokes are used in cycle wheel ?

(10) How can you distinguish between a hard boiled egg and a raw egg by spinning them on the table top.

(11) The cap of a pen can easily be opened with two fingers than one finger, why ?

(12) Why is it difficult to revolve a stone by tieing it to a larger rope than a shorter rope ?

(13) Explain how a swimmer jumping from a height is able to increase the number of loops in the air .

(14) Can an object be in pure translation as well as pure rotation ?

(15) The torque of the weight of any body about any vertical axis is zero. Is it always correct ?

(16) The torque of a force $\vec{F}$ about a point is defined as $\vec{\tau} = \vec{r} \times \vec{F}$. Suppose $\vec{r}$, $\vec{F}$ and $\vec{\tau}$ are all non-zero. Is $\vec{r} \times \vec{\tau} \parallel \vec{F}$ always true ? Is it ever true ?

(17) If the resultant torque of all the forces acting on a body is zero about a point, is it necessary that it will be zero about any other point ?

(18) If the angular momentum of a body is found to be zero about a point, is it necessary that it will also be zero about a different point ?

(19) Can a single force applied to a body change both its translational and rotational motion ?

(20) Can you think of a body that has the same moment of inertia for all possible axes ? For all axes passing through a certain point ? What point ?

D. UNSOLVED PROBLEMS :

(1) Calculate the angular momentum of earth due to rotation about its own axis. Given mass of earth = 5.98×10^{27} g, radius of earth = 9.37×10^6 m.

(2) A horizontal disc rotating about a vertical axis, passing through its centre makes 100 revolutions per minute. A small piece of wax of mass 10g falls vertically on the disc and adheres to it at a distance of 9cm. from the axis. If the number of revolutions per minute is thereby reduced to 90, calculate the M.O.I. of the disc.

(3) A circular ring and its eight spokes are made out of the same wire. The total mass is 2 kg and the radius is 16 cm. Find the moment of inertia and radius of gyration about an axis through its centre and perpendicular to its plane.

(4) Prove that for an earth satellite, the ratio of its velocity at apogee (when farthest from earth) to its velocity at perigee (when nearest to earth) is equal to the inverse ratio of its distance from apogee and perigee.

(5) Show that the axis about which the moment of inertia of a body is the minimum must pass through C.O.G.

(6) If the angular momentum of a body increases by 20% what is the percentage change in K.E. of rotation.

(7) A thin uniform rod of mass 'm' and length 'l' pivoted freely at its base, is allowed to fall from a vertical position. Calculate its angular velocity and acceleration, when it has turned through 60^0.

(8) A bicycle wheel of radius 0.3 m has a rim of mass 1.0 kg and 50 spokes, each of mass 0.01 kg. What is its M.O.I. about its axis of rotation ?

(9) A solid cylinder of mass 15 kg, 0.3m diameter is pivoted about a horizontal axis through its centre, and a rope wrapped around the surface of the cylinder carries at its end a block of mass 8kg. (a) How far does the block descend in 5s, starting from rest ? (b) what is the tension in the rope ? (c) what is the force exerted by the bearing on the cylinder ?

(10) Find the time taken by a sphere to roll down an inclinded plane 5 m long having a slope 1 in 20.

(11) An automobile engine with 250 hp has constant angular velocity of 600 rev/min. Calculate the torque developed.

(12) A body with a M.O.I. 4 kg m^2 is rotating with an angular velocity of 5 rad/s when a torque of 10 Nm is applied for 10 s. Calculate the work done by the torque.

(13) A 20 g. coin of diameter 2cm is spinning about a vertical diameter at a fixed point on a table top at 5 rev/s. Calculate the angulare momentum of the coin about its C.O.M.

(14) Two particles of masses m_1 and m_2 are joined by a light rigid rod of length 'r'. The system rotates at an angular speed ω about an axis through the C.O.M. and perpendicular to the rod. Show that the angular momentum of the system is L = $\mu r^2 \omega$, where $\mu = \frac{m_1 m_2}{m_1 + m_2}$, is the reduced mass.

(15) A wheel of M.O.I. 0.5 kg -m^2 and radius 20.0 cm is rotating about its axis at an angular speed of 20.0 rad/s. It picks up a stationary particle of mass 200 g at its edge. Find the new angular speed of the wheel.

E. LONG ANSWER TYPE QUESTIONS :

(1) Give the concept of centre of mass. Obtain an expression for the position vector of centre of mass.

(2) Define torque. Explain how the effect of axis of rotation is relevant for describing physical rotations. Establish the relation between torque and angular acceleration.

(3) Define moment of inertia. Derive an experession for M.O.I of a circular disc about one of its diameter.

(4) Define M.O.I. Give its physical significane. Deduce expression for M.O.I. of an annular disc about an axis passing through its centre and perpendicular to its plane.

(5) Define angular momentum. Deduce an expression for angular momentum of a body rotating about a given axis.

(6) State and prove law of conservation of angular momentum. Give few live examples.

(7) (a) Obtain an expression for kinetic energy of a body executing pure rotation.

(b) Deduce expression for it in case of circular ring rotating about its diameter.

F. Fill in the Blank Type

1. A ball of mass 1 gm released down an inclined plane describes a circle of radius 10cm in the vertical plane on reaching the bottom. The minimum height of the inclined plane is..........
2. The unit of angular momentum in C.G.S. system are ....
3. The relation in rotatory motion that is analogous to $\vec{p} = m\vec{\vartheta}$ in uniform circular motion is ....
4. An electron of mass 9.1×10^{-31}kg moves in a circular orbit of radius 0.5×10^{-10}m at a speed of 2.2×10^{6} m/s. The angular momentum of electrons is ....
5. The duration of day and night if the earth shrinks to half the present radius is ....

G. True-False Type

1. Spokes are fitted in a cycle wheel to make it more strong.
2. It is much easier to revolve a stone by tieing it to a longer string than by tieing it to a shorter string.
3. A ladder is more apt to slip when you are high up on it than when you just begin to climb.
4. A fly wheel is used in railway engine to assure the running of engine smoother and steadier.
5. There may be no mass at the centre of mass of a system.

ANSWERS

A. Multiple Choice :

(1) iii, (2) ii, (3) i, (4) ii, (5) ii, (6) iv, (7) iii, (8) i, (9) ii, (10) iv, (11) i, (12) i, (13) iv, (14) iv, (15) iv.

B. Very Short Type :

1. $M L^2 T^{-1}$
2. M.O.I.
3. S.I. - N-m, L.G.S. - dyne-cm
4. $M L^2$
5. Kg. m^2; gm. cm^2
6. 4 times
7. 5×10^{-7} kg. m^2/s
8. duration increases
9. See text
10. zero
11. $(m_1 m_2)\vec{R} = m_1\vec{r}_1 + m_2\vec{r}_2$

    In non-rel. limit

    $$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}$$
12. See text
13. Torque
14. See text
15. See text
16. See text
17. No
18. No
19. 'ω' decreases
20. Hollow
21. disc of less density

C. Short Answer Type :

1. When no net external torque acts on it.
2. $I = Mr^2 = 1000$ gm. cm^2
3. $I = \frac{L^2}{2E_k}$
4. $I_1\,\omega_1 = I_2\,\omega_2$

$$\Rightarrow \frac{2}{5}MR_1^2\omega_1 = \frac{2}{5}MR_2^2\omega_2$$

$$\Rightarrow \omega_2 = \frac{R_1^2}{R_2^2}\omega_1 = 4\omega_1$$

$$\Rightarrow T_2 = 2\pi/\omega_2 = \frac{2\pi}{4\omega_1} = \frac{1}{4}T_1 = 6\,\text{hr}$$

5. Angular velocity increases. As the man moves towards the centre, M.O.I. decreases ($I_2 < I_1$). Since $I_2\omega_2 = I_1\omega_1 \Rightarrow \omega_2 > \omega_1$.

6. I (M.O.I.) ; 8 (angular velocity); $\vec{\tau}$ (torque), and $1/2\ I w^2$ (rotational K.E.)

7. $\vec{L} = \hat{i}L_x + \hat{j}L_y + \hat{k}L_z$

$$L_x = \hat{i}.\vec{L} = \hat{i}.(\vec{r}\times\vec{p}) = (\vec{r}\times\vec{p})_x = yp_z - Zp_y$$

$$L_y = \hat{j}.\vec{L} = (\vec{r}\times\vec{p})_y = zp_x - xp_z$$

$$L_z = \hat{k}.\vec{L} = (\vec{r}\times\vec{p})_z = xp_y - yp_x$$

8. $L = I_1\omega_1 = I_2\omega_2$

$$\Rightarrow \omega_2 = \frac{I_1}{I_2}\omega_1 = \frac{I_1}{I_2}\omega$$

$I_1 = MR^2,\ I_2 = MR^2 + 2mR^2 = (M+2m)R^2$

$$\therefore \omega_2 = \frac{MR^2}{(M+2m)R^2}\omega = \frac{M\omega}{M+2m}$$

D. Unsolved Problems :

1. 1.53×10^{34} kg. m^2 s^{-1}

2. 7290 gm. cm^2

3. 0.32 kg. m^2 ; 12.7 cm

[Hints :

Mass of the system = $(8r + 2\pi r)\lambda = 2$kg

$\Rightarrow \lambda = 0.875$ kg m-1, mass of ring = $2\pi r\lambda = 0.88$ kg

Mass of each spoke = $r\lambda = 0.14$ kg

$I_{ring} = Mr^2 = 0.0225$ kg. m^2

$I_{spoke} = (1/3\ m'r^2) \times 8 = 0.0099$ kg. m^2

$I_{total} = 0.032$ kg. m^2]

4. [Hints : Let r_a and r_p be distances from apogee and perigee and ϑ_a, ϑ_p be velocities at these places respectively. Let 'm' be mass of the satellite. Then

$$L = m\vartheta_a r_a = m\vartheta_p r_p \Rightarrow \frac{\vartheta_a}{\vartheta_p} = \frac{r_p}{r_a}\ ]$$

5. [Consider an axis through any point 'o' and 'a' parallel axis through C.O.G. Let the distance between two be l. Then

$$I_o = I_G + Ml^2, \quad \frac{dI_0}{dl} = 2Ml$$

For Io to be minimum $\frac{dI_0}{dl} = 0 \Rightarrow l = 0$.

ie. The axis must pass through the C.O.G.]

6. 44 %

7. [Applying conservation of energy principle

$$M.g.\frac{1}{2} = Mg.\frac{1}{2}\sin 30 + \frac{1}{2}I\omega^2$$

$$\Rightarrow \frac{Mgl}{4} = \frac{1}{2}\left(\frac{1}{3}Ml^2\right)\omega^2$$

$$\Rightarrow \omega = \sqrt{\frac{3g}{2l}}$$

Torque $\tau = Mg\frac{1}{2}\cos 30 = I\alpha = \frac{1}{3}Ml^2\alpha$

$$\Rightarrow \alpha = \frac{3\sqrt{3}}{4l}g\ ]$$

8. 0.105 kg. m2

9. (a) 63.2 m, (b) 37.9 N, (c) 185 N
[See worked out example 6.23

10. 5.345 s

11. 2968.24 N.m

12. 1750 J

13. 157.08 gm. cm^2/s

14.

15. 19.7 rad/s

F. (1) 25 cm (2) erg. sec. (3) $\vec{L} = I\vec{\omega}$ (4) 1.0×10^{-34} kg.m^2 S^{-1} (5) 6hr

G. (1) False (2) False (3) True (4) True (5) True