7

Gravitation

PLANETARY MOTION :

Ancient astronomers of China, India, Greek and Babylonian studied the variation in position of the planets in the sky over many years. After modifying the theories and improving the available data the principle underlying motion of heavenly bodies were formed. Universal laws of Gravitation as suggested by Sir Isas Newton was its product. Greek Philosophers Aristotle, Hipparchus suggested the first theory about the solar system called the Geocenric Theory. This theory was extended by Greek astronomer Ptolemy around 140 AD. According to this theory Moon, Sun and other stars revolve around the earth in simple circles while planets move in more complex orbits revolving in small circles called epicycles. Moon moved in an orbit nearest the earth. He explained the loop formed by a planet when it moves westward for a short time. This theory argreed with the naked eye observations of that period and was accepted till fourteenth century.

In 1543 Nichola Copernicus a Polish astronomer developed heliocentric theory according to which every planet including earth revolves round the Sun in its orbit. Also earth rotates about its axis once each day. Indian mathematician and astronomer Aryabhat of fifth century AD also said that earth rotates about its axis. TYCHO BRACHE of 16th century a Danish astronomer made naked eye observation of planets and German scientist JOHANNES KEPLER modified copernicus theory incorporating BRACHE's observations. Kepler modified motion of the planets into three statements which enabled observers to make accurate mathematical prediction of planetary position. Keplers statement took the form of three empirical laws of motion. Kepler announced his first two laws in 1609 and third in 1619.

7.1. Kepler's Laws of Planetary Motion :

The three laws are :

(1) The Law of Orbits :

Every planet moves in an elliptical orbit around the sun with the sun at one focus.

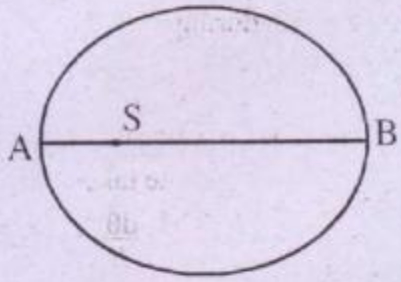


Fig. 7.1

The point A in the planets orbit which is closest to Sun S is called Perihelion and the point B along the major axis in the orbit which is farthest from Sun is called aphelion. The angular momentum of planet about an axis through the sun is constant in time. It means that the orbit of planet is in a fixed plane.

(2) Law of Area :

The radius vector from sun to planet (i.e. the imaginary line that joins a planet to sun) sweeps out equal area in equal interval of time.

S be the position of Sun. $\vec{r}$ is the position vector of planet relative to sun at time t while at A. $\vec{r} + d\vec{r}$ is the position vector of planet while at B after time t + dt relative to Sun.

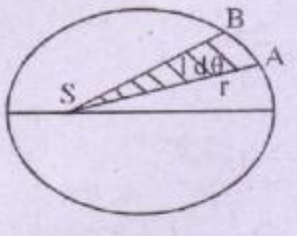


Fig. 7.2

Area swept in time dt by the

Planet at Sun = Area of shaded portion ASB

$$= \frac{1}{2} SA.AB$$

$$= \frac{1}{2} r.\, r\, d\theta$$

$$= \frac{1}{2} r^2\, d\theta$$

where $d\theta = \angle ASB$ i.e the angle that the planet sweeps at sun during time dt.

Areal velocity of planet

$$= \frac{\text{Area swept}}{\text{time taken}}$$

$$= \frac{1}{2} r^2 \frac{d\theta}{dt} = \frac{1}{2} r^2 \omega \qquad ...(7.1.1)$$

where ω = angular speed of planet about Sun.

If T is Time period of revolution of planet around the Sun

$$T = \frac{\text{Area swept}}{\text{Areal velocity of planet}}$$

$$= \frac{\pi ab}{\frac{1}{2} r^2 \omega} = \frac{2\pi ab}{r^2 \omega} \qquad ...(7.1.2)$$

where a and b are the semimajor axis and semiminor axis of the orbit of planet.

i.e. from 2nd law

$$\frac{1}{2} r^2 \omega = \text{constant} \qquad ...(7.1.3)$$

(3) The Law of Period :

The square of orbital period of any planet is directly proportional to the cube of the semimajor axis of the orbit.

If T_1 and T_2 are the orbital periods of two planets moving around Sun and a_1 and a_2 are semimajor axes of their orbits, then from 3rd law

$$\frac{T_1^2}{T_2^2} = \frac{a_1^3}{a_2^3} \qquad ...(7.1.4)$$

7.2: Newton's Laws of Gravitation :

It states that every material particle in the universe attracts every other material particle with a force which is directly proportional to the product of their masses and inversely proportional to the square of the distance between them. The force is directed along the line joining the two particles.

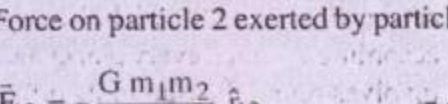


Fig. 7.3

m_1 = mass of particle 1

m_2 = mass of particle 2

$\vec{F}_{12}$ = Force on particle 2 exerted by particle 1

$$\vec{F}_{12} = -\frac{G\, m_1 m_2}{r_{12}^2}\, \hat{r}_{12} \qquad ...(7.2.1)$$

The negative sign indicates that gravitational force is attractive i.e. Force on particle 2 due to particle 1 is directed towards particle 1.

$\hat{r}_{12}$ is the unit vector in the direction of vector directed from particle 1 to the position of particle 2 ($\vec{r}_{12}$)

G is the universal Gravitational constant. It is a scalar constant. The dimension of G is $M^{-1}L^{3}T^{-2}$. The standard values of G as experimentally determined are as follows :

$G = 6.673 \times 10^{-11} Nm^2kg^{-2}$ in SI system

$= 6.673 \times 10^{-8}$ dyne cm^2g^{-2} in CGS system

The universal gravitational constant is numerically equal to the force of attraction between two unit masses at unit distance apart.

Nature of gravitational force

(i) It is a very weak force and is always attractive in nature.

(ii) It acts along the line joining the two particles. Thus it is a central force.

(iii) It is a noncontact type of force and is independent of intervening medium.

(iv) It varies inversely as square of distance between the particles. If the distance between the particles is infinity then the force between them is zero.

7.2(a) Derivation of Newton's law of gravitation from Kepler's 3rd law :

Sun is very large body compared to mass of planet and can be regarded as fixed in position. The distance between Sun and planet is very large compared to the size of Sun and planet. The orbit of a planet around Sun can be taken as circular except for the planet mercury. This asumption is true as circle is a special case of ellipse.

The planet of mass m revolves around the Sun in a circular orbit of radius r.

Let ϑ be the linear speed of planet.

Then the magnitude of centipetal acceleration acting on the planet moving around the Sun is

$$a_c = \frac{\vartheta^2}{r} \quad ...(7.2.2)$$

If T be time period of revolution of planet

Then $T = \frac{2\pi r}{\vartheta}$

$$\vartheta = \frac{2\pi r}{T} \quad ...(7.2.3)$$

Hence $a_c = \frac{4\pi^2 r^2}{T^2 . r}$

$$= \frac{4\pi^2}{r^2T^2}.r^3 = \frac{1}{r^2}\left(\frac{4\pi^2 r^3}{T^2}\right) \quad ...(7.2.4)$$

From Keplers 3rd law

$T^2 \alpha r^3$

i.e $\frac{r^3}{T^2} = K = \text{constant} \quad ...(7.2.5)$

Using (7.2.5) in (7.2.4)

$$a_c = \frac{1}{r^2}.K \quad ...(7.2.6)$$

i.e. force on the planet of mass m due to Sun

$$F_{SP} = ma_c = m.\frac{K}{r^2} \quad ...(7.2.7)$$

or $F_{SP} \, \alpha \, \frac{1}{r^2} \quad ...(7.2.8)$

Thus the attractive force between the planet and Sun is inversely proportional to the square of distance between them.

Also from equation (7.2.7)

$$F_{SP} \, \alpha \, m \quad ...(7.2.9)$$

i.e. the gravitational force between Sun and planet is proportional to the mass of planet.

According to Newton the planet must exert a force on Sun whose magnitude is proportional to mass of Sun M.

i.e. $F_{PS} \alpha M$

From Newton's 3rd law of motion

Magnitude of F_{SP} and F_{PS} are same.

i.e. $F_{SP} \alpha M$ ...7.2.9(a)

Combining (7.2.8), (7.2.9) and 7.2.9(a)

$$F_{SP} \alpha \frac{mM}{r^2}$$

$$F_{SP} = \frac{G\,m\,M}{r^2} \text{ where G is a constant.}$$

$$= F_{PS}$$

This is the mathematical form of Newtons laws of gravitation.

7.2(b) : Derivation of Keplers 3rd law from Newtons laws of Gravitation assuming planets orbits to be circular.

A planet of mass m moves around Sun of mass M in a circular orbit of radius r with linear speed ϑ (Fig. 7.4)

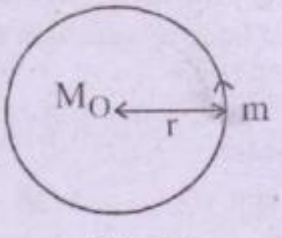


Fig. 7.4

ϑ is assumed to be constant and Sun is assumed to be stationary.

The gravitational force between the Sun and planet provides the centripetal force necessary to keep the planet moving in circular orbit.

i.e $$\frac{G\,m\,M}{r^2} = \frac{m\vartheta^2}{r}$$

or $$\frac{G\,M}{r^2} = \frac{\vartheta^2}{r} = \omega^2 r \qquad ...7.2.9(b)$$

where $\omega = \frac{\vartheta}{r}$ = angular speed of planet about an axis of rotation passing through the Sun.

Also $$\omega = \frac{2\pi}{T} \qquad ...7.2.9(c)$$

Using 7.2.9(c) in 7.2.9(b)

$$\frac{G\,M}{r^2} = \frac{4\pi^2}{T^2}.r$$

or $$T^2 = \frac{4\pi^2}{GM}.r^3$$

$$\Rightarrow T^2 = K.r^3$$

where $K = \frac{4\pi^2}{GM}$ is a constant and has same value for all planet.

Hence which is Keplers 3rd law.

7.3. Acceleration due to gravity :

If a body falls vertically towards the earth its velocity continuously increases i.e. it falls with an acceleration. Such motion is due to the attraction of earth on the body. When a body is situated near earth or on the earths surface then the force on the body due to earth is called gravity. The acceleration produced on the moving body due to gravity is called the acceleration due to gravity and is denoted by 'g'. The standard value of g is taken at sea level and at 45^0 latitude.

It is 9.806 m/s^2 or 980.6 cm/s^2

Suppose a body of mass m is placed on the surface of earth of mass M and radius R.

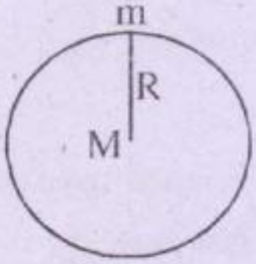


Fig. 7.5

Magnitude of Gravitational force of attraction between earth and the body or Gravity

$$F = \frac{GMm}{R^2} \qquad ...(7.3.1)$$

The weight of body placed on the surface of earth

$$F = mg \qquad ...(7.3.2)$$

Then from (7.3.1) and (7.3.2)

$$mg = \frac{GMm}{R^2}$$

or $$g = \frac{GM}{R^2} \qquad ...(7.3.3)$$

Thus acceleration due to gravity depends upon mass of earth and radius of earth.

7.4 Mass of earth :

The mass of earth can be calculated using standard values in equation (7.3.3)

$$M = \frac{gR^2}{G} \qquad ...(7.4.1)$$

$$= \frac{9.8 m/s^2 \times (6.38 \times 10^6 m)^2}{6.67 \times 10^{-11} Nm^2 kg^{-2}}$$

$$= 5.98 \times 10^{24} \text{ kg}$$

7.5 Density of Earth

Taking the earth to be homogeneous sphere.

The volume of earth = $V = \frac{4}{3}\pi R^3$ ...(7.5.1)

If ρ is the average density of earth, then

$$M = \frac{4}{3}\pi R^3 \rho \qquad ...(7.5.2)$$

From equation (7.3.3.)

$$g = \frac{GM}{R^2} = \frac{G.\frac{4}{3}\pi R^3 \rho}{R^2}$$

$$= \frac{4}{3} G \pi R \rho$$

i.e. $\rho = \frac{3g}{4G\pi R}$; using standard values

$$= \frac{3 \times 9.8}{4 \times 6.67 \times 10^{-11} \times 3.14 \times 6.38 \times 10^6}$$

$$= 5498 \text{ kg/m}^3$$

7.6: Mass of Sun :

Earth of mass 'm' revolves around Sun of mass M in a circular orbit of radius R. ϑ be the linear speed of earth in the orbit. Magnitude of Gravitational force of attraction on planet of mass m due to Sun =

$$F = \frac{GMm}{R^2} \qquad ...(7.6.1)$$

Centripetal force acting on the planet =

$$F_c = \frac{m\vartheta^2}{R} \qquad ...(7.6.2)$$

Since gravitational force provides the necessary centripetal force to the planet in the orbit

$$\frac{GMm}{R^2} = \frac{m\vartheta^2}{R}$$

or $$\frac{1}{R} = \frac{\vartheta^2}{GM} \quad ...(7.6.3)$$

If T is the Time period of revolution of planet

$$T = \frac{2\pi R}{\vartheta}$$

or $$T^2 = \frac{4\pi^2 R^2}{\vartheta^2}$$

or $$\vartheta^2 = \frac{4\pi^2 R^2}{T^2} \quad ...(7.6.4)$$

Substituting (7.6.4) in (7.6.3)

$$\frac{1}{R} = \frac{4\pi^2 R^2}{T^2 GM}$$

or $$M = \frac{4\pi^2 R^3}{T^2 . G}$$

Time period of revolution of earth around Sun = 1 year = 3×10^7 s

Radius of earth's orbit around Sun = 1.5×10^{11}m

$$M = \frac{4 \times (3.14)^2 \times (1.5 \times 10^{11})^3}{6.67 \times 10^{-11} \times (3 \times 10^7)^2}$$

$$= 2.0 \times 10^{30} \text{ kg}$$

7.7. Variation of acceleration due to gravity:

The value of acceleration due to gravity changes with altitude (i.e. as height of body increases from surface of earth) or below the surface of earth. It also changes with latitude. Its value on the surface of earth is different because of shape of earth.

7.7.(a) Variation of acceleration due to gravity with altitude :

Consider earth to be a spherical body of radius R and mass M. A body of mass m is placed at A which is at a height h above the surface of earth Fig. 7.6 where the acceleration due to gravity is g_1.

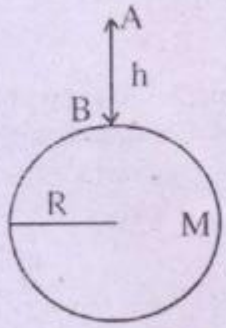


Fig.7.6

Using equation (7.3.3)

$$g_1 = \frac{GM}{(R+h)^2} \quad ...(7.7.1)$$

Let g be the acceleration due to gravity at a point B (i.e. on the surface of earth).

From equation (7.3.3.)

$$g = \frac{GM}{R^2} \quad ...(7.7.2)$$

Dividing (7.7.1) by (7.7.2)

$$\frac{g_1}{g} = \frac{R^2}{(R+h)^2} = \frac{R^2}{\left\{R(1+\frac{h}{R})\right\}^2}$$

$$= \frac{R^2}{R^2(1+\frac{h}{R})^2} = \frac{1}{(1+\frac{h}{R})^2}$$

$$= (1+\frac{h}{R})^{-2}$$

Expanding the RHS by Binomial theorem and neglecting higher powers of $\frac{h}{R}$

$$\frac{g_1}{g} = 1 - \frac{2h}{R}$$

or $$g_1 = g\left[1 - \frac{2h}{R}\right]$$

$$= g - \frac{2hg}{R} \qquad ...(7.7.3)$$

Equation (7.7.3) shows that acceleration due to gravity decreases as the height of A increases above the surface of earth.

From equation (7.7.3)

$$g - g_1 = \frac{2h}{R} g$$

or $$g - g_1 \propto h$$

i.e. the change in acceleration due to gravity at a place from that on the surface on the earth is proportional to height 'h'.

7.7.(b): Effect of depth :

Assuming that earth is a spherical surface having uniform density ρ.

The acceleration due to gravity on the surface of earth 'g' as given by equation (7.3.3)

$$g = \frac{GM}{R^2} \qquad ...(7.7.4)$$

where M is the mass of earth and R is the radius of earth

$$M = \text{volume of earth x density}$$

$$= \frac{4}{3}\pi R^3 \rho \qquad ...(7.7.5)$$

Substituting (7.7.5) in (7.7.4)

$$g = \frac{G\frac{4}{3}\pi R^3 \rho}{R^2}$$

$$= \frac{4}{3} G\pi R\rho \qquad ...(7.7.6)$$

Suppose a body is placed at a depth 'd' below the surface of earth C where the acceleration due to gravity is g'.

The magnitude of gravitational pull on the body at C is due to the pull of shaded portion of earth on the body.

If M' is the mass of the shaded portion of earth, then

$$M' = \frac{4}{3}\pi (R-d)^3 \rho$$

Using eqn (7.3.3)

$$g' = \frac{GM'}{(R-d)^2} = \frac{G\frac{4}{3}\pi(R-d)^3\rho}{(R-d)^2}$$

$$= \frac{4}{3}\pi G(R-d)\rho \qquad ...(7.7.7)$$

Dividing (7.7.7) by (7.7.6)

$$\frac{g'}{g} = \frac{\frac{4}{3}\pi G(R-d)\rho}{\frac{4}{3}\pi GR\rho} = \frac{R-d}{R}$$

$$= 1 - \frac{d}{R}$$

$$g' = g\left(1 - \frac{d}{R}\right) \qquad ...(7.7.8)$$

Here acceleration due to gravity decreases with depth also. If d = R i.e. the acceleration due to gravity at centre of earth

$$g_{centre} = g\left(1 - \frac{R}{R}\right) = 0$$

Suppose acceleration due to gravity at height A

is $$g_1 = \left(1 - \frac{2h}{R}\right) g$$

and acceleration due to gravity at depth C

$$g' = g\left(1 - \frac{d}{R}\right)$$

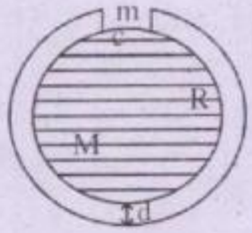


Fig. 7.7

In order to find the heigh and depth above & below earths surface where acceleration due to gravity will have the same value.

i.e $\quad g_1 = g'$

or $\quad \left(1-\frac{2h}{R}\right)=\left(1-\frac{d}{R}\right)g$

or $\quad 1-\frac{2h}{R}=1-\frac{d}{R}$

or $\quad d = 2h \qquad ...(7.7.9)$

i.e. acceleration due to gravity at a heigh 'h' is same as acceleration due to gravity at a depth d provided d = 2h if h << R.

7.7.(c) Shape of Earth :

Earth is not a perfect sphere. It bulges at the equator and is flattened at the poles. If R_e is the radius of earth at the equator and R_p is the radius of the earth at the poles.

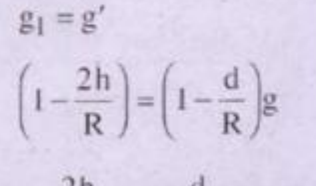


Fig. 7.8

Then from equation (7.3.3)

ie. $\quad g=\frac{GM}{R^2}$

$$g_e=\frac{GM}{R_e^{\,2}} \text{ and } g_p=\frac{GM}{R_p^{\,2}}$$

Since $R_e > R_p$ the acceleration due to gravity at the poles will be greater than at the equator. Hence the value of acceleration due to gravity increases as one moves from equator to the poles on the surface of earth.

7.7.(d) Effect of Latitude :

The angle which the line joining a point on the surface of earth makes with the equitorial plane is called the latitude at that point. It is 0^0 for a point at the equator and 90^0 at the poles.

Consider a body of mass m situated at A whose latitude is θ. Join OA where O is the centre of earth. If E is a point on the equator then $\angle EOA = \theta$.

In the absence of rotational motion of earth about its axis the weight mg of the body will be along AO.

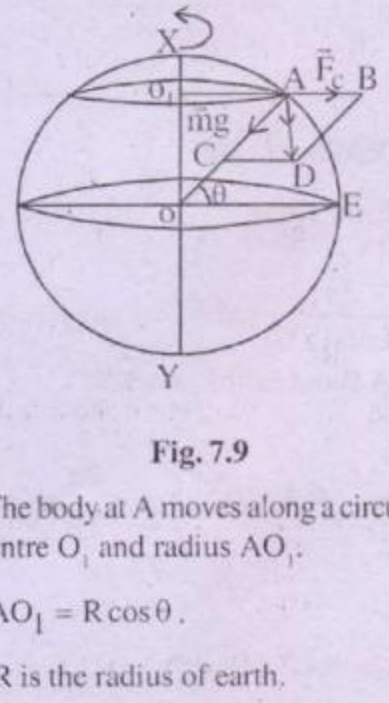


Fig. 7.9

The body at A moves along a circular path with centre O_1 and radius AO_1.

$$AO_1 = R\cos\theta.$$

where R is the radius of earth.

The magnitude of centrifugal force on the body at A

$$= m\,\omega^2(AO_1) = mR\,\omega^2\cos\theta.$$

Let us represent the centrifugal force by $\overrightarrow{AB}$ and the weight mg by $\overrightarrow{AC}$. The resultant of $\overrightarrow{AB}$ and $\overrightarrow{AC}$ will give the apparent weight mg_1 of the body which will be represented by the diagonal $\overrightarrow{AD}$.

$$AD = \left\{(AC)^2 + (AB)^2 + 2(AC).(AB).\cos(180-\theta)\right\}^{1/2}$$

$$mg_1 = \left\{(mg)^2 + (mR\omega^2\cos\theta)^2 + 2.(mg)(mR\omega^2\cos\theta.(-\cos\theta)\right\}^{1/2}$$

$$\text{or } g_1 = \left[g^2 + R^2\omega^4\cos^2\theta - 2gR\omega^2\cos^2\theta\right]^{1/2}$$

$$= g\left[1 + \frac{R^2\omega^4\cos^2\theta}{g^2} - \frac{2R\omega^2\cos^2\theta}{g}\right]^{1/2} \quad ...(7.7.10)$$

But R = Radius of earth

$= 6.4 \times 10^6$ m

$\omega = \frac{2\pi}{24\times60\times60}$ rad S^{-1} = angular speed of body at A about earths axis XY

$$\therefore \quad \frac{R\omega^2}{g} = \frac{6.4\times10^6\times\left(\frac{2\pi}{24\times60\times60}\right)^2}{9.8}$$

$$= 3.5\times10^{-3}$$

As $\frac{R\omega^2}{g}$ is very small. Higher powers of it will be still smaller and can be neglected.

$$\text{i.e.} \quad g_1 = g\left[1 - \frac{2R\omega^2\cos^2\theta}{g}\right]^{1/2}$$

$$= g\left[1 - \frac{1}{2}.\frac{2R\omega^2\cos^2\theta}{g} + ...\right]$$

Expanding by Binomial theorem and neglecting higher powers of $\frac{R\omega^2}{g}$.

$$\text{i.e} \quad g_1 = g\left[1 - \frac{R\omega^2\cos^2\theta}{g}\right] \quad ...(7.7.11)$$

$$= g - R\omega^2\cos^2\theta \quad ...(7.7.12)$$

Thus g_1 is less then 'g' which is due to rotation of earth about its axis.

At the equator $\theta = 0$

$$g_{eq} = g - R\omega^2 \quad ...(7.7.13)$$

At the poles $\theta = 90^\circ$

$$g_p = g - R\omega^2\cos^2 90$$

$$= g \quad ...(7.7.14)$$

Hence due to rotational motion of earth g is maximum at the pole and decreases as latitude of the place decreases.

7.8 Inertial mass and Gravitational mass :

According to Newtons Second Law of motion when a force F is applied on a body of mass 'm', the body moves with an acceleration 'a'.

$$|\vec{F}| = m|\vec{a}|$$

$$m = F/a \quad ...(7.8.1)$$

This mass is called the inertial mass of the body. The same force produces different acceleration in different bodies. If the force acting on a body is increased, the acceleration also increases and F/a is constant for the body. The inertial mass of a body is the ratio of force applied to the acceleration produced in it. It measures the ability of a body to resist in producing acceleration during motion when an external force acts on it. It is proportional to the quantity of matter in the body and is independent of size and shape of the body. When a body moves with speed ϑ which is comparable to speed of light C, then the inertial mass increases according to the relation

$$m = \frac{m_o}{\sqrt{1 - \frac{\vartheta^2}{C^2}}} \qquad ...(7.8.2)$$

where m_o = The inertial mass of body when at rest.

But according to Newton's laws of gravitation the force of attraction between two bodies is proportional to their masses. Hence mass can be treated as that property by virtue of which bodies exerts forces of attaction on each other.

The magnitude of gravitational pull on a body of mass 'm' placed on the surface of earth of radius R and mass M is given by

$$F = \frac{GMm}{R^2}$$

$$\text{or} \quad m = \frac{F.R^2}{GM} \qquad ...(7.8.3)$$

The mass of the body so determined gives the measure of gravitational mass of the body.

In order to know whether inertial mass and gravitational mass of a body are same or different let us consider a body of inertial mass m_i and gravitational mass m placed at a distance 'r' from the centre of earth of mass M.

The magnitude of gravitational pull on the body by the earth

$$F = \frac{GMm}{r^2} \qquad ...(7.8.4)$$

If the body is released then it moves towards the centre of earth with acceleration 'a'.

From Newtons 2nd law

$$F = m_i a \qquad ...(7.8.5)$$

From (7.8.4) and (7.8.5)

$$a = \frac{GM}{r^2}\left(\frac{m}{m_1}\right) \qquad ...(7.8.6)$$

This acceleration 'a' according to Galileo is constant. Also $\frac{GM}{r^2}$ is constant in this case.

i.e. from eqn (7.8.6)

$$a = K\left(\frac{m}{m_1}\right)$$

$$\text{where } K = \frac{GM}{r^2} = \text{constant}$$

$$\frac{m}{m_1} = \text{constant}$$

$$\text{or} \quad m \propto m_1$$

Thus inertial mass and gravitational mass of the body are proportional to each other. The units are so selected that $m = m_i$. That is inertial mass of the body is equal to gravitational mass.

7.9 Gravitational field :

Gravitational field exists between two masses. Gravitational interactions are called actions at a distance and can be explained from field concept. A material body in space sets up gravitational field. The space surrounding a material body in which its gravitational force of attraction can be measured is called gravitational field. It is a vector field.

The intensity of Gravitational field or gravitational field strength or gravitational field of the body at any point in its field is the force experienced by a unit mass (called test mass) placed at that point.

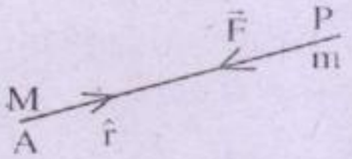


Fig. 7.10

Suppose a test mass 'm' is placed at a point P in a gravitational field. It experiences a gravitational force $\vec{F}$.

Then gravitational field strength at P is

$$\vec{E} = \frac{\vec{F}}{m}$$

i.e $$|\vec{E}| = \frac{|\vec{F}|}{m}$$

Dimension of $|\vec{E}|$ is $M^0L^1T^{-2}$. Units of E are N/kg in SI and Dyne/g in CGS. Gravitational force at P where test mass m is placed is

$$\vec{F} = -\frac{GMm}{r^2}\hat{r}$$

Hence gravitational field at the point P

$$\vec{E} = \frac{\vec{F}}{m} = -\frac{GM}{r^2}\hat{r} \qquad ...(7.9.1)$$

Negative sing indicates that the field is attractive and is directed towards M.

7.10 Gravitational Potential Energy :

Suppose a test mass m is moved between points A and B in a gravitational field. Then the difference of gravitational potential energy of test mass is given as the negative of work done by gravitational force $\vec{F}$ on m.

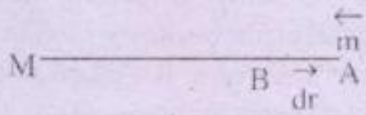


Fig. 7.11

i.e $$U_B - U_A = -W_{AB} = -\int_A^B \vec{F}.d\vec{r} \qquad ...(7.10.1)$$

If the point A is at infinity then $U_A = 0$

i.e $$U_B = -\int_\infty^B \vec{F}.d\vec{r} \qquad ...(7.10.2)$$

Thus gravitational potential energy of test mass m at any point in a gravitational field is the negative of work done on m by the gravitational field when it is moved from infinity to that point.

$$U(r) = -\int_\infty^r \vec{F}.d\vec{r} \qquad ...(7.10.3)$$

7.11 Gravitational Potential :

The gravitational potential at a point is the Gravitational potential energy per unit mass at that point in gravitational field.

$$V(r) = \frac{U(r)}{m} = -\int_\infty^r \frac{\vec{F}}{m}.d\vec{r}$$

$$= -\int_\infty^r \vec{E}.d\vec{r} \qquad ...(7.11.1)$$

In fig. 7.10 a body of mass M is placed at A. The test mass m is at p in the gravitational field of M.

Gravitational force on test mass

$$\vec{F} = -\frac{GMm}{r^2}\hat{r}$$

i.e. Gravitational potential energy of test mass at p

$$= U(r) = -\int_{\infty}^{r} \vec{F}.d\vec{r}$$

$$= -\int_{\infty}^{r} -\frac{GMm}{r^2}\hat{r}.d\vec{r}$$

$$= GMm\int_{\infty}^{r}\frac{dr}{r^2}$$

$$= GMm\left[-\frac{1}{r}\right]_{\infty}^{r}$$

$$U(r) = -\frac{GMm}{r} \qquad ...(7.11.2)$$

Hence gravitational potential at p

$$= V(r) = -\frac{U(r)}{m} = -\frac{GM}{r} \qquad ...(7.11.3)$$

Negative sign in the expression shows that the gravitational potential energy is due to the attractive force. The gravitational potential energy becomes zero when $r = \infty$. This is the maximum value of gravitational potential energy.

$$\text{Also } E = -\frac{dV}{dr} \qquad ...(7.11.4)$$

7.12 Satellites :

A natural satellite is a heavenly body revolving around a planet in a close and stable orbit. An artificial satellite is one which is put in the orbit around a planet by man. India launched some of the artifical satellites such as Aryabhatta, Bhasicara, INSAT series, GSLV.

7.12(a) Orbital speed and Time period of a Satellite :

A satellite is usually carried by a rocket and at a particular height above the earths surface it is released along a horizontal direction with high speed. The satellite then revolves around the earth in its orbit.

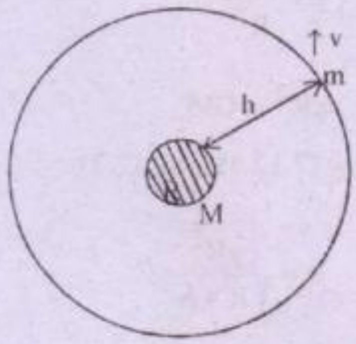


Fig. 7.12

Suppose a satellite of mass 'm' is revolving around the earth of mass M. Earth is assumed to be spherical. R be the radius of earth. It moves at height h above the surface of earth. Let ϑ be the orbital speed of the satellite.

The centripetal force acting on the satellite is provided by the gravitational force of attraction between the earth and satellite.

i.e The magnitude of gravitational force of attraction between earth and satellite =

$$F = \frac{GMm}{(R+h)^2} \qquad ...(7.12.1)$$

Magnitude of centripetal force acting on the satellite

$$= \frac{m\vartheta^2}{R+h} \qquad ...(7.12.2)$$

Then from equation (7.12.1) and (7.12.2)

$$\frac{GMm}{(R+h)^2} = \frac{m\vartheta^2}{R+h}$$

$$\text{i.e} \quad \vartheta = \sqrt{\frac{GM}{R+h}} \qquad ...(7.12.3)$$

Thus orbital speed of satellite is independent of mass of satellite but depends upon radius of orbit (R+h) and mass of earth around which it revolves.

If g is the accleration due to gravity on the surface of earth, then

$$g = \frac{GM}{R^2} \qquad ...(7.12.4)$$

$$\text{or} \quad gR^2 = GM \qquad ...(7.12.4(a))$$

Substituting (7.12.4) in (7.12.3)

$$\vartheta = \sqrt{\frac{gR^2}{R+h}}$$

$$\vartheta = R\sqrt{\frac{g}{R+h}} \qquad ...(7.12.4b)$$

If the satellite is revolving close to the surface of earth then h can be neglected compared to R.

Then equation (7.12.3) becomes

$$\vartheta = \sqrt{\frac{GM}{R}} \qquad ...(7.12.4c)$$

Substituting equation 7.12.4(a)

$$\vartheta = \sqrt{\frac{gR^2}{R}}$$

$$\vartheta = \sqrt{gR} \qquad ...7.12.4(d)$$

Substituting the standard values

$$\vartheta = \sqrt{9.8 \times 6.38 \times 10^6} = 7920 \text{ m s}^{-1}$$

This is the orbital speed of a satellite around earth where $h << R$.

Time Period of Satellite :

It is the time taken by the satellite to go once round the earth. Let it be T.

$$T = \frac{\text{circumference of orbit}}{\text{orbital speed}} = \frac{2\pi(R+h)}{\vartheta} \qquad ...(7.12.5)$$

Substituting eqn. 7.12.4(b) in (7.12.5)

$$T = \frac{2\pi(R+h)}{R\sqrt{\frac{g}{R+h}}} = \frac{2\pi}{R}\sqrt{\frac{(R+h)^3}{g}} \qquad ...7.12.5(a)$$

Substituting eqn. (7.12.3) in 7.12.5(a)

$$\vartheta = \frac{2\pi(R+h)}{\sqrt{GM}}$$

$$= 2\pi\sqrt{\frac{(R+h)^3}{GM}} \qquad ...7.12.5(b)$$

If the satellite is close to earth then $h << R$, and equation 7.12.5 (a) becomes

$$T = \frac{2\pi}{R}\sqrt{\frac{R^3}{g}} = 2\pi\sqrt{\frac{R}{g}} \qquad ...7.12.5(c)$$

and eqn. 7.12.5(b) becomes

$$T = 2\pi\sqrt{\frac{R^3}{GM}}$$

$$= 2\pi R\sqrt{\frac{R}{GM}} \qquad ...7.12.5(d)$$

7.12(b) Height of Satellite above earths surface :

From equation 7.12.5 (a)

$$T = \frac{2\pi}{R}\sqrt{\frac{(R+h)^3}{g}}$$

Squaring both sides

$$T^2 = \frac{4\pi^2}{R^2}\frac{(R+h)^3}{g}$$

or $$(R+h)^3 = \frac{T^2R^2g}{4\pi^2}$$

or $$R+h = \left(\frac{T^2R^2g}{4\pi^2}\right)^{1/3}$$

or $$h = \left(\frac{T^2R^2g}{4\pi^2}\right)^{1/3} - R \quad ...7.12.5(e)$$

Equation 7.12.5(e) gives the height of satellite above the surface of earth.

7.12(c) Geo-stationary satellite :

A satellite revolving around the earth in a circular orbit so that the orbit is coplanar with the equational plane of earth and the sense of rotation (anticlockwise) as well as the peirod of rotation is the same as that of the earth (about its axis)is called a geo-stationary satellite. Such satellites will appear to be stationary to the observer on the earth. These satellites are used for weather forcasting, TV broadcasting, etc.

The angular speed of this satellite

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{86400s} = 7.3\times10^{-5}\,\text{rad/s}$$

$T = 24$ hrs $= 86400$ s

Using eqn. (7.12.3)

$$\vartheta = \sqrt{\frac{GM}{R+h}}$$

But $\vartheta = \omega(R+h)$

or $$\omega(R+h) = \sqrt{\frac{GM}{R+h}}$$

Squarring $$\omega^2(R+h)^2 = \frac{GM}{(R+h)}$$

or $$\omega^2 = \frac{GM}{(R+h)^3}$$

or $$(R+h)^3 = \frac{GM}{\omega^2}$$

or $$R+h = \left(\frac{GM}{\omega^2}\right)^{1/3} \quad ...7.12.5(f)$$

Using standard values of G, M and ω

$$R+h = \left(\frac{6.67\times10^{-11}\times5.98\times10^{24}}{(7.3\times10^{-5})^2}\right)^{1/3}$$

$$= 42\times10^6\,\text{m}$$

i.e $$h = 42\times10^6\,\text{m} - R$$

$$= 42\times10^6\,\text{m} - 6.4\times10^6\,\text{m}$$

$$= 35.6\times10^6\,\text{m}$$

The stationary satellite will be at a height 35.6×10^6 m from the surface of earth which is about more than six times the radius of earth. Such orbits are called as Geostationary orbit or synchronous orbit or the parking orbit.

7.13 Escape Velocity :

The escape velocity from a planet (say Earth) is the minimum velocity with which an object should be projected from the planet so that it does not return back to the planet.

A body of mass m is projected up from the earth at speed ϑ_e.

The kinetic energy of the body is given by $$K = \frac{1}{2}m\vartheta_e^2$$

The potential energy of the body =

$$U = -\frac{GMm}{R}$$

where M is the mass of earth

and R is the radius of earth

Total mechanical energy of the projective on the surface of earth

$$E = K + U$$

$$= \frac{1}{2}m\vartheta_e^2 + \left(-\frac{GMm}{R}\right)$$

$$= \frac{1}{2}m\vartheta_e^2 - \frac{GMm}{R} \quad ...(7.13.1)$$

As the body moves up its velocity reduces. Hence the kinetic energy reduces. The potential energy increases by same amount as the total energy is conserved. At infinite distance from the surface of earth, the potential energy increases to its maximum value i.e. potential energy is zero. After this there can be no further change in potential energy or in kinetic energy of the body. Hence if some kinetic energy is still available the rocket will continue to move with constant speed.

i.e E=K+U should be positive to escape gravitational pull of earth.

Hence $\frac{1}{2}m\vartheta_e^2 - \frac{GMm}{R} \geq 0$ ...(7.13.2)

The equality sign holds good when the particle just escapes the gravitational influence of the earth i.e. $\vartheta_e = \vartheta_{min}$

$$\frac{1}{2}m\vartheta_{min}^2 - \frac{GMm}{R} = 0$$

or $\vartheta_{min} = \sqrt{\frac{2GM}{R}}$ ...(7.13.3)

From equation 7.12.4(a)

$$GM = gR^2$$

i.e $\vartheta_{min} = \sqrt{\frac{2gR^2}{R}}$

$$\vartheta_{min} = \sqrt{2gR} \quad ...(7.13.4)$$

$$\vartheta_{min} = \sqrt{2 \times 9.8 \times 6.4 \times 10^6} \, m/s$$

$$\vartheta_{min} = 11.2 \times 10^3 \, m\, s^{-1}$$

If a body is projected with speed $11.2 \times 10^3 m/s$ from the surface of earth then it will escape the gravitational pull of earth. It is independent of mass of escaping body.

From eqn. 7.12.4(d)

$$\vartheta_{orbital} = \sqrt{gR} \quad \text{if } h << R$$

i.e $\vartheta_{min} = \sqrt{2}.\ \vartheta_{orbital}$

or $\vartheta_{escape} = \sqrt{2}\ \vartheta_{orbital}$ ...(7.13.5)

If a body is projected with a velocity ϑ then

(i) $\vartheta = \vartheta_{orbital}$, the body will be in a circular orbit around the earth.

(ii) $\vartheta < \vartheta_{orbital}$, the body will move in elliptical path and return to earth.

(iii) $\vartheta_{escape} > \vartheta > \vartheta_{orbital}$, the body will move in elliptical path and revolve around the earth.

(iv) $\vartheta = \vartheta_{escape}$, the body moves in a parabolic path and never returns to earth.

(v) $\vartheta > \vartheta_{escape}$, the body moves in a hyperbolic path and escapes from the earth.

7.14 PARKING OBIT :

If the period of artificial satellite in the orbit is exactly equal tot eh period of earth as it turns about its axis and the satellite moves coplanar with equitorial plane of the earth as well as the direction of rotation of the satellite and earth are same in sense then the satellite will appear to be stationary to earth. The orbit of such satellite is clled as parking orbit. The height above the earths surface of the parking orbit is about 36 x 10^6 m.

7.14(a) Weightlessness in Artificial Satellite :

Consider an artificial satellite of mass m revolving round the earth of mass M with orbital speed ϑ in a circular orbit. Let R be the radius of the orbit of satellite.

The gravitational pull of earth on satellite provides the centripetal force acting on the satellite.

i.e. $$\frac{GMm}{R^2} = \frac{m\vartheta^2}{R}$$

or $$\frac{GM}{R^2} = \frac{\vartheta^2}{R} \qquad ...(7.14.1)$$

If m_1 be the mass of the person sitting in the sattelite then forces acting on it are the force of gravity acting on the person $\frac{GMm_1}{R^2}$ and the reaction N acting in the opposite direction perperndicular to the floor of satellite. Hence the net force $\frac{GMm_1}{R^2} - N$ will act towards the centre of orbit. This will be equal to the centripetal force acting on the person.

i.e $$\frac{GMm_1}{R^2} - N = \frac{m_1\vartheta^2}{R}$$

or $$\frac{GM}{R^2} - \frac{N}{m_1} = \frac{\vartheta^2}{R} \qquad ...(7.14.2)$$

Substituting (7.14.1) in (7.14.2)

$$-\frac{N}{m_1} = 0$$

or $N = 0$

Thus reaction on the person is zero. Hence he will experience weightlessness. This can also be experienced in a free falling lift.

SOLVED NUMERICALS :

Example 7.1 The planet jupiter of mass 2 x 10^{27} kg revolves round the Sun of mass 2 x 10^{30} kg in a circular orbit of radius 7.8 x 10^{11} m. Calculate

(i) the orbital velocity of Jupiter in its orbit.

(ii) Time taken to complete one revolution.

Soln.

$$F = \frac{GM_1M_2}{R^2}$$

M_1 = Mass of Sun = 2 x 10^{30} kg

M_2 = Mass of Jupiter = 2 x 10^{27} kg

R = 7.8 x 10^{11} m

i.e $$F = \frac{6.67 \times 10^{-11} \times 2 \times 10^{30} \times 2 \times 10^{27}}{(7.8 \times 10^{11})^2}$$

$$= 4.276 \times 10^{23} N$$

ϑ = orbital velocity of Jupiter

$$F = \frac{m_2\vartheta^2}{R}$$

or $$\vartheta = \sqrt{\frac{FR}{M_2}} = \sqrt{\frac{4.276 \times 10^{23} \times 7.8 \times 10^{11}}{2 \times 10^{27}}}$$

$= 1.29 \times 10^4$ m/s

T = Time taken to complete one revolution around Sun

$$T = \frac{2\pi R}{\vartheta} = \frac{2 \times 3.14 \times 7.8 \times 10^{11}}{1.29 \times 10^4}$$

$= 37.95 \times 10^7$ seconds.

Example 7.2 Two particles of equal mass move in a circle of radius r under the action of their mutual gravitational attraction. Find the speed of each particle if its mass is m.

Soln.

The particles will always remain diametrically opposite so that the force on each particle will be directed along the radius. Considering the circulation of one particle.

$$\frac{m\vartheta^2}{r} = \frac{Gmm}{(2r)^2}$$

i.e $\vartheta = \sqrt{\frac{Gm}{4r}}$

Example 7.3 In a certain region of space the gravitational field is given by $E = -(K/r)$. Assuming the reference point to be at $r = r_0$ with $V = V_0$, Find the gravitational potential.

Soln.

Gravitational Internsity or field E is negative gradient of Gravitational potential.

i.e. $E = -\frac{dV}{dr}$ Here $E = -\frac{K}{r}$

i.e. $-\frac{K}{r} = -\frac{dV}{dr}$

or $dV = K\frac{dr}{r}$

Integrating $\int_{V_0}^{V} dV = K\int_{r_0}^{r} \frac{dr}{r}$

or $V - V_0 = K \log\frac{r}{r_0}$

or $V = K \log\frac{r}{r_0} + V_0$

Example 7.4 The mass of a planet is three times that of the earth and diameter of the planet is also three times that of the earth. What is the acceleration due to gravity on the surface of the planet.

Soln.

Value of g on the earths surface $g = 9.8\ m/s^2$

$$g = \frac{GM}{R^2} \qquad g_p = \frac{GM_p}{R_p^2}$$

M = Mass of earth M_p = Mass of planet

R = Radius of earth R_p = Radius of Planet

But $\frac{M_p}{M} = 3$ and $\frac{R}{R_p} = \frac{1}{3}$

$$\frac{g_p}{g} = \frac{M_p}{M} \cdot \frac{R^2}{R_p^2}$$

$$= 3 \times \left(\frac{1}{3}\right)^2 = \frac{1}{3}$$

i.e $g_p = g/3$

$$= \frac{9.8}{3} = 3.267\ m/s^2$$

Example 7.5 Calculate the escape velocity on the surface of moon. Given the radius of the moon $= 1.7 \times 10^6$ m. Mass of moon $= 7.35 \times 10^{22}$ kg. $G = 6.67 \times 10^{-11}\ Nm^2/kg^2$.

Soln.

The escape velocity is given by $\vartheta_e = \sqrt{\frac{2GM}{R}}$

where M = Mass of the moon

R = radius of the moon

i.e. $$\vartheta_c = \sqrt{\frac{2 \times 6.67 \times 10^{-11} \times 7.35 \times 10^{22}}{1.7 \times 10^6}}$$

$$= 2.4 \times 10^3 \text{m s}^{-1}$$

Example 7.6 If the radius of the earth becomes two times the actual value then by what factor would its density be changed to keep the acceleration due to gravity g the same ?

Soln.

We know $g = \frac{GM}{R^2}$

Let ρ be the density of earth

$$M = \frac{4}{3}\pi R^3.\rho$$

i.e. $$g = \frac{GM}{R^2}$$

$$= \frac{G\frac{4}{3}\pi R^3 \rho}{R^2}$$

$$= \frac{4}{3}\pi GR\rho.$$

If R becomes 2R then ρ must become $\rho/2$ so that there will be no change in 'g' since $\frac{4}{3}, \pi, G$ are constants.

Example 7.7 A satellite is revolving in circular orbit at a height of 1000 km from the surface of earth. Calculate the orbital velocity and time of revolution. Given radius of earth = 6380 km, mass of earth = 6×10^{24} kg, G = 6.67×10^{-11} Nm^2/kg^2.

Soln.

Orbital velocity of satellite is

$$V = \sqrt{\frac{GM}{R+h}}$$ Here R = 6380 km

h = 1000 km

$$= \sqrt{\frac{6.67 \times 10^{-11} \times 6 \times 10^{24}}{7380 \times 10^3}}$$

$$= 7.35 \times 10^3 \text{ms}^{-1}$$

T = Time of revolution

$$= \frac{2\pi(R+h)}{V}$$

$$= \frac{2 \times 3.14 \times (7380) \times 10^3}{7.35 \times 10^3}$$

$$= 6305 \text{ s}$$

Example 7.8 The radius of earths orbit is 1.5 x 10^8 km and that of mass is 2.5 x 10^{11} m. Calculate time taken by the Mars to complete one revolution around Sun.

Soln.

From Kepler 3rd law

$$\frac{T_1^2}{T_2^2} = \frac{R_1^3}{R_2^3}$$

T_1 = Period of revolution of Mars

R_1 = Radius of Mars orbit

T_2 = Period of revolution of earth = 1 year

R_2 = Radius of earths orbit = 1.5 x 10^8 km = 1.5 x 10^{11}m

$$T_1 = T_2\left(\frac{R_1}{R_2}\right)^{3/2}$$

$$= 1 \times \left(\frac{2.5 \times 10^{11}}{1.5 \times 10^{11}}\right)^{3/2}$$

$$= \left(\frac{5}{3}\right)^{3/2}$$

$$\log T_1 = \frac{3}{2} \log \frac{5}{3}$$

$$T_1 = 2.152 \text{ year}$$

Example 7.9 How much below the surface of earth does acceleration due to gravity become 2% of its value at the earth's surface.

Given radius of earth = 6.4×10^6 m.

Soln.

We know

$$\frac{g'}{g} = 1 - \frac{d}{R}$$

where g' = acceleration due to gravity at a depth 'd' below the earths surface.

$$\text{Here } \frac{g'}{g} = \frac{2}{100}$$

$$\text{i.e. } \frac{2}{100} = 1 - \frac{d}{R}$$

$$\frac{d}{R} = 1 - \frac{2}{100}$$

$$= \frac{98}{100} = 0.98$$

$$d = R \times 0.98$$

$$= 6.4 \times 10^6 \times 0.98$$

$$= 6.272 \times 10^6 \text{ m}$$

Example 7.10 Find the value of g at a height of 500 km above the surface of earth. Given Radius of earth = 6.4×10^6 m. and 'g' at the surface of earth = 9.8 m s^{-2}.

Soln.

As h = 500 km is negligible as compared to the radius of earth

$$g_1 = g\left(1 - \frac{2h}{R}\right)$$

where g_1 = acceleration due to gravity at a height 500 km above earths surface

$$g_1 = 9.8\left(1 - \frac{2 \times 500}{6400}\right)$$

$$= 9.8\left(1 - \frac{1000}{6400}\right)$$

$$= 9.8 \times \frac{5400}{6400}$$

$$= 9.8 \times 0.83$$

$$= 8.134 \text{ m S}^{-2}$$

Example 7.11 If the earth is assumed to be a perfect sphere of radius 6.4×10^6 m rotating about its axis in 24 hours how much would the acceleration due to gravity differ from the poles to the equator ?

Soln.

$$g' = g\left(1 - \frac{R\omega^2}{g} \cos^2 \theta\right)$$

where g = acceleration due to gravity on earths surface in the absence of rotational motion of earth.

At the pole, $\theta = 90^\circ$

$$g_{pole} = g\left(1 - \frac{R\omega^2}{g}\cos^2 90\right)$$

$$g_{pole} = g$$

At the equator $\theta = 0$

$$g_{equator} = g\left(1 - \frac{R\omega^2}{g}\cos^2 0\right)$$

$$= g - R\omega^2$$

i.e. $$g_{equator} = g_{pole} - R\omega^2$$

or $$g_{pole} - g_{equator} = R\omega^2$$

$$= 6.4 \times 10^6 \times \left(\frac{2\pi}{24 \times 60 \times 60}\right)^2$$

$$= 3.39 \times 10^{-2} \text{ m s}^{-2}$$

Example 7.12 Calculate the height above the earths surface at which the value of acceleration due to gravity is half the value on the earths surface.

Given Radius of earth = 6400 km

Soln.

$$\frac{g_1}{g} = \left(\frac{R}{R+h}\right)^2$$

But $\frac{g_1}{g} = \frac{1}{2}$ where g_1 = acceleration due to gravity at height h above the surface of earth.

i.e $$\left(\frac{R}{R+h}\right)^2 = \frac{1}{2}$$

or $$\frac{R}{R+h} = \frac{1}{\sqrt{2}}$$

or $$R + h = \sqrt{2}R$$

$$h = \sqrt{2}\,R - R$$

$$= R(\sqrt{2} - 1)$$

$$= R \times 0.414$$

$$= 6400 \times 0.414$$

$$= 2649.6 \text{ km}$$

Example 7.13 At what angular speed the earth have to rotate for the weight of an object at the equator to be zero ? What would be the duration of one day then ?

Soln.

At the equator $\theta = 0$

$$g_1 = g\left(1 - \frac{R\omega^2}{g}\cos^2\theta\right)$$

$$= g\left(1 - \frac{R\omega^2}{g}\cos^2 0\right)$$

$$= g\left(1 - \frac{R\omega^2}{g}\right)$$

$$= g - R\omega^2$$

where R = Radius of earth

$= 6.4 \times 10^6$ m

g_1 = acceleration due to gravity on earths surface at latitude θ.

$$g_1 = g - R\omega^2 = 0$$

or $$\omega = \sqrt{g/R} = \sqrt{\frac{9.8}{6.4 \times 10^6}}$$

$$= 1.2 \times 10^{-3} \text{ rad S}^{-1}$$

The duration of one day will be equal to the time period of rotation.

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{1.2 \times 10^{-3}} \text{ seconds}$$

$$= 1.41 \text{ hours.}$$

Example 7.14 The radius of a planet is 3440 km. The acceleration due to gravity on its surface is 3.7 ms^{-2}. Find the average density of the planet.

Soln.

$$\text{Volume of the planet} = V_p = \frac{4}{3}\pi R_p{}^3$$

where $R_p = 3440 \text{ km} = 3.44 \times 10^6 \text{ m}$

$$\text{Mass of the planet} = M_p = \frac{g_p R_p{}^2}{G}$$

Hence average density of the planet

$$\rho = \frac{M_p}{V_p} = \frac{3g_p}{4\pi R_p G}$$

$$= \frac{3 \times 3.7}{4 \times 3.14 \times 3.44 \times 10^6 \times 6.67 \times 10^{-11}}$$

$$= 3834 \text{ kg m}^{-3}$$

Example 7.15 The gravitational field at a point situated at a distance of 8000 km from the centre of earth is 6 N/kg. Calculate the gravitational potential at this point.

Soln.

Gravitational field at a point = E

$$E = \frac{GM}{r^2} = 6.0 \text{ N kg}^{-1}$$

Gravitational Potential at that point = V

$$V = -\frac{GM}{r}$$

$$= -E \times r$$

$$= -6.0 \times 8000 \times 10^3$$

$$= 4.8 \text{ N M kg}^{-1}$$

Example 7.16 A rocket is fired from the earth towards the Sun. At what point on its path is the gravitation force on the rocket is zero ?

Given mass of Sun = 2×10^{30} kg

mass of earth = 6×10^{24} kg

orbital radius of earth = 1.5×10^{11} m

Neglect the effect of other planets.

Soln.

Gravitational force on the rocket will be zero at a point where the forces on the rocket due to the Sun and the earth become equal and opposite.

Suppose m = mass of rocket

M' = mass of Sun

M = mass of earth

r = Distance of earth from Sun

x = Distance of rocket from earth.

Magnitude of gravitational force on rocket due

$$\text{to Sun} = \frac{GM'm}{(r-x)^2}$$

Magnitude of gravitational force on rocket due

$$\text{to earth} = \frac{GMm}{x^2}$$

$$\text{i.e} \quad \frac{GM'm}{(r-x)^2} = \frac{GMm}{x^2}$$

$$\text{or} \quad \frac{M'}{M} = \frac{(r-x)^2}{x^2}$$

$$\text{or} \quad \sqrt{\frac{M'}{M}} = \frac{r-x}{x}$$

or $r - x = x.\sqrt{\frac{M'}{M}} = x.\sqrt{\frac{2\times10^{30}}{6\times10^{24}}}$

$= 577.35\ x$

or $r = x + 577.35x$

or $1.5 \times 10^{11} = x\ (1+577.35)$

or $x = \frac{1.5\times10^{11}}{578.35} = 2.59\times10^{8}\ m$

Example 7.17 If the radius of earth is 6000 km what will be the weight of 120 kg body if taken to a height of 2000 km above sea level ?

Soln.

$$\frac{g_1}{g} = \left(\frac{R}{R+h}\right)^2$$

where g_1 = acceleration due to gravity at a height h above earth's surface.

$$\frac{g_1}{g} = \frac{6000\times6000}{(6000+2000)^2} = \frac{9}{16}$$

$$g_1 = \frac{9}{16}.g$$

$$mg_1 = \frac{9}{16}.mg$$

$$= \frac{9}{16}\times120 \text{ kg wt}$$

$$= 67.5 \text{ kg wt.}$$

i.e. the weight of the body will be 67.5 kg wt.

Example 7.18 Compare the gravitational acceleration of the earth due to attraction of the sun with that of attraction of moon.

Given

Mass of moon $= M_m = 7.35 \times 10^{22}$ kg

Mass of Sun $= M_S = 1.99 \times 10^{30}$ kg

Distance of Sun from earth =

$r_{es} = 1.49\times10^{11}\ m$

Distance of moon from earth =

$r_{em} = 3.84\times10^{8}\ m$

Soln.

Suppose M_e = Mass of earth

g_{es} = acceleration of earth due to attraction of Sun

g_{em} = acceleration of earth due to attraction of moon

$$M_e.g_{es} = \frac{G\,M_e\,M_s}{(r_{es})^2}$$

or $g_{es} = \frac{G\,M_s}{(r_{es})^2}$ ...(i)

$$M_e g_{em} = \frac{G\,M_e M_m}{(r_{em})^2}$$

or $g_{em} = \frac{G\,M_m}{(r_{em})^2}$ ...(ii)

Dividing (i) by (ii)

$$\frac{g_{es}}{g_{em}} = \frac{M_s}{M_m}.\frac{(r_{em})^2}{(r_{es})^2}$$

$$= \frac{1.99\times10^{30}}{7.35\times10^{22}}\times\frac{(3.84\times10^{8})^2}{(1.49\times10^{11})^2}$$

$$= 1798$$

i.e. $g_{es} : g_{em} :: 1798 : 1$

Example 7.19 Two satellites A and B of same mass are revolving around the earth at attitudes R and 3R respectively, where R is the radius of earth.

Taking their orbits to be circular calculate the ratio of their kinetic and potential energies.

Soln.

Distance of satellite A from centre of earth =

$$R + R = 2R$$

Distance of satellite B from centre of earth =

$$3R + R = 4R$$

$$(\text{Potential Energy})_A = -\frac{GMm}{2R}$$

$$(\text{Potential Energy})_B = -\frac{GMm}{4R}$$

where M = Mass of Earth
m = mass of each satellite A and B.

$$\frac{(P.E)_A}{(P.E)_B} = \frac{2}{1} \quad ...(1)$$

If ϑ_A and ϑ_B are the speed of satellites A & B respectively in the orbits,

Then $$\frac{m\vartheta_A{}^2}{2R} = \frac{GMm}{(2R)^2}$$

or $$\vartheta_A^2 = \frac{GM}{2R}$$

Similarly $$\vartheta_B^2 = \frac{GM}{4R}$$

i.e. $$\frac{(K.E)_A}{(K.E)_B} = \frac{\frac{1}{2}m\frac{GM}{2R}}{\frac{1}{2}m\frac{GM}{4R}} = \frac{2}{1} \quad ...(2)$$

Equation (1) and (2) gives the required ratios.

SUMMARY

Keplers law of planetary motion :

(i) *Law of elliptical orbit :*

Every planet moves in an elliptical orbit around the Sun with the Sun at one of the foci.

(ii) *Law of area :*

The radius vector from Sun to planet sweeps out equal area in equal interval of time. $\frac{1}{2}R^2\omega$ = constant.

(iii) *The law of period :*

The square of orbital period of any planet is directly proportional to the cube of the semimajor axis of the orbit.

$$\frac{T_1^2}{T_2^2} = \frac{a_1^3}{a_2^3}$$

Newtons laws of gravitation :

It states that every particle in this universe attracts every other particle with a force that varies directly as the product of their masses and inversely as the square of distance between them.

$$\vec{F}_{12} = -G\frac{m_1m_2}{r_2}\hat{r}_{12}$$

$\hat{r}_{12}$ is the unit vector in the direction of vector directed from particle 1 to the position of particle 2. Negative sign indicates that Gravitatioanl force is an attractive force.

Dimensional formula of G = $M^{-1}L^3T^{-2}$

Standard value of G = $6.67 \times 10^{-11} Nm^2 kg^{-2}$

Acceleration due to gravity(g) :

It is the acceleration with which a body moves towards the earth due to gravitational pull of the earth. g = 9.805 m s^{-2}

Variation of g with attitude :

$$g_1 = g\left(1 - \frac{2h}{R}\right) \quad \text{if } h << R$$

$$g_1 = g\left[\frac{R^2}{(R+h)^2}\right] \quad \text{for any value of h}$$

Variation of g with depth :

$$g' = g\left(1-\frac{d}{R}\right)$$

At the centre of earth i.e. if d = R

$$g' = 0$$

Variation of g due to shape of earth :

$g_{pole} > g_{equator}$ on the surface of earth

as $R_{pole} < R_{equator}$

Inertial mass :

$$m = \frac{F}{a}$$

Gravitation mass :

$$m = \frac{FR^2}{GM}$$

Variation of g with latitude :

$$g_l = g\left[1-\frac{R\omega^2\cos^2\theta}{g}\right]$$

$g_{equator} = g - R\omega^2$ as $\theta = 0$

$g_{pole} = g$ as $\theta = 90$

i.e. $g_{pole} > g_{equator}$

Gravitational field :

The space surrounding a material body in which its gravitational force of attraction can be measured is called gravitational field. It is a vector field.

Intensity of Gravitational field :

$$\vec{E} = \frac{\vec{F}}{m}$$

It is the force experienced by a unit mass placed at that point.

$$\vec{E} = -\frac{GM}{r^2}\hat{r}$$

negative sign indicates that the field is attractive.

Gravitational potential energy :

Gravitational Potential energy of test mass m at any point in a gravitational field is the negative of work done on m by the gravitational field when it is moved from infinity to that point

$$U(r) = -\int_{\infty}^{r}\vec{F}.d\vec{r}$$

Gravitational Potential :

Gravitational potential at a point is the Gravitational potential energy per unit mass at that point in gravitational field.

$$V(r) = \frac{U(r)}{m}$$

$$= -\int_{\infty}^{r}\vec{E}.d\vec{r}$$

$$= -\frac{GM}{r}$$

$$E = -\frac{dV}{dr}$$

Gravitational potential has maximum value of zero at infinity.

Orbital speed of satellite ϑ :

$$\vartheta = \sqrt{\frac{GM}{R+h}} = R\sqrt{\frac{g}{R+h}}$$

M = mass of earth

R = Radius of earth

h = height of satellite above earths surface.

If the satellite is close to earth i.e. h << R,

then $\vartheta = \sqrt{gR}$

$= 7920 \text{ m S}^{-1}$

Time period of satellite :

$$T = \frac{2\pi}{R}\sqrt{\frac{(R+h)^3}{g}}$$

$$= 2\pi\sqrt{\frac{(R+h)^3}{GM}}$$

If $h << R$

$$T = 2\pi\sqrt{\frac{R}{g}}$$

$$= 2\pi R\sqrt{\frac{R}{GM}}$$

The height of satellite from earths surface :

$$h = \left(\frac{T^2R^2g}{4\pi^2}\right)^{1/3} - R$$

Stationary Satellite or Geostationary Satellite

It is at a height of 35.6×10^6 m from the surface of earth. The orbit of satellite around earth is coplanar with equatorial plane of earth. The sense of rotation and period of rotatioan is same as that of the earth about its axis.

The orbits of stationary satellite are called parking orbits or synchronous orbit.

Escape velocity :

It is the minimum velocity with which if an object is projected from earth then it does not return back.

$$\vartheta_{escape} = \sqrt{\frac{2GM}{R}}$$

$$= \sqrt{2gR}$$

$$= 11.2 \times 10^3 \text{ m S}^{-1}$$

Weightlessness in artificial satellite :

If the reaction on the person is zero when moving in a satellite the weightlessness can be experienced.

MODEL QUESTIONS

A. MULTIPLE CHOICE QUESTIONS

1. The minimum velocity that must be imparted to a body at the surface of earth to escape is given by

   a) $\sqrt{2R_e g}$ b) $\sqrt{\frac{GM_e}{R_e}}$

   c) $\sqrt{\frac{2gGM_e}{R_e}}$ d) $\sqrt{2GM_eR_e}$

   JEE 1996

2. If speed of rotation of earth increases, the weight of a body on earths surface becomes

   a) zero b) smaller

   c) greater d) remains unaffected

3. A satellite in vacuum

   a) is kept in the orbit by remote control

   b) is kept in the orbit by solar energy

   c) derives energy from gravitational field

   d) does not require any energy for orbiting

4. The speed of a satellite depends upon

   a) mass of satellite

   b) material of satellite

   c) height of satellite above earths surface

   d) all of the above

5. The force of attraction between two unit point masses separated by unit distance is called

   a) Gravitational filed strength

   b) Gravitational potential

   c) Acceleration due to gravity

   d) Universal gravitational constant

6. If the radius of the earth were to shrink by one percent, its mass remaining same, the value of g' on the earths surface would

   a) increase by 0.5%

   b) increase by 2%

   c) decrease by 0.5%

   d) decrease by 2%

7. The value of acceleration due to gravity on the surface of the earth is g. If the diameter of the earth becomes double of its present value and the mass remains unchanged the value of acceleration due to gravity on the surface of the earth would become

   a) g/2 b) g/4

   c) 2g d) 4g

8. Weightlessness experienced by a spaceship is due to

   a) absence of inertia

   b) absence of gravity

   c) absence of accelerating force

   d) free fall of spaceship

9. The ratio of inertial mass to gravitational mass is equal to

   a) $\frac{1}{2}$ b) 1

   c) 2 d) No fixed number

10. A missile is launched with a velocity less than the escape velocity. The sum of its kinetic and potential energies is

    a) positive b) negative

    c) zero d) may be positive or negative

11. Where is the intensity of the gravitational field of the earth maximum ?

a) centre of earth b) Equator
c) Poles d) same everywhere

12. Suppose there is a planet about the Sun at a distance 4 times the present distance of the earth. What will be the duration of the year of the planet in terms of the year on the earth?

a) 2 years b) 4 years
c) 8 years d) 16 years

B. VERY SHORT ANSWER QUESTIONS :

1. Give the dimension of universal Gravitational constant 'G'. 1986(S), 1989 (S)
2. Whether gravitational force is a central force ?
3. What is the magnitude of acceleration due to gravity at the centre of earth ? 1998
4. How does the acceleration due to gravity vary with attitude? 1996,1999
5. Which is more- 'g' at the foot or top of a mountain? 2000
6. What is the reason for weak gravity on the moon? 1995(S)
7. Show that the acceleration due to gravity is the same for all bodies on the surface of earth. 1990(S)
8. If earth stops rotating about its own axis what will happen to the weight of a body at the equator? 1989(A)
9. At what places on the surface of earth the acceleration due to gravity is greatest? 1988(S)
10. How much is the acceleration due to gravity at the centre of earth? 1987(A)
11. What will happen to the time period of simple pendulum when moved from equator to the pole of the earth?
12. The mass of moon is $\frac{1}{9}$ that of the earth. The value of gravitational force of the earth on the moon is F newton. What is the gravitational force of the moon on the earth? 1995
13. Give the value of escape velocity of a body from earth. 1994
14. What is the relation between orbital velocity and escape velocity for a given orbit? 1991(S)
15. What forces act on an artifical satellite moving in an orbit round the earth? 1990(S)
16. Mention 2 (two) applications of stationary satellites.
17. What do you mean by Parking orbits?
18. Why is there no atmosphere on the moon?
19. Why does Hydrogen escape faster from earths atmosphere than oxygen?
20. Will it be possible to put an artificial satellite into orbit in such a way that it will always remain directly over Bhubaneswar?
21. What is weightlessness? 1989

C. SHORT ANSWER QUESTIONS :

1. State Newtons laws of Gravitation. 1990, 1993(S), 1996(S)
2. State Keplers third law for planetary motion. 1991(S), 1995
3. Distinguish between 'g' and 'G' and give their units. 1992
4. Although Newtons law of gravitation is true why is it that the attraction of a body towards the earth is visible whereas that of earth towards the body is not visible? 1999
5. Show that Keplers 2nd law is a consequence of law of conservation of angular momentum.

6. How many days will there be in a year when the distance of earth from Sun reduces to $\frac{1}{4}$th of its value.

7. Obtain an expression for orbital velocity of an earth satellite moving close to earths surface. 1997

8. Find out an expression for the radius of an earth satellite. 1996(S)

9. Prove that the orbital velocity of all bodies circulating close to the surface of earth is the same. 1988(S)

10. What is escape velocity? Give the expression for it. 1985

11. Find the time period of a satellite moving in a circular orbit of radius R around the earth. 1985(S)

12. What will happen to the oscillation of a simple pendulum inside an artificial satellite? Explain your answer.

13. If a satellite moving around the earth suddenly looses height what changes will be found in its time period ?

14. How an artifical satellite is launched from the surface of earth to revolve around it in an orbit ?

15. A heavier and a lighter bodies are to be projected out of gravitational pull of the earth. Should they have same or different escape velocities ?

16. Earth is closest to the Sun on December 22 and farthest from it on July 22. When will earths speed in its orbit around Sun will be faster ?

17. What should be the kinetic energy of the body to project it from surface of earth to infinity ?

18. Explain why air friction increases the velocity of satellite ?

19. Distinguish between inertial mass and gravitational mass. 1994

20. If one digs a hole on the earths surface and goes into the hole what changes will occur in the weight of a body ? 1989

21. Derive an expression for the acceleration due to gravity on the surface of earth assuming that earth is spherical. 1988(S)

22. Can one say the situation of weightlessness as masslessness ?

23. If R is radius of earth then at what height above its surface the weight of a mass will become half of his weight on earths surface?

24. What will happen to an object dropped from earths satellite?

25. Two satellites A and B of same mass are orbiting the earth at altitudes R and 3R respectively where R is the radius of the earth. Taking their orbits to be circular. Find the ratio of their potential energies. 2000

26. A body weighs 64N on the surface of the earth. What is the gravitational force on it due to the earth at a height equal to the radius of earth? 1996

D. NUMERICAL PROBLEMS :

1. Calculate the acceleration due to gravity on the surface of moon if the radius is $\frac{1}{4}$th and mass is $\frac{1}{81}$th of the earth. (g on the earth = 9.8 mS^{-2}) 1998(S)

2. If two planets move round a star with mean radii of their orbits in the ratio 2:1 what will be the ratio of their time periods?. 1989(S)

3. The radius of earth's orbit is 1.5×10^8 km and that of venus is 1.1×10^8 km. In how many years does venus complete one revolution round the Sun? 1993(A)

4. The moon revolves round the earth 13 times per year. If the ratio of distance of earth from Sun to distance of moon from earth is 392. Find the ratio of mass of Sun to mass of earth.

Hint : $T = \frac{2\pi R}{\sqrt{GM/R}}$ or $T \alpha \frac{R^{3/2}}{M^{1/2}}$ or

$T^2 \alpha R^3/M$ JEE 1999

5. An artificial satellite is moving in a circular orbit around the earth with a speed equal to half the magnitude of escape velocity from earth. Show that the height of the satellite above earth surface is R.

JEE 1996

Hint : $\vartheta_o = \frac{1}{2}\vartheta_e$ or $\sqrt{\frac{GM}{R+h}} = \frac{1}{2}\sqrt{\frac{2GM}{R}}$

6. The earth suddenly shrinks such that its radius becomes half. Calculate by how many hours the length of the day decreases. JEE 2000

Hint : L = angular momentum =

$\frac{2}{5}MR^2 \times \frac{2\pi}{T}$ or $\frac{T'}{24} = \frac{R'^2}{R^2}$ or $\frac{T'}{24} = \frac{1}{4}$ or

$T' = 6$ hrs.

7. The mass of moon is $\frac{1}{81}$ of earths mass and its radius is 1/4 that of earth. If the escape velocity from the earths surface is 11.2 km/s what will be its value at moon?

8. At what height above the surface of the earth (radius R) will the acceleration due to gravity be reduced by 0.1% ?

Hint : $g' = g(1 - \frac{2h}{R})$ or $\frac{g - g'}{g} = 0.1\% = \frac{1}{1000}$

9. Assuming that the gravitational potential energy of a body at infinite distance away from earth is zero. What percentage of the potential energy is gained by the body in rising through a height equal to the radius of earth ?

Hint : Gravitation P.E. at surface of earth

$= -\frac{GMm}{R}$. G.P.E. at a height $R = -\frac{GMm}{2R}$

10. If the diameter of the earth becomes 2 times its present value and its mass remains unchanged then how would the weight of an object on the surface of the earth be affected ?

Hint : $W = mg = \frac{GMm}{R^2}$

11. Suppose the radius of the earth were to shrink by 1%, its mass remaing same would the acceleration due to gravity on the earths surface increase or decrease and by what percent ?

Hint : $g = \frac{GM}{R^2}$; $dg/g = -\frac{2dR}{R}$

12. What gravitational force acts between two 60 kg men 1metre apart (a) on the earth (b) on the moon ? Given $G = 6.67 \times 10^{-11}$ Nm^2/kg^2.

13. The Mount Everest is 8848m above sea level. Find the acceleration due to gravity at mount Everest given the value of g at the surface of earth is 9.8 m S^{-2} and mean radius of the earth is 6.37×10^6m. Neglect earths rotation about its axis.

14. How much energy must be given to a 100 kg rocket missile to carry it from the surface of earth into free space ? Given mass of earth = 6×10^{24} kg, $G = 6.67 \times 10^{-11}$ Nm^2 kg^{-2}. Radius of earth = 6.4×10^6 m.

15. The escape velocity of a body on the surface of earth is 11.2 km S^{-1}. A body is projected away with twice this speed. What is the speed of the body at infinity?

    Hint : $\frac{1}{2}m\vartheta^2+\left(-\frac{GMm}{R}\right)=\frac{1}{2}m\vartheta'^2+0$

    Here $\vartheta=2\vartheta_e$.

16. Three uniform spheres each having mass m and radius r, are kept in such a way that each touches the other two. Find the magnitude of the gravitational force on any sphere due to the other two.

E. LONG ANSWER QUESTIONS :

1. State Newton's laws of gravitation Deduce this law from Keplers third law.

   1987

2. State Kepler's laws. Derive Newtons laws of gravitation from them. 1994(S) 1996(S)

3. State the laws of Gravitation. Obtain an expression for the gravitational potential due to mass M at a distance R. Show that T^2/R^3 is constant for a satellite orbiting the earth where T and R have usual meaning. 1996

4. Obtain an expression for orbital velocity of a satellite. Discuss its importance in launching of satellites. 1994

5. Find an expression for the velocity of escape from the earths gravitational field. 1990,1992

6. Derive an expression for the Time period of a satellite.

7. What is geostatic satellite ? Obtain expression for its characteristic velocity and distance from earth. 1997

8. Discuss the variation of acceleration due to gravity with altitude and depth.

9. What do you mean by latitude at a place. Discuss how acceleration due to gravity varies with latitude with relevant theory.

10. Prove that the distance below the surface of earth where 'g' will have the same value is two times the distance one has to cover above the surface of earth.

F. Fill in the Blank Type

1. The acceleration due to gravity .... with increase in height and depth.

2. If earth shrinks to half of its radius, its mass remaining same, the weight of an object on earth will change...... times.

3. The escape velocity on earth is 11.2 km/sec. Its value for a planet having double the radius and 8 times the mass of earth....... m/s.

4. The velocity of satellite in the parking orbit is ....

5. The height of the geostationary orbit above the surface of earth is.......

6. If the density of planet is increased, then the acceleration due to gravity at its surface will ....

G. True - False Type

1. If earth sudenly stops rotating about its axis, then the value of 'g' will be same at all the places.

2. Although the mass of moon is 10% of the mass of earth but the gravitational pull of earth on moon is equal to that of moon on earth.

3. Earth has an atmosphere but the moon does not.

4. If an earth satellite moves to a lower orbit, there is same dissipation of energy but the satellite speed increases.

5. Moon travellers tie heavy weight at their back before landing on the moon.

6. The escape velocity from the surface of Jupiter is less than that from earth's surface.

ANSWERS

A. MULTIPLE CHOICE QUESTIONS :

1. a, 2. b, 3. d, 4. c 5. d
6. b 7. b 8. d 9. b 10. b
11. c 12. c

D. NUMERICAL PROBLEMS :

1. $1.9\ mS^{-2}$
2. $2\sqrt{2} : 1$
3. 0.628 yrs
4. 67153 : 1
6. 18 hrs
7. 2.5 km / s
8. R / 2000
9. 50%
10. $\frac{1}{4}$ of its previous value
11. decreases, 2%
12. (a) 2.4×10^{-7} N (b) 2.4×10^{-7} N
13. $9.773\ m\ S^{-2}$
14. 6.253×10^{9} J
15. 19.4 km/S
16. $\sqrt{3}Gm / 4r^2$

F. (1) decreases (2) Four (3) 22.4×10^3 (4) 3.1 km/sec (5) 36×10^3 km (6) Increase

G. (1) True (2) True (3) True (4) True (5) True (6) False.

8

Projectile Motion

PROJECTILE MOTION :

A projectile is an object which is lauched into the space without motive power of its own and moves freely under the action of gravity and air resistance. For short flights the curvature of earth can be neglected. The acceleration due to gravity is along vertically downward direction and is regarded as uniform. The air resistance can be neglected if the speed of projectile is not very large.

The analysis of projectile motion is convenient by considering the velocity of probjectile as the resultant of a horizontal and a vertical component. Neglecting the air resistance the horizontal component of the velocity remains unchanged throughout the motion. Since the gravitational force is along vertically downward direction it produces an acceleration in that direction only. The vertical component decreases continuously till it is zero while the projectile moves upward. It increases as the projectile moves in downward direction.

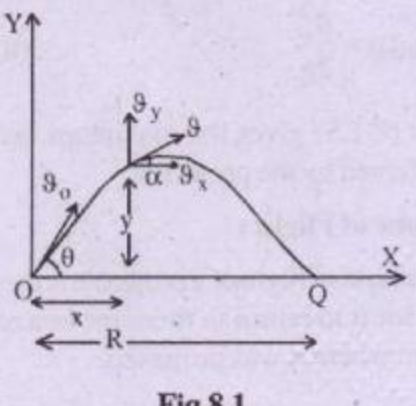


Fig.8.1

Suppose a body is projected with a velocity ϑ_o in XY plane making an angle θ with the horizontal from the point O say the origin.

As the projectile moves from the origin it covers distance along horizontal direction due to horizontal component $\vartheta_o \cos\theta$ of the velocity of projection and along vertical direction due to the vertical component $\vartheta_o \sin\theta$.

The acceleration along horizontal direction

$$a_x = 0$$

The acceleration along vertical direction

$$a_y = -g$$

8.1(a) Velocity of the projectile at time t :

Suppose at time t the projectile is at p where its velocity is $\vec{\vartheta}$. The reactangular components of velocity at p are $\vec{\vartheta}_x$ and $\vec{\vartheta}_y$.

$$\left|\vec{\vartheta}_x\right| = \vartheta_o \cos\theta$$

As the horizontal component remains unchanged when air resistance is neglected.

Initial velocity along the vertical direction (i.e. at t = 0) = $\vartheta_o \sin\theta$.

Acceleration acting on the body = $-g$

Vertical component of velocity at p = ϑ_y

Time taken to reach p = t

Using the relation V = u + at

$$\vartheta_y = \vartheta_o \sin\theta + (-g)t$$
$$= \vartheta_o \sin\theta - gt$$

i.e. the magnitude of velocity ϑ at P. is

$$\vartheta = \sqrt{\vartheta_x^2 + \vartheta_y^2}$$
$$= \sqrt{(\vartheta_o \cos\theta)^2 + (\vartheta_o \sin\theta - gt)^2} \quad ...(8.1.1)$$

Suppose the velocity ϑ at P makes an angle α with the horizontal direction.

$$\tan\alpha = \frac{\vartheta_y}{\vartheta_x}$$
$$= \frac{\vartheta_o \sin\theta - gt}{\vartheta_o \cos\theta}$$

or $$\alpha = \tan^{-1}\left[\frac{\vartheta_o \sin\theta - gt}{\vartheta_o \cos\theta}\right] \quad ...(8.1.2)$$

This gives the direction of velocity at P with the horizontal direction.

8.1.(b) Position of Projectile at P :

Suppose the projectile while at P has covered a horizontal distance x from 0 and vertical height y.

Then $$x = \vartheta_o \cos\theta . t \quad ...(8.1.3)$$

Using the equation

$$S = ut + \frac{1}{2}at^2$$
$$y = (\vartheta_o \sin\theta).t + \frac{1}{2}(-g)t^2$$
$$y = (\vartheta_o \sin\theta).t - \frac{1}{2}gt^2 \quad ...(8.1.4)$$

Equations (8.1.3) and (8.1.4) gives the position of projectile after t seconds.

8.1(c) Maximum height attained :

It is the greatest height reached by the projectile from the horizontal projection level.

In fig. 8.2 the highest point reached by the projectile during the motion is M. Let H be the vertical height of M from level OX.

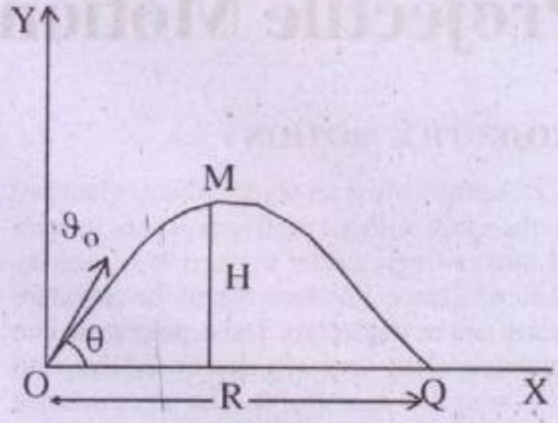


Fig. 8.2

Vertical component of velocity at M = 0

using the relation $\vartheta^2 = u^2 + 2as$

$$0 = (\vartheta_o \sin\theta)^2 + 2(-g)H$$

or $$H = \frac{\vartheta_o^2 \sin^2\theta}{2g} \quad ...(8.1.5)$$

If the projectile is fired vertically $\theta = 90^o$

$$H_{max} = \frac{\vartheta_o^2}{2g} \quad ...(8.1.6)$$

Equation (8.1.5) gives the maximum vertical height attained by the projectile.

8.1(d) Time of Flight :

The time of flight of a projectile is the time required for it to return to the same horizontal level from where it was projected.

Let T be the time of flight. The vertical height of projectile when it reaches the horizontal level from where it is projected

$$y = 0$$

Using the equation of motion

$$S = ut + \frac{1}{2}at^2$$

$$0 = (\vartheta_o \sin\theta)T + \frac{1}{2}(-g)T^2$$

or $(\vartheta_o \sin\theta).T = \frac{1}{2}gT^2$

or $T = \dfrac{2\vartheta_o \sin\theta}{g}$ ...(8.1.7)

The equation (8.1.7) gives the time of flight. Suppose T_1 is the time taken to reach the highest point M.

Vertical component of velocity at M = 0

Acceleration acting on the body = −g

Using the equation

$$\vartheta = u + at$$

$$0 = (\vartheta_o \sin\theta) + (-g)T_1$$

or $T_1 = \dfrac{\vartheta_o \sin\theta}{g}$ ...(8.1.8)

Comparing (8.1.7) and (8.1.8)

Time of ascent of projectile = $T_1 = \dfrac{T}{2}$

...(8.1.9)

8.1(e) Horizontal Range :

It is the distance a projectile travels horizontally before returning to the original elevation ie OQ = R = Horizontal range

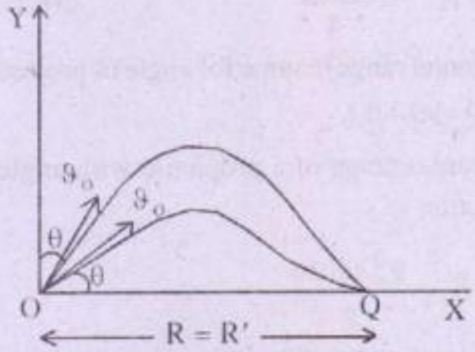


Fig. 8.3

The horizontal range R is the horizontal distance covered by the projectile with uniform velocity $\vartheta_o \cos\theta$ in a time which is equal to the time of flight.

$$R = \vartheta_o \cos\theta . T$$

$$= \vartheta_o \cos\theta . \frac{2\vartheta_o \sin\theta}{g} \text{ (using 8.1.7)}$$

$$R = \frac{\vartheta_o^2 \sin 2\theta}{g} \quad ...(8.1.10)$$

Equation (8.1.10) gives the horizontal range of projectile for a given velocity of projection, the horizontal range will be maximum when $\sin 2\theta = 1$.

i.e. $\sin 2\theta = \sin 90$

i.e. $\theta = 45^o$

Then $R_{max} = \dfrac{\vartheta_o^2}{g}$ ...(8.1.11)

If $\theta = 45^o$

Then from eqn. (8.1.5)

$$H = \frac{\vartheta_o^2 \sin^2 45}{g}$$

$$= \frac{\vartheta_o^2}{4g} = \frac{R_{max}}{4}$$

i.e $H = \frac{R_{max}}{4}$ ...(8.1.12)

Horizontal range is same for angle of projection θ and $(90-\theta)$

Horizontal range of a projectile with angle of projection θ.

$$R = \frac{\vartheta_o^2 \sin 2\theta}{g}$$

Horizontal range of a projectile with angle of projection $(90-\theta)$

$$R' = \frac{\vartheta_o^2 \sin 2(90-\theta)}{g}$$

$$R' = \frac{\vartheta_o^2 \sin(180-2\theta)}{g}$$

$$R' = \frac{\vartheta_o^2 \sin 2\theta}{g} \quad ...(8.1.13)$$

Comparing equation (8.1.10) and (8.1.13)

$$R = R' \quad ...(8.1.14)$$

Hence the horizontal range is same corresponding to angle of projection θ and $(90-\theta)$ with the horizontal direction.

8.1(f) The Path of Projectile

If the projectile is at P after time t Fig 8.1 where x is the horizontal distance covered and y is the vertical distance moved.

ϑ_o is the initial velocity with which the body is launched.

Then $x = (\vartheta_o \cos\theta)t$

or $t = \frac{x}{\vartheta_o \cos\theta}$ ...(8.1.15)

Using equation

$$S = ut + \frac{1}{2}at^2$$

$$y = (\vartheta_o \sin\theta)t + \frac{1}{2}(-g)t^2$$

Substituting (8.1.15) for t in the above equation

$$y = (\vartheta_o \sin\theta).\frac{x}{\vartheta_o \cos\theta} - \frac{1}{2}g\frac{x^2}{\vartheta_o^2 \cos^2\theta}$$

$$= x\tan\theta - \frac{1}{2}g\frac{x^2}{\vartheta_o^2 \cos^2\theta} \quad ...(8.1.16$$

The equation (8.1.16) is of the form $y = Ax - Bx^2$, where $A = \tan\theta$ and

$$B = \frac{g}{2\vartheta_o^2 \cos^2\theta}$$

A and B are constants. Equation (8.1.16) represents a parabola. Hence the path of a projectile moving under gravity in medium of negleigible air resistance is a parabola.

Initital energy at the time of lauching of projectile which is kinetic energy $= \frac{1}{2}m\vartheta_o^2$ ...(8.1.17)

Initial potential energy = mg. o = 0 ...(8.1.18)

Potential energy at the highest point = mgH

$$= mg.\frac{\vartheta_o^2 \sin^2\theta}{2g}$$

$$= \frac{1}{2}m\vartheta_o^2 \sin^2\theta \quad ...(8.1.19)$$

Kinetic energy will be minimum at the highest point (but not zero due to the presence of horizontal component)

$$(KE)_{Highest\ point} = \frac{1}{2}m\vartheta_o^2 \cos^2\theta \quad ...(8.1.20)$$

i.e $(PE)_{Highest\ point} + (KE)_{Highest\ point}$

$$= \frac{1}{2}m\vartheta_o^2 \sin^2\theta + \frac{1}{2}m\vartheta_o^2 \cos^2\theta$$

$$= \frac{1}{2}m\vartheta_o^2 \quad ...(8.1.21)$$

Hence mechanical energy is conserved.

8.2 Projectile Fired Horizontally from an elevation :

The angle between direction of launching and horizontal $\theta = 0$ i.e. the projectile is fired horizontally with the velocity ϑ_o.

Initial vertical component of velocity

$$= \vartheta_o \sin 0 = 0$$

Initital Horizontal component of velocity

$$= \vartheta_o \cos 0 = \vartheta_o$$

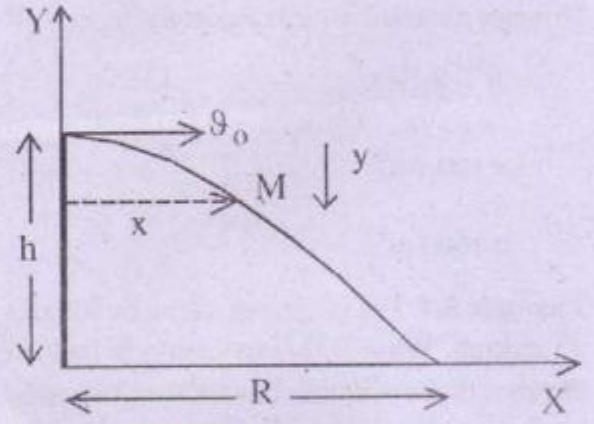


Fig. 8.4

If T = time taken by projectile to reach the ground when launched from an elevation h horizontally,

Using equation $S = ut + \frac{1}{2}at^2$

$$-h = 0.T + \frac{1}{2}(-g)T^2$$

or $\quad T^2 = \frac{2h}{g}$

or $\quad T = \sqrt{\frac{2h}{g}}$ ...(8.2.1)

Equation (8.2.1) gives the time of flight of the projectile fired horizontally from an elevation. The range of projectile

R = Initital horizontal component of velocity x time of flight

$$R = \vartheta_o \times \sqrt{\frac{2h}{g}} \quad ...(8.2.2)$$

Suppose the projectile is at M after time 't' where it has covered horizontal distance x and vertical height covered is y.

Then $x = \vartheta_o.t$

i.e $\quad t = \frac{x}{\vartheta_o}$ ...(8.2.3)

Using the relation

$$S = ut + \frac{1}{2}at^2$$

$$-y = 0 + \frac{1}{2}(-g)t^2$$

$$= -\frac{1}{2}gt^2$$

Substituting (8.2.3)

$$y = \frac{1}{2}g\left(\frac{x}{\vartheta o}\right)^2$$

$$y = \frac{1}{2}g\frac{x^2}{\vartheta_o^2} \quad ...(8.2.4)$$

This is the equation of parabola symmetric about Y-axis.

SOLVED NUMERICALS :

Example 8.1 A ball is projected at an angle 45^0 to the horizontal. If the horizontal range is 10m. Find the maximum height attained by the ball. ($g = 10\ mS^{-2}$)

The horizontal range $R = \frac{\vartheta_o^2 \sin 2\theta}{g}$

$$= \frac{\vartheta_o^2 \sin(2 \times 45)}{g}$$

$$= \frac{\vartheta_o^2}{g}$$

or $\vartheta_o^2 = R.g$

$= 10 \times 10$

$\vartheta_o = 10 \text{ m S}^{-1}$

The maximum height $H = \frac{\vartheta_o^2 \sin^2 \theta}{2g}$

$$= \frac{(10)^2 \sin^2 45}{2 \times 10}$$

$$= \frac{(10)^2 (\frac{1}{\sqrt{2}})^2}{2 \times 10}$$

$$= \frac{10}{4} = 2.5\text{m}$$

Example 8.2 What is the least velocity with which a cricket ball can be thrown 80m ?

The horizontal range $R = \frac{\vartheta_o^2 \sin 2\theta}{g}$

Maximum horizontal Range = $R_{max} = \frac{\vartheta_o^2}{g}$

$$80 = \frac{\vartheta_o^2}{g}$$

or $\vartheta_o^2 = 80 \times g$

$\therefore \vartheta_o = \sqrt{80 \times 9.8} = 28 \text{ m S}^{-1}$

Example 8.3 An aeroplane flying horizontally with a speed of 360 km hr^{-1} releases a bomb at a height of 490m from the ground. When and where will be bomb strike the ground ?

Time taken by the bomb to reach the ground is t.

$$S = ut + \frac{1}{2}at^2$$

$$490 = 0 + \frac{1}{2} \times 9.8.t^2$$

$$t^2 = 100$$

$$t = 10 \text{ S.}$$

Distance travelled in the horizontal direction =R

$$R = \vartheta_o \times t$$

$$= 100 \times 10 \quad (\because \vartheta_o = \frac{360 \times 10^3}{60 \times 60} \text{m/s} = 10^2 \text{m/s})$$

$$= 1000 \text{ m.}$$

Example 8.4 The ceiling of a long building is 25 m high. What is the maximum horizontal distance that a ball thrown with speed 40 m S^{-1} can go without hitting the ceiling of the building ?

The ball is thrown at an angle θ with the horizontal (Say)

Then maximum height $H = \frac{\vartheta_o^2 \sin^2 \theta}{2g}$

or $25 = \frac{(40)^2 \sin^2 \theta}{2 \times 9.8}$

or, $\sin^2 \theta = \frac{25 \times 2 \times 9.8}{40 \times 40}$

$\sin \theta = 0.5534$

$\theta = 33.6^o$

The horizontal range $R = \frac{\vartheta_o^2 \sin 2\theta}{g}$

$$= \frac{(40)^2 \sin(2 \times 33.6)}{9.8}$$

$$= \frac{40 \times 40 \sin 67.2}{9.8}$$

$$= 150.5\text{m}$$

Example 8.5 A shot leaves a gun at the rate of 160 mS^{-1}. Calculate the greatest distance to which it could be projected and the height to which it would rise. Given $g = 10\ m\ S^{-2}$

The horizontal range $R = \frac{\vartheta_o^2 \sin 2\theta}{g}$

R is maximum if $\theta = 45^o$

$$R_{max} = \vartheta_o^2 / g = \frac{(160)^2}{10} = 2560\text{m}$$

Maximum height attained = H

$$H = \frac{\vartheta_o^2 \sin^2 \theta}{2g}$$

$$H = \frac{(160)^2 \times \left(\frac{1}{\sqrt{2}}\right)^2}{2 \times 10} = 640\text{m}$$

Example 8.6 A ball is projected from the ground with a speed $25\ m\ S^{-1}$. Two seconds later it just clears a wall 5 m high. Find (i) the angle of projection, (ii) the maximum height attained.

How far beyond the wall the ball again hits the ground ? Given $g = 10\ m\ S^{-2}$.

Let θ be the angle of projection

The vertical component will be $25 \sin\theta\ m\ S^{-1}$

Then using $S = ut + \frac{1}{2}at^2$

ie. $5 = (25 \sin\theta).2 + \frac{1}{2}(-10)(2)^2$

or $50 \sin\theta = 25$

or $\sin\theta = \frac{1}{2}$

or $\theta = 30^o$

The maximum vertical height moved = H

$$H = \frac{\vartheta_o^2 \sin^2 \theta}{2g} = \frac{(25)^2 \times \left(\frac{1}{2}\right)^2}{2 \times 10} = 7.8\text{m}$$

Time of flight of the ball = T

$$T = \frac{2\vartheta_o \sin\theta}{g} = \frac{2 \times 25 \times \frac{1}{2}}{10} = 2.5\ S$$

Time left after clearing the wall = $2.5 - 2$

$= 0.5$ S

Horizontal component of velocity of projection

$= 25 \cos\theta\ m\ S^{-1}$

Horizontal distance travelled beyond the wall

$= 25 \cos 30 \times 0.5$

$= 10.8$ m.

Example 8.7 A bomb is dropped from an aeroplane when it is directly above the taget at a height of 1960 m. The aeroplne is flying horizontally with a velocity of 360 km h^{-1}. By how much distance will the bomb miss the target ?

Horizontal velocity of aeroplane =

$$\vartheta = \frac{360 \times 1000}{3600} = 100\ m\ S^{-1}$$

Time taken by the bomb to hit the ground = t

Initial vertical component of velocity = 0

acceleration = $9.8\ m\ S^{-2}$

Vertical height = 1960 m

$$S = ut + \frac{1}{2}at^2$$

$$1960 = 0 + \frac{1}{2}(9.8)t^2$$

$$t^2 = 400$$

$$t = 20\ S$$

Horizontal distance covered by the bomb = R

$$R = 100\ m\ S^{-1} \times 20\ S$$

$$= 2000\ m.$$

Thus the bomb will miss the target by a distance of 2000 m.

Example 8.8 A hunter aims his gun at a monkey sitting on a tree. At the instant the bullet leaves the barrel the moneky drops. Will the bullet hit the monkey ?

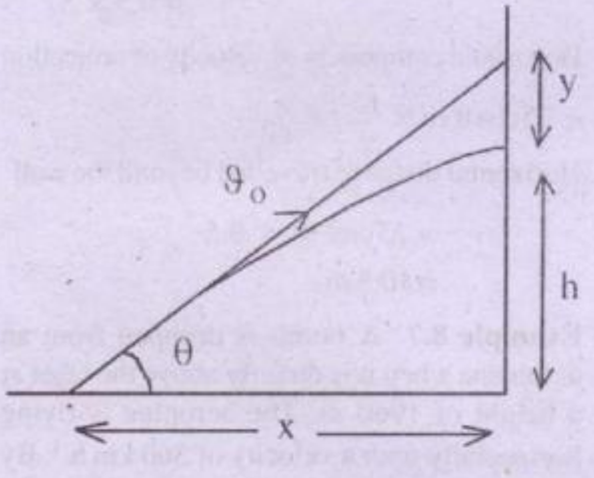


Fig. 8.5

Suppose the bullet is fired with velocity ϑ_o making an angle θ with the horizontal. Horizontal component of velocity of bullet =

$$\vartheta_o \cos\theta$$

Vertical component of velocity of bullet =

$$\vartheta_o \sin\theta$$

Let x be the horizontal distance between hunter and the foot of the tree.

Then $\quad x = \vartheta_o \cos\theta . t$

or $\quad t = \dfrac{x}{\vartheta_o \cos\theta} \qquad ...(1)$

where t is the time taken by the bullet to travel distance x.

Vertical distance through which the monkey falls during time t is y.

Using $\quad S = ut + \frac{1}{2}at^2$

$$y = 0 + \frac{1}{2}gt^2$$

The bullet is at a vertical height h after t seconds.

Using the relation $\quad S = ut + \frac{1}{2}at^2$

$$h = \vartheta_o \sin\theta . t + \frac{1}{2}(-g)t^2$$

$$= \vartheta_o \sin\theta . \frac{x}{\vartheta_o \cos\theta} - \frac{1}{2}g\frac{x^2}{\vartheta_o^2 \cos^2\theta}$$

using equation (1)

$$= x\tan\theta - \frac{1}{2}gt^2 \qquad ...(2)$$

$$\tan\theta = \frac{y+h}{x}$$

or $\quad y + h = x\tan\theta$

$$y = x\tan\theta - h$$

$$= x\tan\theta - (x\tan\theta - \frac{1}{2}gt^2)$$

using equation (2)

$$= \frac{1}{2}gt^2$$

Hence the bullet will hit the monkey.

SUMMARY :

Projectile :

A projectile is an object which is launched into the space without motive power of its own and moves freely under the action of gravity and air resistance.

Projectile fired at an angle θ with the horizontal.

(i) **Velocity at any instant 't'**

magnitude of velocity

$$\vartheta = \sqrt{(\vartheta_o \cos\theta)^2 + (\vartheta_o \sin\theta - gt)^2}$$

$$= \sqrt{\vartheta_o^2 + g^2t^2 - 2\vartheta_o gt\sin\theta}$$

Direction of ϑ

α = angle that ϑ makes with horizontal

$$= \tan^{-1}\left(\frac{\vartheta_o \sin\theta - gt}{\vartheta_o \cos\theta}\right)$$

(ii) **Time of ascent**

$$T_1 = \frac{\vartheta_o \sin\theta}{g}$$

(iii) **Time of flight**

$$T = \frac{2\vartheta_o \sin\theta}{g}$$

(iv) **Maximum height reached**

$$H = \frac{\vartheta_o^2 \sin^2\theta}{2g}$$

(v) **Horizontal range**

$$R = \frac{\vartheta_o^2 \sin 2\theta}{g}$$

for a given velocity of projection R will be maximum if $\theta = 45^o$

(vi) **Equation of trajectory**

$$y = x\tan\theta - \frac{gx^2}{2\vartheta_o^2 \cos^2\theta}$$

Projectile fired Horizontally

(i) **Horizontal range :** $R = \vartheta_o\sqrt{\frac{2h}{g}}$

(ii) **Time of flight :** Time of descent

$$T = \sqrt{\frac{2h}{g}}$$

(iii) **Equation of trajectory**

$$y = \frac{1}{2} g \frac{x^2}{\vartheta_o^2}$$

$$= K x^2$$

MODEL QUESTIONS

A. MULTIPLE CHOICE :

1. A ball is projected from the top of a tower at an anlge 60^0 with the vertical. What happens to the vertical component of velocity ?

   a) increases continuously
   b) decreases continuously
   c) remains unchanged
   d) first decreases and then increases.

2. A projectile is fired with a velocity of 10 m S^{-1} at an angle of 60^0 with the horizontal. Its velocity at the highest point is

   a) zero b) 5 m S^{-1}
   c) 8.66 m S^{-1} d) 10 m S^{-1}

3. A ball is thrown upwards and it returns to ground describing a parabolic path. Which of the following remains constant?

   a) kinetic energy of ball
   b) speed of ball
   c) horizontal component of velocity
   d) vertical component of velocity

4. Three particles A, B and C are projected from the same point with same initial speeds making angles 30^0, 45^0 and 60^0 respectively with the horizontal. Which of the following statements is correct?

   a) A, B, C have unequal ranges
   b) ranges of A& C are equal & less than that of B
   c) Ranges of A & C are equal but greater than B
   d) A,B,C have equal ranges.

5. A cricket ball is hit at 45^0 to the horizontal with a Kinetic energy K. The kinetic energy at the highest point is

   a) 0 b) K / 2
   c) $K/\sqrt{2}$ d) k

6. The angle of projection, for which the horizontal range and maximum height of a projectile are equal is

   a) 45^0 b) $\theta = \tan^{-1} 4$
   c) $\theta = \tan^{-1}(0.25)$ d) none of these

7. A body is projected horizontally from a height of 5m. It reaches the ground at a horizontal distance of 10m. The speed of the particle when it reaches the ground is ($g = 10$ m S^{-2})

   a) $10 mS^{-1}$ b) $10\sqrt{2} mS^{-1}$
   c) $20\ mS^{-1}$ d) $20\sqrt{2}\ mS^{-1}$

8. A particle is thrown horizontally from the top of a tower of height h with a velocity $\sqrt{2gh}$. It strikes the ground at a distance from the foot of the tower equal to

   a) h/2 b) 2h/3
   c) h d) 2h

B. VERY SHORT ANSWER QUESTIONS :

1. What is the shape of the path of a projectile ? 1990,1998

2. At what angle should a projectile be projected so that it could cover the maximum horizontal range ? 1999

3. A body is dropped from an aeroplane moving with uniform velocity. What will be the shape of its path of fall ? 1991

4. Write the shape of the trajectory of a projectile in the mathematical form naming the terms in it. 1989(S)

5. The range of a projectile depends upon the angle of projection. Is it possible to have more than one angle of projection for the same range ?

6. Mention two examples of projectile motion.

7. What is the velocity of the projectile when it is at the highest point ?

8. Radius remaining unchanged if the mass of the earth becomes half the present value, then what will be the acceleration due to gravity ? 1996(S)

9. Two projectiles are projected simultaneously from a point with the same speed. Their angles of projection are 25^0 and 65^0 respectively. Compare the ranges covered. 2000

C. SHORT ANSWER QUESTIONS :

1. What are the factors on which range of a projectile depend ?

2. A small body is projected at an angle of 45^0 to the horizontal with kinetic energy K. What will be its kinetic energy at the top most point in its path of flight ?

3. A person sitting in a running train throws a ball vertically upward. What will be the path described by the ball ?

4. A body is dropeed from the top of a tower and another thrown horizontally with speed 300ms. Which one will reach the ground first and why ?

D. NUMERICAL PROBLEMS :

1. How may metres will a bomb released from an aeroplane fall in 10s after it is released. Assume $g = 9.8\ mS^{-2}$. 1990(S)

2. A particle is projected from the surface of earth vertically upward with a speed equal to half its escape velocity from earth. Taking earth to be a sphere of radius R calculate the height to which the particle rises above the surface of earth.

JEE 1997

Hint : K.E. of projection = change in P.E.

$$\frac{1}{2}m\left(\frac{\vartheta_e}{2}\right)^2 = mgR/(1+\frac{R}{h})$$

3. A shell is fired from a cannon with a velocity $\vartheta\ mS^{-1}$ at an angle θ with the horizontal direction. At the highest point of its path it explodes into two pieces of equal mass. One of the piece retraces its path into cannon. Find the speed of other piece immediately after explosion.

JEE 2000

Hint : Horizontal component of velocity of shell = $\vartheta \cos\theta$

Velocity of one piece is - $\vartheta.\cos\theta$.

ϑ_1 = Velocity of other, m = mass of shell

$m\vartheta\cos\theta = -\frac{m}{2}\vartheta\cos\theta + \frac{m}{2}\vartheta_1$

4. A body is projected horizontally from the top of a cliff with a velocity of $9.8\ m\ S^{-1}$. What time elapses before the horizontal and vertical velocities become equal? Given $g = 9.8\ m\ S^{-2}$.

5. An aeroplane is flying horizontally at a height of 490 m with a velocity of 360 kmh^{-1}. A bag containing food is to be dropped to the Jawans on the ground. How far them the bag be released so that it fall over them?

6. A projectile is thrown horizontally with a velocity of 49 m/s from the top of a tower. If it strikes the ground at an angle of 45^0, find the height of the tower and the velocity with which it strikes the ground.

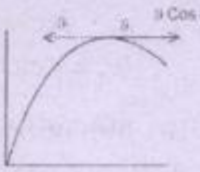
7. Two tall buildings are 200m apart. Calculate the speed with which a stone must be thrown horizontally from a window 540 m above the ground in one building so that it will enter a window 50m above the ground in other?

8. An object of mass 10 kg is projected with velocity 20 mS^{-1} at an angle 60^0 to the horizontal. At the highest point of its path the object explodes into two fragments of 5 kg each. The fragments separate horizontally after the explosion. The explosion releases internal energy so that the kinetic energy of the system is doubled. Calculate the velocity of the two fragments. JEE 1996

Hint : Velocity at highest point = $\vartheta \cos 60$ $= \vartheta/2 = 10$ m/s

Momentum before explosion =

$2m.\frac{\vartheta}{2} = m\vartheta$



2m = 10 kg

Here $m_1 = m_2 = m = 5$ kg

KE before explosion =

$$\frac{1}{2}2m.(\frac{1}{2}\vartheta)^2 = \frac{m\vartheta^2}{4}$$

ϑ_1 and ϑ_2 are the velocity after explosion

$m\vec{\vartheta}_1 + m\vec{\vartheta}_2 = m\vec{\vartheta}$ or $\vec{\vartheta}_1 + \vec{\vartheta}_2 = \vec{\vartheta}$

Also $(\vartheta_1 - \vartheta_2)\hat{i} = \vartheta\hat{i}$

$$\Rightarrow \vartheta_1 - \vartheta_2 = \vartheta = 20$$

$$\frac{1}{2}m\vartheta_2^2 + \frac{1}{2}m\vartheta_1^2 = 2\left(\frac{m\vartheta^2}{4}\right) = \frac{m\vartheta^2}{2}$$

$$\vartheta_2^2 + \vartheta_1^2 = \vartheta^2$$

9. A particle is projected with velocity $2\sqrt{gh}$ so that it just clears two walls of equal height h which are at distance 2h from each other. Prove that the time of flight between the two walls is $2\sqrt{h/g}$.

2000

Hint : $\vartheta_0 = 2\sqrt{gh}$; $h = \vartheta_0 \sin\theta . t - \frac{1}{2}gt^2$

or $gt^2 - 2\vartheta_0 \sin\theta \, t + 2h = 0$ or

$$t = \frac{\vartheta_0 \sin\theta}{g} \pm \frac{\sqrt{\vartheta_0^2 \sin^2\theta - 2gh}}{g}$$

$$t_2 - t_1 = \frac{2\sqrt{\vartheta_0^2 \sin^2\theta - 2gh}}{g}$$

$$= \frac{2\sqrt{4gh\sin^2\theta - 2gh}}{g}$$

$$= \sqrt{\frac{8h}{g}}.\sqrt{2\sin^2\theta - 1} \quad (1)$$

Also $2h = \vartheta_0 \cos\theta . (t_2 - t_1)$

$$= 2\sqrt{gh}.\cos\theta.\sqrt{\frac{8h}{g}}.\sqrt{2\sin^2\theta - 1}$$

or $\cos\theta = \frac{1}{2}$ (2)

Using (2) from (1)

$t_2 - t_1$ = Time of flight between two walls

$$= 2\sqrt{h/g}$$

E. LONG ANSWER QUESTIONAS :

1. Establish the expression for the range and maximum height attained by a projectile. Prove that speed remaining constant the range is same for two angles of projection. 1995

2. A projectile is launched with an initial velocity 'ϑ' making an angle θ with the horizontal. Find an expression for the maximum height to which it will rise and show that its trajectory is a parabola.

3. What is a projectile ? Derive expression for the time of flight and horizontal range of a projectile. 1990, 1992

4. If a projectile is released with a velocity 'u' at an angle θ with the horizontal what is the maximum height to which it will rise?

   Calculate the horizontal range covered by a stone thrown with velocity 30 mS^{-1} inclined at an angle 45^0 with the horizontal. Assume $g = 9.8\ mS^{-2}$.

F Fill in the Blank Type

1. For angles of projection which exceed or fall short of 45^0 by equal amounts, the ranges are ....

2. The direction of motion of a projectile at the highest point of its trajectory becomes.....

3. In a javeline throw a person who throws at an angle of 45^0 to the ground has greater probability of ....

4. A marble 'A" is dropped vertically. Another identical marble 'B' is projected horizontally from the same point at the same instant. Both A and B will reach the ground at the .... instant.

5. A stone is thrown at angle of 45^0 to the horizontal. It rises to a maximum height of 10m. Its horizontal range is ....

6. For a projectile projected at an angle......... the maximum height and horizontal range are equal.

G. True - False Type

1. The horizontal velocity component retains its initial value throughout the flight in projectile motion.

2. In long jump a person who jumps at an angle 45^0 to ground has greater probability of winning.

3. A projectile fired from the ground follows a parabolic path. The speed of the projectile is minimum at the top of the path.

4. A body subjected to uniform acceleration always moves in a straight line.

5. The path of one projectile as seen from another projectile is a straight line.

ANSWERS

A. MULTIPLE CHOICE QUESTIONS :

1. d	2. b	3. c	4. b
5. b	6. b	7. b	8. d

D. NUMERICAL PROBLEMS :

1. 490 m	2. $h = \dfrac{R\vartheta_e^2}{8gR - \vartheta_e^2}$
3. $3\vartheta\cos\theta$	4. 1 S
5. 1000 m	6. 122.5 m
7. 20 m S^{-1}	8. $\vartheta_2 = 20$ m S^{-1}, $\vartheta_1 = 0$

F. (1) Equal (2) horizontal (3) winning (4) same (5) 40m. (6) $\tan^{-1}4$.

G. (1) True (2) True (3) True (4) False (5) True.

9

Oscillations

In chapters 4 and 6 it has been shown that a constant force / torque acting on a body induces uniformly accelerated (linear / angular) motion. Such uniformly accelerated motion has been considered in chapters 3 and 6. But when forces/torque varying in magnitude and/or direction act on a body, non-uniformly accelerated motion is induced. Study of such motion is tedious. However, there is one common and important type of non-uniformly accelerated motion, called periodic motion, which can be analysed with little labour.

9.1 Periodic Motion

A motion which repeats itself again and again at regular intervals of time is called periodic motion.

The regular interval of time in which the motion is repeated is called its **time period** (T).

Some examples of periodic motion are :

(i) Motion of earth around sun (ii) motion of electron around nucleus (iii) motion of hands of a clock (iv) occurrence of day and night (v) motion of a pendulum (vi) motion of balance-wheel of a watch (vii) motion of a swing (viii) vibration of a loaded spring (ix) vibration of prongs of a tuning fork. (x) tórsional oscillation of a disc. etc.

9.2 Oscillatory Motion (Vibratory Motion)

A motion that repeats itself again and again about its mean position of rest such that it remains confined within well-defined limits (called extreme positions) on either side of the equilibrium (mean) position is called **Oscillatory Motion.**

Thus oscillatory motion is a periodic motion about a mean position. Some examples of oscillatory motion are :

(i) motion of a loaded spring (ii) motion of a pendulum (iii) motion of a swing (iv) torsional oscillation of a disc (v) motion of balance-wheel of a watch (vi) beating of heart of a human body etc. We note that all oscillatory motions are periodic but all periodic motions are not oscillatory. For example motion of earth around sun is periodic but not oscillatory, whereas motion of a swing is oscillatory as well as periodic.

Graphs representing oscillatory motion can be of various types, some of which are given below for the sake of illustration.

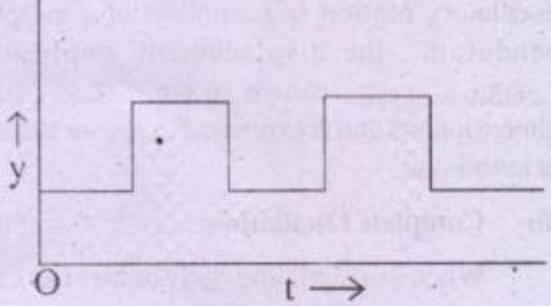


Fig. 9.1 (a)

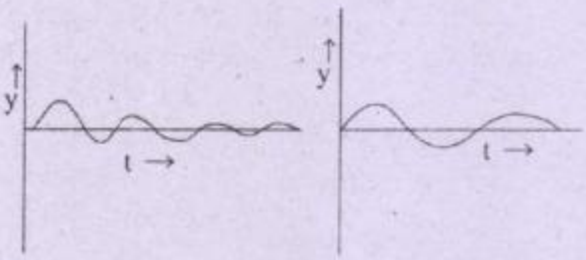


Fig. 9.1.(b) Fig. 9.1 (c)

Oscillatory motions are characterised by physical quantities like amplitude, time period, complete oscillation, frequency etc. which are discussed below.

(i) Amplitude :

The maximum displacement of an oscillating particle from its equilibrium position is called its displacement amplitude.

Sometimes one speaks about velocity amplitude, which means maximum value of velocity.

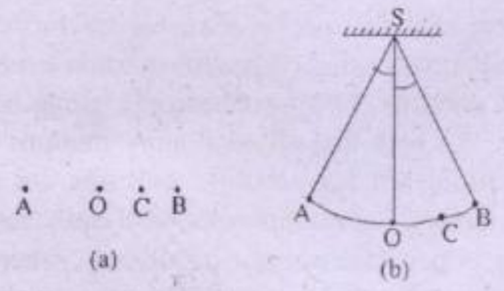


Fig. 9.2

In case of linear oscillatory motion (e.g. Oscillation of a spring) the displacement amplitude (OA=OB, shown in fig. 9.2 (a)) has dimension L and expressed in meter in S.I unit or cm. in C.G.S. units. In case of angular oscillatory motion (e.g. motion of a simple pendulum , the displacement amplitude ($\angle OSA = \angle OSB$, shown in fig. 9.2 (b)) is dimensionless and is expressed in deg. or radian or revolution.

(ii) Complete Oscillation

When an oscillating body comes back to its initial state after elapse of minimum time, one complete oscillation is said to have been executed.

For example when the body covers the path CB+BA+AC (A ← O → C → B, B → A, A → C) or CA+AB+BC (A ← O ← C, A → B, C ← B) etc. One oscillation is completed. (see fig. 9.2)

(iii) Time period

The minimum time taken by the oscillating body to complete one oscillation is called its **time period** (T).

It has the dimension T and is expressed in second(s) or its multiple.

(iv) Frequency

The number of complete oscillations executed per second by the oscillating body is called **Frequency** (f or υ).

Since time period T corresponds to one complete oscillation, so frequency is equal to the reciprocal of the time period i.e. $f = \frac{1}{T}$. Hence it has the dimension T^{-1} and its unit is s^{-1} in C.G.S. unit and Hertz in S.I. unit

1 Hz = 1 cycle / sec

9.3. Periodic Function

A function f(t) is said to be periodic in time if its value is regained after definite intervals of time i.e.

If

$$f(t) = f(t+T) = f(t+2T) = ... = f(t+nT) \quad ...(9.3.1)$$

where 'n' is an integer, then f(t) is said to be a periodic function with time period T, e.g.

$$f(t) = \sin(2\pi t / T), g(t) = \cos(2\pi t / T)$$

are periodic functions as

$$\sin\frac{2\pi t}{T}=\sin\frac{2\pi}{T}(t+T)=...=\sin\frac{2\pi}{T}(t+nT)$$

and

$$\cos\frac{2\pi t}{T}=\cos\frac{2\pi}{T}(t+T)=...=\cos\frac{2\pi}{T}(t+nT)$$

The sum of two or more periodic functions with same time period is also periodic.

e.g. $y=A\sin\frac{2\pi}{T}t+B\cos\frac{2\pi}{T}t+f(t)$

$$=A\sin\frac{2\pi}{T}(t+nT)+B\cos\frac{2\pi}{T}(t+nT)+f(t+nT)$$

where f(t) is a periodic function of period T.

9.4 Harmonic Motion

An oscillatory motion which can be represented by a sine or cosine function of time is called a **harmonic motion.**

For example displacement in harmonic (oscillatory) motion can be expressed as

(i) $y=A(t)\ \sin\frac{2\pi}{T}t=A(t)\sin\omega t$

(ii) $y=A(t)\ \cos\frac{2\pi}{T}t=A(t)\cos\omega t$

(iii) $y=A(t)\sin\frac{2\pi}{T}t+B(t)\cos\frac{2\pi}{T}t$ etc. ...(9.4.1)

In case of harmonic motion the amplitude may or may not depend on time. If the amplitude is constant, then it is called **simple harmonic motion.**

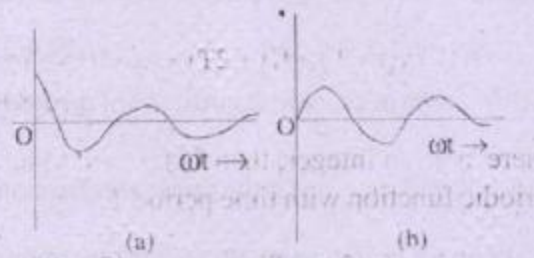


Fig. 9.3

9.5 Simple Harmonic Motion (SHM)

An oscillatory motion which can be represented by a sine or cosine function of time with constant amplitude is called S.H.M. (This is the mathematical definition of S.H.M.)

Thus if 'y' denotes displacement (linear / angular) suffered by the body executing SHM, then

$$y=A\sin\frac{2\pi}{T}t \quad ...(a)$$

or $$y=A\cos\frac{2\pi}{T}t \quad ...(b)$$

or $$y=A\sin\left(\frac{2\pi}{T}t+\phi\right) \quad ...(c)$$

or $$y=A\cos\left(\frac{2\pi}{T}t+\phi\right) \quad ...(9.5.1)(d)$$

The above definition yields on differentiation w.r.to time 't'.

$$\frac{dy}{dt}=\dot{y}=A\left(\frac{2\pi}{T}\right)\cos\left(\frac{2\pi}{T}t+\phi\right) \quad ...(a)$$

and

$$\frac{d^2y}{dt^2}=\ddot{y}=-A\left(\frac{2\pi}{T}\right)^2\sin\left(\frac{2\pi}{T}t+\phi\right) \quad ...(9.5.2(b)$$

This implies

Linear acceleration $a=-\left(\frac{2\pi}{T}\right)^2 x=-\omega^2 x$ ...(9.5.3)

Angular acceleration $\alpha=-\left(\frac{2\pi}{T}\right)^2\theta=-\omega^2\theta$ ...(9.5.4)

and

Force $F=ma=-m\omega^2 x=-kx$ ...(9.5.5)

Toque $\tau = I\alpha = -I\omega^2\theta = -C\theta$ ...(9.5.6)

where 'I' is the M.O.I.

Equations (9.5.3) & (9.5.4) and (9.5.5) & (9.5.6) lead to two more physical definitions as discussed below.

(a) Dynamical definition of S.H.M.

A motion in which a particle is subjected to a (restoring) force (/ torque) which is directly proportional to displacement (linear/ angular) of the particle from the mean (equilibrium) position and is always directed towards the equilibrium position is called a simple harmonic motion (SHM).

Thus if $\vec{F}$ be the (restoring) force on a particle when its (linear) displacement is $\vec{x}(= x\hat{i})$ from the mean position, then according to the definition for SHM (linear)

$$\vec{F} = -k\,\vec{x} \quad ...(9.5.7)$$

$$|\vec{F}| \propto x \quad ...(9.5.8)$$

The constant K is called the <u>force constant</u>. In case of angular SHM

$$\vec{\tau} = -C\vec{\theta} \quad ...(9.5.9)$$

where 'C' is called torque per unit angular displacement. The negative sign in (9.5.7) and (9.5.9) indicates that force (/torque) and displacement (linear/angular) are oppositely directed and force is always directed towards equilibrium position.

(b) Kinematic definition of SHM

A motion in which acceleration (linear/ angular) of a particle is directly proportional to displacement (linear/angular) from its mean position and is always directed towards mean position is called SHM.

If $\vec{a}$ be linear acceleraton of a particle when its linear displacement is $\vec{x}$ from its mean position, then according to definition for SHM (linear)

$$\vec{a} = -\omega^2\vec{x} \quad ...(9.5.10)$$

$$|\vec{a}| \propto x \quad ...(9.5.11)$$

Similarly in case of angular SHM

$$\vec{\alpha} = -\omega^2\,\vec{\theta} \quad ...(9.5.12)$$

From equations (9.5.5) and (9.5.6) we find that

$$\text{for linear motion } \omega^2 = \frac{K}{m} \quad ...(9.5.13)$$

and

$$\text{for angular motion } \omega^2 = \frac{C}{I} \quad ...(9.5.14)$$

The physical meaning of ω will become clear after solution of equations (9.5.10) or (9.5.12).

9.6. Kinematics of SHM (linear)

Consider a body of mass 'm' executing SHM along x-direction. Therefore the linear acceleration of the body must be given by (9.5.10)

$$\text{i.e. } \vec{a} = -\omega^2\vec{x} \quad ...(9.6.1)$$

where $\omega^2 = k/m$, k being force constant.

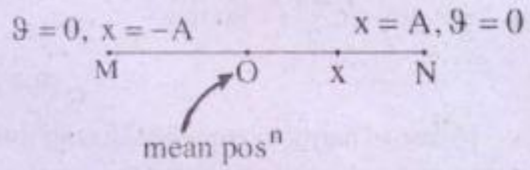


Fig. 9.4

Equation (9.6.1), is re-written in scalar form as

$$a = -\omega^2 x \quad ...(9.6.2)$$

Here 'x' is the positional coordinate of the body at time 't', and $a = \frac{d^2x}{dt^2}$ is the linear acceleration.

Equation (9.6.2) has to be solved to obtain expression for velocity, displacement etc.

Although there are different ways to solve this, we shall mention below two methods, giving liberty, to readers to pick up one as they like.

(a) Method I

According to definition of acceleration and velocity

$$\bar{a} = \frac{d\bar{\vartheta}}{dt}$$

$$\bar{\vartheta} = \frac{d\bar{x}}{dt}$$

$$\Rightarrow \quad \bar{\vartheta}.\frac{d\bar{\vartheta}}{dt} = \bar{\vartheta}.a = \bar{a}.\bar{\vartheta} = -\omega^2\bar{x}.\frac{d\bar{x}}{dt}$$

$$\Rightarrow \quad \frac{1}{2}\frac{d}{dt}(\bar{\vartheta}.\bar{\vartheta}) = -\omega^2\,\frac{1}{2}\,\frac{d}{dt}(\bar{x}.\bar{x}) \quad ...(9.6.4)$$

Integrating both sides of (9.6.3) and imposing the physical conditions

i) at $x = \pm A$, $\vartheta = 0$

ii) at $x = x$, $\vartheta = \vartheta$

We obtain

$$\int_0^{\vartheta}\frac{d}{dt}(\vartheta^2)dt = -\omega^2\int_A^{x}\frac{d}{dt}(x^2)dt$$

$$\Rightarrow \quad \vartheta^2\Big|_o^{\vartheta} = -\omega^2\left(x^2\Big|_A^{x}\right)$$

$$\Rightarrow \quad \vartheta^2 = -\omega^2\left(x^2 - A^2\right) = \omega^2(A^2 - x^2)$$

$$\Rightarrow \quad \vartheta = \omega\sqrt{A^2 - x^2} \quad ...(9.6.4)$$

Since $\bar{\vartheta} = \frac{d\bar{x}}{dt}$ and ϑ & $\bar{x}$ are in same direction, so

$$\vartheta = \frac{dx}{dt} = \omega\sqrt{A^2 - x^2}$$

$$\Rightarrow \quad \frac{dx}{\sqrt{A^2 - x^2}} = \omega\, dt \quad ...(9.6.5)$$

Again integrating equation (9.6.5) and imposing the physical conditions

i) at $t = 0$, $x = x_0$ = initial position

ii) at $t = t$, $x = x$, $\vartheta = \vartheta$

We obtain

$$\int_{x_0}^{x}\frac{dx}{\sqrt{A^2 - x^2}} = \omega\int_0^{t}dt$$

giving

$$\sin^{-1}\left(\frac{x}{A}\right)\Bigg|_{x_0}^{x} = \omega\left(t\Big|_0^{t}\right)$$

$$\Rightarrow \quad \sin^{-1}\left(\frac{x}{A}\right) - \sin^{-1}\left(\frac{x_0}{A}\right) = \omega t \quad ...(9.6.6)$$

Putting $\sin^{-1}\left(\frac{x_0}{A}\right) = \phi$ ...(9.6.7)

One obtains

$$\sin^{-1}\left(\frac{x}{A}\right) = \omega t + \phi$$

$$\Rightarrow \quad x = A\sin(\omega t + \phi) \quad ...(9.6.8)$$

Equation (9.6.8) gives the displacement of the particle executing SHM and equation (9.6.4) gives the velocity of the body.

From the above analysis, we find that at $x = 0$, the mean (equilibrium) position, the body assumes maximum velocity $\vartheta_{max} = A\omega$, and at $x = \pm A$, the extreme positions, the body attains zero velocity.

(b) Method II

By definition $\bar{a} = \frac{d\bar{\vartheta}}{dt} = \frac{d}{dt}(\frac{d\bar{x}}{dt})$

$$\Rightarrow \quad \bar{a} = \frac{d^2\bar{x}}{dt^2} \quad ...(9.6.9)$$

Therefore equation (9.6.1) reduces to

$$\frac{d^2x}{dt^2} = -\omega^2 x$$

$$\Rightarrow \quad \frac{d^2x}{dt^2} = -\omega^2 x \qquad ...(9.6.10)$$

Equation (9.6.10) is a second order homogeneous linear differential equation. Its solution is obtained by putting

$$x = e^{\beta t}$$

so that eqn. (9.6.10) reduces to

$$\left(\beta^2 + \omega^2\right)e^{\beta t} = 0$$

$$\Rightarrow \quad \beta^2 + \omega^2 = 0$$

i.e. $\beta = \pm i\omega$, where $i = \sqrt{-1}$ ...(9.6.11)

Therefore the general solution to (9.6.10) is

$$x = D_1 e^{i\omega t} + D_2 e^{-i\omega t} \qquad ...(9.6.12)$$

Where D_1 and D_2 are arbitrary constants.

Putting $D_1 = \frac{A}{2i} e^{i\phi}$

$$D_2 = -\frac{A}{2i} e^{-i\phi} = D_1^*$$

eqation (9.6.12) reduces to

$$x = \frac{A}{2i}\left[e^{i(\omega t+\phi)} - e^{-i(\omega t+\phi)}\right]$$

giving

$$x = A \sin(\omega t + \phi) \qquad ...(9.6.13)$$

The constants A and ϕ are determined from the conditions of motion. If at $t = 0$, $x = x_0$, then

$$\frac{x_0}{A} = \sin\phi$$

$$\therefore \quad \phi = \sin^{-1}\left(\frac{x_0}{A}\right) \qquad ...(9.6.14)$$

We also note that the maximum value of x is A. Hence A can be identified as the displacement amplitude, which is a measurable quantity.

We also obtain, on differentiation

$$\vartheta = \frac{dx}{dt} = A\omega \cos(\omega t + \phi)$$

$$= A\omega \sin(\omega t + \phi + \pi/2) \qquad ...(a)$$

$$a = \frac{d^2x}{dt^2} = -A\omega^2 \sin(\omega t + \phi)$$

$$= A\omega^2 \sin(\omega t + \phi + \pi) \qquad ...(b)$$

and

$$\vartheta = A\omega\sqrt{1 - \frac{x^2}{A^2}} = \omega\sqrt{A^2 - x^2} \qquad ...(9.6.15)(c)$$

From (9.6.15) it is easy to note that $\vartheta = 0$ at $x = \pm A$ and $\vartheta = \vartheta_{max} = A\omega$ at $x = 0$, the mean (equilibrium) position. We also further note that the displacement and velocity have a phase difference of $\pi/2$ (velocity leading in phase by $\pi/2$). The phase difference between velocity and acceleration is also $\pi/2$ (accl^n leading velocity).

It is now necessary to discuss the physical significance of the terms occurring in (9.6.8) or (9.6.15).

(i) Amplitude (Displacement amplitude)

The maximum value of displacement is $\pm$ A. Hence 'A' occurring in (9.6.15) or (9.6.8) denotes the (displacement) amplitude.

(ii) Phase

The argument ($\omega t + \phi$), tells about the state of the oscillating body, hence is called the phase or phase factor.

(iii) Initial Phase (Phase constant)

Since at t=0, the value of phase factor is ϕ, so 'ϕ' is called initial phase.

(iv) Time Period

The minimum time after which the oscillating body returns to its previous state is called time period (T).

Therefore $x(t) = x(t+T)$

$$\Rightarrow \quad A\sin(\omega t + \phi) = A\sin[\omega(t+T)+\phi]$$

$$= A\sin[\omega t + \phi + \omega T]$$

$$\Rightarrow \quad \omega T = 2\pi \qquad \text{(minimum finite value)}$$

$$\text{Thus } T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}} \qquad ...(9.6.16)$$

The eqn. (9.6.16) implies that 'ω' should be identified as angular frequency i.e. the rate at which angular displacement is executed. Eqn. (9.6.16) also implies that in S.H.M. the time period is independent of amplitude, but depends on the mass of the body and force constant (k).

(v) Frequency

As defined earlier, it is the number of complete vibrations executed per second and is given as

$$\nu = \frac{1}{T} = \frac{\omega}{2\pi} \qquad ...(9.6.17)$$

Ex. 9.6.1 A particle executes SHM along a straight line. If the velocities at distances 4 cm and 5cm from the mean position are 13 cm/s and 5 cm/s respectively, find the period and amplitude.

Soln.

$$\vartheta = \omega\sqrt{A^2 - x^2}$$

$$\Rightarrow \quad \vartheta_1 = 13\ \text{cm/s} = \omega\sqrt{A^2 - 4^2} \qquad ...(1)$$

$$\vartheta_2 = 5\ \text{cm/s} = \omega\sqrt{A^2 - 5^2} \qquad ...(2)$$

$$\Rightarrow \quad \frac{\vartheta_1}{\vartheta_2} = \frac{13}{5} = \sqrt{\frac{A^2 - 16}{A^2 - 25}}$$

$$\Rightarrow \quad 169(A^2 - 25) = 25(A^2 - 16)$$

$$\Rightarrow \quad 144A^2 = 169 \times 25 - 25 \times 16 = 25 \times 153$$

$$\Rightarrow \quad 12A = \sqrt{25 \times 153} = 16.846\ \text{cm}$$

$$A = 5.15\ \text{cm} \qquad ...(3)$$

Using (3) in (2) $\omega = 4$

So time period $T = 2\pi/\omega = 1.57$s

Ex. 9.6.2 A particle is executing SHM along a straight line. When the distances of the particle from mean position are x_1 and x_2 the corresponding velocities are ϑ_1 and ϑ_2. Show that the time period is $2\pi\sqrt{\frac{x_2^2 - x_1^2}{\vartheta_1^2 - \vartheta_2^2}}$ and amplitude is $\sqrt{\frac{\vartheta_1^2.x_2^2 - \vartheta_2^2.x_1^2}{\vartheta_1^2 - \vartheta_2^2}}$

Soln.

$$\vartheta = \omega\sqrt{A^2 - x^2}$$

$$\Rightarrow \quad \vartheta_1 = \omega\sqrt{A^2 - x_1^2},$$

$$\Rightarrow \quad \vartheta_2 = \omega\sqrt{A^2 - x_2^2}$$

$$\Rightarrow \quad \frac{\vartheta_1^2}{\vartheta_2^2} = \frac{A^2 - x_1^2}{A^2 - x_2^2}$$

$$\Rightarrow \quad \vartheta_1^2(A^2 - x_2^2) = \vartheta_2^2(A^2 - x_1^2)$$

$$\Rightarrow \quad A^2(\vartheta_1^2 - \vartheta_2^2) = \vartheta_1^2 x_2^2 - \vartheta_2^2 x_1^2$$

$$\Rightarrow \quad A = \sqrt{\frac{\vartheta_1^2 x_2^2 - \vartheta_2^2 x_1^2}{\vartheta_1^2 - \vartheta_2^2}}$$

Again

$$\vartheta_1^2 - \vartheta_2^2 = \omega^2(A^2 - x_1^2) - \omega^2(A^2 - x_2^2)$$

$$\Rightarrow \quad \vartheta_1^2 - \vartheta_2^2 = \omega^2(x_2^2 - x_1^2)$$

$$\Rightarrow \quad \omega = \sqrt{\frac{\vartheta_1^2 - \vartheta_2^2}{x_2^2 - x_1^2}}$$

Hence time period $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{x_2^2 - x_1^2}{\vartheta_1^2 - \vartheta_2^2}}$

Ex. 9.6.3. The maximum velocity of a body undergoing SHM is 0.04 m/s and its acceleration at 0.02 m from the mean position is 0.06 m/s^2. Find its amplitude and period of vibration.

Soln.

$$\vartheta = \omega\sqrt{A^2 - x^2}$$

$$\vartheta_{max} = \omega A = 0.04 \text{m/s}$$

$$\text{accl}^n = |a| = \omega^2 x$$

At x =0.02 m, $|a| = \omega^2 (0.02)$ m/s^2 = 0.06 m.s^{-2}

$$\Rightarrow \quad \omega^2 = 3 \Rightarrow \omega = \sqrt{3}\ s^{-1}$$

Hence $A = \frac{\vartheta_{max}}{\omega} = \frac{0.04 \text{m/s}}{\sqrt{3}\,s^{-1}} = 0.023$ m

Time period $T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{3}} = 3.627$ s.

Ex. 9.6.4 A particle moves obeying equation f + 4x = 0, where 'x' is its instantaneous displacement and 'f' instantaneous acceleration. The maximum value of x is 20 x 10^{-2} m. How much time the particle will take to move from x = 0.02m to x= 0.08 m ?

Soln.

$f = -4x \Rightarrow$ It is SHM and $\omega = 2$

Therefore $x = A \sin \omega t$ with $A = 20 \times 10^{-2}$ m

$x_1 = A \sin \omega t_1$, $x_2 = A \sin \omega t_2$

$$\Rightarrow \quad \omega(t_2 - t_1) = \sin^{-1}\left(\frac{x_2}{A}\right) - \sin^{-1}\left(\frac{x_1}{A}\right)$$

$$\Rightarrow \quad t_2 - t_1 = \frac{1}{\omega}\left[\sin^{-1}\left(\frac{x_2}{A}\right) - \sin^{-1}\left(\frac{x_1}{A}\right)\right]$$

$$= \frac{1}{2}\left[\sin^{-1}\left(\frac{0.08}{0.2}\right) - \sin^{-1}\left(\frac{.02}{0.2}\right)\right]$$

$$\Rightarrow \quad t_2 - t_1 = 0.156 \text{ s}$$

Ex. 9.6.5 A particle executes SHM of period 2.0 s. Find the time taken by the particle to cover half the amplitude from its mean position.

Soln.

$$x = A \sin \omega t = A \sin \frac{2\pi t}{T}$$

$$\Rightarrow \quad \frac{A}{2} = A \sin\frac{2\pi t_0}{T}$$

$$\Rightarrow \quad \frac{2\pi t_0}{T} = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$$

$$\Rightarrow \quad t_0 = \frac{T}{12} = \frac{2}{12}s = \frac{1}{6}s$$

But x = 0 at t= 0. So time interval = $\frac{1}{6}$ s.

Ex. 9.6.6 A particle executes SHM of period 2.0 s. Find the time taken by the particle to cover half the amplitude from the extreme position.

Soln.

$$x = A\sin\frac{2\pi t}{T}, \text{ with } T = 2.0 \text{ s}$$

$$(x)_{extreme} = A = A\sin\frac{2\pi t_1}{T}$$

$$\Rightarrow \quad \frac{2\pi t_1}{T} = \sin^{-1}(1) = \frac{\pi}{2}$$

$$\Rightarrow \quad t_1 = \frac{T}{4}$$

Now when $x = \frac{A}{2} = A\sin\frac{2\pi t_2}{T}$

$$\Rightarrow \quad \frac{2\pi t_2}{T} = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$$

$$\Rightarrow \quad t_2 = \frac{T}{12}$$

Hence time interval $t_1 - t_2 = \frac{T}{4} - \frac{T}{12} = \frac{T}{6}$

$$\Rightarrow \quad t_1 - t_2 = \frac{2}{6}s = \frac{1}{3}s$$

Ex. 9.6.7 If the maximum velocity and maximum acceleration of SHM are numerically equal, find its time period.

Soln.

$$\vartheta = \omega\sqrt{A^2 - x^2}$$

$$a = \omega^2 x$$

$\vartheta_{max} = \omega A$, $a_{max} = \omega^2 A$

Given $a_{max} = \vartheta_{max}$

$\Rightarrow \quad \omega A = \omega^2 A$

$\Rightarrow \quad \omega = 1$

time period $T = \frac{2\pi}{\omega} = 2\pi$ Sec.

Ex. 9.6.8 A particle executes SHM with frequency ν. What is the frequency with which its K.E. oscillates ?

Soln.

$$x = A \sin \omega t$$

$$\vartheta = \frac{dx}{dt} = A\omega \cos \omega t$$

$$K.E. = \frac{1}{2}m\vartheta^2 = \frac{1}{2}mA^2\omega^2\cos^2 \omega t$$

$$\frac{1}{2}mA^2\omega^2\left(\frac{1+\cos 2\omega t}{2}\right)$$

$$\Rightarrow \quad K.E. = \frac{1}{4}mA^2\omega^2(1+\cos 2wt)$$

The frequency of oscillation of K.E. is

$$\nu' = \frac{1}{T'} = \left(\frac{2\pi}{2\omega}\right)^{-1} = \frac{2\omega}{2A} = 2\nu$$

Thus K.E. oscillates with frequency 2ν.

Ex. 9.6.9 A pendulum clock keeps correct time at a place where g = 9.8 m/s^2. When the clock is moved to a mountain top it loses 3.0 s per day. Compute the value of 'g' at the mountain top.

Soln.

$$T_c = 2\pi\sqrt{\frac{\ell}{g}} = 2\pi\sqrt{\frac{\ell}{9.8}}$$

At the mountain top

$$T = 2\pi\sqrt{\frac{\ell}{g'}}$$

The time for one complete oscillation = 2s.

$$\Rightarrow \quad T_c = 2s = 2\pi\sqrt{\frac{\ell}{9.8}}$$

At the mountain top the time shown by the clock shall be

$$t_m = \frac{24\times 60\times 60}{T} \times 2s$$

Given $t_m = 86397$s

$$\therefore \frac{24\times60\times60}{2\pi\sqrt{\ell/g'}}\times2\pi\sqrt{\frac{\ell}{g}} = 86397$$

$$\Rightarrow \quad \sqrt{\frac{g'}{g}} = \frac{86397}{86400}$$

$$\Rightarrow \quad g' = \left(\frac{86397}{86400}\right)^2 g$$

$$\Rightarrow \quad g' = 9.7996597 \ m/s^2$$

Ex. 9.6.10 How much time will a seconds pendulum gain in one hour if its length decreases by 1 % in winter.

Soln.

$$T_0 = 2s = 2\pi\sqrt{\frac{\ell}{g}}$$

$$T = 2\pi\sqrt{\frac{\ell - 0.01\times\ell}{g}} = 2\pi\sqrt{\frac{\ell}{g}}\left(\sqrt{0.99}\right)$$

$$\Rightarrow \quad T = 2\times\sqrt{0.99} \ s$$

The seconds pendulum makes

$$n_c = \frac{3600}{2} = 1800 \text{ oscillation in 1 hr.}$$

So new time shown shall be

$$t = 1800 \times 2 \times \sqrt{0.99} = 3582 \ s.$$

Hence it loses 18 s. per hour.

Ex. 9.6.11 The balance wheel of a watch vibrates with an angular amplitude π radians and a period of 0.5 s. Find (a) the maximum angular speed of the wheel (b) the angular speed of the wheel when its displacement is $\pi/2$ radians (c) the angular acceleration of the wheel when its displacement is $\pi/4$ radians.

Soln.

Angular displacement $\theta = A\sin\omega t$

Angular speed

$$\Omega = \frac{d\theta}{dt} = A\omega\cos\omega t = \omega\sqrt{A^2-\theta^2}$$

Time period $T = \frac{2\pi}{\omega}$

Angular accl^n $\alpha = -\omega^2\theta$

(a) $\Omega_{max} = \omega A = \frac{2\pi}{T} = \frac{2\pi}{0.5s}.\pi \text{ rad} = \frac{2\pi^2}{0.5} \text{ rad/s}$

$$\Rightarrow \quad \Omega_{max} = 39.48 \ \text{rad/s}$$

(b) $\Omega = \frac{2\pi}{T}\sqrt{A^2-\theta^2}$

when $\theta = \pi/2$, $\Omega = \frac{2\pi}{0.5}\sqrt{\pi^2 - \left(\frac{\pi}{2}\right)^2}$

$$\Rightarrow \quad \Omega = \frac{2\pi}{0.5}\times\frac{\sqrt{3}}{2}\pi = 2\sqrt{3}\ \pi^2 \ \text{rad/s}$$

$$\Rightarrow \quad \Omega = 34.19 \ \text{rad/s}$$

(c) $\alpha = \left(\frac{2\pi}{T}\right)^2\theta$

when $\theta = \frac{\pi}{4}$, $\alpha = \left(\frac{2\pi}{0.5}\right)^2.\frac{\pi}{4} = 4\pi^3 \text{ rad/s}^2$

$$\Rightarrow \quad \alpha = 124.62 \ \text{rad/s}^2$$

Ex. 9.6.12 A pendulum clock keeps correct time at a place where $g = 9.81$ m/s^2. It is taken to a place where $g = 9.8$ m/s^2. Calculate how much time it will lose or gain per day.

Soln.

Time period is given by

$$T = 2\pi\sqrt{\frac{\ell}{g}}$$

So when 'g' decreases T increases; hence it will lose time. Let the pendulum clock lose x seconds per day. There are 856400 seconds in a day. A correct clock will make 86400 beats (vibrations) in a day. Its time period is 2 seconds. Then the pendulum at the 2nd place shall make (86400-x) vibrations. Hence

$$(86400 - x)\,T' = 2 \times 86400 \text{ sec.}$$

$$\Rightarrow \quad (86400 - x)\, 2\pi\sqrt{\frac{\ell}{g'}} = 2\pi\sqrt{\frac{\ell}{g}} \times 86400$$

$$\Rightarrow \quad 86400 - x = \sqrt{\frac{g'}{g}} \times 86400$$

$$\Rightarrow \quad x = 86400\left(1-\sqrt{\frac{g'}{g}}\right) = 86400\left(1-\sqrt{\frac{9.8}{9.81}}\right)$$

$$\Rightarrow \quad x = 44.05 \text{ s}$$

Ex. 9.6.13 A simple pendulum 2 m long is arranged in an elevator. What will be its period when (a) elevator is moving up with uniform velocity of 2 m/s (b) elevator is moving up with uniform acceleration of 2 m/s^2 (c) elevator is going down with uniform acceleration of 2 m/s^2.

Soln.

When the elevator is moving up with an acceleration 'a', its relative acceleration w.r.to ground is g+a. So

$$\text{Time period } T_1 = 2\pi\sqrt{\frac{\ell}{g+a}} \qquad ...(1)$$

when the elevator is moving down with an accl^n. an its relative accl^n. w.r.to the ground is $g-a$. Hence time period is

$$T_2 = 2\pi\sqrt{\frac{\ell}{g-a}} \qquad ...(2)$$

when it moves with uniform velocity, time period is

$$T = 2\pi\sqrt{\frac{\ell}{g}}$$

So,

(a) $T = 2\pi\sqrt{\frac{2}{9.8}} = 2.848 \text{ s}$

(b) $T_1 = 2\pi\sqrt{\frac{2}{9.8+2}} = 2.59 \text{ s}$

(c) $T_2 = 2\pi\sqrt{\frac{2}{9.8-2}} = 3.18 \text{ s}$

9.7 Graphical representation of SHM (linear)

It is instructive to study graphically the variation of displacement, velocity and acceleration (in SHM) with time. For this equations in (9.6.15) provide the necessary guideline. If one chooses initial time (t=0) when the body passes through mean position (x = 0), then $\phi = 0$, $x = A \sin \omega t$, $\vartheta = A\omega \cos\omega t = A\omega \sin\left(\omega t + \frac{\pi}{2}\right)$, $a = -A\omega^2 \sin \omega t = A\omega^2 \sin(\omega t + \pi)$. The values of x, ϑ and a for different values of time 't' are listed in table 9.1 and their variation with time is shown in fig. 9.5.

Table . 9.1

t	ωt	x	ϑ	a
0	0	0	$A\omega$	0
T/4	$\pi/2$	A	0	$-A\omega^2$
T/2	π	0	$-A\omega$	0
3T/4	$3\pi/2$	-A	0	$A\omega^2$
T	2π	0	$A\omega$	0
5T/4	$5\pi/2$	A	0	$-A\omega^2$
3T/2	3π	0	$-A\omega$	0

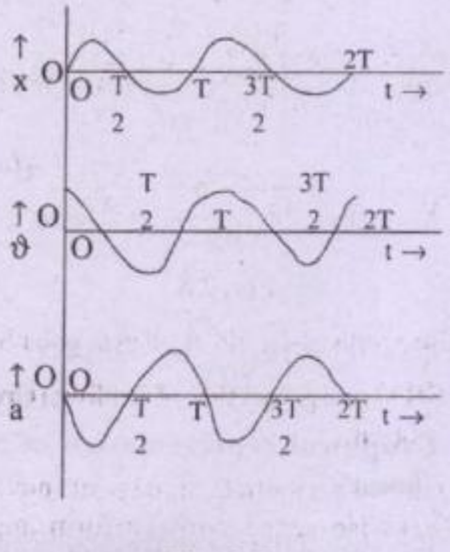


Fig. 9.5

It also follows from eqn. 9.6.15 (b) that

$$\frac{\vartheta^2}{\omega^2 A^2} + \frac{x^2}{A^2} = 1$$
...(9.7.1)

implying that the graph between speed ϑ and displacement 'x' shall be an ellipse, as shown in fig. 9.6.

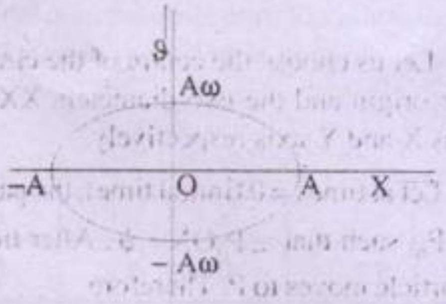


Fig. 9.6

Also eqn. (9.6.15) shows that the graph a ~ x shall be a straight line passing through the origin and with a negative slope (Fig. 9.7)

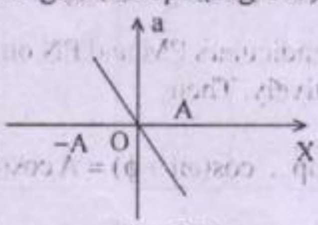


Fig. 9.7

9.8 Kinematics of SHM (angular)

As discussed in sec. 9.5 when a body executes angular SHM.

$$\vec{\alpha} = -\omega^2 \vec{\theta}$$

Since by definition

$$\text{angular velocity } \vec{\Omega} = \frac{d\vec{\theta}}{dt}$$

So proceeding in the same manner as in sec. 9.6 we obtain

$$\theta = A\sin(\omega t + \phi) \quad \text{...(a)}$$

$$\Omega = A\omega\cos(\omega t + \phi) = A\omega\sqrt{1 - \frac{\theta^2}{A^2}} \quad \text{..(b)}$$

$$\alpha = -A\omega^2 \sin(\omega t + \phi) = -A\omega^2\theta \quad \text{...(c)}$$

$$T = \frac{2\pi}{\omega}, \; v = \frac{\omega}{2\pi} \quad \text{...(9.8.1) (d)}$$

i.e. θ, Ω, α takes the place of x, ϑ and a respectively. One also draws conclusions similar to the conclusions drawn in Sec. 9.7.

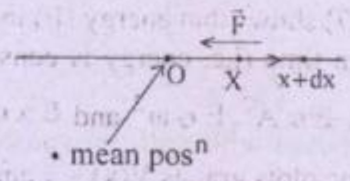
9.9. Energy in SHM

Consider a body of mass 'm', executing SHM along a straight line. The force acting on it is given by (See eqn. 9.5.7)

$$\vec{F} = -k\vec{x} \quad \text{...(9.9.1)}$$

The work done by the force during displacement from x to x+dx is

$$dW = \vec{F}.\vec{dx} = -k\vec{x}.\vec{dx} = -kx\, dx \quad \text{...(9.9.2)}$$



Therefore the work done by force $\vec{F}$ in displacing it from x = 0 to x + dx is

$$W = \int_0^x -kx \; dx = -\frac{1}{2}kx^2 \quad ...(9.9.3)$$

Let V(x) be P.E. of the body when the body is at x and V(0) be P.E. of the body when it is at mean position. As change in P.E. is equal to the negative of the work done by the force, so

$$\Delta V = -W = V(x) - V(0) = \frac{1}{2}kx^2 \quad ...(9.9.4)$$

If one measures P.E. w.r.to the P.E. at the mean position ie. choosing V(0) = 0, eqn. (9.9.4) reduces to

$$V(x) = \frac{1}{2}kx^2 = \frac{1}{2}m\omega^2 x^2 \quad ...(9.9.5)$$

Eqn. (9.9.5) gives the P.E. of the body executing SHM when at displaced position x. The kinetic energy of the body is given by

$$E_k(x) = \frac{1}{2}m\vartheta^2 = \frac{1}{2}m\omega^2(A^2 - x^2) \quad ...(9.9.6)$$

(We have used 9.6.15 to obtain r.h.s of 9.9.6)

Hence the total energy of the body executing SHM is given by

$$E = P.E. + K.E. = \frac{1}{2}m\omega^2 x^2 + \frac{1}{2}m\omega^2(A^2 - x^2)$$

$$\Rightarrow \quad E = \frac{1}{2}m\omega^2 A^2 \quad ...(9.9.7)$$

Eqn. (9.9.7) shows that energy (E) in SHM is constant in time (i.e. energy is conserved in SHM) but $E \propto A^2, E \propto \omega^2$ and $E \propto m$.

If one plots graphs $V(x) \sim x$ and $E_k \sim x$, the curves are as shown in fig. 9.8.

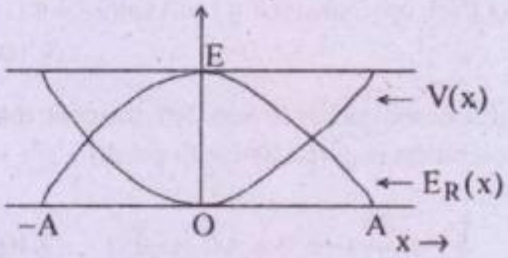


Fig. 9.8

* (These results also hold good in angular SHM)

9.10. SHM as a projection of Uniform circular motion.

Consider a particle of mass 'm' moving in anticlockwise sense with uniform angular velocity 'ω', along a circle of radius 'A'. (see fig. 9.9)

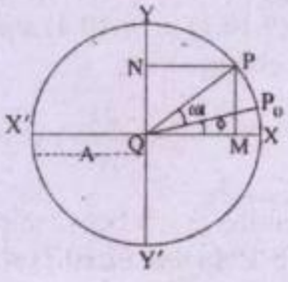


Fig. 9.9

Let us choose the centre of the circle to be the origin and the two diameters XX' and YY' as X and Y-axis respectively.

Let at time t = 0 (initial time), the particle be at P_0, such that $\angle P_0OX = \phi$. After time 't' the particle moves to P. Therefore

$$\angle POP_0 = \omega t$$

and

$$\angle POX = \angle POP_0 + \angle P_0OX = \omega t + \phi$$

Draw perpendiculars PM and PN on X and Y-axis respectively. Then

$$x = \overline{OM} = \overline{op} \; . \; \cos(\omega t + \phi) = A\cos(\omega t + \phi) \quad ...(9.10.1)$$

$y = \overline{ON} = \overline{op} \cdot \sin(\omega t + \phi) = A\sin(\omega t + \phi)$ ...(9.10.2)

As discussed earlier in Sec. 5.6, the centripetal acceleration is given by (with r = A)

$$\bar{a}_c = \frac{\vartheta^2}{A}(-\hat{e}_r) = A\omega^2(-\hat{e}_r) \quad ...(9.10.3)$$

Resolving $\bar{a}_c$ along X and Y-axis

$$a_{cx} = -a_c\cos(\omega t + \phi) = -A\omega^2\cos(\omega t + \phi) \quad ...(9.10.4)$$

$$a_{cy} = -a_c\sin(\omega t + \phi) = -A\omega^2\sin(\omega t + \phi) \quad ...(9.10.5)$$

Using eqn. (9.10.1) in (9.10.4) and (9.10.2) in (9.10.5) we obtain

$$a_{cx} = -\omega^2 x \quad ...(9.10.6)$$

$$a_{cy} = -\omega^2 y \quad ...(9.10.7)$$

Equations (9.10.6) and (9.10.7) show that the projections of uniform circular motion on a diameter of the circle is simple harmonic and the displacement is given by

$x = A\cos(\omega t + \phi)$.

OR

$y = A\sin(\omega t + \phi)$

These yield velocity ϑ as

$\vartheta = \omega\sqrt{A^2 - x^2}$

or $\vartheta = \omega\sqrt{A^2 - y^2}$

9.11. Examples of SHM

SHM occurs in a large number of physical situations. We shall consider few such examples.

(a) Simple Pendulum :

A simple pendulum consists of a heavy point mass, suspended from a rigid support by an in-extensible, weightless string and capable of oscillating in a vertical plane about a horizontal axis.

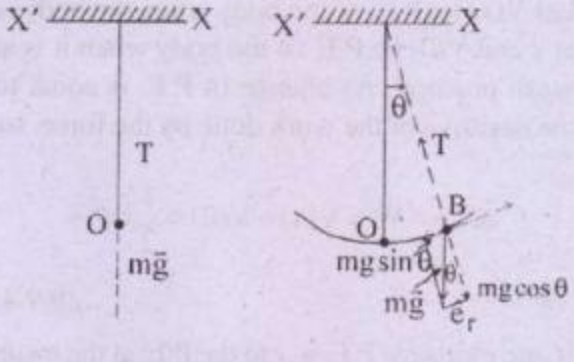


Fig. 9.10

Figure 9.10 shows a simple pendulum in which a small bob of mass 'm' is suspended from a rigid support XX' by an in-extensible string. 'O' is the equilibrium position of the bob. The net force at equilibrium position is zero. The bob is allowed to oscillate in the vertical plane (XZ-plane).

Let at any time 't', the bob be at B. The forces acting on the bob are

i) The tension $\bar{T}$ acting along the string towards S; with $\bar{T} = T(-\hat{e}_r)$

ii) The weight $m\bar{g}$ acting vertically downward with $m\bar{g} = -mg\ \hat{z}$

Therefore the net force acting on the bob is

$$\bar{F} = \bar{T} + m\bar{g} \quad ...(911.1)$$

Resolving $m\bar{g}$ along radial and tangential direction (see fig. 9.10 (b)) we express

$$m\bar{g} = mg\cos\theta\hat{e}_r - mg\sin\theta\hat{e}_t \quad ...(9.11.2)$$

We then use (9.11.2) in (9.11.1) and obtain

$$\bar{F} = (T - mg\cos\theta)(-\hat{e}_r) - mg\sin\theta\hat{e}_t$$

$$= F_r(-\hat{e}_r) + F_t\hat{e}_t \quad ...(9.11.3)$$

The force component $F_r(-\hat{e}_r)$ provides the necessary centripetal force to the bob to move in a vertical circle of radius 'ℓ'. The velocity of the bob in the vertical circle is $\bar{\vartheta} = r\omega\,\hat{e}_t = \ell\omega\,\hat{e}_t$.

As it is shown below $\omega = \sqrt{\frac{g}{\ell}}$, so $\bar{\vartheta} = \sqrt{g\ell}\;\hat{e}_t$,

which ensures $\vartheta = \sqrt{g\ell} < \sqrt{2g\ell}$, the necessary condition for the motion to be oscillatory (see eqn. 5.6.54). The tangential force component is

$$\vec{F}_t = F_t\hat{e}_t = -mg\,\sin\theta\;\hat{e}_t$$

$$\Rightarrow \quad F_t = -mg\,\sin\theta = ma_t \qquad ...(9.11.4)$$

where 'a_t' is the tangential acceleration given by eqn. (5.4.7) as

$$a_t = \ell\frac{d\Omega}{dt} = \ell\frac{d^2\theta}{dt^2} \qquad ...(9.11.5)$$

Using eqn.(9.11.5) in eqn. (9.11.4) we obtain

$$m\ell\frac{d^2\theta}{dt^2} + mg\,\sin\theta = 0$$

$$\Rightarrow \quad \frac{d^2\theta}{dt^2} + \frac{g}{\ell}\sin\theta = 0 \qquad ...(9.11.6)$$

When the oscillation are small, so that $\sin\theta = \theta$, eqn. (9.11.6) reduces to

$$\frac{d^2\theta}{dt^2} + \frac{g}{\ell}\theta = 0 \qquad ...(9.11.7)$$

Since $\alpha = \frac{d^2\theta}{dt^2} = -\frac{c}{I}\theta = -\omega^2\theta$ (see 9.5.14)

So, $\sqrt{\frac{g}{\ell}} = \omega$ = angular frequency ...(9.11.8)

equation (9.11.9) reduces to the form

$$\frac{d^2\theta}{dt^2} + \omega^2\theta = 0 \qquad ...(9.11.9)$$

and this is the differential equation to angular SHM. Its solution is

$$\theta = A\sin(\omega t + \phi) \qquad ...(9.11.10)$$

where 'ϕ' is the initial phase and 'A' is the displacement amplitude. The time period 'T' is given by

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{\ell}{g}} \qquad ...(9.11.11)$$

and frequency

$$\upsilon = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{g}{\ell}} \qquad ...(9.11.12)$$

Conclusions (Laws of Simple pendulum)

i) Time period 'T' is independent of amplitude of oscillation, so long as length is constant. (law of isochronism)

ii) $T \,\alpha\, \sqrt{\ell}$ (law of length)

iii) $T \,\alpha\, \frac{1}{\sqrt{g}}$ (law of acceleration)

iv) 'T' is independent of mass of the bob.

Seconds Pendulum :

A simple pendulum whose time period is two seconds is called a seconds pendulum. So for a seconds pendulum

$$2 = 2\pi\sqrt{\frac{\ell_s}{g}}$$

$$\Rightarrow \quad \ell_s = \frac{g}{\pi^2} \qquad ...(9.11.13)$$

Eqn. (9.11.13) gives the length of a seconds pendulum.

(b) **Massless loaded spring**

Consider a **massless** spring of length L, suspended from a rigid support.

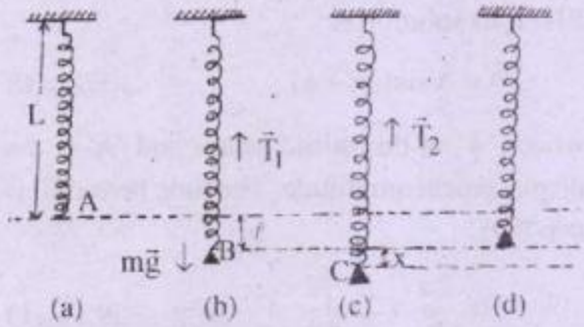


Fig. 9.11

Let a weight $m\vec{g}$ be loaded at the lower (free) end of the spring as a result of which the spring gets extended by a length 'ℓ'. At this position (fig. 9.11(b)) a tension $\vec{T_1}$ acts upwards along the spring. This tension T_1 is proportional to the extension i.e.

$$T_1 \alpha \ell$$

$$\Rightarrow \quad T_1 = k\ell \qquad ...(9.11.14)$$

where 'k' is the spring constant. For equilibrium

$$\vec{T_1} + m\vec{g} = 0 = (T_1 - mg)\hat{j}$$

$$\Rightarrow \quad T_1 = mg \qquad ...(9.11.15)$$

So eqns (9.11.14) and (9.11.15) give

$$mg = k\ell$$

If now the spring is further pulled down (applying a force $\vec{f}$) by a distance x (see fig. 9.11(c)), the new tension 'T_2' developed along the spring is given by

$$T_2 = k(\ell + x) \qquad ...(9.11.17)$$

Since $T_2 > T_1$ (=mg), so on removal of the force $\vec{f}$, the spring tends to move back to position B, which is its equilibrium position. On release, the force acting on the spring in the position C is

$$\vec{F} = \vec{T_2} + m\vec{g} = (T_2 - mg)\hat{j}$$

$$= (T_2 - k\ell)\hat{j} = kx\hat{j} \quad \text{(Using 9.11.16)}$$

$$\Rightarrow \quad \vec{F} = -k\vec{x} \quad (\because \vec{x} = -x\hat{j}) \qquad ...(9.11.18)$$

Equation (9.11.18) shows that force is directly proportional to displacement and oppositely directed. Hence the motion shall be SHM. Using (9.11.16) in r.h.s

$$F = -kx = -\frac{mg}{\ell}x$$

$$\Rightarrow \quad a = -\frac{g}{\ell}x \qquad ...(9.11.19)$$

This shows that time period of oscillation shall be

$$T = 2\pi\sqrt{\frac{\ell}{g}} = 2\pi\sqrt{\frac{m}{k}} \qquad ...(9.11.20)$$

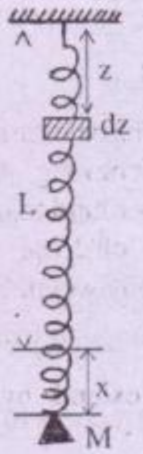
However it should be noted that 'ℓ' is the elongation of the spring by the load.

(c) **Massive loaded spring**

Consider an element dz at a distance z from the fixed end. If the free end of spring is depressed through 'x' then the element dz at z, is depressed by $\frac{Z}{L}$ x. So velocity of the element dz, shall be

$$\vartheta(z) = \frac{Z}{L}.\frac{dx}{dt}$$

Let 'λ' be mass per unit length of spring. Hence K.E. of the system shall be



$$E_k = \frac{1}{2}M\left(\frac{dx}{dt}\right)^2 + \int_0^L \frac{1}{2}\lambda dz.\left(\frac{Z}{L}.\frac{dx}{dt}\right)^2$$

$$= \frac{1}{2}M\left(\frac{dx}{dt}\right)^2 + \frac{1}{2}\lambda.\frac{L^3}{3L^2}\left(\frac{dx}{dt}\right)^2$$

$$E_k = \frac{1}{2}(M + \frac{m}{3})\left(\frac{dx}{dt}\right)^2 \qquad ...(9.11.21)$$

P.E. of the spring is $V = \frac{1}{2}kx^2$

Therefore total energy of the spring while oscillating is

$$E = \frac{1}{2}\left(m + \frac{m}{3}\right)\left(\frac{dx}{dt}\right)^2 + \frac{1}{2}kx^2 \qquad ...(9.11.22)$$

Since energy is to the conserved, so

$$\frac{dE}{dt} = 0 = \left(M + \frac{m}{3}\right)\frac{d^2x}{dt^2} + kx\frac{dx}{dt}$$

$$\Rightarrow \left(M + \frac{m}{3}\right)\frac{d^2x}{dt^2} + kx = 0$$

$$\Rightarrow \frac{d^2x}{dt^2} = -\frac{k}{(M + \frac{m}{3})}x \qquad ...(9.11.23)$$

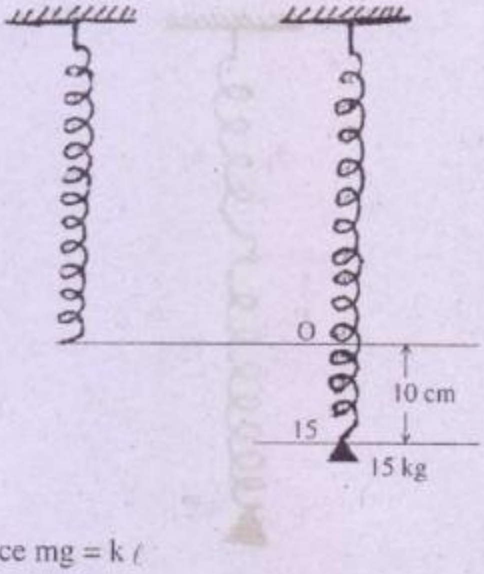
This indicates time period of oscillation is

$$T = 2\pi\sqrt{\frac{M + \frac{m}{3}}{k}} \qquad ...(9.11.24)$$

Ex. 9.11.1 A spring balance scale, reading from 0 to 15 kg is 10 cm long. A body suspended from the spring is found to oscillate vertically with a frequency of 2 osc. per second. How much does the body weigh ?

Soln.

The spring extends by 10 cm for a load of 15 kg.



Hence $mg = k\ell$

$\Rightarrow \quad 15 \times 9.8 = k \times 0.1$

$\Rightarrow \quad k = 15 \times 98$

Time period

$$T = 2\pi\sqrt{\frac{m}{k}} = \frac{1}{2} \text{ (Given)}$$

$$\Rightarrow \sqrt{\frac{m}{k}} = \frac{1}{4\pi}$$

$$m = \frac{k}{16\pi^2} = \frac{15 \times 98}{16\pi^2} = 9.31 \text{ kg.}$$

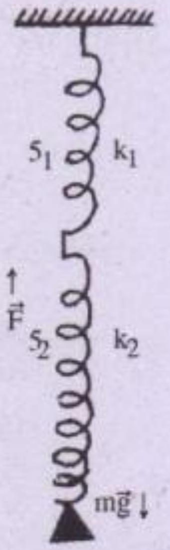
Ex. 9.11.2 Two light springs of force constants k_1 and k_2 are joined together and the combination is attached to a mass 'm' at one end and the other end is clamped to a fixed support. Show that the frequency of oscillation is

$$\upsilon = \frac{1}{2\pi}\sqrt{\frac{k_1k_2}{(k_1 + k_2)m}}$$

Soln.

$$\vec{F}_1 = \vec{F}_2 = \vec{F} = -k_1\vec{x}_1 = -k_2\vec{x}_2$$

where x_1 and x_2 are the extensions of spring S_1 and S_2 respectively.



Now
$$k_1\vec F = -k_1k_2\vec x_2$$
$$k_2\vec F = -k_1k_2\vec x_1$$
$$\Rightarrow \quad (k_1+k_2)\vec F = -k_1k_2(\vec x_1+\vec x_2)$$
$$\Rightarrow \quad \vec F = -\frac{k_1k_2}{k_1+k_2}(\vec x_1+\vec x_2)$$

The load mg is depressed by $\vec x = \vec x_1 + \vec x_2$ and

$\vec F = -mg$

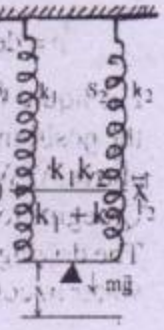
So $\vec F = -m\vec g = -\frac{k_1k_2}{k_1+k_2}\vec x$

when it is further depressed by a force $\vec f$ through a distance $\vec y$, we find that the restoring force developed is $\vec F'$ given by

$$\vec F' = -\frac{k_1k_2}{k_1+k_2}(\vec x+\vec y)$$

On release, the force acting on the system shall be

$$\vec F'' = \vec F' + m\vec g = -\frac{k_1k_2}{k_1+k_2}(\vec x+\vec y) + \frac{k_1k_2}{k_1+k_2}\vec x$$



$$\Rightarrow \quad = -\frac{k_1k_2}{k_1+k_2}\bar y = -k_e\bar y$$

Thus the effective spring constant is k_e

$$k_e = \frac{k_1k_2}{k_1+k_2}$$

Time period

$$T = 2\pi\sqrt{\frac{m}{k_e}}$$

frequency $\quad \upsilon = 2\pi\sqrt{\frac{k_e}{m}} = 2\pi\sqrt{\frac{k_1k_2}{(k_1+k_2)m}}$

<u>**Ex. 9.11.3**</u> Two light springs of Ex. 9.11.2 are connected in parallel as shown below. Find the frequency of oscillation.

<u>**Soln.**</u>

$$\vec F_1 = -k_1\vec x$$
$$\vec F_2 = -k_2\vec x$$
$$m\vec g + \vec F_1 + \vec F_2 = 0$$
$$\Rightarrow \quad m\vec g = (k_1+k_2)\vec x \qquad ...(1)$$

If the load is further depressed by 'y', by applying a force $\vec f$, then

$$\vec F_1' = -k_1(\vec x+\vec y) \qquad ...(2)$$
$$\vec F_2' = -k_2(\vec x+\vec y) \qquad ...(3)$$

On releasing, the net force acting on the system is

$$\vec F' = m\vec g + \vec F_1' + \vec F_2' = m\vec g - k_1(\vec x+\vec y) - k_2(\vec x+\vec y)$$

Using (1)

$$\vec F' = -(k_1+k_2)\ \vec y = -k_e\ \vec y \qquad ...(4)$$

It implies motion is SHM and time period

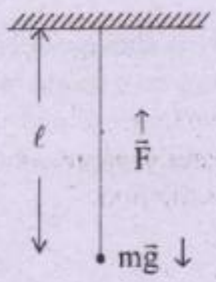
$$T = 2\pi\sqrt{\frac{m}{k_e}} = 2\pi\sqrt{\frac{m}{k_1 + k_2}}$$

and frequency

$$\upsilon = \frac{1}{2\pi}\sqrt{\frac{k_1 + k_2}{m}} \qquad ...(5)$$

Ex. 9.11.4 A point mass 'm' is suspended at the end of a massless wire of length 'ℓ' and cross-section A. If Y is the young's modulus for the wire, obtain the frequency of oscillation for the SHM along the vertical line.

Soln.



Deforming force = $m\vec{g}$

Elastic restoring force = $\vec{F} = -m\vec{g}$

Let elongation be = x

Then $Y = \dfrac{mg/A}{x/\ell}$

$$\Rightarrow \quad \frac{x}{\ell} = \frac{mg}{AY}$$

$$\Rightarrow \quad m\vec{g} = \frac{YA}{\ell}\vec{x} \qquad ...(1)$$

$$\text{and } \vec{F} = -m\vec{g} = -\frac{YA}{\ell}\vec{x} \qquad ...(2)$$

If the wire is further depressed through y by applying extra force $\vec{f}$, then restoring force developed is

$$\vec{F}' = -\frac{YA}{\ell}(\vec{x} + \vec{y}) \qquad ...(3)$$

On releasing, the net force acting on the mass is

$$\vec{F}_N = m\vec{g} + \vec{F}' = +\frac{YA}{\ell}\vec{x} - \frac{YA}{\ell}(\vec{x} + \vec{y})$$

$$\Rightarrow \quad \vec{F}_N = -\frac{YA}{\ell}\vec{y} \qquad ...(4)$$

Hence the motion is SHM and time period is

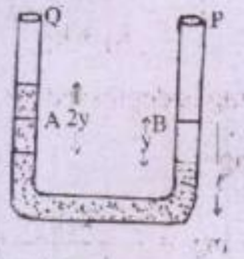
$$T = 2\pi\sqrt{\frac{m}{YA/\ell}} = 2\pi\sqrt{\frac{m\ell}{YA}} \qquad ...(5)$$

and frequency

$$\upsilon = \frac{1}{2\pi}\sqrt{\frac{YA}{m\ell}} \qquad ...(6)$$

Ex. 9.11.5 Find the expression for the time period of oscillation of liquid in a U-tube, if the liquid in one column is depressed and then released.

Soln.



Let A and B be the equilibrium positions of liquid in the two limbs of U-tube.

Mass of the liquid in the two limbs is

$$m = (S \times 2\ell)\rho \qquad ...(1)$$

where S = area of cross-section

ℓ = length of liquid column

ρ = density of liquid

Let liquid in limb P be depressed by 'y' from the position B. Then liquid in limb Q rises by y above A. When released, the liquid in column Q begins to fall while that in P, begins to rise. The driving force is due to the weight of excess water in column Q of height 2y.

Hence driving force

$$\vec{F} = -(S.2y.\rho\ g)\hat{j} \qquad ...(2)$$

This force acts on mass 'm' of water. Hence acceleration $\vec{a} = \frac{\vec{F}}{m} = \frac{-(s.2y.p.g)\hat{j}}{s.2\ell.p} = -\frac{g}{\ell}y\,\hat{j}$

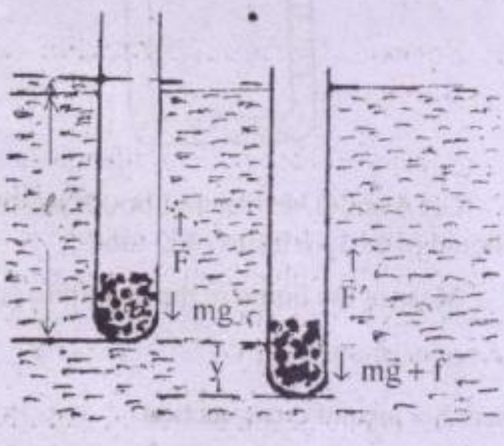
$$\Rightarrow \quad \vec{a} = -\frac{g}{\ell}\vec{y} \qquad ...(3)$$

Hence motion is SHM and time period T is

$$T = 2\pi\sqrt{\frac{\ell}{g}} \qquad ...(4)$$

Ex. 9.11.6 A weighted glass cylinder is floating in a liquid with 'ℓ' of its length immersed. It is pushed down some distance and then released. Compute the time period of vibration.

Soln.



For equilibrium

$$s\ell\rho g\ \hat{j} - mg\ \hat{j} = 0 \qquad ...(1)$$

$$\Rightarrow s\ell\rho g = mg$$

$$\ell = \frac{m}{s\rho}$$

When the cylinder is pushed down by 'y', by applying extra force $\vec{f}$, we have net force acting

$$mg\ \hat{j} + \vec{f} = S(\ell + y)\rho g\ \hat{j} = \vec{F}' \qquad ...(2)$$

On release the net force acting upward is

$$F_N\ \hat{j} = \vec{F}_N = S(\ell + y)\rho g\ \hat{j} - mg\ \hat{j}$$

Using (1) on r.h.s

$$\vec{F}_N = F_N\ \hat{j} = +S\ \rho g y\ \hat{j} = -\ S\ \rho g\vec{y} \quad ...(3)$$

Then the motion is SHM and time period is

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{m}{s\rho g}} = 2\pi\sqrt{\frac{\ell}{g}} \qquad ...(4)$$

Ex. 9.11.7 Two bodies are alternately hung on the same spring to vibrate with frequency $\frac{1}{3}$ Hz and $\frac{2}{3}$ Hz. respectively. Find the ratio of their masses.

Soln.

$$T_1 = 2\pi\sqrt{\frac{m_1}{k}} \quad \Rightarrow \upsilon_1 = \frac{1}{2\pi}\sqrt{\frac{k}{m_1}}$$

$$T_2 = 2\pi\sqrt{\frac{m_2}{k}} \quad \Rightarrow \upsilon_2 = \frac{1}{2\pi}\sqrt{\frac{k}{m_2}}$$

$$\Rightarrow \frac{\upsilon_1}{\upsilon_2} = \sqrt{\frac{m_2}{m_1}}$$

$$\Rightarrow \frac{1/3}{2/3} = \sqrt{\frac{m_2}{m_1}} = \frac{1}{2}$$

$$\therefore \frac{m_2}{m_1} = \frac{1}{4}$$

$$\Rightarrow m_1 : m_2 = 4 : 1$$

9.12 Damped Harmonic Motion

We have discussed earlier that when a body is displaced from its equilibrium position and released, so that a force proportional to its displacement and always directed towards the equilibrium position acts on it, the body executes simple harmonic oscillations with a frequency determined by the characteristics of the system (e.g. in case of simple pendulum it depends on length of simple pendulum, in case of loaded spring it depends on extension etc.) Such oscillations are called **free oscillations** and the frequency is called natural frequency. In such oscillations (as discussed earlier) there is no dissipation of energy.

But in actual practice, dissipative forces do operate and the amplitude of vibration gradually decreases (e.g. in the laboratory, the amplitude of simple pendulum gradually decreases, amplitude of oscillations of a spring gradually decreases). Such oscillations, where amplitude gradually decreases, due to presence of dissipative (dampting) forces are called damped oscillations (vibrations). The damping force is a function of speed of the moving system and is directed opposite to the velocity. The damping force may be a complicated function of speed. In several cases of practical interest the damping force is directly proportional to speed ($\vec{F}_{\alpha} = -b\vec{\vartheta}$). In such cases the equation of motion becomes

$$m\frac{d\vartheta}{dt} = -kx - b\vartheta$$

$$\Rightarrow \quad m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0 \qquad ...(9.12.1)$$

Detail analysis of eqn. (9.12.1) (beyond the scope of the text) reveals that (i) if the damping is small, there is still oscillatory motion (fig. 9.13 (a)) but with decreasing amplitude and a frequency different from the natural frequency.

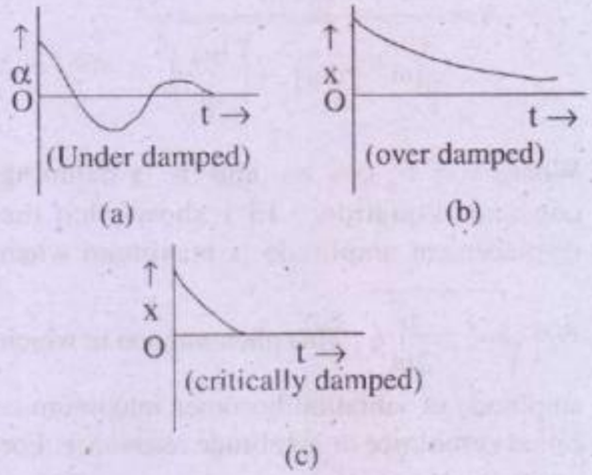


Fig. 9.13

On the otherhand if the damping is very large then no oscillation takes place, but it slowly returns to its equilibrium position (fig. 9.13. (b)). There exists a particular value of damping for which the body returns to its equilibrium position quickly without executing oscillations (see fig. 9.134.(c)).

9.13 Forced Harmonic Motion and Resonance

A damped harmonic oscillation can be maintained if the energy loss is compensated by feeding energy from an external source. This is achieved by subjecting the system to a periodic force, in the direction of motion of the damped harmonic oscillator.

The phenomenon of vibration of a body under the influence of a periodic external force whose frequency (ω) is different from the natural frequency (ω_0) of the body is called forced vibration. The system ultimately oscillates with the frequency of the periodic external force with a constant amplitude (A) given as

$$A = \frac{F_0 / m}{\sqrt{\left(\omega^2 - \omega_0^2\right)^2 + \left(\frac{b\omega}{m}\right)^2}} \quad ...(9.13.1)$$

Where $F = F_0 \cos wt$, and 'b' is damping constant. Equation 9.13.1 shows that the displacement amplitude is maximum when $\omega = \sqrt{\omega_0^2 - \frac{b^2}{2m^2}}$. This phenomenon in which amplitude of vibration becomes maximum is called **resonance** or amplitude resonance. For small damping when $\omega = \omega_0$, resonance occurs.

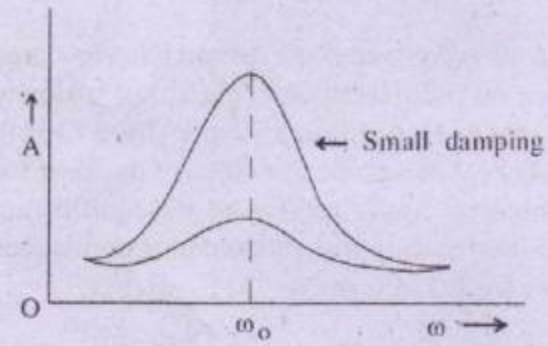


Fig.9.14

This phenomenon can be demonstrated by considering a number of simple pendulums suspended from the same rigid support as shown in fig. 9.15

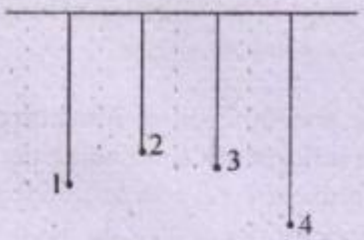


Fig. 9.15

If pendulum 1 is set into vibration, then it is seen that pendulum 3 vibrates in resonance (as frequency of 1 & 3 are same) while 2 and 4 vibrate in forced vibration.

Summary

1. A motion which repeats itself after a regular interval of time is called a periodic motion. The time interval after which the motion is repeated is called its time period.

2. The periodic motion in which the body moves to and fro on the same path about a fixed point is called oscillatory motion or oscillation.

3. Simple Harmonic motion (SHM) is a special type of oscillation in which the particle oscillates on a straight line such that the acceleration of the particle is always directed towards a mean position and its magnitude is propertional to the displacement of the particle from the mean position.

$$a = -\omega^2 x$$

where $\omega = \frac{2\pi}{T} = 2\pi f$ is the angular frequency.

4. The force on the particle executing SHM at any instant is $F = -kx$

K is the force constant and x is the displacement at that instant. $K = mw^2$

5. a) Displacement of the particle in SHM at any instant t is

$$x = A \sin(\omega t + \phi)$$

where A is the amplitude and is the phase constant.

b) Velocity of the particle at any instant is

$$V = \frac{dx}{dt} = A\omega \sin(\omega t + \varphi + \pi/2)$$

or $V = \omega . \sqrt{A^2 - x^2}$

c) Acceleration at any instant

$$a = \frac{d\upsilon}{dt} = -\omega^2 x$$

(6) At the mean position x=0 and hence

$V_{max} = A\omega$

$a_{min} = 0$

At the extreme position $x = \pm A$. Hence

$V_{min} = 0$

and $a_{max} = \pm\ \omega^2 A$

7. The kinetic energy $E_k(x)$, the potential energy $V(x)$ and total mechanical energy E of a particle in SHM at any instant is

$$E_k(x) = \frac{1}{2} mw^2 (A^2 - x^2)$$

$$V(x) = \frac{1}{2} mw^2 x^2$$

$$E = E_k(x) + V(x) = \frac{1}{2} mw^2 A^2$$

We note that total mechanical energy of a particle in SHM is a constant while K.E. and P.E. depend on the position of the particle.

8. Time period (T) of a particle in SHM is

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{x}{a}}$$

where x is the displacement and a is the acceleration of the particle at any instant.

9. **Simple Pendulum**

A simple pendulum is a heavy point mass suspended from a rigid support by an in extensible, weightless and flexible thread.

Time period (T) of a simple pendulum

$$T = 2\pi\sqrt{\frac{l}{g}}$$

l is the effective length of the pendulum.

10. A simple pendulum whose time period is 2 sec. is called a seconds pendulum.

11. In case of a mass less loaded spring, time period

$$T = 2\pi\sqrt{\frac{m}{k}}$$

Where m is the mass of the load and K is the spring constant.

12. (i) When a body is displaced from its equilibrium position and left to itself to oscillate, it is called free oscillation and the frequency with which the body oscillates is called its natural frequency.

(ii) The oscillation where amplitude gradually decreases is called damped oscillation.

Every natural free oscillation is damped due to the presence of dissipative forces.

(iii) The oscillation of a body under the influence of a periodic force, whose frequency (ω) is different from the natural frequency (ω_0) of the body is called forced oscillation.

(iv) Resonance is a special case of forced oscillation in which frequency of the external periodic force coincides with the natural frequency of the body.

Physical quantity	Dimensional formula	SI unit
1. Force constant or spring constant (k)	$[MT^{-2}]$	NM^{-1}
2. Frequency (f) and angular frequency(ω)	$[T^{-1}]$	Hz (or s^{-1})

MODEL QUESTIONS

A. Multiple Choice Type Questions :

1. The displacement of an oscillator is given by $x = 5 \sin(10\pi t+\pi)$. The phase of the particle at t = 1 second is

   (i) π (ii) 5π

   (iii) 10π (iv) 11π

2. In SHM, the acceleration of a particle at its mean position is

   (i) zero

   (ii) maximum

   (iii) between zero and maximum

   (iv) unpredictable

3. The equation of a SHM with amplitude 5 cm and period 0.5 s is

   (i) $y = 5 \sin 2\pi (t-0.5)$

   (ii) $y = 0.5 \sin (2/5\pi)\ t$

   (iii) $y = 5 \sin 2\pi t$

   (iv) $y = 5 \sin 4\pi t$

4. What is the ratio of P.E. & K.E. of a body in SHM at a point where displacement is 1/3 rd of its amplitude

   (i) 8 : 1 (ii) $1 : 2\sqrt{2}$

   (iii) 1 : 9 (iv) 1 : 8

5. Which of the two sets of quantities below are related to each other ?

   (i) velocity & phase

   (ii) velocity & amplitude

   (iii) phase & amplitude

   (iv) frequency & time period

6. A particle executes SHM with a frequency υ. Its K.E. oscillates with a frequency

   (i) $\upsilon/2$ (ii) υ

   (iii) 2υ (iv) 4υ

7. How will the period of oscillation of a simple pendulum be affected if it is moved from the surface of earth to mine

   (i) It will increase

   (ii) decrease

   (iii) no change

   (iv) becomes zero

8. A simple pendulum of period T has a metal bob which is negatively charged. If it is allowed to oscillate above a positively charged metal plate, its period will

   (i) remain equal to T

   (ii) become less than T

   (iii) become greater than T

   (iv) become infinite

9. Two body-spring systems oscillate at frequencies n_1 and n_2 such that $n_1 = 2n_2$. If the force constants of the two springs are same, then masses m_1 and m_2 of the bodies are in the ratio

   (i) 1 : 2 (ii) 1 : 4

   (iii) 1 : 8 (iv) 1 : 16

10. A simple pendulum with a bob of mass m swings with an angular amplitude of 40^0. When its angular displacement is 20^0, the tension in the string is

    (i) mg

    (ii) $mg \cos 20^0$

    (iii) more than $mg \cos 20^0$

    (iv) less than $mg \cos 20^0$

11. A body is executing SHM of amplitude 1 cm. Its velocity while passing through the mean position is 10 cm s^{-1}. Its angular frequency is

    (i) 100 units (ii) $50/\pi$ units

    (iii) $40/3\pi$ units (iv) 10 units

12. A particle moves on X-axis according to equation $x = A+B \sin \omega t$. The motion is SHM with amplitude

    (i) A (ii) B

    (iii) A+B (iv) $\sqrt{A^2+B^2}$

13. The displacement of a particle is given by $\vec{r} = A(\hat{i}\cos\omega t + \hat{j}\sin\omega t)$. The motion of the particle is

    (i) simple harmonic
    (ii) on a straight line
    (iii) on a circle
    (iv) with constant acceleration

14. The motion a particle is given by $x = A \sin \omega t + B \cos \omega t$. The motion of the particle is

    (i) not SHM
    (ii) SHM with amplitude A+B
    (iii) SHM with amplitude (A+B) / 2
    (iv) SHM with amplitude $\sqrt{A^2+B^2}$

15. The distance moved by a particle in SHM in one time period is

    (i) A (ii) 2 A

    (iii) 4 A (iv) zero

B. Very Short Answer Type Questions :

1. Define amplitude.
2. In case of an oscillating simple pendulum what is the work done by tension in the string ?
3. By what factor the time period of a simple pendulum will change if its length is halved ?
4. Write the expression for velcoity of a particle executing SHM, at any instant of time.
5. What do you mean by 'phase' of a particle ?
6. Will the acceleration of a body executing linear SHM be zero anywhere in its path ?
7. How will the time shown by pendulum clock change when it is taken to the bottom of a mine ?
8. What is the relation between acceleration and displacement in SHM ?
9. What is the acceleration of a body executing SHM, when its velocity is maximum ?
10. How phase of a particle is measured in terms of time period ?
11. Give positions of maximum velocity and maximum acceleration .
12. The equation of SHM is given as $y = 10 \sin(3t + \pi)$. What is its time period ?
13. Does the time period in SHM depend on displacement ?
14. Write the expression for total energy in SHM.
15. Can simple pendulum experiment be performed in satellite ?
16. Write the expression for the time period of a massless loaded spring.

C. Short Answer Type Questions :

1. At what position the K.E. and P.E. of a simple pendulum same ?
2. Calculate the percentage of K.E. and P.E. when displacement is one half of amplitude.
3. Calculate the length of a seconds pendulum.
4. Calculate the force constant of a spring which is stretched by 0.1 m, when a mass of 0.5 kg is hung from it.
5. A body executing SHM with amplitude of 2 cm makes $30/\pi$ vibrations per minute. What is the maximum velocity of the body during the motion ?

6. A particle executing SHM has period 20 s, and amplitude 10 cm. Calculate its maximum acceleration and maximum velocity.

7. Define time period and frequency. Write relation between them.

8. Length of a seconds pendulum is decreased by 1%, calculate the gain or loss in time per day.

9. A spring has a force constant k, and a mass M is suspended from it. The spring is cut into half and the same mass is suspended from one of the halves. Is the frequency of vibration the same before and after the spring is cut ? How are the frequency related ?

10. How does the mass of a spring affect the time period of oscillation of a loaded vertical spring ?

11. When the motion of a spring is simple harmonic ?

12. On what factors does the period of simple pendulum depend ?

13. One clock is based on an oscillatory spring, the other on a pendulum. Both are taken to Mars. Will they keep the same time there that they keep on earth ? Will they agree with each other ? Explain Mars has mass 0.1 times that of earth and radius half as great.

14. At what point in the motion of a simple pendulum is the string tension greatest ? Least ?

15. A pendulum is mounted on an elevator that accelerates upward with constant acceleration. Does the period increase, decrease or remain same ?

16. If a pendulum clock is taken to a mountain top. Does it gain or lose time ? Explain.

17. When the displacement of SHO is half as its amplitude, what fraction of the total energy is the kinetic energy ?

18. Show that both K.E. and P.E. of a particle executing SHM with frequency υ oscillate with frequency 2υ .

Conceptual :

1. A hollow sphere filled with water is suspended by a thread and is made to oscillate. If water begins to leak out from the bottom of the sphere, how is the time period affected ? Not affected

2. A person goes to bed at sharp 10.00 pm everyday. Is it an example of periodic motion ? If yes, what is the time period ? If no, why ? Yes

3. A particle executing SHM comes to rest at the extreme positions. Is the resultant force on the particle zero at these positions according to Newton's first law ? No accl is ......

4. A small creature moves with constant speed in a vertical circle on a bright day. Does its shadow formed by the sun on a horizontal plane move in a simple Harmonic motion? Yes

5. Can the P.E. in SHM be negative ? Will it be so if we choose zero potential energy at some point other than mean position ?

6. The energy of a system in SHM is given by $E = 1/2\ m\ \omega^2 A^2$. Which of the following two statements is more appropriate ?

   (i) The energy is increased because the amplitude is increased.

   (ii) The amplitude is increased because the energy is increased.

7. The force acting on a particle moving along X-axis is $F = -K(x - \vartheta_0 t)$, where K is a positive constant. An observer moving at a constant velocity ϑ_0 along the X-axis looks at the particle. What kind of motio does he find for the particle ?

8. A particle moves on X-axis according to the equation $x = x_0 \sin^2 \omega t$. Is the motion simple harmonic ? If yes what is its time period ?

D. Numerical Problems :

1. Determine the maximum velocity and maximum acceleration of a S.H.O. having frequency 20 Hz. and amplitude 2 mm.

2. Maximum velocity and maximum acceleration of a S.H.O. are 20 cm/s and 2 m/s^2 respectively. Evaluate its amplitude and frequency.

3. If the length of a seconds pendulum is decreased by 2 %, find the gain or loss of time in one day.

4. A body executing SHM has an acceleration of 4 cm/s^2 when its displacement is 1 cm. What is the time period of SHM ? Find the maximum value of acceleration if amplitude is 5 cm.

5. A particle executes SHM along X-axis with frequency 1 Hz. Its speed at equilibrium position is 0.3 m/s. Determine its amplitude and write the expression for its displacement.

6. A block lies on a horizontal surface which executes SHM of period 1 s horizontally. What is the maximum amplitude for which the block does not slide if 'μ' between block and surface is 0.4 ?

7. A block is placed on a horizontal surface, that executes SHM with amplitude 5 cm. What is the maximum frequency on which the block does not slide if $\mu = 0.8$?

8. The total energy of a particle of mass 800 gm, executing SHM is 15.9 J. At t= 0 its displacement is $\hat{i}$ (4.3 m) and velocity is $-\hat{i}(3.2 m/s)$. Determine its period and amplitude. Write down the expression for its displacement.

9. What is the percentage change in time period of a simple pendulum taken to a place where g increases by 4% ?

10. How much time will the seconds pendulum gain in one hour if its length decreases by 1 % in winter ?

11. A spring hangs freely from a ceiling. It elongates by 2cm when a block is attached to its free end. If the block is subsequently pulled downwards by 5 mm and then released it starts oscillating. Determine the amplitude and frequency.

12. A 100 gm block vibrates with frequency 1 Hz at the end of a vertical spring. Determine the spring constant and amplitude if the total energy is 0.05 J.

13. Determine the potential energy and kinetic energy of the last problem (No.12) when it is 2 cm. away from the mean position.

14. The scale of a spring balance, reading from 0 to 15 kg is 15 cm long. A body suspended from the balance is found to oscillate vertically with a frequency 1.5 Hz. What is the weight of the body ?

15. A body of mass 5 kg hangs from a spring and oscillates with a period of 0.5 s. How much will the spring shorten when the body is removed ?

16. A particle executes SHM of time period T. Find the time taken by the particle to go directly from its mean position to half the amplitude.

17. Consider a particle moving in SHM according to the equation $x = 2\cos(50\pi t + \tan^{-1} 0.75)$, where 'x' is in cm. and t in seconds. The motion is started at t= 0
    (a) when does the particle come to rest for the first time ?
    (b) when does the acceleration have its maximum magnitude for the first time ?
    (c) when does the particle come to rest for the second time ?

18. The balance wheel of a watch vibrates with an angular amplitude of π radians and with a period of 0.5 s.
    (a) Find its maximum angular velocity
    (b) Find its angular velocity when its displacement is one-half its amplitude.
    (c) Find its angular acceleration when its displacement is 45^0.
19. A point is moving in SHM about a fixed point 'O'. Its distance from 'o' at a certain time is 1 cm and 1 sec later its distance from 'o' is 5 cm. After yet another second its distance from 'o' is again 5 cm. Find the time period.
20. A particle executes SHM with period T about a fixed point 'o'. It passes through a point P with velocity ϑ along OP. Show that the time that elapses when it again comes to P. is given by

$$t=\frac{T}{\pi}\tan^{-1}\left(\frac{T.\vartheta}{2\pi.op}\right).$$

E. Long Answer Type Questions :

1. What do you mean by SHM ? How is SHM connected to uniform circular motion ?
2. Define SHM. Hence derive an expression for the displacement of a particle executing SHM . Prove that its amplitude is a constant quantity.
3. Define S.H.M. Derive expression for displacement and velocity of a body executing SHM. Prove that velocity is maximum while passing through Its mean position.
4. What is a simple pendulum ? Derive an expression for its time period.
5. A massless spring is hung vertically from a rigid support. A mass 'm' is attached at its free end. The mass is slightly depressed and then released. Show that the motion is simple harmonic and find its time period.

F. Fill in the Blank Type

1. The magnitude of acceleration of a particle in SHM is .... at the end points.
2. The total energy of a particle in SHM equals the .... at the end points and the .... at the mean position.
3. Two simple harmonic motions are represented by the equations $y_1 = 10\sin(3\pi t + (\pi/4))$ and $y_2 = 5(\sin 3\pi t + \sqrt{3}\cos 3\pi t)$. Then amplitudes are in the ratio.
4. The restoring force in SHM is .... in magnitude, when the particle is instantaneously at rest.
5. The time period of a particle in SHM depends in general on ...., but is indepnedent of ....

G. True - False Type

1. Acceleration is proportional to displacement. This condition is not sufficient for motion to be simple harmonic.
2. One of the two clocks on earth is controlled by a pendulum and the other by a spring. If these clocks be taken to the moon, then both will show accurate time.
3. The bob of a simple pendulum is a ball full of water. If a fine hole is made in the bottom of the ball, then the time period will no more remain constant.
4. In a SHM kinetic energy and potential energy become equal when the displacement is $\frac{1}{\sqrt{2}}$ times the amplitude.
5. In a SHM when the displacement is one half the amplitude the kinetic energy is three-fourth of the total energy.

ANSWERS

A. MULTIPLE CHOICE TYPE QUESTIONS :

1. (iv), 2. (i), 3. (iv), 4. (iv), 5. (iv), 6. (iii), 7. (i), 8. (ii), 9. (ii), 10. (iii), 11. (iv), 12. (ii), 13. (iii), 14. (iv), 15. (iii).

B. VERY SHORT ANSWER TYPE QUESTIONS :

1. See text
2. No work
3. $\frac{1}{\sqrt{2}}$
4. $\vartheta = \omega\sqrt{A^2 - x^2}$
5. See text
6. Yes, at mean position
7. Increases
8. $a = -\omega^2 x$, $a \propto x$
9. Zero
10. Multiplying by $2\pi / T$
11. Velocity maximum at mean position, accleration maximum at extreme position.
12. $T = 2\pi / 3$
13. No
14. $1/2\, m w^2 A^2$
15. No
16. $T = 2\pi\sqrt{\frac{\ell}{g}}$

C. SHORT ANSWER TYPE :

1. At a distance $\frac{1}{\sqrt{2}}$ times the amplitude w.r.to mean position i.e. $x = A / \sqrt{2}$.
2. K.E. 75 %, P.E. 25%
3. $\frac{g}{\pi^2} \approx 100\text{cm}$
4. $K = 49\ \text{kg/s}^2$
5. $\vartheta_{max} = 2\text{cm}/s$
6. $\vartheta_{max} = \pi\ \text{cm/s} = 3.14\ \text{cm/s}$

   $a_{max} = \frac{\pi^2}{10}\text{cm/s}^2$
7. $\upsilon = \frac{1}{T}$
8. 435. 267 per day
9. [Hints : Spring constant $k = \frac{Y.A}{L}$, so when halved new spring constant

   $k' = \frac{Y.A}{L/2} = 2\frac{YA}{L} = 2k$, $\upsilon = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$, $\upsilon' = \frac{1}{2\pi}\sqrt{\frac{k'}{m}} = \frac{1}{2\pi} = \sqrt{2}\upsilon$]
10. $T = 2\pi\sqrt{\frac{M + (m/3)}{K}}$, M = load, m = mass of spring
11. When elastic reaction force is proportional to elongation ie as long as Hooke's law is obeyed.

12. Length, g.
13. [Hints : For spring, $T = 2\pi\sqrt{m/k}$ ie gravity has no effect. Hence spring clock shall show same time as on earth. In case of pendulum clock $T = 2\pi\sqrt{l/g}$, ie. gravity dependent. Since

$$\frac{g_e}{g_m} = \frac{GM_e/R_e^2}{GM_m/R_m^2} = \frac{M_e}{M_m} \cdot \frac{R_m^2}{R_e^2} = \frac{1}{0.1}\left(\frac{1}{2}\right)^2 = \frac{1}{0.4} \Rightarrow g_m = 0.4g_e \Rightarrow g_e > g_m.$$

Hence for pendulum clock $T_e < T_m$ i.e. period in Mars is greater than on earth.]
14. Maximum at mean position, minimum at extreme position.
15. T decreases
16. Loses time
17. 75%

CONCEPTUAL :

8. Yes, $T = \pi / \omega$

D. NUMERICAL PROBLEMS :

1. 25.1 cm/s, 31.58 m/s^2
2. 2 cm ; 1.59 Hz
3. gain of 877.18 s
4. π sec; 20 cm/s^2
5. 4.77 cm; $x(t) = 0.3/2\pi \sin(2\pi t) = 4.77 \sin(2\pi t)$
6. 9.93 cm

[Hints : $a_{max} = \omega^2 A; F_l = ma_{max} = m\omega^2 A = \mu mg$

$$\Rightarrow A = \frac{\mu g}{\omega^2} = \frac{\mu g T^2}{4\pi^2} = \frac{0.4 x 980 x 1}{4 x 2} = 9.93 cm$$

7. 1.99 Hz
8. 4.97 s, 5m, $x = 5 \sin(1.26\,t + 0.33\pi)$
9. 2 % decrease
10. 18 s
11. 5 mm, 3.52 Hz.
12. 3.94 N/m, 15.9 cm
13. 0.00079 J ; 0.0491 s
14. 11.03 kg
15. 6.2 cm
16. T / 12
17. 0.036 s ; 0.036 s; 0.056 s
18. 39.48 rad/s; 34.19 rad/s; 124.025 rad/s^2
19. $T = \dfrac{2\pi}{\cos^{-1}\left(\frac{3}{5}\right)}$
20.

F. (1) greatest (2) potential energy, kinetic energy (3) 1:1 (4) maximum (5) Force constant and mass, amplitude, initial phase and total energy.

G. (1) True (2) False (3) True (4) True (5) True.

10

Wave Motion

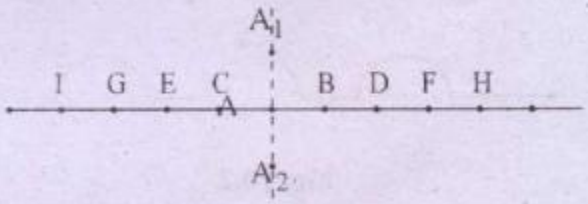
Wave Motion :

The study of mechanics shows that moving particles carry energy with themselves wherever they go. Thus energy is transported from one place to another due to body motion or bulk motion of the material particle.

But in nature we also observe that energy is transferred from one place to another without body motion or bulk motion. For example sound energy of a speaker to a listener, light energy from sun to earth, are transferred without body motion. When a speaker speaks, the disturbance produced in air near the lips, travels in air, but the air itself does not move. Such a phenomenon is called wave-motion.

10.1. Waves :

Consider the effect of dropping a stone into a pond of water. It is observed that ripples (circular rings of upraised water) spread out from the place of disturbance. The height of the ripples gradually decreases as it moves away from the place of disturbance. It is also observed that a leaf floating on the surface of water does not move with the ripples, but moves up and down at the same spot. This indicates that the moving ripples donot carry the floating objects with them. It means water particles are not carried with the ripples. But, however, one feels the impact of the disturbance when the ripples reach him. This implies that the ripples must be carrying the energy of the disturbance, The transport of energy without body motion of the particles of the medium can be explained as follows :



When the stone strikes at A, the energy is transferred to the particle at A; and it vibrates up and down. Due to force of cohesion between various particles (molecules) vibrations of A get communicated to B, C, D, E, F etc. As a result, the disturbance spreads to all sides. Thus one defines :

"A wave is a disturbance that travels through a medium due to repeated periodic vibration of the particles of the medium about their mean (equilibrium) position, transporting energy but not matter."

The direction of energy transport is called the direction of propagation of the wave. The velocity with which energy is transported is called as wave-velocity. But the velocity with which particles of the medium vibrate is called particle velocity.

10.2 Wave pulse :

It is a disturbance travelling through a medium for a short duration. It has a beginning and an end.

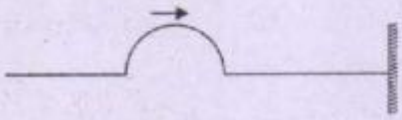


Fig. 10.1

For example when we shake a string a wave pulse moves for a short duration.

10.3 Continuous Wave :

When a disturbance travels through a medium for a reasonable extended duration, then it is called a continuous wave or wave train. For example in the example given above if we shake the string continuously for a pretty long time then a wave train is set up in the string.

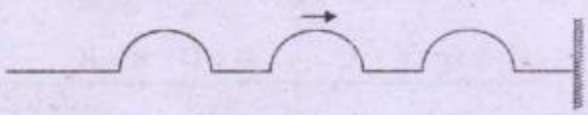


Fig. 10.2

10.4 Mechanical Wave :

Waves which require a material medium for its propagation is called a mechanical wave.

The requirements of a mechanical wave are

i) A source of disturbance

ii) An elastic medium for propagation

iii) Isotropy of the medium

Sound waves, seismic waves etc. are mechanical waves. But light waves, radio waves, micro waves, etc. donot belong to this category.

10.5 Types of Waves :

There are two types of wave-motion (a)longitudinal wave motion (b) transverse wave motion.

(a) Longitudinal Wave motion :

A wave motion in which particles of the medium oscillate parallel to the direction of propagation is called a *longitudinal wave*.

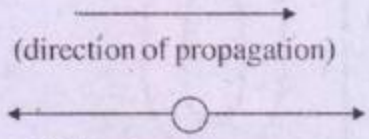


Fig. 10.3

For example (i) a vibrating tuning fork produces longitudinal waves (ii) sound wave in air are longitudinal waves. The following characteristic are associated with a longitudinal wave.

(i) Alternate compressions and rerefactions are created.

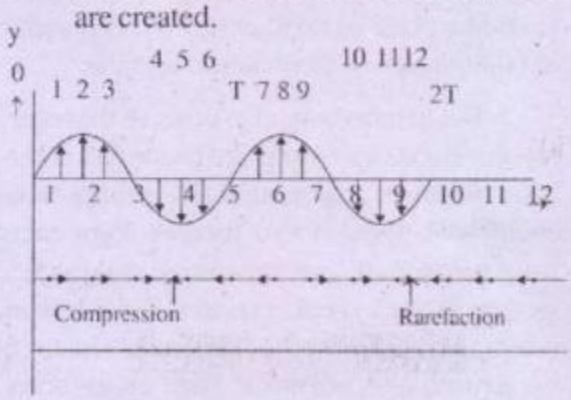


Fig. 10.4

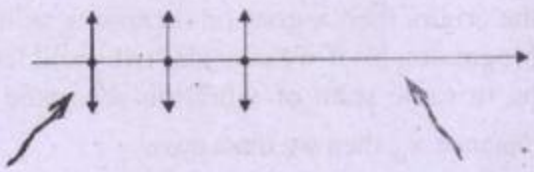
(ii) They can propagate through solids, liquids and gaseous medium.

(iii) They can travel only through elastic medium, and velocity of wave depends on the elastic coefficient (Y or K) and density of medium

$$\vartheta = \sqrt{Y/\rho} \text{ or } \sqrt{B/\rho}$$

(b) Transverse Wave :

A wave motion in which particles of the medium osciallate perpendicularly to the direction of propagation is called transverse wave-motion.



Oscillation of a particle   dir[n] of propagation

Fig. 10.5

For example wave set up in a plucked string is a transverse wave. The following characteristics are associated with a transverse wave :

i) Crests and troughs are created, causing a change in shape (shear strain) of the medium. (A crest is a portion of the medium raised above the normal position of the medium and a trough is a portion of the medium, depressed below the normal position of the medium).

ii) They can only travel in solid medium, as liquid and gaseous medium cannot sustain shear strain.

iii) Polarisation is a special phenomenon exhibited by a transverse wave.

Thus from the above discussion we note the following general characteristics of a wave motion.

i) It is a disturbance travelling in a medium.

ii) In a wave-motion there is no body-motion of particles of the medium.

iii) Energy is transported from one part to another.

iv) It can travel through only in an elastic medium.

v) Velocity of a particle of the medium is different from the velocity of the wave.

10.6 Equation of a progressive wave :

As discussed earlier a wave is a disturbance that travels in a medium due to repeated periodic vibration of the particles of the medium about their mean position of rest. When a continuous wave travels in a medium the particles of the medium are set into vibrations. Therefore the study of a wave is equivalent to the study of the vibrations of the particles of the medium at different positions at different times.

A function which describes the vibrations of the particles of the medium at different positions at different times is therefore called a wave function or equation to a progresive wave.

So in general the equation to a progressive wave travelling in x-direction shall be of the form

$$y = f(x,t) \qquad ...(10.6.1)$$

where 'x' gives the position of a particle in the medium, 't' gives time and 'y' denotes the displacement of a particle of the medium. Since the motion is assumed to be periodic, so

$$y = f(x,t) = f(x + m\lambda, t) = f(x, t + nT) \qquad ...(10.6.2)$$

where λ is a distance, such that particles separated by this distance vibrate in same phase, T is the time period of vibration of the particles and m,n are integers.

For simplicity we assume that the vibrations are simple harmonic. Then the vibration of a particle at 'O' (assumed to be origin) at any time 't' shall be given by

$$y = A \sin \omega t \qquad ...(10.6.3)$$

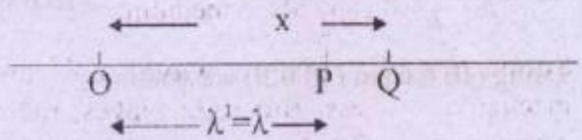


Fig. 10.6

where we have started counting time from the instant when the disturbance reaches the particle at 'O'. On receiving energy of disturbance, the particle at 'O' starts vibrating and comes back to its initial state (at t =0) after one time period of vibration (= T). Now let P be a point at a distance λ', where the disturbance reaches after a time T. Then the particle at 'P' just starts vibrating. Hence now at this instant (t = T), the particle at O and at P are in the same state of vibration. Therefore the quantity λ (included in eqn 10.6.2) and λ' defined above must be one and same (i.e. $\lambda = \lambda'$). If ϑ be the velocity of the wave, then according to above discussion

$$\lambda = \lambda' = \vartheta\ T \qquad ...(10.6.4)$$

One calls this distance 'λ' as wave-length. Thus "wavelength is the minimum distance between two particles of the medium which are in the same state of vibration and which is covered by the wave in a time equal to the time period of vibration of the particles of the medium." Thus over a distance λ the phase difference is 2π. Hence the phase difference between two particles of the medium separated by a distance 'x', must be $\Delta\phi = 2\pi x/\lambda$. Therefore the state of vibration of a particle 'Q' at a distance 'x' from 'O' is given by

$$y = A\sin(\omega t \pm \Delta\phi) = A\sin(\omega t \pm \tfrac{2\pi x}{\lambda}) \qquad ...(10.6.5)$$

From (10.6.4) we have

$$\lambda = \vartheta T = \vartheta.\frac{2\pi}{\omega}$$

$$\Rightarrow \omega = \frac{2\pi}{\lambda}\vartheta \qquad ...(10.6.6)$$

Using (10.6.6) in (10.6.5) we obtain

$$y = A\sin\frac{2\pi}{\lambda}(\vartheta t \pm x) \qquad ...(10.6.7)$$

Equation (10.6.7) describes the vibration of the particles of the medium at any time t and at any postion x. Hence eq. (10.6.7) is the equation to a progressive wave. Now it remains to decide about the sign (+ or −) in eqn. (10.6.7). For this consider a wave travelling to the right of the origin, then x goes on increasing as time progresses. So if we consider two particles to be in same state of vibration separated by distance x_0 then we must have

$$y = A\sin\frac{2\pi}{\lambda}(\vartheta t \pm x)$$

$$= A\sin\frac{2\pi}{\lambda}[\vartheta(t + t_0) \pm (x + x_0)]$$

This implies that this can be possible only when '-ve' sign is chosen. So we write

$$y = A\sin\frac{2\pi}{\lambda}(\vartheta t - x) \qquad ...(10.6.8)$$

as representing a wave travelling to the right of the origin.

Similar considerations show that a wave travelling to the left of the origin is given by

$$y = A\sin\frac{2\pi}{\lambda}(\vartheta t + x) \qquad ...(10.6.8)(a)$$

In equations (10.6.7) and (10.6.8), A is called the amplitude (displacement amplitude), and $\phi = \frac{2\pi}{\lambda}(\vartheta t \pm x)$ is called the phase factor (or phase). The locus of all points (in a medium in which the wave travels) which are in same phase of vibration is called wave front.

*We also note that a wave travels in a medium without any change in its shape. So the phase $\phi' = \frac{2\pi}{\lambda}(\vartheta t \pm x)$ should always remain unchanged as the wave progresses. ie

$$\frac{d\phi}{dt} = 0 = \frac{2\pi}{\lambda}(\vartheta \pm \frac{dx}{dt})$$

$$\Rightarrow \vartheta \pm \frac{dx}{dt} = 0 \qquad ...(10.6.9)$$

Since ' ϑ ' corresponds to magnitude only, (so always positive) and dt is always positive, so one must choose '-ve' sign i.e. write

$\phi = \frac{2\pi}{\lambda}(\vartheta t - x)$ when the wave travels to the right of the origin and choose '+ve' sing ie write

$\phi = \frac{2\pi}{\lambda}(\vartheta t + x)$ when the wave travels to the left of the origin. "Thus the conclusions (10.6.7) and (10.6.8) are also explained."*

10.6(a) Relation between Particle velocity and Wave Velocity :

The speed with which a particle of the medium moves along or perpendicular to the direction of propagation during its vibration is called particle velocity.

Since displacement of a particle during its vibration is given by

$$y = A \sin \frac{2\pi}{\lambda}(\vartheta t \pm x)$$

So particle velocity u is given by

$$u = \frac{dy}{dt} = A.\ \frac{2\pi\vartheta}{\lambda}.\ \cos \frac{2\pi}{\lambda}(\vartheta t \pm x) \quad ...(10.6.10)$$

This implies

$$u_{max} = \frac{2\pi\vartheta}{\lambda} A \quad ...(10.6.11)$$

The quantity ' ϑ ' is called the wave velocity. This is the speed with which a point on the wave front travels, which is also the speed with which energy is transferred from one place to another. Equation (10.6.10) and (10.6.11) give the relation between particle velocity and wave velocity.

10.6(b) Other forms of Wave Function :

In Sec. 10.6 it has been shown that

$$y = A.\ \sin \frac{2\pi}{\lambda}(\vartheta t \pm x)$$

represents a progressive wave. However one can make use of the relations

$$\vartheta = \nu\lambda = \text{wave velocity}$$

$$\omega = 2\pi\nu = \text{angular frequency}$$

$$\nu = \frac{1}{T} = \text{frequency}$$

$$T = \frac{2\pi}{\omega} = \text{Time-period}$$

$$k = \frac{2\pi}{\lambda} = \text{wave-vector}$$

and write other forms like

(i) $y = A \sin\left(\frac{2\pi\vartheta}{\lambda}\right)\left(t \pm \frac{x}{\vartheta}\right)$ ...(10.6.12)

(ii) $y = A \sin(2\pi\nu)\left(t \pm \frac{x}{\vartheta}\right) = A \sin\omega\left(t \pm \frac{x}{\vartheta}\right)$ ...(10.6.13)

(iii) $y = A \sin 2\pi\left(\frac{t}{T} \pm \frac{x}{\lambda}\right)$ ...(10.6.14)

(iv) $y = A \sin\left(\frac{2\pi t}{T} \pm \frac{2\pi x}{\lambda}\right) = A \sin(\omega t \pm kx)$ ....(10.6.15)

Equations (10.6.12) to (10.6.15) can be also given by replacing Sine with Cosine function.

Sometimes it is useful to represent a progressive wave in terms of pressure. We can arrive at it as follows :

$$y = A \sin \frac{2\pi}{\lambda}(\vartheta t - x)$$

$$\text{strain produced} = \frac{dy}{dx} = -\frac{2\pi}{\lambda} A \cos \frac{2\pi}{\lambda}(\vartheta t - x)$$

Since Bulk modulus $B = \frac{P}{(-dV/V)} = \frac{P}{(-dy/dx)}$

So

$$P = \frac{2\pi}{\lambda} A.B. \cos \frac{2\pi}{\lambda}(\vartheta t - x)$$

$$\Rightarrow \quad P = P_m \cos \frac{2\pi}{\lambda}(\vartheta t - x)$$

$$\Rightarrow \quad P = P_m \sin\left[\frac{2\pi}{\lambda}(\vartheta t - x) + \frac{\pi}{2}\right] \quad ...(10.6.16)$$

Thus pressure wave and displacement wave have a phase difference of $\pi/2$.

10.6(c) Energy in Progressive wave :

As said earlier energy is transported by a wave. To compute this let us consider (for simplicity) a simple harmonic wave given as

$$y = A \sin \frac{2\pi}{\lambda}(\vartheta t - x) \quad ...(10.6.17)$$

This equation describes the vibration of a particle of the medium at position 'x' and at time 't'. The particle velocity of this particle is then given by

$$u = \frac{dy}{dt} = A.\frac{2\pi\vartheta}{\lambda}.\cos\frac{2\pi}{\lambda}(\vartheta t - x) \quad ...(10.6.18)$$

So the K.E. of this particle at position 'x' and time 't' is given by

$$E_k(x,t) = \frac{1}{2} mu^2 = \frac{1}{2} m\left(\frac{2\pi A\vartheta}{\lambda}\right)^2 \cos^2\frac{2\pi}{\lambda}(\vartheta t - x)$$

where 'm' is the mass of each particle.

The <u>time average</u> of this K.E. is given as

$$\langle E_k(x,t)\rangle_t = \overline{E}_k = \frac{1}{2} m\left(\frac{2\pi A\vartheta}{\lambda}\right)^2 .\frac{1}{T}\int_0^T \cos^2\frac{2\pi}{\lambda}(\vartheta t - x)dt$$

$$\Rightarrow \quad \overline{E}_k = \frac{1}{4} m.\left(\frac{2\pi A\vartheta}{\lambda}\right)^2 \quad ...(10.6.19)$$

Eqn. (10.6.17) gives accleration a of the vibrations particle as

$$a = \frac{d^2y}{dt^2} = -\left(\frac{2\pi\vartheta}{\lambda}\right)^2 y = -\omega^2 y \quad ...(10.6.20)$$

Therefore force acting on the vibrating particle is $F = ma = -m\omega^2 y = -ky$...(10.6.21)

Hence the work done by this force in displacing the particle through 'y' from its equilibrium (mean) position is given by

$$W = \int_0^y K\, y\, dy = \frac{1}{2} K y^2$$

$$\Rightarrow \quad W = \frac{1}{2} m\left(\frac{2\pi\vartheta}{\lambda}\right)^2 .\sin^2\frac{2\pi}{\lambda}(\vartheta t - x) \quad ...(10.6.22)$$

This is stored as the potential energy of the particle. Therefore

$$E_p(x,t) = \frac{1}{2} m\left(\frac{2\pi\vartheta}{\lambda}\right)^2 .\sin^2\frac{2\pi}{\lambda}(\vartheta t - x)$$

and

$$\langle E_p(x,t)\rangle_t = \overline{E}_p = \frac{1}{2} m\left(\frac{2\pi\vartheta}{\lambda}\right)^2 \left\langle \sin^2\frac{2\pi}{\lambda}(\vartheta t - x)\right\rangle$$

$$\Rightarrow \quad \overline{E}_p = \frac{1}{4} m\left(\frac{2\pi\vartheta}{\lambda}\right)^2 A^2 \quad ...(10.6.23)$$

Therefore average energy of particle of the medium is

$$\overline{E} = \overline{E}_p + \overline{E}_k = \frac{1}{2} m\left(\frac{2\pi\vartheta}{\lambda}\right)^2 A^2 \quad ...(10.6.24)$$

If 'n' be number particles per unit volume of a medium, then average energy per unit volume is

$$\overline{E}_\rho = \frac{1}{2} mn \left(\frac{2\pi\vartheta}{\lambda}\right)^2 A^2 \qquad ...(10.6.25)$$

Putting mn = ρ = density of the medium we have

$$\overline{E}_\rho = \frac{1}{2}\rho \left(\frac{2\pi\vartheta}{\lambda}\right)^2 A^2 = \frac{1}{2}\rho\omega^2 A^2$$

$$= 2\pi^2 \rho v^2 A^2 \qquad ...(10.6.26)$$

10.6(d) Intensity of a wave : (I)

Intensity of a wave is defined as the average rate at which a wave transmits energy per unit area held normal to the direction of propagation.

i.e. $$I = \frac{1}{a}\left\langle \frac{dE}{dt} \right\rangle = \frac{1}{a}\frac{\langle \Delta E \rangle}{\Delta t} \qquad ...(10.6.27)$$

Where $\langle \Delta E \rangle$ is the average energy flowing through area 'a', held normal to the direction of propagation, in a time interval dt.

Consider a section of the medium of cross-sectional area 'a', perpendicular to the direction of propagation. The disturbance leaving cross-section AB (see fig. 10.7) covers a distance ϑdt, in time dt.

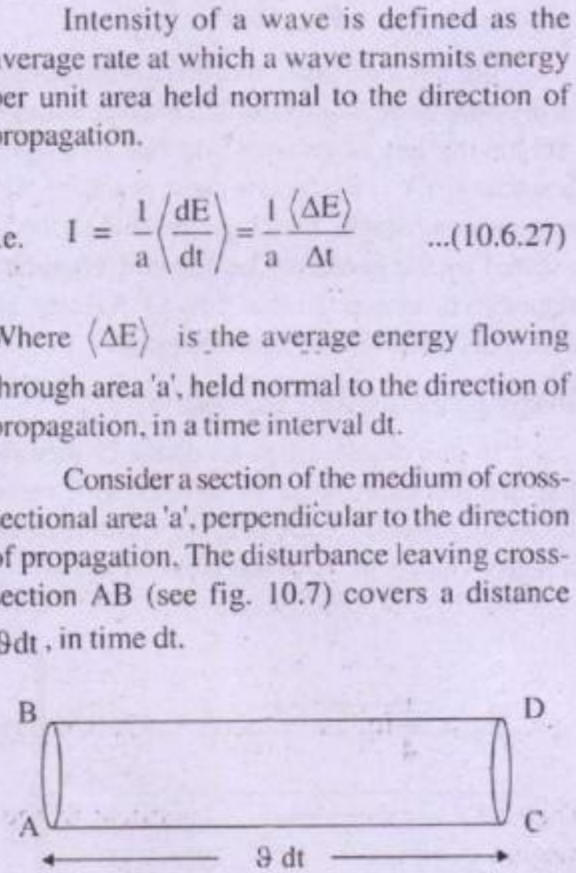


Fig.10.7

Therefore average energy flowing in time Δt through the cross-section of area 'a' is given by

$$\langle \Delta E \rangle = \overline{E}_\rho \, dV = \frac{1}{2}\rho\,\omega^2 A^2 . a.(\vartheta dt) \qquad ...(10.6.28)$$

This leads to intensity of the wave as

$$I = \frac{1}{a}\frac{\langle \Delta E \rangle}{\Delta t} = \left(\frac{1}{2}.\rho\,\omega^2 A^2\right)\vartheta$$

i.e.

$$I = \left(\frac{1}{2}.\rho\,\omega^2 A^2\right)\vartheta = \overline{E}_\rho.\vartheta \qquad ...(10.6.29)$$

Eqn. (10.6.29) shows that intensity depends on various factors, ρ, ω, A and ϑ as given below

$I \;\alpha\; \rho$

$\alpha\; \omega^2$

$\alpha\; A^2$

$\alpha\; \vartheta$

Eqn.(10.627) gives the dimension of intensity as $[I] = M\ T^{-3}$, and its units are $watt / m^2 \equiv Joule / (m^2.s)$ in S.I. units; erg / $(cm^2.s)$ in C.G.S. units.

Velocity of longitudinal wave in elastic medium

The speed of a wave travelling in an elastic medium depends on the elastic property and density of the medium.

Let $\vartheta \;\alpha\; E_c^a$

$\alpha\; \rho^b$

where E_c is the proper elastic coefficient e.g. if it is an extended medium like air E_c = B (the bulk modulus) if it is a rod then E_c = Y (Young's modulus).So by the law of combination of variables

$$\vartheta = K\, E_c^a\, \rho^b \qquad ...(10.6.30)$$

where 'K' is a dimensionless constant whose value shall be determined from the experiment. Equating the dimension on both sides of eqn.

10.6.30, we have

$$L\,T^{-1} = (M\,L^{-1}\,T^{-2})^a\,(M\,L^{-3})^b$$

$$\Rightarrow \quad -a-3b = 1 \qquad ...(10.6.31)$$

$$-2a = -1 \qquad ...(10.6.32)$$

$$a+b=0 \qquad ...(10.6.33)$$

From eqn. (10.6.32) $a = \frac{1}{2}$ and from (10.6.33) $b = -a = -\frac{1}{2}$. Therefore

$$\vartheta = K\sqrt{\frac{E_c}{\rho}} \qquad ...(10.6.34)$$

Experiment shows that k = 1, so

$$\vartheta = \sqrt{E_c/\rho} \qquad ...(10.6.35)$$

when longitudinal wave travels in a fluid

$$\vartheta = \sqrt{\frac{B}{\rho}}, \qquad B = \text{Bulk modulus} \qquad ...(10.6.36)$$

When longitudinal wave travels in rod, in a bar etc.

$$\vartheta = \sqrt{\frac{Y}{\rho}} \qquad ...(10.6.37)$$

10.7 Reflection of Wave :

It is the phenomenon by virtue of which incident energy is sent back to the same medium by an interface separating two media from each other.

The phenomenon of reflection of transverse wave and longitudinal wave should be considered separately.

A. Transverse Wave :

When a transverse wave is incident on a boundary separating two media, the nature of the reflected wave depends on the nature of the reflecting medium.

<u>(i) Reflection on a rigid boundary</u>

When the reflecting medium is rigid (fixed) there occurs a phase change of π in the displacement wave on reflection. Crests return as troughs as shown in fig. 10.8

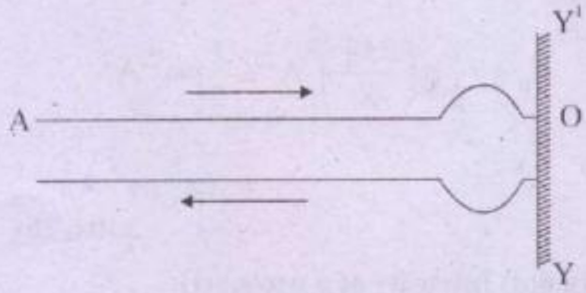


Fig. 10.8

But it is to be noted that there is no change in pressure wave. For example, consider a string AO having one of its ends attached to a rigid boundary YY'. The wave crest reaching 'O' returns as a trough. This happens due to force exerted by the boundary on the end 'O' in the opposite direction, so that end 'O' remains at rest. This generates an inverted pulse.

<u>(ii) Reflection on free boundary</u> :

In this case there is no phase-change in the displacement wave on reflection. Crests return as crests from the boundary. But there is a phase-change of π in the pressure wave.

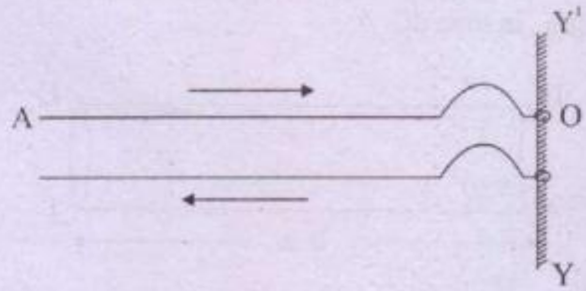


Fig.10.9

For example consider a string AO, with end 'O' tied to a ring capable of sliding without friction on a rod. When a wave crest reaches 'O', it pulls the ring upwards and the crest returns as a crest.

However quite often the end point is neither completely fixed nor completely free. For example consider a light string attached to a heavier string (See fig. 10.10 (a)). If a wave pulse, generated in the light string, moves towards the junction, a part of it is reflected and a part is transmitted on the heavier string. The reflected wave is inverted w.r.to the original one.

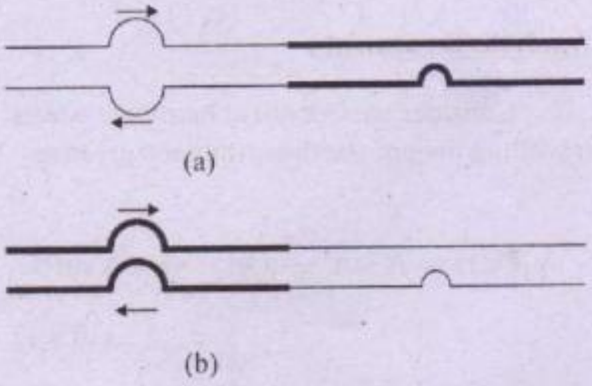


Fig.10.10

On the otherhand, if a wave pulse, generated on the heavy string, travel towards the junction a part will be reflected and a part will be transmitted. But there is no inversion in the wave shape (see fig. 10.10 (b)).

Thus in general, if a wave enter a region where the wave velocity is smaller, the reflected wave is inverted. If it enters a region where the wave-velocity is larger, the reflected wave is not inverted. The transmitted wave is never inverted.

B. Longitudinal Wave :

Reflection of longitudinal waves also depends on the nature of reflecting boundary.

(i) Rigid boundary :

In this case there is no phase change in the pressure wave. A compression returns as a compression and a rarefaction as a rarefaction (see fig. 10.11)

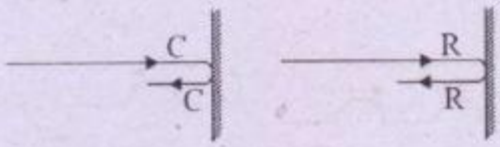


Fig.10.11

(ii) Free Boundary :

In this case there is a phase-change of π in the pressure wave. A compression returns as a rarefaction and vice-versa. (See fig. 10.12)

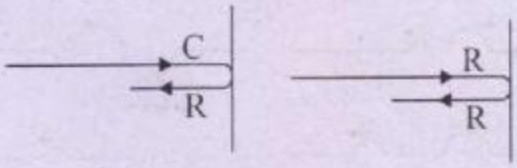
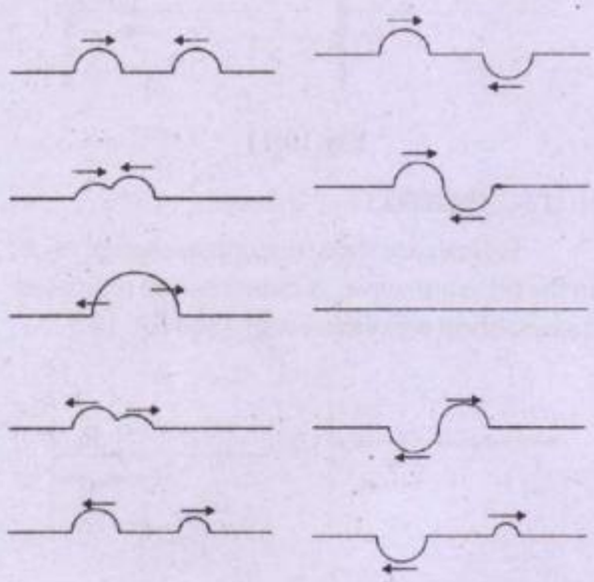


Fig.10.12

However sometimes, we neither come across completely rigid or completely free boundary. Then in such cases " a longitudinal wave travelling from a region of higher velocity (denser medium) to that of lower velocity (rarer medium) undergoes a phase reversal while that going from lower velocity region to that of higher velocity, does not do so, in case of reflection.

10.8 Superposition of waves and Superposition Principle

It is observed that when two waves, after overlapping, recedes from each other, they exhibit their original shape, speed and energy. But in the region of overlapping, the amplitude of resultant wave is changed. For example consider a string being held by two persons at the two ends and snapping their hands to start a wave pulse each. The pulses travel at the same speed although their shapes depend on how the persons snap their hands. The pulses travel towards each other, overlap and recede from each other.



(a) (b)

Fig.10.13

It is seen that the shape of the pulses, as they emerge after overlap, are indentical to their original shapes, showing as if there was no overlap. Such observations indicate the following principle to have been obeyed, during overlapping; known as superposition principle.

"When two or more waves travelling in a medium superpose on each other, they behave independent of each other and the resultant wave-fuction is the algebraic sum of the individual wave functions."

ie. $y = y_1 + y_2 + ...$ ...(10.8.1)

OR

"When two or more waves travelling in a medium superpose on each other, they behave independent of each other and the resultant displacement of a particle of the medium at any time is equal to vector sum of the individual displacements

i.e. $\vec{y} = \vec{y}_1 + \vec{y}_2 + ...$ ...(10.8.2)

10.9 Stationary (Standing wave) :

When two identical harmonic waves travelling in same medium but in opposite directions superpose on each other, the resulting wave pattern is called standing (stationary) wave; and is characterised by formation of nodes, antinodes and no energy flow.

This phenomenon is common to transverse as well as longitudinal waves.

Analytic Treatment :

Consider two identical harmonic waves travelling in opposite directions and given as

$$y_1(x,t) = A \sin \frac{2\pi}{\lambda}(\vartheta t - x) \equiv A \sin \alpha$$

...(10.9.1)

$$y_1(x,t) = A \sin \left[\frac{2\pi}{\lambda}(\vartheta t + x) + \delta\right] \equiv A \sin \beta$$

...(10.9.1)

where $\alpha = 2\pi(\vartheta t - x)/\lambda$, $\beta = 2\pi(\vartheta t + x)/\lambda + \delta$ and δ is the initial phase difference at time t = 0 and x = 0. The value of δ depends on the boundary condition (e.g. one may choose initial time (t= 0) at x= 0, when the two waves are in same phase then $\delta = 0$, or one may choose the origin at a place when they differ in phase by π and start counting time from that instant or one may choose such that at x = 0 and L, y = 0 for all t, so that $\delta = \pi$)

Then according to principle of superposition the wave function of the resultant wave is (as per eqn. 10.8.1) gives as

$$y \doteq y_1 + y_2 = 2A \sin \frac{\alpha + \beta}{2} . \cos \frac{\beta - \alpha}{2}$$

...(10.9.3)

Giving

$$y(x,t) = 2A\sin\left(\frac{2\pi}{\lambda}\vartheta t + \frac{\delta}{2}\right).\cos\left(\frac{2\pi}{\lambda}x + \frac{\delta}{2}\right) \quad ...(10.9.4)$$

Thus the resultant wave function is simple-harmonic but not in the form of a travelling wave. It shows that different particles of the medium vibrate with different amplitude but with same period.

Vibration at different Positions :

Equation (10.9.4) can be re-written as

$$y(x,t) = B(x)\sin\left(\frac{2\pi}{\lambda}\vartheta t + \frac{\delta}{2}\right) \quad ...(10.9.5)$$

where, $B(x) = 2A\cos\left(\frac{2\pi}{\lambda}x + \frac{\delta}{2}\right)$ ...(10.9.6)

gives the amplitude of displacement at position 'x' and at any time t. Equation (10.9.6) shows that at any arbitrary time t, the amplitude B(x) is maximum when

$$\cos\left(\frac{2\pi}{\lambda}x + \frac{\delta}{2}\right) = \pm 1$$

$$\Rightarrow \quad \frac{2\pi}{\lambda}x + \frac{\delta}{2} = n\pi,\ n = 0,1,2,3...\text{etc.}$$

$$\Rightarrow \quad x = \left(n - \frac{\delta}{2\pi}\right)\frac{\lambda}{2} \quad ...(10.9.7)$$

Particles at the positions given by eqn. (10.9.7) undergo maximum displacement compared to particles at other positions at that instant. These positions are called antinodes. Thus "antinodes" are the points in the medium at which the particles suffer maximum displacement compared to other particles at any time. Eqn. (10.9.7) also shows that these positions depend on the initial phase difference (δ) (i.e. phase difference at t = 0 and x = 0). For example (See fig. 10.14)

i) If $\delta = 0$, $x = 0, \frac{\lambda}{2}, \lambda, \frac{3\lambda}{2}$ .... etc correspnd to the positions of antinodes.

ii) If $\delta = \pi$, $x = \frac{\lambda}{4}, \frac{3\lambda}{4}, \frac{5\lambda}{4}$ ... etc. correspond to the positions of antinodes.

But in any case the distance between two consecutive antinodes is $\frac{\lambda}{2}$.

Eqn. (10.9.6) also shows that at any arbitrary time 't', the amplitude of displacement B(x) is minimum (= 0), when

$$\cos\left(\frac{2\pi}{\lambda}x + \frac{\delta}{2}\right) = 0$$

$$\Rightarrow \quad \frac{2\pi}{\lambda}x + \frac{\delta}{2} = (2n+1).\frac{\lambda}{4} \quad ...(10.9.8)$$

Eqn. (10.9.8) gives the positions at which the particles of the medium are permanently at rest. These positions are called as nodes. Thus "nodes" are the points in the medium at which the particles are permanently at rest. The position of nodes also depend on the initial phase difference (δ). For example (see fig. 10.14)

(i) If $\delta = 0$, then $x = \frac{\lambda}{4}, \frac{3\lambda}{4}, \frac{5\lambda}{4}$ .... etc. correspond to positions of nodes.

(ii) If $\delta = \pi$, then $x = 0, \frac{\lambda}{2}, \lambda, \frac{3\lambda}{2}$ .... etc. correspond to positions of nodes.

But in any case the distance between a node and nearest antinode is $\frac{\lambda}{4}$ and distance between two consecutive nodes is $\frac{\lambda}{2}$.

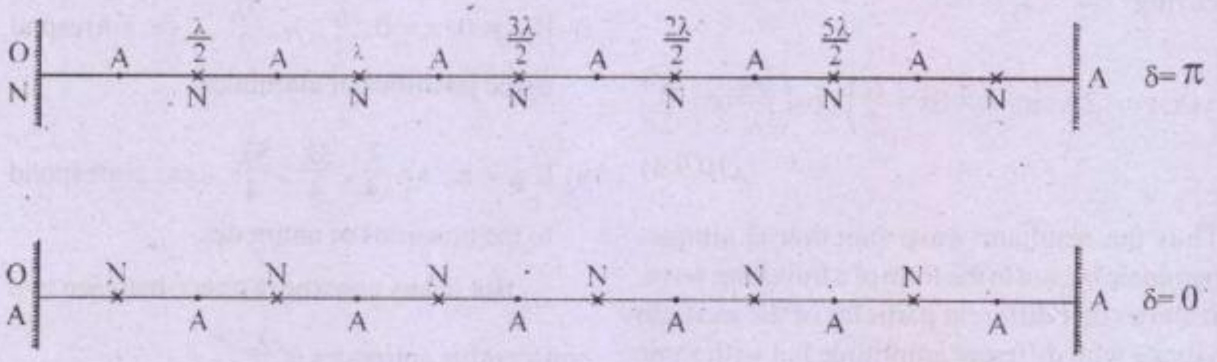


Fig.10.14

Vibration at different times :

Equation 10.9.4 shows that when

$$\sin\left(\frac{2\pi}{\lambda}\vartheta t + \frac{\delta}{2}\right) = 0$$

the wavefunction is zero everywhere; and this is satisfied whenever

$$\frac{2\pi}{\lambda}\vartheta t + \frac{\delta}{2} = n\pi, \qquad n = 0,1,2,3... \text{ etc.} \qquad ...(10.9.9)$$

$$\Rightarrow \quad \frac{2\pi t}{T} + \frac{\delta}{2} = n\pi$$

$$\Rightarrow \quad t = \left(n - \frac{\delta}{2\pi}\right)\frac{T}{2} \qquad ...(10.9.10)$$

Where 'T' is the time-period of vibration of each particle of the medium. At times, corresponding to eqn. (10.9.10) particles everywhere are at rest. These instants are called stationary instants. The stationary instants also depend on the initial phase difference (δ). For example

i) If $\delta = 0$, then at $t = 0, \frac{T}{2}, T, \frac{3T}{2}$ .... etc the particles are at rest everywhere.

ii) If $\delta = \pi$, then at $t = \frac{T}{4}, \frac{3T}{4}, \frac{5T}{4}$ ... etc. the particles are at rest everywhere.

But in any case the time interval between two consecutive stationary instants is T / 2.

Equation 10.9.4 further gives that whenever

$$\sin\left(\frac{2\pi}{\lambda}\vartheta t + \frac{\delta}{2}\right) = \pm 1$$

the wavefunction, everywhere, attains maximum value, and this is satisfied when

$$\frac{2\pi}{\lambda}\vartheta t + \frac{\delta}{2} = (2n+1)\frac{\pi}{2}$$

$$\Rightarrow \quad t = \left(2n + 1 - \frac{\delta}{\pi}\right)\frac{T}{4} \qquad ...(10.9.11)$$

Eqn. (10.9.11) gives instants when particles everywhere undergo maximum displacement suitable to their positions. These instants also depend upon initial phase difference (δ). Thus

i) If $\delta = 0$, then at $t = \frac{T}{4}, \frac{3T}{4}, \frac{5T}{4}$ .... etc. the particles everywhere suffer maximum displacement.

ii) If $\delta = \pi$, then at $t = \frac{T}{2}, \frac{3T}{2}, \frac{5T}{2}$ ... etc. the particles everywhere suffer maximum displacement.

Graphical Treatment :

Case. I. $\delta = \pi$, $y_1 = A \sin \frac{2\pi}{\lambda}(\vartheta t - x)$,

$y_2 = -A \sin \frac{2\pi}{\lambda}(\vartheta t + x)$

(a) $t = 0$ $x = 0$ $y_1 = 0$ $y_2 = 0$ $y = 0$

$x = \frac{\lambda}{4}$ $y_1 = -A$ $y_2 = -A$ $y = -2A$

$x = \frac{\lambda}{2}$ $y_1 = 0$ $y_2 = 0$ $y = 0$

$x = \frac{3\lambda}{4}$ $y_1 = A$ $y_2 = A$ $y = 2A$

$x = \lambda$ $y_1 = 0$ $y_2 = 0$ $y = 0$

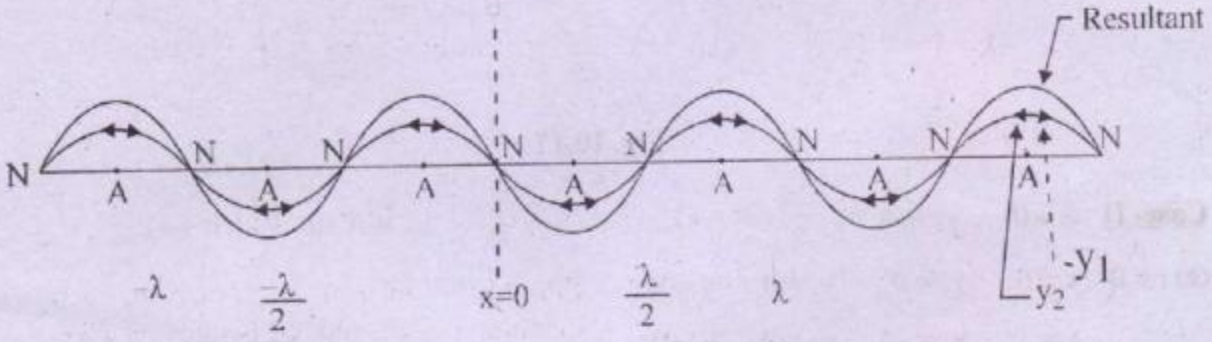


Fig. 10.15

(b) $t = \frac{T}{4}$ $x = 0$ $y_1 = A$ $y_2 = -A$ $y = 0$

$x = \frac{\lambda}{4}$ $y_1 = 0$ $y_2 = 0$ $y = 0$

$x = \frac{\lambda}{2}$ $y_1 = -A$ $y_2 = A$ $y = 0$

$x = \frac{3\lambda}{4}$ $y_1 = 0$ $y_2 = 0$ $y = 0$

$x = \lambda$ $y_1 = A$ $y_2 = -A$ $y = 0$

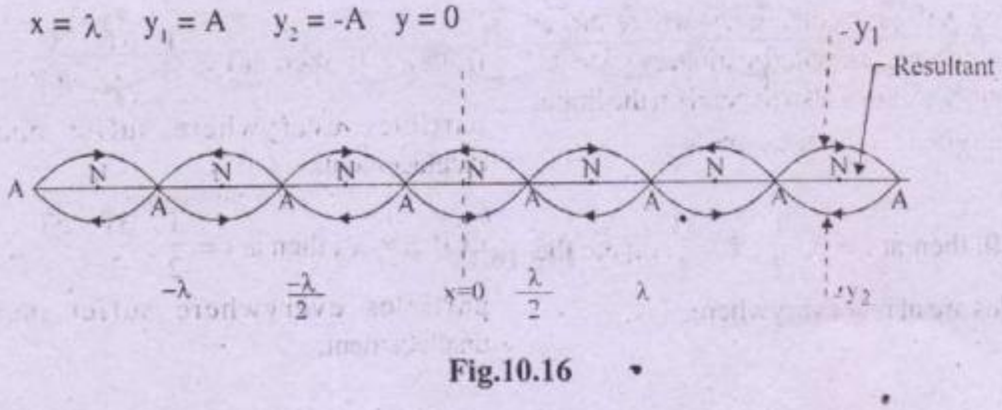


Fig.10.16

(c) $t = \frac{T}{2}$ $x = 0$ $y_1 = 0$ $y_2 = 0$ $y = 0$

$x = \frac{\lambda}{4}$ $y_1 = A$ $y_2 = A$ $y = 2A$

$x = \frac{\lambda}{2}$ $y_1 = 0$ $y_2 = 0$ $y = 0$

$x = \frac{3\lambda}{4}$ $y_1 = -A$ $y_2 = -A$ $y = -2A$

$x = \lambda$ $y_1 = 0$ $y_2 = 0$ $y = 0$

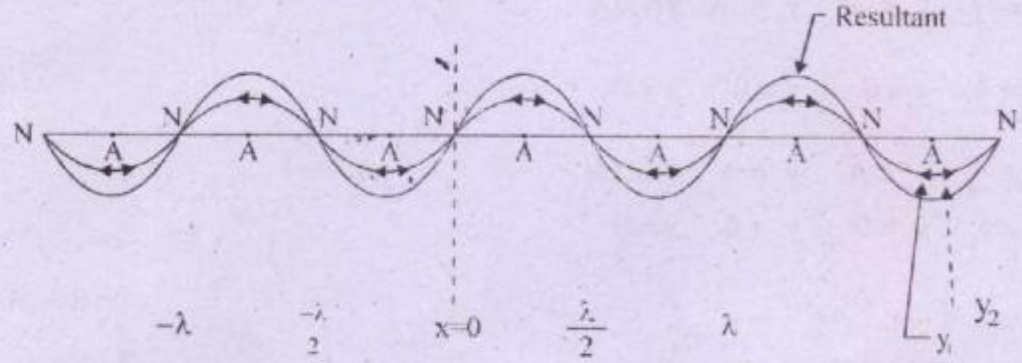


Fig. 10.17

Case- II $\delta = 0$ $y_1 = A \sin \frac{2\pi}{\lambda}(\vartheta t - x)$ $y_2 = A \sin \frac{2\pi}{\lambda}(\vartheta t + x)$

(a) $t = 0$ $x = 0$ $y_1 = 0$ $y_2 = 0$ $y = 0$

$x = \frac{\lambda}{4}$ $y_1 = -A$ $y_2 = A$ $y = 0$

$x = \frac{\lambda}{2}$ $y_1 = 0$ $y_2 = 0$ $y = 0$

$x = \frac{3\lambda}{4}$ $y_1 = A$ $y_2 = -A$ $y = 0$

$x = \lambda$ $y_1 = 0$ $y_2 = 0$ $y = 0$

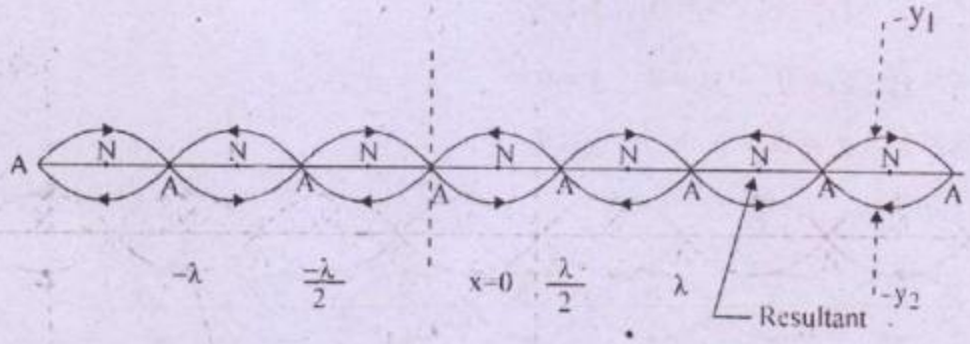


FIg.10.18

(b) $t = \frac{T}{4}$ $x = 0$ $y_1 = A$ $y_2 = A$ $y = 2A$

$x = \frac{\lambda}{4}$ $y_1 = 0$ $y_2 = 0$ $y = 0$

$x = \frac{\lambda}{2}$ $y_1 = -A$ $y_2 = -A$ $y = -2A$

$x = \frac{3\lambda}{4}$ $y_1 = 0$ $y_2 = 0$ $y = 0$

$x = \lambda$ $y_1 = A$ $y_2 = A$ $y = 2A$

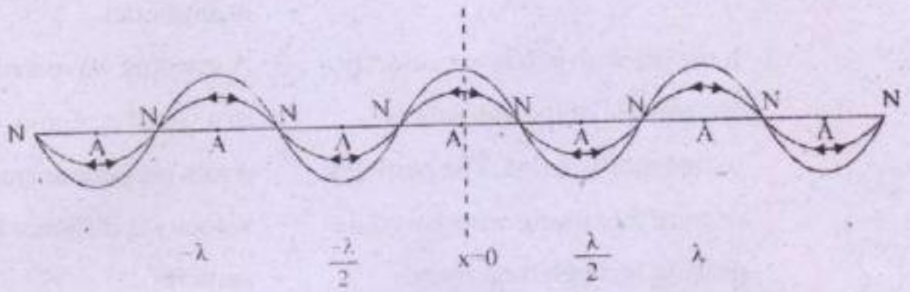


Fig.10. 19

(c) $t = \frac{T}{2}$ $x = 0$ $y_1 = 0$ $y_2 = 0$ $y = 0$

$x = \frac{\lambda}{4}$ $y_1 = A$ $y_2 = -A$ $y = 0$

$x = \frac{\lambda}{2}$ $y_1 = 0$ $y_2 = 0$ $y = 0$

$x = \frac{3\lambda}{4}$ $y_1 = -A$ $y_2 = A$ $y = 0$

$x = \lambda$ $y_1 = 0$ $y_2 = 0$ $y = 0$

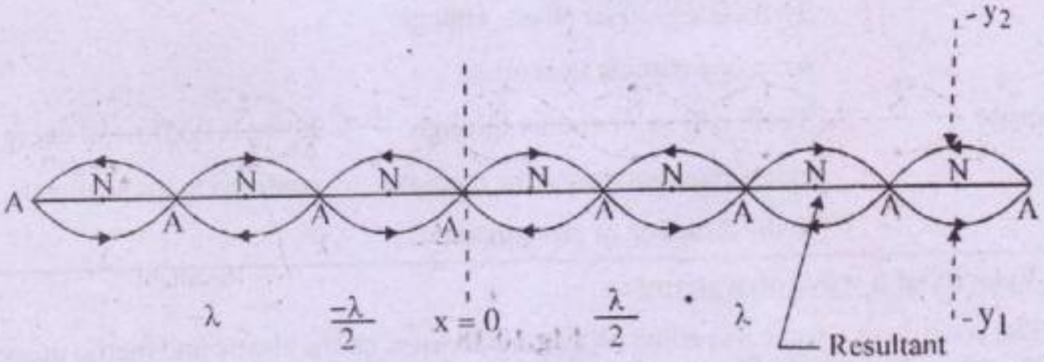


Fig.10. 20

Distinction between Progressive and Standing wave

Features	Progressive Wave	Standing Wave
1. Production	1. A continuous disturbance in a medium generates a progressive wave	1. Superposition of two identical waves, travelling in opposite direction in a medium, generates a standing wave.
2. Amplitude	2. It is same for all particles of the medium	2. It is different for different particles It is zero at nodes and maximum at antinodes.
3. Velocity	3. It moves with a definite velocity. No particle of the medium is permanently at rest. The particles acquire maximum velocity while passing through their mean position.	3. A standing wave remains confined to a specific region. Particles at nodes are permanently at rest. The velocity is different for different particles.
4. Frequency	4. All particles of the medium vibrate with same frequency	4. All particles, except those at nodes vibrate with same frequency.
5. Wave length	5. The wavelength is equal to the minimum distance between two particles vibrating in phase and is covered by the wave in one time period of vibration.	5. The wave length is equal to twice the distance between two consecutive nodes / or antinodes.
6. Phase	6. All particles within one wave length have different phases. There is a gradual phase-change from one particle to another.	6. All particles between two adjacent nodes are in same phase.
7. Energy	7. There is flow of energy through every cross-section of the medium in the direction of propagation.	7. There is no flow of energy. It is confined to the segments.

10.10 Velocity of a wave on a string :

The velocity of a wave travelling on a string depends on the elastic and inertia properties of the string. Thus it depends upon the tension (F) applied on the string and mass per unit length of the string.

Let $\vartheta \; \alpha \; F^a$

$\alpha \; \mu^b$

where 'F' is the tension applied and 'μ' is the mass per unit length of the string. Then

$$\vartheta = K F^a \mu^b \qquad ...(10.10.1)$$

where 'K' is a dimensionless constant, whose value shall be determined from the experiment. Equating the dimension on both sides

$$L T^{-1} = (M L T^{-2})^a \; (M L^{-1})^b$$

$$\Rightarrow \quad a - b = 1 \qquad ...(10.10.2)$$

$$a + b = 0 \qquad ...(10.10.3)$$

$$-2a = -1 \qquad ...(10.10.4)$$

From (10.10.4) we have $a = \frac{1}{2}$ and from (10.10.3) $b = -a = -\frac{1}{2}$. Therefore

$$\vartheta = K\sqrt{\frac{F}{\mu}} \qquad ...(10.10.5)$$

Experiments show that K = 1. Hence

$$\vartheta = \sqrt{\frac{F}{\mu}} \qquad ...(10.10.6)$$

10.10(a) Transverse Vibration of String :

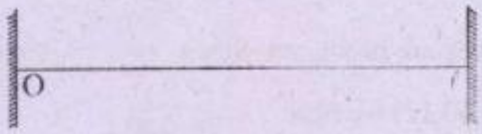
Consider a string of length ℓ, stretched and kept fixed at the ends $x = 0$ and $x = \ell$. When the string is made to vibrate, waves produced on it suffer multiple reflections at the ends. Due to multiple reflections at the ends waves going in positive x-direction interfere to give a resultant wave

$$y_1 = A\sin\frac{2\pi}{\lambda}(\vartheta t - x) = A\sin\alpha \qquad ...(10.10.7)$$

Similarly, waves travelling in the negative x-direction interfere to give the resultant wave

$$y_2 = A\sin\left[\frac{2\pi}{\lambda}(\vartheta t + x) + \delta\right] = A\sin\beta \qquad ...(10.10.8)$$

where 'δ' is the phase difference to be determined from the boundary conditon.



These two waves superpose on each other to give the resultant wave function

$$y = y_1 + y_2 = 2A\sin\frac{\alpha+\beta}{2}\cos\frac{\beta-\alpha}{2}$$

giving

$$y = 2A\sin\left(\frac{2\pi}{\lambda}\vartheta t + \frac{\delta}{2}\right).\cos\left(\frac{2\pi}{\lambda}x + \frac{\delta}{2}\right) \qquad ...(10.10.9)$$

Thus standing waves are formed. But the ends $x = 0$ and $x = \ell$ are fixed, hence these must be nodes. So with $x = 0$, $y = 0$ we have

$$0 = 2A\sin\left(\frac{2\pi}{\lambda}\vartheta t + \frac{\delta}{2}\right).\cos\left(\frac{\delta}{2}\right) \text{ for all t.}$$

This implies $\cos\frac{\delta}{2} = 0$ ie $\delta = \pi$.

Therefore equation 10.10.9 reduces to

$$y = -2A\sin\left(\frac{2\pi}{\lambda}x\right).\cos\left(\frac{2\pi}{\lambda}\vartheta t\right) \qquad ...(10.10.10)$$

Again $y = 0$ for $x = \ell$, for all t. Therefore

$$\sin\frac{2\pi}{\lambda}\ell = 0$$

$$\Rightarrow \quad \frac{2\pi}{\lambda}\ell = n\pi \quad , n = 0,1,2,3... \text{ etc.} \qquad ...(10.10.11)$$

$$\Rightarrow \quad \ell = n\frac{\lambda}{2} \qquad ...(10.10.12)$$

Eqn. (10.10.12) shows that if the length of the string is an integral multiple of $\lambda/2$, standing waves are produced. Since $\nu = \frac{\vartheta}{\lambda}$, so using (10.10.12) we find

$$\nu = \frac{n}{2\ell}.\vartheta \qquad ...(10.10.13)$$

But the velocity of a transverse wave on a vibrating string is given by (10.10.6) as $\sqrt{F/\mu}$. So using this value in eqn. 10.10.13 we obtain

$$\nu = \frac{n}{2\ell}\sqrt{F/\mu} \qquad ...(10.10.14)$$

and

$$y = -2A\sin\left(\frac{n\pi x}{\ell}\right)\cos(2\pi\nu t) \qquad ...(10.10.15)$$

Normal mode of Vibration :

When a string vibrates according to eqn.(10.10.15) with frequency ν, given by eqn. (10.10.14) it is said to vibrate in a normal mode.

For fundametal mode n = 1 and the equation to the standing wave is

$$y = -2A\cos(2\pi\upsilon t)\sin\left(\frac{\pi x}{\ell}\right) \qquad ...(10.10.16)$$

with frequency $\nu = \frac{1}{2\ell}\sqrt{\frac{F}{\mu}}$ ...(10.10.17)

Then the amplitude of vibration is zero at $x = 0$ and $x = \ell$; and is maximum at $x = \ell/2$. Thus the fundamental mode of vibration is characterised by nodes at $x = 0$ and $x = \ell$ and antinode at $x = \ell/2$ (see fig. 10.21 (a))

In the first overtone, also called as second harmonic n = 2, so that

$$y = -2A\cos(2\pi\nu t)\sin(2\pi x/\ell) \qquad ...(10.10.18)$$

with frequency $\upsilon = \frac{2}{2\ell}\sqrt{F/\mu}$ ...(10.10.19)

This shows that it has nodes at $x = 0,\ \ell/2,\ \ell$ and antinodes at $x = \ell/4$ and $3\ell/4$ (See fig. 10.21. (b))

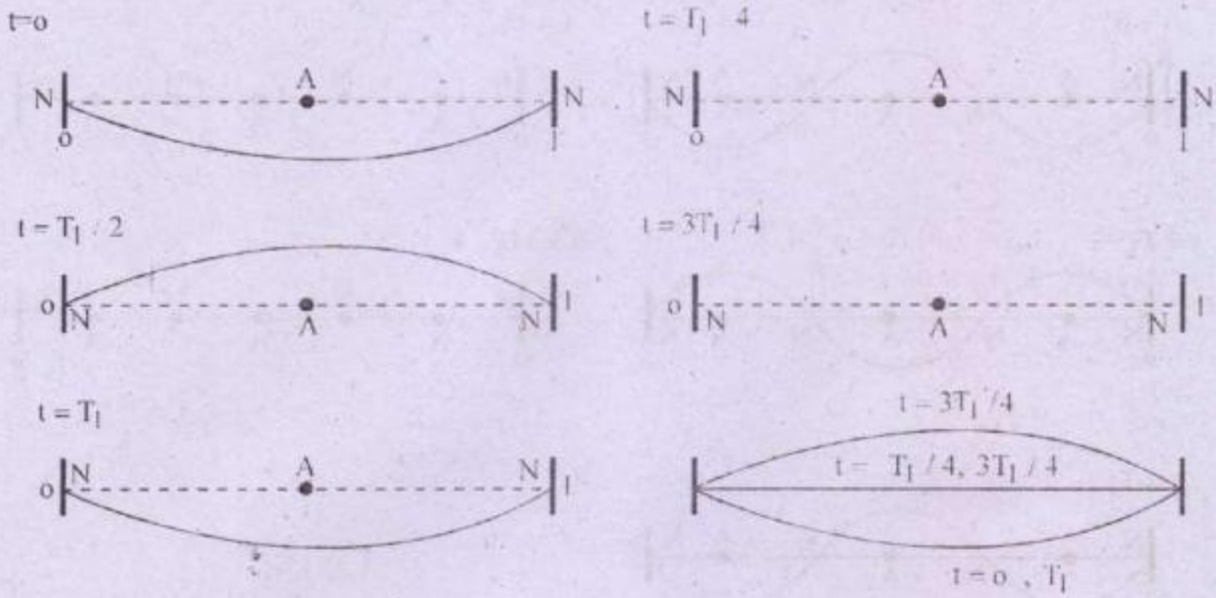


Fig.10.21 (a) (Sequence of vibration in fundamental mode)

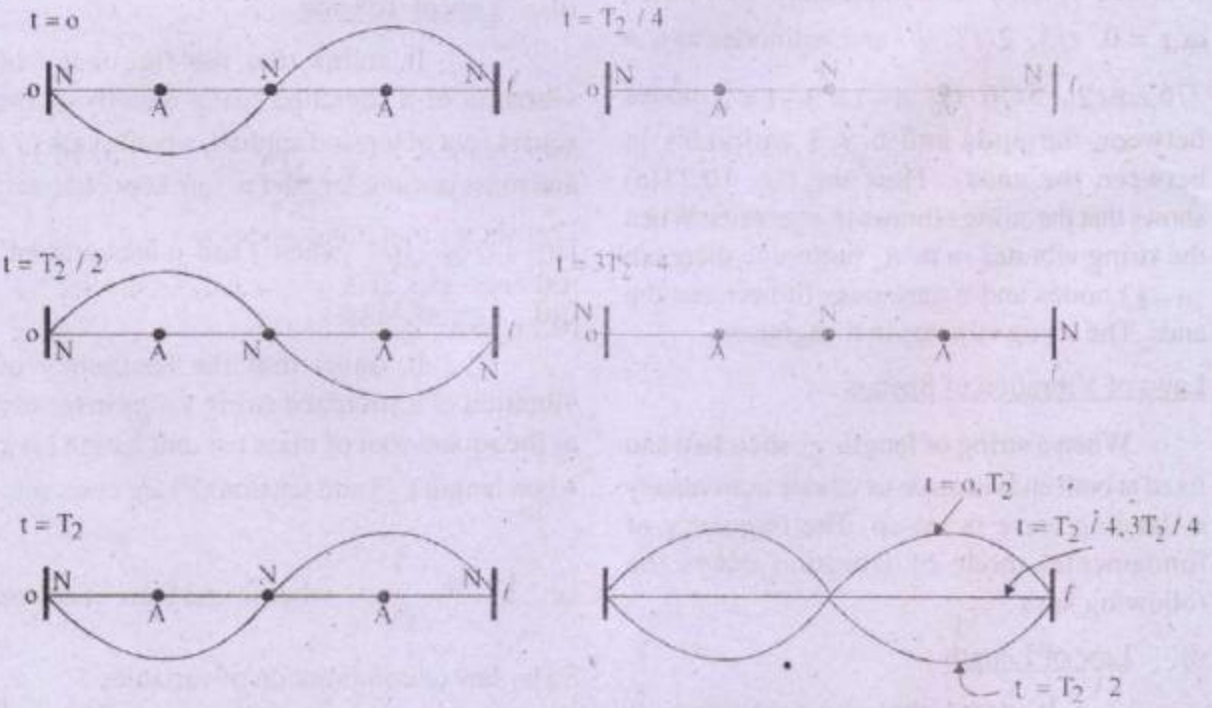


Fig.10.21 (b) (Sequence of vibration in first overtone)

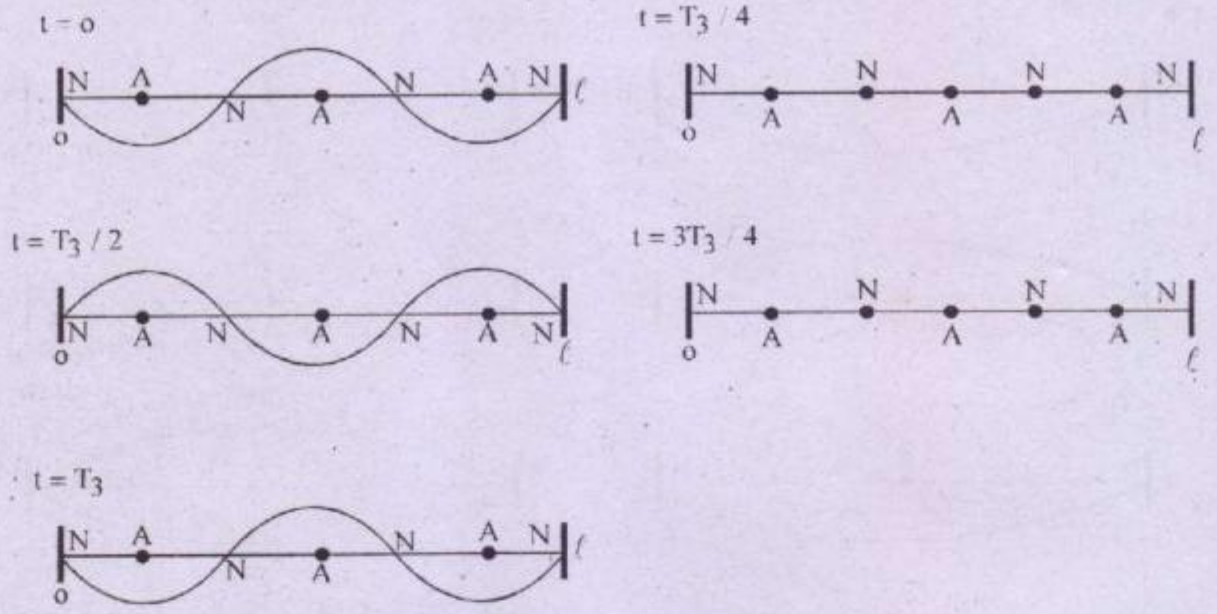


Fig.10.21. (c)

Considering in a similar manner we would find that for n = 3 ie in the second overtone or third harmonic, nodes are formed at $x = 0,\ \ell/3,\ 2\ell/3,\ \ell$ and antinodes at $x = \ell/6,\ \ell/2,\ 5\ell/6$ (ie $n-1=3-1=2$ nodes between the ends and n = 3 antinodes in between the ends). Thus the fig. 10.21(a) shows that the string vibrates in segments. When the string vibrates in its n_{th} harmonic there are $(n-1)$ nodes and n antinodes in between the ends. The string vibrates in n-segments.

Laws of Vibration of Strings :

When a string of length ℓ, stretched and fixed at both ends is made to vibrate transversely a standing wave is set up. The frequency of fundamental mode of vibration obeys the following laws :

(i) Law of Length :

It states that the frequency of vibration of a stretched string varies inversely as length, when tension (F), and mass per unit length (μ) are constant

i.e. $\nu \propto \frac{1}{\ell}$, when F & μ are constant

(ii) Law of Tension :

It states that the frequency of vibration of a stretched string directly as the square root of tension applied, when length (ℓ) and mass per unit length (μ) are kept constant.

i.e $\nu \propto \sqrt{F}$ when ℓ and μ are constant.

(iii) Law of Mass :

It states that the frequency of vibration of a stretched string varies inversely as the square root of mass per unit length (μ); when length (ℓ) and tension (F) are constant.

i.e. $\nu \propto \frac{1}{\sqrt{\mu}}$ when ℓ and F are constant

So by law of combination of variables

$$\nu \propto \frac{1}{\ell}\sqrt{F/\mu}$$

Experiment shows that when the string vibrates in its fundamental mode

$$\nu = \frac{1}{2\ell}\sqrt{F/\mu}$$

and when it vibrates in its nth harmonic

$$\nu = \frac{n}{2\ell}\sqrt{F/\mu}$$

Sometimes the law of mass is broken into

the laws (as $\sqrt{\mu} = \sqrt{\frac{\pi D^2}{4}.1.\rho} = \sqrt{\frac{\pi}{4}}.\ D.\ \sqrt{\rho}$)

(a) Law of Diameter

It states that $\nu \alpha \frac{1}{D}$, when ℓ, F & ρ are constant

(b) Law of Density

It states that $\nu \alpha \frac{1}{\sqrt{\rho}}$, when ℓ, F & D are constant.

Solved Examples

Ex.10.1 The displacement of a wave is represented by $y = 0.25 \times 10^{-3} \sin(500t - 0.25x)$, where y, t, x are in cm, Sec and m respectively. Deduce (i) amplitude (ii) time period (iii) angular frequency (iv) wavelength. What is the phase of wave at x = 1.2 cm and t = 0.1 s.

Soln.

Given

$y = 0.25 \times 10^{-3} \sin(500t - 25x)$ with y in Cm and x in m. So expressing x in cm

$$y = 0.25 \times 10^{-3} \sin(500t - 25x)$$

$$\equiv A\sin\frac{2\pi}{\lambda}(\vartheta t - x)$$

$$\Rightarrow \quad \frac{2\pi}{\lambda} = 25 \quad \Rightarrow \quad \lambda = \frac{2\pi}{25} = 0.251\text{cm}$$

$$\Rightarrow \quad A = 0.25 \times 10^{-3}\text{cm}$$

$$\Rightarrow \quad \frac{2\pi}{\lambda}.\vartheta = 500 \quad \Rightarrow \quad 25\vartheta = 500$$

$$\Rightarrow \quad \vartheta = 20 \text{ cm/s}$$

Time period

$$T = \left(\frac{\vartheta}{\lambda}\right)^{-1} = \left[20/\left(\tfrac{2\pi}{25}\right)\right] \bigg/ \left(\tfrac{500}{2\pi}\right)^{-1} = \frac{\pi}{250}\text{s}$$

Angular frequency

$$\omega = \frac{2\pi}{T} = 2\pi \times \frac{250}{\pi} = 500 \text{ Hz}$$

Phase $\phi = 500t - 25x$

$$= (500 \times 0.1 - 25 \times 1.2) \text{ rad}$$

$$= (50 - 30) \text{ rad} = 20 \text{ rad}$$

Ex.10.2 A progressive wave of frequency 550 Hz is travelling with a speed of 360 m/s. How far apart are two points 60^0 out of phase ?

Soln.

Given $\nu = 550$ Hz, $\vartheta = 360$ m/s

$$\Rightarrow \quad \lambda = \frac{\vartheta}{\nu} = \frac{360}{550}\text{m}$$

for λ distance apart phase difference is 2π (=360^0). So for 60^0 phase difference distance apart is

$$\Delta = \frac{\lambda}{360} \times 60 = \frac{\lambda}{6} = \frac{360}{550} \times \frac{1}{6}\text{m} = \frac{6}{55}\text{m}$$

Ex.10.3 The amplitude of a wave disturbance propagating in the positive x-direction is given by

$$y = \frac{1}{1+x^2} \text{ at time } t = 0$$

and by

$$y = \frac{1}{1+(x-1)^2} \text{ at time } t = 2.0 \text{ s}$$

where x and y are in meters . The shape of the wave disturbance does not change during the propagation. What is the velocity of the wave ?

Soln.

$$\text{Given } y = \frac{1}{1+x^2} \text{ at } t = 0$$

$$y = \frac{1}{1+(x-1)^2} \text{ at } t = 2.0 \text{ s}$$

$\Rightarrow$ for $\Delta x = 1\text{m}$, $\Delta t = 2.0 \text{ s}$

$$\Rightarrow \text{speed } \vartheta = \frac{\Delta x}{\Delta t} = \frac{1}{2} \text{m/s} = 0.5/\text{s}.$$

Ex.10.4 The equation for a wave travelling in x-direction on a string is

$$y = (3.0 \text{ cm}) \sin\left[(3.14 \text{ cm}^{-1})x - (314 \text{ s}^{-1})t\right]$$

(i) Find the maximum velocity of a particle of the string (ii) Find the acceleration of a particle at x = 6.0 cm, at time t = 0.11 s.

Soln.

Given

$$y = (3.0 \text{ cm}) \sin\left[(3.14 \text{ cm}^{-1})x - (314 \text{ s}^{-1})t\right]$$

(i) Particle velocity

$$u = \frac{\partial y}{\partial t} = -(3.0 \text{ cm}) \times 314 \cos\left[(3.14 \text{ cm}^{-1})x - (314 \text{ s}^{-1})t\right]$$

$$u_{max} = 3 \times 314 \text{ cm/s} = 9.42 \text{ m/s}$$

(ii) Accl^n. of a particle

$$a = \frac{\partial^2 y}{\partial t^2} = -(3)(314)^2 \sin[3.14x - 314t]$$

$\therefore$ at x = 6.0 cm, t = 0.11 s

$$a = -(3)(314)^2 \sin[3.14 \times 6 - 314 \times 0.11]$$

$$= -(3)(314)^2 \sin[6\pi - 11\pi] = 0$$

$\therefore a = 0$

Ex.10.5 Speed of transverse wave, going on a wire of length 40 cm and mass 4.0 gm is 60 m/s. The area of cross-section of the wire is 1.0 mm^2 and its Young's modulus is 16×10^{11} N/m^2. Find the extension of the wire over its natural length.

Soln.

Mass per unit length

$$\mu = \frac{4}{40} \text{gm/cm} = \frac{1}{10} \text{gm/cm}$$

$$\text{Speed } \vartheta = \sqrt{F/\mu} = 60\text{m/s} = 60 \times 10^2 \text{cm/s}$$

$$\Rightarrow \frac{F}{\mu} = 36 \times 10^6 \text{cm}^2/\text{s}^2$$

$$\Rightarrow F = 36 \times 10^6 \times 0.1 \text{ gm.cm/s}^2$$

$$= 36 \times 10^5 \text{ dynes}$$

i.e F = 36 N.

$$\text{Young's modulus } Y = \frac{F/A}{\Delta\ell/\ell}$$

$$\Rightarrow \frac{\Delta\ell}{\ell} = \frac{F}{A.Y} = \frac{36\text{N}}{1 \times 10^{-6} \times 16 \times 10^{11}}$$

$$\Rightarrow \quad \Delta\ell = \frac{36\times10^5}{1\times10^{-6}\times16\times10^{11}}\times40 = 90\text{cm}$$

Ex.10.6 Two waves passing through a given region are represented by

$$y_1 = (1.0\text{cm})\sin\left[(3.14\ \text{cm}^{-1})x - (157\ \text{s}^{-1})t\right]$$

$$y_2 = (1.5\text{cm})\sin\left[(1.57\ \text{cm}^{-1})x - (314\ \text{s}^{-1})t\right]$$

Find the displacement of the particle at x = 4.5 cm at time t = 5. ms.

Soln.

Given

$$y_1 = (1.0\text{cm})\sin\left[(3.14\ \text{cm}^{-1})x - (157\ \text{s}^{-1})t\right]$$

$$\Rightarrow \quad y_1 = (1.0\text{cm})\sin\left[\pi x - 100\tfrac{\pi}{2}t\right]$$

$$y_2 = (1.5\text{cm})\sin[1.57x - 314t]$$
$$= 1.5\ \text{cm}\ \sin\left[\tfrac{\pi}{2}x - 100\pi t\right]$$

$\therefore$ At $t = 5\ \text{ms} = 5\times10^{-3}\text{s}$, $x = 4.5$ cm

$$y_1 = (1.0\text{cm})\sin\left[4.5\pi - 100\times\tfrac{\pi}{2}\times5\times10^{-3}\right]$$

$$= \sin\left[4.5\pi - \tfrac{\pi}{4}\right] = \sin\left[\tfrac{17\pi}{4}\right]\text{cm}$$

$$y_1 = \sin\left[4\pi + \tfrac{\pi}{4}\right] = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} = 0.707\text{cm}$$

$$y_2 = 1.5\sin\left[\tfrac{\pi}{2}\times4.5 - 100\pi\times5\times10^{3}\right]$$

$$= 1.5\sin\left[\frac{9}{4}\pi - \frac{\pi}{2}\right] = 1.5\sin\left(\frac{7\pi}{4}\right)$$

$$= 1.5\sin\left[2\pi - \tfrac{\pi}{4}\right] = -1.5\sin\tfrac{\pi}{4} = \frac{-1.5}{\sqrt{2}} = -1.061\text{cm}$$

Net displacement

$$\therefore \quad y = y_1 + y_2 = (0.707 - 1.061)\text{cm} = -0.354\text{cm}$$

Ex. 10.7 The vibration of a string fixed at both ends are described by the equation

$$y = (5.0\ \text{mm})\sin\left[(1.57\ \text{cm}^{-1})x\right].\sin\left[(314\ \text{s}^{-1})t\right]$$

(i) What is the maximum displacement of the particle at x = 5.66 cm ?

(ii) What are the wave-length and wave speeds of the two tranverse waves that combine to give the above vibration ?

(iii) What is the velocity of the particle at x = 5.66 cm at time t = 2.0 s ?

(iv) If the length of hte string is 10.0 cm, locate the nodes and the antinodes. How many loops are formed in the vibration ?

Soln.

Given

$$y = (5.0\ \text{mm})\sin\left[(1.57\ \text{cm}^{-1})x\right].\sin\left[(314\ \text{s}^{-1})t\right]$$

This is the standing wave formed by

$$y_1 = (2.5\ \text{mm})\cos\left[(1.57\ \text{cm}^{-1})x - (314\ \text{s}^{-1})t\right]$$

and

$$y_2 = -(2.5\ \text{mm})\cos\left[(1.57\ \text{cm}^{-1})x + (314\ \text{s}^{-1})t\right]$$

(i) At x = 5.66 cm,

$$y = (5\times10^{-1}\text{cm})\times\sin\left[\tfrac{\pi}{2}\times5.66\right]\times\sin[100\pi t]$$

$$\Rightarrow \quad y = 0.255\ \sin(100\ \pi\ t)$$

$$\therefore \quad y_{max} = 0.255\ \text{cm} = 2.55\ \text{mm}$$

(ii) $\frac{2\pi}{\lambda} = 1.57 = \frac{\pi}{2}$

$\Rightarrow \lambda = 4$ cm

$\frac{2\pi}{\lambda}\vartheta = 100\pi$

$\Rightarrow \vartheta = \frac{100\pi}{\pi/2} = 200 \text{cm/s}$

(iii) $u = \frac{\partial y}{\partial t} = (5\times10^{-1})(100\pi)$

$\sin\left(\frac{\pi}{2}x\right) \cos(100\pi t)$

at x = 5.66 cm, t = 2.0 s

$u = 0.5 \text{x} 100\pi \times \sin\left(\frac{\pi}{2}\times 5.66\right) \cos(100\pi\times 2)$

$u = 79.96$ cm/s.

(iv) Nodes occur where y = 0

$\Rightarrow \sin\left(\frac{\pi}{2}x\right) = 0$

$\Rightarrow \frac{\pi}{2}x = n\pi$

$x = 2n$ cm $\qquad n = 0, 1, 2...$ etc

Thus nodes occur at 0, 2, 4, 6, 8 and 10 cm. and antinodes occur at 1, 3, 5, 7, 9 cm. The string vibrates in 5 loops.

Ex.10.8 A steel wire of length 64 cm weighs 5 g. If it is stretched by a force of 8N, what would be the speed of a transverse wave passing on it ?

Soln.

Mass per unit length $\mu = \frac{5}{64}$ g/cm

Tension F = 8 N = 8 x 10^5 dynes

Speed $\vartheta = \sqrt{\frac{F}{\mu}} = \sqrt{\frac{8\times10^5}{(564)}}$

$= 3200 \text{ cm/s} = 32 \text{ m/s}$.

Ex.10.9 A wave pulse is travelling on a string with speed ϑ in the positive x-direction. The shape of the string at t = 0 is given by g(x) = A sin (x/a), where A and a are constants (a) What are the dimensions of A and a, (b) Write the equation of the wave for a general time 't', if the wave speed is ϑ.

Soln.

(a) A has the dimension of length

a has the dimension of length

(b) $g(x,t) = A\sin\left(\frac{x}{a} - \frac{\vartheta}{a}t\right)$

Ex.10.10 A uniform string of mass 0.5 kg is stretched between two rigid supports, separated by a distance of 2 m. The tension in the string is 10 N. Determine the speed, the fundamental frequemncy and the corresponding wavelength of the wave along the string.

Soln.

Given $\ell = 2$ m, m = 0.5 kg

=> mass per unit length

$\mu = \frac{0.5}{2} \text{kg/m} = 0.25 \text{ kg/m}$

Tension F = 10 N

$\therefore \vartheta = \sqrt{F/\mu} = \sqrt{\frac{10}{0.25}} \text{m/s}$

$\Rightarrow \vartheta = 20 \text{ m/s}$

Fundamental frequency

$\nu_1 = \frac{\vartheta}{2\ell} = \frac{20}{2\times2} = 5 \text{ Hz}$

Wavelength $\lambda_1 = \frac{\vartheta}{\nu_1} = \frac{20}{5} = 4\text{m}$

Ex.10.11 Compute the speed of longitudinal wave in air

Soln.

Mean molecular mass of air = 28.8 gm/mol

$= 28.8 \times 10^{-3}$ kg / mol

Also $\gamma = 1.40$, $R = 8.314$ J. mol^{-1} $^0K^{-1}$ at $T = 300^0$ K

$$\therefore \quad \vartheta = \sqrt{\frac{\gamma P}{\rho}} = \sqrt{\frac{\gamma PV}{\rho V}} = \sqrt{\frac{\gamma RT}{M}}$$

$$= \sqrt{\frac{1.4 \times 8.314 \times 300}{28.8 \times 10^{-3}}} \text{ m / s}$$

$$\Rightarrow \quad \vartheta = 348.2 \text{ m / s}$$

Ex.10.12 A wave is propagating on a long stretched string along its length taken as the positive x-axis. The wave equation is given as

$$y = y_o e^{-\left(\frac{t}{T} - \frac{x}{\lambda}\right)^2}$$

where $y_o = 4$ mm, $T = 1.0$ S, $\lambda = 4$ Cm.

(a) Find the velocity of the wave

(b) Find the function f (t) giving the displacement of the particle at $x = 0$

(c) Find the function g (x) giving the shape g (x) of the string at $t = 0$

(d) Plot the shape g(x) of the string at $t = 0$

(e) Plot the shape of the string at $t = 5.0$ S.

Soln.

Given

$$y = y_0 e^{-\left(\frac{t}{T} - \frac{x}{\lambda}\right)^2} = y_0 e^{-\frac{1}{\lambda^2}\left(\frac{\lambda}{T}t - x\right)^2}$$

$$\equiv f(\vartheta t - x)$$

$$\Rightarrow \text{(a)} \quad \vartheta = \frac{\lambda}{T} = \frac{4 \text{ cm}}{1 \text{ s}} = 4 \text{ cm / s}$$

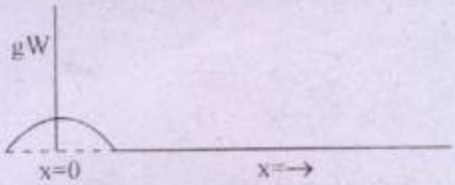
(b) Putting $x = 0$

$$f(t) = y_0 e^{-(t/T)^2}$$

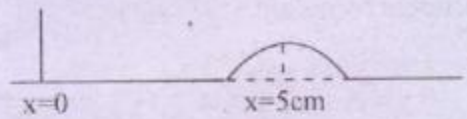
(c) Putting $t = 0$

$$g(x) = y_0 e^{-x^2/\lambda^2}$$

(d)



(e)



Summary

1. In a wave motion energy is transmitted from one place to another.

2. A wave which requires a material medium for its propagation is called a mechanical wave.

A mechanical wave may be longitudinal or transverse. e.g. Sound wave.

3. A wave which does not require a material medium for its propagation is called an electromagnetic wave.

E.g : Light, x-ray, γ-ray, radio waves etc.

4. (i) Wave length (λ) : It is the minimum distance between two particles of the medium vibrating in same phase.

(ii) Wave number (n) : Reciprocal of wave length (λ)

$n = \frac{1}{\lambda}$, $k = 2\pi\ n = \frac{2\pi}{\lambda}$ is the angular wave number.

(iii) Speed of a wave

$$V = \frac{\lambda}{T} = \lambda f = \frac{\omega}{k}$$

$\nu = \frac{1}{T}$ is the frequency of the wave.

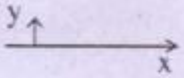
ω : angular frequency = $2\pi\nu$

5. Phase difference (ϕ) between two particles of the medium having a path difference 'x' is

$$\phi = \frac{2\pi}{\lambda}.x$$

6. Equations of a progressive wave in different forms are :

i) $y = A \sin \frac{2\pi}{\lambda}(vt \pm x)$



ii) $y = A \sin \omega \left(t \pm \frac{x}{v}\right)$

iii) $y = A \sin 2\pi \left(\frac{t}{T} \pm \frac{x}{\lambda}\right)$

iv) $y = A \sin (\omega t \pm kx)$

7. Intensity of a wave is the energy crossing unit area of the medium in unit time.

Intensity $I = 2\pi^2\nu^2\rho V A^2$

where ν = frequency

A - Amplitude

V - Velocity of the wave

$\rho \rightarrow$ density of the medium

8. Speed of longitudined wave in an elastic medium.

$$V = \sqrt{\frac{E}{\rho}}$$

E is the modulus of elasticity and ρ is the density of the medium.

. In case of longitdinal wave travelling in a fluid E = B, Bulk modulus.

$$\therefore V = \sqrt{\frac{B}{\rho}}$$

. In case of longitudinal wave travelling in a solid bar.

E = Y, Youngs modulus.

$$\therefore V = \sqrt{\frac{Y}{\rho}}$$

9. Speed of transverse wave in a string :

$$V = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{T}{\rho A}}$$

where T is the tension acting along the string and μ is the mass per unit length of the string.

$\mu = \rho A$, ρ being the density and A the cross sectional area.

10. When two identical harmonic waves travelling in same medium but in opposite directions superpose on each other, the resulting wave pattern is called standing or stationary wave.

Antinodes : The points in the medium at which pasticles suffer maximum displacement compared to other particles at any time.

Nodes : The points in the medium at which the particles are permanently at rest.

. Distance between two cansecutive nodes or two consecutive antindes is $\frac{\lambda}{2}$.

. Distance between a node and its nearest antinode is $\frac{\lambda}{4}$.

11. **Transverse vibrations in stretched string :**

The frequencies of different modes of

vibrations of a streched string are given by

$$f_n = \frac{n}{2L}\sqrt{\frac{T}{\mu}} \quad n = 1, 2, 3......$$

L is the length, μ is the mass per unit length and T is the tension acting on the string.

n=1 : $f_1 = \frac{1}{2L}\sqrt{\frac{T}{\mu}}$: Fundamental frequency or 1st harmonic.

n=2 : $f_2 = 2 f_1$: 1st over tone or 2nd harmonic. and so on.

12. Laws of transverse vibrations in a stretched string :

Law of length $f \alpha \frac{1}{L}$

Law of tension $f \alpha \sqrt{T}$

Law of mass $f \alpha \frac{T}{\sqrt{\mu}}$

Physical quantity	Dimensional Formula	SI unit
1. Wave length (λ)	[L]	m
2. Wave number (n)	$[L^{-1}]$	m^{-1}
3. Intensity of wave (I)	$[MT^{-3}]$	$w.m^{-2}$

MODEL QUESTIONS

A Multiple Choice Type Questions :

1. A sine wave is travelling in a medium. The minimum distance between two particles, which always have same speed is

   (i) $\lambda/4$ (ii) $\lambda/3$
   (iii) $\lambda/2$ (iv) λ

2. A sine wave is travelling in a medium. A particle has zero displacement at a certain instant. The particle nearest to it having zero displacement is

   (i) $\lambda/4$ (ii) $\lambda/3$
   (iii) $\lambda/2$ (iv) λ

3. Which of the following represent a wave travelling along the z-axis.

   (i) $x = A\sin(ky - wt)$
   (ii) $x = A\sin(kz - wt)$
   (iii) $y = A\sin kz \cos wt$
   (iv) $y = A\cos ky \sin wt$

4. The equation $y = A\sin^2(kx - \omega t)$ represents a wave motion with

   (i) amplitude A, frquency $\omega/2\pi$
   (ii) amplitude A/ 2, frequency ω/π
   (iii) amplitude 2A, frequency $\omega/4\pi$
   (iv) does not represent a wave motion.

5. Which of the following is a mechanical wave

   (i) Radio waves
   (ii) Light waves
   (iii) x-rays
   (iv) Sound waves

6. A cork floating in a calm pond executes SHM of frequency ν, when a wave generated by a boat passes by it. The frequency of the wave is

   (i) ν (ii) 2ν
   (iii) $\nu/2$ (iv) $\nu\sqrt{2}$

7. Two strings A and B are of same material. The radius of A is double that of B. If same tensions are applied to keep the strings stretched, what would be ratio of the speed of a transverse wave $(\vartheta_A/\vartheta_B)$ on the strings.

   (i) 1/2 (ii) 2
   (iii) 1/4 (iv) 4

8. Two waves represented by $y = a\sin(wt - kx)$ and $y = a\cos(wt - kx)$ are superposed. The resultant wave has an amplitude

   (i) a (ii) $a\sqrt{2}$
   (iii) 2a (iv) 0

9. Two periodic waves of amplitudes A_1 and A_2 pass through a region. If $A_1 > A_2$, the difference in the maximum and minumum resultant amplitude possible is

   (i) $2A_1$ (ii) $2A_2$
   (iii) A_1+A_2 (iv) $A_1 - A_2$

10. Two waves of equal amplitude A and equal frequency ν travel in a medium in same direction. The amplitude of the resultant wave is

    (i) 0 (ii) A
    (iii) 2A (iv) between 0 & 2A

11. The fundamental frequency of a string is proportional to
    - (i) inverse of its length
    - (ii) diameter
    - (ii) tension
    - (iv) density
12. Longitudinal waves cannot
    - (i) have a unique amplitude
    - (ii) transmit energy
    - (iii) have a unique wave velocity
    - (iv) be polarised.
13. A wave going in a solid
    - (i) must be longitudinal
    - (ii) may be longitudinal
    - (iii) must be transverse
    - (iv) may be transverse.
14. A wave going in a gas
    - (i) must be longitudinal
    - (ii) may be longitudinal
    - (iii) must be transverse
    - (iv) may be transverse.
15. A transverse wave travels along z-axis. The particles of the medium must move
    - (i) along z-axis
    - (ii) along x-axis
    - (iii) along y-axis
    - (iv) in the xy-plane

B. Very Short Answer Type Questions :

1. What is a wave ?
2. What is transferred in the propagation of a wave ?
3. If stationary waves are formed in a medium, is there any transfer of energy through the medium ?
4. Write the relation between ϑ, f, λ.
5. What is the type of sound wave produced when we talk ?
6. Why cannot transverse wave travel in a gas ?
7. Define wavelength.
8. What is the distance between two consecutive nodes ?
9. Which exhibits polarisation, longitudinal or transverse ?
10. Out of frequency, wavelength, velocity, which one does not change during refraction ?
11. Define nodes.
12. Define antinodes.
13. Write the equation to a progressive wave.
14. What is the phase difference between the prongs of a vibrating tuning fork ?
15. Write the expression for wave-velocity in an elastic medium.
16. What is the relation between wavelength (λ) and wave number ($\bar{\lambda}$) ?
17. What is the phase relationship between particles lying in two consecutive segments?
18. What happens to the frequency of transverse vibration of a string if its diameter is doubled and the tension is increased 4 times ?
19. What is the distance between a compression and a nearest rarefaction in longitudinal wave ?
20. What is the distance between a node and nearest antinode ?
21. What type of wave is produced in sitar ?
22. Write expression for intensity of a wave.
23. Ratio of amplitudes of two waves is 3:4. What is the ratio of their intensities ?

24. What type of wave is produced when a tuning fork vibrates ?

25. What type of waves are the ripples on a pond of water longitudinal or transverse ?

26. A progressive wave is represented by an equation $y = c_1 \sin (c_2 x + c_3 t)$. If c_1, c_2, c_3 are all positive in which direction is the wave travelling ?

27. What is the phase change when a longitudial wave travelling in a denser medium gets reflected from a rarer medium?

C. Short Answer Type Questions :

1. State the laws of transverse vibration of string.

2. Distinguish between longitudinal and transverse wave.

3. Give two important distinctive fretures of progressive and stationary wave.

4. A progressive wave is represented by $y = 10 \sin (100\pi t - \frac{x}{10})$. If x is in meter and t in seconds, find the velocity and time period.

5. State the principle of superposition.

6. What is the effect of propagation of transverse wave in a medium ?

7. State the basic principle on which the working of a sonometer is based .

8. Show that the particle speed car never be equal to the wave speed in a sin: wave if the amplitude is less than wavelength divided by 2π.

9. Show that for a wave travelling on a string

$$\frac{y_{max}}{u_{max}} = \frac{u_{max}}{a_{max}}$$

where y denotes displacement, u denotes particle speed and a denotes particle acceleration.

10. What is the smallest positive phase constant which is equivalent to 7.5π ?

11. Are torsional waves longitudinal or transverse ?

12. Is it possible to have a longitudinal wave on a stretched string ? A transverse wave on a steel rod ?

13. Two wave pulses identical in shape but inverted with respect to each other are produced at the two ends of a stretched string. At an instant when the pulses reach the middle, the string becomes completely straight. What happens to the energy of the two pulses?

14. What kinds of energy are associated with waves on a stretched string ? How could such energy be detected experimentally?

D. Unsolved Problems :

1. The speed of radio-waves in vaccum is $3 \times 10^8 \, m/s$. Find the wavelength for (i) an AM radiostation with frequency 1000 kHz (ii) an FM radiostation with frequency 100 M Hz.

2. The equation of a certain progressive wave is $y = 2 \sin 2\pi (\frac{t}{0.01} - \frac{x}{30})$ where x and y are in centimeter and t is seconds. What are (i) amplitude (ii) wave length (iii) frequency (iv) speed of propagation.

3. The equation of a wave travelling on a string stretched along x-axis is given by

$$y = A\, e^{-\left(\frac{x}{a} - \frac{t}{T}\right)^2}$$

(a) Write the dimension of A, a and T (b) Find the wave speed (c) In which direction the wave is travelling ? (d) Where is the maximum pulse located at t = T ? at t=2T?

4. The displacement of the particle at x= 0 of a stretched string carrying a wave in the positive x-direction is given by $f(t) = A \sin(t/T)$. The wave speed is ϑ. Write the wave equation.

5. Write down the equaiton to a wave travelling in negative x-direction and having amplitude 0.01 m, frequency 550 Hz, and a speed 330 m/s.

6. A string with linear mass density 0.10 kg/m is stretched under a tension of 30 N. A transverse single wave of amplitude 0.05 m and angular frequency 88 rad/s travels along the string from left to right. (a) what is the speed of the wave? (b) what is the average power of the wave?

7. Calculate the speed of mechanical waves in rail road tracks made of steel. If the frequency of the source is 240 Hz, what is the wave length of the waves in steel?

Given $Y_{steel} = 2 \times 10^{11} N/m^2$; $\rho = 7.8 \times 10^3 kg/m^3$.

8. Write the equation for a wave travelling in the negative direction along the X-axis and having an amplitude 0.01 m, frequency 200 vib / s, and a speed of 300 m/s.

9. Two waves, travelling in the same direction through the same region, have equal frequencies, wavelengths and amplitudes. If the amplitude of each wave is 4 mm and the phase difference between the waves is 90^0. What is the resultant amplitude ?

10. A longitudinal wave propagates in a steel bar having density 7.0 gm / cm^3 and Young's modulus 2 x 10^{11} Pa. (a) What is the wave speed? (b) By what factor is this speed greater than the speed in air at 20^0C ?

11. A stationary wave is represented by $y = 4\sin(0.2\pi x)\cos(200\pi t)$ where displacement (y) is in cm. What is the distance between a node and next antinode ?

12. Two waves are represented at a certain point by equations

$$y_1 = 10 \sin 2000\pi t$$

and

$$y_2 = 10 \sin(2000\pi t + \frac{\pi}{2})$$

Find the frequency and amplitude of the resultant wave.

13. A transverse wave discribed by

$$y = (0.02m)\sin\left[(1.0m^{-1})x + (30s^{-1})t\right]$$

propagates on a stretched string having a linear mass density of 1.2 x 10^{-4} kg/m. Find the tension in the string.

14. One end of a stretched rope is given a periodic transverse motion with a frequency of 10 Hz. The rope is 50 m long, has total mass 0.5 kg and is stretched with a tension of 400 N.

(a) Find wave speed and wavelength

(b) If the tension is doubled, how must the frequency be changed to maintain the same wavelength ?

15. A metal wire has these properties.

Coefficient of linear expansion = $1.5 \times 10^{-5} (^0C)^{-1}$

Young's modulus = 2.0×10^{11} Pa

Density = 9.0×10^3 kg / m^3

At each end there are rigid supports. If the tension is zero at 20^0 C, what will be the speed of a transverse wave at 8^0 C.

E. Long Answer Type Questions :

1. What do you mean by wave motion. Deduce the equation to a progressive wave.
2. Define wavelength of a wave and establish a relation between velocity and wavelength.
3. Prove analytically, that the intensity of a wave, at any point, varies direcrtly as the squate of its amplitude at that point.
4. Discuss analytically the formation of stationary waves.
5. Discuss graphically the formation of stationary waves.
6. Discuss the various points of distinction between progressive wave and stationary wave.

F. Fill in the Blank Type

1. A travelling wave has frequency υ and the particle displacement amplitude A. For the wave the particle velocity amplitude is .... and the particle acceleration amplitude is ....
2. The waves are : y = a sin (wt - kx) and y = a cos (wt - kx). Phase difference between them is ....
3. The equation : y = a cos (wt + kx) represents a .... wave, whose wavelength is ...., frequency is .... and velocity is ....
4. $y = 0.5 \sin 2\pi\ (0.1x + 2t)$, represents a wave equation in which the distances are measured in meters and time in seconds, the wave speed is .... m/s.
5. The equation to a progressive wave is given by $\frac{d^2y}{dt^2} = \frac{k}{\rho}\left(\frac{d^2y}{dx^2}\right)$, the velocity of the wave is ....

G. True - False Type

1. A wave is represented by the equation y = 0.5 sin (10t + x)m. It is a travelling wave propagating along the +x direction with velocity 10ms^{-1}
2. During the propagation of a progressive wave in a material medium, there is no net transfer of energy.
3. Waves produced by a motor boat sailing in water are both congitudinal and transverse waves.
4. Ocean waves hitting a beach are always found to be nealy normal to the shore.
5. Waves produced in a cylinder containing a liquid by moving its piston back and forth are longitudinal waves.

ANSWERS

A. Multiple Choice Type Questions :

1. (iii), 2. (iii), 3. (ii), 4. (ii),[Hints : $y = A\sin^2(kx - wt) = \frac{A}{2}\{1 - \cos 2(kx - wt)\}$]

5. (iv), 6. (i), 7. (i), 8. (ii), 9. (ii), 10. (iv), 11. (i), 12. (iv), 13. (i) & (iv), 14. (i), 15. (iv)

B. Very Short Answer Type Qestions :

1. See text
2. energy
3. No
4. $\vartheta = f\lambda$
5. Audible sound
6. gas cannot sustain a shear strain
7. See text
8. $\lambda / 2$
9. transverse
10. frequency
11. See text
12. See text
13. See text
14. π
15. $\vartheta = \sqrt{y/\rho}$ or $\sqrt{B/\rho}$
16. $\bar{\lambda} = \frac{1}{\lambda}$
17. λ - phase difference
18. No change
19. $\lambda / 2$
20. $\lambda / 4$
21. stationary transverse wave
22. See text
23. 9 : 16
24. longitudinal
25. transverse
26. - ve x- direction
27. π

C. Short Answer Type Questions :

1. See text
2. See text
3. See text
4. $\vartheta = 3.14 \times 10^3 \, m/s$, $T = \frac{1}{50} s$
5. See text
6. See text
7. Resonance
10. 1.5 π

Conceptual Type :

11. Transverse

D. Unsolved Problems

1. (i) 300 m , (ii) 3 m

2. (i) 2 cm, (ii) 30 cm, (iii) 100 Hz, (iv) 3000 cm/s

3. (a) [A] = L, [a]=L, [T]=T (b) $\vartheta = a/T$ (c) positive x-dirn

(d) at t = T, x = a and at t = 2T, x = 2a

4. $f(x,t) = A\sin\left(\frac{t}{T} - \frac{x}{\lambda}\right)$

5. $y = 0.01\sin\left(1100\pi t + \frac{10\pi x}{3}\right)$

6. (a) $10\sqrt{3}$ m/s (b) 16.77 watt

7. 5063.7 m/s ; 21.1 m

8. $y = 0.01\sin\left(400\pi t + \frac{4}{3}\pi x - \phi\right)$

9. $4\sqrt{2}$ mm

10. 5345.22 m / s; 15.58

11. 2.5 cm

12. 1000 Hz, 14.14 Units

13. 0.108 N

14. (a) 200 m/s; 20m (b) increase by $\sqrt{2}$ i.e. $10\sqrt{2}$ Hz

15. 63.25 m/s

[Hints : $\frac{\Delta\ell}{\ell} = \alpha t = 1.5 \times 10^{-5} \times 12 = 18 \times 10^{-5}$

$$\rho = \frac{m}{A.\ell} = \frac{\mu}{A} \Rightarrow \frac{1}{A} = \frac{\rho}{\mu}$$

$$\therefore \quad Y = \frac{F/A}{\Delta\ell/\ell} = \frac{\frac{F}{\mu}.\rho}{\Delta\ell/\ell}$$

$$\Rightarrow \quad \frac{F}{\mu} = \frac{Y.(\Delta\ell/\ell)}{\rho}$$

$$\vartheta = \sqrt{\frac{F}{\mu}} = \sqrt{\frac{Y.(\Delta\ell/\ell)}{\rho}} = \sqrt{\frac{2\times10^{11}\times18\times10^{-5}}{9\times10^{3}}} = 63.25 \text{ m/s}$$]

F. (1) $2\pi\upsilon A$, $4\pi^2\upsilon^2 A$ (2) $\pi/2$ (3) progressive, $2\pi/k$, $w/2\pi$, w/k (4) 20 m/s (5) $\sqrt{k/\rho}$

G. (1) False (2) True (3) True (4) True (5) True

11

Sound

Psychologically, sound is said to be the sensation produced when proper disturbance reaches the ear. Physically sound is the stimulus capable of producing the sensation of sound.

Usually motion of a vibrating source sets up waves in the surrounding elastic medium. Whether the vibration of the source is transverse or longitudinal the wave set up in the surrounding medium is a longitudinal wave whose frequency is same as the frequency of vibration of source. If the frequency is within the range 20 Hz to 20,000 Hz, then the ear is sensitive to it and is called audible (sonic) sound. The waves with frequency $\upsilon < 20$ Hz are called infrasonics and those with $\upsilon > 20{,}000$ Hz are called ultrasonics.

11.1 Speed of Sound Wave (Newton's formula)

When a longitudinal wave travels in an elastic medium compressions and rare- factions are created in the surrounding medium. The compressions and rarefactions move forward with speed ϑ from the vibrating source. This speed is called wave-speed of the longitudinal wave. The speed ϑ must depend on the density ρ of the medium and proper coefficient of elasticity E_c, e.g. in case of bar $E_c = Y$ (Young's modulus) and in case of an extended fluid $E_c = B$ (Bulk modulus).

For convenience and simplicity we adopt dimensional method and write

$$\vartheta \propto E_c^{a}$$

$$\vartheta \propto \rho^{b}$$

This gives, by Law of Combination of Variables

$$\vartheta = K E_c^{a} \rho^{b} \qquad ...(11.1.1)$$

where 'K' is a dimensionless constant. As done in sec. 10.6 one finds

$$\vartheta = \sqrt{\frac{E_c}{\rho}} \qquad ...(11.1.2)$$

When sound wave travels in air medium E_c = B (Bulk modulus), so

$$\vartheta = \sqrt{\frac{B}{\rho}} \qquad ...(11.1.3)$$

Newton assumed that when sound wave travels in an air or gaseous medium the temperature of various layers are nearly constant i.e. isothermal conditions are maintained. Therefore

$$pV = \text{constant}$$

$$\Rightarrow \; pdV + Vdp = 0$$

giving

$$P = \frac{dp}{-(dv/v)} \qquad ...(11.1.4)$$

But by definition r.h.s of eqn (11.1.4) represents bulk modulus. So under **isothermal conditions** B = p and eqn. (11.1.3) reduces to

$$\vartheta = \sqrt{\frac{P}{\rho}} \qquad ...(11.1.5)$$

Eqn. (11.1.5) is called "Newton's formula" for speed of sound wave in a gaseous medium.

The derivation of Newton's formula is based on the assumption of isothermal conditions during propagation of sound wave and ideal gas condition. However Newton's formula does not agree with experimental results.

Laplace's Correction :

Laplace introduced modifications in Newton's formula suggesting that (i) when sound waves travel in a gaseous medium compressions and rarefactions take place rapidly and (ii) the gas being poor conductor of heat, the heat developed in compression zone cannot flow to the rarefaction zone. As a result of the above isothermal conditions are not maintained rather abiabatic conditions are maintained. Hence

$$PV^{\gamma} = \text{constant} \qquad ...(11.1.6)$$

On differentiation it gives

$$\gamma P = \frac{dp}{-(dv/v)} = B \qquad ...(11.1.7)$$

Using eqn. (11.1.7) in (11.1.3)

$$\vartheta = \sqrt{\frac{\gamma P}{\rho}} \qquad ...(11.1.8)$$

Equation (11.1.8) is the Laplace's corrected formula for speed of sound wave.

Factors affecting speed of Sound :

The speed of sound in a gaseous medium is given by Laplace corrected formula (11.1.8) as

$$\vartheta = \sqrt{\frac{\gamma P}{\rho}}$$

where P is the pressure, ρ is the density and γ is the adabatic gas constant. Using ideal gas equation PV = RT, eqn. (11.1.8) can be re-written as

$$\vartheta = \sqrt{\frac{\gamma P}{\rho}} = \sqrt{\frac{\gamma PV}{\rho V}} = \sqrt{\frac{\gamma RT}{M}} \qquad ...(11.1.9)$$

where T is the absolute temperature and M is the grammalecular mass.

(i) **Effect of Pressure :**

As seen from eqn. (11.1.9) change of pressure has no effect on the speed of sound.

(ii) **Effect of density :**

Eqn. (11.1.9) shows that speed of sound in a gas varies inversely as the square root of its density i.e.

$$\frac{\vartheta_2}{\vartheta_1} = \sqrt{\frac{\rho_1}{\rho_2}}$$

(iii) **Effect of temperature :**

It also follows from eqn. (11.1.9) that speed of sound varies directly as the square root of absolute temperature i.e.

$$\vartheta \ \alpha \ \sqrt{T}$$

Therefore,

$$\frac{\vartheta_t}{\vartheta_0} = \sqrt{\frac{273+t}{273}} = \left(1+\frac{t}{273}\right)^{\frac{1}{2}} \qquad ...(11.1.10)$$

This gives

$$\vartheta_0 = \vartheta_t\left(1+\frac{t}{273}\right)^{-\frac{1}{2}} = \vartheta_t\left(1-\frac{t}{546}\right) \qquad ...(11.1.11)$$

and $$\frac{\vartheta_t - \vartheta_0}{\vartheta_0 t} = \frac{1}{546} = \alpha = 1.832\times10^{-3}\ {}^{\circ}c^{-1} \qquad ...(11.1.12)$$

The physical quantity α defined in (11.1.12) is called the temperature coefficient of speed of sound. It is defined as the change in speed of sound per unit speed at 0^0C per degree rise of temperature.

(iv) Effect of humidity :

Since the density of water vapour is less than the density of dry air, so density of moist air (ρ_m) is less than the density of dry air (ρ_d)

i.e. $\rho_d > \rho_m$

Therefore from equ. (11.1.9) we obtain

$$\frac{\vartheta_d}{\vartheta_m} = \sqrt{\frac{\rho_m}{\rho_d}} < 1$$

$$\Rightarrow \quad \vartheta_m > \vartheta_d$$

i.e. speed of sound in moist air is more than speed of sound in dry air.

Ex. 11.1.1 Calculate the speed of sound waves in rail road tracks made of steel : (Given $Y_{steel} = 2 \times 10^{11}$ N/m^2; $\rho = 7.8 \times 10^3$ Kg /m^3). If the frequency of source is 240 Hz, what is the wave length of the waves in steel ?

Soln.

$$\vartheta = \sqrt{\frac{Y}{\rho}} = \sqrt{\frac{2\times10^{11}\ \ N/m^2}{7.8\times10^3\ \ kg/m^3}}$$

$$\vartheta = 5064 m/s$$

$$\lambda = \frac{\vartheta}{\upsilon} = \frac{5064}{240} = 21.1 m \qquad \text{(Ans)}$$

Ex. 11.1.2 Calculate the speed of sound in mercury if its Bulk modulus is 2.7×10^{10} N / m^2 and its density is 13.6×10^3 kg / m^3.

Soln.

$$\vartheta = \sqrt{\frac{B}{\rho}} = \sqrt{\frac{2.7\times10^{10} N/m^2}{13.6\times10^3 kg/m^3}} = 1409 m/s$$

Ex. 11.1.3 Calculate the speed of sound waves in an atmosphere of helium at 0^0c. and 1 atm. pressure. Note that 4 gm. of helium under STP has a volume of 22.4 litre. For He, $\gamma = 1.67$.

Soln.

$$\vartheta = \sqrt{\gamma P_0/\rho_0} = \sqrt{\gamma P_0 V_0 / M}$$

$$= \sqrt{\frac{1.67\times13.6\times76\times980\times22.4\times1000}{4}}$$

$$= 97328.838 \text{ cm/s}$$

$$\vartheta = 973.29 \text{ m/s} \qquad \text{(Ans)}$$

Ex. 11.1.4 Calculate the speed of sound waves at 30^0 C in (a) hydrogen (b) helium (c) oxygen. Given R = 8.31 J mol^{-1} K^{-1} and $\gamma = 1.41$ for H_2, O_2, and 1.67 for He.

Soln.

$$\vartheta = \sqrt{\frac{\gamma RT}{M}}$$

(a) For hydrogen

$$\vartheta = \sqrt{\frac{1.41\times8.31\times303}{2\times10^{-3}}} = 1332.3 m/s$$

(b) For helium

$$\vartheta = \sqrt{\frac{1.67\times8.31\times303}{4\times10^{-3}}} = 1025.3 m/s$$

(c) For oxygen

$$\vartheta = \sqrt{\frac{1.41\times8.31\times303}{16\times10^{-3}}} = 471.05 m/s$$

Ex. 11.1.5 At what temperature will the velocity of sound in a gas be double its value at 0^0 C.

Soln.

$$\vartheta \ \alpha \ \sqrt{T}$$

$$\Rightarrow \quad \frac{\vartheta}{\vartheta_0} = \sqrt{\frac{T}{T_0}} = \sqrt{\frac{T}{273}} = \sqrt{\frac{273+t}{273}} = 2$$

$$\Rightarrow \frac{273+t}{273} = 4$$

$$\Rightarrow \quad t = 4\times273 - 273 = 819^0 C$$

Ex. 11.1.6 A tuning fork produces a sound wave of wavelength 2m in air at 0^0 C. What will be the wavelength of sound in air at 10^0 C ?

Soln.

$$\vartheta = \vartheta_0\left(1+\frac{t}{273}\right)^{\frac{1}{2}}$$

$$\Rightarrow \quad \upsilon\lambda = \upsilon\lambda_0\left(1+\frac{t}{273}\right)^{\frac{1}{2}}$$

$$\Rightarrow \quad \lambda = \lambda_0\left(1+\frac{t}{273}\right)^{\frac{1}{2}}$$

$$= 2\left(1+\frac{10}{273}\right)^{\frac{1}{2}}$$

$$\lambda = 2.036 \text{ m}$$

Ex. 11.1.7 Show that the velocity of sound in a gas of $\gamma = 1.41$ is 0.68 C, where 'C' is the rms speed of its molecules.

Soln.

$$\vartheta = \sqrt{\frac{\gamma P}{\rho}}$$

According to Kinetic theory $P = \frac{1}{3}\rho C^2$

$$\Rightarrow \quad \sqrt{\frac{P}{\rho}} = \frac{C}{\sqrt{3}}$$

$$\therefore \vartheta = \sqrt{\frac{\gamma C^2}{3}} = \sqrt{\frac{1.41}{3}}\ C$$

$$\Rightarrow \quad \vartheta = 0.685\ C$$

Ex. 11.1.8 Show that the speed of sound increases by 0.61 m/s per ^{0} C rise in temperature. (velocity of sound at 0^0 C is 333 m/s)

Soln.

$$\vartheta_t = \vartheta_0\left(1+\frac{t}{273}\right)^{\frac{1}{2}} \simeq \vartheta_0\left(1+\frac{t}{546}\right)$$

$$\Rightarrow \quad \frac{d\vartheta_t}{dt} = \frac{\vartheta_0}{546}$$

If $dt = 1^0$ C

$$d\vartheta_t = \frac{\vartheta_0}{546} = \frac{332 m/s}{546} = 0.61 m/s$$

Table 11.1

(Speed of sound in different media)

Medium	**Speed (m/s)**	**Medium**	**Speed (m/s)**
Air (dry 0^0C)	332	Copper	3560
Air (20^0 C)	343	Aluminium	5100
CO_2 (0^0C)	259	Iron	5120
H_2 (0^0C)	1284	Steel	5945
O_2 (0^0C)	317	Glass (pyrex)	5170
Water (0^0C)	1402	Vulcanised rubber	54
Sea water	1522		
Methyl alcohol	1143		
Blood (37^0 C)	1570		

Equation to a travelling sound wave :

The equation to a progressive sound wave is given as (similar to 10.6.7, 10.6.16)

$$y = A \sin \frac{2\pi}{\lambda}(\vartheta t \pm x) \qquad ...(11.1.13)$$

OR

$$P = P_m \sin\left[\frac{2\pi}{\lambda}(\vartheta t \pm x) + \frac{\pi}{2}\right] \qquad ...(11.1.14)$$

where $P_m = \frac{2\pi}{\lambda}$ A. B $= \frac{\omega}{\vartheta}$ AB; B being Bulk modulus.

Intensity of a sound wave :

As worked out in Sec. 10.6, (eqn. 10.6.29) the intensity of sound wave is given as

$$I = \frac{1}{2}\rho\ \omega^2 A^2 \vartheta \qquad ...(11.1.15)$$

Since $P_m = \frac{\omega}{\vartheta}$ AB and $\vartheta = \sqrt{\frac{B}{\rho}}$

So we obtain $P_m^2 = \frac{\omega^2}{\vartheta^2} A^2 B^2 = \frac{\omega^2}{\vartheta^2} A^2 \vartheta^4 \rho^2$

$\Rightarrow \quad P_m^2 = \omega^2 A^2 \vartheta^2 \rho^2$

$\Rightarrow \quad \frac{P_m^2}{2\rho\vartheta} = \frac{1}{2}\rho\omega^2 A^2 \vartheta^2 = I$

$$\text{Thus } I = \frac{P_m^2}{2\rho\vartheta} \qquad ...(11.1.16)$$

11.2 Musical Sound :

The sound which is produced by a series of similar pulses following each other regularly and rapidly at equal intervals of time without any sudden change in amplitude is called as musical sound.

A musical sound produces a pleasing effect. Musical sound having a single frequency is called a tone and that having a number of frequencies is called note . In a musical sound, the tone having the lowest frequency is called the fundamental or prime tone and higher ones are called overtones .

For example, sound produced by a tuning fork, sitar, organ pipe, are musical sound.

Noise :

The sound which is produced by a series of dissimilar pulses following each other irregularly and slowly at unequal intervals of time, with sudden change in amplitude is called noise.

For example, sound produced by firing of a gun, clapping, falling of a metal plate are noise.

Characteristics of musical sound :

A musical sound is characterised by one objective property (i) intensity and three subjective properties (ii) loudness (iii) pitch (iv) quality.

*(i) **Intensity :***

As discussed earlier in Sec. 10.6 and Sec.11.1, intensity of sould wave at any point is defined as the amount of energy flowing per unit normal area around that point per unit time.

The lowest intensity of an audible sound for a normal ear is $I_0 = 10^{-12}$ W/m², and is called Threshold intensity for hearing. The minimum intensity of sound which causes pain to a normal ear is 1 W/m², and is called Threshold intensity of pain.

Intensity of sound depends on

(a) Amplitude of vibration as ($I \propto A^2$)

(b) Area of the vibrating surface as ($I \propto s$)

(c) Distance from the source as $\left(I \propto \frac{1}{r^2}\right)$

(d) Density of the medium as $(I \propto \rho)$

(ii) ***Loudness :***

Loudness of sound is the degree of sensation of sound produced in an ear.

Thus the loudness depends on (a) intensity of sound and (b) response of the ear. Therefore loudness is not entirely a physical quantity. It is partly subjective.

Weber and Fechner's study gives the relation between loudness (L) and intensity (I) as

$$L \propto \log I$$

$$\Rightarrow \quad L = K \log_{10} I \quad ...(11.2.1)$$

where K is a constant of proportionality. If I_0 be threshold intensity and L_0 be corresponding loudness for a normal ear, then

$$L_0 = K \log_{10} I_0 \quad ...(11.2.2)$$

Fron eqn. (11.2.1) and (11.2.2) one obtains

$$\beta = L - L_0 = K \log_{10} (I / I_0) \quad ...(11.2.3)$$

The quantity $\beta = L - L_0$, defined in eqn (11.2.3) is called the intensity level. The constant of proportionality 'K' depends upon the choice of units. When K = 1, the unit for intensity level is 'bel'. Then

$$\beta = \log_{10} (I / I_0) \text{bels} \quad ...(11.2.4)$$

Eqn. (11.2.4) shows that if $I = 10 I_0$, then $\beta = 1$ bel. Hence **'intensity level of sound is said to be 1 bel if its intensity (I) is 10 times the threshold intensity (I_0).**

On the otherhand, when one takes K = 10, the unit of intensity level is expressed in "deci bel". Then

$$\beta = 10 \log_{10} (I / I_0) \text{decibels} \quad ...(11.2.5)$$

Equations (11.2.4) and (11.2.5) give

1 bel = 10 decibel

Further eqn (11.2.5) implies that **"intensity level of a sound is said to 1 decibel if its intensity is $(10)^{1/10}$ times the threshold intensity".**

(iii) ***Pitch :***

Pitch of a sound is the sensation that enables one to differentiate between voices and classify them as high or low. For example we say that buffalovoice is of low pitch, a male voice has higher pitch and a female voice has still higher pitch. It is a subjective quantity and cannot be measured by any instrument. Pitch is related to the objective quantity "frequency", more particularly the dominant frequency present in the sound. Experiments reveal that pitch depends on the fundamental (dominant) frequency and not on the overtones. But there is no one-to-one correspondence. For a pure tone of constant intensity the pitch becomes higher as the frequency is increased. But the pitch of a pure tone of constant frequency becomes lower as the intensity level is raised.

(iv) ***Quality (Timbre) :***

It is a common experience that sound notes of same pitch and loudness coming from a flute and a violin are different. Thus there exists a property which distinguishes two notes of same pitch and loudness. This property is called **quality.**

It is observed that two notes of same pitch and loudness have different wave forms. Thus one can associate quality with wave form. Hence quality depends on the number of overtones present, the nature and relative intensities of the overtones.

Ex.11.2.1 Calculate the intensity of the wave $y = 0.002 \sin (4000 t - 10 x)$. Density of the medium $\rho = 1.29$ kg/m^3.

Soln.

$$I = \frac{1}{2}\rho\ \omega^2 A^2 \vartheta$$

Given y = 0.002 Sin (4000 t - 10 x)

$$\Rightarrow \quad A = 0.002$$

$$\omega = 4000 = 2\pi\upsilon$$

$$k = \frac{2\pi}{\lambda} = 10$$

$$\vartheta = \frac{\omega}{k} = \frac{2\pi\upsilon}{2\pi/\lambda} = \upsilon\lambda = \frac{4000}{10} = 400 \text{ m/s}$$

$$\therefore I = \frac{1}{2} \times 1.29 \times (4000)^2$$

$$\times (0.002)^2 \times 400 \text{ watt/m}^2$$

$$I = 165.12 \times 10^2 \text{ watt/m}^2$$

Intensity level

$$\beta = L - L_0 = \log_{10}(I/I_0) \text{bel}$$

$$= \log_{10}\left(\frac{165.12 \times 10^2}{10^{-12}}\right) \text{bel}$$

$$\beta = 16.22 \text{ bel}$$

Ex. 11.2.2 A tuning fork of frequency 5/2 Hz causes air particles to vibrate with amplitude 2 x 10^{-6} m. What energy current density (intensity) does it produce in air ? Velocity of sound in air = 340 m/s. Density of air = 1.3 kg/m³.

Soln.

Given A = 2 x 10^{-6} m

$$\upsilon = 512 \text{Hz}$$

$$\vartheta = 340 \text{m/s}$$

$$\rho = 1.3 \text{ kg/m}^3$$

$$I = \frac{1}{2}\rho\omega^2 A^2 \vartheta = 2\pi^2 \rho \upsilon^2 A^2 \vartheta$$

$$= 2 \times \pi^2 \times 1.3 \times (512)^2$$

$$\times (2 \times 10^{-6})^2 \times 340 \text{ watt/m}^2$$

$$= 9.15 \times 10^{-3} \text{ watt/m}^2$$

Ex. 11.2.3 The intensity level of a class room is 50 dB if only one person is talking. What will be the intensity level if 100 persons are talking simultaneously?

Soln.

Given intensity level of a classroom when one person is talking as 50 dB.

$$\text{i.e } \beta = 10 \log_{10}(I/I_0) = 50$$

$$\Rightarrow \quad \log_{10}(I/I_0) = 5$$

When 100 persons are talking

$$\beta' = 10 \log_{10}\left(\frac{100I}{I_0}\right)$$

$$= 10\left[\log_{10} 10^2 + \log_{10}(I/I_0)\right]$$

$$= 20 + 10 \log (I/I_0)$$

$$= 20 + 50 = 70 \text{ dB} \qquad \text{(Ans)}$$

Ex. 11.2.4 The intensity of 10 violines is 10 times that of a single violin. What is the change in dB in the sound level if all the violins are playing simultaneously ?

Soln.

$$\beta = 10 \log_{10} (I/I_0) \text{dB}$$

$$\beta' = 10 \log_{10} (10\, I/I_0) \text{dB}$$

$$= 10 [\log_{10} 10 + \log_{10} (I/I_0)]$$

$$\Rightarrow \quad \beta' = 10 + 10 \log_{10} (I/I_0) = (10 + \beta) \text{ dB}$$

$$\therefore \quad \beta' - \beta = 10 \text{ dB} \qquad \text{(Ans)}$$

Ex.11.2.5 If the intensity level is 100 dB. Calculate the amount of energy falling on the ear of area 5 cm².

Soln.

$$\beta = 100\,dB = 10\log_{10}(I/I_0)$$

$$\Rightarrow \log_{10}(I/I_0) = 10$$

$$\Rightarrow I = 10^{10} I_0 = 10^{10} \times 10^{-12}\,watt/m^2$$

$$I = 10^{-2}\,watt/m^2$$

So energy falling on area A per second is

$$E = I.A. = 10^{-2}\,\frac{watt}{m^2} \times 5 \times 10^{-4}\,m^2$$

$$E = 5 \times 10^{-6}\,Js^{-1}. \quad \text{(Ans)}$$

Ex. 11.2.6 Show that if P_1 and P_2 are the pressure amplitude of two sound waves, the difference in intensity level of the waves is

$$\beta_2 - \beta_1 = 20\log_{10}\left(\frac{P_2}{P_1}\right)$$

Soln.

We have $I = \frac{P_m^2}{2\rho\vartheta}$

where 'P_m' is pressure amplitude, ρ density and ϑ speed.

$$\therefore I_1 = \frac{P_1^2}{2\rho\theta}, \quad \therefore I_2 = \frac{P_2^2}{2\rho\theta}$$

$$\Rightarrow \beta_1 = K\log_{10}(I_1/I_0);\ \beta_2 = K\log_{10}(I_2/I_0)$$

$$\Rightarrow \beta_2 - \beta_1 = K\log_{10}(I_2/I_1) = K\log_{10}(P_2^2/P_1^2)$$

$$\beta_2 - \beta_1 = 2K\log_{10}(P_2/P_1)$$

When intensity level is measured in <u>decibel</u> $K = 10$

$$\therefore \beta_2 - \beta_1 = 20\log_{10}(P_2/P_1)$$

Ex.11.2.7 A source of sound emits a total power of 10 watt, uniformly in all directions. At what distance from the source is the sound level 100 db.

Soln.

At distance r m; $I = \frac{10\text{ watt}}{(4\pi r^2)m^2}$

$$\therefore \beta = 10\log_{10}\left(\frac{I}{I_0}\right) = 10\log_{10}\left(\frac{I}{10^{-12}}\right)$$

$$= 10\log_{10}\left(\frac{10/4\pi r^2}{10^{-12}}\right)$$

$$\Rightarrow \beta = 10\log_{10}\left(\frac{10^{13}}{4\pi r^2}\right) = 100\text{db}$$

$$= 10\log_{10}(10^{10})$$

$$\Rightarrow \frac{10^{13}}{4\pi r^2} = 10^{10}$$

$$\Rightarrow 4\pi r^2 = 10^3$$

$$\Rightarrow r = 8.92\ \text{m.}$$

11.3 Interference of Sound Waves :

The phenomenon of superposition of two waves of same frequency, same wavelength and same state of vibration (polarisation) and travelling in same direction in a given medium with no phase difference or constant phase difference is called **Interference**. It is characterised by re-distribution of energy as a result of reinforcement and cancellation of waves.

Conditions of Interference

(i) Two sources must emit similar waves continuously - condition for sustained interference.

(ii) The amplitudes of the two waves must be equal or nearly equal - condition for good contrast.

(iii) The two sources should be situated close to each other.

(iv) The two sources should be coherent i.e. they should be in same state of vibration and have no or constant phase difference.

Analytical Treatment :

Consider two waves, emanating from two sources (say two tuning forks as shown in fig.11.1) and travelling in positive X- direction. Let them be represented as

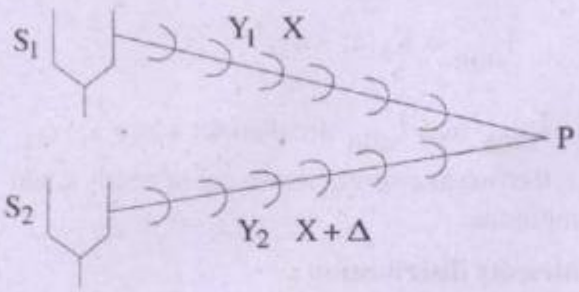


Fig.11.1

$$y_1(x,t) = a_1 \sin\frac{2\pi}{\lambda}(\vartheta t - x) \qquad ...(11.3.1)$$

$$y_2(x,t) = a_2 \sin\left[\frac{2\pi}{\lambda}(\vartheta t - x) + \phi\right] \qquad ...(11.3.2)$$

at the point P, where they superpose on each other. The phase difference ϕ is

$$\phi = \phi_O + \phi_p$$

where, ϕ_0 is the initial phase difference which may be zero or have a constant value, and ϕ_p is the phase difference arising due to the path difference (Δ). Then by the principle of superposition, the resultant wave function is given by

$$y\,(x, t) = y_1\,(x, t) + y_2\,(x, t)$$

$$= a_1 \mathrm{Sin}\frac{2\pi}{\lambda}(\vartheta t - x) + a_2 \mathrm{Sin}\left[\frac{2\pi}{\lambda}(\vartheta t - x) + \phi\right]$$

$$\Rightarrow \quad y\,(x,t) = (a_1 + a_2 \mathrm{Cos}\phi)\mathrm{Sin}\frac{2\pi}{\lambda}(\vartheta t - x)$$

$$+(a_2 \mathrm{Sin}\phi)\mathrm{Cos}\frac{2\pi}{\lambda}(\vartheta t - x) \qquad ...(11.3.3)$$

Putting

$$a_1 + a_2 \mathrm{Cos}\phi = A\mathrm{Cos}\theta \qquad ...(11.3.4)$$

$$a_2 \mathrm{Sin}\,\phi = A \,\mathrm{Sin}\,\theta \qquad ...(11.3.5)$$

so that

$$A = \sqrt{a_1^2 + a_2^2 + 2a_1 a_2 \mathrm{Cos}\,\phi} \qquad ...(11.3.6)$$

$$\tan\theta = \frac{a_2 \mathrm{Sin}\,\phi}{a_1 + a_2 \mathrm{Cos}\,\phi} \qquad ...(11.3.7)$$

equation (11.3.3.) reduces to

$$y\,(x,t) = A\,\mathrm{Sin}\left[\frac{2\pi}{\lambda}(\vartheta t - x) + \theta\right] ...(11.3.8)$$

Thus the resultant wave pattern is also sinusoidal, (similar to the interfering ones) Hence the intensity (I) of the resultant wave form is given (as per eqn. 10.6.29) as

$$I = \frac{1}{2}\rho\omega^2 A^2 \vartheta \equiv K_O A^2$$

$$\Rightarrow \quad I = K_0\,(a^2 + a_2^2 + 2a_1 a_2\,\mathrm{Cos}\,\phi) ...(11.3.9)$$

Equation (11.3.9) shows that the intensity of resultant waveform (at P) depends upon the amplitudes a_1 and a_2 of interfering waves and the phase difference ϕ.

Case - I (Constructive interference) :

Intensity (I) is maximum when A^2 is maximum. Eqn. (11.3.6) shows that A^2 is maximum when $\cos\phi = +1$

i.e. $\phi = 2n\pi$, n= 0,1,2,..... etc. ...(11.3.10)

Then $A_{max} = a_1 + a_2$ and $I_{max} = K_0\,(a_1 + a_2)^2$ ...(11.3.11)

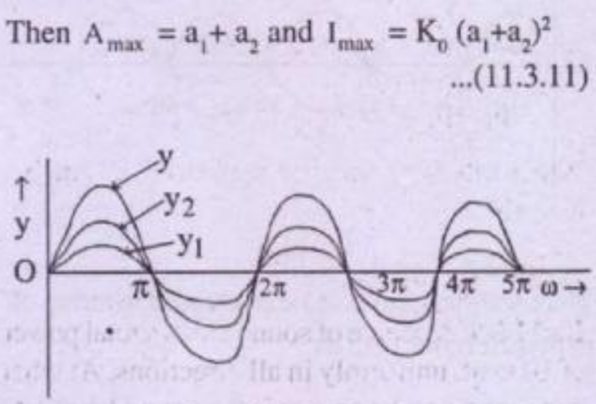


Fig. 11.2

Equation (11.3.10) is the condition for constructive interference. But $\phi = \phi_0 + \phi_p$. However if we start counting time from the instant when the vibrating sources are in same phase, then $\phi_0 = 0$ and $\phi = \phi_p = \frac{2\pi}{\lambda}\Delta$. Then eqn. (11.3.10) reduces to

$$\frac{2\pi}{\lambda}\Delta = 2n\pi$$

$$\Rightarrow \Delta = n\pi,\ n = 0, 1, 2, 3..... \text{etc.} \quad ...(11.3.12)$$

Eqn. (11.3.12) implies that when the two waves arrive at the point (P) with zero or integral multiple of λ, path difference (Δ), there is constructive interference.

Case II (Destructive interference)

Intensity (I) is minimum when A^2 is minimum. Now A^2 is minimum when $\text{Cos}\phi = -1$.

i.e. $\phi = (2n+1)\pi$, n= 0, 1, 2, ...(11.3.13)

Then

$$A^2_{min} = a_1^2 + a_2^2 - 2a_1a_2 = (a_1 - a_2)^2$$

$$\Rightarrow A_{min} = a_1 \sim a_2$$

$$I_{min} = K_0(a_1 - a_2)^2 \quad ...(11.3.14)$$

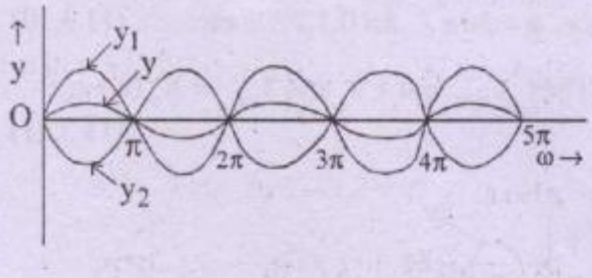


fig.11.3

The condition can also be expressed in terms of path difference (Δ) as

$$\frac{2\pi}{\lambda}\Delta = (2n+1)\pi$$

$$\Rightarrow \Delta = (2n+1)\pi/2,\ n=0, 1, 2,... \text{etc.} \quad ...(11.3.15)$$

From eqns. (11.3.11) and (11.3.14) we find that

$$I_{max} = K_0(a_1 + a_2)^2$$

$$I_{min} = K_0(a_1 - a_2)^2$$

So I_{max} and I_{min} are distinct when $a_1 \simeq a_2$. i.e. the two interfering waves are of nearly equal amplitude.

Intensity distribution :

As given in eqn. (11.3.9), the intensity of resultant wave is given as

$$I = K_0\ (a_1^2 + a_2^2 + 2a_1a_2\text{Cos}\ \phi)$$

For distinct maxima and minima it is required that $a_1 \sim a_2 = \delta$, be very small and is most distinct when $\delta = 0$ (i.e. $a_1 = a_2$) Now putting $a_1 - a_2 = \delta$ $a_2 = a$ and retaining upto first order in δ, we obtain

$$I = K_0\left[(a+\delta)^2 + a^2 + 2(a+\delta)a\ \text{Cos}\phi\right]$$

$$\simeq K_0\left[2a^2(1+\text{Cos}\phi) + 2a\delta\ (1+\text{Cos}\phi)\right]$$

$$= K_0\left[4a^2\ \text{Cos}^2\frac{\phi}{2} + 4a\delta\ \ \text{Cos}^2\frac{\phi}{2}\right]$$

$$I = K_0.4a\ (a+\delta)\ \ \text{Cos}^2(\frac{\phi}{2}) \quad ...(11.3.16)$$

This shows that for maximum $\phi = 2n\pi$, hence

$$I_{max} = K_0.4a(a+\delta)$$

and for minimum $\phi = (2n+1)\pi$, hence

$$I_{min} \mp 0$$

Thus for distinct interference pattern, the leading order contribution to intensity can be taken to be

$$I = K_0 . 4a^2 . \cos^2(\frac{\phi}{2}) \quad ...(11.3.17)$$

A graphical representation of variation of intensity (I) with phase difference ϕ is as shown in fig. 11.4.

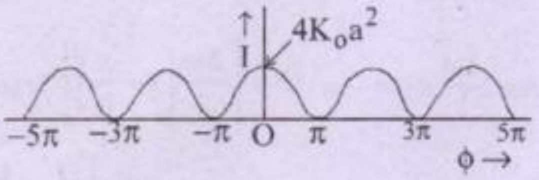


Fig.11.4

It is to be noted that had there been no interference the intensity at any point would have been $K_0a^2 + K_0a^2 = 2K_0a^2$. The maximum value $4K_0a^2$ and minimum value zero, when there is interference indicates the energy of $2K_0a^2$ has flown from region of minimum to region of maximum. Thus there is redistribution of energy.

Experimental Demonstration :

Interference of sound can be experimentally demonstrated by using a quincke's tube. It consists of two U-tubes such that one can slide into the other. Through one

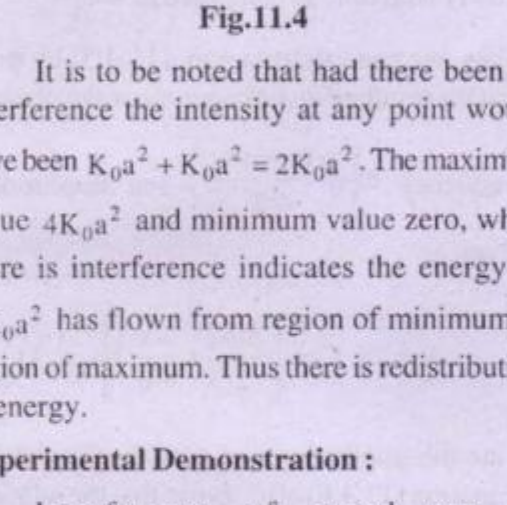


Fig.11.4(a)

of the opening (O_1) in APB tube sound wave is injected into Quincke's tube. The wave is divided into two parts near O_1, and travel distance O_1BPAO_2 and O_1DQCO_2. By sliding the tube, the path difference is varied and maximum, minimum sounds are heard near the opening O_2.

Ex. 11.3.1 Two waves of same frequency but of amplitude in the ratio 1 : 3 are superposed. What is the ratio of maximum to minimum intensity ?

Soln.

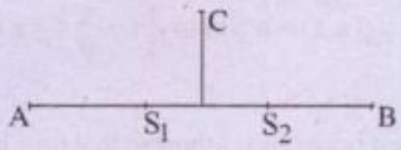
$$a_1 : a_2 = 1 : 3 \Rightarrow a_2 = 3a_1$$

$$A_{max} = a_1 + a_2 = 4a_1$$

$$A_{min} = a_2 - a_1 = 2a_1$$

$$\therefore \frac{I_{max}}{I_{min}} = \frac{A^2 max}{A^2 min} = \frac{16}{4} = \frac{4}{1}$$

Ex.11.3.2 The sources S_1 and S_2 have the same frequency and same phase. Also $S_1S_2 = \lambda / 2$. What type of interference will occur at A, B and C.



Soln.

At 'A' path difference

$$\Delta = S_2A - S_1A = S_1S_2 = \lambda / 2$$

Hence phase difference between waves reaching at A is $\phi = \frac{2\pi}{\lambda} . \frac{\lambda}{2} = \pi$

Hence at 'A' destructive interference will take place.

Also at B, $\Delta = S_1B - S_2B = S_1S_2 = \frac{\lambda}{2}$.

Hence destructive interference at B.

At C, $\Delta = S_1C - S_2C = 0 \Rightarrow \phi = 0$.

Hence constructive interference at C.

11.4 Beats :

When two sound waves of slightly different frequencies, with equal or nearly equal amplitudes, moving with same speed in same direction in a medium superpose on each other, periodic variations of intensity of sound at a place is observed.

This phenomenon of periodic variations of the intensity of the wave (resultant wave) resulting due to the superposition of two waves of equal or nearly equal amplitude, but with slightly different frequencies, and moving with same speed in the same direction in a medius is called beats.

Analytic Treatment

Consider two waves of slightly different frequencies and travelling in the same direction with same speed (ϑ). Let them be given as

$$y_1(x,t) = a\ \text{Sin}\ \omega_1\left(t - \frac{x}{\vartheta}\right) = a\ \text{Sin}\ \omega_1\alpha$$

$$y_2(x,t) = a\ \text{Sin}\ \omega_2\left(t - \frac{x}{\vartheta}\right) = a\ \text{Sin}\ \omega_2\alpha \quad ...(11.4.1)$$

Where it is assumed (chosen) that time is counted from the instant when the two waves pass the origin in phase, ie $\phi = 0$ at $x = 0, t = 0$.

Then by the principle of superposition, the resultant wave pattern shall be

$$y(x,t) = y_1(x,t) + y_2(x,t)$$

$$= a\ \text{Sin}\ \omega_1\alpha + a\ \text{Sin}\ \omega_2\alpha$$

$$\Rightarrow \quad y(x,t) = 2a \sin\frac{\omega_1 + \omega_2}{2}\alpha . \cos\frac{\omega_1 - \omega_2}{2}\alpha \quad ...(11.4.2)$$

Let $\omega_1 - \omega_2 = \Delta\omega = 2\pi(\upsilon_1 - \upsilon_2) = 2\pi(\Delta\upsilon)$ ...(11.4.3)

and $\frac{\omega_1 + \omega_2}{2} = \omega$ ...(11.4.4)

Then since $\omega_1 \approx \omega_2$ so $\Delta\omega << \omega$ and we have

$$y(x,t) = 2a\ \text{Cos}\left(\frac{\Delta\omega\alpha}{2}\right)\text{Sin}\ (\omega\alpha)$$

$$\Rightarrow \quad y(x,t) = 2a\ \text{Cos}\left[\frac{\Delta\omega}{2}\left(t - \frac{x}{\vartheta}\right)\right]\text{Sin}\left[\omega\left(t - \frac{x}{\vartheta}\right)\right]. \quad ...(11.4.5)$$

As $\frac{\Delta\omega}{2} << \omega$ so $\text{Cos}\left[\frac{\Delta\omega}{2}\left(t - \frac{x}{\vartheta}\right)\right]$ varies slowly with time as compared to $\text{Sin}\left[\omega\left(t - \frac{x}{\vartheta}\right)\right]$.

Thus, we can interpret eqn. (11.4.5) by saying that the resultant disturbance is a wave of angular frequency $\omega\left(= \frac{\omega_1 + \omega_2}{2}\right)$ and amplitude 'A' given as

$$A = 2a\ \text{Cos}\left[\frac{\Delta\omega}{2}\left(t - \frac{x}{\vartheta}\right)\right] \quad ...(11.4.6)$$

Thus the amplitude varies periodically with time. Equation (11.4.6) also shows that the amplitude at different points shall be different, although shall be varying with same period. Therefore let us now concentrate our attention to a particular point (x) and study the variation of amplitude with time. Let us define

$$B = \text{Sin}\left[\omega\left(t - \frac{x}{\vartheta}\right)\right] \quad ...(11.4.7)$$

then y(x,t) = A. B

We now plot A and B w.r.to time (shown in fig11.5) and then plot y (x,t)

** . One can also derive the same conclusion by noting that

$$I = K_0A^2 = K_0 4a^2 \cos^2 \frac{\Delta\omega}{2}\left(t - \frac{x}{\vartheta}\right)$$

$$\Rightarrow \quad I = K_0 2a^2\left[1 + \cos \Delta\omega\left(t - \frac{x}{\vartheta}\right)\right]$$

i.e. the intensity varies with frequency $2\pi / \Delta\omega = |\upsilon_2 - \upsilon_1|$.

Graphical Analysis :

Consider two sources 'S_1' and 'S_2' having frequency $n_1 = 12$ and $n_2 = 15$. The block (a) when superimposed on block (b) one obtains block(c).

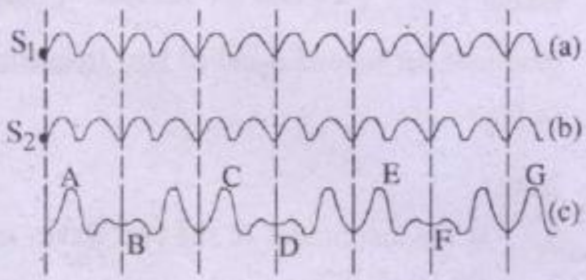


Fig.11.6(a)

One observes that constructive interference occus at C, E and G, while destructive interference takes place at B,D and F. Thus the sound increases three times and decreases three times in one second, i.e. there are three beats in one second. Thus the beat frequency is equal to the difference in the frequencies of the two sources.

Determination of frequency of tuning fork

By Beats Method

Suppose a tuning fork of unknown frequency 'n' is sounded together with a tuning fork of known frequency n_0. Let N be number of beats heard per second. Then

$$n \sim n_0 = N$$

If $n > n_0$ then $n - n_0 = N$ and $n = n_0 + N$...(A)

If $n < n_0$ then $n_0 - n = N$ and $n = n_0 - N$ ...(B)

To decide whether (A) is true or (B) is true, a little wax is attached to one of the prongs of the known tuning fork. Then its new frequency becomes $n_0' = n_0 - \Delta n$. When the unknown tuning fork is sounded with this loaded known tuning fork, then again beats will be heard, but now the beat frequency N' shall be different from N.

If $n > n_0$, the new beat frequency $N' = n - (n_0 - \Delta n) = N + \Delta n$ i.e. beat frequency shall be increased and eqn (A) has to be applied to get the unknown frequency.

If $n < n_0$ then $N' = (n_0 - \Delta n) - n = N - \Delta n$ i.e. beat frequency shall be decreased and eqn (B) shall be applied to get the unknown frequency.

Ex. 11.4.1 Two tuning forks A and B produce 4 beats per second when sounded together. On loading B, the number of beats per second becomes 2. Find the frequency of B if that of A is 510 Hz.

Soln.

$$\upsilon_A \sim \upsilon_B = 4$$

If $\upsilon_A > \upsilon_B$ then $\upsilon_A - \upsilon_B = 4$

$$\Rightarrow \upsilon_B = \upsilon_A - 4$$

If $\upsilon_A < \upsilon_B$ then $\upsilon_B - \upsilon_A = 4$

$$\Rightarrow \upsilon_B = \upsilon_A + 4$$

On loading B, its frequency decreases so that

$$\upsilon_B' = \upsilon_B - \delta$$

New beats shall be

(i) $\upsilon_A - \upsilon_B' = \upsilon_A - (\upsilon_B - \delta) = 4 + \delta$

(ii) $\upsilon'_B - \upsilon_A = (\upsilon_B - \delta) - \upsilon_A = 4 - \delta$

Since new beat frequency is 2 so case (ii) is true. Hence $\upsilon_B = \upsilon_A + 4 = 510 + 4 = 514$ Hz.

Ex. 11.4.2 Two waves of sound of wavelength 1m and 1.02 m in a gas produce 60 beats in 10s. Calculate the velocity of sound in the gas.

Soln.

Let ϑ be velocity of sound

Then $\upsilon_1 = \frac{\vartheta}{\lambda_1} = \frac{\vartheta}{1}$

$$\upsilon_2 = \frac{\vartheta}{\lambda_2} = \frac{\vartheta}{1.02}$$

$\therefore$ Beat frequency $= \frac{60}{10} = 6$

$$\therefore \upsilon_1 - \upsilon_2 = 6$$

$$\Rightarrow \frac{\vartheta}{1} - \frac{\vartheta}{1.02} = 6$$

$$\Rightarrow \vartheta\left(\frac{1.02-1}{1.02}\right) = 6$$

$$\Rightarrow \vartheta = \frac{6\times1.02}{0.02} = 306 \text{ m/s}$$

Ex. 11.4.3 A fork of unknown frequency gives 4 beats per second when sounded with another of frequency 256. The fork is now loaded with a piece of wax and again 4 beats are heard. Calculate the frequency of the unknown fork.

Soln.

Let unknown tuning fork be A

known tuning fork be B

$$\therefore \upsilon_A \sim \upsilon_B = 4$$

$$\upsilon_A = 256 \pm 4 = 260 \text{ or } 252$$

On loading A its frequency decreases. But the no. of beats is again 4, implying its frequency is again 260 or 252. So the only way is its frequency should be 260.

Ex. 11.4.4 Two tuning forks A and B when sounded together produces 4 beats per second. The frequency of A is 256. When B is loaded with wax and sounded again with A, produces 8 beats. Calculate the frequency of B.

Soln.

Given $\upsilon_A = 256$, and no. of beats = 4

$\therefore \upsilon_B = 256 \pm 4$ i.e. $\upsilon_B = 260$ or 252

when 'B' loaded with wax no. of beats = 8

$$\therefore \upsilon'_B = 264 \text{ or } 248$$

But $\upsilon'_B < \upsilon_B$ So $\upsilon'_B = 248$

Hence $\upsilon_B = 260$ or 252.

Ex. 11.4.5 Two tuning forks A and B, when sounded together produce 5 beats / Sec. The frequency of B is 512 Hz. If one arm of A is filed then the no. of beats / Sec. increases. Find the frequency of A.

Soln.

Given $\upsilon_B = 512$ Hz.

No. of beats per sec. = 5

$$\therefore \upsilon_A = 512 \pm 5 = 517 \text{ or } 507$$

when filed $\upsilon'_A > \upsilon_A$. The no. of beats is increased i.e. $(5+\delta)$

$$\therefore \upsilon'_A = 512 \pm (5+\delta) = 517+\delta \text{ or } 507-\delta$$

But $\upsilon'_A > \upsilon_A \Rightarrow \upsilon'_A = 517+\delta$

Hence $\upsilon_A = 517$

11.5 Vibration of air column :

If vibrations are set up at one end of a long straight pipe, longitudinal progressive waves proceed through the air column, get reflected at the other end and moves back. As a result stationary waves are set up due to superposition of original and reflected wave.

The stationary waves thus formed creates a musical sound. Musical instruments like flute, clarionet etc. work on this principle.

Organ Pipe :

An organ pipe consists of a mouth M (very narrow). In front of the mouth there is a wedge W, which changes the direction of wind blown through mouth M. The wind strikes a sharp edged lip L; causing vibrations.

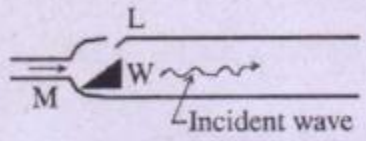


Fig.11.7

Organ-pipes could be of two types :

(i) Open organ pipe (ii) Closed organ pipe.

11.5(a) Open Organ Pipe :

An open organ pipe is one whose both ends are open. Molecules of air at the open ends can move freely. Therefore a displacement antinode (pressure node) is formed at each open end.

The incident wave gets reflected at the open end. Superposition of incident wave and reflected wave gives rise to standing wave with displacement antinodes (pressure nodes) at the open ends.

Since the reflection takes place at a rarer medium, so incident wave and reflected waves are given as (with the point of reflection as x = 0)

$$y_i(x,t) = A \sin\frac{2\pi}{\lambda}(\vartheta t + x) \quad ...(11.5.1)$$

$$p_i(x,t) = p_m \cos\frac{2\pi}{\lambda}(\vartheta t + x) \quad ...(11.5.2)$$

and

$$y_r(x,t) = A \sin\frac{2\pi}{\lambda}(\vartheta t - x) \quad ...(11.5.3)$$

$$p_r(x,t) = p_m \cos\frac{2\pi}{\lambda}(\vartheta t - x) \quad ...(11.5.4)$$

where 'y' denotes displacement, p-denotes pressure and subscripts i, r stand for incident and reflected waves. Then by principle of superposition the resultant wave pattern is obtained as

$$y(x,t) = y_i(x,t) + y_r(x,t)$$

$$\Rightarrow \quad y(x,t) = 2A \sin\left(\frac{2\pi}{\lambda}\vartheta t\right) \cos\left(\frac{2\pi}{\lambda}x\right) \quad ...(11.5.5)$$

and $p(x,t) = p_i(x,t) + p_r(x,t)$

$$\Rightarrow \quad p(x,t) = -2p_m \sin\left(\frac{2\pi}{\lambda}\vartheta t\right) \sin\left(\frac{2\pi}{\lambda}x\right) \quad ...(11.5.6)$$

Eqn. (11.5.5) shows that displacement antinodes are likely to occur at $x = 0, \frac{\lambda}{2}, \lambda, \frac{3\lambda}{2}, 2\lambda ...$ etc. and displacement nodes are likely to occur at $x = \frac{\lambda}{4}, \frac{3\lambda}{4}, \frac{5\lambda}{4}$ ... etc. But the other end x = L has to be also a displacement antinode (pressure node). i.e. p (0,t) = 0= p (L,t) and y(0,t) = y (L,t) = maximum. Therefore it follows from (11.5.6) that

$$\sin\left(\frac{2\pi}{\lambda}L\right) = 0$$

$$\Rightarrow \quad \frac{2\pi L}{\lambda} = n\pi, \; n = 1,2,3,.... \text{ etc.}$$

$$\Rightarrow \quad \lambda = \frac{2L}{n} \qquad ...(11.5.7)$$

As the value of λ depends upon the 'integer n, so we write

$$\lambda_n = \frac{2L}{n}, \; n = 1, 2, 3, ... \text{ etc.} \qquad ...(11.5.8)$$

and thus λ_n corresponds to the wavelength of different modes of vibration. The condition (11.5.8) along with the positions $x = 0$, $\frac{\lambda}{2}$, λ, $\frac{3\lambda}{2}$, ... etc. will indicate the positions of antinodes and hence the nodes.

When the air column vibrates with wavelength λ_n the corresponding frequency is

$$\upsilon_n = \frac{\vartheta}{\lambda_n} = n.\frac{\vartheta}{2L} \qquad ...(11.5.9)$$

This shows that the minimum frequency with which the open organ pipe can vibrate is $\upsilon_1 = \frac{\vartheta}{2L}$, which is obtained by putting n = 1.

(i) ***Fundamental Mode :***

If the air column vibrates with a frequency $\upsilon_1 = \frac{\vartheta}{2L}$, then it is said to be vibrating in its fundamental mode. In such situation

$$y_i(x,t) = A \text{ Sin}\left(\frac{2\pi t}{T_1} + \frac{2\pi}{\lambda_1}x\right) = A \text{ Sin}\left(\frac{2\pi t}{T_1} + \frac{\pi}{L}x\right)$$

$$y_r(x,t) = A \text{ Sin}\left(\frac{2\pi t}{T_1} - \frac{2\pi}{\lambda_1}x\right) = A \text{ Sin}\left(\frac{2\pi t}{T_1} - \frac{\pi}{L}x\right)$$

This leads to the following sequence of vibrations.

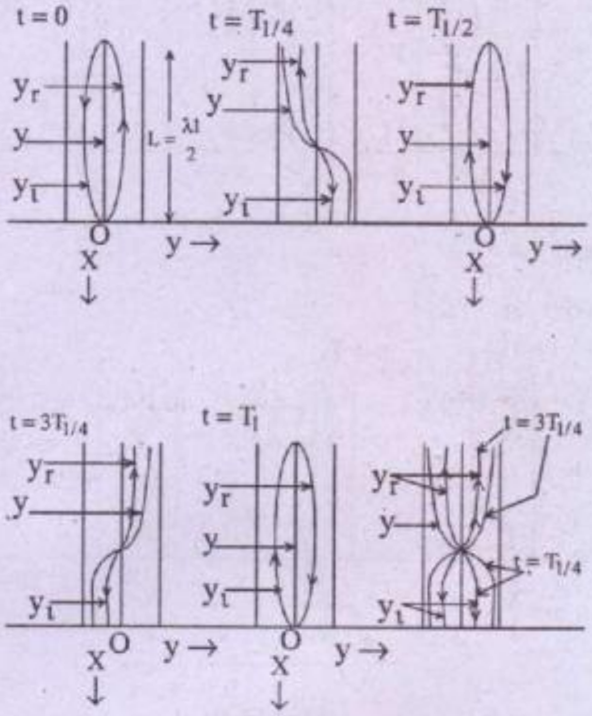


Fig.11.8

Thus in the fundamental mode displacement antinode (pressure nodes) are formed at x = 0 and L (ie at open ends) and displacement node (pressure antinode) at x = L/2.

(ii) ***1st Overtone :***

If the air column vibrates with frequency $\upsilon_2 = 2\frac{\vartheta}{2L}$ (i.e. n= 2), then it is said to be vibrating in its first overtone mode. In such situation

$$y_i = A \text{ Sin}\left(\frac{2\pi t}{T_2} + \frac{2\pi}{\lambda_2}x\right) = A \text{ Sin}\left(\frac{2\pi t}{T_2} + \frac{2\pi}{L}x\right)$$

$$y_r = A \text{ Sin}\left(\frac{2\pi t}{T_2} - \frac{2\pi}{\lambda_2}x\right) = A \text{ Sin}\left(\frac{2\pi t}{T_2} - \frac{2\pi}{L}x\right)$$

The sequence of vibrations are then as given below

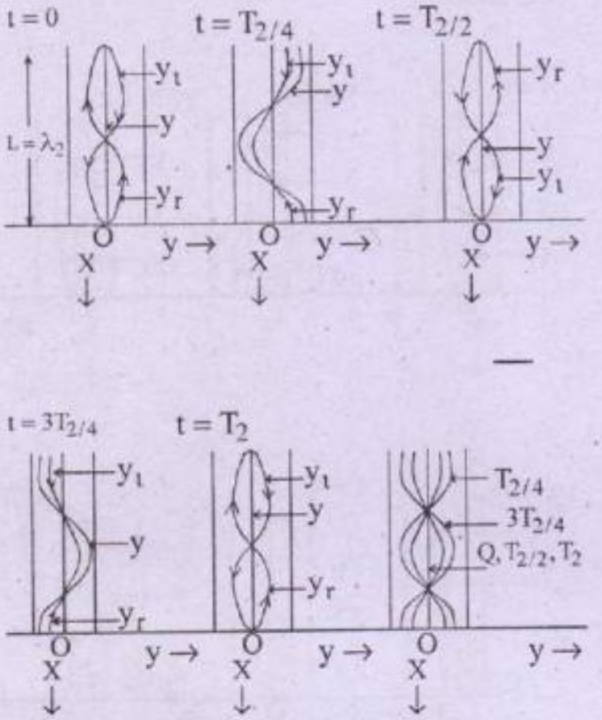


Fig. 11.9

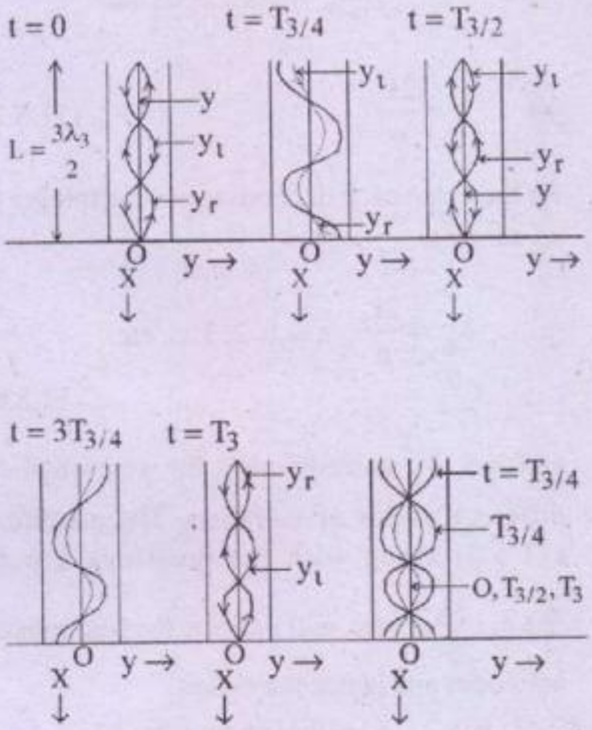


fig. 11.10

Thus in the 1st overtone mode displacement antinodes (pressure nodes) are formed at x = 0, L/2, and L while displacement nodes (pressure antinodes) at x = L/4, 3L/4.

*(iii) **2nd Overtone :***

If the air column vibrates with frequency $\upsilon_3 = 3\frac{\vartheta}{2L}$ (ie n = 3), then it is said to be vibrating in its <u>second overtone</u>. In such situation

$$y_i = A\ \text{Sin}\left(2\pi\frac{t}{T_3}+\frac{2\pi}{\lambda_3}x\right) = A\ \text{Sin}\left(\frac{2\pi t}{T_3}+\frac{3\pi}{L}x\right)$$

$$y_r = A\ \text{Sin}\left(\frac{2\pi t}{T_3}-\frac{2\pi}{\lambda_3}x\right) = A\ \text{Sin}\left(\frac{2\pi t}{T_3}-\frac{3\pi}{L}x\right)$$

The sequence of vibrations are then as given below.

Thus in the 2nd overtone mode displacement antinode (pressure nodes) are formed at x = 0, L/3, 2L/3, L and displacement nodes at x = L/6, L/2 and 5L/6.

Other overtones are similarly described. It is noticed that the natural frequencies are in the ratio 1 : 2 : 3 : 4 : 5 ... etc. So all harmonies are present in an open pipe. This is the reason why musical sound due to an open organ pipe is of better qual ity.

11.5(b) ClosedOrgan Pipe :

A closed organ pipe is one which is closed at one end and open at the other end.

When a vibrating source is kept at the open end, the wave travelling inside the air column gets reflected at the closed end. The incident wave and reflected wave superpose on each other and standing waves are created with displacement node (pressure antinode) at closed end (x=0) and displacement antinode (pressure node) at the open end(x= - L).

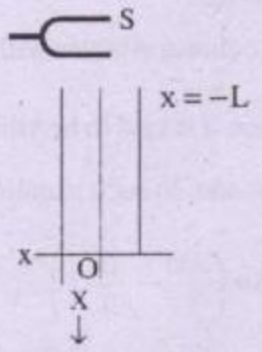


Fig. 11.11

Since the reflection takes place at a rigid boundary, the incident wave and reflected wave are given (in terms of displacement) as

$$y_i(x,t) = A\ \text{Sin}\frac{2\pi}{\lambda}(\vartheta t - x) \quad ...(11.5.10)$$

$$y_r(x,t) = -A\ \text{Sin}\frac{2\pi}{\lambda}(\vartheta t + x) \quad ...(11.5.11)$$

OR (in terms of pressure) as

$$p_i(x,t) = p_m\ \text{Cos}\frac{2\pi}{\lambda}(\vartheta t - x) \quad ...(11.5.12)$$

$$p_r(x,t) = p_m\ \text{Cos}\frac{2\pi}{\lambda}(\vartheta t + x) \quad ...(11.5.13)$$

where y denotes displacement, p denotes pressure and subscripts i, r denote incident and reflected waves. Then by the principle of superposition, the resultant wave is given by

$$y(x,t) = y_i(x,t) + y_r(x,t)$$

$$\Rightarrow \quad y(x,t) = 2A\ \text{Cos}\frac{2\pi}{\lambda}\vartheta t.\ \text{Sin}\frac{2\pi}{\lambda}x \quad ...(11.5.14)$$

and

$$p(x,t) = p_i(x,t) + p_r(x,t)$$

$$\Rightarrow \quad p(x,t) = 2p_m\ \text{Cos}\frac{2\pi}{\lambda}\vartheta t.\ \text{Cos}\frac{2\pi}{\lambda}x \quad ...(11.5.15)$$

Equation (11.5.14) shows that displacement nodes are likely to occur at $x = 0, \frac{\lambda}{2}, \lambda, \frac{3\lambda}{2}, 2\lambda$ ... etc. and displacement antinodes are likely to occur at $x = \frac{\lambda}{4}, \frac{3\lambda}{4}, \frac{5\lambda}{4}$ ... etc. It is also required that the open end $(x = -L)$ should be a pressure node.

i.e. at $x = -L$ $p(x,t) = 0$

$$\Rightarrow \quad \text{Cos}\left(\frac{-2\pi}{\lambda}L\right) = \text{Cos}\frac{2\pi L}{\lambda} = 0$$

$$\Rightarrow \quad \frac{2\pi L}{\lambda} = (2n-1)\frac{\pi}{2} \quad n = 1, 2, 3 \text{ ...... etc.}$$

$$\Rightarrow \quad \lambda_n = \frac{4L}{2n-1}, \quad n = 1, 2, 3 \text{ ..... etc.} \quad ...(11.5.16)$$

(We have used λ_n as it depends upon the interger n). It is also easily observed that at $x = 0, \frac{\lambda}{2}, \lambda$ ... etc. pressure antinodes and at $x = \frac{\lambda}{4}, \frac{3\lambda}{4}, \frac{5\lambda}{4}$ ... etc. pressure nodes are formed. When condition (11.5.16) is satisfied a displacement antinode is formed at the open end.

When the air column vibrates with wavelength λ_n, the corresponding frequency is

$$\upsilon_n = \frac{\vartheta}{\lambda_n} = \frac{2n-1}{4L}\vartheta = (2n-1)\frac{\vartheta}{4L} \quad ...(11.5.17)$$

$$n = 1, 2, 3, \text{.... etc.}$$

Eqn. (11.5.17) shows that the minimum frequency with which the air column can vibrate is $\upsilon_1 = \frac{\vartheta}{4L}$, and is obtained by putting n = 1 in eqn. (11.5.17)

(i) *Fundamental Mode :*

If the air column vibrates with a frequency

$\upsilon_1 = \frac{\vartheta}{4L}$, then it is said to be vibrating in its fundamental mode. In such situation

$$y_i = A\ \text{Sin}\left(\frac{2\pi t}{T_1} - \frac{\pi}{2L}x\right)$$

$$y_r(x,t) = -A\ \text{Sin}\left(\frac{2\pi t}{T_1} + \frac{\pi}{2L}x\right)$$

Then the sequence of vibration are as given below.

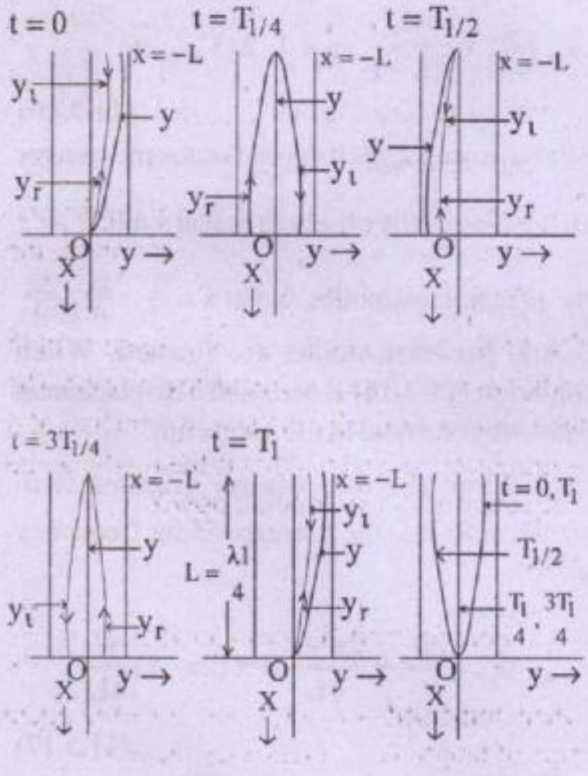


Fig. 11.12

Thus in the fundamental mode of vibration displacement antinode (pressure nodes) is formed at x = -L (open end) and displacement node (pressure antinode) is formed at x = 0 (closed end).

(ii) *1st Overtone :*

If the air column vibrates with frequency

$\upsilon_2 = 3.\frac{\vartheta}{4L}$, then it is said to be vibrating in its first overtone mode. In such situation

$$y_i = A\ \text{Sin}\left(\frac{2\pi t}{T_2} - \frac{3\pi}{2L}x\right)$$

$$y_r = -A\ \text{Sin}\left(\frac{2\pi t}{T_2} + \frac{3\pi}{2L}x\right)$$

Then the sequence of vibrations are as shown below.

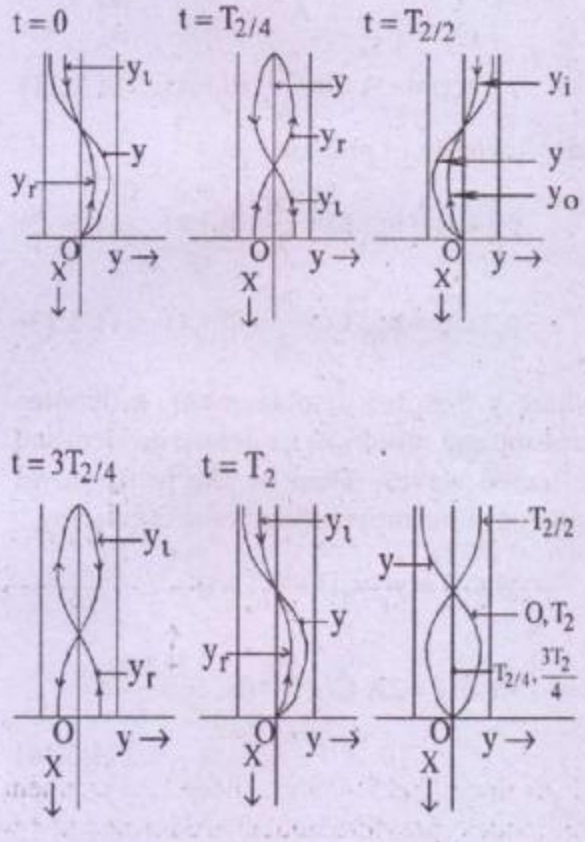


Fig. 11.13

Thus in the 1st overtone mode displacement antinodes (pressure nodes) are formed at x = L/3, L, and displacement nodes (pressure antinodes) at x = 0, 2L/3.

(iii) 2nd Overtone :

If the air column vibrates with frequency $\upsilon_3 = 5\frac{\vartheta}{4L}$, then it is said to be vibrating in its 2nd overtone. In such situation

$$y_i = A\,\text{Sin}\left(\frac{2\pi t}{T_3} - \frac{5\pi}{2L}x\right)$$

$$y_r = -A\,\text{Sin}\left(\frac{2\pi t}{T_3} + \frac{5\pi}{2L}x\right)$$

Then the sequence of vibrations are as shown below.

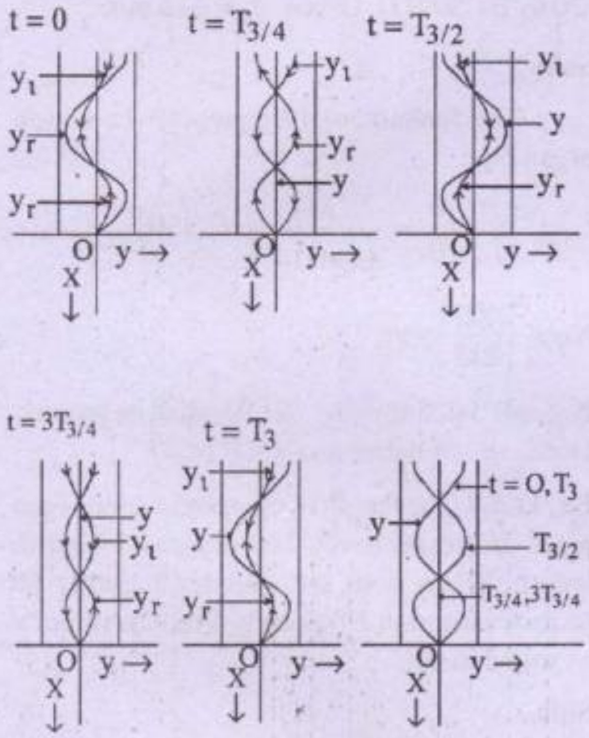


Fig. 11.14

Thus in the 2nd Overtone mode displacement antinodes (pressure nodes) are formed at x = L/5, 3L/5 and L and displacement nodes (pressure antinodes) at x = 0, 2L/5, 4L/5.

Other overtones can be similarly studied by putting n = 4, 5, 6... etc. It is noticed that the frequencies of various modes (over tones) are in the ratio 1 : 3 : 5 : 7 ... etc. Thus all harmonics are not present. Therefore the quality of sound in closed organ pipe is poorer compared to open-organ pipe.

We also note that

$$\frac{(\upsilon_1)\text{open organ}}{(\upsilon_1)\text{closed organ}} = \frac{\vartheta/2L}{\vartheta/4L} = \frac{2}{1}$$

This shows that the frequency of fundamental mode of open organ pipe is twice that of the closed organ pipes.

Factors affecting Pitch and Quality of note emitted by pipes :

(i) Pitch of a pipe increases due to increase in temperature. This happens due to increase in speed of sound and increase in length of pipe, the later being relatively smaller.

(ii) As the diameter of pipe increases the frequency decreases. This happens due to end correction i.e. $\ell_{tone} = \ell + x = \ell + 0.30$, as reflection of sound from an open end takes place at a small distance 'x' above the mouth of the tube.

Ex. 11.5.1 A cyclindrical tube, open at both ends, has a fundamental frequency 'n' in air. The tube dipped vertically in water so that half of it is immersed in water. What is the fundamental frequency of the air column now ?

Soln.

For open ended pipe of length L, $n = \frac{\vartheta}{2L}$

when immersed in water, it acts as closed organ pipe of length L / 2. Hence frequency

$$n' = \frac{\vartheta}{4(L/2)} = \frac{\vartheta}{2L}$$

$\therefore\ n' = n$

Ex.11.5.2 What is the length of an air-column closed in a pipe that will be in resonance with a vibrating tuning fork frequency 264 Hz, if the speed of sound in air = 330 m/s ?

Soln.

As given n = 264 Hz.

Frequency of closed organ pipe $n = \frac{\vartheta}{4L} = 264$

$$\Rightarrow \quad L = \frac{\vartheta}{4 \times 264} = \frac{330}{4 \times 264} = 0.3125 m$$

$$\Rightarrow \quad L = 31.25 \text{ cm} \qquad \text{(Ans)}$$

Ex.11.5.3 An open organ pipe of length 100cm emits sound of same fundamental frequency as that of a closed organ pipe. What is the length of the closed organ pipe ?

Soln.

Frequency of open organ pipe

$$\upsilon_{01} = \frac{\vartheta}{2L_0} = \frac{\vartheta}{200}$$

Frequency of closed organ pipe

$$\upsilon_{c1} = \frac{\vartheta}{4L_c}$$

As given $\upsilon_{01} = \upsilon_{c1}$

$$\Rightarrow \quad \frac{\vartheta}{200} = \frac{\vartheta}{4L_c}$$

$$\Rightarrow \quad L_c = 50 cm \qquad \text{(Ans)}$$

Ex. 11.5.4 Find the frequency of first over tone in a closed pipe of length 37.5 cm. The velocity of sound = 350 m/s.

Soln.

Frequency of 1st over tone is given by

$$\upsilon_2 = 3.\frac{\vartheta}{4L} = 3 x \frac{350}{4 x 35.5 x 10^{-2}} = 700 Hz$$

Ex. 11.5.5 How many harmonics can be heard by a person with normal hearing range (20 Hz to 20 K Hz) in case of an open pipe of length 40 cm ? (ϑ_0 = 344 m/s)

Soln.

Fundamental frequency of open pipe

$$\upsilon_1 = \frac{\vartheta}{2L} = \frac{344 m/s}{2 x 40 x 10^{-2} m} = 430 Hz$$

Since harmonics are in the ratio 1 : 2 : 3 : 4... etc. so no. of harmonics in the range 20 Hz to 20 K Hz is

$$n = \frac{20000}{430} = 46.5 = 46$$

Ex. 11.5.6 Consider a closed organ pipe of length 40 cm. How many harmonics can be heard by a person with a normal hearing range (20 to 20000 Hz). Given ϑ_0 = 344 m/s.

Soln.

The fundamental frequency of a closed organ pipe

$$\upsilon_1 = \frac{\vartheta}{4L} = \frac{344}{4 x 40 x 10^{-2}} = 215 Hz$$

Now $\frac{20000}{215} = 93$

But only 1st, 3rd, 5th..... 91,93, shall be present. Hence no. of harmonics shall be 47.

Ex. 11.5.7 The length of open and closed organ pipes are respectively 160 cm and 75 cm in length. When both are sounded together 70 beats are heard in 10 seconds. Find the velocity of sound in air.

Soln.

For open organ pipe $\upsilon_{10} = \frac{\vartheta}{2L} = \frac{\vartheta}{2 x 160}$

For closed organ pipe $\upsilon_{1c} = \frac{\vartheta}{4L} = \frac{\vartheta}{4 x 75}$

Now no. of beats per sec = $\frac{70}{10} = 7$

$$\upsilon_{1c} - \upsilon_{10} = 7 = \frac{\vartheta}{300} - \frac{\vartheta}{320} = \frac{\vartheta(20)}{300 x 320}$$

$$\Rightarrow \quad \vartheta = \frac{300\text{x}320\text{x}7}{20}\text{cm/s} = 336\text{x}10^{-2}\text{cm/s}$$

$$\Rightarrow \quad \vartheta = 336 \text{ cm/s}$$

Ex. 11.5.8 The third overtone of a closed pipe is found to be in unison with the first overtone of an open pipe. Find the ratio of the lengths of the pipes.

Soln.

Frequency of 3rd overtone of closed pipe is

$$\upsilon_4 = 7\frac{\vartheta}{4L_c}$$

frequency of 1st overtone of open pipe is

$$\upsilon_2 = 2.\frac{\vartheta}{2L_0}$$

As given

$$7.\frac{\vartheta}{4L_c} = 2.\frac{\vartheta}{2L_0}$$

$$\Rightarrow \quad \frac{L_c}{L_0} = \frac{7}{4}$$

Ex. 11.5.9 Calculate the length of a narrow tube closed at one end, which will resonate with a tuning fork of frequency 510 Hz, if the velocity of sound is 340 m/s.

Soln.

$$\upsilon_{1c} = \frac{\vartheta}{4L_c} = 510\text{Hz}$$

$$\Rightarrow \quad L_c = \frac{\vartheta}{4\times510} = \frac{340}{4\times510}\text{ m}$$

$$L = 0.167 \text{ m} = 16.7 \text{ cm}$$

Ex. 11.5.10 Two organ pipes give 5 beats per second when sounded together in air at 20^0 C. How many beats would be produced by them at 40^0 C ?

Soln.

$$\upsilon_1(t) = \frac{\vartheta(t)}{2L_1} = \frac{\vartheta_0\left(1+\frac{t}{273}\right)^{\frac{1}{2}}}{2L_1} = \upsilon_1(o)\left(1+\frac{t}{273}\right)^{\frac{1}{2}}$$

$$\upsilon_2(t) = \frac{\vartheta_0}{2L_2}\left(1+\frac{t}{273}\right)^{\frac{1}{2}} = \upsilon_2(o)\left(1+\frac{t}{273}\right)^{\frac{1}{2}}$$

$$\therefore \upsilon_1(20) - \upsilon_2(20) = (\upsilon_1(o) - \upsilon_2(o))\left(1+\frac{20}{273}\right)^{\frac{1}{2}} = 5$$

$$\upsilon_1(40) - \upsilon_2(40) = (\upsilon_1(o) - \upsilon_2(o))\times\left(1+\frac{40}{273}\right)^{\frac{1}{2}}$$

$$= \frac{5\left(1+\frac{40}{273}\right)^{\frac{1}{2}}}{\left(1+\frac{20}{273}\right)^{\frac{1}{2}}}$$

$$\upsilon_1(40) - \upsilon_2(40) = 5.168$$

11.6 Doppler's Effect :

It is a common experience that when a train emitting whistle approaches, the passengers on the platform hear sound of gradually increasing pitch and as it passes away, the passengers listen to sound fading away gradually. Similarly when a person in a car passes away from a stationary train, emitting whistle, the person hears sound with pitch gradually weakening. We also observe that when wind blows, the observer listens to sound of different pitch, either increased or decreased.

This phenomenon of apparent change in frequency of a wave, due to relative motion between source, observer, and medium is called **Doppler's Effect**.

We consider these cases one by one.

I. Source in Motion

Consider a source emitting waves of frequency υ_0 . Let ϑ be the velocity of the wave w.r.to the medium in which the wave travels. Let υ_s be the velocity of the source w.r.to the medium.

(a) Source approaching observer :

Since the source emits waves of frequency υ_0, so it sends similar pulses at intervals of $T = 1/\upsilon_0$.

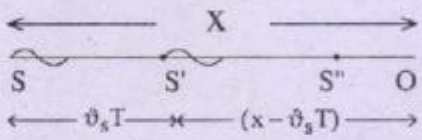


Fig. 11.15

Suppose the separation between source 'S' and observer 'O' at initial time t = 0, is x when a pulse is emitted from the source. The next pulse (2nd pulse) is emitted by the source after a time T at S'. During this time the source has moved through a distance

$$SS' = \vartheta_s T \qquad ...(11.6.1)$$

Thus the 2nd pulse is emitted at a distance $(x - \vartheta_s T)$ from the observer. The first pulse shall take time t_1 given as

$$t_1 = x/\vartheta \qquad ...(11.6.2)$$

and reach the observer at time $t = t_1$. The second pulse will take time $t_2' = \dfrac{x-\vartheta_s T}{\vartheta}$ and reach the observer at time t_2 given as

$$t_2 = \frac{x-\vartheta_s T}{\vartheta} + T \qquad ...(11.6.3)$$

The third pulse is emitted at S" after a time 2T, and SS" = $2\vartheta_s T$. Thus the third pulse is emitted at a distance $x - 2\vartheta_s T$ from the observer and hence will take time $t_3' = (x - 2\vartheta_s T)/\vartheta$ and reach the observer at time t_3,

$$t_3 = \frac{x-2\vartheta_s T}{\vartheta} + 2T \qquad ...(11.6.4)$$

Therefore the time interval between two consecutive pulses as detected by the observer is

$$T' = t_2 - t_1 = \left(T + \frac{x-\vartheta_s T}{\vartheta}\right) - \frac{x}{\vartheta}$$

$$= T - \frac{\vartheta_s}{\vartheta} T = t_3 - t_2$$

giving

$$T' = \frac{\vartheta - \vartheta_s}{\vartheta} T \qquad ...(11.6.5)$$

$$\Rightarrow \quad \frac{1}{T'} = \frac{\vartheta}{\vartheta - \vartheta_s} \frac{1}{T}$$

$$\Rightarrow \quad \upsilon' = \frac{\vartheta}{\vartheta - \vartheta_s} \upsilon_0 \qquad ...(11.6.6)$$

Thus when the source approaches the observer the apparent frequency is greater than the actual frequency. The apparent wavelength λ' which is equal to the distance between two consecutive pulses (w.r.to the observer) is $\vartheta T - \vartheta_s T = (\vartheta - \vartheta_s)T$ (i.e. appears to decrease)

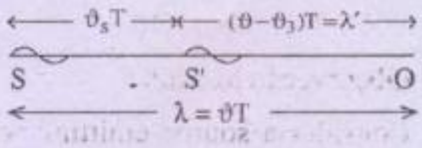


Fig.11.16

(b) Source moving away from observer :

When the source 'S' moves away from the observer 'O', the situation is shown below in fig. 11.17

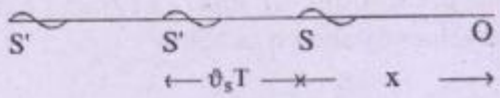


Fig. 11.17

It is easily perceived that

$$t_1 = \frac{x}{\vartheta}$$

$$t_2 = \frac{x + \vartheta_s T}{\vartheta} + T$$

$$t_3 = \frac{x + 2\vartheta_s T}{\vartheta} + 2T$$

giving

$$T' = t_2 - t_1 = T + \frac{\vartheta_s}{\vartheta} T = t_3 - t_2$$

$$\Rightarrow \quad T' = \left(1 + \frac{\vartheta_s}{\vartheta}\right) T$$

$$\Rightarrow \quad \upsilon' = \frac{\vartheta}{\vartheta + \vartheta_s} \upsilon_0 \qquad ...(11.6.7)$$

Thus when the source moves away the frequency appears to decrease and the apparent wavelength $\lambda' = (\vartheta + \vartheta_s)T$, appears to increase.

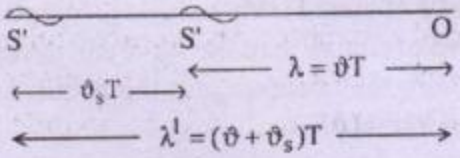


Fig.11.18

II. Observer in Motion :

Consider a source emitting waves of frequency υ_0. Let ϑ be its velocity w.r.to the medium, in which the wave is propagating. Let ϑ_0 be the velocity of the observer w.r.to the medium.

(a) Observer approaching Source :

As the source is stationary similar pulses are emitted from the same point at regular intervals of T $(= 1/\upsilon_0)$. These pulses travel with a speed ϑ in the medium and at any instant the separation between two consecutive pulses is $\lambda = \vartheta T$.

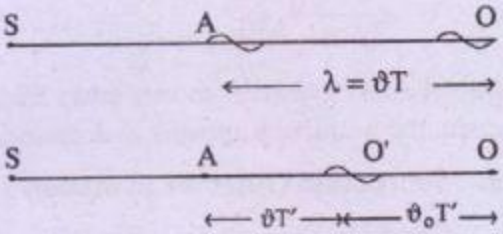


Fig.11.19

Suppose the observer receives a pulse at time t = 0. Then the next pulse (at this instant) must be at A, at a distance $\lambda = \vartheta T$ from the observer O. As the observer is moving towards the source, so the observer receives the second pulse after time T' at position O'; such that

$$OO' = \vartheta_0 T'$$

$$AO' = \vartheta T'$$

and $\vartheta T' + \vartheta_0 T' = \vartheta T$

$$\Rightarrow \quad T' = \frac{\vartheta}{\vartheta + \vartheta_0} . T$$

$$\Rightarrow \quad \upsilon' = \frac{\vartheta + \vartheta_0}{\vartheta} . \upsilon_0 \qquad ...(11.6.8)$$

Thus when the observer approaches the source the frequency appears to increase.

(b) Observer moving away from source :

When the observer moves away from source, the situation is shown below.

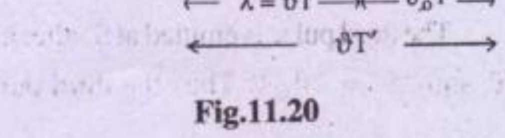


Fig.11.20

Proceeding in a similar manner it is easily perceived that

$$AO = AO' - OO'$$

$$\Rightarrow \quad \vartheta T = \vartheta T' - \vartheta_0 T'$$

$$\Rightarrow \quad T' = \frac{\vartheta}{\vartheta - \vartheta_0} . T$$

$$\Rightarrow \quad \upsilon' = \frac{\vartheta - \vartheta_0}{\vartheta} . \upsilon_0 \qquad ...(11.6.19)$$

Thus when the observer moves away from the source, the frequency appears to decrease.

III. Source and Observer in motion :

Consider a source S, emitting waves of frequency υ_0. Let ϑ be the velocity of the wave w.r.to the medium. Let υ_s be the velocity of source and ϑ_0 be velocity of observer, both measured w.r.to the medium.

Then the apparent frequency of the source as observed by the observer shall be

$$\upsilon' = \frac{\vartheta'}{\lambda'} \qquad ...(11.6.20)$$

where ϑ' is the relative velocity of the wave w.r.to the observer and λ' is the observed wavelength (apparent wavelength).

$$\text{Now } \vartheta' = \vartheta \pm \vartheta_0 \qquad ...(11.6.21)$$

with $\vartheta' = \vartheta + \vartheta_0$, when the observer is moving towards the source; and $\vartheta' = \vartheta - \vartheta_0$, when the observer is moving away from the source.

$$\text{Also } \lambda' = (\vartheta \pm \vartheta_s)T \qquad ...(11.6.22)$$

with $\lambda' = (\vartheta + \vartheta_s)T$, when source moves away from observer; and $\lambda' = (\vartheta - \vartheta_s)T$, when source approaches the observer.

Therefore

$$\upsilon' = \frac{\vartheta \pm \vartheta_0}{(\vartheta \pm \vartheta_s)T} = \frac{\vartheta \pm \vartheta_0}{\vartheta \pm \vartheta_s} \upsilon_0 \; ** \qquad ...(11.6.23)$$

The predictions of eqn. (11.6.23) can be conveniently shown as below.

S $\xrightarrow[\vartheta_s]{}$ $\quad$ $\xleftarrow[\vartheta_0]{}$ O $\quad$ i) $\upsilon' = \frac{\vartheta + \vartheta_0}{\vartheta - \vartheta_s} \upsilon_0$

S $\xrightarrow[\vartheta_s]{}$ $\quad$ O $\xrightarrow[\vartheta_0]{}$ $\quad$ ii) $\upsilon' = \frac{\vartheta - \vartheta_0}{\vartheta - \vartheta_s} \upsilon_0$

$\xleftarrow[\vartheta_s]{}$ S $\quad$ $\xleftarrow[\vartheta_0]{}$ O $\quad$ iii) $\upsilon' = \frac{\vartheta + \vartheta_0}{\vartheta + \vartheta_s} \upsilon_0$

$\xleftarrow[\vartheta_s]{}$ S $\quad$ O $\xrightarrow[\vartheta_0]{}$ $\quad$ iv) $\upsilon' = \frac{\vartheta - \vartheta_0}{\vartheta + \vartheta_s} \upsilon_0$

IV. Source, Observer and Medium in motion :

If the motion of the medium is in the direction of line joining source and observer then ϑ shall be replaced by $\vartheta + \vartheta_\omega$; whereas if the converse occurs, then ϑ shall be replaced by $\vartheta - \vartheta_\omega$.

$$\text{Thus } \upsilon' = \frac{(\vartheta \pm \vartheta_\omega) \pm \vartheta_0}{(\vartheta \pm \vartheta_\omega) \pm \vartheta_s} \upsilon_0 \qquad ...(11.6.24)$$

Limitations of Doppler effect

If the observer moves away from the source of sound with a speed greater than the speed of sound, then the wave can never catch up with the observer and the above formula cannot be applied.

<u>NOTE</u>

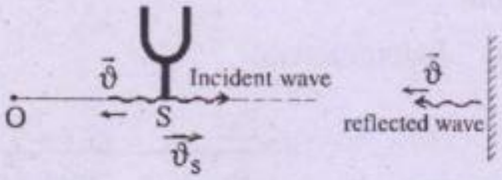
** The eqn. (11.6.23) $\upsilon' = \frac{\vartheta \pm \vartheta_0}{\vartheta \pm \vartheta_s} \upsilon_0$ can be simply written as

$$\upsilon' = \frac{\vartheta - \vartheta_0}{\vartheta - \vartheta_s} \upsilon_0$$

if we take the <u>direction from source to observer</u> as positive direction and treat ϑ_0 and ϑ_s as

Ex. 11.6.3 A tuning fork of frequency 440 Hz approaches a wall with a velocity 4 m/s. What will be the number of beats between the direct and reflected sound if the velocity of sound is equal to 332 m/s.

Soln.



The observer 'o' is receiving waves from two sources (i) direct wave from the source (ii) reflected wave from the wall.

Let the frequency of the source be υ_0.

The apparent frequency of the reflected wave

$$\upsilon' = \frac{\vartheta}{\vartheta - \vartheta_s}\upsilon_0$$

The apparent frequency of directed wave is

$$\upsilon'' = \frac{\vartheta}{\vartheta + \vartheta_s}\upsilon_0$$

Hence the no. of beats per second is

$$n = \upsilon' - \upsilon'' = \vartheta\left(\frac{1}{\vartheta - \vartheta_s} - \frac{1}{\vartheta + \vartheta_s}\right)\upsilon_0$$

$$n = \vartheta \frac{2\vartheta_s}{\vartheta^2 - \vartheta_s^2}.\upsilon_0$$

$$\Rightarrow \quad n = 332 \times \frac{2 \times 4}{(332)^2 - (4)^2} \times 440$$

$$\Rightarrow \quad n = 10.6$$

Ex. 11.6.4 An engine blowing a whistle of frequency 128 Hz moves with a velocity of 54 km/ hr towards a hill from which a well defined echo is heard by the driver. if velocity of sound in air is 330 m/s, calculate the frequency of echo.

Soln.

Given velocity of source (engine) = 54 km/hr = 15 m/s considering the hill as an observer, the frequency of echo shall be

$$\nu_e = \frac{\vartheta}{\vartheta - \vartheta_s}\nu_0$$

$$= \frac{330}{330 - 15}.128 = 134.1 \text{ Hz}$$

Now the driver shall perceive this echo with a frequency

$$\upsilon'_e = \frac{\vartheta + \vartheta_0}{\vartheta}\upsilon_e$$

where ϑ_0 = velocity of the engine

$$\upsilon'_e = \frac{330 + 15}{330} \times 134.1 \text{ Hz}$$

$$\Rightarrow \quad \upsilon'_e = 140.2 \text{ Hz} \qquad \text{(Ans)}$$

Ex. 11.6.5 A policeman on duty detects a drop of 15% in the pitch of the horn of a motor car as it crosses him. If the velocity of sound is 330 m/ s, calculate the speed of the car.

Soln.

Let υ_0 be frequency of the source.

As car approaches the frequency is

$$\upsilon' = \frac{\vartheta}{\vartheta - \vartheta_s}\upsilon_0$$

as car recedes

$$\upsilon'' = \frac{\vartheta}{\vartheta + \vartheta_s}\upsilon_0$$

Therefore $\frac{\upsilon''}{\upsilon'} = \frac{85}{100} = \frac{\vartheta - \vartheta_s}{\vartheta + \vartheta_s}$

$$\Rightarrow \quad \frac{85}{100} = \frac{330 - \vartheta_s}{330 + \vartheta_s}$$

$$\Rightarrow \quad \frac{185}{100} = \frac{660}{330 + \vartheta_s}$$

$$\Rightarrow \quad 330 + \vartheta_s = \frac{660 \times 100}{185}$$

$$\vartheta_s = \frac{660 \times 100}{185} - 330$$

$$\Rightarrow \quad \vartheta_s = 26.76 \text{ m/s} \qquad \text{(Ans)}$$

Summary

1. Sound is a form of energy that propagates in a medium as a longitudinal wave. It is produced in a material medium by a vibrating source.

2. Human ear is sensitive to sound waves in the frequency range from 20Hz to 20KHz. The waves with frequency $\nu < 20Hz$ are called infrasonics and the waves with frequency $\nu >$ 20KHz are called ultrasonics.

3. Newtons formula for velocity of sound in a gaseous medium is

$$v = \sqrt{\frac{p}{\rho}}$$

where p is the pressure and ρ is the density of the medium. This formula, however, does not agree with experimental results.

4. Laplace's Correction

$$V = \sqrt{\frac{\gamma p}{\rho}}$$

where $\gamma = \frac{C_p}{C_v}$

For one mole of ideal gas : $V = \sqrt{\frac{\gamma RT}{M}}$

5. a) Effect of density : $\frac{V_1}{V_2} = \sqrt{\frac{\rho_2}{\rho_1}}$

b) Effect of temperature : $\frac{V_1}{V_2} = \sqrt{\frac{T_1}{T_2}}$

where T_1 and T_2 are absolute temperatures.

$$V_t = V_0(1+\frac{t}{273})^{1/2} \simeq V_0(1+\frac{t}{546})$$

6. a) Constructive interference occurs where phase difference between two interfering waves is even multiple of π or path different between them is an integral multiple of λ.

i.e. $\varphi = 2n\pi \quad n = 0, 1, 2.....$

or $\Delta x = n\lambda \quad n = 0, 1, 2 ......$

b) For destructive interference

$\varphi = (2n + 1)\pi \quad n = 0, 1, 2.....$

or $\Delta x = (n + \frac{1}{2})\lambda \quad n = 0, 1, 2.....$

7. The periodic variations in the intensity of the wave due to superposition of two waves having equal or nearly equal amplitude but with slightly different frequencies and moving with same speed in the same direction in a medium are called beats.

8. Beat frequency $\nu_b = |\nu_1 - \nu_2|$

9. Organ pipes are wind instruments in which standing waves for sound can be produced in its air column.

a) Open organ pipe :

Frequence of n th harmonic :

$$f_n = n\frac{v}{2l} \qquad n = 1, 2, 3.....$$

V = speed of sound and l is the length of the pipe.

b) Closed organ pipe :

Frequency of n th harmonic :

$$f_n = (2n-1)\frac{v}{2l} \cdot \quad n = 1, 2, 3.....$$

10. The apparent change in frequency of wave due to relative motion between source and observer is called Dopplers effect. If ν_0 is the frequency of wave emitted by the source, then apparent frequency ν' is

$$\nu' = \left(\frac{V \pm V_0}{V \pm V_s}\right) V_0$$

where V_0 and V_s are the velocities of observer and source relative to the medium.

When the motion of the source or observer is towards the other, the sign of its speed should give an upward shift in frequency. When the motion of the source or observer is away from the other, sign of its speed must give a downward shift in frequency.

MODEL QUESTIONS

A. **Multiple Choice Type Questions :**

1. The temperature at which speed of sound in air is double its speed at 0^0 C is

   (i) 1^0 C (ii) 2^0 C

   (iii) 819^0 C (iv) 981.9^0 C

2. Which of the following properties of sound is affected by change in air temperature ?

   (i) wavelength (ii) frequency

   (iii) amplitude (iv) intensity

3. Sound waves are travelling in a medium whose adiabatic elasticity is E_H and isothermal elasticity is E_T. The velocity of sound waves is proportional to

   (i) E_T (ii) $\sqrt{E_H}$

   (iii) $\sqrt{E_T}$ (iv) E_T / E_H

4. Sound travels in rocks in the form of

   (i) longitudinal elastic waves only

   (ii) transverse elastic wave only

   (iii) both longitudinal and transverse elastic waves

   (iv) non-elastic waves.

5. The fundamental frequency of an open pipe is 'f'. Higher frequencies which the pipe can produce are

   (i) 2f, 3f, 4f, 5f,......

   (ii) 2f, 4f, 6f, 8f,......

   (iii) 3f, 5f, 7f, ...

   (iv) (n+1) f / 2, where n is integer

6. Two tuning forks can produce distinct beats if their frequencies are

   (i) equal

   (ii) slightly different

   (iii) widely different

   (iv) of unconditional values.

7. Water in a reasonance tube is replaced by oil up to the same level. How will its frequency of resonance change ?

   (i) increase

   (ii) decrease

   (iii) no change

   (iv) may increase or decrease depending on the density of oil.

8. When a source of sound wave and an observer approach each other in static air medium, then the wavelength of sound wave appears to the observer to

   (i) increase

   (ii) decrease

   (iii) no change

   (iv) first increase and then decrease.

9. The velocity of sound wave of frequency 200 vib/s when compared with the velocity of another sound wave of frequency 400 vib/s at a given place is

   (i) twice (ii) half

   (iii) same (iv) four times

10. Decibel is

    (i) a musical instrument

    (ii) a musical note

    (iii) a measure of sound level

    (iv) the wavelength of noise.

11. Sound travels fastest in
    (i) metal (ii) air
    (iii) vaccum (iv) none of the above
12. The velocity of sound wave is maximum in
    (i) iron (ii) water
    (iii) mercury (iv) air
13. Distance between successive nodes is
    (i) λ (ii) $\lambda/2$
    (iii) $\lambda/4$ (iv) 2λ
14. A sound has intensity level of 30 decibel. Its intensity in watt / cm^2 is
    (i) 10^{-10} (ii) 10^{-11}
    (iii) 10^{-13} (iv) 10^{-15}
15. For every 1^0 C rise in temperature, the velocity of sound waves
    (i) increases by 61 cm/s
    (ii) decreases by 61 cm/s
    (iii) increase by 61 m/s
    (iv) decreases by 61 m/s

B. Very Short Answer Type Questions :

1. Two sound waves of frequencies 135 and 140 Hz are superposed on each other. What is the number of beats produced per second ?
2. What happens to the apparent frequency of a note if the source of sound approaches the listener ?
3. Is ticking of a clock musical or noise ?
4. Why is the roaring of lion different from the sound of mosquito ?
5. Why do two musical sounds of same loudness and pitch produce different impressions on the human ear ?
6. Define a decibel .
7. How can you conclude that the star is coming towards earth ?
8. What is the effect of change of pressure on velocity of sound ?
9. At what temperature is the velocity of sound in air is twice its velocity at 0^0 C.
10. Give the ratio of the frequency of the fundamental and overtones produced in an open pipe .
11. A vertical rod is hit at one end horizontally. What kind of wave propagates ?
12. A vertical rod is hit at one end vertically. What kind of wave propagates ?
13. What kind of sound is produced by clapping ?
14. Write an equation to represent sound produced by clapping.
15: An open organ pipe of length L vibrates in its fundamental mode. Where the pressure vibration is maximum ?
16. What happens to frequency of an organ pipe when temperature increases ?

C. Short Answer Type Questions :

1. Differentiate between musical sound and noise.
2. Name the characteristics of musical sound.
3. What are the factors upon which pitch of a note depends ?
4. How does velocity of sound depend on humidity ?
5. Explain why music given by open organ pipe is sweeter and richer than the music given by a closed organ pipe.
6. The speed of sound in air at 27^0 C is 330 m/s. Calculate the speed at 127^0 C.

7. A sound source and a listener are both at rest on earth, but a strong wind is blowing. Is there a Doppler effect ?

8. Two wires, A and B are tied up between any two points. The diameter, tension applied and density of B are twice those of A. Calculate the frequency of B relative to that of A.

9. Write down Newton's formula for speed of sound in air medium. Also write Laplace's corrected formula.

10. Show that the second resonance length of a closed organ pipe is three times the first resonance length.

11. If you are walking on the moon, con you hear the sound of your own foot steps ?

12. Why does your voice sound different over telephone than in person ?

13. Two tuning forks have identical frequencies, but one is stationary while the other is mounted on a rotating record turn table. What does a listener hear ?

D. Unsolved Problems :

1. If the speed of sound in air is 332 m/s at N.T.P., find the speed of sound at 30^0C and 0.7 m of mercury pressure.

2. Calculate the speed of sound in steel, given Y steel = 2×10^{11} N/m^2 and density of steel = 7.8×10^3 kg / m^3.

3. How long will it take for sound waves to travel a distance '*l*' between points A and B if the air temperature between them varies linearly from T_1 to T_2 ? The speed of sound at 0^0 C is C_0.

4. The fundamental frequency of an open pipe is 500 Hz. Determine the length of the pipe, if the speed of sound is 340 m/s.

5. Two sound waves of wave length 100 cm and 102 cm respectively produce 6 beats per second. Determine the speed of sound in the medium.

6. A train approaches a stationary observer at a speed of 75 kms / hr sounding a whistle of frequency 1000 Hz. What will be the apparent frequency of the whistle to the observer ? (speed of sound = 332 m/s)

7. A man is standing beside a railway line listening to the whistle of a passing train. The whistle which has a frequency of 1000 Hz suffers an apparent change of 100 Hz. What is the speed of the train ? (speed of sound 333.3 m/s)

8. The frequency of an organ pipe at 40^0 C is 256 Hz. What will be its frequency at 20^0 C ?

9. Two tuning forks A and B give 18 beats in 2 seconds. A resonates with a closed column of air 15 cm long and B with an open column 30.5 cm long. Calculate their frequencies.

10. A tuning fork of frequency 440 Hz is to be mounted on a wooden box with one end open to reinforce its sound. What would be the length of the sounding box ? (speed of sound in air = 332 m/s)

11. The first overtone of an open pipe and the fundamental tone of a closed pipe give 5 beats per second when sounded together. If the length of the closed pipe is 25 cm, what are the possible length of the open pipe ? (speed of sound in air = 340 m/s)

12. A siren having a ring of 200 holes is making 132 revolutions per minute and found to emit a note which is an octave higher than that of a tuning fork. Find the frequency of the latter.

13. Calculate the intensity of a note of frequency 1000 Hz if the amplitude of vibration is 10^{-9} cm. Density of air = 1.3 kg/m^3 and speed of sound = 340 m/s.

14. An observer on a railway station platform observed that as the train passed the station at 96 km/hr., the frequency of the whistle appeared to drop by 400 Hz. Find the frequency of the whistle. (speed of sound = 1200 km/hr.)

15. The ratio of the apparent frequencies of the horn of a car when approaching and receding stationary observer is 11 : 9. What is the speed of the car, if the velocity of sound in air is 300 m/s.

E. Long Answer Type Questions :

1. Define Newton's formula for speed of sound in air. Explain how Laplace corrected it ?
2. Give the analytical treatment of interference and establish the conditions for constructive and destructive interferences.
3. Give analytical treatment of beats. Explain what would happen if the difference of frequencies of two sources is large.
4. Explain the phenomenon of beats. How can you determine the frequency of an unknown tuning fork by method of beats ?
5. Derive an expression for the fundamental frequency of an open organ pipe. Also find the frequencies of the overtones.
6. Derive expressions for the fundamental and overtones of a closed organ pipe.
7. What is Doppler's effect ? Derive an expression for the apparent frequency when source is in motion.
8. What is Doppler's effect ? Derive an expression for the apparent frequency, when observer is in motion.
9. What is Doppler's effect ? Derive an expression for the apparent frequency, when source and observer are both in motion.

F. Fill in the Blank Type

1. For a wave travelling from one medium to another .... remains unchanged.
2. The velocity of sound in air is not affected by the change in...........
3. At temperature .... the speed of sound in air becomes double of its speed at 0^0C.
4. On an increase of 1^0C in temperature the speed of sound in air increases by ....
5. The speed of sound in moist air is .... than the speed in dry air.
6. Decibel is ....
7. A sound has intensity level of 30 decibel. Its intensity in watt / cm^2 is ....

G. True - False Type

1. The speed of sound in water is higher than that in air.
2. The changes in temperature have no effect on the speed of sound.
3. The changes in air pressure have no effect on the speed of sound.
4. The ratio of velocity of sound in hydrogen gas (γ =7/5) to that in helium gas (γ =5/3) at the same temperature is $\sqrt{21}$ /5
5. A baloon is filled with hydrogen gas. For sound waves it acts as a convex lens.
6. Velocity of sound is generally greater in solids than in gases at NTP.
7. The decrease in speed of sound at high elevation is due to low pressure.

ANSWERS

A. Multiple Choice Type Questions :

1. (iii), **2.** (i), **3.** (ii), **4.** (iii), **5.** (i), **6.** (ii), **7.** (iii), **8.** (ii), **9.** (iii), **10.** (iii), **11.** (i), **12.** (i), **13.** (ii), **14.** (iii), **15.** (i).

B. Very Short Answer Type Questions :

1. 5
2. Increases
3. musical
4. sound of mosquito has higher pitch and less loudness than that of a lion
5. distinguished due to quality or overtones
6. See text
7. The shift of wavelengths in the spectrum of light coming from star, towards violet end of spectrum indicates that the start is coming towards earth.
8. No change
9. 819^0 C
10. 1 : 2 : 3 : 4 : 5 .....
11. Longitudinal
12. longitudinal
13. noise
14. $y = \sum_n p_n \sin(\omega_n t - k_n x)$
15. At L / 2 distance from either end
16. increases

C. Short Answer Type Questions :

1. See text
2. See text
3. frequency, intensity level
4. speed of sound in moist air is more. Hence as humidity increases, speed of sound increases.
5. Open organ pipe contains all the overtones of fundamental note. On the other hand closed organ pipe contains only odd overtones.
6. $\vartheta_t = \sqrt{\frac{\gamma RT}{M}}, \frac{\vartheta_{27}}{\vartheta_{127}} = \sqrt{\frac{300}{400}} \Rightarrow \vartheta_{127} = \sqrt{\frac{4}{3}}.\vartheta_{27} \Rightarrow \vartheta_{127} = \sqrt{\frac{4}{3}} x 330 m/s = 381.05 m/s$
7. No
8. $\frac{\upsilon_B}{\upsilon_A} = \frac{\frac{1}{2\ell}\sqrt{\frac{4}{\pi}}.\sqrt{\frac{T_B}{\rho_B D_B^2}}}{\frac{1}{2}\sqrt{\frac{4}{\pi}}\sqrt{\frac{T_A}{\rho_A D_A^2}}} = \frac{1}{2}$
9. $\vartheta = \sqrt{\frac{P}{\rho}}, \ \vartheta = \sqrt{\frac{\gamma P}{\rho}}$
10. $\ell_1 = \lambda / 4, \ell_2 = 3\lambda / 4 \Rightarrow \ell_2 = 3\ell_1$

D. Unsolved Problems

1. 349 . 8 m/s
2. 5064 m/s
3. $\frac{2\ell}{C_0}\frac{\sqrt{T_0}}{\sqrt{T_1}+\sqrt{T_2}}$.

[Hints $\alpha = \frac{t_2 - t_1}{\ell}$

$\frac{T_1}{A}\ dx\ \frac{T_2}{B}$

$$dt = \frac{dx}{C_x}\ ;\ C_x = C_0\left(1+\frac{t}{273}\right)^{\frac{1}{2}} = C_0\left(1+\frac{t_1+\alpha x}{273}\right)^{\frac{1}{2}}$$

$$\Rightarrow\ C_x = \frac{C_0}{\sqrt{273}}(T_1+\alpha x)^{\frac{1}{2}} = \frac{C_0}{\sqrt{T_0}}(T_1+\alpha x)^{\frac{1}{2}}$$

$$\therefore dt = \frac{dx}{C_x} = \frac{\sqrt{T_0}}{C_0}\frac{dx}{(T_1+\alpha x)^{\frac{1}{2}}}$$

$$\therefore t = \frac{\sqrt{T_0}}{C_0}\int_0^{\ell}\frac{dx}{(T_1+\alpha x)^{\frac{1}{2}}} = \frac{\sqrt{T_0}}{C_0}.\frac{2\ell}{T_2-T_1}\left(T_2^{\frac{1}{2}}-T_1^{\frac{1}{2}}\right)$$

$$\Rightarrow\ t = \frac{\sqrt{T_0}}{C_0}.\frac{2\ell}{T_2^{1/2}+T_1^{1/2}} = \frac{2\ell}{C_0}.\frac{\sqrt{T_0}}{\sqrt{T_1}+\sqrt{T_2}}$$]

4. 0.34 m = 34 cm
5. 306 m/s
6. 1067 Hz
7. 16.7 m/s
8. 247.69 Hz
9. 594 Hz, 540 Hz
10. 18.86 cm
11. 98.55 cm or 101.49 cm
12. 220 Hz
13. 0.87×10^{-12} watt / m^2
14. 2484 Hz
15. 30 m/s

F. (1) frequency (2) pressure (3) 819°C (4) 0.61 m/s. (5) larger (6) a measure of sound level (7) 10^{-13}

G. (1) True (2) False (3) True (4) False (5) False (6) True (7) False

12

Elasticity

12.1 Molecular Structure of Matter :

Matter is made of molecules and atoms. Molecules are made of atoms; and electromagnetic (e.m.) force between atoms is responsible for the molecular structure. The inter-molecular force which is also an e.m. force binds the molecules and thus a material is constituted.

12.2 Inter atomic and Intermolecular force :

The general feature of inter atomic and intermolecular force are nearly similar. These features are well illustrated by the curve shown in fig. 12.1.

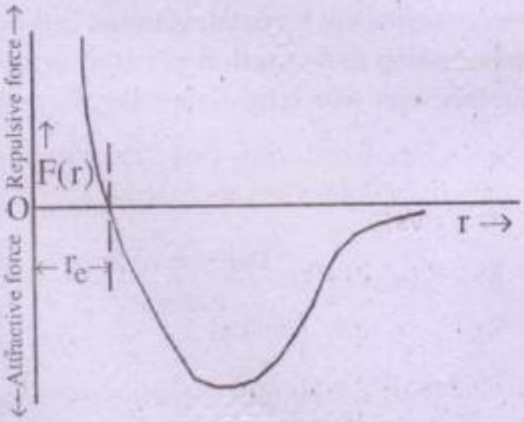


fig 12.1

(Intermolecular force Vs. distance of separation)

The curve indicates the following significant features :

i) The force remains attractive up to a separation 'r_e' called as <u>equilibrium distance.</u> Its value is ~ 0.74×10^{-10} m.

ii) The force becomes repulsive when the separation between molecules falls below the equilibrium distance (r_e).

The potential energy curve corresponding to the molecular force is similar to inter molecular force curve and is shown in fig. 12.2.

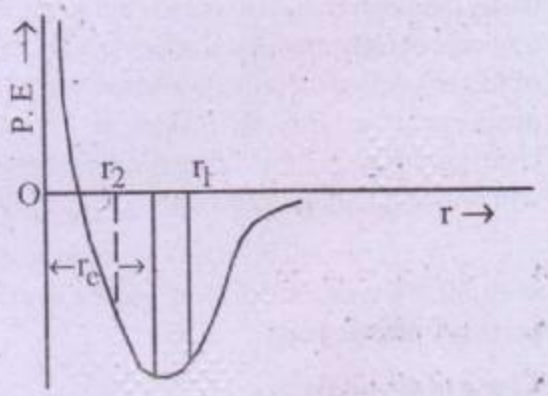


fig 12.2

(Potential energy Vs. distance of separation)

The curve in fig. 12.2 indicates that :

i) P.E. is minimum when separation distance 'r' between molecules equals the equilibrium distance 'r_e'. At this separation the force between molecules is zero.

ii) For $r > r_e$ the P.E. increases from the minimum value and approaches zero as $r \to \infty$. In this region force remains attractive.

iii) For $r < r_e$, the P.E. also increases from the minimum value and approaches ∞ as $r \to 0$. In this region a repulsive force comes into play.

12.3 Elasticity :

When a force or torque acts on a body, not free to move, a change in shape or size (or both) i.e. deformation of the body takes place, due to rearrangement of its constituent particles. Depending on the response of the body to such forces or torque, the materials from which they are made, can be classified as (i) rigid body (ii) elastic body, (iii) plastic body.

A body which does not undergo any deformation, under the application of force of any value, is called a perfectly **rigid body**. But such bodies are seldom found in nature.

A body which undergoes deformation under the application of a force but regains its original configuration, soon after the withdrawal of force is called a perfectly **elastic body**. The property of a body to restore its original configuration, when deforming forces are withdrawn is called **elasticity**.

A body which retains its deformation even after the withdrawal of forces is called perfectly **plastic body.**

Cause of elasticity :

In a body large number of molecules are bound together by molecular force and the molecules take up such sites in the body that it remains in the minimum energy state. At this equilibrium state the distance between any two molecules is the equilibrium distance 'r_e'.

When by application of force the distance of separation 'r' is increased (i.e. $r_1 > r_e$) the molecules will have a tendency to come back to the original state. Similarly when by application of force 'r' is decreased (i.e. $r_2 < r_e$) the molecules will have a tendency to separate out to distance r_e.

The tendency of the molecules to remain at equilibrium distance apart is the cause of elasticity. **An elastic (reaction) force thus comes into play and opposes the deforming force.**

i.e. elastic reaction force ($\vec{R}$)

= - (Deforming force ($\vec{F}$)

$$\vec{R} = -\vec{F} \qquad ...(12.3.1)$$

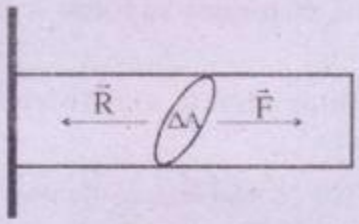


fig. 12.3

12.4 Stress (Γ)

It is defined as the elastic reaction force (restoring force) developed in a body, undergoing deformation per unit area of a surface over which the deforming force acts.

Since restoring force is numerically equal to the deforming force (eqn. 12.3.1), so

$$\text{Stress}(\Gamma) = \frac{\text{Deforming force}}{\text{Area}} = \frac{F}{A} \qquad ...(12,4.1)$$

The stress may be classified as (i) **Normal stress or longitudinal stress** and (ii) **Tangential stress or shearing stress.**

(i) Normal Stress :

The stress is called a normal stress if the restoring force (or deforming force) acts at right-angles to the surface and is defined as

$$\Gamma_n = \frac{F_n}{A} \qquad ...(12.4.2)$$

When F_n is uniform over the surface area A.

But if F_n is not uniform then.

$$\Gamma_n = \lim_{\Delta A \to 0} \frac{\Delta F_n}{\Delta A} = \frac{dF_n}{dA} \quad ...(12.4.3)$$

gives the stress at a point.

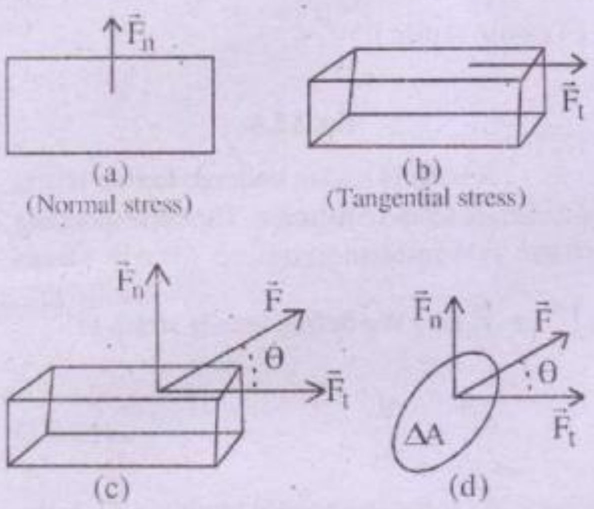


fig. 12.4

(ii) **Tangential Stress :**

The stress is called a tangential stress if the restoring force (deforming force) acts tangentially to the surface and is given by

$$\Gamma_t = \frac{F_t}{A} \qquad ...(12.4.4)$$

when F_t is uniform over the surface area A. If F_t is not uniform, then

$$\Gamma_t = \lim_{\Delta A \to 0} \frac{\Delta F_t}{\Delta A} = \frac{dF_t}{dA} \quad ...(12.4.5)$$

gives the tangential stress at a point

However when the deforming force is applied in an arbitrary fashion, as shown in fig 12.4 (c) and (d), then

$$\vec{F} = \vec{F}_n + \vec{F}_t \qquad ...(12.4.6)$$

Where, $F_n = F \sin\theta$, $F_t = F \cos\theta$

giving

$$\Gamma_n = \frac{F_n}{A} = \frac{F \sin\theta}{A} \qquad ...(12.4.7)$$

$$\Gamma_t = \frac{F_t}{A} = \frac{F \cos\theta}{A} \qquad ...(12.4.8)$$

Stress has dimension of $ML^{-1}T^{-2}$ and its units are (i) Nm^{-2} or Pa in S.I.

Ex. 12.4.1 A load of 5.0kg. is suspended from a ceiling with the help of a steel wire of adius 2.0 mm. Find the tensile stress developed in the wire when equilibrium is achieved.

Soln.

Load $F_n = 5kg = 5 \times 9.8\ N$

Stress $\Gamma_\ell = F_n/A = F_n/\pi r^2$

$$= \frac{5 \times 9.8}{\pi \times (2 \times 10^{-3})^2} = 3.9 \times 10^6\ N/m^2$$

Ex. 12.4.2 A rectangular black of metal has dimension (0.4m x 0.4m x 0.5m). Its lower surface is fixed. A tangential force $1.6 \times 10^{+7}$ N is applied to the upper surface. Compute the shear stress.

Soln.

A = area of upper surface = (0.4m) x (0.4m) = $1.6 \times 10^{-1} m^2$

Shear stress = $F_t/A = 1.6 \times 10^{+7} / 1.6 \times 10^{-1}$

$= 10^8 N/m^2$

12.5 Strain

The deforming force acting on a body causes a change in the configuration (length, volume or shape) of the body; and strain is a measure of this deformation. (change in configuration).

The strain is defined as the ratio of change in particular dimension of the body to the original dimension of the body.

$$\text{Strain } (\in) = \frac{\text{Change in dimension}}{\text{Original dimension}} \quad ...(12.5.1)$$

Thus strain is **dimension - less** and has **no unit.**

Types of Strain :

Depending on the mode of application of the deforming force different types of strains are caused. The commonly known three types are (i) Longitudinal strain (ii) Volume (Bulk) strain and (iii) shear strain.

(i) Longitudinal Strain :

When deforming force acts normally along one direction only longitudinal strain occurs. Longitudinal strain can be of two types : (a) Tensile (or elongation) strain (b) compressional strain.

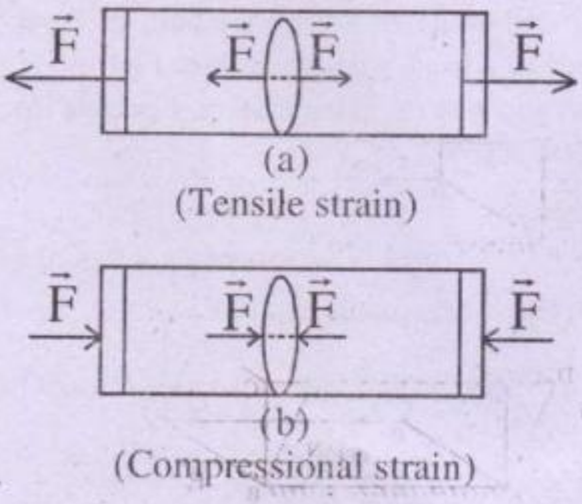


fig. 12.5

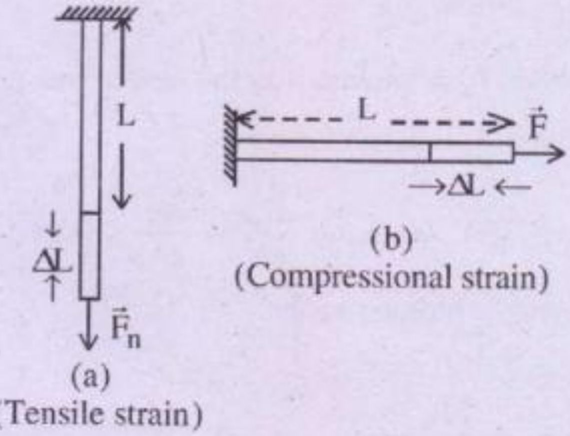


fig. 12.6

A body is said to undergo tensile strain, if its length tends to increase. The corresponding stress is sometimes called tensile stress ($\Gamma_n^+ = F_n/A$) We define tensile strain as

$$\in_\ell^+ = \frac{\Delta L}{L} \quad ...(12.5.2)$$

where ΔL is the increase in length and L is the original length.

On the other hand a body is said to undergo compressional strain if its length tends to decrease. The corresponding stress is sometimes called compressional stress $\left(\Gamma_n^- = F_n/A\right)$. We define compressional strain as

$$\in_\ell^- = \frac{-\Delta L}{L} \quad ...(12.5.3)$$

Where $(-\Delta L)$ is the decrease in length and L is original length.

It is to be noted that only solids can have longitudinal strain, Fluids cannot sustain longitudinal strain.

(ii) Volume strain :

When the deforming force acts uniformly and normally at every point on the surface of a body the volume strain occurs. The corresponding stress is sometimes called volume stress $\left(\Gamma_{\theta} = F_n/A\right)$.

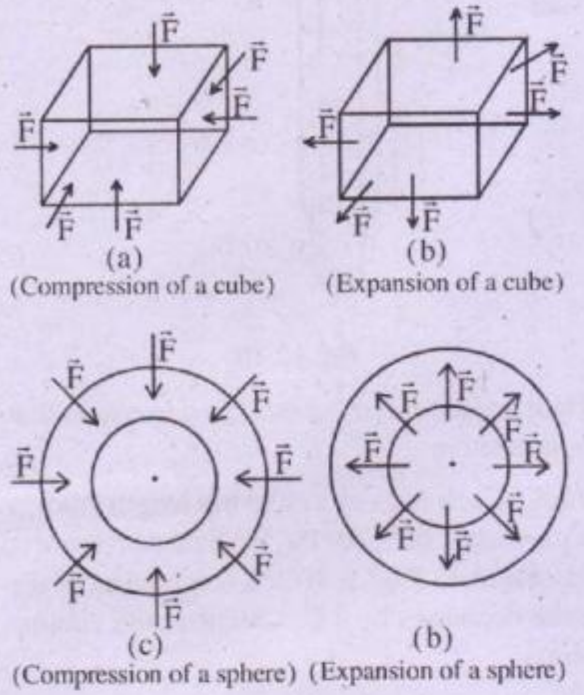


fig. 12.7

The volume strain is defined as

$$\epsilon_V = \frac{\Delta V}{V} \quad ...(12.5.4)$$

where ΔV, the change in volume (is positive for an expansion and is negative for compression) and V is the original volume.

(iii) Shear strain :

When a deforming force is applied tangentially to a surface, with opposite face (surface) fixed, the shear strain (due to change in shape) occurs. The corresponding stress is called shear stress $\left(\Gamma_t = F_t/A\right)$

The shear strain ($\in_s$) is defined as the angle (θ) through which a line originally perpendicular to the fixed plane turns.

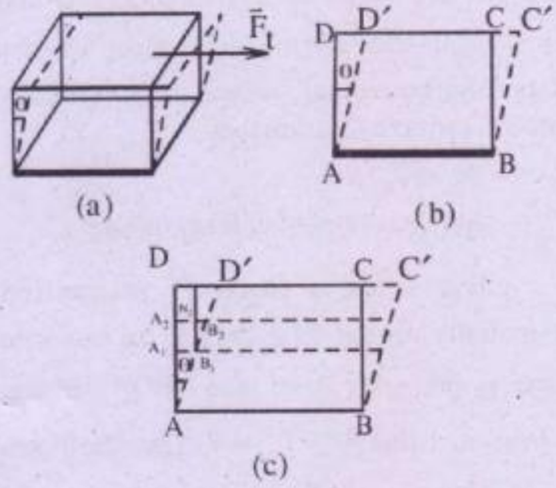


fig.12.8

$$\text{Since, } \tan\theta = \frac{DD'}{AD} = \frac{N_2B_2}{B_1N_2} = \frac{N_2B_2}{A_1A_2} \approx \theta \quad ...(12.5.5)$$

(when θ is small), so shear strain can be defined as the relative displacement between two layers separated by unit distance.

Cause of shear strain :

To understand how shear strain occurs we consider a rectangular slab ABCDPQRS, with surface ABQP fixed on a horizontal plane. (see fig. 12.9(a)). Then two forces

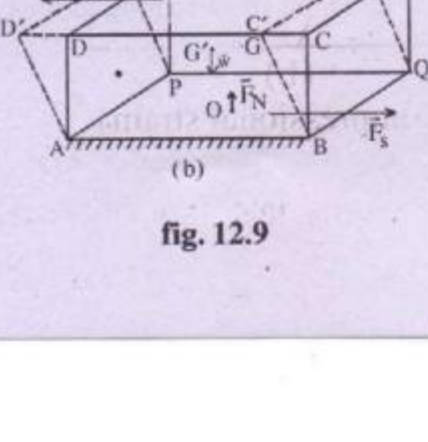


fig. 12.9

(i) $\vec{W} = m\vec{g}$, the weight of the body acts vertically downward through its C.O.G. 'G' and

(ii) $\vec{F}_N$, the normal reaction of the supporting horizontal surface, acts vertically upward, through O, such that

$$\vec{W} + \vec{F}_N = 0$$

and their lines of action concide.

But when a force $\vec{F}_t$ is applied tangentially on the face DCRS, an opposite force $\vec{F}_S$ acts over fixed face ABQP (see fig. 12.9(b)) such that $\vec{F}_t + \vec{F}_s = 0$. Thus the forces now acting are $\vec{W}$, $\vec{F}_n$, $\vec{F}_t$ and $\vec{F}_s$. Since now C.O.G. has been shifted to G', the lines of action of $\vec{W}$ and $\vec{F}_N$ do not coincide. As a result $(\vec{W}, \vec{F}_N)$ and $(\vec{F}_t, \vec{F}_s)$ constitute couples and balance each other. The rectangular slab remains in equilibrium in the deformed state $ABC'D'$ $PQR'S'$. Thus the shear is produced due to the couple $(\vec{F}_t, \vec{F}_s)$ and remains in equilibrium due to the balancing effect of $(\vec{W}, \vec{F}_N)$.

(iv) **Lateral strain :**

When a deforming force tends to elongate a body in one direction, there occurs a contraction in the perpendicular directions and vice versa. This leads to lateral strain and is defined as

$$\epsilon_p = \frac{\text{Change in perpendicular dir}^n}{\text{Original dimension in perpendicular dir}^n}$$

$$\Rightarrow \epsilon_p = \frac{\Delta r}{r} \quad ...(12.5.6)$$

The ratio of lateral strain to longitudinal strain is called poisson's ratio (σ)

$$\text{i.e. } \sigma = \frac{\text{Lateral strain}}{\text{Longitudinal strain}} = \frac{\epsilon_p}{\epsilon_\ell} \quad ...(12.5.7)$$

$$\Rightarrow \sigma = -\frac{\Delta r / r}{\Delta L / L} \quad ...(12.5.8)$$

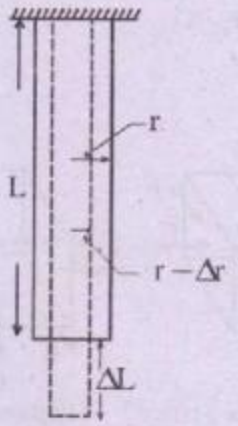


fig. 12.10

Here negative sign is introduced to ensure that σ is positive.

12.5.1 Each edge of a cube has length 2.0m at a pressure 1.02×10^5 Pa. When the pressure is increased to 3.02×10^5 Pa, each edge of the cube decreases by 1%. Calculate the volume strain.

Soln.

Given each edge of cube = 2.0m

$\Rightarrow$ Original volume $V = 2^3 = 8 m^3$

New edge length = 2 - 0.01 x 2 = 1.98 m

New volume $V' = (1.98)^3 m^3$

Change in volume $\Delta V = (8 - (1.98)^3)\, m^3$

$= 0.2376\, m^3$

Volume strain $= \frac{\Delta V}{V} = 0.0297$

12.5.2 A steel wire of length 6 m and cross - sectional area $10^{-6} m^2$ is elongated by 6×10^{-3} m. Compute the tensile strain.

Soln.

Given $\ell = 6m$

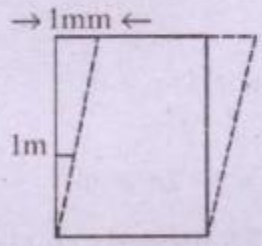
$\Delta \ell = 6 \times 10^{-3}$ m

Tensile strain $\frac{\Delta \ell}{\ell} = \frac{6 \times 10^{-3}}{6} = 10^{-3}$

12.5.3 A rectangular block of size (0.5m x 0.5m x 1m) is fixed at its lower surface. When a tangential force is applied its upper surface gets displaced by 1mm. Compute the shear strain.

Soln.

$$\tan\phi = \phi = \frac{1 \times 10^{-3}}{1} = 10^{-3}$$



12.6 Hooke's law :

Hooke's Law states that for small deformations, stress is directly proportional to strain.

i.e. $\frac{\text{stress}}{\text{strain}}$ = constant (for small strain)

This constant of proportionality is known as elastic modulus or modulus of elasticity. It is a measure of the susceptibility of a body to be deformed and depends on the nature of the material.

Since strain is dimensionless and unitless so dimension of elastic modulus is same as that of stress i.e. $ML^{-1}T^{-2}$; and its units are Nm^{-2} (≡ Pa) in S.I. system and dyne cm^{-2} in C.G.S. system.

Types of elastic modulus :

There are three commonly known elastic moduli, corresponding to the three types of strain (described in sec. 12.5).

(i) Young's modulus (Y) (for longitudinal strain)

(ii) Bulk modulus (B) (for volume strain)

(iii) Rigidity modulus (n) (for shear strain)

[A fourth elastic modulus is the Axial modulus. This arises when deforming forces are applied in such a manner that lateral strain is not allowed to occur in perpendicular direction. However this is beyond the scope of this book.]

(i) Young's Modulus : (Y)

It is defined as the ratio of longitudinal stress to longitudinal strain (for small strains) within proportional limit.

$$\text{i.e. } Y = \frac{\text{long . stress}}{\text{long . strain}}$$

$$\Rightarrow Y = \frac{F_n / A}{\Delta L / L} \quad ...(12.6.1)$$

The quantity force per unit extension (i.e. $F_n / \Delta L = Y.A/L$) is called **stiffness** or force constant. This shows that stiffness (or force constant) is directly proportional to the area of cross-section and inversely proportional to the length.

(ii) Bulk modulus (B)

It is defined as the ratio of the volume (Bulk) stress to volume (Bulk) strain (for small strains) within proportional limit.

$$\text{i.e. } B = \frac{\text{Bulk stress}}{\text{Bulk strain}} = \frac{P}{(-\Delta V / V)} \quad ...(12.6.2)$$

The negative sign makes B positive when volume actually decreases on applying pressure.

$$\text{But} \quad B = \frac{P}{(\Delta V / V)} \quad ...(12.6.3)$$

when volume increases on applying pressure (e.g. filling of air into a balloon).

Quite often the change in volume is measured corresponding to change in pressure. Then

$$B = \frac{\Delta P}{(-\Delta V / V)} = -V\frac{dP}{dV} \quad ...(12.6.4)$$

Compressibility 'K' is defined as the reciprocal of Bulk modulus.

$$\text{i.e. } K = \frac{1}{B} = -\frac{1}{V}\frac{dV}{dP} \quad ...(12.6.5)$$

As solids and liquids are almost incompressible hence K is small (i.e. B large) for solids and liquids. But gases are easily compressible, hence K is large (B small) for gases.

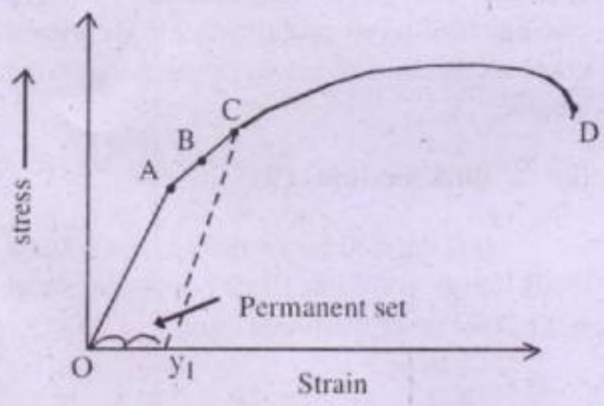
(iii) Rigidity modulus : (n)

It is defined as the ratio of shear stress to shear strain within proportional limit.

$$\text{i.e } n = \frac{\text{Shear stress}}{\text{Shear strain}} = \frac{F_t / A}{\theta} \quad ...(12.6.6)$$

It is also sometimes called **torsional modulus.**

Fig :



A typical Stress ~ Strain Curve for a ductile metal

A ⟶ Proportional limit

B ⟶ Elastic limit or yield point

D ⟶ Fracture point

O to B ⟶ Elastic region

B to D ⟶ Plastic region

Relation among elastic constants (Y,B,n,σ)

The three elastic moduli Y,B,n and Poisson's ratio σ are related to one another. It can be shown that

$$Y = 2n(1 + \sigma) \quad ...(12.6.7)$$

$$Y = 3B(1 - 2\sigma) \quad ...(12.6.8)$$

Solving eqns. (12.6.7) and 12.6.8) we obtain

$$\frac{1}{B} + \frac{3}{n} = \frac{9}{Y} \quad ...(12.6.9)$$

We also observe that, since Y, n, B are all positive so

$$\frac{Y}{2n} = 1 + \sigma \geq 0 \quad ...(12.6.10)$$

$$\frac{Y}{3B} = 1 - 2\sigma \geq 0 \quad ...(12.6.11)$$

Relations (12.6.10) and (12.6.11) imply that

$$-1 < \sigma < 0.5 \quad ...(12.6.12)$$

Thus relation (12.6.12) shows that σ can assume both negative and positive values. But actually it could assume either positive value (if we define σ by (12.5.8) or negative value (if we define $\sigma = \frac{\Delta r / r}{\Delta L / L}$). This is because a body cannot expand laterally when it is stretched longitudinally.

The values of Y,B,n, and σ are listed for some common materials in table 12.2. Table 12.1 lists compressibilities of few liquids.

Liquid	Compressibility (K) 10^{-11} m^2/N
Carbon disulphide	64
Ethyl alcohol	110
Glycerine	21
Mercury	3.7
Water	49

Table 12.1

Material	Y in$10^{11}Nm^{-2}$	B in$10^{11}Nm^{-2}$	n in$10^{11}Nm^{-2}$	σ
Aluminium	0.7	0.78	0.26	0.35
Bismuth	0.318	0.294	0.122	0.32
Brass	1.1	1.07	0.415	0.328
Bronze	0.81	0.435	0.34	0.19
Cadmium	0.499	0.416	0.192	0.30
Constantin	1.63	1.637	0.611	0.334
Copper	1.3	1.4	0.483	0.345
Glass	0.738	0.4	0.31	0.19
Gold	0.8	1.66	0.277	0.422
Iron	2.0	1.45	0.8	0.27
Lead	0.16	0.46	0.056	0.442
Manganin	1.24	1.24	0.465	0.333
Mercury	—	0.27	—	—
Nickel	2.02	1.76	0.77	0.309
Platinum	1.69	2.49	0.61	0.387
Quartz	0.52	0.14	0.3	—
Rubber	0.05	—	0.00016	0.48
Tungsten	3.6	2.0	1.5	0.2
Wood	0.1	—	—	—

Table 12.2

Ex. 12.6.1 What force is required to stretch a wire of cross-section 0.5 cm^2 to double its length. Given

$$Y = 2 \times 10^{11}\ N/m^2$$

Soln. $Y = \frac{F/A}{\ell/L} \Rightarrow F = Y.A\frac{\ell}{L}$

Since wire is elongated to double its length, so elongation $\ell = L$

$$\therefore F = Y.A.\frac{L}{L} = Y.A. = 2 \times 10^{11}\frac{N}{m^2} \times 0.5 \times 10^{-4} m^2$$

$$\Rightarrow F = 10^7 N$$

Ex. 12.6.2 Identical springs of steel and copper are equally stretched. Compare the work done for each spring.

Soln. $W = \frac{1}{2}\frac{Y.A}{L}.\ell^2$

Since springs are identical A, L are same and as equally stretched so ℓ is same for both.

$$\therefore \frac{W_{steel}}{W_{copper}} = \frac{\frac{1}{2}Y_s.\frac{A}{L}.\ell^2}{\frac{1}{2}Y_c.\frac{A}{L}.\ell^2} = \frac{Y_s}{Y_c}$$

Since $Y_S > Y_C$, so more work is done for steel spring than for copper spring.

Ex.12.6.3 The length of a wire is increased by 2% when loaded with 2kg weight. What is the strain produced ? If the area of cross-section is 0.01 mm^2 calculate the Young's modulus.

Soln. Let original length be L.

Change in length is 2% $\Rightarrow \Delta L = \frac{2}{100}L = .02L$

$$\therefore \text{Strain produced} = \frac{\Delta L}{L} = \frac{0.02L}{L} = 0.02$$

$$\text{Stress applied} = \frac{2 \times 9.8}{0.01 \times 10^{-6}} N/m^2$$

$$Y = \frac{2 \times 9.8}{0.01 \times 10^{-6}} / 0.02$$

$$= \frac{2 \times 9.8}{0.01 \times 0.02 \times 10^{-6}} = 19.6 \times 10^{10}$$

$$Y = 1.96 \times 10^{11} N/m^2$$

Ex.12.6.4 The length of a wire is cut to half of its original length (i) what will be the effect on increase in its length under a given load. (ii) what will be the effect on the maximum load it can bear.

Soln.

Let original length = L
Load applied = F

(i) elongation $\ell = \frac{F/A}{Y}.L = \frac{F.L}{Y.A}$

If the wire is cut to half its length, keeping the load same, then new elongation ℓ' is given by

$$\ell' = \frac{F}{Y.A}.\frac{L}{2} = \frac{\ell}{2}$$

i.e. increase in length is also halved

(ii) Original strain $= \frac{\ell}{L}$

New strain $= \frac{\ell'}{L/2} = \frac{\ell/2}{L/2} = \frac{\ell}{L}$

i.e the strain remains same. Since the stress also remains same so no change will occur for the maximum load it can bear.

Ex.12.6.5 Y of brass is 9×10^{10} N/m^2, and n of brass is 4×10^{10} N/m^2. Calculate the Bulk modulus for brass.

Soln. We have

$$\frac{1}{B} + \frac{3}{n} = \frac{9}{Y}$$

$$\Rightarrow \frac{1}{B} = \frac{9}{Y} - \frac{3}{n} = \frac{9}{9 \times 10^{10} N/m^2} - \frac{3}{4 \times 10^{10} N/m^2}$$

$$= \frac{1}{10^{10} N/m^2}\left(1-\frac{3}{4}\right) = \frac{0.25}{10^{10} N/m^2}$$

$$\Rightarrow B = \frac{10^{10} N/m^2}{0.25} = 4x10^{10} N/m^2$$

Ex.12.6.6 Y. of copper is 13×10^{10} N/m^2 and its poisons ratio is 0.33. Calculate the Bulk modulus.

Soln. $Y = 3B(1-2\sigma)$

$$\Rightarrow B = \frac{Y}{3(1-2\sigma)} = \frac{13x10^{10} N/m^2}{3(1-2x0.33)}$$

$$\Rightarrow B = \frac{13x10^{10}}{3x0.34} N/m^2$$

$$\Rightarrow B = 12.745 \times 10^{10} N/m^2$$

Ex.12.6.7 What will be the density of copper under a pressure of 10,000 N/cm^2. (Given density of copper is 8.96×10^3 kg/m^3, Bulk modulus of copper is 1.4×10^{10} N/m^2).

Soln. $B = \frac{P}{\Delta V / V}$

$$\Rightarrow \Delta V = \frac{P}{B}.V = \frac{10{,}000 N/cm^2}{1.4 \times 10^{10} N/m^2}.V$$

$$\Delta V = \frac{10^4 x 10^4 N/m^2}{1.4x10^{10} N/m^2}.V = \frac{V}{140}$$

New volume $V' = V - \Delta V = V - \frac{V}{140} = \frac{139}{140}V$

New density $\rho' = \frac{M}{V'} = \frac{M}{\frac{139}{140}V}$

$$= \frac{M/V}{(139/140)} = \frac{\rho}{(139/140)}$$

$$\Rightarrow \rho' = \frac{8.96 \times 10^3 x 140}{139} kg/m^3$$

$$= 9.02x10^3 kg/m^3$$

Ex.12.6.8 The elastic limit of steel is 8×10^8 N/m^2 and its Young's modulus is 2×10^{11} N/m^2. Find the maximum elongation of a half-meter steel wire that can be given without exceeding the elastic limit.

Soln.

elastic limit = 8×10^8 N/m^2 = maximum stress

$$Y = 2 \times 10^{11} N/m^2$$

$$Y = \frac{stress}{strain} = \frac{max.stress}{max.strain}$$

$$\Rightarrow \text{max strain} = \frac{\text{max stress}}{Y} = \frac{8x10^8 N/m^2}{2x10^{11} N/m^2}$$

$$\Rightarrow \text{max.strain} = 4x10^{-3} = \frac{(\Delta L)max}{L}$$

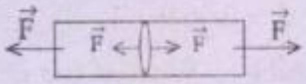
$$\Rightarrow (\Delta L)_{max} = 4x10^{-3} xL$$

$$= 4 x 10^{-3} x 0.5m$$

$$(\Delta L)_{max} = 2x10^{3} m = 0.2cm$$

Ex.12.6.9 Two persons pull a rope towards themselves with equal force of 100 N; on the rope. Find the Young's modulus of the material of the rope if it extends in length by 10cm. Original length of rope is 2m and area of cross-section is 2 cm^2.

Soln.



Here force of 2F = 2 x 100 N act over every cross-section.

Given ΔL = 1 cm, L = 2m = 200cm, A = 2cm^2

$$Y = \frac{F.L}{A.\Delta L} = \frac{(2x100x10^5)}{2x1} \times 200 = 2 \times 10^9$$ dyne/cm^2

Ex.12.6.10 A wire of diameter 1 mm and length 2 m is hung from a ceiling. A load of 5 kg is hung at the free end, calculate the extension produced. What minimum diameter is allowed so that elastic limit is not exceeded. (Elastic limit = 1.5×10^9 dyne/cm², Y = 1.1×10^{12} dyne/cm²).

Soln.

Given

diameter d = 1mm = 0.1 cm

original length L = 2 m = 200cm

F = 5kg wt. = 5x9.8 N = $5 \times 9.8 \times 10^5$ dynes

$$\therefore \frac{\Delta L}{L} = \frac{F}{A}.\frac{1}{Y} = \frac{F}{\pi d^2/4}.\frac{1}{Y}$$

$$\Rightarrow \Delta L = \frac{4F}{\pi d^2}.\frac{1}{Y}.L$$

$$= \frac{4x5x9.8x10^5}{\pi x(0.1)^2} x \frac{1}{1.1x10^{12}} x200$$

$$\Rightarrow \Delta L = 0.1134 \text{ cm}$$

Elastic limit = maximum stress

$$= \frac{F}{\pi d'^2/4} = \frac{4F}{\pi d'^2}$$

$$\Rightarrow d'^2 = \frac{4F}{\pi} x \frac{1}{\text{elastic limit}} = \frac{4x5x9.8x10^5}{\pi} x \frac{1}{1.5x10^9}$$

$$d'^2 = 4.159 \times 10^{-3}$$

$$d' = 0.0645$$

Ex.12.6.11 What is the density of ocean water at a depth, where the pressure is 80 atmosphere, if its density at the surface is 1.03×10^3 kg/m³. Comprehensibility of water is 45.8×10^{-11} Pa⁻¹, 1 atmosphere = 1.013×10^5 Pa.

Soln.

Given

$$\text{Comprehensibility} = \frac{1}{B}$$

$$= 45.8 \times 10^{-11} / Pa^{-1}$$

$$= 45.8 \times 10^{-11} \text{ m}^2/\text{N}$$

Pressure P = 80 atoms = $80 \times 1.013 \times 10^5$ N/m².

consider a mass M of water whose volume is V near surface and V' at the depth.

$$\text{Then } \rho = \frac{M}{V} = 1.03 \times 10^3 \text{ kg/m}^3$$

$$\rho' = \frac{M}{V'}$$

$$\text{Change in volume } \Delta V = V' - V = M\left(\frac{1}{\rho'} - \frac{1}{\rho}\right)$$

$$\text{Vol. strain} = -\frac{\Delta V}{V} = -\frac{M}{V}\left(\frac{1}{\rho'} - \frac{1}{\rho}\right) = -\rho\left(\frac{1}{\rho'} - \frac{1}{\rho}\right)$$

$$= 1 - \frac{1}{\rho'}$$

$$\text{Now } B = \frac{P}{-\Delta V/V} \Rightarrow \frac{1}{B} = \frac{-\Delta V/V}{P}$$

$$\Rightarrow \frac{1}{B} = \frac{1-\frac{1}{\rho'}}{P}$$

$$\Rightarrow 1 - \frac{\rho}{\rho'} = \frac{P}{B}$$

$$\Rightarrow \rho' = \frac{\rho}{1-\frac{P}{B}}$$

$$= \frac{1.03x10^3 \text{ kg/m}^3}{1-(80x1.013x10^5)(45.8x10^{-11})}$$

$$\Rightarrow \rho' = 1.034 \times 10^3 \text{ kg/m}^3$$

Ex.12.6.12 Find the increase in pressure required to diminish the volume of water sample by 0.01% (Given Bulk modulus of water is 2.1×10^9 N/m²)

Soln.

$$B = \frac{\Delta P}{\Delta V / V}$$

$$\Rightarrow \quad \Delta P = B.\frac{\Delta V}{V} = 2.1 \times 10^9 \frac{N}{m^2} x \frac{0.01}{100}$$

$$\Delta P = 2.1 \times 10^5 \frac{N}{m^2} = 2.1 x 10^5 Pa.$$

Ex.12.6.13 An Indian rubber cord 12m long is suspended vertically. How much does it stretch under its own weight ? Density of rubber is 1.5 g/cm^3, Y = 5x10^6 g wt/cm^2.

Soln.

Given L = 12.m = 1200 cm

density ρ = 1.5 gm/cm^3

Y = 5 x 10^6 gm wt/cm^2

= 980 x 5 x 10^6 dyne/cm^2

Let A be its cross - section

Its own weight W = A L ρ g = F

Since weight W acts through C.O.G.

Hence effective original length = $\frac{L}{2}$

$$\therefore Y = \frac{F/A}{\Delta L/(L/2)} = \frac{L\rho g}{\Delta L/(L/2)}$$

$$= \frac{L^2 \rho g}{2,\Delta L}$$

$$\Rightarrow \Delta L = \frac{L^2 \rho g}{2Y} = \frac{(1200)^2 x 1.5 x 980}{2 x 980 x 5 x 10^6} = 0.216 cm$$

Ex.12.6.14 An Indian rubber cube of side 7 cm has one side fixed while a tangential force equal to the weight of 200 kg is applied tó the opposite face. Find the shearing strain produced, and the distance through which the strained side moves. Modulus of rigidity for rubber is 2x10^7 dyne/cm^2.

Soln.

Given each side L = 7 cm

F = 200 kg wt = 200 x 9.8 N

= 200 x 9.8 x 10^5 dynes

n = 2 x 10^7 dyne/cm^2

$$n = \frac{F/A}{\theta}$$

$$\Rightarrow \text{shearing strain } \theta = \frac{F}{An} = \frac{200 x 9.8 x 10^5}{7^2 x 2 x 10^7}$$

$$\theta = \frac{9.8}{49} = 0.2 \text{ radian}$$

Now $\theta = \frac{\ell}{L} \Rightarrow \ell = L\theta = 7x0.2 = 1.4$ cm

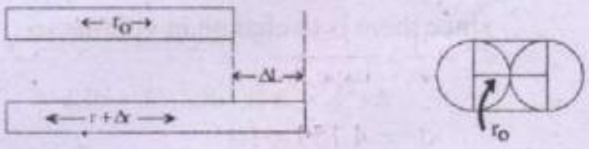
Ex.12.6.15 Y of iron is 2 x 10^{11} N/m^2 and interatomic spacing between two molecules is 3 x 10^{-10} m. Find the inter-atomic force constant.

Soln.

$F' = K.\Delta r$ = Interatomic force

where Δr is the increase in interatomic distance

K is force constant.



$$\text{long. strain} = \frac{\Delta L}{L} = \frac{\Delta r}{r_0}$$

Chain of atoms in cross -sectional area r_0^2 is only one. No. of chains in unit cross-sectional area is $\frac{1}{r_0^2}$.

$$\therefore F' = \frac{\text{Total applied force}}{\text{Total no of chains in cross-sectional area A}}$$

$$\Rightarrow F' = \frac{\text{Total force applied}}{\text{Total no of chains in cross-sectional area A}}$$

$$\Rightarrow F' = \frac{F}{A/r_0^2} = r_0^2\frac{F}{A}$$

$$\Rightarrow K.\Delta r = r_0^2\frac{F}{A}$$

$$\Rightarrow K = \frac{r_0^2}{\Delta r}.\frac{F}{A} = r_0\frac{F/A}{\Delta r/r_0} = r_0\frac{F/A}{\Delta L/L}$$

$$\Rightarrow K = r_0.Y. = 3x10^{-10}mx2x10^{11}N/m^2$$

$$\Rightarrow K = 60\ N/m.$$

Ex.12.6.16 There is no change in the volume of a wire undergoing longitudinal extension. Calculate the poisons ratio.

Soln.

Let initial radius be = r

initial length be = L

initial volume $V = \pi r^2 L$

final length $= L + dL$

final radius $= r - dr$

$\therefore$ final volume $= \pi(r - dr)^2(L + dL)$

since there is to change in volume so

$$\pi r^2 L = \pi(r-dr)^2\ (L+dL)$$

$$= \pi r^2 L\left(1-\frac{dr}{r}\right)^2\left(1+\frac{dL}{L}\right)$$

$$\Rightarrow 1 \cong \left(1-\frac{2dr}{r}\right)\left(1+\frac{dL}{L}\right)$$

$$\cong 1 + \frac{dL}{L} - \frac{2dr}{r} \text{ (retaining upto 1st order)}$$

$$\Rightarrow \frac{dL}{L} = 2\frac{dr}{r}$$

$$\Rightarrow \frac{1}{2} = \frac{dr/r}{dL/L} = \sigma \text{ Poisson's ratio}$$

$\therefore \sigma = 0.5$

Ex.12.6.17 A steel wire 2 mm in diameter is just stretched between two fixed points at a temperature of 20^0C. Determine its tension when its temperature falls to 10^0C. Linear expansivity of steel = $11 \times 10^{-6}/^0$K. Young's modulus = 2×10^{11} N/m^2.

Soln.

We have $\ell_2 = \ell_1[1 + \alpha(t_2 - t_1)]$

$$\Rightarrow \frac{\ell_2 - \ell_1}{\ell_1} = \alpha(t_2 - t_1) = \text{long strain}$$

stress = Y. strain = Y. $\alpha(t_2 - t_1)$

$\Rightarrow$ Tension = stress x area = Y.A. $\alpha(t_2 - t_1)$

$$= Y.\frac{\pi}{4}d^2\alpha(t_2 - t_1)$$

$$\Rightarrow \text{Tension} = (2x10^{11})\left(\frac{\pi}{4}\right)(2 \times 10^{-3})^2 \times (11 \times 10^{-6})(10^0)$$

Tension = 69.12.N

Ex.12.6.18 A load of 7.6 kg. hangs from the lower end of a steel wire which is rigidly clamped at the upper end. When the load is immersed in water. the length of the wire changes by 1 mm. Calculate the length of the wire.

($Y_S = 2 \times 10^{11}$ N/m^2, d = 0.4 mm, ρ_{load} = 7600 kg/m^3, g = 9.8 m/s^2)

Soln.

$$Y = \frac{Mg/(\pi d^2/4)}{\ell/L} = \frac{4MgL}{\pi d^2\ell}$$

$$\Rightarrow W = Mg = \left(\frac{\pi d^2 y}{4L}\right)\ell = k\ell$$

When immersed in water the weight of load in water is

$$W' = K\ell'.$$

$$W' = W - V.\rho_\ell.g = \left(W - \frac{W}{\rho}.\rho_\ell\right) = W\left(1 - \frac{\rho_\ell}{\rho}\right)$$

$$\Rightarrow \rho = \frac{W\rho_\ell}{W - W'} = \frac{1000\,K\ell}{K\ell - K\ell'} = \frac{1000\,\ell}{\ell - \ell'}$$

$$\Rightarrow \ell - \ell' = \frac{1000\,\ell}{\rho} = \frac{1000\,\ell}{7600}$$

$$\Rightarrow 1 \times 10^{-3} = \frac{1000 \times \ell}{7600}$$

$$\Rightarrow \ell = \frac{7600 \times 10^{-3}}{1000} = 0.0076 = 7.6 \times 10^{-3}\,m.$$

Now $Mg = K\ell$.

$$\Rightarrow K = \frac{Mg}{\ell} = \frac{7.6 \times 9.8}{7.6 \times 10^{-3}} = 9.8 \times 10^{3}$$

$$\Rightarrow \frac{\pi d^2 Y}{4L} = 9.8 \times 10^{+3}$$

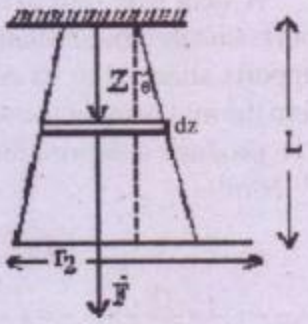
$$\Rightarrow L = \frac{\pi d^2 Y}{4 \times 9.8 \times 10^3} = \frac{\pi.(0.4 \times 10^{-3})^2 \times 2 \times 10^{11}}{4 \times 9.8\ 10^3}$$

$$L = 2.56\ m.$$

Ex.12.6.19 A body of mass 3.14 kg is suspended from one end of a wire of length 10.0 m. The radius of the wire is changing uniformly from 9.8×10^{-4} m at one end to 5.0×10^{-4} m at the other end. Find the change in length of the wire. What will be the change in length if the ends are interchanged. ($Y = 2 \times 10^{11}\ N/m^2$).

Soln.

Consider an element 'dz' at a depth 'z' from the fixed end. Then change $d\ell$ in length (extension) of this element is given by



$$Y = \frac{F/A}{d\ell/dz} = \frac{F dz}{d\ell.A}$$

$$\Rightarrow d\ell = \frac{F dz}{AY} = \frac{F\,dz}{(\pi r^2)Y} \qquad ...(1)$$

$$d = d_1 + 2z\tan\theta$$

$$\Rightarrow r = r_1 + z\tan\theta \qquad ...(2)$$

$$\therefore\ A = \pi r^2 = \pi(r_1 + Z\tan\theta)^2$$

$$\therefore\ d\ell = \frac{F\,dz}{\pi Y (r_1 + Z\tan\theta)^2}$$

Total change in length

$$\int_0^{\ell} dl = \int_0^{L} \frac{F\,dz}{\pi Y (r_1 + Z\tan\theta)^2}$$

$$= \frac{F}{\pi Y}\int_0^{L} \frac{dz}{(r_1 + Z\tan\theta)^2}$$

$$r_1 + Z\tan\theta = r,\ r_2 = r_1 + L\tan\theta$$

$$\Rightarrow \tan\theta\ dz = dr$$

$$\therefore \ell = \frac{F}{\pi Y}\int_{r_1}^{r_2} \frac{\cot\theta\,dr}{r^2} = \frac{F}{\pi Y}\cot\theta\left(-\frac{1}{r}\Big|_{r_2}^{r_1}\right)$$

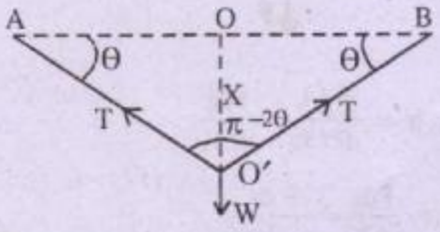
$$= \frac{F}{\pi Y}\cot\theta\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

$$= \frac{F}{\pi Y}\cot\theta\,\frac{r_2 - r_1}{r_1 r_2} = \frac{F}{\pi Y}\,\frac{L}{r_1 r_2}$$

On interchanging r_1 and r_2 'ℓ' remains same.

Ex.12.6.20 A steel wire of length 1 m and diameter 4 mm is stretched horizontally between two rigid supports attached to its ends. What load hung from the mid point of the wire would be required to produce a depression of 1cm. ($Y = 2 \times 10^{11}\ N/m^2$).

Soln.



Suppose length of the wire = 2ℓ

Depression oo' = x

T = tension along the string

W = load applied.

Then $W = 2T\cos\left(\frac{\pi}{2}-\theta\right) = 2T\sin\theta = 2T.\frac{x}{\ell'}$

Where stretched length = $2\ell'$

Now $\ell'^2 = x^2+\ell^2 = \ell^2\left(1+\frac{x^2}{\ell^2}\right)$

$$\ell' = \ell\left(1+\frac{x^2}{\ell^2}\right)^{\frac{1}{2}} \cong \ell\left(1+\frac{1}{2}\frac{x^2}{\ell^2}\right)$$

$$\ell' = \ell + \frac{x^2}{2\ell}$$

$$2\ell' = 2\ell + \frac{x^2}{\ell}$$

$$2\ell' - 2\ell = \text{change in length} = \frac{x^2}{\ell}.$$

Strain produced $= \frac{2\ell'-2\ell}{2\ell} = \frac{x^2/\ell}{2\ell} = \frac{x^2}{2\ell^2}$

Now $\frac{1}{\ell'} = \frac{1}{\ell}\left(1+\frac{x^2}{\ell^2}\right)^{-\frac{1}{2}} = \frac{1}{\ell}\left(1-\frac{1}{2}\frac{x^2}{\ell^2}\right)$

$$\frac{1}{\ell'} = \frac{1}{\ell} - \frac{x^2}{2\ell^3} \cong \frac{1}{\ell}.$$

Therefore $W = 2T\frac{x}{\ell'} = 2T\frac{x}{\ell}$

$$\Rightarrow T = \frac{W\ell}{2x}$$

$$\text{Stress} = \Rightarrow \frac{T}{\pi r^2} = \frac{W\ell}{2\pi r^2 x}$$

$$\therefore Y = \frac{\text{stress}}{\text{strain}} = \frac{W\ell/2\pi r^2 x}{x^2/2\ell^2}$$

$$\Rightarrow Y = \frac{2W\ell^3}{2\pi r^2 x^3} = \frac{W\ell^3}{\pi r^2 x^3}$$

If M be mass to be suspended then W = Mg

$$\Rightarrow Y = \frac{Mg\,\ell^3}{\pi r^2 x^3}$$

$$\Rightarrow M = \frac{\pi r^2 x^3}{g\ell^3}Y. = \frac{\pi d^2 x^3}{4g\,\ell^3}Y$$

$$\Rightarrow M = \frac{\pi(4\times10^{-3})^2(10^{-2})^3}{4(9.8)(0.5)^3} \times 2 \times 10^{11}\ \text{kg}$$

M = 2.05 kg.

12.7 Variation of longitudinal strain with longitudinal stress

For small deformation longitudinal strain is directly proportional to longitudinal stress. But for large deformations the relation of longitudinal stress and strain is very complicated. The curve obtained, when stress is plotted against strain, depends on the nature of the material. To have some idea about this, we describe stress-strain curve for two representative materials, a metal wire and a rubber piece.

(a) Stress - strain curve for metal wire

If we study the longitudinal strain produced in a wire due to normal deforming force, then the stress - strain curve will be as shown in fig. 12.11 & 12.12

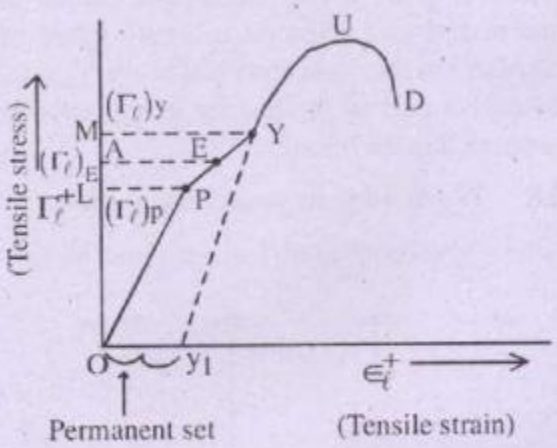


fig. 12.11

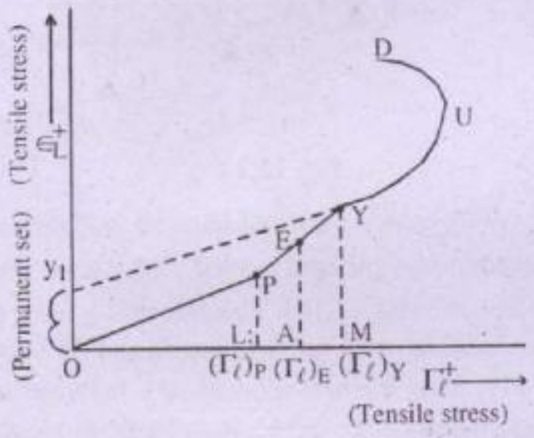


fig. 12.12

The curve can be divided into five distinctive segments (i) OP, (ii) PE, (iii) EY, (iv) YU, (v) UD.

i) **Segment OP**

This segment is a straight line, indicating that stress is directly proportional to strain over the region. The point 'P' on the curve is called **proportional limit.** In this region Hooke's law remains valid.

Proportional limit is the maximum stress up to which stress-strain relation remains linear.

ii) **Segment PE**

This segment is not linear. In this region stress is not proportional to strain (i.e. Hooke's law is not obeyed). But when the stretching force is removed, the wire acquires its original length. This means the wire still remains elastic. If the stress is increased beyond E, and then removed, the wire does not regain its original length. So point 'E' corresponds to the maximum stress up to which the wire remains elastic. This point E is called **elastic limit.**

Elastic limit of a material is the maximum stress (corresponding to a deforming force) up to which the material remains perfectly elastic.

Elastic limit depends on the property of the material. Elastic limit and proportional limit are different but not far apart. In some cases elastic limit and proportional limit are same.

iii) **Segment EY**

This segment corresponds to stress $\Gamma_\ell > (\Gamma_\ell)_{max}$. When $\Gamma_\ell > (\Gamma_\ell)_{max}$ strain increases much more rapidly. This continues until $\Gamma_\ell = (\Gamma_\ell)_y$. If the stress is removed, the wire does not regain its original length. Thus the behaviour of the wire is now plastic and retains a permanent set (OY_1), when the stress is removed at Y. The point 'Y' on the curve is called yield point.

iv) **Segment YU**

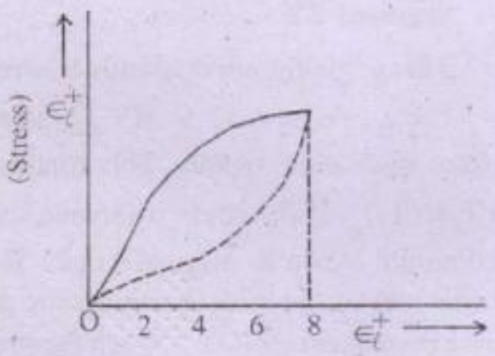
When stress $\Gamma_\ell > (\Gamma_\ell)_y$, the strain increases less rapidly compared to the region EY. This phenomenon continues until stress $\Gamma_\ell = (\Gamma_\ell)_u$, the ultimate stress or breaking stress. Material which has larger plastic range (Portion between elastic limit and ultimate stress) is called a more ductile material e.g. lead, copper. Material which has less plastic range is called a brittle material. Material which has larger (breaking stress) ultimate stress is considered more strong.

v) Segment UB

When the stress exceeds the ultimate stress, local constriction arises any where on the wire; and strain goes on increasing, even if the deforming force is reduced ($\Gamma_\ell < (\Gamma_\ell)_u$). Ultimately the wire breaks down.

(b) Stress - strain curve for rubber :

The stress - strain curve for galvanised rubber is shown is fig.12.13. One finds that no portion of the curve is linear, implying stress is not proportional to strain and Hooke's law is not obeyed. But the substance is elastic, even when it is stretched to several times its original length. In this sense rubber is more elastic than steel. However the magnitude of stress for a given strain is much larger in case of steel wire than in case of rubber. This means large internal forces appear if the steel wire is deformed. In this sense steel is more elastic than rubber.



Another important feature to be noted is that when deforming forces are removed the original curve is not retraced; although the sample finally assumes its original length. The work done in stretching is more than the work done in returning to its original length. Thus a certain amount of energy is absorbed in the cycle and this appears as. heat. This energy is equal to the area bounded by the two curves.

This phenomenon of non-coincidence of stress-strain curve for increasing and decreasing stress is called as elastic hysterisis.

Elastic hysterisis has an important application in shock absorbers. If a padding of vulcanised rubber is given between a vibrating system and say a flat board, the rubber is compressed and released in every cycle of vibration. An energy is absorbed in every cycle, and only a part of the energy of vibration is transmitted to the board.

12.8 Work done in stretching a wire :

(Elastic potential energy of a strained body)

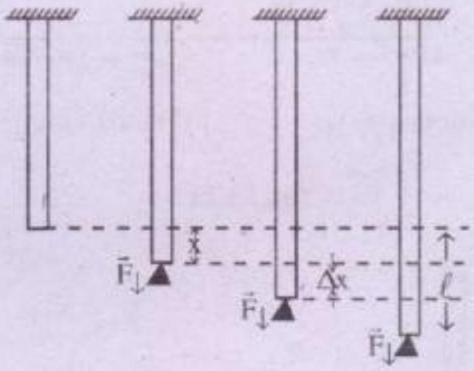


fig. 12.14

Consider a wire of length L and radius r, fixed at one end and loaded with a load $\vec{F}$ at the free end (fig.12.14). The deforming force F is so chosen that the wire is stretched very slowly. This ensures that at any instant the external deforming force equals tension in wire.

Let the wire be stretched through 'x' at any instant. Then at this instant

$$\text{Stress } \Gamma_\ell^+ = \frac{F}{A} = \frac{F}{\pi r^2}$$

$$\text{Strain } \in_\ell^+ = \frac{x}{L}$$

giving

$$Y = \frac{\Gamma_\ell^+}{\in_\ell^+} = \frac{F/A}{x/L} = \frac{FL}{Ax}$$

$$\Rightarrow F = \frac{YA}{L}x \qquad \text{...(12.8.1)}$$

If the wire is further stretched by dx the work done by external deforming force F, against the tension is

$$dW = F.\,dx = \frac{YA}{L}\ x\ dx \quad ...(12.8.2)$$

Therefore the total work done by external force in stretching the wire through ℓ is

$$W = \int_0^{\ell} \frac{YA}{L} x\, dx = \frac{1}{2}\frac{YA}{L}\ell^2 \quad ...(12.8.3)$$

This work is stored in the wire as elastic potential energy. Thus elastic potential energy of the wire is

$$U = \frac{1}{2}\cdot\frac{YA}{L}\cdot\ell^2$$

$$= \frac{1}{2}\cdot\left(\frac{YA}{L}\ell\right)\cdot\ell$$

$$= \frac{1}{2}\text{ (Max. stretching force) (Max. extension)} \quad ...(12.8.4)$$

Also we can re-write (12.8.4) as

$$U = \frac{1}{2}\cdot\left(\frac{Y\ell}{L}\right)\left(\frac{\ell}{L}\right)(AL)$$

$$= \frac{1}{2}\text{ (stress) (strain) (volume)} \quad ...(12.8.5)$$

So from eqn. (12.8.5) one obtains elastic potential energy per unit volume as

$$u = \frac{U}{V} = \frac{1}{2}\text{ (stress) (strain)} \quad ...(12.8.6)$$

Note: Although eqn. (12.8.6) has been deduced for longitudinal strain, it is true for any type of strain.

12.9 Applications of elasticity

Elastic property of materials plays an important role in a number of ways in our day to day activities.

i) Thickness of metallic ropes used in cranes is decided on the basis of elastic limit and factor of safety.

$$\text{Breaking stress} = \frac{\text{Breaking load}}{\pi r^2}$$

Breaking weight = Breaking stress x πr^2

A rope is never loaded beyond $\frac{1}{3}$rd of its breaking load.

ii) Any metallic part of a machinery is never subjected to a stress beyond the elastic limit of the material.

iii) A knowledge of elastic moduli, helps in selecting materials for high pressure tools like plier, screw driver etc.

iv) A bridge is declared unsafe after long use as it loses its elastic strength.

v) A hollow shaft (axle) is stronger than a solid shaft mode of same and equal material.

[Torque required to produce unit twist in a solid cylinder of radius r, length ℓ is $\tau = n\pi r^4 / 2\ell$.

For a hollow cylinder of external radius r_2 and internal radius r_1, length ℓ and same mass is $\tau' = n\pi(r_2^4 - r_1^4)/2\ell$

since $m = \pi r^2 \ell\,\rho = \pi(r_2^2 - r_1^2).\,\ell\,\rho$

$$\Rightarrow r^2 = r_2^2 - r_1^2$$

$$\text{Hence } \frac{\tau'}{\tau} = \frac{r_2^4 - r_1^4}{r^4} = \frac{r_2^2 + r_1^2}{r^2}$$

Since $r_2^2 - r_1^2 = r^2$

so $r_2^2 + r_1^2 > r^2$

$$\Rightarrow \frac{\tau'}{\tau} > 1 \Rightarrow \tau' > \tau$$

[i.e. Torque required to twist a hollow cylinder is greater than the torque necessary to twist a solid cylinder of same mass, length and material through the same angle. Hence a hollow shaft shall be stronger than a solid shaft.]

This is the reason why electric poles are given hollow structure.

vi) A study of bending of beams shows that while designing a beam to support load (roofs, bridges) the depth should be larger than the breadth.

Summary

1. Stress is defined as the deforming force per unit area. Stress may be classified as a) Normal stress and b) shearing stress. Normal stress may be called a tensile stress when it causes stretching of a rod and it may be called a compressional stress. when it causes compression of the rod.

2. Strain is the fractional change in the dimension of the body. Strain is of three types. Longitudinal strain ($\frac{\Delta L}{L}$), shear strain and volume strain.

3. **Hookes law :**

If deformation is small, the stress in a body is propertional to the Corresponding strain.

$$\frac{\text{Stress}}{\text{Strain}} = \text{E (Modulus of elasticity)}$$

There are some materials like rubber which are elastic but does not obey Hookes law.

4. a) Youngs modulus $Y = \frac{F/A}{\Delta L/L}$

b) Bulk modulus $B = -\frac{P}{(\Delta V/V)}$

c) Modulus of rigidity $n = \frac{F_t/A}{\theta}$

5. Poissons ratio (σ) is defined as the ratio of lateral strain over longitudinal strain.

$$\sigma = -\frac{\Delta d/d}{\Delta l/l}$$

6. Relations between elastic Constants (Y, B, n, σ)

a) $Y = 2n(1+\sigma)$

b) $Y = 3B(1-2\sigma)$

c) $\frac{1}{B}+\frac{3}{n}=\frac{9}{y}$

7. Bulk modulus is the common proporty of all the three states of matter. Where as Youngs modulus and modulus of rigidity are relevant only to solids.

8. Reciprocal of Bulk modulus is called compressibility (k). $K = \frac{1}{B}$.

9. Elastic potential energy of a strained rod is $U = \frac{1}{2}$ maximum stretching force x (maximum extension)

Energy density

$$u = \frac{1}{2} \text{stress x strain}$$

10.

Physical quantity	Dimensional Formula	SI unit
1. Stress	$[ML^{-1}T^{-2}]$	Nm^{-2}
2. Strain	Dimensionless	No unit
3. Modulus of elasticity (Y, B or n)	$[ML^{-1}T^{-2}]$	$N m^{-2}$
4. Poissons ratio	Dimensionless	No unit
5. Compressibility (k)	$[M^{-1}L^{1}T^{-2}]$	$N^{-1}m^{2}$

MODEL QUESTIONS

A. Multiple Choice Questions :

1. The following four wires are made of same material. Which of these will have largest extension when the same tension is applied ?
   i) $\ell = 1\text{m},\ d = 0.1\text{mm}$
   ii) $\ell = 0.75\text{m},\ d = 0.75\text{ mm}$
   iii) $\ell = 0.5\text{m},\ d = 0.5\text{ mm}$
   iv) $\ell = 0.25\text{ m},\ d = 0.25\text{ mm}$.

2. What is the length of a hanging wire of density (d) which will just break under its own weight ?
   i) $\frac{\text{stress}}{\text{gd}}$ ii) stress x dg
   iii) $\frac{\text{stress x g}}{\text{d}}$ iv) $\frac{\text{stress x d}}{\text{g}}$

3. The extension of wire by application of a load is 3.0 mm. The extension in a wire of same material, same length but half the radius by same load is
   i) 12.0 mm ii) 0.75 mm
   iii) 6.0 mm iv) 3.0 mm.

4. When an elastic material with Young's modulus Y is subjected to a stretching stress S, the elastic energy stored per unit volume of the material is
   i) ½ YS ii) ½ S^2Y
   iii) $\frac{S^2}{2Y}$ iv) $\frac{S}{2Y}$

5. In order to have an appreciable extension of a wire the wire should be
   i) long and thin
   ii) Thick block of any cross-section
   iii) Thick block of rectangular cross-section.
   iv) Short thin wire

6. Young's modulus of a material is equal to the stress which will increase the length L to
   i) $L + \frac{L}{2}$ ii) $L + \frac{L}{4}$
   iii) $L + \frac{L}{6}$ iv) 2L

7. The product of _____ and ____ is two times the strain energy per unit volume.
   i) stress, strain ii) stress, stress
   iii) strain, strain iv) None of the above

8. The relation between Y, n, and B is
   i) $\frac{3}{Y} = \frac{1}{n} + \frac{9}{B}$ ii) $\frac{1}{Y} = \frac{8}{n} + \frac{3}{B}$
   iii) $\frac{9}{Y} = \frac{8}{n} + \frac{3}{B}$ iv) $\frac{9}{Y} = \frac{3}{n} + \frac{1}{B}$

9. Factor of safety is given by
   i) yielding stress/yielding strain
   ii) Working stress/yielding stress
   iii) Breaking stress/working stress
   iv) all of the above

10. The extension in a steel wire is ℓ. If a similar wire of twice length and twice radius is taken the extension for same load shall be
   i) ℓ ii) 2ℓ
   iii) $\ell/2$ iv) 4ℓ

11. Iron is preferred over copper to make spring because
   i) Iron is cheap
   ii) Iron is easily oxidised
   iii) Iron is more elastic than copper
   iv) Iron is less elastic than copper

12. Theoretical values of poisson's ratio lie between
   i) 0 and 0.5 ii) 0.5 and 1.0
   iii) -1.0 and 1.0 iv) -1.0 and 0.5

13. There is no change in volume of a wire undergoing extension. the value of poisson's ratio is
    i) 0.25 ii) 0,5
    iii) 0.75 iv) 1.0
14. The unit of spring constant is
    i) N/m ii) N/m^2
    iii) N/kg iv) N/kg^2
15. A force of one newton doubles the length of a cord of cross-sectional area 1 cm^2, Young's modulus of elasticity is
    i) 10^5 dyne/cm^2
    ii) 2×10^5 dyne/cm^2
    iii) 3×10^5 dyne/cm^2
    iv) 4×10^6 dyne/cm^2

B. Very Short Answer Type Questions :

1. Define Young's modulus.
2. Give dimension of Y.
3. Give unit of Y.
4. Define elastic limit.
5. State Hooke's law.
6. Write relation between Y, n, B.
7. Write relation between Y, n, σ.
8. Write relation between Y, B, σ.
9. Define modulus of rigidity (n).
10. Define poissons ratio (σ).
11. What are the theoretical limiting values of σ.
12. Name a quantity which is dimensionally similar to stress.
13. Name a quantity which is dimensionally similar to strain.
14. What is the value of rigidity modulus of liquid.
15. Which is more elastic, steel or rubber.
16. Which is more elastic glass or rubber.
17. Which is more elastic air or water.
18. How does Y change with temperature.
19. Define Bulk modulus of elasticity.

C. Short Answer Type Questions :

1. Explain elasticity on the basis of molecular theory.
2. Explain the term elastic limit
3. Show graphically the variation of intermolecular force with distance of separation between two molecules.
4. Show the stress-strain curve for a wire under normal deforming force.
5. What force is required to stretch a wire of cross section 1 cm^2 to double its length (given $Y = 2 \times 10^{11}$ N/m^2).
6. Explain why a spring is made of steel not of copper.
7. Identical springs of steel and copper are equally stretched. On which more work is done.
8. What happens to the work done in stretching the wire ?
9. The length of a wire is increased by 1% when loaded with 2 kg. wt. What is the strain produced.
10. The length of a wire is cut into half of its original length. (i) what will be the effect on increase of length under the given load. (ii) What will be the effect on the maximum load it can bear ?
11. Y - for brass is 10×10^{10} N/m^2 and n for brass is 3.5×10^{10} N/m^2. Calculate Bulk modulus for brass.
12. What will be the density of copper under a pressure of 10,000 N/cm^2. (density of copper is 8.96×10^3 Kg/m^3; $B_{copper} = 1.4 \times 10^{10}$ N/m^2).

13. A thick rope of rubber of density 1.5 kg/m^2, Young's modulus 5×10^6 N/m^2 and length 8 m is hung from the ceiling of a room. Calculate the increase in its length due to its own weight.

14. Steel is more elastic than rubber.

    A body is said to be more elastic if it retains its elastic behaviour for a larger deforming force i.e. its elastic limit is more. In case of rubber it losses its elastic behaviour even after application of small amount of load. i.e. it has smaller elastic limit. On the other hand a steel wire remains elastic for larger loads i.e. its elastic limit is more. Hence steel is more elastic.

15. Relation between elastic-limit and moduli of elasticity.

    Consider a steel wire and a similar rubber wire. If same load is applied. Then

$$F/A = Y_S \frac{(\Delta L)_S}{L} = Y_R \frac{(\Delta L)_R}{L}$$

Since $(\Delta L)_R > (\Delta L)_S$ so $Y_S > Y_R$

We know that elastic limit of rubber < elastic limit of steel

So $Y_S > Y_R \Leftrightarrow$ (elastic limit)$_S$ > (elastic limit)$_R$.

16. i) Perfect rigid body $Y = \infty$ ($\because \Delta L = 0$)

    ii) Perfect elastic body $Y = \infty$ ($\because F = \infty$)

    iii) Perfect plastic body $Y = 0$ ($\because F = 0$)

    iv) $(n)_{liquid} = 0$

    v) $(Y)_{air} = 0$

17. Dependence of elastic moduli on temperature.

$$Y_t = Y_0 (1 - \alpha t)$$

$$n_t = n_0 (1 - \alpha' t)$$

Elastic moduli decrease due to increase of temperature. This arises due to weakening of bonding forces.

D. Unsolved Problems

1. A load of 2kg produces an extension of 1mm in a wire of length 3m and diameter 1mm. Calculate Young's modulus of the material of the wire. ($g = 9.8$ m/s^2).
2. A uniform steel wire of density 7800 kg/m^3 is 2.5 m long; and weight 15.6×10^{-3} kg. It extends by 1.25 mm, when loaded by 8 kg. Calculate Young's modulus for steel.
3. A steel rail 100m long and 40 cm^2 in area of cross-section changes in length by 1 cm between summer and winter. If it is laid in winter, what tension develops in summer.
4. A steel wire of cross-section area 0.5 mm^2 is held between two fixed supports. If the tension in the wire is negligible and it is just taut at 20^0C. Determine the tension when the temperature falls at 0^0C. ($Y = 2.1 \times 10^{11}$ N/m^2, $\alpha = 12 \times 10^{-6}/^0$C).
5. Two wires of same material and length are stretched with equal force. Find the ratio of their elongation if their radii are in the ratio 1:2.
6. Two wires of same material and same radius are stretched by equal forces. Find the ratio of their elongation if their lengths are in the ratio 1:2.
7. A copper wire of 2.0 m long and 0.5 mm in diameter supports a mass of 10 kg. It is stretched by 2.38 mm. Calculate Young's modulus of the wire.
8. A wire of length 3 m, diameter 0.4 mm and Young's modulus 8×10^{10} N/m^2, is suspended from a point and supports a heavy cylinder of volume 10^{-3} m^3; at its lower end. Find the decrease in length when the metal cylinder is immersed in a liquid of density 800 kg/m^3.

9. Find the greatest length of steel wire that can hang vertically without breaking (Breaking stress of steel = 7.9×10^8 N/m^2, density of steel = 7900 kg/m^3).

10. A steel wire of 2 mm diameter is just stretched between two points at a temperature of 25^0 C. Determine the tension when temperature falls to 15^0 C (linear expansivity of steel = $11 \times 10^{-6}/^0K$, $Y_{steel} = 2.1 \times 10^{11}$ N/m^2).

11. A wire of length 1 m and radius 1 mm is welded to another wire of length 2m and radius 2 mm. The free end of the first is clamped and a load of 5 kg is applied at the free end of the second wire. What is the total increase of compound wire ? (Y of both wire = 2×10^{11} N/m^2).

12. A uniform pressure P is exerted on all sides of a solid cube at temperature t^0C. By what amount should the temperature be raised in order to bring its volume back to the value it had before the pressure was applied. (Cubical expansivity = γ , Bulk modulus = B).

13. Two springs have their force constants as K_1 and K_2 ($K_1 > K_2$). On which spring is more work done (a) when their lengths are increased by same amount (b) when they are stretched by same force.

14. A steel wire of length 1 m and diameter 3 mm is stretched horizontally between supports attached at its ends. What load hung from the mid-point of the wire would be required to produce a depression of 1 cm ? ($Y = 2 \times 10^{11}$ N/m^2, $g = 9.8$ m/s^2).

15. A uniform spring whose unstretched length is ℓ has a force constant K. The spring is cut into two pieces of unstretched lengths ℓ_1 and ℓ_2 where $\ell_1 = n\ell_2$ and n is an integer. What are the force constants K_1 and K_2 of the two pieces in terms of n and K.

E. Long Answer Type Questions :

1. a) State Hooke's law. Define moduli of elasticity. Find the relation among them.

   b) Due to a stress of 20 N/m^2, the percentage increase in the length of a wire is 0.01. Calculate Young's modulus.

2. Define stress and strain. Draw stress-strain graph for a wire under a normal load. Explain the terms proportional limit, elastic limit, yield point. ultimate stress.

3. Define stress and strain. Describe their types. State Hooke's Law and hence the elastic moduli.

F. Fill in the Blank Type

1. The stress required to double the length of a wire of Young's modulus Y is ....

2. A wire of length L and cross sectional area A is made of a material of Young's modulus Y. If the wire stretched by an amount x, the the workdone is ....

3. The steel is .... elastic than rubber

4. The longitudinal strain in a metal bar is 0.05. If the poisson's ratio for a metal is 0.25, the lateral strain is........

5. A wire of length L and cross-sectional area A is made of a material of Young's Y. If the wire is stretched by the amount x, the work done is ....

G. True - False type

1. Rubber is more elastic than glass.

2. Young's modulus increases with rise of temperature.

3. Springs are made of steel and not of copper.

4. A cable is cut to half its original length. This change has no effect on the maximum load, the cable can support.

5. The stretching of a coil spring is determined by shear modulus.

ANSWERS

A. Multiple Choice Type Questions :

1.(i) **2.**(i) **3.**(i) **4.**(iii) **5.**(i) **6.**(iv) **7.**(i) **8.**(iv) **9.**(iii) **10.**(iii) **11.**(iii) **12.**(ii) **13.**(ii) **14.**(i) **15.**(i)

B. Very Short Type Questions :

1. See text
2. $ML^{-1}T^{-2}$
3. N/m^2, $dyne/cm^2$
4. See text
5. See text
6. See text
7. See text
8. See text
9. See text
10. See text
11. $-1 < \sigma < 0.5$
12. Pressure
13. Specific gravity
14. Zero
15. Steel
16. Glass
17. Water
18. Decreases with rise of temperature
19. See text

C. Short Answer Type Questions :

1. See text
2. See text
3. See text
4. See text
5. 2×10^7 N
6. $Y_{steel} > Y_{copper} \Rightarrow$ Elastic limit of steel > Elastic limit of copper
7. $U = \frac{1}{2}$ stress x strain $= \frac{1}{2} Y \in^2$. Since $Y_s > Y_c$, so more work is done in steel spring
8. Stored as elastic P.E.
9. .01
10. (i) Increase in length will be halved (ii) No change
11. $\frac{1}{B}+\frac{3}{n}=\frac{9}{Y} \Rightarrow \frac{1}{B}=\frac{9}{Y}-\frac{3}{n} = 4.286 \times 10^{-12}\ m^2/N$ $B = 2.333 \times 10^{11}\ N/m^2$
12. $\Delta V = V' - V = M\left(\frac{1}{\rho'}-\frac{1}{\rho}\right) \Rightarrow -\frac{\Delta V}{V} = \frac{-M}{V}\left(\frac{1}{\rho'}-\frac{1}{\rho}\right) \Rightarrow -\frac{\rho}{\rho'}+1=\frac{P}{B} \Rightarrow \rho' = \frac{\rho}{1-\frac{P}{B}}$

$$\rho' = \frac{8.96 \times 10^3\ kg/m^3}{1-\frac{10^8 N/m^2}{1.4 \times 10^{10} N/m^2}} = 9.024 \times 10^3\ kg/m^3$$

13. $Y = \frac{AL\ dg / A}{\Delta L / (L/2)} \Rightarrow \Delta L = 9.41 \times 10^{-5}$ m

D. Unsolved Problems :

1. 7.5×10^{10} N/m^2
2. 19.6×10^{10} N/m^2
3. 76×10^3N
4. 25.2 N
5. 4 : 1
6. 1 : 2
7. 4.2×10^{11} N/m^2
8. 2.339×10^{-3} m
9. 1.02×10^4 m
10. 72.6 N
11. 11.7×10^{-5} m
12. $P / \gamma B$

    Hints : stretching force is same but stress different.

13. (i) $W_1 > W_2$, (ii) $W_1 < W_2$

    Hints : $W = \frac{1}{2} Kx^2$

14. 1.15 kg
15. $\frac{n+1}{n}$ K, $(n+1)$ K

F. (1) Y (2) $\frac{1}{2}$Y.A.L. (3) more (4) 0.0125 (5) $\frac{Y.Ax^2}{2L}$

G. (1) False (2) False (3) True (4) True (5) True

13

Hydrostatics

13.1 Fluids :

Liquids and gases together are called fluids, as they have the ability to flow.

A liquid has definite volume but does not possess definite shape. It takes the shape of the container and cannot sustain shearing stress. Liquids are nearly incompressible and thus density of liquids is nearly independent of variation in pressure. Again liquids could also be viscous and non-vicous. If there exists a force between two consecutive layers in a direction other than the direction normal to the surface of contact, then liquid is said to be viscous, otherwise non-viscous.

Gases donot possess definite volume or shape. They are compressible, hence their density depends on pressure.

13.2(a) Density (Mass density) :

The mass density or simply density of a homogeneous substance is defined as the mass per unit volume of the substance.

$$\text{(Rho) } \rho = \frac{m}{V} \qquad ...(13.2.1)$$

where, 'm' is mass of 'V' volume of substance. When the material is not homogeneous the local density $\rho(\vec{r})$ is defined as

$$\rho(\vec{r}) = \lim_{\Delta V \to 0} \frac{\Delta m}{\Delta V} \qquad ...(13.2.2)$$

where Δm is mass of a small element of volume ΔV around the point, defined by position vector $\vec{r}$.

Thus density is a scallar quantity. Its dimension is ML^{-3}. Its units are kg/m^3 in S.I. system and g / cm^3 in C.G.S. system. $1\ kg/m^3 = 10^{-3}\ g / cm^3$.

Density of water at 4^0 C is defined to be $1 g/ cm^3$ or $10^3\ kg / m^3$.

As discussed in Sec. 13.1, since solids and liquids are almost incompressible, their densities are nearly constant. But as gases are compressible their densities vary appreciably with change in temperature and pressure. Therefore in case of gases it is necessary to state the temperature and pressure while quoting value of density.

(b) Weight density :

It is defined as the weight per unit volume of the homogeneous substance

$$\vec{D} = \frac{\vec{W}}{V} \qquad ...(13.2.3)$$

where 'W' is the weight of 'V' volume of substance. When the substance is not homogeneous, one defines local weight density $\vec{D}(\vec{r})$ as .

$$\vec{D}(\vec{r}) = \lim_{\Delta V \to 0} \frac{\Delta \vec{W}}{\Delta V} \qquad ...(13.2.4)$$

where ΔW is the weight of a small element of volume ΔV at the point, defined by position vector $\bar{r}$.

The quantity, weight density, is useful when we are concerned with the effects of weight, while density (mass density) is useful when mass is to be considered.

Weight density is a vector quantity. Its dimensions are $ML^{-2}T^{-2}$. Its units are N/m^3, kg wt / m^3 in S.I. system; and dyne / cm^3, g wt / cm^3 in C.G.S. system.

Relation between D and ρ

From eqn. (13.2.3)

$$\vec{D} = \frac{\vec{W}}{V} = \frac{m\vec{g}}{V} = \rho\vec{g} \quad ...(13.2.5)$$

Thus mass density remains constant at all places, whereas weight density varies from place to place due to variation of g.

(c) Relative density (specific gravity) (ρ_r)

Relative density or specific gravity of a substance is defined as the ratio of its density (ρ_s) to that of water (ρ_w) at 4^0 C.

i.e. $\rho_r = \dfrac{\text{density of substance at } t^0C}{\text{density of water at } 4^0C}$

$$\Rightarrow \quad \rho_r = \frac{\rho_{s(t^0C)}}{\rho_w(4^0C)} \quad ...(13.2.6)$$

[Here water at 4^0 C is taken as a standard substance, as water assumes maximum density at 4^0 C]

If we consider 'V' volume of substance at t^0 C and 'V' volume of water at 4^0 C; then from (13.2.6) we obtain

$$\rho_r = \frac{\rho_s(t^0C) \times V(t^0C) \times g}{\rho_w(4^0C) \times V(4^0C) \times g}$$

$$\Rightarrow \quad \rho_r = \frac{\text{Weight of 'V' volume of substance at } t^0C}{\text{Weight of 'V' volume of water at } 4^0C} \quad ...(13.2.7)$$

So relative density can also be defined as the "ratio of weight of 'V' volume of substance to the weight of 'V' volume of water at 4^0 C."

Also from the defining eqn (13.2.6) we obtain

$$\rho_r = \frac{\rho_s(t^0C)}{\rho_w(4^0C)} = \frac{\rho_s(t^0C)}{\rho_w(t^0C)} \times \frac{\rho_w(t^0C)}{\rho_w(4^0C)} \quad ...(13.2.8)$$

$$\Rightarrow \quad \rho_r = \rho_r(t).\rho_w(t) \text{ in C.G.S. system}$$

$$= 10^{-3}\,\rho_r(t).\rho_w(t) \text{ in S.I. system.} \quad ...(13.2.9)$$

We also note from (13.2.6) that specific gravity is truly a relative density, measured w.r.to a standard substance (water at 4^0 C). As there is nothing to do with gravity so relative density gives a more correct and precise concept than specific gravity.

It also follows from eqn. (13.2.7) and (13.2.8)

$$\rho_r = \frac{D_s(t)V}{D_w(4)V} = \frac{D_s(t^0C)}{D_w(4^0C)} \quad ...(13.2.10)$$

i.e. relative density is also the ratio of the weight density of the substance to the weight density of water at 4^0 C.

Relative density is a scalar quantity and has no dimension or unit.

Relation between ρ_r and ρ.

We note from (13.2.6) that

$$\rho_r = \frac{\rho_s(t^0C)}{\rho_w(4^0C)}$$

Since in C.G.S. system $\rho_w(4^0C) = 1\ g/cm^3$ and in S.I. system $\rho_w(4^0C) = 10^3\ kg/m^3$, so

$$\rho_r = \rho_s\ (\rho_s \text{ in C.G.S. unit})$$
$$= 10^{-3}\ \rho_s\ (\rho_s \text{ in S.I. unit})$$

i.e. in C.G.S. system relative density (Sp. gravity) is numerically equal to the density of the substance.

Table 13.1
(Density of Substances)

Substance	Density in gm/cm3
Aluminium	2.7
Brass	8.44 - 8.7
Carbon, graphite	2.25
Copper	8.89
Germanium	5.46
Glass	2.4 - 2.8
Gold	19.3
Ice	0.917
Iron	7.85
Lead	11.3
Silicon	2.42
Silver	10.5
Tungsten	19.3
Uranium	18.7
Wood	0.8
Zinc	7.1
Alchol	0.79
Ether	0.74
Gasoline	0.68
Mercury	13.595
Water at 4^0 C	1.0
Water at 20^0 C	0.998
Air	1.293×10^{-3}
CO_2	1.997×10^{-3}
H_2	0.99×10^{-3}
He	0.178×10^{-3}
N_2	1.251×10^{-3}
O_2	1.429×10^{-3}

Ex.13.2.1 Estimate the mass of air in a room of length 4m, breadth 3m, and height 3m at 20^0 C. When density of air is 1.29 kg/m^3. What is the weight of air ?

Soln.

Volume of the room = V= 4x3x3 m^3 = 36 m^3

Density of air ρ = 1.29 kg/m^3

Mass of air = ρ V= 1.29 x 36 kg = 46.44 kg

Wt. of air = ρ Vg= 46.44 x 9.8 N = 455.1 N

Ex.13.2.2 Equal volumes of two liquids of sp. gr. ρ_1 and ρ_2 are mixed together and they donot react. What is the sp. gr. of the mixture ?

Soln.

Let volume of each liquid = V cm^3

Total volume of the liquid mixture = 2V cm^3

Total mass of the mixture = $(\rho_1 V + \rho_2 V)$ g

$$= (\rho_1 + \rho_2)V\ g$$

$$\text{Sp. gr. of the mixture} = \frac{(\rho_1 + \rho_2)V.g}{2V.g}$$

$$= \frac{1}{2}(\rho_1 + \rho_2)$$

Ex.13.2.3 Equal weight of two liquids of densities ρ_1 and ρ_2 are mixed together and they donot react. What is the density of the mixture ?

Soln.

Let mass of each liquid = m

Total mass of mixture = 2m

Volume of the mixture =

$$\frac{m}{\rho_1} + \frac{m}{\rho_2} = m\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right)$$

$$\text{Density of mixture} = \frac{2m}{m\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right)} = \frac{2\rho_1\rho_2}{\rho_1 + \rho_2}$$

Ex.13.2.4 When equal masses of two metals are mixed together, the sp.gr. of the alloy is 3. But when equal volume of the same two metals are mixed togther the sp.gr. of the alloy is 4. Calculate the sp.gr of each metal.

Soln.

Let the sp.gr. of metals be ρ_1 and ρ_2 C.G.S. units.

When equal masses of two metals are mixed, Total volume of the mixture shall be =

$$\frac{m}{\rho_1}+\frac{m}{\rho_2}$$

Total mass of the mixture = 2m

Hence sp.gr. of the mixture =

$$\frac{2mg}{\left(\frac{m}{\rho_1}+\frac{m}{\rho_2}\right)g}=\frac{2\rho_1\rho_2}{\rho_1+\rho_2}$$

$$\Rightarrow \frac{2\rho_1\rho_2}{\rho_1+\rho_2}=3 \quad ...(1)$$

When equal volume of metals are mixed, let the total vlume = V + V= 2V

Total mass = $V\rho_1 + V\rho_2$

$$\text{Sp. gr. of mixture} = \frac{(V\rho_1+V\rho_2)g}{2Vg}=\frac{\rho_1+\rho_2}{2}$$

$$\Rightarrow \frac{\rho_1+\rho_2}{2}=4 \quad ...(2)$$

From (1) and (2)

$$\frac{2\rho_1\rho_2}{8}=3$$

$$\Rightarrow \rho_1\rho_2=12 \quad ...(3)$$

$$\Rightarrow \rho_1-\rho_2=\sqrt{(\rho_1+\rho_2)^2-4\rho_1\rho_2}=\sqrt{8^2-4\times 12}$$

$$=\sqrt{16}=4$$

$$\therefore \rho_1-\rho_2=4$$

$$\rho_1+\rho_2=8$$

$$\Rightarrow 2\rho_1=12$$

$$\Rightarrow \rho_1=6$$

$$\rho_2=2$$

$$\therefore \rho_1=6\ \text{gm/cm}^3,\ \rho_2=2\ \text{gm/cm}^3$$

Sp.gr. $\rho_1=6,\ \rho_2=2$

Ex.13.2.5 If the volume of a body increases by 1 %, what is the percentage change in its density ?

Soln.

$$\rho=\frac{m}{V}$$

$$\Rightarrow \log\rho=\log m-\log V$$

Differentiating

$$\frac{d\rho}{\rho}=-\frac{dV}{V}$$

$$100\times\frac{d\rho}{\rho}=-\frac{dV}{V}\times 100=-1\%$$

13.3 Pressure :

When a body rests on a support it exerts a force, equal to its weight, on the supporting surface and the supporting surface also exerts an equal and opposite force on the body.

The force with which a body pushes another body in contact with it is called as thrust.

Similarly when a liquid comes in contact with a solid, it also exerts a thrust on the surface of solid and vice versa. The thrust of liquid on solid surface and solid surface on liquid are equal and opposite. For a liquid at rest this thrust must be perpendiuclar to the surface in contact. Because if the force is not perpendicular, then the component (see fig. 13.1) $F_{SL}\sin\theta$ shall tend to push the liquid and the liquid shall tend to flow.

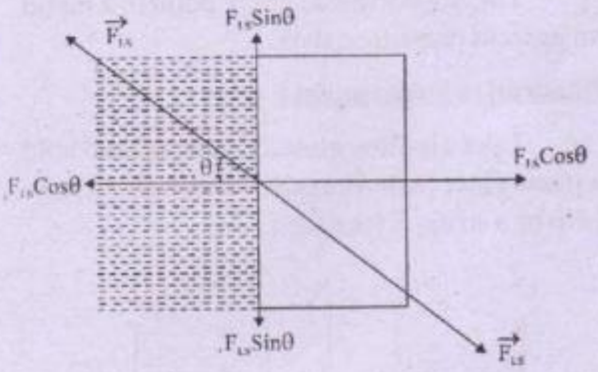


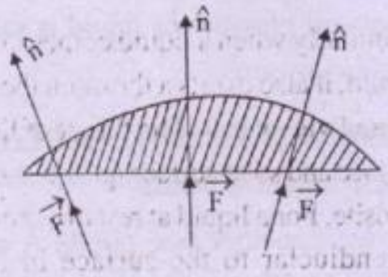
Fig.13.1

The effect of this thrust depends on the area of contact. For example (i) When we stand on a heap of sand our feet sink into the sand. But when we lie down on the heap of sand the body does not go deep. (ii) Similarly it is very easy to steetch with a sharp needle than with a blunt needle and (iii) It is easy to cut with a sharp knife than with a blunt knife.

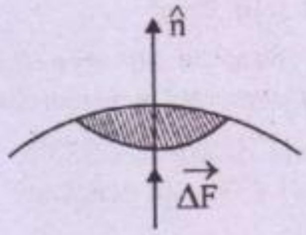
Thus the above discussion implies that the ratio of force and area upon which it acts needs to be considered, for gauging the effect of a force. Further since for a fluid the thrust must act perpendicularly to a surface in contact, so we define **"Pressure is the normal force per unit area"**

i.e. $P = \frac{F_n}{A}$

where F_n denotes the (uniform) force along the direction normal to the area A.



(a)



(b)

Fig. 13.2

i.e. $F\,\hat{n} = PA\hat{n}$

$$\Rightarrow \vec{F} = P\vec{A} \qquad ...(13.3.1)$$

If $\vec{F}$ is not uniform, then

$$\vec{\Delta F} = <P> \vec{\Delta A}$$

giving average value of pressure $<P>$

$$\text{i.e. } <P> = \frac{\Delta F}{\Delta A} \qquad ...(13.3.2)$$

The local value of pressure (ie pressure at a point) P is defined as

$$P = \lim_{\Delta A \to 0} \frac{\Delta F}{\Delta A} = \frac{dF}{dA} \qquad ...(13.3.3)$$

Pressure is a scalar quantity since force on a surface ΔA, resulting from a pressure depends on how surface ΔA is oriented.

<u>Dimension of Pressure</u>

$$[P] = \frac{[F]}{[A]} = \frac{MLT^{-2}}{L^2} = ML^{-1}T^{-2}$$

<u>Unit of Pressure</u>

S.I. Units : $N/m^2 \equiv$ Pa (Pascal)

C.G.S. Units : dyne / cm^2

Sometimes we use units like 1 atmosphere (1atm), 1 bar, 1 torr (named after Torricelli)

1 atm = 76 cm of Hg = 1.01×10^6 dyne / cm^2
= 1.01×10^5 N/m^2

1 bar = 10^5 N/m^2 = 10^6 dyne/cm^2 (used extensively in meteorology)

1 torr = 1 mm of Hg = 1.333×10^2 N/m^2
= 1.333×10^3 dyne/cm^2

1 atm = 760 torr.

* **Since units like atm, torr, depend on value of g which is not constant, so these units are gradually not preferred.**

13.4 Upward Pressure of a Liquid :

At any point inside a liquid there exists a pressure acting upwards which is equal and opposite to that acting downwards at that point

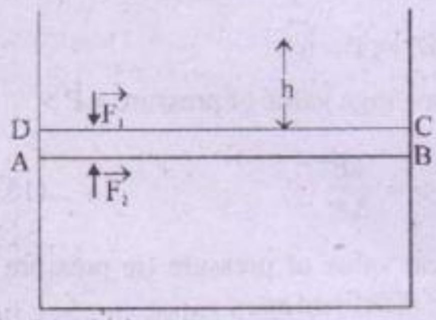


Fig.13.3

Consider some liquid contained in a beaker (see fig. 13.3). Let ABCD be a horizontal strip of liquid having an area A. The force $\vec{F}_1$ due to weight of the liquid lying above it and atmospheric pressure, acts vertically downwards. For equilibrium of the strip ABCD, a force $\vec{F}_2$ must act on the strip, such that

$$\vec{F}_1 + \vec{F}_2 = 0$$

$$\Rightarrow \quad \vec{F}_1 = -\vec{F}_2 \text{ and } |\vec{F}_2| = |\vec{F}_1|$$

Thus the force $\vec{F}_2$ must act vertically upwards, being equal in magniftude to that of $\vec{F}_1$. This implies that

$$F_1 = P_1A = F_2 = P_2A$$

$$\Rightarrow \quad P_1 = P_2$$

This shows that at every point in a liquid an upward pressure exists.

Illustrative Experiment :

Take a hollow glass cylinder (c) and hold a plane glass plate G against one end, with the help of a string S (see fig 13.4)

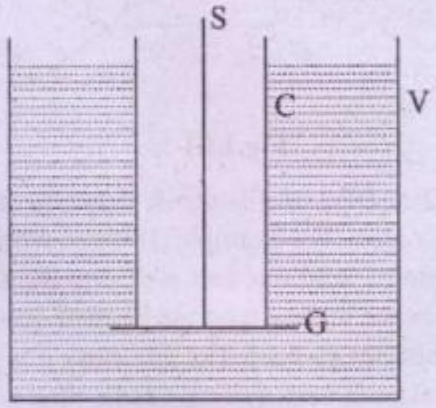


Fig. 13.4

Take it into a vessel and dip it into the liquid inside the vessel. The plate G remains in position even if the string is released. This indicates the existence of the upward thrust (or pressure) inside the liquid. Now pour coloured water gradually into the cylinder (c). It will be noticed that the glass plate G shall remain in position so long as the liquid levl in the cylinder remains below the level of the liquid in the vessel. As soon as the levels in the cylinder and vessel become equal 'G' falls down. This implies that the upward pressure of liquid is same as that acting downwards.

13.5 Variation of Pressure And Pressure due to height of liquid

Let us consider two points A and B separated by a small vertical height dy. Imagine horizontal areas ΔA_1 and ΔA_2 containing points A and B respectively, such that $\Delta A_1 = \Delta A_2 = \Delta A$; and they form the base and top surfaces of a liquid cylinder abcd. (see fig. 13.5).

The forces acting on the liquid cylinder abcd are :

(i) $\vec{F}_1 = F_1\hat{y} = P.\Delta A\hat{y}$; acting vertically upwards due to upward thrust of liquid on the bottom surface of the cylinder abcd.

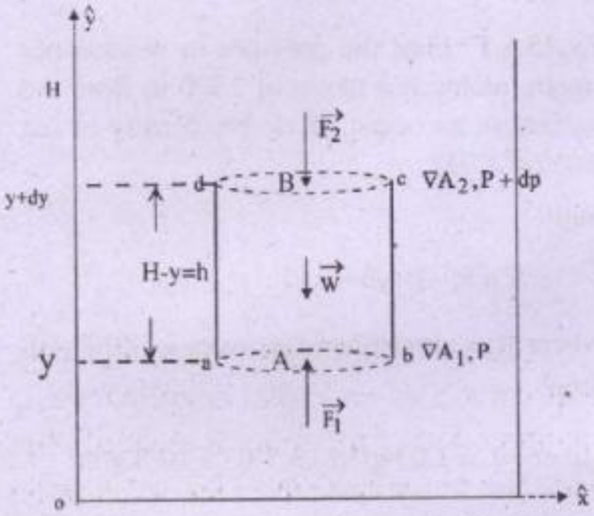


Fig.13.5

(ii) $\vec{F}_2 = F_2(-\hat{y}) = -(p+dp).\Delta A\hat{y}$; acting vertically downwards, due to liquid above the top surface of the cylinder, and atmospheric pressure.

(iii) $\vec{W} = W(-\hat{y}) = -(\Delta A.dy)\,\rho\, g\,\hat{y}$; acting vertically downwards through the C.O.G. of the liquid cylinder abcd, due to the weight of the liquid cylinder. Here 'ρ' is the density of liquid.

Since the liquid is at rest, so

$$\vec{F}_1 + \vec{F}_2 + \vec{W} = 0$$

$$\Rightarrow \quad P.\,\Delta A\,\hat{y} - (P + dP)\Delta A\hat{y} - \Delta A.dy.\,\rho\, g\,\hat{y} = 0$$

$$\Rightarrow \quad dP = -\rho\, g\, dy \qquad ...(13.5.1)$$

Equation (13.5.1) gives the variation of pressure as one moves up from bottom towards the free surface; and it is observed that as we move downwards from the surface the pressure shall go on increasing.

Now the presure 'P' at a point 'y' above the bottom surface is given by

$$\int_{P_a}^{P} dP = -\int_{H}^{y} \rho(y)\, g(y)\, dy \qquad ...(13.5.2)$$

where 'P_a' corresponds to atmospheric pressure on the free surface (y = H). If we assume density ρ and acceleration due to gravity 'g' to be constant, then equation (13.5.2) gives

$$P - P_a = -\rho\, g(y - H) = \rho\, g\,(H - y) = \rho gh$$

$$\Rightarrow P = P_a + \rho gh \qquad ...(13.5.3)$$

Where 'h' is the depth below the free surface and as said above 'P_a' is the atmospheric pressure over the free surface.

Equation (13.5.3) leads to the following conclusions :

(i) Pressure at any point in a liquid increases monotonically with depth 'h' below the free surface. Hence (a) pressure is same at the same level everywhere (b) pressure is exerted equally in all directions at the same level (c) pressure is independent of the shape and size of the containing vessel. (d) liquid seeks its own level every where in a communicating vessel (see fig.13.6)

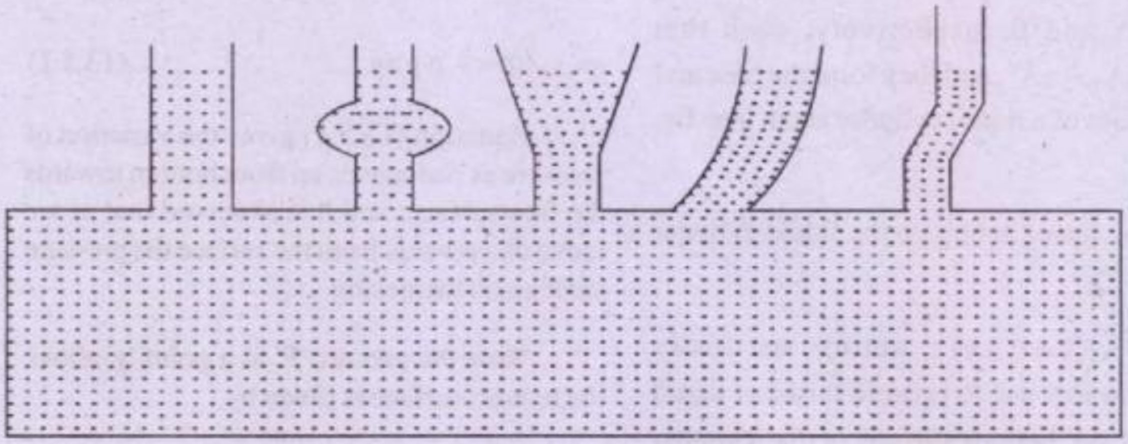


Fig. 13.6

(ii) Pressure at any point increases monotonically with density ρ.

(iii) Pressure also increases monotonically with acceleration due to gravity (g).

and

(iv) If the pressure over the free surface is increased by P_s, then the total pressure at any point within the liquid also increases by P_s. This leads to the Pascal's Law, stated as **"Pressure applied to an enclosed fluid is transmitted undininished to every portion of the fluid and walls of the container."** Pascal's law finds application in **Hydraulic press** (see fig. 13.7)

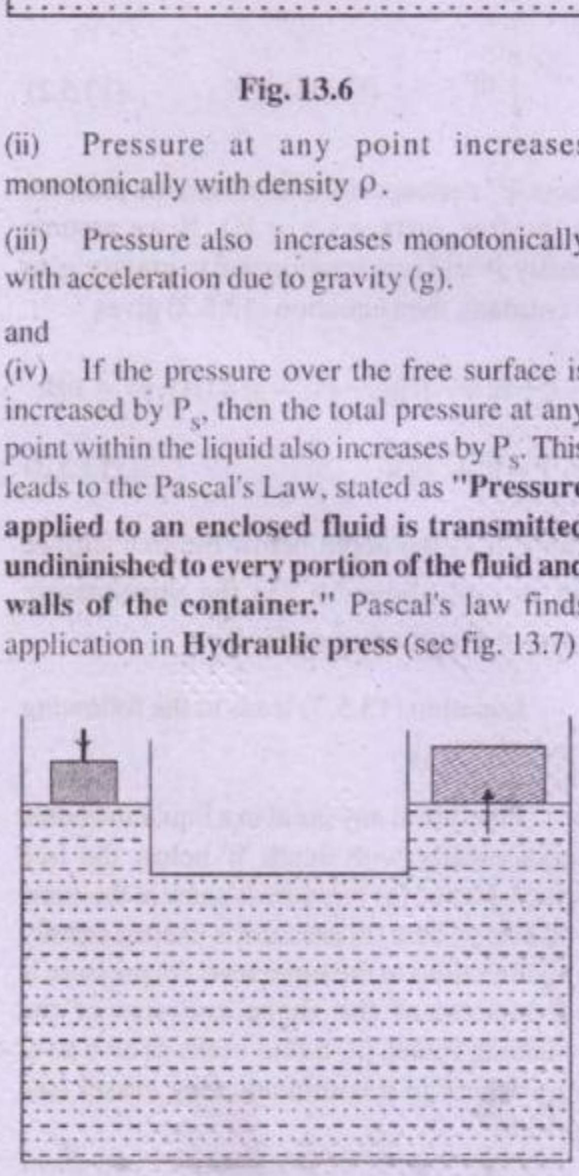


Fig.13.7
(Hydraulic Press)

Ex.13.5.1 Find the pressure in newton per square meter at a depth of 1500 m from the surface of an ocean. (Relative density of sea water = 1.03)

Soln.

$$P = P_a + \rho_r gh$$

where P_a = atmospheric pressure = 1.013×10^3 N/m^2

Given $\rho_r = 1.03$ g/cm^3 = 1.03×10^3 kg/m^3

$$\therefore P = 1.013 \times 10^5 \frac{N}{m^2} + 1.03 \times 10^3 \times 9.8 \times 1500 \frac{N}{m^2}$$

$$= 1.013 \times 10^5 \frac{N}{m^2} + 1.499 \times 10^7 \frac{N}{m^2}$$

$$= 1.01 \times 10^7 \frac{N}{m^2} + 1.499 \times 10^7 \frac{N}{m^2}$$

$$P = 1.509 \times 10^7 N/m^2$$

Ex.13.5.2 Water is filled in a flask up to a height of 20 cm. The bottom of the flask is circular with radius 10 cm. If the atmospheric pressure is 1.01×10^5 Pa, find the force exerted by water on the bottom (Take g = 10 m/s^2, density of water = 10^3 kg/m^3)

Soln.

$$P = P_a + \rho_l gh$$

$$= 1.01\times10^5\frac{N}{m^2}+10^3\frac{kg}{m^2}\times10\frac{m}{s^2}\times20\times10^{-2}m$$

$$= 1.01\times10^5\frac{N}{m^2}+2\times10^3\frac{N}{m^2}$$

$$\Rightarrow \quad P = 1.03\times10^5\frac{N}{m^2}$$

So force $F = P.A = P.\ \pi r^2 =$

$$1.03\times10^5\frac{N}{m^2}\times\pi\times(10\times10^{-2}m)^2$$

$$F = 3235.84\ N.$$

Ex.13.5.3 A man weighing 60 kg is standing on a floor. The area of his feet in contact with the floor is $1.8 \times 10^{-2}\ m^2$. Calculate the pressure on the floor.

Soln.

Weight of man $W = 60 \times g\ N = 60 \times 9.8\ N$
Area of the feet $A = 1.8 \times 10^{-2}\ m^2$

$$\text{Pressure on the floor} = \frac{W}{A} = \frac{60\times9.8}{1.8\times10^{-2}}N/m^2$$

$$P = 326.7 \times 10^2\ N/m^2$$

Ex.13.5.4 A fresh water lake is 25 m deep. The pressure on the surface of the lake is equal to the atmospheric pressure (1.013×10^5 Pa). Calculate the (a) gauge pressure and (b) the absolute pressure at the bottom of the lake.

Soln.

Absolute Pressure $P = P_a + \rho gh$

$$\Rightarrow \quad P = 1.013 \times 10^5\frac{N}{m^2}+10^3\times9.8\times25\frac{N}{m^2}$$

$$= 1.013 \times 10^5\frac{N}{m^2}+2.45\times10^5\frac{N}{m^2}$$

$$= 3.463 \times 10^5\frac{N}{m^2}$$

Gauge pressure $= P - P_a = 2.45 \times 10^5\ N/m^2$.

Ex.13.5.5 The area of cross-section of two arms of a hydraulic press are $1\ cm^2$ and $10\ cm^2$ respectively. A force of 5 N is applied on the water in the thinner arm what load it can lift.

Soln.

Force applied in the thinner arm = 5N

$$\Rightarrow \quad \text{Pressure applied} = \frac{5N}{1\times10^{-4}m^2} =$$

$$5\times10^4 N/m^2$$

Hence upward force in the larger arm =

$$5\times10^4\frac{N}{m^2}\times(10\times10^{-4})m^2$$

$$\text{Load it can lift} = \frac{50}{g}kg = \frac{50}{9.8} = 5.102kg$$

Ex.13.5.6 What is the hydrostatic blood pressure difference between the head and the foot of a 2m tall man standing straight ? (The density of blood is $1.06 \times 10^3\ kg/m^3$)

Soln.

$$P - P_a = h\rho g$$

$$\Rightarrow \quad P - P_a = 2m\times(1.06\times10^3 kg/m^3)\times(9.8m/s^2)$$

$$= 2.08 \times 10^4\ N/m^2$$

Ex.13.5.7 Calculate the maximum height over which a liquid can be siphoned at atmospheric pressure of 76 cm of mercury. (Given sp. rg. of liquid = 0.6)

Soln.

$$h\rho g = 76\text{ cm of Hg} = 76 \times 13.6 \times 980$$

$$\Rightarrow \quad h = \frac{76\times13.6\times980}{0.6\times980} = \frac{76\times13.6}{0.6}$$

$$\Rightarrow \quad h = 1722.67\ cm.$$

Ex.13.5.8 How high would water rise in the pipes of a building, if water pressure gauge shows the pressure of the ground level to be 270 k Pa.

Soln.

$P - P_a =$ Gauge pressure $= 270$ k Pa $= 270 \times 10^3$ N/m^2

$\therefore P - P_a = h.\rho_w \cdot g$

$$\Rightarrow \quad h = \frac{P - P_a}{\rho_w \cdot g} = \frac{270 \times 10^3 N/m^2}{10^3 \frac{kg}{m^3} \times 9.8 \frac{m}{s^2}}$$

$h = 27.55$ m

13.6 Atmospheric Pressure :

Earth is surrounded by an ocean of air, called atmosphere, spreading upto a height of about 200 km. This atmosphere exerts pressure on the surface of bodies, lying on earth's surface. This thrust of air on any body on earth's surface acts perpendicularly to the surfaces of the bodies.

If we consider a small surface area ΔA, in contact with air, near earth's surface and $\vec{\Delta F}$ be the force (thrust) exerted on this area, then atmospheric pressure P_a, near earth's surface is given by

$$P_a = \lim_{\Delta A \to 0} \frac{|\vec{\Delta F}|}{\Delta A}$$

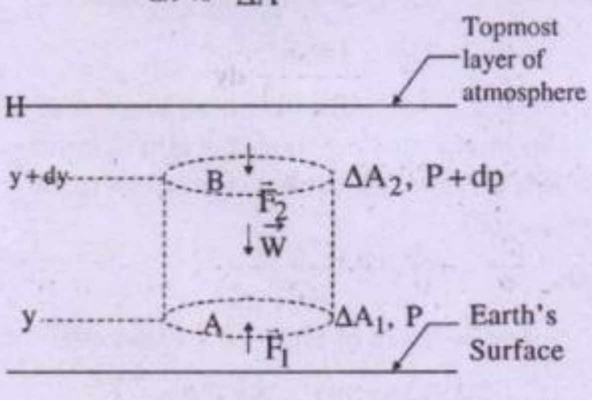


Fig.13.8

Imagine horizontal areas ΔA_1. Containing point A and ΔA_2 containing point B. Let $\Delta A_1 = \Delta A_2 = \Delta A$ and form the base and top surfaces of the air cylinder of height dy. Let the pressure at level y and y+dy be P and P+dP respectively. Then proceeding as in Sec. 13.5 (for liquid), we obtain (as in eqn. 13.5.1)

$$dp = -\rho\, g\, dy \qquad ...(13.6.1)$$

where 'ρ' is the density of air and 'g' is acceleration due to gravity at height y above earth's surface. In general $\rho = \rho(y)$; $g = g(y)$, so

$$dP = -\rho(y)\, g(y)\, dy \qquad ...(13.6.2)$$

So the atmospheric pressure near the earth's surface is given by

$$\int_{P_a}^{0} dp = -\int_{0}^{H} \rho(y)\, g(y)\, dy \qquad ...(13.6.3)$$

The upperlimit on l.h.s is taken to be zero, as pressure at the topmost layer of atmosphere must be zero. Equation (13.6.3) gives atmospheric pressure p_a as

$$P_a = \int_{0}^{H} \rho(y)\, g(y)\, dy \qquad ...(13.6.4)$$

The pressure 'P' at any height y above earth's surface is given by

$$\int_{P_a}^{P} dp = -\int_{0}^{y} \rho(y)\, g(y)\, dy$$

giving

$$P = P_a - \int_{0}^{y} \rho(y)\, g(y)\, dy \qquad ...(13.6.5)$$

Equation (13.6.5) gives the variation of atmospheric pressure with height (y) above earth's surface. This can be exactly studied if we know variations of ρ and g with height y, above the earth's surface. So we consider the following cases :

(i) **ρ, g nearly constant**

In this case the integration on r.h.s of eqn. (13.6.5) can be easily carried out and we obtain

$$P = P_a - \rho g \int_0^y dy = P_a - \rho g y \qquad ...(13.6.6)$$

This implies that as we go up vertically the pressure should decrease linearly. This actually happens upto small heights above earth's surface.

(ii) **g is constant, ρ vary with y**

Since $\rho = \frac{m}{V} = \frac{mP}{PV} = \frac{mP}{nRT}$ and 'T' is assumed to remain constant, so $\rho \propto P$. Hence if ρ_a be density of air near earth's surface and $\rho(y)$ be density of air at height y, then

$$\frac{\rho_a}{\rho(y)} = \frac{P_a}{P}$$

giving

$$\rho(y) = \frac{\rho_a}{P_a} . P \qquad ...(13.6.7)$$

Using eqn. (13.6.7) in eqn. (13.6.2) we obtain

$$dp = -\frac{\rho_a}{P_a} P g \, dy$$

$$\Rightarrow \quad \int_{P_a}^{P} \frac{dp}{P} = -\frac{g\rho_a}{Pa} \int_0^y dy$$

$$\Rightarrow \quad \log(P/P_a) = -\frac{g\rho_a}{P_a} y$$

$$\Rightarrow \quad P = P_a \, e^{-\rho_a g_a y/P_a} \qquad ...(13.6.8)$$

Equation (13.6.8) shows that pressure should decrease exponentially as we move up from earth's surface.

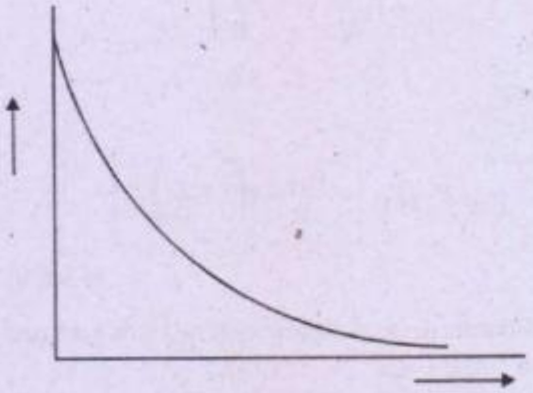


Fig. 13.9

(iii) **ρ and g both vary with height**

We know from our knowledge on gravitation that

$$g = \frac{GM}{(R+y)^2}$$

where R is radius of earth and y is the height above the surface of earth. Then

$$dP = -\rho(y) \; g(y) \; dy$$

$$= -\left(\frac{\rho_a}{P_a} P\right)\left(\frac{GM}{(R+y)^2}\right) dy$$

$$\Rightarrow \quad \frac{dP}{P} = -\frac{\rho_a}{P_a} \cdot \frac{GM}{(R+y)^2} \, dy$$

Integrating both sides

$$\int_{P_a}^{P} \frac{dP}{P} = -\frac{\rho_a}{P_a} . GM \int_0^y \frac{dy}{(R+y)^2}$$

$$\log(P/P_a) = -\frac{\rho_a}{P_a} . GM \left(-\frac{1}{R+y} \Big|_0^y \right)$$

$$= -\frac{\rho_a}{P_a} . \frac{GM}{R^2} \frac{y}{1+\frac{y}{R}}$$

$$= -\frac{\rho_a g_a}{P_a} y\left(1+\frac{y}{R}\right)^{-1}$$

Giving

$$P = P_a \exp\left[-\frac{\rho_a g_a}{P_a} y\left(1+\frac{y}{R}\right)^{-1}\right] \quad ...(13.6.9)$$

But atmosphere almost extends to 8 kms and R, radius of earth ~ 6380 km. Hence $\frac{y}{R} << 1$, therefore $y\left(1+\frac{y}{R}\right)^{-1} \approx y$, and eqn. (13.6.9) boils down to

$$P = P_a \exp\left[-\frac{\rho_a g_a}{P_a} y\right] \quad ...(13.6.10)$$

Equation (13.6.10) is same as (13.6.8) implying variation of g with y has little effect on the variation of atmospheric pressure.

__Height of atmosphere :__

From eqn. (13.6.8) or (13.6.10) we have

$$P = P_a \exp\left[-\frac{\rho_a g_a}{P_a} y\right]$$

But $\frac{\rho_a g_a}{P_a} \approx 1.251\times10^{-6}\text{cm}^{-1}$

Therfore $\exp\left[-\frac{\rho_a g_a}{P_a} y\right] \approx 1-\frac{\rho_a g_a}{P_a} y$

Hence

$$P = P_a\left[1-\frac{\rho_a g_a}{P_a} y\right] \quad ...(13.6.11)$$

At the topmost level of atmosphere (y= H) pressure P = 0, Therefore

$$1-\frac{\rho_a g_a}{P_a} H = 0$$

$$\Rightarrow \quad H \approx \frac{P_a}{\rho_a g_a} \quad ...(13.6.12)$$

Equation (13.6.12) gives

$$H = \frac{76\times13.6}{1.293\times10^{-3}}\text{cm} \approx 7.994\times10^5\text{cm} = 7.994\text{Km}$$

Ex. 13.6.1 The density of air in the atmosphere decreases with height and can be expressed by the relation

$$\rho = \rho_0\ e^{-\alpha h}$$

where ρ_0 is the density at sea level, α is a constant and h is the height. Calculate the atmospheric pressure at sea level assuming 'g' to be constant ($g = 9.8\ m/s^2$, $\rho_0 = 1.3\ kg/m^3$, $\alpha = 1.2\times10^{-4}\ m^{-1}$)

Soln.

$$\int_{P_a}^{0} dp = -\int_0^H \rho(y)\ g(y)\ dy$$

Since 'g' is constant

$$P_a = g\int_o^H \rho(y)\ dy = g\rho_0 \int_o^H e^{-\alpha y} dy$$

$$= \frac{\rho_0 g}{-\alpha}\left(e^{-\alpha y}\Big|_o^H\right) = \frac{\rho_0 g}{-\alpha}(e^{-\alpha H}-1)$$

$$\Rightarrow \quad P_a = \frac{\rho_0 g}{\alpha}\left(1-e^{-\alpha H}\right) \approx \frac{\rho_0 g}{\alpha}$$

Ex.13.6.2 Calculate the atmospheric pressure at a height of about 6 km from sea level.

Soln.

(i) Assuming 'g' to be constant but density varying

$$P = P_a \exp\left[-\frac{\rho_a g_a}{P_a} y\right]$$

$$= P_a \exp\left[-\frac{1.3\frac{kg}{m^3}\times 9.8\frac{m}{s^2}\times 6\times 10^3 m}{1.013\times 10^5 N/m^2}\right]$$

$$= P_a \times 0.47 = 0.476 \times 10^5 N/m^2 \approx$$

(ii) Assuming ρ and g both vary

$$P = Pa \exp\left[-\frac{\rho_a g_a}{P_a}\left(y - \frac{y^2}{R}\right)\right]$$

$$= P_a \exp\left[-\frac{1.3\times 9.8\times 6\times 10^3}{1.013\times 10^5}\left(1-\frac{6\times 10^3}{6.38\times 10^6}\right)\right]$$

$$= P_a \times 0.4705 = 0.4766 \times 10^5 N/m^2$$

$$P = 0.477 \times 10^5 N/m^2$$

Ex.13.6.2 The density of air near earth's surface is 1.3 kg/m^3, and the atmospheric pressure is $1.0 \times 10^5 N/m^2$. If the atmosphere had uniform density, same as that near the surface, what would be the height of the atmosphere to exert the same pressure ? What do you learn from this calculation ?

Soln.

$$\int_{P_a}^{0} dP = -\int_0^H \rho(y)\ g(y)\ dy$$

$$P_a = \int_0^H \rho(y)\ g(y)\ dy$$

$$= \rho . g \int_0^H dy = \rho\, g\, H$$

$$\therefore H = \frac{P_a}{\rho g} = \frac{1.0\times 10^5 N/m^2}{1.3 kg/m^3 \times 9.8 m/s^2}$$

$$H = 0{,}07849 \times 10^5 m = 7849 m.$$

This calculation shows that such an atmospheric height cannot contain Mount Everest i.e. Mount Everest shall be outside the atmosphere which is not a fact. Hence 'ρ' varies.

13.7 Barometer (Principle) :

Torricelli devised a simple instrument 'barometer' to measure atmospheric pressure.

A barometer consists of a glass tube of nearly 1m. length, with one end sealed. It is filled with mercury. Then keeping the thumb at the open end, it is inverted and kept vertically in another vessel, containing mercury (see fig.13.10). The mercury within the tube falls down and remains at level A, leaving a vaccum, called Torricelli Vaccum) above it.

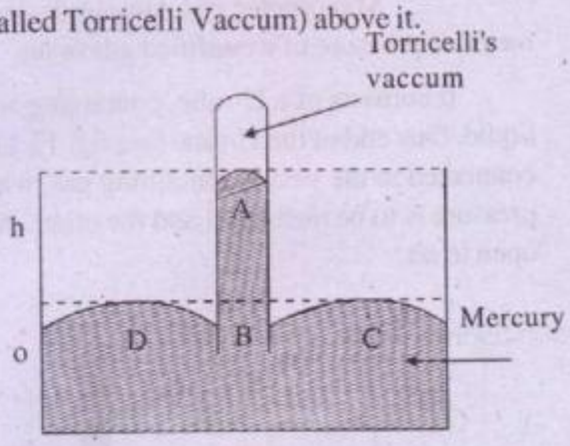


Fig.13.10

Since pressure in a liquid is same at one level, so pressure at B, C and D are same.

i.e $P_B = P_C = P_D = P_a$ = Atmospheric pressure.

Pressure at A is zero (i.e. $P_A = 0$), as there is vaccum above it and the end is sealed, (so that no communication with atmosphere). So using eqn (13.5.1) we obtain

$$\int_{P_B = P_a}^{P_A = 0} dp = -\int_0^h \rho(y)\ g(y)\ dy$$

As ρ and g can be taken as constant over this small height h, so we obtain

$$-P_a = -\rho g \int_0^h dy = -\rho gh$$

$$\Rightarrow \quad P_a = \rho gh \qquad ...(13.7.1)$$

This is the guiding relation for a liquid barometer. It is seen experimentally that for mercury barometer h = 76 cm. Therefore

$$P_a = h\rho g = 76 \times 13.6 \times 980 \text{ dyne/cm}^2$$

$$\Rightarrow \quad P_a = 1.013 \times 10^6 \text{ dyne/cm}^2$$

$$= 1.013 \times 10^5 \text{ N/m}^2$$

13.8 Manometer :

Manometer is a simple device to measure pressure of a confined gas or air.

It consists of a U-tube, containing some liquid. One end of the U-tube (see fig. 13.11) is connected to the vessel containing gas, whose pressure is to be measured and the other end is open to air.

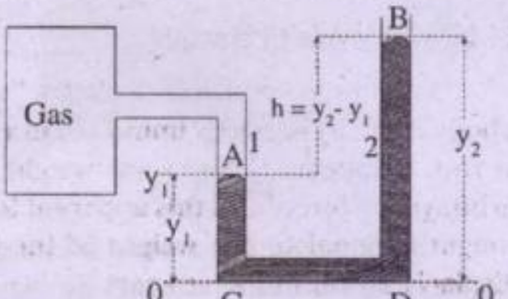


Fig.13.11

It is observed that the liquid levels in the two limbs 1 and 2, are different. The difference in the level is a measure of the pressure of the gas as shown below.

From eqn. (13.5.1) it follows that

$$\int_{P_C}^{P_A} dp = -\rho g \int_0^{y_1} dy$$

giving

$$P_A - P_C = -\rho g y_1 \qquad ...(13.8.1)$$

Similarly

$$P_B - P_D = -\rho g y_2 \qquad ...(13.8.2)$$

But $P_C = P_D$ as C and D are at same level

$P_B = P_a$ = atmospheric pressure

$P_A = P$ = pressure due to confined gas.

Hence from (13.8.1) and (13.8.2) we obtain

$$(P_A - P_C) - (P_B - P_D) = \rho g(y_2 - y_1)$$

$$\Rightarrow \quad (P - P_C) - (P_a - P_C) = \rho g(y_2 - y_1)$$

$$\Rightarrow \quad P - P_a = \rho g(y_2 - y_1) = \rho gh \qquad ...(13.8.3)$$

$$\Rightarrow \quad P = P_a + \rho gh \qquad ...(13.8.4)$$

Equation (13.8.4) gives the pressure of the confined gas. Sometimes the pressure difference ($P - P_a$) is called **Gauge Pressure.**

Ex.13.8.1 A mercury manometer is connected to a gas tank (as shown in fig. 13.W.1) compute the pressures at A, B and within the tank.

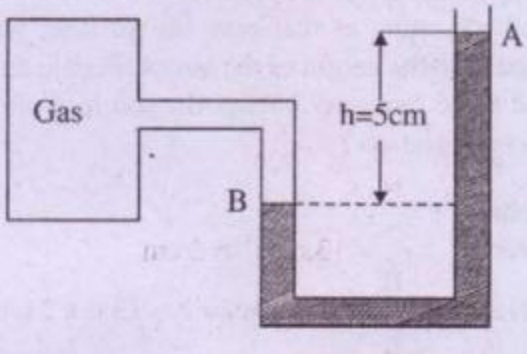


Fig.13.W.1

Soln.

$P_A = P_a$ = atmospheric Pressure = 1.013×10^5 N/m²

$P_B = P_a + \rho gh =$

$$1.013 \times 10^5 \frac{N}{m^2} + 5 \times 10^{-2} m \times 13.6 \times 10^3 \frac{kg}{m^3} \times 9.8 \text{ ms}^{-2}$$

$$= 1.013 \times 10^5 \frac{N}{m^2} + 0.06664 \times 10^5 \frac{N}{m^2}$$

$$P_B = 1.07964 \times 10^5 \frac{N}{m^2}$$

Pressure within the tank is same as that at B.

Ex.13.7.2 The liquids shown in fig. 13.W.2 in the two arms are mercury (sp.gr. = 13.6) and water. If the difference of heights of mercury column is 2 cm. Find the height 'h' of water column.

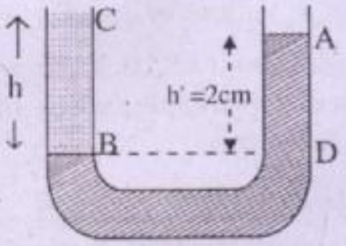


Fig. 13.W.2

Soln.

Pressure at B = $P_B = P_a + h\rho_w g$

Pressure at D = $P_D = P_a + h'\rho g$

Since $P_D = P_B \Rightarrow h\,\rho_w g = h'\rho g$

$$\Rightarrow \quad h = \frac{\rho}{\rho_w}.h' = \rho_r h'$$

Given $\rho_r = 13.6,\ h' = 2$ cm

$\therefore$ Height of water column = h = 13.6 x 2 cm

=> h = 27.2 cm

13.9 Buoyancy :

When a body is wholly or partly immersed in a fluid at rest, the fluid exerts forces on the body, in a direction perpendicular to every element of the surface of the body. The force on an element of the body is the product of pressure at the point and the element of area. The resultant of all these forces is called buoyant force and it acts vertically upwards.

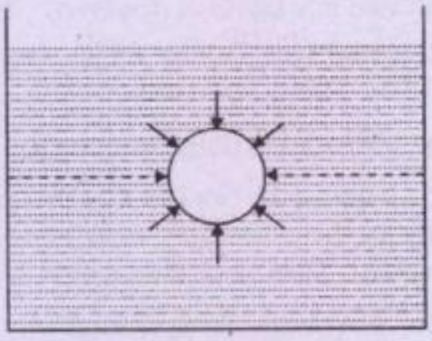


Fig. 13.12

This phenomenon is called buoyancy. The upward direction of buoyant force can be explained as follows. The portions of lower hemisphere are at greater depths than the portions of upper hemisphere. Threfore the pressure over the lower hemisphere is more than the pressure over the upper hemisphere. Hence the net upward force on the lower hemisphere is more than the net downward force on the upper hemisphere. This explains the upward nature of buryancy.

13.10 Archimede's Principle

Archimede's principle states that "**when a body is wholly or partly immersed in a fluid at rest, it appears to lose some weight (due to buoyancy force) and this apparent loss of weight is equal to the weight of the fluid displaced by the immersed part**".

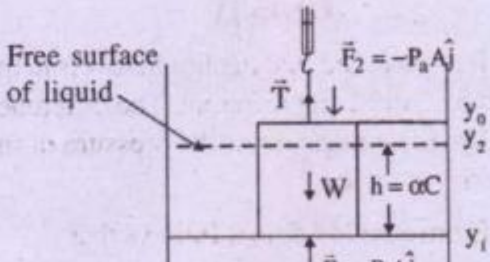


Fig. 13.13

For simplicity consider a rectangular block, suspended from a spring balance and immersed in a liquid of density ρ_ℓ. The block could be partially immersed (as shown in fig. 13.13) or wholly immersed.

Case I (Block Partially immersed)

Suppose the block (a x b x c) is immersed partly so that a fraction α of its height c is immersed. Then immersed height is $h = \alpha c$. The forces acting on the block are :

i) $\vec{W} = -mg\hat{j}$, the weight of the block acting downward, vertically through C.O.G.

ii) $\vec{F}_2 = -F_2\hat{j} = -P_aA\hat{j}$, due to atmospheric pressure P_a acting on the upper surface of the block of area A = ab; in the vertically downward direction.

iii) $\vec{F}_1 = F_1\hat{j} = P_1A\hat{j}$, due to upward pressure of the liquid lying below the block and acting on the lower surface in vertically upward direction.

iv) $\vec{T} = T\hat{j}$, acting along the string in the vertically upward direction.

Since the block is at rest, therefore

$$\vec{W} + \vec{F}_1 + \vec{F}_2 + \vec{T} = 0$$

$$\Rightarrow \quad -mg\hat{j} + P_1A\hat{j} - P_aA\hat{j} + T\hat{j} = 0$$

$$\Rightarrow \quad T = mg + (P_a - P_1)A \qquad ...(13.10.1)$$

Using eqn. (13.5.3) we obtain

$$P_1 = P_a + h\rho_\ell g$$

$$\Rightarrow \quad P_1 - P_a = h\rho_\ell g \qquad ...(13.10.2)$$

Using eqn (13.10.2) in r.h.s of eqn. (13.10.1) we obtain apparent weight $W_a \equiv T$ as

$$W_a = mg - (h\rho_\ell g)A$$

$$= W_0 - \alpha\, cA\, \rho_\ell g \qquad (\because h = \alpha C)$$

$$= W_0 - \alpha\, cab\, \rho_\ell g$$

$$= W_0 - \alpha V_0 \rho_\ell g = W_0 - \alpha W_{\ell 0}$$

$$= W_0 - V\rho_\ell g = W_0 - W_\ell$$

i.e. $W_a = W_0 - W_\ell = W_0 - \alpha W_{\ell 0}$ ...(13.10.3)

where we have put (i) V_0 = abc as the volume of the block (ii) $V = \alpha V_0$ as the fraction of volume immersed. (iii) W_ℓ, as the weight of velume of liquid displaced and (iv) $W_{\ell 0}$, as the weight of V_0 volume of liquid.

Equation (13.10.3) is a proof of Archimede's principle, when the body is partially immersed.

Case II (Block fully immersed)

When the block is fully immersed $\alpha = 1$, equation (13.10.3) reduces to

$$W_0 = W_0 - W_{\ell 0} \qquad ...(13.10.4)$$

This again is a proof of Archimei's principle when it is fully immersed.

Experimental Verification :-

A hollow cylinder H and a solid cylinder S, such that the internal volume of H and external volume of S are same; are taken. The solid cylinder S is hung below the hollow cylinder H and their weight W_1 is taken, with the help of a physical balance. Then the solid cylinder is fully immersed in water kept in a beaker. Then weight W_2 is required to counter

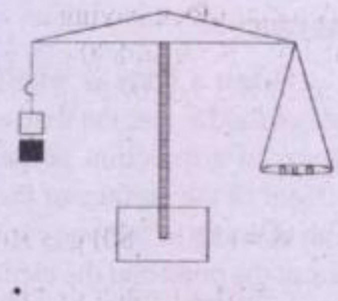


Fig. 13.14

balance H and S. It is noticed that $W_2 < W_1$, confirming that a body loses weight when immersed. Then water is gradually poured into the hollow cylinder until it is just filled up. Then weight W_1 has to be again put on the pan for counter balancing. This proves that the weight lost is equal to the weight of equal volume of water.

Application of Archimede's Principle :

i) Determination of density of a homogeneous body

Consider a homogeneous body of volume V_0 and density ρ, submerged in a liquid of density ρ_ℓ. Then the apparent weight W_a of the body is given by

$$W_a = W_0 - V_0\rho_\ell g$$

$$\Rightarrow \quad W_0 - W_a = V_0\rho_\ell g = \frac{\rho_\ell}{\rho} W_0$$

$$\Rightarrow \quad \rho = \frac{W_0}{W_0 - W_a} \cdot \rho_\ell \qquad ...(13.10.5)$$

This helps in determining density of the body.

ii) Testing of purity of substance :

As discussed in (i) above, if calculated density does not match with standard value for a given substance, then it will be suspected to be impure.

iii) Explains floatation of bodies.

Ex.13.10.1 A metal block having an internal cavity weighs 110 g in air and 80 g in water. If the density of metal is 5.5 g/cm^3. Find the volume of the cavity.

Soln.

Weight loss = (110 – 80) g = 30 g wt

= wt. of water displaced

$= V \cdot \rho_w \cdot g$

Since $\rho_w = 1$ g/cm^3

So V = external volume of the block = 30 cm^3.

Now volume of the metal in the block =

$$\frac{110\text{g}}{5.5\text{g}/\text{cm}^3} = 20 \text{ cm}^3$$

$\therefore$ Volume of cavity = (30 – 20) cm^3 = 10 cm^3.

Ex.13.10.2 A crown made of gold and copper weights 210 g in air and 198 g. in water. Find the weight of gold in crown. (Given $\rho_{gold} = 19.3\text{g}/\text{cm}^3$, $\rho_{cu} = 8.5\text{g}/\text{cm}^3$)

Soln.

Let mass of gold = m_1 g

mass of copper = m_2 g

$\therefore m_1 + m_2 = 210 \text{ g} = M$

Volume of gold $V_1 = \frac{m_1}{\rho_1}$

Volume of copper $V_2 = \frac{m_2}{\rho_2}$

Net volume $V = V_1 + V_2 = \frac{m_1}{\rho_1} + \frac{m_2}{\rho_2}$

Loss of weight = weight of displaced water

=> $(210 - 198)\text{ g} = (V_1 + V_2) \cdot \rho_w$

=> $(V_1 + V_2) \cdot \rho_w = V = 12 \text{ cm}^3$

$$=> \left(\frac{m_1}{\rho_1} + \frac{m_2}{\rho_2}\right)\rho_w = 12 = M - M'$$

$$=> \frac{m_1}{\rho_1} + \frac{M - m_1}{\rho_2} = 12 = \frac{M - M'}{\rho_w}$$

$$=> \frac{M - M'}{\rho_w} - \frac{M}{\rho_2} = m_1\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right)$$

$$=> m_1 = \frac{\rho_1\rho_2}{\rho_2 - \rho_1}\left(\frac{M - M'}{\rho_w} - \frac{M}{\rho_2}\right)$$

$$m_1 = \frac{\rho_1\rho_2}{\rho_1-\rho_2}\left(\frac{M}{\rho_2}-\frac{M-M'}{\rho_w}\right)$$

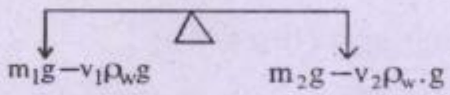
$$= \frac{19.3\times 8.5}{19.3-8.5}\left(\frac{210}{8.5}-\frac{210-198}{1}\right)$$

$\Rightarrow m_1 = 186.107$ g

$m_2 = 23.893$ g

Ex.13.11.3 Two metal blocks are suspended from the arms of a balance. When immersed in water, these blocks are in equilibrium. The mass of one piece is 35 g and its density is 5 g/cm^3. If the density of other piece is 8 g/cm^3 then what is the mass of this piece.

Soln.



$$(m_1 - V_1\rho_w)g = (m_2 - V_2\rho_w)g$$

$$\Rightarrow \quad m_1(1-\tfrac{1}{\rho_1}\rho_w) = m_2(1-\rho_w/\rho_2)$$

$$\Rightarrow \quad m_2 = \frac{m_1(\rho_1-\rho_w)/\rho_1}{(\rho_2-\rho_w)/\rho_2} = m_1.\frac{\rho_2}{\rho_1}.\frac{\rho_1-\rho_w}{\rho_2-\rho_w}$$

$$\Rightarrow \quad m_2 = 35\times\frac{8}{5}\times\frac{5-1}{8-1} = 35\times\frac{8}{5}\times\frac{4}{7} = 32\text{g}$$

Ex.13.11.4 Determine the quality of pure gold in 50 g of alloy of gold and copper. Given $\rho_{gold} = \rho_1 = 19\text{g/cm}^3$, $\rho_{copper} = \rho_2 = 9\text{g/cm}^3$ $\rho_{alloy} = \rho = 16\text{g/cm}^3$.

Soln.

Let m_1 be mass of gold in the alloy

m_2 be mass of copper in the alloy

Mass of alloy $= M = m_1 + m_2 = 50$ g

Volume of gold $= V_1 = \frac{m_1}{\rho_1}$

Volume of copper $= V_2 = \frac{m_2}{\rho_2}$

Volume of alloy $= V = V_1 + V_2$

density of alloy $\rho = \frac{M}{V} = \frac{m_1+m_2}{V_1+V_2}$

$$V = \frac{M}{\rho} = \frac{m_1}{\rho_1}+\frac{m_2}{\rho_2} = \frac{m_1}{\rho_1}+\frac{M-m_1}{\rho_2}$$

$$\Rightarrow \quad \frac{M}{\rho}-\frac{M}{\rho_2} = m_1\left(\frac{1}{\rho_1}-\frac{1}{\rho_2}\right)$$

$$\Rightarrow \quad \frac{M(\rho_2-\rho)}{\rho\,\rho_2} = \frac{m_1(\rho_2-\rho_1)}{\rho_1\,\rho_2}$$

$$\Rightarrow \quad \frac{M(\rho_2-\rho)}{\rho} = \frac{m_1(\rho_2-\rho_1)}{\rho_1}$$

$$m_1 = M.\frac{\rho_1}{\rho}.\frac{\rho_2-\rho}{\rho_2-\rho_1} = 50\times\frac{19}{16}\times\frac{9-16}{9-19}$$

$$= 50\times\frac{19}{16}\times\frac{(-7)}{(-10)}$$

$$= \frac{5\times 19\times 7}{16} = 41.563\text{ g}$$

$m_1 = 41.563$ g

$\Rightarrow \quad m_2 = 8.437$ g

13.11 Flotation of Bodies :

Consider an arbitrary volume V of liquid at rest, being bounded by surface S.

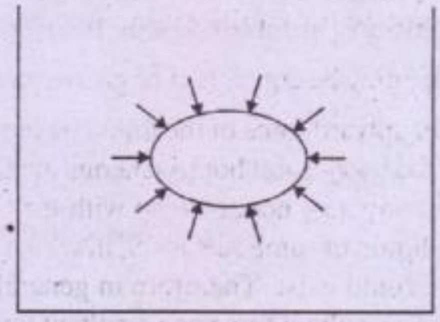


Fig. 13.15

Forces $\overrightarrow{\Delta F_1}, \overrightarrow{\Delta F_2}, \ldots \overrightarrow{\Delta F_n}$ etc. act on each element of surface due to pressure of liquid, existing near the surface and its weight $m_\ell g = V.\rho_\ell.g$ acts vertically downwards through C.O.G. 'G'. Since the fluid is at rest.

$$\overrightarrow{\Delta F_1} + \overrightarrow{\Delta F_2} + \ldots + \overrightarrow{\Delta F_n} + m\vec{g} = 0 \quad \ldots(13.11.1)$$

Resolving the forces along the horizontak (X-dir^n) and vertical (Y-dir^n)

$$\overrightarrow{\Delta F_{1x}} + \overrightarrow{\Delta F_{2x}} + \ldots\ldots + \overrightarrow{\Delta F_{nx}} = 0 \quad \ldots(13.11.2)$$

and

$$(\overrightarrow{\Delta F_{1y}} + \overrightarrow{\Delta F_{2y}} + \ldots\ldots + \overrightarrow{\Delta F_{ny}}) + mg(-\hat{j}) = 0 \quad \ldots(13.11.3)$$

This implies

$$\vec{F}_y = \overrightarrow{\Delta F_{1y}} + \overrightarrow{\Delta F_{2y}} + \ldots\ldots + \overrightarrow{\Delta F_{ny}}) = mg\ \hat{j} \quad \ldots(13.11.4)$$

where $\vec{F}_y$ is the net force acting vertically upward and passing through a point, called Centre of Buoyancy (C.O.B). For equilibrium the line of action of $\vec{F}_y$ and $\overrightarrow{mg}$ must pass through C.O.G. i.e. C.O.G. and C.O.B. must lie on the same vertical line or coincide.

Now when volume V is replaced by a body of same surface S, the submerged body need not be in equilibrium. Because now (i) $|m\vec{g}|$ may be equal, less or greater than $|\vec{F}_y|$ (the net upward force of the liquid on the body) (ii) If the body is not homogeneous the C.O.G. of the body may not coincide with the C.O.G. of the liquid of same surface S, in which case a torque could exist. Therefore in general there may be a resultant force or a resultant torque as a result of which it may move up or down; or rotate or remain in equilibrium. The following cases may arise.

i) $mg > F_y\ (= V\rho_\ell g)$ => body will sink

ii) $mg < F_y (= V\rho_\ell g)$ => body will float being fractionally immersed so that $mg = V_f\ \rho_\ell g$, where $V_f = \alpha V$ = fraction of volume immersed.

iii) $mg = F_y\ (= V\ \rho_\ell g)$ => body will float being fully immersed anywhere inside the liquid.

Conditions of floating & equilibrium of floating bodies

i) Weight of the body must be equal to the weight of fluid displaced by the immersed part of the body.

ii) C.O.B. and C.O.G. must lie on the same vertical line or coincide.

iii) The **metacentre**, (a point at which a vertical line through displaced position B′ of C.O.B., intersects the line joining the C.O.B. and C.O.G.) should lie above C.O.G.

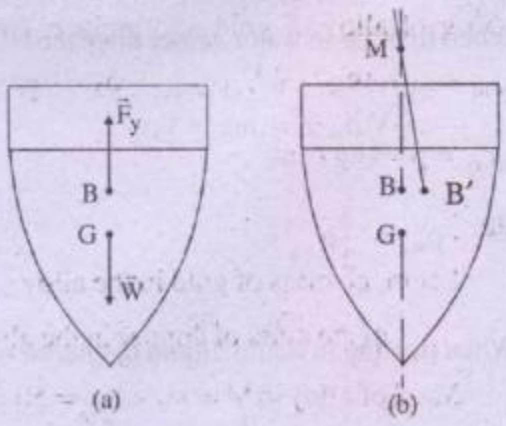


Fig. 13.16

Ex.13.11.1 An ice cube floats in a glass of water filled to the brim. What percentage of the volume of ice is below the surface of water ?

Soln.

Let m be mass of ice, V_i be its volume.

Let a fraction 'α' of its volume be immersed while floating. Then volume of water displaced $V = \alpha V_i$

For floating

$$V.\rho_w.g = V_i\,\rho_i\,g$$

$$\Rightarrow \quad \alpha V_i\,\rho_w = V_i\,\rho_i$$

$$\Rightarrow \quad \alpha = \frac{\rho_i}{\rho_w}$$

% of volume immersed =

$$\frac{V}{V_i}\times 100 = 100\times\alpha = 100\frac{\rho_i}{\rho_w}$$

$$= 100\times\frac{0.9}{1} = 90\%$$

Ex.13.11.2 A body floats with 1/3 rd of its volume outside water and 3/4 the of its volume outside another liquid. Find the density of the other liquid.

Soln.

Let the body be of mass m and volume V

When floating in water, water displaced=2/3 V

$$\therefore \frac{2}{3}V.\rho_w.g = mg = V\rho g$$

$$\Rightarrow \quad \rho_w = \frac{3}{2}\rho \qquad ...(1)$$

When floating in liquid, liquid displaced = $\frac{1}{4}V$

$$\therefore \quad \frac{1}{4}V.\rho_\ell.g = V\rho g$$

$$\Rightarrow \quad \rho_\ell = 4\rho \qquad ...(2)$$

From (1) & (2)

$$\frac{\rho_\ell}{\rho_w} = \frac{4\rho}{\frac{3}{2}\rho} = \frac{8}{3}$$

$$\rho_\ell = \frac{8}{3}\rho_w = \frac{8}{3}\text{gm / cm}^3 \qquad ...(3)$$

Ex.13.11.3 A piece of wood of relative density 0.25 floats in a pool containing oil of relative density 0.81. What is the fraction of volume of the wood above the surface of the oil.

Soln.

Let mass of wood = m g

$$\text{volume of wood} = V = \frac{m}{0.25}\text{cm}^3$$

Let α fraction of it be immersed while floating

so liquid displaced = $\alpha V = \alpha.\frac{m}{0.25}\text{cm}^3$

For floating

$$V_\ell.\rho_\ell.g = mg$$

$$\Rightarrow \quad \alpha.\frac{m}{0.25}\times 0.81 = m$$

$$\Rightarrow \quad \alpha = \frac{0.25}{0.81} = 0.30864$$

fraction lying above = $1 - \alpha = 0.691$

Ex.13.11.4 A cube of wood floating in water supports a mass of 200 g resting at the centre of its top surface. When the mass is removed it rises 2 cm. Find the volume of the cube.

Soln.

Let the volume of the wood be = V
mass of wood = m

Let its each side be = a

Then $V = a^3$ ...(1)

When the block is alone floating let the block be immersed by x cm.

Then immersed volume = $a^2.x$ = volume of water displaced.

For floating

$$a^2x.\rho_w.g = mg = a^3.\rho.g$$

$$\Rightarrow \quad x = \frac{\rho a}{\rho_w} = \rho a \text{ cm} \quad ...(1)$$

when mass 200 g is placed on it, it further sinks by 2cm. So volume of water displaced = a^2 (x+2)

Now for floating

$$a^2(x+2).\rho_w.g = (m+200)g$$

$$\Rightarrow \quad x+2 = \frac{m+200}{a^2.\rho_w} = \frac{m+200}{a^2}$$

$$= \frac{m}{a^2} + \frac{200}{a^2} = \frac{a^3.\rho}{a^2} + \frac{200}{a^2}$$

$$x + 2 = \rho a + \frac{200}{a^2} \quad ...(2)$$

Using (1) in (2)

$$2 = \frac{200}{a^2}$$

=> $a^2 = 100$

giving a = 10 cm

So volume of the cube = 10^3 cm^3 = 10^{-3} m^3.

13.12 Pressure difference and Buoyant force in accelerated fluids

Eqn. 13.5.3 and 13.11.4 were derived by assuming that the fluid under consideration is in equilibrium in an intertial frame. If there is deviation from this, these equations need to be modified. We shall discuss a few special cases to throw some light on this aspect.

A. Liquid placed in an elevator

(a) Pressure difference

Suppose a beaker, containing some liquid is placed in an elevator which is moving with an acceleration $\vec{a}$. Let A and B be two points separated by a small distance dy. Imagine horizontal areas ΔA_1 and ΔA_2 containing A and B respectively, such that $\Delta A_1 = \Delta A_2 = \Delta A$. Let us imagine a cylinder with ΔA_1 as base and ΔA_2 as top surface. (see fig. 13.17)

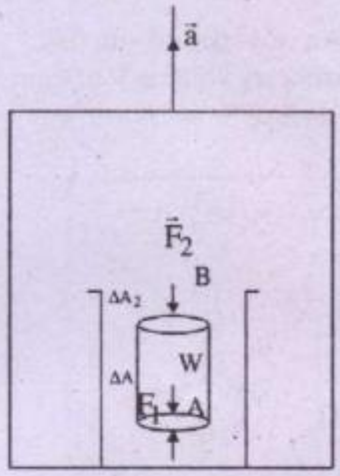


Fig. 13.17

Then forces acting are

(i) $\vec{F}_1 = F_1\hat{y} = P_1\,\Delta A_1\,\hat{y} = P\,\Delta A\,\hat{y}$

(ii) $\vec{F}_2 = F_2(-\hat{y}) = -P_2\,\Delta A_2\,\hat{y} = -(P + \Delta P)\,\Delta A\,\hat{y}$

(iii) $\vec{W} = w(-\hat{y}) = -\,\Delta A.dy.\rho(y).g(y).\,\hat{y}$

Where 'ρ' is the density of the liquid.

Under the action of these forces the liquid is accelerating. Therefore

$$\vec{F}_1 + \vec{F}_2 + \vec{W} = m\vec{a} = \Delta A.dy.\rho(y)\,\vec{a}$$

$$\Rightarrow \quad P\Delta A\,\hat{y} - (P + dP)\Delta A\hat{y} - \Delta A.dy.\rho(y)g(y)\,\hat{y}$$

$$= \Delta A.dy.\rho(y)\,\vec{a}$$

$$\Rightarrow \quad -dP\,\hat{y} = \rho(y)(g(y)\hat{y} + \vec{a})dy \quad ...(13.2.1)$$

If the elevator is accelerating upwards, then

$$\vec{a} = a\,\hat{y}$$

and eqn. (13.12.1) reduces to

$$dp = -\rho(y)(g(y)+a)dy \qquad ...(13.12.2)$$

On the otherhand if the elevator is accelerating downwards, than $\vec{a} = -a\,\hat{y}$ and eqn. (13.12.1) gives

$$dp = -\rho(y)(g(y)-a)dy \qquad ...(13.12.3)$$

(b) Buoyant force :

As discussed in Sec. 13.11. consider an arbitrary volume V of liquid, being bounded by surface S.

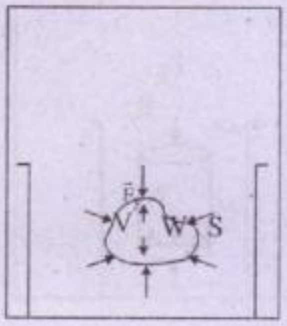


Fig. 13.18

So $\quad \Delta\vec{F}_1 + \Delta\vec{F}_2 .... + \Delta\vec{F}_n + \vec{W} = m\vec{a}$

As there is no horizontal motion

$$\Rightarrow \quad \Delta\vec{F}_{1x} + \Delta\vec{F}_{2x} .... = 0$$

and

$$\Delta\vec{F}_{1y} + \Delta\vec{F}_{2y} + ... + \Delta\vec{F}_{ny} + m\vec{g} = m\vec{a}$$

$$\Rightarrow \quad \vec{F}_y = -m\vec{g} + m\vec{a} \qquad ...(13.12.14)$$

Since $\vec{F}_y = F_y\hat{y}, \quad m\vec{g} = -mg\hat{y}$

So $\quad F_y\hat{y} = mg\hat{y} + m\vec{a} = m(g\hat{y} + \vec{a}) \quad ...(13.12.5)$

(i) If the elevator is moving upwards, then $\vec{a} = a\hat{y}$ and

$$F_y = m(g+a) \qquad ...(13.12.6)$$

(ii) If the elevator is moving down, then $\vec{a} = -a\hat{y}$ and

$$F_y = m(g-a) \qquad ...(13.12.7)$$

B. <u>Free surface of a liquid in horizontal acceleration</u> :

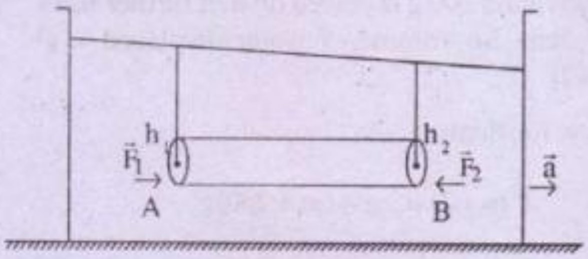


Fig. 13.19

Consider a liquid placed in a beaker let A and B be two points in the liquid on the same horizontal line AB and separated by a distance x. Imagine a liquid cylinder with face areas ΔA, containing A and B.

The forces acting on the liquid cylinder along the horizontal direction are

(i) $\vec{F}_1 = P_1.\Delta A\,\hat{x}$, towards right on the face A due to liquid, lying to its left.

(ii) $\vec{F}_2 = -P_2\Delta A\,\hat{x}$, towards left on the face B, due to liquid, lying to its right.

Therefore

$$\vec{F}_1 + \vec{F}_2 = m\vec{a}$$

$$\Rightarrow \quad P_1\Delta A\hat{x} - P_2\Delta A\,\hat{x} = \Delta A.x.\rho.\vec{a}$$

$$\Rightarrow \quad (P_1 - P_2)\,\hat{x} = x\rho\vec{a} \qquad ...(13.12.8)$$

(i) If the liquid is on a frame accelerating in $\hat{X}$ - dirn, then $\vec{a} = a\hat{x}$ and

$$P_1 - P_2 = x\rho a \qquad ...(13.12.9)$$

(ii) If the liquid is on a frame accelerating in -ve x - dirn, then $\vec{a} = -a\hat{x}$ and

$$P_1 - P_2 = -x\rho a \qquad ...(13.13.10)$$

Eqns. (13.12.9) & (13.12.10) imply that $P_1 \neq P_2$

As there is no vertical acceleration

$$P_1 = P_a + h_1 \rho g$$

$$P_2 = P_a + h_2 \rho g$$

Therefore

$$P_1 \sim P_2 = (h_1 \sim h_2)\rho g$$

$$\Rightarrow \quad \frac{h_1 \sim h_2}{x} = \frac{a}{g} = \tan\theta \qquad ...(13.12.11)$$

where 'θ' is the inclination of the free surface with the horizontal line.

C. Principle of Hydrometer :

Let 'm' be mass of hydrometer.

Then $mg = V.\ \rho_\ell .g$ ...(13.12.12)

where 'V' is volume of liquid desplaced

ρ_ℓ is density of liquid

Eqn. (1) gives

$$V = \frac{m}{\rho_\ell} \qquad ...(13.12.13)$$

Let 'ℓ_w' be length of stem immersed when immersed in water, then

$$V_0 + a.\ell_w = V_w = \frac{m}{\rho_w} \qquad ...(13.12.14)$$

Let 'ℓ_s' be the length of stem immersed, when immersed in staandard liquid, then

$$V_0 + a.\ell_s = V_s = \frac{m}{\rho_s} \qquad ...(14.12.15)$$

From (13.12.14) & (13.12.15)

$$a(\ell_w - \ell_s) = m\left(\frac{1}{\rho_w} - \frac{1}{\rho_s}\right)$$

$$\Rightarrow \quad m = \frac{a(\ell_w - \ell_s).\rho_w \rho_s}{\rho_s - \rho_w} \qquad ...(13.12.16)$$

Now using (13.12.16) in (13.12.14)

$$V_0 = \frac{m}{\rho_w} - a\ell_w = \frac{a(\ell_w - \ell_s).\rho_s}{\rho_s - \rho_w} - a\ell_w$$

$$= \frac{a}{\rho_s - \rho_w}[(\ell_w - \ell_s)\rho_s - \ell_w(\rho_s - \rho_w)]$$

$$V_0 = \frac{a(\ell_w \rho_w - \ell_s \rho_s)}{\rho_s - \rho_w} \qquad ...(13.12.17)$$

For any other liquid let length immersed be 'ℓ'. Then

$$V_0 + a\ell = \frac{m}{\rho}$$

$$\Rightarrow \quad \frac{a(\ell_w \rho_w - \ell_s \rho_s)}{\rho_s - \rho_w} + a\ell = \frac{a(\ell_w - \ell_s)}{\rho_s - \rho_w} \times \frac{\rho_w \rho_s}{\rho}$$

$$\Rightarrow \quad \frac{\ell_w \rho_w - \ell_s \rho_s}{\rho_s - \rho_w} + \ell = \frac{\rho_w \rho_s}{\rho(\rho_s - \rho_w)}(\ell_w - \ell_s)$$

$$\Rightarrow \quad \frac{\ell_w \rho_w - \ell_s \rho_s}{\rho_s - \rho_w} \times \frac{\rho_s - \rho_w}{\rho_s \rho_w(\ell_w - \ell_s)} + \ell \frac{\rho_s - \rho_w}{\rho_w \rho_s(\ell_w - \ell_s)} = \frac{1}{\rho}$$

$$\Rightarrow \quad \frac{\ell_w \rho_w - \ell_s \rho_s}{\rho_w \rho_s(\ell_w - \ell_s)} + \frac{\rho_s - \rho_w}{\rho_w \rho_s(\ell_w - \ell_s)}.\ell = \frac{1}{\rho}$$

...(13.12.18)

Summary

1. Pressure is the normal force per unit area.

2. Pressure due to a liquid column at a depth 'h' is

$$P = P_a + \rho gh$$

where p_a is the atmospheric pressure and ρ is the density of the liquid. P-Pa is called the gauge pressure and P is the absolute pressure.

3. a) Pressure of a liquid is the same at all points having same depth.

b) The shape of the vessel does not affect the pressure.

4. **Archimede's Principle :**

When a body is wholly or partly immersed in a fluid at rest, it appears to lose some weight and this apparent loss of weight is equal to the weight of the fluid displaced by the immersed part of the body.

5. **Floatation of bodies**

A body floats in equilibrium when the net force and net torque acting on the body is zero. This is possible if

a) the weight of the body is equal to the weight of the fluid displaced by the immersed part of the body and

b) weight acting at the centre of gravity (G) of the body and upthrust acting at the centre of buoyancy (B) lie on one line, with G below B.

Physical quantity	Dimensional Formula	SI unit
Pressure (P)	$[ML^{-1}T^{-2}]$	Nm^{-2} or pascal (p_a)

MODEL QUESTIONS

A. MULTIPLE CHOICE TYPE :

1. A liquid can easily change its shape but a solid cannot because
   (i) the density of a liquid is smaller than that of a solid
   (ii) the forces between the molecules is stronger in solid than in liquids
   (iii) the atoms combine to form bigger molecules in solid.
   (iv) the average separation between the molecules is larger in solids.

2. Consider the equaitons

   $$P = \lim_{\Delta A \to 0} \frac{\Delta F}{\Delta A} \text{ and } P_1 - P_2 = \rho g z$$

   In an elevator accelerating upward
   (i) both the equations are valid
   (ii) the first is valid but not the second
   (iii) the second is valid but not the first
   (iv) both are valid.

3. The three vessels shown in fig. 13.Q.1 have same base area. Equal volumes of a liquid are poured in the three vessels. The force on the base will be
   (i) maximum in vessel A
   (ii) maximum in vessel B
   (iii) maximum in vessel C
   (iv) equal in all the vessels.

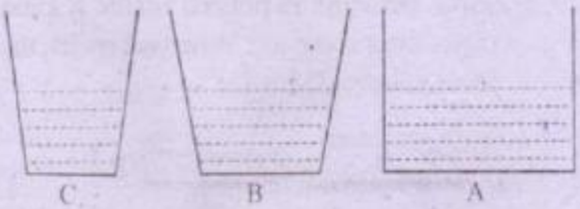


Fig. 13.Q.1

4. Equal mass of three liquids are kept in three identical cylindrical vessels A, B and C. The densities are ρ_A, ρ_B, ρ_C with $\rho_A < \rho_B < \rho_C$. The force on the base will be
   (i) maximum in vessel A
   (ii) maximum in vessel B
   (iii) maximum in vessel C
   (iv) equal in all vessels.

5. A beaker containing a liquid is kept in a big closed jar. If the air inside the jar is continuously pumped out, the pressure in the liquid near the bottom of the liquid will
   (i) increase
   (ii) decrease
   (iii) remain constant
   (iv) first increase and then decrease.

6. The pressure in a liquid at two points in the same horizontal plane are equal. Consider an elevator accelerating upward and a car accelerating on a horizontal road. The above statement is correct in
   (i) the car only
   (ii) the elevator only
   (iii) both of them
   (iv) neither of them

7. Suppose the pressure at the surface of mercury in a barometer tube is P_1 and the pressure at the surface of mercury in the cup is P_2.
   (i) $P_1 = 0$, P_2 = atmospheric pressure
   (ii) P_1 = atmospheric pressure, $P_2 = 0$
   (iii) $P_1 = P_2$ = atmospheric pressure
   (iv) $P_1 = P_2 = 0$

8. A barometer kept in an elevator accelerating upward reads 76 cm. The air pressure in the elevator is
   (i) 76 cm (ii) < 76 cm
   (iii) > 76 cm (iv) zero

9. A 20 N metal block is suspended by a spring balance. A beaker containing some water is placed on a weighing machine which reads 40 N. The spring balance is now lowered so that the block gets immersed in water. The spring balance now reads 16 N. The reading of the weighing machine will be

(i) 36 N (ii) 60 N
(iii) 44 N (iv) 56 N.

10. A metal cube is placed in an empty vessel. When water is filled in the vessel, so that the cube is completely immersed in the water, the force on the bottom of the vessel in contact with the cube

(i) will increase
(ii) will decrease
(iii) will remain the same
(iv) will become zero.

11. A solid floats in a liquid in a partially dipped position.

(i) the solid exerts a force equal to its weight on the liquid
(ii) the liquid exerts a force of buoyancy on the solid which is equal to the weight of the solid
(iii) the weight of the displaced liquid equals the weight of the solid
(iv) the weight of the dipped part of the solid is equal to the weight of the displaced liquid.

12. Two solids A and B float in water. 'A' floats with half of its volume immersed and 'B' floats with 2/3 of its volume immersed. Compare the densities of A and B

(i) 4 : 3 (ii) 2 : 3
(iii) 3 : 4 (iv) 1 : 3

13. A cork is pushed into water kept in a jar inside a satellite rotating around earth. What will happen to the cork when immersed

(i) slowly rise up
(ii) quickly rise up
(iii) remain stable
(iv) stick to the wall of the jar.

14. How much length of mercury column could exert pressure equal to that of 136 cm of water column of same cross - section

(i) 1 cm (ii) 10 cm
(iii) 136 cm (iv) 100 cm

15. A piece of ice is floating in a glass of water and another in a glass of milk. When ice melts the level (s)

(i) in the glasses will rise
(ii) of milk will rise and that of water will fall
(iii) of milk will rise and that of water will remain unchanged
(iv) in two glasses will fall.

16. Two identical beakers A and B are taken. A is completely filled with water and B filled with water but it has a piece of cork floating in water. The beaker

(i) B will weigh more than A
(ii) A will weigh more than B
(iii) The difference in weight is equal to the weight of cork
(iv) Both will have same weight

17. Some kerosine is poured inside a glass containing some ice. When ice melts, the level of kerosine will

(i) Increase
(ii) remain unchanged
(iii) decrease
(iv) first decrease and then increase

18. It is easier to swim in sea water than in river because
    (i) sea has large quantity of water
    (ii) sea water has waves
    (iii) see water pushed the man to the shore quickly
    (iv) sea water is more dense than river water.

19. A balloon filled with air is weighed so that it just floats in water. It is just pushed down to certain depth and then released. The balloon will
    (i) come up again slowly to its former position
    (ii) remain in the position it is left
    (iii) sink to the bottom
    (iv) emerge out of the liquid quickly.

20. A block of wood weighs 4 N in air and 3 N when immersed in a liquid. The buoyant force is
    (i) 1 N (ii) 0 N
    (iii) 3 N (iv) 4 N

21. A block of steel has size 5 cm x 5cm x 5cm. It is weighed in water. If the relative density of steel be 7, its apparent weight shall be
    (i) 6 x 5 x 5 x 5 g
    (ii) 4 x 4 x 4 x 7 g
    (iii) 5 x 5 x 5 x 7 g
    (iv) 4 x 4 x 4 x 6 g

22. The density of ice is 920 kg/m^3. Density of water is 1030 kg/m^3. Percentage of volume of an ice berg submerged is
    (i) 9.20 (ii) 10.3
    (iii) 89 (iv) 19.3

B. VERY SHORT ANSWER TYPE :

1. If density of water is 1 g/cm^3 in C.G.S. system. Express it in S.I. system.
2. Define specific gravity.
3. Write the expression for liquid pressure.
4. State Archimede's principle.
5. A block of ice is floating in a liquid of specific gravity 2 contained in a beaker, when ice melts completely what happens to the liquid level.
6. A man is sitting in a boat which is floating in a pond. If the man drinks some water from the pond, will the level of water in the pond change ?
7. A boat with iron scrap is floating in a lake. If the iron is thrown into the lake what will happen to water level of the lake.
8. Which is heavier, one kg of iron or one kg of cotton ?
9. A piece of ice is floating in water contained in a vessel. Inside ice there is a bubble of air. Will the level of water in the vessel change when the ice melts ?
10. A submarine is 50m below the surface of the ocean, with its weight exactly balanced by the buoyant force. If it descends to 100m, does the buoyant force increase, decrease or remain the same ?
11. Define torr.
12. Define bar.
13. What is the practical unit of pressure in meteorological science ?
14. Give the factors affecting atmospheric pressure.
15. What does the fall in barometer reading indicate ?
16. Why do fluids exert pressure on the wall of containing vessel ?
17. What happens to barometer reading if a drop of water is introduced in a mercury barometer tube ?
18. What is buoyancy ?

C. SHORT ANSWER TYPE :

1. Explain why pressure is treated as a scalar quantity, even though it is defined as normal force per unit area.
2. Explain why the blood pressure in a human body is greater near the feet than at the brain .
3. Explain why the force required by a man to move his limbs in water is less than the force for the same movement in air.
4. What is the effect of temperature on specific gravity ?
5. Explain why water cannot be used in a barometer ?
6. Explain why it is easier to swim in a sea water than in ordinary pond water ?
7. Although an averageman experiences a thrust of 2×10^5 N due to atmospheric pressure, why one does not feel it ?
8. Why passengers in an aeroplane are advised to remove ink from their fountain pen ?
9. To empty an oil tin two holes are made, why ?
10. Explain why straws are used to take soft drinks ?
11. A piece of ice floats in a liquid. What will happen when it melts.
12. A block of ice is floating in a liquid of sp.gr. 12 ; contained in a beaker. When ice melts completely what happens to the liquid level ?
13. Twenty persons are sitting in aboat, which is floating in a tank. If each of them drink some water from the tank, will the level of water in the tank change ?
14. A boat with iron scrap is floating in a lake. If the iron is thrown into the tank, what will happen to water level in the lake ?
15. A piece of ice is floating in water, contained in a vessel. Inside ice there is a bubble of air. Will the level of water in the vessel change when ice melts ?

D. UNSOLVED PROBLEMS :

1. A rectangular tank 2m deep, 5m broad and 10m long is filled with water. Calculate the thrust on each sides and on the base (density of water = 1000 kg/m^3, g= 9.8 m/S^2)
2. A cube floating on mercury has 1/4 of its volume submerged. If enough water is added to cover the cube, what fraction of its volume will remain immersed in mercury ?
3. A block of wood weighs 4 kg and has a relative density 0.6. It is to be loaded with lead so that it will float completely immersed. What weight of lead is needed ? (a) If the lead is on top of the block (b) if the lead is attached below the block ? (density of lead = 11.34×10^3 kg/m^3)
4. A spherical iron shell floats almost completely submerged in water. If outer diameter is 0.5 m and the relative density of iron is 8, find the inner diameter.
5. A cylinder of iron floats vertically and fully immersed in a vessel containing water and mercury. Find the ratio of the length of cylinder immersed in water to that in mercury. (Relative density of mercury = 13.6 and that of iron = 7.78)
6. A stem of a common hydrometer is cyclindrical and the highest graduation corresponds to relative density 1.0 and the lowest to 1.3. What relative density corresponds to a point exactly mid-way between the division ?

7. The stem of hydrometer is divided into 100 equal parts; it reads 0 in water and 100 in a liquid of relative density 0.8. Find the relative density of liquid in which it reads 50.

8. An ornament weighing 36 g in air, weighs only 34 g in water. Assuming that some copper is mixed with gold to prepare the ornament. Find the amount of copper in it. Specific gravity of gold is 19.3 and that of copper is 8.9.

9. A cubical block of ice floating in water has to support a metal piece weighing 0.5 kg. What can be the minimum edge of the block so that it does not sink in water ? (sr.gr. of ice = 0.9)

10. A cube of ice floats in water and partly in Kerosene oil. Find the ratio of the volume of ice immersed in water to that in k. oil. (sp.gr. of k-oil is 0.8 and that of ice is 0.9)

11. A cubical block of wood weighing 200 g has a lead piece which will just allow the block to float in water. Specific gravity of wood is 0.8 and that of lead is 11.3.

12. A hollow spherical body of inner and outer radii 6 cm and 8 cm respectively floats half submerged in water. Find the density of the material of the sphere.

E. LONG ANSWER TYPE :

1. What do you mean by pressure ? Derive an expression for pressure exerted by liquid column of height h.

2. (a) State and prove Archimede's principle.

   (b) Show how you can verify it experimentally.

3. What are the conditions of floatation of bodies in static equilibrium ? Explain.

4. Explain the working of a lactometer, giving its necessary principle.

5. Explain with necessary theory, the working of a barometer.

6. What is a manometer ? Explain its working with necessary theory.

F. Fill in the Blanks Type

1. The pressure at all points lying at the same height in a liquid is ....

2. The value of g at a place decreases by 2% the barometric height of mercury at that place ....

3. A block of ice is floating in a liquid of specific gravity 1.2 contained in a beaker. When the ice melts completely the liquid level would ....

4. A piece of cork is embeded inside an ice block which floats in water. When ice melts the level of water will ....

5. A cubical wooden block of density 500 kg/m^3 floats in water in a tank. Oil is poured into the tank until the surface of the oil is in level with the upper surface of the block. The desity of oil is .... x 10^3 kg/m^3.

G. True False Type

1. A piece of ice floats in water. The level of water remains unchanged when the ice melts completely.

2. A plastic bag weigh the same when empty as when filled with air at atmospheric pressure.

3. A man is sitting in a boat which is floating on a pond. If the man drinks same water from the pond, the level of water in the pond decreases.

4. A baloon filled with helium does not rise in air indefinitely but halts after a certain height (neglect wind).

5. Air passengers prefer a ball pen over a fountain pen.

ANSWERS

A. MULTIPLE CHOICE TYPE :

1. (ii), 2. (ii), 3. (iii), 4. (iv), 5. (ii), 6. (ii), 7. (i), 8. (iii),

9. (iii) [Hints : - The upward thrust on the block by the liquid is 4 N. So the block exerts a thrust of 4 N on the wats. Hence weight increases by 4N i.e. weighing machine will record 44 N.]

10. (iii), 11. (i), (ii) & (iii), 12. (iii), 13. (iii), 14. (ii), 15. (iii), 16. (iv), 17. (iii), 18. (iv), 19.(iii), 20. (i), 21. (i), 22. (iii).

B. VERY SHORT ANSWER TYPE :

1. 10^3 kg / m^3
2. See text
3. $P = h.\rho.g$, ρ = density, h = liquid height, g = accln due to gravity.
4. See text
5. Liquid level rises
6. No
7. Water level will go down
8. One kg of iron
9. No change in the level
10. Remain same
11. See text
12. See text
13. See text
14. See text
15. Rain
16. Due to collision of fluid molecules on the walls
17.
18. See text

C. SHORT ANSWER TYPE :

1. See text
2. Height of the blood column is quite large at the feet than at brain.
3. The upthrust of water is more than that of air. Therefore the limb appears to lose more weight in water, hence the force required is less.
4. $\text{sp.gr} = \dfrac{\rho(t^0c)}{\rho_w(4^0c)}$, so as temperature increases $\rho(t)$ decreases, hence sp.gr. decreases.
5. $P_a = h.\rho_\ell.g \Rightarrow h_w = \dfrac{P_a}{g} = 10.34$ m i.e. tube shall ~ 10 m long, which is inconvenient.
6. Density of sea water is more than that of ordinary pond water. Therefore buoyant force is more in case of sea water. Hence it is easier to swim in sea water.

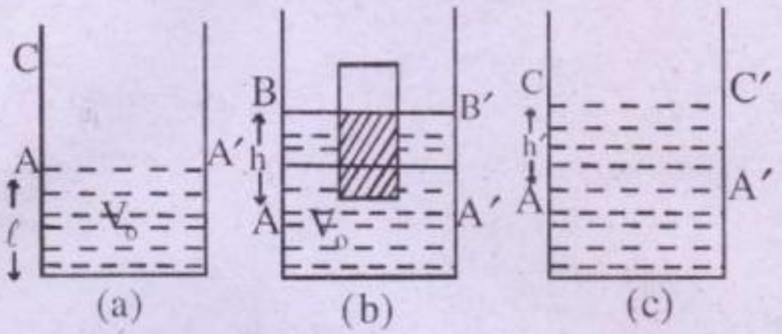
7. In a human body blood circulates under a pressure caused by the regular pumping out of blood due to heart thrombing. Hence external pressure being less than this pressure one does not feel the force due to atmospheric pressure.

8. At high attitudes atmospheric pressure is low. Hence air and ink within the pin kept under normal atmospheric pressure shall try to come out to the low pressure zone.

9. By making two holes and tilting the tin air rushes into the tin through one hole and thus atmospheric pressure now acts on the oil, which forces oil to flow out.

10. While taking soft drinks with a straw one tries to draw out air from the straw, thus causing low pressure within the straw. Hence atmospheric pressure acting on the drink in the bottle forces the liquid into the straw and thus helps in drinking.

11. Let mass of ice be = m, volume of ice $V_i = m/\rho_i$



Let the vol. of liquid be = $V_0 = \ell A$

where 'A' is base area of the container 'C'

Original level of liquid is AA'

When ice block floats level of liquid is BB'

Vol. of liquid desplaced V = Vol. of ice immersed

$$= \alpha V_i = \alpha m/\rho_i \quad .... (1)$$

For floating

$$V.\rho_\ell.g = V_i.\rho_i.g$$

$$\Rightarrow \quad \frac{V}{V_i} = \frac{\rho_i}{\rho_\ell} = \alpha \quad ...(2)$$

Now $(h+\ell)A - \alpha V_i = V_0$

$$\Rightarrow \quad h = \frac{\alpha V_i}{A} = \frac{\alpha}{A}.\frac{m}{\rho_i} \quad ...(3)$$

Using (2) in (3) for α

$$h = \frac{1}{A}\cdot\frac{m}{\rho_\ell} \quad ...(4)$$

when ice melts completely, the volume arising out of melting is

$$V' = \frac{m}{\rho_w} \quad ...(5)$$

So new volume of liquid is $= V_0 + V'$

Let this level be CC', then

$$(h' + \ell)A = V_0 + V'$$

$$\Rightarrow \quad h'A = V' = \frac{m}{\rho_w}$$

$$\Rightarrow \quad h' = \frac{1}{A}\cdot\frac{m}{\rho_w} \quad ...(6)$$

From (4) and (6)

$$\frac{h}{h'} = \frac{\frac{1}{A}\frac{m}{\rho_\ell}}{\frac{1}{A}\frac{m}{\rho_w}} = \frac{\rho_w}{\rho_\ell}$$

$$\Rightarrow \quad \frac{h'}{h} = \frac{\rho_\ell}{\rho_w} \quad ...(7)$$

Eqn. (7) shows that :

(i) If $\rho_\ell > \rho_w$, then $h' > h$ ie liquid level will rise

(ii) If $\rho_\ell < \rho_w$, then $h' < h$ ie liquid will fall

(iii) If $\rho_\ell = \rho_w$, then $h' = h$ ie level will remain same.

12. Level will rise as $\rho_\ell > \rho_w$.

13. No. On drinking water say m g, the wt. of the boat increases by m g. Hence for floating it will further displace water equivalent to m gs, tending to raise the level.

Original volume $= V_0$

Let the volume drunk be $V = \frac{m}{\rho_w}$

Tending to decrease the level by $h = \frac{m/\rho_w}{A}$, Extra volume of water displaced = V

This tends to increase the level by $h' = \frac{m/\rho_w}{A}$

=> $h = h'$ i.e level will remain unaffected.

14. Water level will go down.

Volume of water displaced when iron scrap is in the boat = V.

Mass of iron scrap = m_1

mass of boat = m_2

for floating $V.\ \rho_w . g = (m_1 + m_2) g$

$$\Rightarrow \quad V = \frac{m_1}{\rho_w} + \frac{m_2}{\rho_w} \qquad ...(1)$$

So vol. of water in the lake appears to be Vo + V i.e rise of water level due to floating.

When iron scrap is thrown into lake, change in water volume is $V_1 = \frac{m_1}{\rho_s}$ ...(2)

Where ρ_s is density of iron scrap.

Let now V' be the water displaced by the boat, without iron scrap. Then for floating

$$V'.\ \rho_w . g = m_2 g$$

$$\Rightarrow \quad V' = \frac{m_2}{\rho_w} \qquad ...(2)$$

Now volume of water in the lake shall appear to be =

$$V_0 + V_1 + V' = V_0 + \frac{m_1}{\rho_s} + \frac{m_2}{\rho_w} \qquad ...(4)$$

Old volume = $V_0 + V = V_0 + \frac{m_1}{\rho_w} + \frac{m_2}{\rho_w}$ ...(5)

Thus change in volume = Old vol – New vol.

$$\Rightarrow \quad \Delta V = (V_0 + V) - (V_0 + V_1 + V')$$

$$= \frac{m_1}{\rho_w} - \frac{m_1}{\rho_s} = m_1\left(\frac{1}{\rho_w} - \frac{1}{\rho_s}\right) > 0$$

=> New volume < Old volume

Hence water level will fall.

15. No change in the level.

Let 'm' be mass of ice

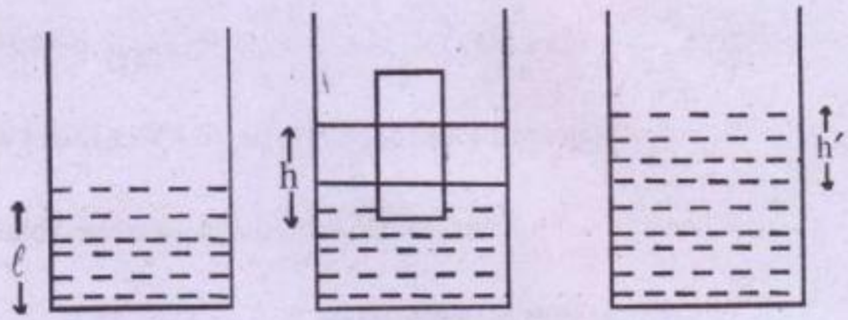
Let 'V_0' be original volume of liquid = ℓ A

V_1 be water displaced by ice.

For floating $V_1.\rho_\ell.g = mg = V_i.\rho_i.g + V_a.\rho_a.g$

Where V_i is actual vol. of ice and V_a is actual volume of air in the ice.

$$\therefore \quad V_1 = \frac{m}{\rho_\ell} = \frac{V_i\rho_i}{\rho_\ell} + \frac{V_a\rho_a}{\rho_\ell} = \alpha(V_i + V_a) = \frac{m}{\rho_\ell} + \frac{m_a}{\rho_\ell}$$



Now $V_0 = (h + \ell)A - \alpha(V_i + V_a)$

$$\Rightarrow \quad hA = \alpha(V_i + V_a) = \frac{V_i\rho_i}{\rho_\ell} + \frac{V_a\rho_a}{\rho_\ell}$$

$$\Rightarrow \quad h = \frac{\alpha(V_i + V_a)}{A} = \frac{1}{A}\left(\frac{V_i\rho_i}{\rho_\ell} + \frac{V_a\rho_a}{\rho_\ell}\right) = \frac{1}{A}\left(\frac{m}{\rho_\ell} + \frac{m_a}{\rho_\ell}\right)$$

When ice melts new volume of liquid is

$$V' = V_0 + V_i' = V_0 + \frac{m}{\rho_w}$$

$$\Rightarrow \quad (h' + \ell)\,A = V_0 + \frac{m}{\rho_w}$$

$$\Rightarrow \quad h'\,A = \frac{m}{\rho_w}$$

$$\Rightarrow \quad h' = \frac{m}{A\rho_w}$$

$$\therefore \quad \frac{h'}{h} = \frac{m/A\rho_w}{\frac{1}{A}\left(\frac{m}{\rho_\ell}+\frac{m_a}{\rho_\ell}\right)} = \frac{m\rho_\ell}{(m+m_a)\rho_w}$$

$$\frac{h'}{h} = \frac{\rho_\ell}{\rho_w}\left(\frac{1}{1+\frac{m_a}{m}}\right) \simeq \frac{\rho_\ell}{\rho_w}$$

$(\because m_a/m \simeq 0)$

(i) So if the liquid is water then $\rho_\ell = \rho_w$ and h' = h implying no change of level.

(ii) If $\rho_\ell > \rho_w$ h' > h ie level will rise

(iii) If $\rho_\ell < \rho_w$ h' < h ie level will fall

D. UNSOLVED PROBLEMS :

1. 9.8×10^4 N, 19.6×10^4 N, 9.8×10^5 N

[Hints : - Average pressure ($\equiv$ Pressure at the centre)

$= 1 \text{ m} \times 10^3 \text{ kg/m}^3 \times 9.8 \text{ m/s}^2 = 9.8 \times 10^3 \text{ N/m}^2$.

Thrust on sides (i) $9.8 \times 10^3 \times (2\text{x}5) = 9.8 \times 10^4$ N

(ii) $9.8 \times 10^3 \times (2 \times 10) = 19.6 \times 10^4$ N

Thrust on bottom Pressure on the bottom x (5×10) m^2

$= (10^3 \text{ kg/m}^3 \times 9.8 \text{ m/s}^2 \times 2 \text{ m}) \times (5 \times 10) \text{ m}^2 = 9.8 \times 10^5$ N]

2. 0.19

[Hints : $\frac{V}{4} \cdot \rho_M \cdot g = W$

$\alpha V \rho_w g + (1-\alpha)V\rho_M g = W = \frac{V}{4}\rho_M g$

$\Rightarrow \quad \alpha = 0.809, \; 1-\alpha = 0.1904$]

3. (a) 2.667 kg. (b) 2.92 kg
4. 0.478 m
5. 54 : 46
6. 1.13

[Hints : m g= $V.\rho_\ell.g \quad \Rightarrow V = \frac{m}{\rho_\ell}$

$V_w = \frac{m}{\rho_w} = V_0 + a\ell_0 \;; \quad V_l = \frac{m}{\rho_l} = V_0 + a\ell_1 \quad (\rho_{lr} = 13)$

$$V_2 = \frac{m}{\rho_2} = V_0 + a(\ell_0 + \ell_1)/2 = \frac{1}{2}\left(\frac{m}{\rho_w} + \frac{m}{\rho_1}\right)$$

$$\Rightarrow \quad \frac{1}{\rho_2} = \frac{1}{2}\left(\frac{1}{\rho_w} + \frac{1}{\rho_1}\right) \Rightarrow \rho_{2r} = 1.13 \quad]$$

7. 0.89

8. 2.225 g

   [Hints : $\alpha V\rho_a + (1-\alpha)V\rho_c = 36$

   loss of wt. = V. ρ_w.g = 2 g wt. => V = 2 cm^3

   => $\alpha = 0.875$, $m_c = (1-\alpha)$. 2. (8.9) = 2.225 g]

9. 17.1 cm

10. 1 : 1

11. 54.85 g

12. 0.865 g/cm^3

F. (1) equal (2) remaiin unchanged (3) rise (4) remain unchanged (5) 0.6

G. (1) True (2) True (3) False (4) True (5) True.

14

Surface Tension

Introduction :

For study of surface phenomena of liquids one needs some knowledge about (a) adhesive force (b) cohesive force (c) molecular range (d) sphere of influence (e) surface film etc. So we give brief notes on the above terms.

The inter-molecular force between two similar molecules is called **cohesive force** and the molecular force between two dissimilar molecules is called **adhesive force.**

The maximum distance between two molecules so that the molecular force between them still remains attractive is called **molecular range** (r_0). It is of the order of 10^{-7} Cm.

The sphere drawn with a molecule as its centre and radius equal to the molecular range of that molecule, is called **sphere of influence** of the molecule at the centre. Any other molecule lying within the sphere of influence experiences a force of attraction due to the molecule at the centre.

The portion of liquid lying between the free surface and a parallel layer at a depth r_0 (= molecular range) below is called **surface film.**

14.1 Surface phenomena :

The free surface of a liquid exhibit several features, which we notice in our day to day life. e.g.

[i] It takes different shapes under different conditions - (some times concave upwards, some times convex upwards, sometimes horizontal etc.)

[ii] A greased steel needle, whose density is much higher compared to that of water, so that buoyant force of water cannot support its weights, floats on still water when gently placed.

[iii] A spring supporting a bent wire gets more stretched when the wire is dipped into a soap solution and raised. The stretching of the spring is proportional to the length of the horizontal portion of the bent wire, to which the spring is attached. (see fig 14.1)

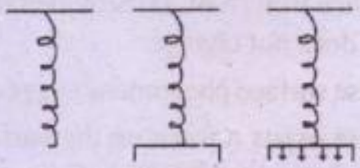


fig 14.1

[iv] A zig-zag thread loop, kept gently on a soap film clinging to a wire gauge takes a circular loop, when the soap film within the thread-loop is pierced. (See fig. 14.2)

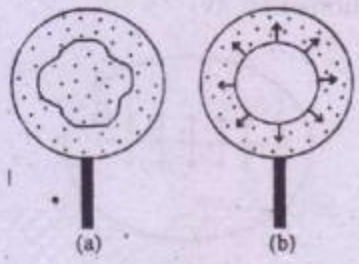


fig. 14.2

[v] Small mercury drop, rain drop, assume spherical shape.

[vi] When a U-shaped wire ABCD with a sliding wire AB, on arms AD and BC (see fig. 14.3) is dipped into a soap solution (so that a soap film clings to it) and brought out, the sliding wire AB, moves upwards.

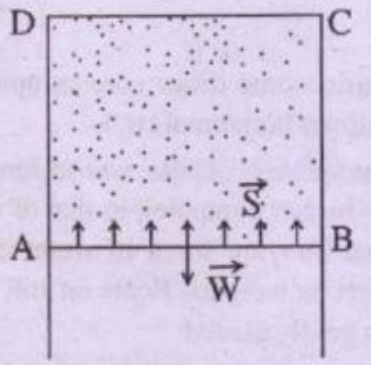


fig. 14.3

This sliding is resisted by loading a weight W on the sliding wire. It is also seen that the same weight W prevents sliding irrespective of the area of the surface film (ABCD) provided length of AB does not change.

These surface phenomena suggest that :

(a) There exists a force on the surface of a liquid (i.e. liquid surface is in a state of stress) such that it exerts a force perpendicular to any line on the surface and acts tangentially to the surface. If liquid surface lies on both sides of the line the force exists on either side. (e.g. phenomenon -iv)

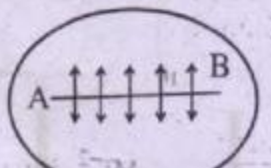


fig. 14.4

(b) This force is perpendicular to the length of the line. (e.g. phenomenon -iii)

(c) The free surface tends to assume minimum surface area (e.g. phenomenon V).

The above observations lead one to define "Surface tension is the magnitude of force per unit length of an imaginary line, drawn on the liquid surface, acting perpendicularly to the line and tangentially to the liquid surface."

Thus $S = \frac{F}{\ell}$ ...(14.1.1)

Where, F is the magnitude of force acting across the line of length ℓ. (on either side)

Unit : It is measured in

i) S. I. Unit - N/m

ii) C.G.S. Unit - dyne / cm

Dimension :

$$[S] = \frac{[F]}{[\ell]} = \frac{MLT^{-2}}{L} = MT^{-2}$$

Factors affecting Surface Tension :

(i) Surface tension of a liquid depends upon the nature of the media in contact with its free surface.

(ii) Surface tension decreases with increase of temperature. This happens because the molecular force decreases and liquid molecules move faster.

(iii) Presence of impurities changes surface tension.

In-organic impurities normally increases surface tension while organic impurity decreases surface tension. (e.g. water with NaCl has S.T. = 84 dyne/Cm. while ordinary water has S.T.= 72 dyne/Cm). Water with detergent lowers surface tension (S.T.)

Ex. 14.1.1 A wire, 10 cm long, is placed horizontally on the surface of water and is gently pulled up with a force of 1.456 X 10^{-2} N to keep the wire in equilibrium. What is the surface tension of water ?

Soln. The film has two surfaces

$$\therefore S = \frac{F}{2\ell} = \frac{1.456\times10^{-2}N}{2\times10\times10^{-2}m}$$

$$= 0.0728 \text{ N/m}$$

Ex. 14.1.2 A ring of 3 cm diameter is dipped and pulled out slowly in the vertically upward direction. If a force of 0.1 N is needed to break the film, calculate the surface tension of the liquid.

Soln. The film has two surfaces

$$\therefore \quad S = \frac{F}{2\times(\pi d)} = \frac{0.1 \text{ N}}{2\times\pi\times 3\times10^{-2}}$$

$$= 0.53 \text{ N/m}$$

14.2 Molecular theory of surface tension :

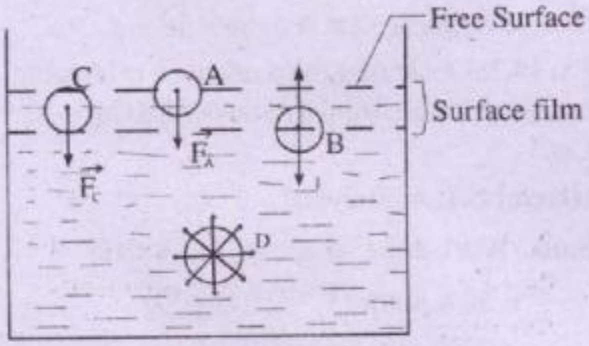


fig. 14.5

The following discussion shall give an idea about the basic cause of surface tension.

Consider a liquid in equilibrium in a vessel. Let A,B,C and D be four molecules (as shown in Fig. 14.5) drawn with their sphere of influence. Molecule D, lying in the interior of the liquid is attracted uniformly from all sides. Hence molecule D experiences no net force. Molecule C, lying within the surface film, experiences a net downward force $(\vec{F}_c)$ as there are less number of molecules in the upper hemisphere than in the lower hemisphere. Molecule B, lying just below the surface film, also experiences no net force as it contains equal numbers of molecules in lower and upper hemisphere. Molecule A, lying on the surface experiences a maximum downward force (F_A), as the upper hemisphere contains no liquid molecules and lower hemisphere contains only liquid molecules.

Thus liquid molecules lying within the surface film experience a net downward force due to inter-molecular forces, this being maximum for molecules lying on the surface. This downward force on surface-molecules tends to compress the surface layer; and thus the liquid surface behaves as a stretched membrane. So work has to be done to stretch the free surface in order to accommodate molecules brought from within the liquid on to the surface, against downward force. This work increases the potential energy of the surface molecules. Since every system, in equilibrium, tries to remain in minimum energy state, so the liquid surface tends to take a shape, so that its surface area is minimum.

14.3 Surface energy :

Potential energy stored in a surface is called (free) surface energy. The free surface energy stored per unit area of the surface is called free surface energy density (u_S).

If under isothermal conditions dW work is required to stretch the surface area by dA amount, then free surface energy density is given by

$$u_S = \frac{dW}{dA} \qquad ...(14.3.1)$$

Thus eqn. (14.3.1) reveals that free surface energy density is defined as the work required to increase the surface area by one unit, under isothermal conditions.

It is a scalar quantity. Its dimension is $[u_s] = \frac{ML^2T^{-2}}{L^2} = MT^{-2}$, same as that of surface tension. It is expressed in erg/Cm^2 in C.G.S. system and joul/m^2 in S.I. units.

Relation between Surface tension & Surface energy :

Consider a U-shaped wire ABCD (fig. 14.6), with AB, capable of sliding on arms AD and BC. The system is dipped into soap solution and raised gently, so that a soap film is formed

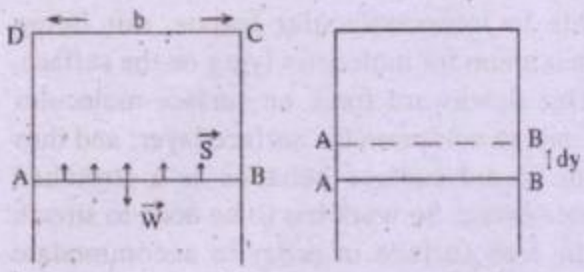


fig. 14.6

Within the region ABCD. The sliding wire begins to move up (towards CD). To keep AB in position, a suitable load W is applied (see fig. 14.6). It will be seen that the same load prevents the motion of AB, irrespective of the area of surface film ABCD, provided length of AB is kept fixed and temperature is maintained constant.

Thus under equilibrium condition

$$W = S(2\ell) = 2s\ell \quad ...(14.3.2)$$

[Since the soap film has two surfaces, so force on the sliding wire is $S\ell + S\ell = S(2\ell)$].

Now if the sliding wire is pulled slowly through a distance $\vec{dy}$, then work done against surface tension is

$$dW = \vec{S}.\vec{dy} = \vec{W}.\vec{dy} = W\,dy = (2s\ell)\,dy$$

$$\Rightarrow dW = S(2\ell\,dy) = SdA \quad ...(14.3.3.)$$

[Where $dA = \ell dy + \ell dy$, as there are two surfaces of the soap film). Therefore eqn. (14.3.3) gives

$$S = \frac{dW}{dA} = u_S \quad ...(14.3.4)$$

Thus surface tension and surface energy density are numerically equal under isothermal conditions.

Ex. 14.3.1 Calculate the work done in blowing a soap bubble of radius 5 mm (S.T. of soap solution = 28 dyne/cm)

Soln. Soap bubble has two surfaces

Hence work done = energy stored

$$W = u_S . 2(4\pi r^2) = 5.8\,\pi r^2$$

$$= 28 \times 8 \times \pi \times (5 \times 10^{-1})^2 \text{ erg}$$

$$W = 175.93 \text{ erg}$$

Ex. 14.3.2 Calculate the work done in blowing up a soap bubble from an initial surface area of 0.5 Cm^2 to a final surface area of 1.10 Cm^2. Given S.T. of soap solution = 30 dyne / Cm.

Soln. Increase in surface area

$$= (1.1 - 0.5)\ Cm^2 = 0.6\ Cm^2$$

Work done

$$= S.\ dA = 30 \times 0.6 \text{ erg} = 18 \text{ erg.}$$

Ex. 14.3.3 Calculate the work done in blowing up a soap bubble from a radius of 2.0 Cm to 3.0 Cm.

Given : S.T. = 30 dyne/Cm

Soln. Work done $= S\ .\ 8\pi\ (r_2^2 - r_1^2)$

$$= 30 \times 8 \times \pi\,(3^2 - 2^2) \text{ erg}$$

$$= 3769\ .\ 91 \text{ erg}$$

Ex. 14.3.4 A water drop of radius 1 Cm is broken into 1000 equal droplets. Calculate the gain in surface energy (S.T. of water = 0.075 N/m).

Soln. The volume of the water drop

$$V_o = \frac{4}{3}\pi r_0^3$$

If 'r' be radius of the droplets then

$$V_o = \frac{4}{3}\pi r_0^3 = \frac{4}{3}\pi r^3 \times 1000$$

$$\Rightarrow r = \frac{r_0}{10} \quad ...(1)$$

Surface energy or the drop $E_S^0 = S.4\pi r_0^2$

Surface energy of the droplets

$$E_S = S \times 4\pi r^2 \times 1000$$

$$\Rightarrow E_s = S \times 4\pi\left(\frac{r_0}{10}\right)^2 \times 1000$$

So gain in Surface energy

$$\Delta E_S = S \times 4\pi r_0^2(10-1))$$

$$= S \times 4\pi r_0^2 \times 9$$

$$= 75 \times 4\pi \times 1^2 \times 9 = 8492 \text{ erg}$$

$$= 8.492 \times 10^{-4} \text{ J}$$

Ex. 14.3.5 Calculate the amount of energy evolved when eight droplets of mercury of radius 1 mm each combine into one.

Soln.

Surface energy of each droplet = $S.4\pi r^2$

Volume of each droplet = $\frac{4}{3}\pi r^3$

Volume of eight droplets = $\frac{4}{3}\pi r^3 \times 8$

Volume of the final drop = $\frac{4}{3}\pi R^3$

$$\Rightarrow \frac{4}{3}\pi R^3 = \frac{4}{3}\pi r^3 \times 8$$

$$\Rightarrow R = 2r$$

Surface Energy of 8-droplets

$$= 8 \times S \times 4\pi r^2$$

Surface energy of final drop

$$= S \times 4\pi R^2$$

$$= S \times 4\pi(2r)^2$$

$$= 4 \times S \times 4\pi r^2$$

So energy evolved

$$= 8.S.4\pi r^2 - 4.S.4\pi r^2$$

$$= 4.\ S.\ 4\pi r^2$$

$$= 4 \times 0{\cdot}55\frac{N}{m} \times 4 \times \pi \times (1 \times 10^{-3})^2$$

$$= 2.765\ 10^{-5} \text{ J}$$

Ex. 14.3.6 Two spherical bubbles of radius 3.0 Cm and 4.0 Cm coalesce to form another spherical bubble. Calculate the surface tension of the bubble. The radius of the bubble formed = 4.498 Cm and atmospheric pressure $P_a = 10^5$ N/m^2

Soln. Let r_1 be radius of 1st bubble, and let it contain n_1 moles of gas. Let r_2 be radius of 2nd bubble and let it contain n_2 moles of gas.

Then pressure inside 1st bubble

$$P_1 = P_a + \frac{4S}{r_1}$$

The pressure inside 2nd bubble

$$P_2 = P_a + \frac{4S}{r_2}$$

Now,

$$\therefore \quad P_1V_1 = n_1RT, \quad V_1 = \frac{4}{3}\pi r_1^3$$

$$P_2V_2 = n_2RT, \quad V_2 = \frac{4}{3}\pi r_2^3$$

Let 'r' be radius of final bubble then

$$PV = (n_1+n_2)\ RT, \text{ with } P = P_a + \frac{4S}{r}$$

and $V = \frac{4}{3}\pi r^3$

$$\therefore \quad PV = (n_1+n_2)\ RT = P_1V_1 + P_2V_2$$

$$\Rightarrow \left(P_a + \frac{4S}{r}\right)\frac{4}{3}\pi r^3 = \left(P_a + \frac{4S}{r_1}\right)\frac{4}{3}\pi r_1^3 + \left(P_a + \frac{4S}{r_2}\right)\frac{4}{3}\pi r_2^3$$

$$\left(P_a+\frac{4S}{r}\right)r^3=\left(P_a+\frac{4S}{r_1}\right)r_1^3+\left(P_a+\frac{4S}{r_2}\right)r_2^3$$

$$\Rightarrow \quad P_a\left(r^3-r_1^3-r_2^3\right)=4S\left(r_1^2-r_2^2-r^2\right)$$

$$\Rightarrow \quad S=\frac{r^3-r_1^3-r_2^3}{r_1^2+r_2^2-r^2}\times P_a$$

$$=\frac{\left[(4.498)^3-(3)^3-(4)^3\right]\times10^{-6}}{\left[3^2+4^2-(4.498)^2\right]\times10^{-4}}\,m\times10^5\ N/m^2$$

$S=0.745$ N/m

Ex. 14.3.7 If a number of droplets, each of radius r, coalesce to form a single drop of radius R, show that the rise in temperature will be given by $\frac{3S}{\rho s}\left(\frac{1}{r}-\frac{2}{R}\right)$ where S = Surface tension, ρ = density of liquid, s = specific heat capacity of liquid in joule per kg per kelvin.

Soln. Let 'n' be no of droplets

Then $n\times\frac{4}{3}\pi r^3=\frac{4}{3}\pi R^3$

$$\Rightarrow R=n^{1/3}r$$

Energy released

$$=n\times 4\pi r^2S-4\pi R^2S$$

$$=n\times4\pi r^2S-4\pi S\times n^{2/3}r^2$$

$$\Rightarrow \Delta E=4\pi r^2S\left(n-n^{2/3}\right)$$

So $4\pi r^2S\left(n-n^{2/3}\right)=\left(\frac{4}{3}\pi r^3\times n\right)\rho.s.\Delta\theta$

$$\Rightarrow \quad \Delta\theta=\frac{3S}{\rho s}\left[\frac{1}{nr}\left(n-n^{2/3}\right)\right]$$

$$=\frac{3S}{\rho s}\left[\frac{1}{r}-\frac{1}{n^{1/3}r}\right]$$

$$\therefore \Delta\theta=\frac{3S}{\rho s}\left[\frac{1}{r}-\frac{1}{R}\right] \quad \text{Ans.}$$

14.4 Angle of contact, Shape of liquid film & Wetting

(a) Angle of contact :

When a liquid surface touches a solid surface, the shape of the liquid surface near the contact is generally curved. "The angle between the tangent planes at the liquid surface and solid surface is called the **angle of contact**," for the given pair of solid and liquid. Thus angle of contact depends upon the shape of the liquid surface.

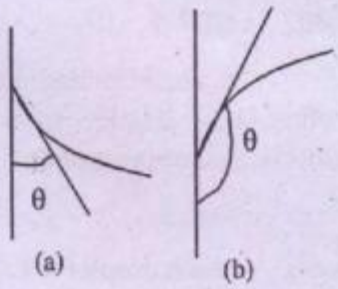


fig. 14.7

(b) Shape of liquid surface :

The shape of liquid surface can be explained on the basis of the basic principle:

"A liquid in equilibrium cannot sustain tangential stress and the resultant force on any small part of the surface must be perpendicular to the surface there."

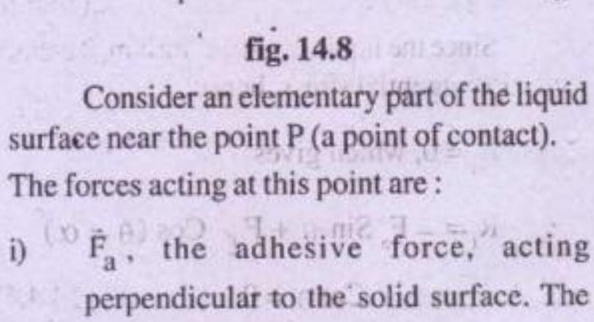


fig. 14.8

Consider an elementary part of the liquid surface near the point P (a point of contact). The forces acting at this point are :

i) $\vec{F}_a$, the adhesive force, acting perpendicular to the solid surface. The magnitude of this force shall depend on the nature of the solid and liquid and their densities.

ii) $\vec{F}_l$, the cohesive force, acting along any direction within the liquid. The magnitude and direction of this force shall depend on the nature of the liquid and its density.

iii) $\vec{W}$, the weight of the liquid, acting vertically downwards.

Therefore resultant force $\vec{R}$ is given by

$\vec{R} = \vec{F}_a + \vec{F}_\ell + \vec{W}$

Resolving the above forces along X-direction and Y-direction (as shown in fig 14.8). We find components of resultant $\vec{R}$ as.

$$R_X = F_a + F_\ell \operatorname{Cos}(90+\alpha) = F_a - F_\ell \operatorname{Sin}\alpha \quad ...(14.4.1)$$

$$R_Y = W + F_\ell \operatorname{Cos}\alpha \quad ...(14.4.2)$$

The resultant $\vec{R}$ of $\vec{F}_a, \vec{w}$ and $\vec{F}_\ell$ makes an angle β with X-direction given by

$$\tan\beta = \frac{R_y}{R_x} = \frac{W + F_\ell \operatorname{Cos}\alpha}{F_a - F_\ell \operatorname{Sin}\alpha} \quad ...(14.4.3)$$

Now resolving the forces along tangential direction $(\hat{e}_t)$ and direction perpendicular to it $(\hat{e}_p)$. (See fig. 14.8), we find

$$R_t = -F_a \sin\theta + F_\ell \operatorname{Cos}(\theta - \alpha) + W \operatorname{Cos}\theta \quad ...(14.4.4)$$

$$R_p = F_a \operatorname{Cos}\theta + F_\ell \operatorname{Sin}(\theta - \alpha) + W \operatorname{Sin}\theta \quad ...(14.4.5)$$

Since the liquid is in equilibrium, it cannot sustain tangential stress, hence

$R_t = 0$, which gives

$$R_t = -F_a \operatorname{Sin}\theta + F_\ell \operatorname{Cos}(\theta - \alpha) + W \operatorname{Cos}\theta = 0 \quad ...(14.4.6)$$

and, $\vec{R} = \vec{R}_p$

Equation (14.4.6) simplifies to -

$$-F_a \operatorname{Sin}\theta + F_\ell \operatorname{Cos}\theta.\operatorname{Cos}\alpha + F_\ell \operatorname{Sin}\theta \operatorname{Sin}\alpha + W\operatorname{Cos}\theta = 0$$

$$\Rightarrow -\operatorname{Sin}\theta(F_a - F_\ell \operatorname{Sin}\alpha) = -\operatorname{Cos}\theta(W + F_\ell \operatorname{Cos}\alpha)$$

$$\Rightarrow \tan\theta = \frac{W + F_\ell \operatorname{Cos}\alpha}{F_a - F_\ell \sin\alpha} \quad ...(14.4.7)$$

Comparing (14.4.3) and (14.4.7) we find that $\theta = \beta$. This implies that the resultant is perpendicular to the liquid surface. We further note that eqn. (14.4.5), on using eqn (14.4.7) gives

$$R_p = F_a \operatorname{Cos}\theta + F_\ell \sin\theta \cdot \operatorname{Cos}\alpha - F_\ell \cdot \operatorname{Cos}\theta \cdot \operatorname{Sin}\alpha + W\operatorname{Sin}\theta$$

$$= \operatorname{Cos}\theta(F_a - F_\ell \operatorname{Sin}\alpha) + \operatorname{Sin}\theta(W + F_\ell \operatorname{Cos}\alpha)$$

$$= \operatorname{Cos}\theta(F_a - F_\ell \sin\alpha) + \operatorname{Sin}\theta(F_a - F_\ell \operatorname{Sin}\alpha)\tan\theta$$

$$= \frac{\operatorname{Cos}^2\theta + \operatorname{Sin}^2\theta}{\operatorname{Cos}\theta}(F_a - F_\ell \operatorname{Sin}\alpha)$$

$$= (F_a - F_\ell \operatorname{Sin}\alpha)\sec\theta$$

$$\Rightarrow R_p = (F_a - F_\ell \operatorname{Sin}\alpha)\sec\theta$$

$$= \sqrt{R_x^2 + R_y^2} = R \quad ...(14.4.8)$$

The above analysis shows that when,

i) $F_a > F_\ell$ (i.e. adhesive force is larger than the cohesive force), so that $F_a > F_\ell \sin\alpha$, eqn. (14.4.7) indicates that $\tan\theta$ is greater than zero. This implies that $0 < \theta < 90$. This in turn implies that resultant $\vec{R}$ lies outside the liquid and the liquid surface must be concave upwards, so that it lies perpendicular to the resultant. The liquid rises near the solid.

As a result of this the liquid spreads over the solid (see fig. 14.9)

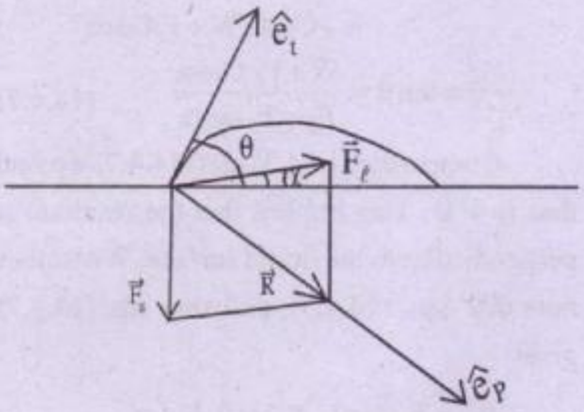


fig. 14.9

(e.g. water spreading on glass surface)

ii) $F_a < F_\ell \sin\alpha$ (which is satisfied when F_ℓ is sufficiently large compared to F_a), $\tan\theta$ is negative. (as given by eqn 14.4.7). This implies that

$180^0 > \theta > 90^0$ and resultant $\vec{R}$ lies within the liquid. This in turn implies that liquid surface must be convex upward; so that it lies perpendicular to the resultant $\vec{R}$. The liquid is depressed near the solid.

As a result of this the liquid does not wet the solid surface.

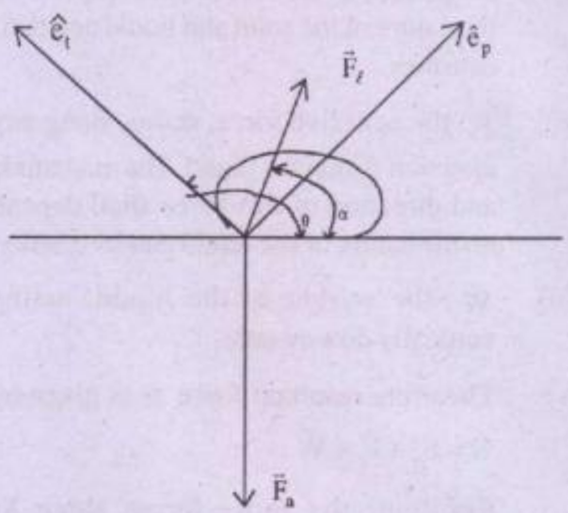


fig 14.10

(e.g. mercury drop on glass plate)

iii) If a solid surface is inserted in a liquid such that it does not project out of the liquid (see fig. 14.11). The adhesive force F_a shall not be perpendicular to the solid.

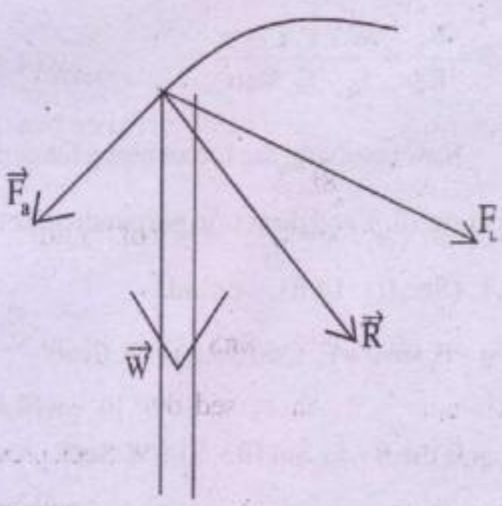


fig. 14.11

The actual angle between solid surface and liquid surface may be different from the angle of contact for the given pair of solid and liquid.

14.5. Excess Pressure : (Pressure difference across a liquid surface)

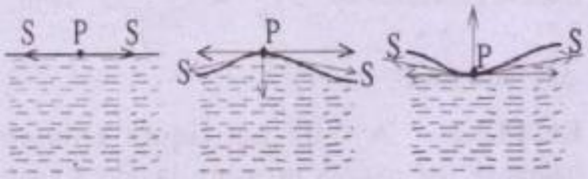


fig. 14.12

Consider any liquid molecule 'P'. One finds that there exists a net force on the concave side. This is the cause of excess pressure on the concave side.

Now we shall find expression for excess pressure in few simple cases.

(a) Liquid drop :

Consider a liquid drop of radius r. If the drop is small, then the effect of gravity may be neglected and the shape may be assumed spherical. Its total surface area A is given as :

$$A = 4\pi r^2 \quad ...(14.5.1)$$

If the radius of the drop increases by dr, isothermally, then the surface area increases by

$$dA = 8\pi r dr \quad ...(14.5.2)$$

The increase in surface energy of liquid drop is then

$$dE_s = u_S dA = SdA = 8\pi S\, rdr \quad ...(14.5.3)$$

This must have increased due to work done against the pressure difference between pressure 'P' inside the drop and pressure 'P_a' outside the drop (which is equal to atmosphre pressure). Therefore

$$dE_s = dW = (P - P_a) \cdot \Delta V \quad ...(14.5.4)$$

Where ΔV = increase in volume = $4\pi r^2 dr$ ...(14.5.5)

So from eqns (14.5.3), (14.5.4) and (14.5.5.) we obtain

$$dE_s = 8\pi srdr = (P - P_a) \cdot 4\pi r^2 dr$$

$$\Rightarrow \Delta P = P - P_a = \frac{2S}{r} \quad ...(14.5.6)$$

giving

$$P = P_a + \frac{2s}{r} \quad ...(14.5.7)$$

Equation (14.5.6) gives 'ΔP' the excess pressure or gauge pressure and eqn (14.5.7) gives the pressure 'P' on the concave side.

(b) Soap bubble :

In case of soap bubble there are two surfaces and there is air inside the bubble. The total surface area is -

$$A = 2 \times 4\pi r^2 \quad ...(14.5.8)$$

If the bubble expands and its radius increases by dr, then increase in its surface area is

$$dA = 16\pi\, rdr \quad ...(14.5.9)$$

So increase in surface energy due to this is

$$dE_s = u_s dA = S.dA = 16\pi\, Srdr \quad ...(14.5.10)$$

This must be equal to the workdone against the excess pressure i.e.

$$dE_s = dW = (P - P_a)\, \Delta V \quad ...(14.5.10)$$

where $\Delta V = 4\pi r^2 dr$ = increase in volume ...(14.5.11)

Using (14.5.11) in (14.5.10) and comparing with (14-5-10) we have

$$dE_s = 16\pi\, Srdr = (P - P_a)\, 4\pi\, r^2 dr$$

$$\Rightarrow \Delta P = P - P_a = \frac{4S}{r} \quad ...(14.5.12)$$

giving

$$P = P_a + \frac{4S}{r} \quad ...(14.5.13)$$

Eqn. (14-5-12) gives excess pressure or gauge pressure and eqn. (14.5.13) gives pressure on the concave side.

Experimental demonstration

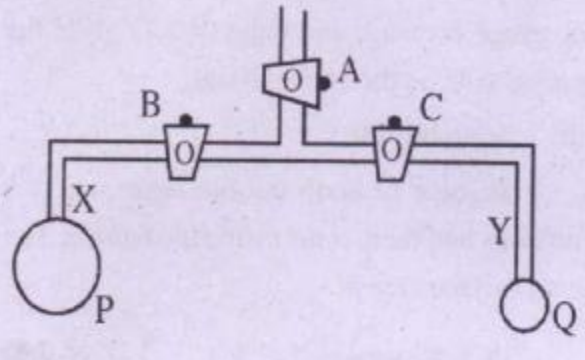


fig. 14.14

Two soap bubbles of different sizes P and Q are made at the two ends x and y respectively (see fig 14.14). The stop cock A is closed and stop-cocks B and C are opened. It is observed that the small bubble 'Q' shrinks and larger bubble 'P' expands. This indicates that air from smaller bubble flows towards the larger bubble, which in turn implies that pressure inside the smaller bubble is more than that inside the larger bubble. This is in accordance with the theoretical inference $\Delta p \propto \frac{1}{r}$.

Ex.14.5.1 Calculate the excess pressure inside a mercury drop of radius 2.0 mm. The surface tension of mercury is 0.464 N/m.

Soln.

Excess pressure $\Delta P = \frac{2S}{r} = \frac{2x0.464}{2x10^{-3}}$ N/m^2

$\Delta P = 464$ N/m^2

Ex. 14.5.2 A 0.02 cm liquid column balances the excess pressure inside a soap bubble of radius 7.5 mm. Determine the density of the liquid S.T of soap solution = 0.03 N/m.

Soln.

Excess pressure inside soap bubble $\Delta P = \frac{4S}{r}$.

Pressure due to liquid column $\Delta P = h\rho g$

Since soap bubble balances the liquid column

$\frac{4S}{r} = h.\rho.g.$

$$\rho = \frac{4S}{r.h.g} = \frac{4 \times 0.03}{7.5 \times 10^{-3}\ 0.02 \times 10^{-2} \times 9.8} \text{ kg/m}^3$$

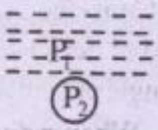
$\rho = 0.082 \times 10^{+5}$ kg/m^3

$= 8.2 \times 10^3$ kg/m^3

Ex. 14.5.3 What would be the pressure inside a spherical air bubble of 0.1 mm radius situated at 1m below the surface of water ? (S.T. of water = 0.07 N/m, atmosphere pressure = 10^5 N/m^2

Soln.

$P_2 - P_1 = \frac{2S}{r}$



$P_1 = P_a + h\rho g$

$P_2 = P_1 + \frac{2S}{r}$

$= P_a + h\rho g + \frac{2S}{r}$

$$= (10^5 + 1\times 10^3 \times 9.8 + \frac{2\times .07}{0.1\times 10^{-3}})\,N/m^2$$

$P_2 = 1.112 \times 10^5\ N/m^2$ = Pressure inside the air bubble.

14.6 Capillarity :

The phenomenon of rise or fall of liquid in a capillary tube (tube of small radius) w.r. to the level of liquid in which the tube is dipped is called as capillarity.

If angle of contact θ is acute ($\theta < 90^0$) then the liquid rises in the tube; while if θ is obtuse ($\theta > 90^0$) the liquid falls inside the tube. This arises due to existence of excess pressure on the concave side of a liquid meniscus, (which is again due to surface tension effect as discussed in sec. 14.5)

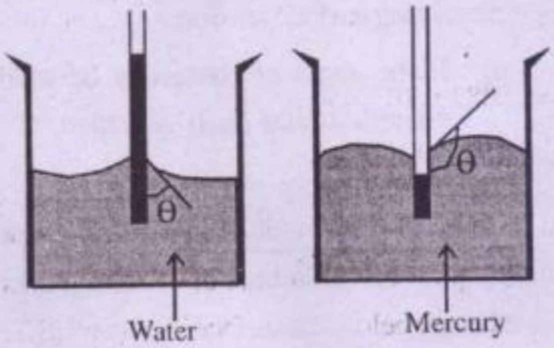


Fig. 14.15

Expression for Capillary-rise

Consider a tube of radius 'r' dipped into a liquid of density 'ρ' and surface tension S. Let the angle of contact between the solid and liquid be θ. When radius of the tube is small, the free surface of the liquid within the tube is nearly speherical.

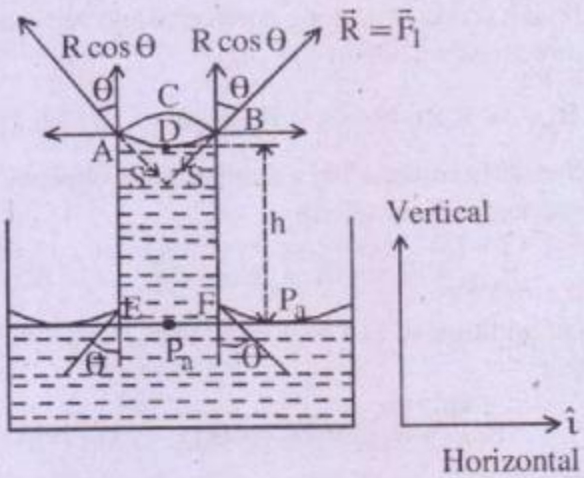


Fig. 14.16

Now let us consider the equilibrium of the part of the liquid raised within the tube (i.e. contained within ADBFE). The forces acting on this liquid are :

i) $\vec{F}_1$, the force exerted by the surface of the tube on the surface ACBD of the liquid.

ii) $\vec{F}_2$, the force exerted by the pressure of air above the surface ACBD. (i.e. due to atmospheric pressure P_a).

iii) $\vec{F}_3$, the force due to upward thrust of the liquid acting at EF.

iv) $\vec{W}$, the weight of the liquid within ADBFE.

The liquid surface meets the tube along the periphery ACBD of length $2\pi r$. If we consider unit length of the periphery, then force due to surface tension is $\vec{S}$, acting along the tangent (see fig. 14.16). Therefore the reaction force of the tube on the liquid is $\vec{R}$, such that

$$\vec{R} = -\vec{S}$$

$$\Rightarrow |\vec{R}| = |\vec{S}| \qquad ...(14.6.1)$$

Now resolving this (reaction) force on the liquid, acting at A along horizontal and vertical directions we obtain

$$\vec{R}_{(A)} = R\sin\theta(-\hat{i}) + R\cos\theta\hat{j}\,. \quad ...(14.6.2)$$

Similarly considering a diametrically opposite element at B we obtain

$$\vec{R}_{(B)} = R\sin\theta\hat{i} + R\cos\theta\,\hat{j} \quad ...(14.6.3)$$

On addition of (14.6.2) and (14.6.3) we find that

$$\vec{R}_{(A)} + \vec{R}_{(B)} = 2R\cos\theta(\hat{j}) \quad ...(14.6.4)$$

The force on the entire periphery ACBD can be calculated by adding forces on diametrically opposite elements; and eqn. (14.6.4) shows that the net force on the liquid by the tube shall be effectively along the (Vertical) $\hat{j}$ - direction.

Therefore

$$\vec{F}_1 = 2\pi r.R\cos\theta\hat{j} = 2\pi r.S.\cos\theta(\hat{j}) \quad ...(14.6.5)$$

The force $\vec{F}_2$ due to atmospheric pressure shall be given by

$$\vec{F}_2 = P_a.\pi r^2(-\hat{j}) \quad ...(14.6.6)$$

The force $\vec{F}_3$, due to upward thrust at EF, is given by

$$\vec{F}_3 = P_a.\pi r^2(\hat{j}) \quad ...(14.6.7)$$

Finally the weight $\vec{W}$ of the raised liquid is given by

$$\vec{W} = \text{Volume of liquid x density x g } (-\hat{j})$$

$$= \left[\pi r^2 h + (\pi r^3 - \frac{2}{3}\pi r^3)\right]\rho.g.(-\hat{j})$$

$$\Rightarrow \vec{W} = \pi r^2\left(h+\frac{r}{3}\right)\rho.g(-\hat{j}) \quad ...(14.6.8)$$

For equilibrium of the raised liquid, we should have

$$\vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \vec{W} = 0 \quad ...(14.6.9)$$

Using eqns. (14.6.5) to (14.6.8) in eqn. (14.6.9) we obtain

$$\left[2\pi r S\cos\theta - P_a.\pi r^2 + P_a\pi r^2 - \pi r^2\left(h+\frac{r}{3}\right)\rho.g\right]\hat{j} = 0$$

$$\Rightarrow 2\pi r\, S\cos\theta = \pi r^2\left(h+\frac{r}{3}\right).\rho.g$$

$$\Rightarrow h = \frac{2S\cos\theta}{r\rho g} - \frac{r}{3} \quad ...(14.6.10)$$

and

$$S = \frac{r\left(h+\frac{r}{3}\right).\rho.g}{2\cos\theta} \quad ...(14.6.11)$$

Equation (14.6.10) gives the capillary rise of a liquid of density ρ, and S.T. S, in a tube of radius r, and eqn. (14.6.11) gives the expression for determining surface tension.

(i) If the angle of contact θ be small enough so that $\cos\theta \approx 1$, then

$$S = \frac{r\left(h+\frac{r}{3}\right)\rho g}{2} \quad ...(14.6.12)$$

and

$$h = \frac{2s}{r\rho g} - \frac{r}{3} \quad ...(14.6.13)$$

(ii) If radius of capillary tube be very small then

$$S = \frac{r.h.\rho.g}{2\cos\theta} \quad ...(14.6.14)$$

and

$$h = \frac{2S\cos\theta}{r\rho g} \quad ...(14.6.15)$$

(iii) If angle of contact θ, as well as capillary tube radius r, are both small, then

$$S = \frac{r.h.\rho.g}{2} \quad ...(14.6.16)$$

and

$$h = \frac{2S}{r.\rho.g} \quad ...(14.6.17)$$

Experimental determination of S.T. :

A capillary tube is fixed on a glass plate with help of fixing wax, and an index pin is also fixed by its side. The glass plate is held by a clamp stand and the height is adjusted so that the index pin just touches the surface of water in the beaker.

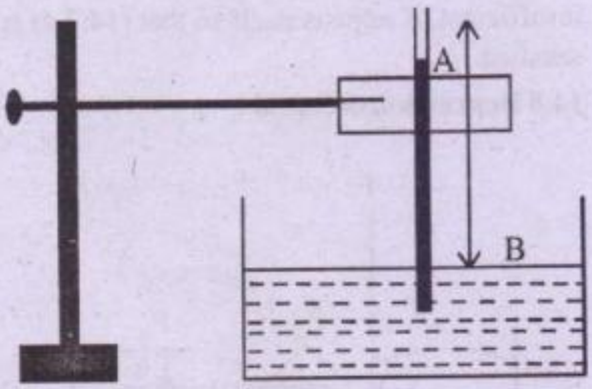


Fig. 14.17

The position A is observed through a travelling microscope and its reading 'L_1' is noted. Then the beaker is removed and lower end B of the index rod is observed through the travelling microscope. Its reading L_2 is noted. The difference $L_2 \sim L_1$ gives the height 'h' of the liquid column in the capillary tube.

Then the radius r of the capillary tube is determined by the travelling microscope. Using the observed values of 'h' and 'r' in eqn. (14.6.12) or (14.6.11) as deemed appropriate, S is determined.

Examples of Capillarity :

i) Blotting paper soaks ink by action of capillarity.

ii) Oil rises in the wick of a lamp by action of capillarity.

iii) Ink flows into the tip of the fountain pen by the action of capillarity.

iv) While constructing a wall we lay a layer of concrete and coal tar at the ground level to stop rise of water from ground by capillary action.

v) After taking bath we soak our body with a porous towel, so that water particles rise through the pores of the towel.

vi) Plants and trees receive water from earth due to capillarity and osmotic action of roots.

vii) We plough land so that distance between two consecutive soil particles increases and capillary action is resisted. This helps in restraining ground water coming to the surface.

Ex. 14.6.1 A capillary tube of radius 0.2 mm is dipped vertically in water. Calculate the height of water column raised in the tube. (S.T. of water = 0.075 N/m, density of water = 10^3 kg/m^3, g= 10m/s^2)

Soln.

$$h = \frac{2S\cos\theta}{r.\rho.g} - \frac{r}{3}$$

For water $\theta \approx 0$

$$\therefore h = \frac{2S}{r\rho g} - \frac{r}{3} = \frac{2x0.075}{0.2x10^{-3}x10^3x10} - \frac{0.2x10^{-3}}{3}$$

$= 0.075m - 0.67 \times 10^{-3} \approx 0.075m$

h = 7.5 cm

Ex. 14.6.2 By how much will the surface of a liquid be depressed in a glass tube of radius 0.2mm if the angle of contact of the liquid is 135^0 and the surface tension is 0.547N/m. Density of liquid is 13.5×10^3 kg/m^3.

Soln.

Neglecting liquid in the meniscus

$$h = \frac{2\,S\,\cos\theta}{r\,\rho\,g}$$

$$= \frac{2 \times 0.547 \times \cos 135}{0.2 \times 10^{-3}\ 13.5 \times 10^3 \times 9.8}\ \text{m}$$

$= 0.029\text{m} = 2.9 \times 10^{-2}\text{m} = 2.9\text{cm}$

14.7 Rise of liquid in a tube of insufficient length :

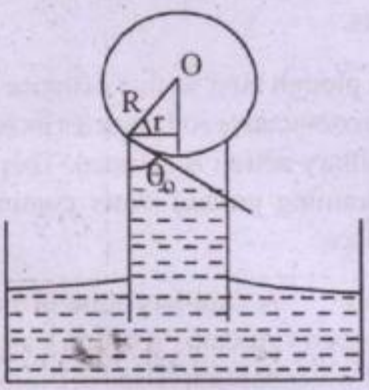


Fig. 14.18

Suppose a capillary tube of radius 'r' when dipped in a certain liquid, the liquid rises to a height h. Now if an identical capillary but of height $\ell < h$, is dipped into the liquid then the liquid does not overflow but adjusts its curvature and takes up a shape such that the tangent to the surface (free surface) makes an angle θ_0 not equal to the angle of contact for the given liquid and capillary tube.

Actually angle θ_0 is such that

$2\pi r\,S\,\cos\theta_0$ = weight of the liquid ...(14.7.1)

with $r = R\ \cos\theta_0$ ...(14.7.2)

where r = radius of the tube

R = radius of curvature of free surface.

From eqn. (14.7.1)

$$2\pi r\,S\,\cos\theta_0 = (\pi r^2 \ell).\rho.g$$

$$\Rightarrow S = \frac{r\,\ell\,\rho\,g}{2\cos\theta_0} \quad ...(14.7.3)$$

Using (14.7.2) on r.h.s of (14.7.3), we obtain

$$S = \frac{R\cos\theta_0.\ell\,\rho\,g}{2\cos\theta_0}$$

$$\Rightarrow R\ell = \frac{2s}{\rho g} = \text{constant} \quad ...(14.7.4)$$

Thus eqn. (14.7.4) shows that when length of the tube is sufficient, 'ℓ' adjusts itself (and assumes the value h); but when 'ℓ' is insufficient, R adjusts itself so that (14.7.4) is satisfied.

14.8 Depression of liquid :

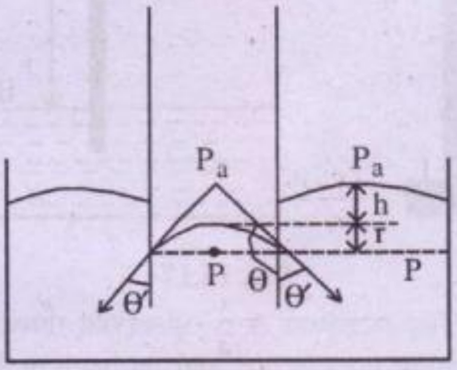


Fig.14.19

We proceed as in sec 14.6. The forces acting are.

i) $\vec{F}_1$, due to S.T. given as

$$\vec{F}_1 = 2\pi r\,S\,\cos\theta'(-\hat{j}) \quad ...(14.8.1)$$

ii) $\vec{F}_2$, due to atmospheric pressure P_a,

given as $\vec{F}_2 = P_a\,(\pi r^2)\,(-\hat{j})$ ...(14.8.2)

iii) $\vec{W}$, due to wt. of the liquid, given as

$$\vec{W} = \frac{2}{3}\pi r^3 . \rho . g(-\hat{j}) \quad ...(14.8.3)$$

iv) $\vec{F}_3$, due to upward thrust, given as

$$\vec{F}_3 = \left[P_a + (h+r)\rho g\right] . \pi r^2 (\hat{j}) \quad ...(14.8.4)$$

For equilibrium

$$\vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \vec{W} = 0$$

$$\Rightarrow 2\pi r \cos\theta' + P_a \pi r^2 + \frac{2}{3}\pi r^3 \rho . g$$

$$= \left[P_a + (h+r)\rho g\right]\pi r^2$$

$$\Rightarrow S = \frac{r.\rho.g\left(h+\frac{r}{3}\right)}{2\cos\theta'} = \frac{r\rho g\left(h+\frac{r}{3}\right)}{-2\cos\theta} \quad ...(14.8.5)$$

($\because$ Angle of contact $\theta = 180 - \theta'$)

Eqn. (14.8.5) shows that the expression remains unaltered (same as obtained in case of rise). This gives depression

$$h = \frac{2s\cos\theta'}{r\rho g} - \frac{r}{3} \quad ...(14.8.6)$$

$$= -\frac{2s\cos\theta}{r\rho g} - \frac{r}{3}$$

Table No. 14.1 (S.T. of Some Liquids)

Liquids	Temp. in 0^0c	S in dyne/cm
Benzene	20	27.6
	50	24.7
Cadmium tetra-chloride	20	26.8
Ethyl alcohol	20	22.3
	50	19.8
Glycerine	20	630
Methyl alcohol	20	22.6
	50	20.1
Mercury	20	465
Soap solution	20	30
Water	10	74.2
	20	72.8
	30	67.9

Table No. 14.2 (Angle of Contact)

Liquid	Solid	Angle of contact in deg.
Water	glass	0
Ethyl alcohol	glass	0
Kerosine	glass	26^0
Mercury	glass	140^0
Methyl iodide	glass	29^0

Summary

1. Surface tension is the magnitude of force per unit length of an imaginary line drawn on the liquid surface acting perpendicular to the line and tangentially to the liquid surface.

$$S = \frac{f}{l}$$

2. Potential energy stored in a surface is called surface energy. The force surface energy stored per unit area of the surface is called free surface energy density.

3. The surface tension and surface energy are numerically equal under isothermal conditions.

4. The angle between the tangent planes at the liquid surface and solid surface is called the angle of contact.

5. a) Excess pressure inside a liquid drop or air bubble.

$$\Delta P = \frac{2s}{r}$$ (s → surface tension
r → radius of the drop)

b) Excess pressure inside a soop bubble.

$$\Delta P = \frac{4s}{r}$$

6. The phenomenon of rise or fall of liquid in a capillary tube with respect to the level of liquid in which the tube is dipped is called as capillarity.

7. If s is the surface tension, r is the radius of the tube and θ is the angle of contact, then

$$S = \frac{r\,(h + \frac{r}{3})\,\rho\, g}{2\cos\theta}$$

a) If radius of capillary tube is very small, then

$$S = \frac{r\,h\,\rho\, g}{2\cos\theta} \quad \text{or} \quad h = \frac{2\,s\cos\theta}{r\,\rho\, g}$$

b) If θ is also very small then $\cos\theta \approx 1$

$$\therefore \quad h = \frac{2\,s}{r\,\rho\, g}$$

Physical quantity	Dimensional Formula	SI unit
1. Surface tension (s)	MT^{-2}	Nm^{-1}
2. Surface energy density	MT^{-2}	$J\,m^{-2}$

MODEL QUESTIONS

A. Multiple Choice Type Questions :

1. A liquid drop breaks into several droplets. Its surface energy
   (a) increases
   (b) decreases
   (c) remains the same
   (d) is unpredictable

2. Soap helps in cleaning clothes because
   (a) it gives strength to the solution
   (b) it reduces surface tension of the solution
   (c) it absorbs the dirt
   (d) chemicals of soap change.

3. In a capillary tube, a liquid rises upto 10 cm. If a capillary tube of same bore and length 5 cm is dipped in liquid then
   (a) a fountain of liquid will be obtained
   (b) there will be no rise of liquid in the tube
   (c) the excess of liquid will slowly ooze out
   (d) the liquid rises to the top of tube and will stay there.

4. 'Vhen a capillary tube is dipped in a liquid, the level of liquid inside the tube rises becauses of
   (a) viscosity
   (b) surface tension
   (c) Osmosis
   (d) Diffusion

5. If two soap bubbles of different radii are in communication with each other
   (a) air flows from larger bubble till the sizes are equal
   (b) air does not flow from one to the other
   (c) air flows from smaller bubble to larger bubble
   (d) none of the above.

6. Two soap bubbles have their radii in the ratio 2:1. The ratio of the excess pressure inside the bubbles is
   (a) 1 :2 (b) 1 : 4
   (c) 2 : 1 (d) 4 : 1

7. The lower end of a capillary tube is at a depth 12 cm and water rises 3 cm. in it. The mouth pressure required to blow an air bubble at lower end will be x cm of water column where x is
   (a) 3 (b) 9
   (c) 12 (d) 15

8. When temperature is increased, surface of a liquid
   (a) increases
   (b) decreases
   (c) remains unchanged
   (d) completely vanishes

9. Which of the following will rise to maximum height in a given capillary tube
   (a) pure water at 4^0 c
   (b) pure water at 2^0 c
   (c) a solution of salt in water at 2^0c
   (d) a solution of salt in water at 4^0c.

10. A needle floats on water surface because of
    (a) surface tension
    (b) lighter weight
    (c) adhesive force
    (d) viscosity

11. What is the effect on size of a soap bubble when some electric charge is given to it ?
    (a) it will not change
    (b) it will decrease
    (c) it will increase
    (d) it will increase on one side only
12. A number of small drops of mercury adiabatically coalesce to form a single drop. The temperature of drop will
    (a) increase
    (b) decrease
    (c) remain same
    (d) depend on size
13. A soap bubble has radius 5cm. If the soap solution has a surface tension 30×10^{-3} N/m, what is the gauge pressure within the bubble ?
    (a) 1 Pa (b) 2 Pa
    (c) 2.4 Pa (d) Zero
14. The spherical shape of rain drops is due to
    (a) density of liquid
    (b) surface tension
    (c) gravity
    (d) atmospheric pressure
15. What is surface tension of boiling water?
    (a) same as at room temperature
    (b) infinity
    (c) zero
    (d) its value depends on pressure.

B. Very Short Answer Type Questions :

1. Why is the shape of a small drop of water spherical ?
2. What is the nature of force that makes a piece of paper stick to another piece of paper by means of gum ?
3. State the principle which explains the absorption of ink by blotting paper.
4. Define surface tension
5. A liquid rises to a particular height in a capillary tube. Will the liquid overflow if the tube is of insufficient length ?
6. How does tree draw water from ground?
7. Is the pressure inside a soap bubble, smaller than, equal to or greater than atmospheric pressure ?
8. How does surface tension vary with temperature ?
9. If a brush is dipped in water and raised, hairs get greezed, why this happens ?
10. How can you distinguish pure water from saline water from the property of capillarity ?
11. What happens when a capillary tube is dipped into a mercury in a container ?
12. What is the difference between bubble and drop ?
13. Under what conditions surface energy density is numerically equal to the surface tension ?
14. Why do we plough our lands ?
15. The pressure just below a surface is less than the atmospheric pressure. What is the shape of the surface ?
16. Write expression for excess pressure inside a soap bubble, water drop and air bubble in a liquid.
17. You have a drop of soap solution and a soap bubble of same radius, which of the two has a greater pressure in it ?
18. What are the units of surface energy ?
19. What are the units of surface tension?
20. If two soap bubbles of radii 1.0 cm and 2.0 cm could be joined by a tube without bursting what would happen ?

21. If more air is pushed into soap bubble what happens to the pressure inside the soap bubble ?

C. Short Answer Type Questions :

1. A water drop has a diameter $2x10^{-5}$ m. Calculate the excess pressure inside, compared to that outside.

   (S.T. of water = 0.072 N/m^2)

2. What is the nature of inter-molecular force at very short distance and at long distance ?

3. Explain why water spreads out but not mercury on a glass plate.

4. Show that S.E. is numerically equal to S.T. at isothermal condition.

5. What will be the rise of water in capillary tube of radius 0.5 mm.

6. Distinguish between cohesive and adhesive force.

7. Calculate the excess pressure inside a soap bubble of radius 0.5cm if S.T. of soap solution is $3.2x10^{-2}$ N/m.

8. If radius of a soap bubble is doubled, then calculate the percentage change in excess pressure inside the soap bubble.

9. Air is pushed into a soap bubble what happens to the pressure inside it. Explain.

10. In the given figure what is the pressure difference between points A and B.

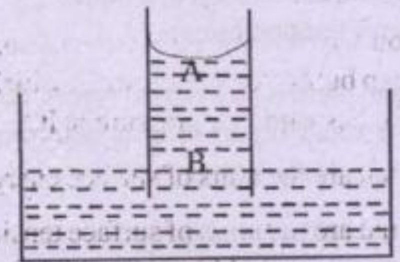


Fig. 14. Q.1

11. Which of the following graphs may represent the relation between capillary rise h and radius of the capillary tube.

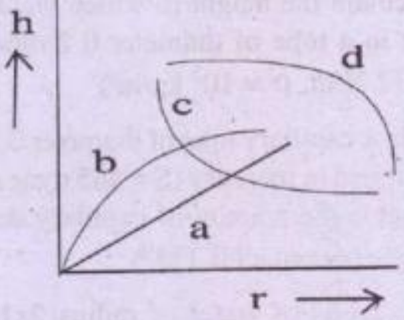


Fig. 14.Q.2

12. When some wax is rubbed on a cloth, it becomes water proof. Explain.

13. The contact angle between pure water and pure silver is 90^0. If a capillary tube made of silver is dipped at one end in pure water, will the water rise in the capillary ?

14. When a glass capillary tube is dipped at one end in water, water rises in the tube. The gravitational potential energy is thus increased. Is it a violation of conservation of energy ?

15. If a mosquito is dipped into water and released, it is not able to fly till it is dry again. Explain.

16. When the size of a soap bubble is increased by pushing more air in it, the surface area increases. Does it mean that the average separation between the surface molecules is increased ?

D. Unsolved Problems :

1. A soap bubble of radius 1 cm. expands into a bubble of radius 2 cm. Calculate the increase in total surface energy if S.T. for soap solution is 25 dyne/cm.

2. How much work must be done in blowing a soap bubble of radius 1cm. of S.T. for soap solution is 25 dyne /cm .

3. Calculate the work done in splitting a water drop of radius 1mm into 1000 similar droplets. (S.T. of water is 70 dyne/cm)

4. Calculare the height to which water will rise in a tube of diameter 0.2 mm ($S = 0.072$ N/m, $\rho = 10^3$ kg/m^3)

5. A glass capillary tube of diameter 0.2 mm is dipped in mercury ($S = 465$ dyne / cm). What is the amount of capillary descent (angle of contact is 135^0)

6. Two drops of water of radius 2×10^{-7} m coalesce. What rise of temperature will result ?

   (S.T. of water = 74×10^{-3} N/m, sp. heat capacity of water = 4200 Jkg^{-1} ok^{-1})

7. To what height can mercury be filled in a vessel without leak occurring if there is a pin hole of diameter 0.1 mm at the bottom of the vessel ?

   (density of mercury = 13.6×10^3kg/m^3, S.T. of mercury = 550×10^{-3}N/m neglect angle of contact)

8. A tube of 1mm bore is dipped into a vessel containing a liquid of density 800 kg/m^3, S.T = 30×10^{-3} N/m, and angle of contact zero. Calculate the rise of liquid in the tube when the tube is held.

   (a) vertical, (b) inclined to the vertical at an angle of 30^0.

9. The end of a capillary tube of radius r is immersed into water of surface tension T and density ρ. What amount of heat will be evolved when water rises in the tube ?

   [Hints : Heat evolved = work done by S.T. - grav. pot.energy stored in the standing column]

10. A clean glass tube of conical bore is dipped vertically into pure water, with apex upwards. The length of the tube is 24 cm. The radii of upper and lower ends are 0.01 cm and 0.04 cm respectively. Calculate the height to which the water rises in the tube.

    (S.T. of water = 75 dyne / cm)

11. Calculate the pressure inside an air bubble of radius 0.01 mm situated at a depth of 30 cm below the free surface of a liquid of density 0.9gm/cm^3 and surface tension 72 dyne/cm. Given atmospheric pressure = 76 cm of Hg.

E. Long Type Questions :

1. Explain the terms surface tension and surface energy of a liquid. Show that surface energy per unit area is numerically equal to surface tension under isothermal conditions.

2. What do you mean by excess pressure ? Derive an expression for excess pressure inside a soap bubble.

3. Derive an expression for excess pressure inside a water drop.

4. Define surface tension and angle of contact. Derive an expression for rise of water in a capillary tube.

5. Define adhesive force and cohesive force. Explain the shape of curves in terms of these forces.

F. Fill in the Blanks Type

1. The surface tension of a liquid .... with rise of temperature.

2. When there is no external forces, the shape of a small liquid drop is determined by .... of the liquid.

3. Two capillaries of different radei are held vertically in a liquid. The height of the liquid will be more in the tube of ....

4. Kerosine oil rises up in the wich of a lantern because of ....

5. A soap bubble has radius 5 cm. If the soap solution has a surface tension 30×10^{-3} N/m, the gauge pressure within the bubble is ....

G. True - False Type

1. When a soap bubble is charged it contracts.

2. If a drop of oil is placed on the surface of water, it will spreed as a thin layer.

3. A liquid does not wet the surface of a solid if the angle of contact is an obtuse one.

4. Soap helps in cleaning the clothes because it reduces the surface tension of the solution.

5. When a drop of water breaks into two droplets of equal size, there is loss in surface energy.

ANSWERS

A. Multiple Choice Type Questions :

1.(a) **2.**(b) **3.**(d) **4.**(b) **5.**(c) **6.**(a) **7.**(d) **8.**(b) **9.**(b) **10.**(a) **11.**(c) **12.**(a) **13.**(c) **14.**(b) **15.**(c)

B. Very Short Type Questions :

1. Due to surface tension the free surface acquires minimum surface area so that it has minimum surface energy.
2. Adhesive force
3. Ink rises through the small pores of the blotting paper- capillary action.
4. See text
5. No overflow, the meniscus will adjust its curvature.
6. Trees draw water from ground through the roots by capillary action.
7. Greater than atmospheric pressre.
8. S.T. decreases as temperature increases.
9. Due to surface tension. When the brush is dipped in water and brought out, a water membrance is present between consecutive hairs. Due to S.T. the membrance tries to shrink. Hence hairs stick together.
10. Saline water has less capillary rise than pure water.
11. The mercury inside capillary falls below the level of mercury in the container.
12. In the case of bubble air is on the concave side and liquid on the convex side, whereas in case of drop just the converse.
13. Isothermal condition.
14. See text.
15. Concave upwards.
16. Water drop $\Rightarrow \frac{2S}{r}$; soap bubble $\Rightarrow \frac{4S}{r}$ air bubble $\Rightarrow \frac{2S}{r}$.
17. Soap bubble.
18. See text.
19. See text.
20. Air from the smaller bubble will flow to the larger bubble.
21. The soap bubble size increases, hence excess pressure and pressure inside it decreases.