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BUSINESS MATHEMATICS AND STATISTICS

Class - XII

(For Class XII Commerce Students)

(Prescribed by the Council of Higher Secondary Education, Odisha)

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FOREWORD

The Odisha State Bureau of Textbook Preparation and Production, Bhubaneswar has made a pioneer attempt to publish text books for Commerce Stream with an excellent team of teachers in different subjects.

The present book **"Business Mathematics and Statistics - Class - XII"** is meant for Class - XII Higher Secondary Commerce students. This book has been written by a team of learned academicians namely Dr. Ranjit Mishra, Dr. R.C Jena, Dr. Banamali Panda, and Dr. P.K Parida and reviewed by Prof. A. P. Nayak. I would like to record my sincere gratitude to all of them for accomplishing this maiden venture in time. The main purpose of developing this text book is to provide a thorough exposure to the students of Commerce in this subject. The book prepared according to the new syallabus prescribed by the CHSE, Odisha, and as per new pattern of questions, shall cater to the needs of young students. I owe my special thanks to Prof. A.P. Nayak for making necessary revision in the text and questions according to new pattern.

I believe that the students and teachers of commerce stream shall welcome and appreciate the book. I would also like to welcome constructive suggestions for further improvement of the book.

Sri Umakanta Tripathy
Director
Orissa State Bureau of Textbook
Preparation and Production,
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PREFACE

Elementary knowledge in Mathematics & Statistics has assumed immense importance in every sphere of human activity. Its application is found in various diversified fields such as business, industry, management, economics, planning, agriculture, &- insurance, sociology, biometry etc. and also in many professional courses. The Odisha State Bureau of Textbook Preparation and Production, Bhubaneswar through its Board of Writers and Reviewers, has presented this new edition of the book, **"Business Mathematics and Statistics Class-XII** for Commerce students to have them a simple and comprehensive exposition to the application of quantitative techniques.

We, the writters are pleased to commend the readers the new edition of this book which is commensurate with the latest syllabus of CHSE, Odisha. The textbook is prepared to fully cater to the needs of the students both in terms of the course content as well as the level of knowledge required to excel in the examination. Further the textbook is thoroughly reviewed by an expert to incorporate the requirements of the students.

The main features of the book are :

- The book has two parts, namely Mathematics and Statistics -
- Simplicity of expression
- Sufficient illustrations to tackle practical problems
- Systematic presentation of subject matter
- Latest course content
- Meaningful focus on new question pattern

The book is unique in its presentation because steps have been taken to keep pace with new syllabus and new pattern of questions. Sufficient Multiple Choice Questions and other objective type questions are provided along with their answers in addition to long questions.

We wish to thank the Odisha State Bureau of Textbook Preparation and Production, Bhubaneswar for its efforts and co-operation in the publication of the book in time.

Any suggestion for improvement of the book will be highly appreciated.

Board of Writers

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SYLLABUS

Business Mathematics and Statistics

Class-XII

Objectives :

- ❖ To enable the students to learn basic concepts of determinants & Matrices;
- ❖ To learn the concept, features, types of Set Theory;
- ❖ To develop an understanding about concept, relations, types and application of functions;
- ❖ To enable the students to learn the concept, methods and applications of Limit, continuity, Derivation and Integration;
- ❖ To help the students in learning the concept, types and calculation of Average and
- ❖ To enable the students to understand the concept, objectives, features and applicationsof measures of dispersion.

COURSE INPUTS :

Unit - I Business Mathematics:

Determinants :-	Upto third order, Minors, Co-factors, properties and Cramer's rule,
Matrices :-	Meaning, Definition, Types, Algebra of matrices, Solving Linear Equation Problems through Matrics.
Set Theory :-	Meaning, Definition, Types and Operations (Union & Intersection)
Functions :-	Meaning and Relations of Functions, Types of Functions and Classification of Functions (excluding Trigonometric Functions)

Unit -II Calculus :

Calculus -1	Limit & Continuity - Meaning, Definition, Methods of Finding Limits, Differentiation
Calculus - II	Integration up to substitution

Unit-III Measure of Central Tendency :-

Meaning, Objectives, Types,of Averages (Mathematical & Positional Averages)
Mathematical Averages : AM, GM, HM (Simple & Weighted)
Positional Averages : Median, Mode, Quartile, Deciles and Percentiles
Relationship of AM, GM, HM, Median, Mode

Unit - IV Measure of Dispersion :-

Meaning, Objectives, Characteristics of dispersion, Measures of Dispersion, (Absolute and Relative) Positional Dispersion : Range, Inter Quartile Range, Quartile Deviation.
Mathematical Dispersion : Mean Deviation, Standard Deviation & Co-efficient of variation.

Unit-V Project work and viva :-

Suggested outline for Project Work

- ❖ Case study on Application of Matrix for solving real life business problems.
- ❖ Report on concept and Rules regarding Matrix, Determining inverse of a matrix by using Elementary Operation Methods and Co-factor Method.
- ❖ Find out the Averages (Mean, Median and Mode) by taking sample from Number of students from any class and Marks secured by them in their annual examination as two variables.
- ❖ Calculate the Mean or Average deviation and co-efficient of Variation form Mean / Median / Mode from by taking example of Life Time in No. of years of two different T.V. sets namely Model A and Model B to find out the average life of each model of these TV and the model having greater uniformly or variability.
- ❖ Calculating Standard Deviation and its co-efficient alongwith its coeffi cient of variation form the data relating to the Profit or Loss made by Engineering Companies in Odisha during the year 201-15.

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CHAPTER - 1

DETERMINANTS

STRUCTURE

1.1 Meaning of determinant

The concept of matrices and determinant has a wide applicability not only in mathematics but also in other diciplines like sociology, Genetics, Engineering etc. Therefore for their wide applicability it is essential to study the fundamentals of matrices and determinants. To learn about determinant one must know the meaning of matrix.

A matrix is a rectangular array of numbers, real or complex. It is usually denoted by capital letters. The numbers inside the rectangular array are known as elements of the matrix. The horizontal lines of elements are called rows and vertical lines of elements are called columns. Let us have an example.

$$A = \begin{bmatrix} 4 & 2 & 6 \\ 3 & 1 & 5 \\ 2 & 0 & 4 \end{bmatrix} \begin{matrix} \text{First row} \\ \text{Second row} \\ \text{Third row} \end{matrix}$$

1st Col. 2nd Col. 3rd Col.

In general, if a matrix A has m rows and n columns, it is expressed as below :

$$A = \begin{bmatrix} a_{11} & a_{12} \cdots\cdots a_{1j} \cdots\cdots & a_{1n} \\ a_{21} & a_{22} \cdots\cdots a_{2j} \cdots\cdots & a_{2n} \\ \vdots & \vdots \quad\quad \vdots & \vdots \\ a_{m1} & a_{m2} \cdots\cdots a_{mj} \cdots\cdots & a_{mn} \end{bmatrix}$$

To understand determinant let us solve the following two linear equation.

We have learnt that a system of algebraic equations can be expressed in the form of matrices. This means, a system of linear equations like.

$a_1x + b_1y = c_1$

$a_2x + b_2y = c_2$

Can be represented as $\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

Now this system of equations has a unique solution or not, is determined by the number $a_1b_2 - a_2b_1$ (Recall that if $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$ or, $a_1b_2 - a_2b_1 \neq 0$, then the system of linear equations has a unique solution). The number $a_1b_2 - a_2b_1$ which determines uniqueness of solution is associated with the matrix

$A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$ and is called the determinant of A or det A. Determinants have wide applications in Engineering, Science, Economics, Social Science etc.

In this chapter, we shall study determinants upto order three only with real entries. Also, we will study various properties of determinants, minors, cofactors & applications of determinants in finding the area of a triangles, adjoint & inverse of a square matrix, concistency and inconsistency of system of linear equtions and solutions of linear equation in two or three variables using inverse of a matrix.

Determinant :

To every square matrix $A = [a_o]$ of order n, we can associate a number (real or complex) called determinant of the square matrix A, Where $a_{ij} = (i, j)^{th}$ element of A.

This may be thought of as a function which associates each square matrix with a unique number (real or complex) called determinant of the square matrix A, where $a_{ij} = (i, j)^{th}$ element of A. This may be thought of as a function which associate each square matrix with a unique number (real or complex). If M is the set of square matrices, K is the set of numbers (real or complex) and f : M→k is defined by f (A) = k, where A ∈ M and k∈K, then f(A) is called the determinant of A. It is also denoted by |A| or det A or Δ.

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then determinant of A is

Written as $|A| = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \det(A)$

Remarks :

(i) For matrix A, |A| is read as determinant of A and not modules of A.

(ii) Only square matrices have determinants.

Determinant of a matrix of order one .

Let A = [a] be the matrix of order 1, then determinant of A is defined to be equal to a.

Determinant of a matrix of order two :

Let $A = \begin{bmatrix} a_{11} & b_{12} \\ c_{21} & d_{22} \end{bmatrix}$ be a matrix of

order 2 x 2, then the determinant of A is

defined as :

$$\det(A) = |A| = \Delta = \begin{vmatrix} a_{11} & a_{12} \\ = a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

Example 1 : Evaluate $\begin{vmatrix} 2 & 4 \\ -1 & 2 \end{vmatrix}$

Solution : We have $\begin{vmatrix} 2 & 4 \\ -1 & 2 \end{vmatrix} = 2(2) - 4(-1)$

$= 4+4$

$= 8$

Example 2 :

Evaluate $\begin{vmatrix} x & x+1 \\ x-1 & x \end{vmatrix}$

Solution :

We have,

$\begin{vmatrix} x & x+1 \\ x-1 & x \end{vmatrix} = x(x) - (x+1)(x-1) = x^2-(x^2-1)$

$= x^2 - x^2+1=1$

1.2 EXPANSION OF DETERMINANTS :

Determinant of a matrix of order 3×3 :

Determinant of a matrix of order three can be determined by expressing it in terms of second order determinants. This is known as expansion of a determinant along a row (or a column). There are six ways of expanding a determinant of order 3 corresponding to each of three rows (R_1, R_2 and R_3) and three columns (C_1, C_2 and C_3) giving the same value as shown below :

Consider the determinant of square matrix

$A=[a_{ij}]_{3\times3}$

i.e., $|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

Expansion along first Row (R_1):

Step-1: Multiply first element a_{11} of R_1 by $(-1)^{(1+1)}[(-1)^{\text{sum of suffixes in } a_{11}}]$ and with the second order determinant obtained by deleting the elements of first row (R_1) and first column (C_1) of |A| as a_{11} lies in R_1 and C_1,

i.e., $(-1)^{1+1} a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$

Step-2: Multiply 2nd element a_{12} of R_1 by $(-1)^{1+2}$ $[(-1)^{\text{sum of suffixes in } a_{12}}]$ and the second order determinant obtained by deleting elements of first row (R_1) and 2nd column (C_2) of |A| as a_{12} lies in R_1 and C_2,

i.e., $(-1)^{1+2} a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$

Step-3: Multiply third element a_{13} of R by $(-1)^{1+3}$ $[(-1)^{\text{sum of suffixes in } a_{13}}]$ & the second order determinant obtained by deleting elements of first row (R_1) and third column (C_3) of |A| as a_{13} lis in R_1 and C_3.

i.e., $(-1)^{1+3} a_{13} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{32} \end{vmatrix}$

Step-4: Now the expansion of determinant of A, that is, |A| written as sum of all three terms obtained in steps 1,2 and 3 above is given by :

Let $A = |A| = (-1)^{1+1} \cdot a_{11} \begin{vmatrix} a_{22} & b_{23} \\ c_{32} & d_{33} \end{vmatrix} +$

$(-1)^{1+2} \cdot a_{12} \begin{vmatrix} a_{21} & b_{23} \\ c_{31} & d_{33} \end{vmatrix} +$

$$(-1)^{1+3} \cdot a_{13} \begin{vmatrix} a_{21} & b_{22} \\ c_{31} & d_{32} \end{vmatrix}$$

or, $|A| = a_{11} (a_{22}a_{33} - a_{32}a_{23}) - a_{12} (a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{22})$

$+ a_{13}(a_{21}a_{32} - a_{31}a_{22})$

$= a_{11}a_{22}a_{33} - a_{11}a_{32}a_{23} - a_{12}a_{21}a_{33} + a_{12}a_{31}a_{23} + a_{13}a_{21}a_{32} - a_{13}a_{31}a_{22}$ – (1)

N.B : We shall apply all four steps together.

Expansion along second row (R_2) :

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Expanding along R_{21} use get :-

$$|A| = (-1)^{2+1}.a_{21} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{2+2}.a_{22} \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$$

$$+(-1)^{2+3} \cdot a_{23} \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$

$= -a_{21}(a_{12}a_{33} - a_{32}a_{13}) + a_{22}(a_{11}a_{33} - a_{31}a_{13})$

$-a_{23}(a_{11}a_{32} - a_{31}a_{12})$

$\Rightarrow |A| = -a_{21}a_{12}a_{33} + a_{21}a_{32}a_{13} + a_{22}a_{11}a_{33} - a_{22}a_{31}a_{13}$

$- a_{23}a_{11}a_{32} + a_{23}a_{31}a_{12}$

$= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31}$

$+ a_{13}a_{21}a_{32} - a_{13}a_{31}a_{22}$.......... (2)

Expansion along first column (C_1) :

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

By expanding along C_1, we get :-

$$|A| = a_{11}(-1)^{1+1}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{21}(-1)^{2+1}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$$

$$+ a_{31}(-1)^{3+1}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{21}(a_{12}a_{33} - a_{13}a_{32}) + a_{31}(a_{12}a_{23} - a_{13}a_{22})$$

$$|A| = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{21}a_{12}a_{33} + a_{21}a_{13}a_{32} + a_{31}a_{12}a_{23} - a_{31}a_{13}a_{22}$$

$$= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{31}a_{22} \ldots\ldots\ldots\ldots\ldots (3)$$

Clearly, Value of |A| in (1), (2) and (3) are equal. It's left as an exercise to the reader to verify that the value of |A| by expanding along R_3, C_2 and C_3 are equal to the value of |A|, Obtained in (1), (2) or (3).

Hence, expanding a determinant along any row or column gives same value.

Remarks

(i) For easier calculation, We shall expand the determinant along the row or column, which contains maximum number of zeroes.

(ii) While expanding, instead of multiplying by $(-1)^{i+j}$, we can multiply by (+1) or (-1) according as (i+j) is even or odd.

(iii) Let $A = \begin{bmatrix} 2 & 2 \\ 4 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}$ then

it is easy to verify that A = 2B. Also,

$|A| = 0-8 = -8$ and $|B| = 0-2=-2$

Observe that, $|A| = \{(4)-2\}$ and $= 2^2 |B|$ or

$|A| = 2^n |B|$, where n=2 is the order of square matrices A and B

In general, if A = KB, where A and B are square matrices or order n, then $|A| = K^n |B|$, where n=1,2,3

Example 3 : Evaluate the determinant.

$$\triangle = \begin{vmatrix} 1 & 2 & 4 \\ -1 & 3 & 0 \\ 4 & 1 & 0 \end{vmatrix}$$

Solution : Note that in the third column, two entries are zero. So expanding along third column (C_3), we get :-

$$\triangle = 4\begin{vmatrix} -1 & 3 \\ 4 & 1 \end{vmatrix} - 0\begin{vmatrix} 1 & 2 \\ 4 & 1 \end{vmatrix} + 0\begin{vmatrix} 1 & 2 \\ -1 & 3 \end{vmatrix}$$

$= 4\ (-1-12) - 0 + 0$

$= -52$

Remark - For solving problems on determinant of 3rd or higher order, expansion method is applied.

Example 5 : Find values of x for which $\begin{vmatrix} 3 & x \\ x & 1 \end{vmatrix} = \begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix}$

Solution :- We have, $\begin{vmatrix} 3 & x \\ x & 1 \end{vmatrix} = \begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix}$

i.e, $3-x^2= 3-8$

$\Rightarrow x^2 = 8$

Hence, $x = \pm 2\sqrt{2}$

1.3 Minors And Co-Factors :

The minor of an element a_{ij} of a determinant A is the determinant obtained from A by deleting the row and column in which the given element occurs. The minor of an element a_{ij} is denoted by M_{ij}. The minor of an element of third order determinant is a second order determinant.

For Example :- If $|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

Then, the minor of the element :

$$a_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

Similarly, minor of the element $a_{32} \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$

$$= a_{11}a_{23} - a_{21}a_{13}$$

$$\therefore M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \text{and}$$

$$M_{32} = \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$$

We find that the minors, are second order determinants.

CO-FACTOR :

The co-factor of an element a_{ij} of determinant A is defined as :

$A_{ij} = (-1)^{i+j}$. Mij; where i is the number of the row & j is the number of the column, in which the given element is present; Mij is the minor of the element a_{ij} and Aij is the co factor of the element aij. Usually, the co-factor is denoted by the corresponding capital letter of the given element. For example :

$$\text{If } |A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$A_{11} = (-1)^{1+1} M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

$$A_{12} = (-1)^{1+2} M_{12} = -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

$$A_{32} = (-1)^{3+2} M_{32} = -\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$$

$$A_{33} = (-1)^{3+3} M_{33} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

Illustration - 1

Evaluate the determinant :

(i) $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ (ii) $\begin{vmatrix} -2 & 4 \\ 3 & -7 \end{vmatrix}$ (iii) $\begin{vmatrix} 4 & 2 \\ 1/2 & 6 \end{vmatrix}$

(iv) $\begin{vmatrix} 4 & -17 \\ 0 & 5 \end{vmatrix}$

Solution :

(i) $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$

(ii) $\begin{vmatrix} -2 & 4 \\ 3 & -7 \end{vmatrix} = -2x(-7) - 3x4 = 14 - 12 = 2$

(iii) $\begin{vmatrix} 4 & 2 \\ 1/2 & 6 \end{vmatrix} = 4x6 - \frac{1}{2}x2 = 24 - 1 = 23$

(iv) $\begin{vmatrix} 4 & -17 \\ 0 & 5 \end{vmatrix} = 4x5 - 0x(-17) = 20 - 0 = 20$

Illustration 2 : Find the minors and co-factors of elements of the following determinant.

$$|A| = \begin{vmatrix} 2 & -3 & 4 \\ 5 & 0 & 6 \\ 7 & 5 & 1 \end{vmatrix}$$

Solution : $M_{11} = \begin{vmatrix} 0 & 6 \\ 5 & 1 \end{vmatrix} = 0x1 - 5x6 = -30;$

$A_{11} = (-1)^{1+1}(-30) = -30$

$m_{12} = \begin{vmatrix} 5 & 6 \\ 7 & 1 \end{vmatrix} = 5x1 - 7x6 = -37;$

$A_{12} = (-1)^{1+2}(-37) = 37$

$M_{13} = \begin{vmatrix} 5 & 0 \\ 7 & 5 \end{vmatrix} = 5x5 - 7x0 = 25; A_{13} = (-1)^{1+3}25 = 25$

$M_{21} = \begin{vmatrix} -3 & 4 \\ 5 & 1 \end{vmatrix} = -3x1 - 5x4 = -23;\ A_{21} = (-1)^{2+1}(-23) = 23$

$$M_{22} = \begin{vmatrix} 2 & 4 \\ 7 & 1 \end{vmatrix} = 2x1 - 7x4 = -26;\ A_{22} = (-1)^{2+2}(-26) = -26$$

$$M_{23} = \begin{vmatrix} 2 & -3 \\ 7 & 5 \end{vmatrix} 2x5 - 7x(-3) = 31; A_{23} = (-1)^{2+3}(31) = -31$$

$$M_{31} = \begin{vmatrix} -3 & 4 \\ 0 & 6 \end{vmatrix} = -3x6 - 0x4 = -18;\ A_{31} = (-1)^{3+1}(-18) = -18$$

$$M_{32} = \begin{vmatrix} 2 & 4 \\ 5 & 6 \end{vmatrix} = 2x6 - 5x4 = -8, A_{32} = (-1)^{3+2}(-8) = 8$$

$$M_{33} = \begin{vmatrix} 2 & -3 \\ 5 & 0 \end{vmatrix} = 2x0 - 5x(-3) = 15; A_{33} = (-1)^{3+3}15 = 15$$

Illustratation 3 : Find minors & co-factors of elements of the determinant

$$\begin{vmatrix} 3 & 5 \\ -2 & 4 \end{vmatrix}$$

Solution: $\det A = \begin{vmatrix} 3 & 5 \\ -2 & 4 \end{vmatrix}$

M_{11} = Minor of the element 3 = $|4| = 4$

M_{12} = Minor of the element 5 = $|-2| = -2$

M_{21} = Minor of the element (-2) = $|5| = 5$

M_{22} = Minor of the element 4 = $|3| = 3$

A_{11} = Co-factor of 3 = $(-1)^{1+1}\,4 = 4$

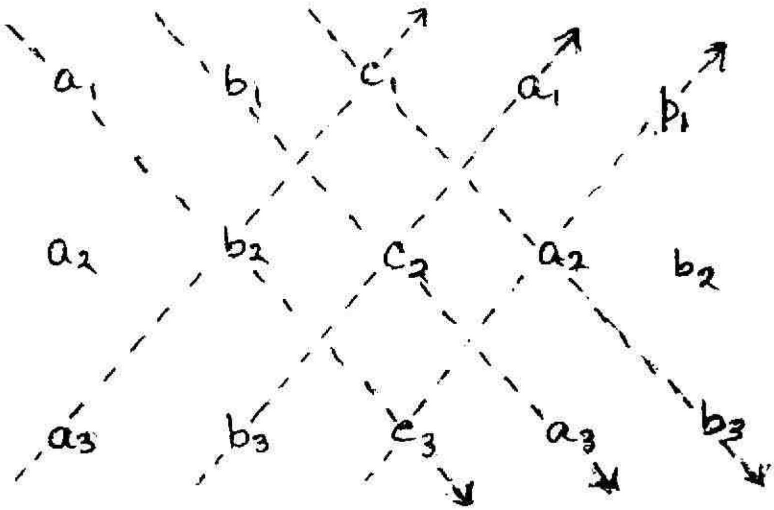
A_{12} = Co-factor of 5 = $(-1)^{1+2}(-2) = 2$

A_{21} = Co-factor of -2 = $(-1)^{2+1}\,5 = -5$

A_{22} = Co-factor of 4 = $(-1)^{2+2}\,13 = 13$

Expansion of Determinant with the Help of Sarrus Diagram :

We can also evaluate a third order determinant by drawing a sarrus diagram. In sarrus diagram first two column are added to the given matrix of the determinant & the elements are connected through arrows heading downwards & are assigned plus sign (+) & those connected upwards are assigned minus sign (–). Let's take the following example and find out the value of determinant through sarrus diagram.

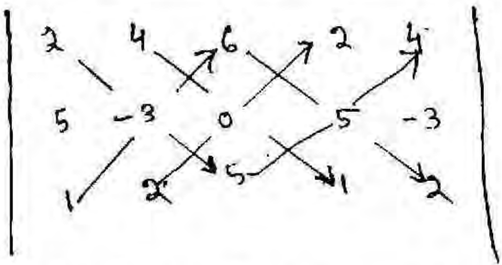


Now, according to the sarrus diagram :-

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1b_2c_3 + b_1c_2a_3 + c_1a_2b_3 - a_3b_2c_1 - b_3c_2a_1 - c_3a_2b_1$$

Illustration - 4 : Use sarrus Diagram and evaluate :-

$$|A| = \begin{vmatrix} 2 & 4 & 6 \\ 5 & -3 & 0 \\ 1 & 2 & 5 \end{vmatrix}$$



Solution : With the help of sarrus diagram are as follows. i.e, the product of the elements which are connected by down-word arrow are calulated first, out of which the product of the elements of the up-word arrow are subtracted. The resulting fiquare is the value of determinant.

Therefore,

$$\begin{aligned}
|A| &= [2x(-3)x5]+(4x0x1)+(6x5x2) \\
&-1(-3)x6-2x0x2-5x5x4 \\
&=-30+0+60+18-0-100 \\
&=-130+78 \\
&=-52
\end{aligned}$$

1.4. Properties of Determinants :

Its often difficult to simplify certain determinents. The properties of determinants mentioned below enable us to simplify the determinants & quickly obtain the results. So, it becomes necessary on our part to carefully observe some important properties of determinants, & apply it to situations intelligently. Though the properties are true for determinants of any order, we restrict its use up to determinants of order three only.

1. The value of a determinant remains unchanged, if its rows and columns are interchanged.

Proof : $\det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

Expanding det A along first column, we have

$\det A = a_{11}(a_{22}.a_{33} - a_{32}.a_{23}) - a_{21}(a_{12}.a_{33} - a_{32}.a_{13}) + a_{31}.(a_{12}.a_{23} - a_{22}.a_{13})$

Changing the rows and columns of det A, we get that :-

$$\det A_1 = \begin{vmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{vmatrix}$$

Now, expanding det A, along first row, we have

$\det A_1 = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{21}(a_{12}a_{33} - a_{13}a_{32}) + a_{31}(a_{12}a_{23} - a_{13}.a_{22})$

Now det A = det A_1, hence, there's no change in the value of determinant.

2. If two rows or columns of a determinant are interchangable, then the value of determinant doesn't change but its sign change.

Proof : Let $\det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

Expanding the above along first column, we have :-

$\det A_1 = a_{11}\ (a_{22}a_{23} - a_{32}a_{23}) - a_{21}\ (a_{12}a_{33} - a_{32}a_{13}) + a_{31}\ (a_{12}a_{13} - a_{22}.a_{13})$

Interchanging 1 st and 3rd column, we have :-

$$\det A_1 = \begin{vmatrix} a_{13} & a_{12} & a_{11} \\ a_{23} & a_{22} & a_{21} \\ a_{33} & a_{32} & a_{31} \end{vmatrix}$$

Expanding det A, along third column, we have

$$\det A_1 = a_{11}(a_{32}a_{23} - a_{22}a_{33}) - a_{21}(a_{32}a_{13} - a_{12}a_{33}) + a_{31}(a_{22}a_{13} - a_{12}.a_{23})$$

$$= -[a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{21}(a_{13}a_{33} - a_{32}a_{13}) + a_{31}(a_{12}a_{23} - a_{22}a_{13})$$

$$= -\det A$$

Hence, the value of the determinant doesnot change, but its sign changes.

3. If any two rows/columns of a determinant are identical, then the value of the deternimant is zero.

Proof : $\det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

Expanding along first row, we have:-

$$\det A = a_{11}(a_{12}a_{33} - a_{32}a_{13}) - a_{12}(a_{11}a_{33} - a_{31}a_{13}) + a_{13}(a_{11}a_{32} - a_{31}.a_{12})$$

$$= a_{11}a_{12}a_{33} - a_{11}a_{32}a_{13} - a_{12}a_{11}a_{33} + a_{12}a_{31}a_{13} + a_{13}a_{11}a_{32} - a_{13}a_{31}a_{12}$$

$$= 0$$

4. If each element of a particular row / column of a determinant is multipled by a constant K, then the value of the determinant gets multiplied by k.

Proof :

$$\text{Let } \det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\det A_1 = \begin{vmatrix} ka_{11} & Ka_{12} & Ka_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Expanding det A_1 along first row, we have :-

$$\det A_1 = ka_{11}(a_{22}a_{33} - a_{32}a_{23}) - ka_{12}(a_{21}a_{33} - a_{31}a_{23}) + ka_{13}(a_{21}a_{32} - a_{31}.a_{22})$$

$$= K\{a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{12}a_{32} - a_{31}a_{22})\}$$

$$= K \det A$$

5. If every element of a row / column of a determinant is expressed as the sum or difference of two terms, then the determinant can be expressed as sum or difference of two determinants, thus,

$$\begin{vmatrix} x_1+a_1 & x_2+a_2 & x_3+a_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix} = \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix} + \begin{vmatrix} a_1 & a_2 & a_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix}$$

Proof : Expending the LHS, we have :-

$$(x_1+a_1)(y_2z_3 - z_2y_3) - (x_2+a_2)(y_1z_3 - z_1y_3) + (x_3+a_3)(y_1z_2 - z_1y_2)$$

$$= x_1(y_2z_3 - z_2y_3) - x_2(y_1z_3 - z_1y_3) + x_3(y_1z_2 - z_1y_2) + a_1(y_2z_3 - z_2y_3) - a_2(y_1z_3 - z_1y_3) + a_3(y_1z_2 - z_1y_2)$$

$$= \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix} + \begin{vmatrix} a_1 & a_2 & a_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix}$$

6. If any row / column of a determinants, a multiple of any other row / column is added or subtracted the value of the original determinant remains the same.

$$\text{Let } \det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Let us multiply a consfant k to third row & add to the first row :-

$$\therefore \det A_1 = \begin{vmatrix} a_{11}+ka_{31} & a_{12}+ka_{32} & a_{13}+ka_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

As per previous properties, we have :

$$\det A_1 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + k\begin{vmatrix} a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + 0$$

= det A ($\because$ two rows are identical in the 2nd determinant)

Illustration 5 : Show that (i) $\begin{vmatrix} 4 & 5 & 6 \\ 2 & 1 & 4 \\ 2 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 4 & 6 & 5 \\ 2 & 4 & 1 \\ 2 & 2 & 1 \end{vmatrix}$

ii) $\begin{vmatrix} 4 & 5 & 6 \\ 2 & 1 & 4 \\ 2 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 4 & 2 & 2 \\ 5 & 1 & 1 \\ 6 & 4 & 2 \end{vmatrix}$

Solution (i) $\begin{vmatrix} 4 & 5 & 6 \\ 2 & 1 & 4 \\ 2 & 1 & 2 \end{vmatrix} = 4(2-4) - 5(4-8) + 6(2-2)$

$= -8 + 20 + 0$

$= 12.$

Now changing c_2 to c_3 and vice - verse, we have :

and $\begin{vmatrix} 4 & 6 & 5 \\ 2 & 4 & 1 \\ 2 & 2 & 1 \end{vmatrix} = 4(4-2) - 6(2-2) + 5(4-8)$

$= 8 + 0 - 20$

$= -12.$

$\therefore \begin{vmatrix} 4 & 5 & 6 \\ 2 & 1 & 4 \\ 2 & 1 & 2 \end{vmatrix} = -\begin{vmatrix} 4 & 6 & 5 \\ 2 & 4 & 1 \\ 2 & 2 & 1 \end{vmatrix}$

(iii) $\begin{vmatrix} 4 & 5 & 6 \\ 2 & 1 & 4 \\ 2 & 1 & 2 \end{vmatrix} \begin{aligned} &= 4(2-4) - 5(4-8) + 6(2-2) \\ &= -8 + 20 + 0 = 12 \end{aligned}$

Now changing columns into Rows, we have

and $\begin{vmatrix} 4 & 2 & 2 \\ 5 & 1 & 1 \\ 6 & 4 & 2 \end{vmatrix} = 4(2-4) - 2(10-6) + 2(20-6)$

$= -8 - 8 + 28 = 12.$

Illustration 6 : Show that :- $\begin{vmatrix} 1 & 2 & 3 \\ 1 & 0 & 0 \\ 1 & 2 & 3 \end{vmatrix} = 0$

Solution : $\begin{vmatrix} 1 & 2 & 3 \\ 1 & 0 & 0 \\ 1 & 2 & 3 \end{vmatrix}$

$=1(0-0) - 2(3-0) + 3(2-0)$

$= 0 - 6 + 6 = 0$

Again since Row 1 and Row 3 are idential, the value of the determinant = 0

Illustration 7 : Show that $\begin{vmatrix} 1x8 & 2x8 & 3x8 \\ 5 & 6 & 7 \\ 8 & 2 & 4 \end{vmatrix} = 8\begin{vmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ 8 & 2 & 4 \end{vmatrix}$

Solution :- $\begin{vmatrix} 1x8 & 2x8 & 3x8 \\ 5 & 6 & 7 \\ 8 & 2 & 4 \end{vmatrix}$

$= 8(24-14)-16(20-56)+24(10-48)$

$= (8x10)+5+6-912$

$= -256$

$= 8 \text{ x } (-32)$

Now, $\begin{vmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ 8 & 2 & 4 \end{vmatrix}$

$=1(24-14)-2(20-56)+$

$3(10-48)=10+72-114$

$= -32$

$\therefore \begin{vmatrix} 1x8 & 2x8 & 3x8 \\ 5 & 6 & 7 \\ 8 & 2 & 4 \end{vmatrix} = 8\begin{vmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ 8 & 2 & 4 \end{vmatrix}$

Illustraction 8 :

Prove that : $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = -b(b-a)(b-c)(c-a)$

Solution - L H S

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 \\ a-b & b-c & c \\ a^2-b^2 & b^2-c^2 & c^2 \end{vmatrix}$$

Replacing c_1–c_2 and c_2–c_3 :-

$$=(a-b)(b-c) \begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & c \\ a+b & b+c & c^2 \end{vmatrix}$$

$= (a–b)\ (b–c)\ [(0–0+\{1(b+c) – (a+b).1]$

$= (a–b)\ (b–c)\ [0–0+b+c–a–b]$

$= –(b–a)\ (b–c)\ (c–a)$ = RHS. (Hence Proved)

Illustration - 9 : Show that $\begin{vmatrix} a & a^2 & 1+a^3 \\ b & b^2 & 1+b^3 \\ c & c^2 & 1+c^3 \end{vmatrix}$

$= (1+abc)\ (b–a)\ (c–a)\ (c–b)$

Solution : $\begin{vmatrix} a & a^2 & 1+a^3 \\ b & b^2 & 1+b^3 \\ c & c^2 & 1+c^3 \end{vmatrix} = \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} + \begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix}$

$$= \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} + abc \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

$$= (-1) \begin{vmatrix} a & 1 & a^2 \\ b & 1 & b^2 \\ c & 1 & c^2 \end{vmatrix} + abc \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

$$= (-1)^2 \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} + abc \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

$$= (1+abc) \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

Now applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we have :

$$= (1+abc) \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix}$$

$$= (1+abc)(b-a)(c-a) \begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 1 & c+a \end{vmatrix}$$

$= (1+abc)\ (b-a)(c-a)\ \{(c+a)-(b+a)\}$

$=(1+abc)\ (b-a)\ (c-a)\ (c-b)$ (proved)

Illustration - 10 : Show that :

$$\begin{vmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{vmatrix} = 0$$ (where ω is one of the imaginary cube roots of unity)

Solution : $$\begin{vmatrix} 1 & w & w^2 \\ w & w^2 & 1 \\ w^2 & 1 & w \end{vmatrix} = \begin{vmatrix} 1+w+w^2 & 1+w+w^2 & 1+w+w^2 \\ w & w^2 & 1 \\ w^2 & 1 & w \end{vmatrix}$$

Replacing R_1 by $R_1+R_2+R_3$

$$= \begin{vmatrix} 0 & 0 & 0 \\ w & w^2 & 1 \\ w^2 & 1 & w \end{vmatrix} (\text{As } 1+w+w^2 = 0)$$

= 0+0+0 (On expanding along first row)

= 0

Illustration 11 : Solve the following :

a) $\begin{vmatrix} 5 & 2 \\ x+1 & 4 \end{vmatrix} = 0$

b) $\begin{vmatrix} 2 & 2 & x \\ -1 & x & 4 \\ 3 & 3 & 3 \end{vmatrix} = 0$

Solutation (a) $\begin{vmatrix} 5 & 2 \\ x+1 & 4 \end{vmatrix} = 0$

$\Rightarrow$ 20- (x+1)x2

$\Rightarrow$ 20 - (2x+2) = 0

$\Rightarrow$ 20- 2x - 2 = 0

$\Rightarrow$ - 2x = -18

$\therefore$ x = 9

(b) $\begin{vmatrix} 2 & 2 & x \\ -1 & x & 4 \\ 3 & 3 & 3 \end{vmatrix} = 0$

$\Rightarrow$ 2 (3x–12)–2(–3–12)+x (–3–3x)=0

$\Rightarrow$ $6x-24+6+24-3x-3x^2=0$

$\Rightarrow$ $6x-24+30-3x-3x^2=0$

$\Rightarrow$ $-3x^2+3x+6=0$

$\Rightarrow$ $x^2-x-2=0$

$\Rightarrow$ $x^2-2x+x-2=0$

$\Rightarrow$ (x–2) (x+1) = 0

$\therefore$ x=2 or x = –1

Illustration 12 : Show that :

$$\begin{vmatrix} x & y & z \\ p & q & r \\ m & n & o \end{vmatrix} = \begin{vmatrix} y & n & q \\ x & m & p \\ z & o & r \end{vmatrix}$$

Solution : LHS = $\begin{vmatrix} x & y & z \\ p & q & r \\ m & n & o \end{vmatrix}$

$= \begin{vmatrix} x & p & m \\ y & q & n \\ z & r & o \end{vmatrix}$ (by changing rows to columns)

$= - \begin{vmatrix} x & m & p \\ y & n & q \\ z & o & r \end{vmatrix}$ (by interchanging C_3 and C_2)

$= \begin{vmatrix} y & n & q \\ x & m & p \\ z & o & r \end{vmatrix}$ (by interchanging R_1 and R_2)

= RHS

$$\therefore \begin{vmatrix} x & y & z \\ p & q & r \\ m & n & o \end{vmatrix} = \begin{vmatrix} y & n & q \\ x & m & p \\ z & o & r \end{vmatrix}$$

1.5 Cramer's Rule :-

Swiss mathematician, Gabriel Cramer provided a simple method to solve a system of linear equations with the help of determinants, which is known as cramer's rule. Thus, cramer's rule is a rule, using determinants, to express the solution of a sytem of linear equations, for which the number of equations is equal to the number of variables.

Let us consider the following two linear equations, with two unknown, x and y :-

$a_1x + b_1y = c_1$

$a_2x + b_2y = c_2$

As we have seen earlier,

$$x=\frac{b_2c_1-b_1c_2}{a_1b_2-a_2b_1} \text{ and } y=\frac{a_1c_2-a_2c_1}{a_1b_2-a_2b_1};\ a_{1,}b_2-a_2b_1=0$$

If we denote $D = a_1b_2-a_2b_1 = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$

$$D_x = b_2c_1-b_1c_2=\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}$$

$$Dy = a_1c_2-a_2c_1=\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$

then, $x=\frac{Dx}{D}$ and $y=\frac{Dy}{D}$, where $D \neq 0$

The above method shows how to solve a system of liner equations using Cramer's rule. The method can be extended to solve a system of 'n' equation with 'n' variables (unknowns). In other words, for solving a system of three liner equations, with three unknowns, we may proceed as below :

Let $a_1x+b_1y+c_1z=d_1$............(i)

$a_2x+b_2y+c_2z=d_2$...........(ii)

$a_3x+b_3y+c_3z=d_3$...........(iii)

be a system of three liner equations with three unknown variable x, y and z. On solving the equations simultaneously (i.e. by addition and subtraction), we get.

$$x=\left[\frac{d_1b_2c_3+d_2b_3c_1+d_3b_1c_2-d_3b_2c_1-d_2b_1c_3-d_1b_3c_2}{a_1b_2c_3+a_2b_3c_1+a_3b_1c_2-a_3b_2c_1-a_2b_1c_3-a_1b_3c_2}\right]$$

and similar other values of y and x. if we denote,

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} Dx = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix} = Dy = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} \text{.................................. (iv)}$$

$$D_z = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

then, as per equation (iv),

$$x = \frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}} = \frac{Dx}{D}$$

Similarlly,

$$y = \frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}} = \frac{Dy}{D}$$

$$z = \frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}} = \frac{D_z}{D}$$

So, we use Cramer's rule & find out the values of variables as below :-

$$x = \frac{Dx}{D}, y = \frac{Dy}{D} \text{ and } z = -\frac{Dz}{D}$$

Let us illustrate application of Cramer's rule to linear equations.

Illustration - 13 : Solve the equations using Cramer’s rule.

$3x + 2y = 7$

$5x - y = 3$

Solution : Using Cramer's rule, we have :-

$$D = \begin{vmatrix} 3 & 2 \\ 5 & -1 \end{vmatrix} = -3 - 10 = -13$$

$$Dx = \begin{vmatrix} 7 & 2 \\ 3 & -1 \end{vmatrix} = -7 - 6 = -13$$

$$Dy = \begin{vmatrix} 3 & 7 \\ 5 & 3 \end{vmatrix} = 9 - 35 = -26$$

$$\therefore x = \frac{Dx}{D} - \frac{-13}{-13} = 1, y = \frac{Dy}{D} = \frac{-26}{-13} = 2$$

Illustration 14 :- Solve the equations using determinants.

$x + 2y + 3z = 6$

$2x + y + z = 4$

$x + y + 2z = 4$

We have $D = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{vmatrix} = 1(2-1) - 2(4-1) + 3(2-1) = 1 - 6 + 3 = -2$

$$Dx = \begin{vmatrix} 6 & 2 & 3 \\ 4 & 1 & 1 \\ 4 & 1 & 2 \end{vmatrix} = 6(2-1) - 2(8-4) + 3(4-4) = 6 - 8 = -2$$

$$Dy = \begin{vmatrix} 1 & 6 & 3 \\ 2 & 4 & 1 \\ 1 & 4 & 2 \end{vmatrix} = 1(8-4) - 6(4-1) + 3(8-4) = 4 - 18 + 12 = -2$$

$$Dz = \begin{vmatrix} 1 & 2 & 6 \\ 2 & 1 & 4 \\ 1 & 1 & 4 \end{vmatrix} = 1(4-4) - 2(8-4) + 6(2-1) = -8 + 6 = -2$$

Applying Cramer's rule, we have :-

$x = \frac{Dx}{D}; y = \frac{Dy}{D}; z = \frac{Dz}{D}$

i.e, $x = \frac{-2}{-2}, y = \frac{-2}{-2}$ and $z = \frac{-2}{-2}$

$\therefore x = 1, y = 1$ and $z = 1$

Illustration - 15 : Examine and solve the following homogeneous equations :-

x+y+z = 0

9x+3y+2z=0

2x+y+2z = 0

Solution : In the homogeneous equations :-

x+y+z = 0

9x+3y+2z=0

2x+y+2z = 0

Where, $D = \begin{vmatrix} 1 & 1 & 1 \\ 9 & 3 & 2 \\ 2 & 1 & 2 \end{vmatrix} = -7 \text{ i.e, } D \neq 0$

$$Dx = \begin{vmatrix} 0 & 1 & 1 \\ 0 & 3 & 2 \\ 0 & 1 & 2 \end{vmatrix} = 0, Dy = \begin{vmatrix} 1 & 0 & 1 \\ 9 & 0 & 2 \\ 2 & 0 & 2 \end{vmatrix} = 0$$

and $Dz = \begin{vmatrix} 1 & 1 & 0 \\ 9 & 3 & 0 \\ 2 & 1 & 0 \end{vmatrix} = 0$

As Dx=Dy=Dz=0, the system of equations is called system of homogeneous equations.

Obviously, $x = \frac{Dx}{D}, y = \frac{Dy}{D}, z = \frac{Dz}{D}$

(ii) $x = \frac{0}{-7}, y = \frac{0}{-7}, z = \frac{0}{-7}$

$\therefore$ x = 0, y=0 and z=0

Illustration - 16 : The cost of 2 books, 1 pen and 2 notebooks is ₹ 375. The cost of 1 book, 1 pen and 1 notebook is ₹ 225. The cost of 9 books, 3 pens and 2 notebooks is ₹ 875, using Cramer's rule, find out the price of each item.

Solution : Let the price of each book, each pen and each notebook be ₹ x, ₹ y, and ₹ z respectively. According to the question, we have :-

$2x+y+2z=375$		$x+y+z = 225$
$x+y+z=225$	or	$9x+3y+2z = 875$
$9x+3y+2z=875$		$2x+y+2z = 375$

Now the determinant of the coefficients of x,y and z is :-

$$D = \begin{vmatrix} 1 & 1 & 1 \\ 9 & 3 & 2 \\ 2 & 1 & 2 \end{vmatrix} = -7$$

$$Dx = \begin{vmatrix} 225 & 1 & 1 \\ 875 & 3 & 2 \\ 375 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 0 \\ 200 & 3 & -1 \\ 150 & 1 & 1 \end{vmatrix}$$

Replacing c_1 by c_1-225c_2 and c_3-c_2 :–

$$= -1(200+150)$$

$$= -350 \text{ .... (i)}$$

$$Dy = \begin{vmatrix} 1 & 225 & 1 \\ 9 & 875 & 2 \\ 2 & 375 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 9 & -1150 & -7 \\ 2 & -75 & 0 \end{vmatrix}$$

Replacing c_2 by c_2-225c_1 and c_3 by c_3-c_1

$$= 1(0-525)$$

$$= -525 \text{ .... (ii)}$$

$$Dz = \begin{vmatrix} 1 & 1 & 225 \\ 9 & 3 & 875 \\ 2 & 1 & 375 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 9 & -6 & 200 \\ 2 & -1 & 150 \end{vmatrix}$$

Replacing c_3 by $c_3 - 225c_2$ and c_2 by $c_2 - c_1$

$= (-900+200)$

$= -700$ .... (iii)

By Cramer's rule we have :-

$$x = \frac{Dx}{D}, y = \frac{Dy}{D}, z = \frac{Dz}{D};$$

$$\therefore x = \frac{-350}{-7}, y = \frac{-525}{-7} \text{ and } z = \frac{-700}{-7}$$

So, $x = 50$, $y = 75$ and $z = 100$

Ans - The costs of each book, each pen and each note book is ₹ 50, ₹ 75 and ₹ 100 respectively.

1.6. Questions

Multiple choice Questions

1. Choose the correct answer from the following alternatives.

(i) The value of $\begin{vmatrix} 3 & 2 \\ 1 & 4 \end{vmatrix}$ is

(a) 14 (b) -10 (c) 10 (d) 12

(ii) Cramare's rule solve a system of liner equations using.

(a) Matrices (b) differentiation (c) Integration (d) determinant

(iii) Under Cramer's rule $x = \frac{Dx}{D}, y = \frac{Dy}{D}, z = \frac{Dz}{D}$

Where D is not equal to :

(a) One (b) two (c) three (d) Zero

(iv) The determinant $\begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 4 & 5 & 6 \end{vmatrix}$ is a determinant of order :

(a) one (b) two (c) three (d) four

(v) The matrix associated with determinant is a

(a) Rectangular matrix (b) Square matrix

(c) Both square and rectangular matrix.

(d) Triangular Matrix

(vi) The minor of the element a_{32} is

$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ is ;

(a) $\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$ (b) $-\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$ (c) $\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$

(d) $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$

(vii) The co-factor of the element a_{32} in

$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ is :-

a) $\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$ (b) $-\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$ (c) $\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$

d) $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$

(viii) The minior and co-factor of 5 in $\begin{vmatrix} 2 & 3 & 4 \\ -1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix}$

are respectively :-

(a) 10, –10 (b) –10, –10 (c) 10, 10 (d) –10, 10

(ix) If all the elements of a row / column of a determinant arte zeroes, the value of the determinant is -

(a) One (b) Two (c) Three (d) Zero

(x) If two rows / column of a determinant are identical, i.e, $\begin{vmatrix} a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

the value of the determinant is :

(a) k (b) a (c) zero (d) Two

(xi) The expansion of the determinant

$\begin{vmatrix} 4 & 5 & 6 \\ -1 & 7 & 3 \\ 2 & 3 & 8 \end{vmatrix}$ along second row is

a) $-1\begin{vmatrix} 5 & 6 \\ 3 & 8 \end{vmatrix} - 7\begin{vmatrix} 4 & 6 \\ 2 & 8 \end{vmatrix} + 3\begin{vmatrix} 4 & 5 \\ -1 & 7 \end{vmatrix}$

b) $1\begin{vmatrix} 5 & 6 \\ 3 & 8 \end{vmatrix} + 7\begin{vmatrix} 4 & 6 \\ 2 & 8 \end{vmatrix} - 3\begin{vmatrix} 4 & 5 \\ -1 & 7 \end{vmatrix}$

c) $5\begin{vmatrix} -1 & 3 \\ 2 & 8 \end{vmatrix} - 7\begin{vmatrix} 4 & 6 \\ 2 & 8 \end{vmatrix} + 3\begin{vmatrix} 4 & 6 \\ -1 & 7 \end{vmatrix}$

d) $3\begin{vmatrix} 5 & 6 \\ 3 & 8 \end{vmatrix} + 7\begin{vmatrix} 4 & 6 \\ 2 & 8 \end{vmatrix} - 1\begin{vmatrix} 4 & 5 \\ -1 & 7 \end{vmatrix}$

(xii) ω is :

(a) Cube-roof of unity (b) Square root of unity

(c) Imaginary cube-roof of unity (d) Unity

(xiii) The minor of the elements of the determinant $\begin{vmatrix} 2 & 3 & 2 \\ 1 & 2 & 1 \\ 3 & 1 & 3 \end{vmatrix}$ are

(a) 5, 0, –5, 7, 0. –7, –2, 0, 2.

(b) –5, 0, 5, 7, 0, –7, 1, 0, 1.

(c) 5, 0, –5, 7, 0, –7, –1, 0, 1.

(d) 5, 0, 5, 7, 0, 7, 1, 0, 1.

xiv) The co-factor of the following determinant

$\begin{vmatrix} 7 & 5 & 4 \\ 2 & 0 & 3 \\ 1 & 2 & 0 \end{vmatrix}$ are :-

(a) –6, 3, 4, –8, –4, –9, 15, –13, –10

(b) 6, 3, 4, 8, 4, 9, 15, 13, 10

(c) –6, –3, 4, –8, –4, 9, 15, –13, –10

(d) –6, –3, –4, –9

Q.2 Answer the following questions in one word / term.

(i) The value of the determinant $\begin{vmatrix} 3 & 3 \\ 1 & 2 \end{vmatrix}$ is :

(ii) If two rows / columns of a determinant are iedentical then, the value of the determinant is :

(iii) If rows of determinant are changed to columns or vice-vera the value of determinant remain :

(iv) The value of $1+\omega+\omega^2$ is always equal to :

(v) The number of rows and number of column of a determinant is always :

(vi) Cramar's rule can be applied only when the value of determinant is :

(vii) If two adjacent rows or column of a determinant are interchanged, the sign of the determinant is :

(viii) If all the element of a row/column of a determinant are zero, then the value of determinant would be.

Q. 3. Fill in the blanks of the following :

(i) The value of the determinant $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ is __________.

(ii) The value of the determinant $\begin{vmatrix} \text{see x} & \tan x \\ \tan x & \sec x \end{vmatrix}$ is __________.

(iii) The value of ω is equal to __________.

(iv) To evaluate the value of determinant under sarrus diagram method, the product of all the elements lies in down-ward arrow are _________.

(v) We generally expand the rows columns having maximum number of ________ for finding the value of determinant.

(vi) Given $A = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$

∴ The co factor of element 3 is __________.

(vii) ∴ The value of determinant $\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix}$ is ________.

(viii) Sarrus diagram can be applied for determinant of order ________ only.

(ix) The determinant which has two rows and two columns, is called a determinant of __________ two.

Q.4 Correct the under lined portion of the following :

(i) A determinant of of order 3 contains <u>6</u> elements.

(ii) A matrix having one element is called a determinant of order <u>two</u>.

(iii) If every element of a third order determinant of | A | is multiplied by 5, then the value of the new determinant in 5| A |.

(iv) The determinant which is obtained by changing rows into the corresponding columns or the columns into the corresponding rows is called <u>inverse</u> of a determinant.

(v) The product of the minor and (–1) i+j of an element is called the <u>minor</u> of that element.

Q.5 Answer the following in one sentence each.

(i) Minor

(ii) Co-factor

(iii) Cramer's rule

(iv) Sarrus expansion

(v) Consistent system of equation.

(vi) Inconsistent system of equation.

6. Evaluate (Answer the following within 30 words each)

(i) $\begin{vmatrix} 4 & 4 \\ 8 & 8 \end{vmatrix}$ (ii) $\begin{vmatrix} 7 & 9 & 15 \\ 0 & 0 & 0 \\ 8 & 7 & 5 \end{vmatrix}$ (iii) $\begin{vmatrix} 1 & 0 & 4 \\ 2 & 0 & 5 \\ 3 & 0 & 6 \end{vmatrix}$

(iv) $\begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+ca & c^2 \end{vmatrix}$ (v) $\begin{vmatrix} 4 & 0 & 0 \\ 2 & 5 & 0 \\ 10 & 0 & 9 \end{vmatrix}$

(vi) $\begin{vmatrix} a & a+1 \\ a+2 & a+3 \end{vmatrix}$ vii $\begin{vmatrix} i^{103} & 3 & i^{101} \\ i^{56} & 5 & i^{54} \\ i^{23} & 7 & i^{21} \end{vmatrix}$ where $i = \sqrt{-1}$

7. **Answer the following within 50 words :-**

Write the minors and co factors of each element of the following determinants.

a)

(i) $\begin{vmatrix} 3 & -2 \\ 1 & 7 \end{vmatrix}$ (ii) $\begin{vmatrix} 3 & -2 & 5 \\ 4 & 0 & 4 \\ 1 & 7 & 6 \end{vmatrix}$

b. Write the minors and co-factors of each element of the second column of each determinant.

(i) $\begin{vmatrix} 2 & 4 & 3 \\ 1 & 2 & 3 \\ 2 & 3 & 1 \end{vmatrix}$ (ii) $\begin{vmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{vmatrix}$

c). (i) Evaluate the determinant by expanding along the first row.

$$\begin{vmatrix} 1 & b & c \\ 0 & a & 0 \\ 0 & 0 & a \end{vmatrix}$$

(ii) Evaluate the determinant by expanding along the second row.

$$\begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix}$$

(iii) Evaluate the determinant by expanding along the third row.

$$\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix}$$

8. Evaluate the determinant by expanding along second column and along second row and show that both the expansions give the same result.

$$\begin{vmatrix} 6 & 5 & -3 \\ 1 & 5 & -2 \\ 2 & 8 & 4 \end{vmatrix}$$

9. Evaluate the determinant with the help of surrus dIagram

(i) $\begin{vmatrix} 5 & 1 & -3 \\ 5 & 3 & -2 \\ 8 & 1 & 4 \end{vmatrix}$ (ii) $\begin{vmatrix} 6 & 1 & -3 \\ 1 & 3 & -2 \\ 2 & 1 & 4 \end{vmatrix}$

10.(a) Show that the value of determinant becomes zero, if any row / column of the determinant consists of zeros only.

b) Show that $\begin{vmatrix} 3+2 & 2 & 5 \\ 2+3 & 1 & 4 \\ 1+4 & 2 & 3 \end{vmatrix} = \begin{vmatrix} 3 & 2 & 5 \\ 2 & 1 & 4 \\ 1 & 2 & 3 \end{vmatrix} + \begin{vmatrix} 2 & 2 & 5 \\ 3 & 1 & 4 \\ 4 & 2 & 3 \end{vmatrix}$

11. Expand the determinant without using properties of determinant.

$$\begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+ca & c^2 \end{vmatrix}$$

12. Using properties of determinants, prove that :-

(a) $\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix} = (a-b)(b-c)(c-a)$

(b) $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$

(c) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)$

(d) $\begin{vmatrix} x & x^2 & yz \\ y & y^2 & zx \\ z & z^2 & xy \end{vmatrix} = (x-y)(y-z)(z-x)(xy+yz+zx)$

(e) $\begin{vmatrix} (b+c)^2 & a^2 & bc \\ (c+a)^2 & b^2 & ca \\ (a+b)^2 & c^2 & ab \end{vmatrix} = (a^2+b^2+c^2)(a+b+c)(b-c)(c-a)(a-b)$

(Hints : take $c_1 = c_1 + c_2 - 2c_3$

(f) $\begin{vmatrix} a^3-x^3 & a^2 & a \\ b^3-x^3 & b^2 & b \\ c^2-x^3 & c^2 & c \end{vmatrix} = (abc-x^3)(a-b)(b-c)(a-c)$

(Hints : Express it as the sum of two sets)

13. Using properties of determinants, prove that :

(a) $\begin{vmatrix} -a^2 & ab & ac \\ ba & -b^2 & bc \\ ac & bc & -c^2 \end{vmatrix} = 4a^2b^2c^2$

(b) $\begin{vmatrix} 0 & ab^2 & ac^2 \\ a^2b & 0 & bc^2 \\ a^2c & b^2c & 0 \end{vmatrix} = 2a^3b^3c^3$ (Hints : take common a, b and c)

(c) $\begin{vmatrix} x^2 & 2xy & y^2 \\ y^2 & x^2 & 2xy \\ 2xy & b^2 & a^2 \end{vmatrix} = (a^3+b^3)^2$

(d) $\begin{vmatrix} 1 & x & x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix} = (1-x^3)^2$

(e) $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3$

f) $\begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix} = 2(a+b+c)^3$ (Hints : apply $c_1+c_2+c_3=c_1$)

14. Prove that : $\begin{vmatrix} x & x+y & x+y+z \\ 2x & 3x+2y & 4x+3y+2z \\ 3x & 6x+by & 10x+6y+2z \end{vmatrix} = x^3$

15. Prove that :- (a) $\begin{vmatrix} 18 & 9 & 6 \\ 12 & 4 & 4 \\ 6 & 8 & 2 \end{vmatrix} = 0$ (b) $\begin{vmatrix} 1 & x & y+z \\ 1 & y & z+x \\ 1 & z & x+y \end{vmatrix} = 0$

(c) $\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix} = 0$ (d) $\begin{vmatrix} a+1 & \omega & \omega^2 \\ \omega & a+\omega^2 & 1 \\ \omega^2 & 1 & a+\omega \end{vmatrix} = 0$

16. Prove that :- $\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = ab+bc+ca+abc$

17. Prove that : $\begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ac & bc & c^2+1 \end{vmatrix} = 1+a^2+b^2+c^2$

(Hints : take common a from R_1 & C_1, b from R_2 & C_2 and c from R_3 & C_3)

18. Final the minor and co-factor of the elements a_{11}, a_{13} and a_{31} of the determinant.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

19. Prove that : $\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = abc(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c})$

20. Prove that : $a\begin{vmatrix} a & b & c \\ a-b & b-c & c-a \\ b+c & c+a & a+b \end{vmatrix} = a^3+b^3+c^3-3abc$

21. Solve the following question of liner equation using determinants.

(i) $3x+4y=11$, $4x+3y=10$ (ii) $2x+y=10$, $4x-3y=0$ (iii) $6x-5y=0$, $y-x=1$

(iv) $x+y=3$, $3x+4y=10$ (v) $3x+y=7$, $2x+3y=21$ (vi) $4x+3y=8$, $6x+7y=17$

22. Solve the following by using Cramer's rule :

(i) $x+2y=249$, $2x-y=103$ (ii) $2x+y-25=0$, $y+3-2x=0$

23. Solve the following system of linear equations by using Cramer's rule.

(i) $6x+y-3z=5$, $x+3y-2z=5$, $2x+y+4z=8$

(ii) $x+2y+3z=6$, $2x+y+z=4$, $x+y+2z=4$

24. Using determinants, solve the following :

(i) $y+z=0$

$x+y-2z=2$

$2x-5y+z=4$

(ii) $\frac{x}{2}+\frac{y}{3}+\frac{z}{5}=3$

$\frac{x}{3}+\frac{2y}{5}+\frac{z}{3}=27$

$\frac{x}{10}+\frac{y}{10}+\frac{z}{10}=9$

25. Using Cramer's rule, solve the following :

$a^2x + b^2y + c^2z = d^2$

$ax + by + cz = d$

$x + y + z = 1$

26. Solve the following by using Cramer's rule :

(i) $x + y + 2z = 5$

$x + 2y + z = 6$

$-x - y + z = -2$

(ii) $7x + y + z = 4$

$2x + 2y - z = 2$

$7x - 3z = 13$

27. Solve using Cramer's rule

$5x + y + z = 10,$

$7x - y + 5z = 21$

$2x + 5y + 2z = 18$

28. The sum of three numbers is 14. If we multiply the first number by 2 and add the second number we get 13. If we add the first number and second to two times the third number, we get 18, find the number.

29. Cost of 110g of sugar, 1 kg of rice and 1 kg of ghee is Rs. 180; cost of 2 kgs of sugar, 1 kg of rice and 1 kg of ghee is Rs. 230. Similarly the cost of 1 kg of sugar, 2 kgs of rice and 2 kg of ghee is Rs. 310. Find the cost of each per kg.

30. The cost of 9 tables and 12 chairs is Rs. 279 and that of 8 tables and 16 chairs is Rs. 312. Find the cost of a chair and a table.

31. The perimeter of a triangle is 45 cm. The longest side exceeds the shortest by 8 cm and the sum of the length of the longest and shortst side is twice the length of the other side. Find the length of the sides.

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Answer

1. (i) c (ii) d (iii) d (iv) c (v) b (vi) a
(vii) b (viii) a (ix) d (x) c
(xi) b (xii) c (xiii) c (xiv) a

2. (i) 3 (ii) Zero (iii) same (iv) Zero
(v) equal (vi) Non-singular.

3. (i) $a_{11}a_{22} - a_{21}a_{12}$ (ii) $Sec^2x - tan^2x$
(iii) $\sqrt[3]{1}$ (iv) added (v) zeroes
(vi) –2 (vii) zero (viii) Three (ix) Order

4. (i) 9 (ii) One (iii) 125A (iv) transpose (v) co-factor

5. (i) The minor of any element is the determinant obtained by deleting the row and the column in which that element lies.

(ii) The product of M_{ij} and $(-1)^{i+j}$ is called the co-factor of that element, where M_{ij} is the minor.

(iii) Cramer's rule is a rule which is applied to solve linear equations under determinant.

(iv) A simple method used for evaluating the determinant of order three with help of arrows is called sarrus diagram.

(v) When there is soluation to the system of equations, the system is called consistent, i.e., $D \neq 0$.

(vi) When there is no solutation to the system of equations, it is called inconstent i.e., $(D = 0)$

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CHAPTER - 2

MATRICES

STRUCTURE :

2.1 Meaning of Matrix

When we have to express a single value, there is hardly any difficulty. But when we have to express several values of several variables in a single arrangement, it becomes complicated. The study of matrices simplifies this difficulty to a great extent.

In a simpler sense, a matrix is a rectangular array (arrangement) of numbers or symbols, arranged in rows and columns. For example,

(i) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ (ii) $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ (iii) $\begin{pmatrix} 7 & 2 \\ 8 & 1 \\ 9 & 0 \end{pmatrix}$

Suppose we have to express the following information in matrix form, then,

	Ram	Shyam	Hari
Age	17	18	16
Marks	90	95	97

$$= \begin{bmatrix} 17 & 18 & 16 \\ 90 & 95 & 97 \end{bmatrix}$$

The numbers of a matrix are called elements of a matrix or entries of a matrix.

Let us analyze the features of the following matrix.

$$\begin{array}{lccc} & & \text{First Column} & \text{Second Column} \\ & & \downarrow & \downarrow \\ \text{First row} & \rightarrow & 7 & 2 \\ \text{Second row} & \rightarrow & 8 & 1 \\ \text{Third row} & \rightarrow & 9 & 0 \end{array}$$

Numbers in this same vertical line are called a column, numbers in the same horizontal line are called a row.

The number of rows and columns of a matrix determines the order of a matrix. In the above example, as there are 3 rows and 2 columns, the order of the matrix is 3×2. The first number of the order represents the number of rows, whereas the latter represents the number of columns.

Each element is an intersectional placement of a particular row & a particular column. In the above example, the element 2 is placed at the intersection of 1st row and 2nd column. Hence, the element may be represented by a_{12}. For general term i is row and j is column, we write the element as a_{ij}.

It is customary to denote a matrix by a capital letter & the matrix by parantheses ( ) or by square [ ] or by curly brackets { }. It should not be expressed by two straight lines like | |, which will otherwise mean a determinant. A matrix consists of 'm' rows 'n' columns is called a matrix of order m×n (read as a m by n matrix). In general, the different elements of the matrix will be represented as below :–

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{1j} & a_{1n} \\ a_{21} & a_{22} & a_{2j} & a_{2n} \\ a_{i1} & a_{i2} & a_{ij} & a_{in} \\ a_{m1} & a_{m2} & a_{mj} & a_{mn} \end{bmatrix}$$

The element a_{ij} appears in the i^{th} row and j^{th} column. The matrix may be noted as $A = \left[a_{ij}\right]_{mn}$ where, m×n indicates the order of the matrix x.

Illustration 1 :

$$A = \begin{pmatrix} 2 & 4 & 9 & 7 \\ 8 & 9 & 2 & 8 \\ 3 & 2 & 0 & 1 \end{pmatrix}$$

From the above matrix, find :–

(i) Number of rows and columns

(ii) Order of the matrix

(iii) Rows and Columns separately

(iv) a_{24}, a_{32}, a_{12}, and a_{23}

Solution :

(i) There are 3 rows and 4 columns.

(ii) The order of matrix is 3×4.

(iii) Rows are (2 4 9 7), (8 9 2 8) and (3 2 0 1)

and columns are $\begin{pmatrix} 2 \\ 8 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 9 \\ 2 \end{pmatrix}, \begin{pmatrix} 9 \\ 2 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 7 \\ 8 \\ 1 \end{pmatrix}$.

(iv) $a_{24'} = 8$, $a_{32} = 2$, $a_{12} = 4$ and $a_{23} = 2$,

Illustration 2 : Construct a 3×2 matrix whose elements are $a_{ij} = 3_i + 2_j$.

Solution :

The 3×2 matrix is represented by :

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$$

As $a_{ij} = 3_i + 2_j$, the respective elements are :

$a_{11} = 3\times1 + 2\times1 = 5$

$a_{12} = 3\times1 + 2\times2 = 7$

$a_{21} = 3\times2 + 2\times1 = 8$

$a_{22} = 3\times2 + 2\times2 = 10$

$a_{31} = 3\times3 + 2\times1 = 11$

$a_{32} = 3\times3 + 2\times2 = 13$

Hence, the required matrix is $\begin{pmatrix} 5 & 7 \\ 8 & 10 \\ 11 & 13 \end{pmatrix}$

2.2 TYPES OF MATRICES :

The different types of matrix may be defined as below :

(I) Row matrix : A matrix is said to be a row matrix, if it has only one row and any number of columns. For example (2 7 0) is a row matrix of order 1×3.

(II) Column matrix : A matrix is said to be a column matrix, if it has only one column and any number of rows. For example $\begin{bmatrix} 2 \\ 0 \\ 7 \end{bmatrix}$ is column matrix of order (3 x 2)

(III) Zero matrix or null matrix : A matrix is said to be a zero or null matrix, if all its elements are zero.

For example, $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ are null matrices of different order. Usually, null matrices are denoted by 0.

(IV) Square matrix : A matrix is said to be a square matrix, if its number of rows is equal to its number of columns.

For example, $\begin{bmatrix} 2 & 3 & 4 \\ 4 & 5 & 8 \\ 1 & 2 & 3 \end{bmatrix}$ is a square matrix of order 3 × 3.

(V) Diagonal matrix : A square matrix $A = [a_{ij}]$ is said to be a diagonal matrix if all its elements other than that of principal diagonal are zero, i.e. if $a_{ij} = 0$ for $i \neq j$.

For example : $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

where the main diagonal elements of A and B are 2, 3, and 1, 2, 3 respectively & other elements are zero.

(VI) Identity or Unit matrix : A square matrix where elements in the main diagonal are all '1' (one) & rest are all zero, is called an identity matrix or unit matrix. For example :

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is an identity matrix of order 3 × 3.

Such identity matrix is denoted as I.

(VII) Scalar matrix : A square matrix whose diagonal elements are equal & all non-diagonal elements are zero is called a scalar matrix. Thus unit matrix & null matrix are also called scalar matrix.

For example, $\begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ is a scalar matrix.

(VIII) Triangular matrix : A square matrix in which all elements above the principal diagonal are zero and rest are non-zero, is called a lower triangular matrix.

For example $\begin{bmatrix} 1 & 0 & 0 \\ 4 & 2 & 0 \\ 7 & 2 & 3 \end{bmatrix}$. Similarly, a square matrix in which all elements below the principal diagonal are zero and rest are non-zero, is called an upper triangular matrix.

For example, $\begin{bmatrix} 1 & 4 & 7 \\ 0 & 2 & 8 \\ 0 & 0 & 3 \end{bmatrix}$ is an upper triangular matrix.

(IX) Singular matrix : If the determinants of a matrix is zero, the matrix is called singular matrix. Similarly, if the determinant of the matrix is not zero, the matrix is known as non-singular matrix.

(X) Sub-matrix : A matrix obtained by deleting any row/rows or column/columns of the matrix is known as a sub-matrix. For example,

if $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$, then $\begin{bmatrix} 2 & 3 \\ 5 & 6 \end{bmatrix}$ is a sub-matrix.

Illustration 3 : Match the columns.

	Column A		**Column B**
(a)	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	(i)	Zero matrix
(b)	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	(ii)	Unit matrix
(c)	$\begin{bmatrix} 2 & 3 & 4 \end{bmatrix}$	(iii)	Upper triangular matrix

(d) $\begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$ (iv) Diagonal matrix

(e) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ (v) Row matrix

(f) $\begin{bmatrix} 1 & 7 & 2 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$ (vi) Column matrix

Solution : (a) ii (d) vi

(b) i (e) iv

(c) v (f) iii

2.3. Algebra of Matrices :-

In this section, we shall deal with the following.

1) Equivalent matrix and equality of matrices.
2) Multiplication of matrix by a scalar.
3) Addition and subtraction of matrices.
4) Multiplication of Matrices.
5) Transpose of a matrix.
6) Adjoint of a matrix.
7) Inverse of a matrix.

Adjoint of a matrix and inverse of a matrix will be death with towards the end of this chapter.

2.3.1 Equivalent of a matrix and Equality of matrices :-

A matrix is said to be equivalent to another matrix, if its number of rows & column are equal to that of the other matrix. For example, $\begin{pmatrix} 1 & 2 & 3 \\ 7 & 8 & 9 \end{pmatrix}_{2\times3}$ is equivalent to

$\begin{pmatrix} 8 & 11 & 12 \\ 10 & 15 & 17 \end{pmatrix}_{2\times3}$ as both have 2 rows and 3 columns. It is denoted by the symbol $\simeq$. The matrices are said to be equal if (i) they are of same order and (ii) each element of a matrix is equal to the corresponding element of the other, i.e., let $A = [a_{ij}]$, $B = [b_{ij}]$, then matrix A is equal to matrix B, if $a_{ij} = b_{ij}$, for all values of i and j.

For example, if $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 4 & 7 \\ 5 & 6 \end{bmatrix}$, then a = 4, b = 7, c = 5 and d = 6.

In other words, if a = 4, b = 7, c = 5 and d = 6, then

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 4 & 7 \\ 5 & 6 \end{bmatrix}$$

Example 3 : Find the values of x, y, z, r, if :

$$\begin{bmatrix} z & 2x+y \\ r & 3x-2y \end{bmatrix} = \begin{bmatrix} 3 & 7 \\ 5 & 0 \end{bmatrix}$$

Solution :

$$\text{As } \begin{bmatrix} z & 2x+y \\ r & 3x-2y \end{bmatrix} = \begin{bmatrix} 3 & 7 \\ 5 & 0 \end{bmatrix}$$

Equating the corresponding elements on both sides

we have, $z = 3$

$r = 5$

$2x + y = 7$

and $3x - 2y = 0$

Solving $2x + y = 7$ and $3x - 2y = 0$, we get $x = 2$ and $y = 3$.

2.3.2 Multiplication of a matrix by a scalar :

If $A = [a_{ij}[$ is a matrix & 'k' is a scalar, then $kA = [ka_{ij}]$, that is, when a matrix is multiplied by a scalar, each element of the matrix gets multiplied by the same scalar. For example, if

$$A = \begin{bmatrix} 4 & 5 \\ 7 & 8 \end{bmatrix}, \text{then } 5A = \begin{bmatrix} 4\times5 & 5\times5 \\ 7\times5 & 8\times5 \end{bmatrix} = \begin{bmatrix} 20 & 25 \\ 35 & 40 \end{bmatrix}$$

Example 4 : If $A = \begin{bmatrix} 2 & -2 & 0 \\ -1 & 2 & 1 \\ 3 & 5 & -7 \end{bmatrix}$, find $-A$.

Solution : $$A = \begin{bmatrix} 2 & -2 & 0 \\ -1 & 2 & 1 \\ 3 & 5 & -7 \end{bmatrix}$$

$$-A = (-1)\,A = \begin{bmatrix} -2 & 2 & 0 \\ 1 & -2 & -1 \\ -3 & -5 & 7 \end{bmatrix}.$$

2.3.3 Addition and subtraction of matrices :

If $A = [a_{ij}]$ and $B = [b_{ij}]$ be two matrices of same order, their sum or difference is another matrix, whose $(ij)^{th}$ element is $a_{ij} + b_{ij}$ or $a_{ij} - b_{ij}$. That is, on addition or subtraction, the corresponding elements are added or subtracted. For example,

If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, and $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$, then

$$A + B = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{bmatrix} = \left[a_{ij} + b_{ij}\right] \text{ and}$$

$$A - B = \begin{bmatrix} a_{11} - b_{11} & a_{12} - b_{12} \\ a_{21} - b_{21} & a_{22} - b_{22} \end{bmatrix} = \left[a_{ij} - b_{ij}\right]$$

Illustration 4 :

If $A = \begin{bmatrix} 2 & 3 \\ 4 & 0 \\ -2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ -2 & 4 \end{bmatrix}$, find $A + B$ and $A - B$.

Solution :

$$A+B=\begin{bmatrix}2 & 3\\ 4 & 0\\ -2 & 1\end{bmatrix}+\begin{bmatrix}1 & 2\\ 3 & 1\\ -2 & 4\end{bmatrix}=\begin{bmatrix}2+1 & 3+2\\ 4+3 & 0+1\\ -2-2 & 1+4\end{bmatrix}=\begin{bmatrix}3 & 5\\ 7 & 1\\ -4 & 5\end{bmatrix}$$

$$A-B=\begin{bmatrix}2 & 3\\ 4 & 0\\ -2 & 1\end{bmatrix}-\begin{bmatrix}1 & 2\\ 3 & 1\\ -2 & 4\end{bmatrix}=\begin{bmatrix}2-1 & 3-2\\ 4-3 & 0-1\\ -2+2 & 1-4\end{bmatrix}=\begin{bmatrix}1 & 1\\ 1 & -1\\ 0 & -3\end{bmatrix}.$$

Properties of matrix addition :

(i) Matrix addition is commutative.

If A and B are matrices of same order, then, $A+B=B+A$

(ii) Matrix addition is associative.

If A, B and C are three matrices of same order, then, $(A+B)+C=A+(B+C)$

(iii) Existence of addition identity.

If O is a null matrix of same order as that of matrix A, then :

$A+0=A=0+A$

(iv) Existence of additive inverse.

To each matrix A, there correspond a matrix B, such that,

$A+B=0=B+A$, where 0 is a null matrix i.e. $B=(-A)$.

For example, if $A=\begin{pmatrix}2 & 1\\ 3 & 1\\ 4 & 2\end{pmatrix}$

then, $A+(-A)=\begin{pmatrix}2 & 1\\ 3 & 1\\ 4 & 2\end{pmatrix}+\begin{pmatrix}-2 & -1\\ -3 & -1\\ -4 & -2\end{pmatrix}=\begin{pmatrix}0 & 0\\ 0 & 0\\ 0 & 0\end{pmatrix}=0.$

Also, $(-A)+A=\begin{pmatrix}0 & 0\\ 0 & 0\\ 0 & 0\end{pmatrix}=0$.

$\therefore\ A+(-A)=0=-A+A.$

Illustration 5 :

If $A=\begin{bmatrix}1 & -1 & 3\\ 2 & 4 & 5\\ 2 & 3 & 6\end{bmatrix}$, $\quad B=\begin{bmatrix}7 & 1 & 2\\ -1 & -2 & -3\\ 4 & 5 & 6\end{bmatrix}$, find $5A-3B$.

Solution : $5A-3B=5\begin{bmatrix}1 & -1 & 3\\ 2 & 4 & 5\\ 2 & 3 & 6\end{bmatrix}+(-3)\begin{bmatrix}7 & 1 & 2\\ -1 & -2 & -3\\ 4 & 5 & 6\end{bmatrix}$

$$=\begin{bmatrix}5 & -5 & 15\\ 10 & 20 & 25\\ 10 & 15 & 30\end{bmatrix}+\begin{bmatrix}-21 & -3 & -6\\ 3 & 6 & 9\\ -12 & -15 & -18\end{bmatrix}$$

$$=\begin{bmatrix}-16 & -8 & -9\\ 13 & 26 & 34\\ -2 & 0 & 12\end{bmatrix}$$

2.3.4 Multiplication of matrices :

The product of a row (row matrix) and a column (column matrix) is defined by :

$$(a_1, a_2, a_3, \ldots\ldots\ldots\ldots, a_n)_{1\times 4}\begin{pmatrix}b_1\\ b_2\\ b_3\\ \vdots\\ \vdots\\ b_n\end{pmatrix}_{4\times 1}=\left(a_1b_1+a_2b_2+a_3b_3+\ldots\ldots+a_nb_n\right)_{1\times 1}$$

which is the sum of products of corresponding elements. For example,

$$\begin{pmatrix}1 & 2 & 3 & 4\end{pmatrix}_{(1\times4)}\begin{pmatrix}2\\3\\0\\1\end{pmatrix}_{(4\times1)} = (1.2+2.3+3.0+4.1) = (12)_{1\times1}.$$

Two matrices A and B can be multiplied only if **number of columns of the first matrix is equal to number of rows of second matrix.** If A is an (m × n) matrix and B is an (n × z) matrix, then the two matrices are multiplicable and AB is an m × z matrix. When the number of columns of the first matrix is not equal to the number of rows of the second matrix, their product is not defined. Let us take an example to understand the computation.

Illustration 6 : If $A = \begin{pmatrix}2 & 3 & 0\\4 & 1 & 2\\1 & 2 & 3\end{pmatrix}$ and $B = \begin{pmatrix}1 & 2\\2 & 4\\3 & 0\end{pmatrix}$ find AB and BA.

Solution : $AB = \begin{pmatrix}2 & 3 & 0\\4 & 1 & 2\\1 & 2 & 3\end{pmatrix}_{3\times3} \cdot \begin{pmatrix}1 & 2\\2 & 4\\3 & 0\end{pmatrix}_{3\times2}$

$$= \begin{pmatrix}2.1+3.2+0.3 & 2.2+3.4+0.0\\4.1+1.2+2.3 & 4.2+1.4+2.0\\1.1+2.2+3.3 & 1.2+2.4+3.0\end{pmatrix}_{3\times2}$$

$$= \begin{pmatrix}8 & 16\\12 & 12\\14 & 10\end{pmatrix}_{3\times2}$$

$$B\cdot A = \begin{pmatrix}1 & 2\\2 & 4\\3 & 0\end{pmatrix}_{3\times2} \cdot \begin{pmatrix}2 & 3 & 0\\4 & 1 & 2\\1 & 2 & 3\end{pmatrix}_{3\times3}$$

B.A is not defined as the number of columns of the first matrix B is not equal to the numbers of rows of the second matrix A.

Properties of matrix multiplication :

(i) Matrix multiplication is associative, that is $(AB)\cdot C = A\cdot(BC)$, whenever all products are defined.

(ii) Matrix multiplication distributes over addition.

That is, $A\cdot(B+C) = AB + AC$, provided the products and sum exist.

(iii) Matrix multiplication is not commutative.

If A and B are two matrices, then $AB \neq BA$.

Illustration 7 : If $A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 3 \\ 4 & 0 & 7 \end{pmatrix}$, $B = \begin{pmatrix} 4 & 1 \\ 2 & 1 \\ 1 & 0 \end{pmatrix}$ and $C = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 5 \end{pmatrix}$

Show that $(AB)\cdot C = A\cdot(BC)$.

Solution : $$AB = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 3 \\ 4 & 0 & 7 \end{pmatrix}\begin{pmatrix} 4 & 1 \\ 2 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 8 & 3 \\ 7 & 2 \\ 23 & 4 \end{pmatrix}$$

$$\text{Now, } (AB)\cdot C = \begin{pmatrix} 8 & 3 \\ 7 & 2 \\ 23 & 4 \end{pmatrix}\begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 5 \end{pmatrix} = \begin{pmatrix} 16 & 6 & 23 \\ 14 & 4 & 17 \\ 46 & 8 & 43 \end{pmatrix}$$

$$BC = \begin{pmatrix} 4 & 1 \\ 2 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 5 \end{pmatrix} = \begin{pmatrix} 8 & 2 & 9 \\ 4 & 2 & 7 \\ 2 & 0 & 1 \end{pmatrix}$$

$$\text{Now } A\cdot(BC) = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 3 \\ 4 & 0 & 7 \end{pmatrix}\begin{pmatrix} 8 & 2 & 9 \\ 4 & 2 & 7 \\ 2 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 16 & 6 & 23 \\ 14 & 4 & 17 \\ 46 & 8 & 43 \end{pmatrix}$$

$\therefore (AB)\cdot C = A\cdot(BC)$.

Illustration 8 : If $\begin{bmatrix} 3x & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} x \\ 3 \end{bmatrix} = 0$, find x.

Solution : $\begin{bmatrix} 3x & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} x \\ 3 \end{bmatrix} = 0$, (Null matrix)

$$\Rightarrow \begin{bmatrix} 3x+2 & 6x \end{bmatrix}_{1\times 2} \begin{bmatrix} x \\ 3 \end{bmatrix}_{2\times 1} = \begin{bmatrix} 0 \end{bmatrix}_{1\times 1}$$

$$\Rightarrow \begin{bmatrix} 3x^2 + 2x + 18x \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$$

$\Rightarrow 3x^2 + 20x = 0$ (Equating the elements on both sides)

$$\Rightarrow x(3x+20) = 0$$

$$\therefore x = 0 \quad \text{or} \quad x = -\frac{20}{3}$$

Illustration 9 : If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 2 & 1 \\ 7 & 8 & 4 \end{bmatrix}$

find AB and BA. Are they equal ?

Solution : As A and B are 3×2 and 2×3 matrices, AB and BA will be 3×3 and 2×2 matrices and hence, both AB and BA are defined.

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 3 & 2 & 1 \\ 7 & 8 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1\times3+2\times7 & 1\times2+2\times8 & 1\times1+2\times4 \\ 3\times3+4\times7 & 3\times2+4\times8 & 3\times1+4\times4 \\ 5\times3+6\times7 & 5\times2+6\times8 & 5\times1+6\times4 \end{bmatrix}$$

$$= \begin{bmatrix} 17 & 18 & 9 \\ 37 & 38 & 19 \\ 57 & 58 & 29 \end{bmatrix} \text{.....................................(1)}$$

$$\text{Now, } BA = \begin{bmatrix} 3 & 2 & 1 \\ 7 & 8 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} = \begin{bmatrix} 3\times1+2\times3+1\times5 & 3\times2+2\times4+1\times6 \\ 7\times1+8\times3+4\times5 & 7\times2+8\times4+4\times6 \end{bmatrix}$$

$$= \begin{bmatrix} 14 & 20 \\ 51 & 70 \end{bmatrix}$$

Now $AB \neq BA$.

2.3.5 Transpose of a matrix and symmetric and skew symmetric matrix :

The matrix obtained by interchanging the rows and columns of a matrix A, is called transpose of the matrix A and is denoted by A' or A^T. That is, if $A = [a_{ij}]_{m\times n}$, then $A^T = [a_{ji}]_{m\times n}$.

Let us take a numerical example.

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & -2 & 2 \end{bmatrix}, A^1 = \begin{bmatrix} 1 & 2 \\ 1 & -2 \\ -1 & 2 \end{bmatrix}$$

If the transpose of a matrix is equal to the matrix itself, then the matrix is called symmetric matrix. For example,

$$\text{If } A = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 6 & 4 \\ 1 & 4 & 7 \end{bmatrix}, \text{ then } A^1 = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 6 & 4 \\ 1 & 4 & 7 \end{bmatrix}$$

As $A = A^1$, A is called symmetric matrix. If $A^1 = -A$, then the matrix A is said to be Skew-symmetric matrix. For example,

$$\text{if} \quad A = \begin{bmatrix} 0 & 4 & 9 \\ -4 & 0 & 2 \\ -9 & -2 & 0 \end{bmatrix}, \text{ then } -A = \begin{bmatrix} 0 & -4 & -9 \\ 4 & 0 & -2 \\ 9 & 2 & 0 \end{bmatrix}$$

$$\text{and} \quad A^1 = \begin{bmatrix} 0 & -4 & -9 \\ 4 & 0 & -2 \\ 9 & 2 & 0 \end{bmatrix}$$

As $A^1 = -A$, A is a skew-symmetric matrix. Similarly, when a matrix A is multiplied with its transpose A^1, if it gives an identity matrix I, the matrix A is known as an orthogonal matrix.

2.4 APPLICATIONS

Illustration 10 : If $A = \begin{bmatrix} 2 & 3 \\ 3 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix}$, find

$2A + 3AB + B^2$.

Solution :

$$2A = 2\begin{bmatrix} 2 & 3 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 6 & 2 \end{bmatrix}$$

$$3AB = 3\begin{bmatrix} 2 & 3 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix} = 3\begin{bmatrix} 3 & 6 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 9 & 18 \\ 3 & 6 \end{bmatrix}$$

$$B^2 = B \times B = \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 2 & 4 \end{bmatrix}$$

$$\text{Now } 2A + 3AB + B^2 = \begin{bmatrix} 4 & 6 \\ 6 & 2 \end{bmatrix} + \begin{bmatrix} 9 & 18 \\ 3 & 6 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 13 & 24 \\ 11 & 12 \end{bmatrix}.$$

Illustration 11 : Show that matrix $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 2 \\ 1 & 4 & 0 \end{bmatrix}$ satisfies the equation

$A^3 - A^2 - 8A + 8I = 0$, where I and 0 are identity and null matrix of the order 3.

Solution :

$$8I = 8\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

$$8A = 8\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & 4 & 0 \end{bmatrix} = \begin{bmatrix} 8 & 16 & 0 \\ 0 & 0 & 16 \\ 0 & 32 & 0 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 2 \\ 10 & 4 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 2 \\ 10 & 4 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

$$A^3 = A^2.A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 2 \\ 10 & 4 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 18 & 4 \\ 0 & 0 & 16 \\ 0 & 32 & 0 \end{bmatrix}$$

Now, $A^3 - A^2 - 8A + 8I$

$$= \begin{bmatrix} 1 & 18 & 4 \\ 0 & 0 & 16 \\ 0 & 32 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 4 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix} - \begin{bmatrix} 8 & 16 & 0 \\ 0 & 0 & 16 \\ 0 & 32 & 0 \end{bmatrix} + \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 1-1-8+8 & 18-2-16+0 & 4-4+0+0 \\ 0+0+0+0 & 0-8+0+8 & 16+0-16+0 \\ 0+0+0+0 & 32+0-32+0 & 0-8+0+8 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$= 0.$

Thus, matrix A satisfies the equation $A^3 - A^2 - 8A + 8I = 0$.

Illustration 12 : If $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 0 \\ 2 & 1 \\ 1 & 5 \end{bmatrix}$, then show that $(AB)^1 = B^1A^1$.

Solution :

$$AB = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 2 & 1 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 10 & 17 \\ 6 & 21 \end{bmatrix}$$

$$AB^1 = \begin{bmatrix} 10 & 6 \\ 17 & 21 \end{bmatrix} \text{...} \quad \text{(i)}$$

$$B^1A^1 = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 1 & 5 \end{bmatrix} \times \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 10 & 6 \\ 17 & 21 \end{bmatrix} \text{..............................} \quad \text{(ii)}$$

$\therefore$ $(AB)^1 = B^1A^1$.

Illustration 13 : If $\begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$, find x, y and z.

Solution :

$$\begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix}_{(3\times3)} \times \begin{bmatrix} x \\ y \\ z \end{bmatrix}_{(3\times1)} = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}_{(3\times1)}$$

$$\Rightarrow \begin{bmatrix} x-y+z \\ 2x+y-3z \\ x+y+z \end{bmatrix}_{(3\times1)} = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}_{(3\times1)}$$

$\therefore$ $x-y+z=4$ .... (i)

$2x+y-3z=0$ .... (ii)

$x+y+z=2$ .... (iii)

Solving the equations (i) and (iii) simultaneously, we have

$x - y + z = 4$

$x + y + z = 2$

$2x + 2z = 6$

$\Rightarrow \quad x + z = 3$

Putting $x + z = 3$, in equation (iii), we have :

$y + 3 = 2$

$\Rightarrow \quad y = -1$

Multiplying 3 in eqn. (iii) & solving simultaneously with eqn. (ii) we have :

$3x + 3y + 3z = 6$

$\underline{2x + y - 3z = 0}$

$5x + 4y = 6$ .... (iv)

Putting $y = -1$ in eqn. (iv) we have : $5x + 4(-1) = 6$

$\Rightarrow \quad 5x = 10$

$\Rightarrow \quad x = 2$

Now, $x + y + z = 2$

$\Rightarrow \quad 2 - 1 + z = 2$

$\Rightarrow \quad z = 1$

$\therefore \quad x = 2, y = -1$ and $z = 1$.

2.5 ADJOINT OF A MATRIX :

The adjoint of a square matrix A $[a_{ij}]$ is defined a matrix obtained by taking the transpose of the matrix (a_{ij}), where A_{ij} is the co-factor of the element a_{ij}, it is denoted by adj A, i.e., Adj A=$[A_{ij}]$.

For example, If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then

Adj $A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$, where A_{11}, A_{12}, ... are cofactors of a_{ij}.

Note : If A is a square matrix, then $A(\text{adj } A) = |A|I = (\text{adj } A).A$

But if A is a singular matrix, the poduct is zero matrix since $|A| = 0$.

Illustration 14 : Find the adjoint of the following matrix, $\begin{bmatrix} 3 & 1 & 0 \\ 0 & 5 & 2 \\ -10 & 4 & 3 \end{bmatrix}$.

Solution : Let $A = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 5 & 2 \\ -10 & 4 & 3 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Now co-factor Aij = $(-)^{i+j}$ minor of aij

$$\therefore \quad A_{11} = (-1)^{1+1} \begin{vmatrix} 5 & 2 \\ 4 & 3 \end{vmatrix} = 7$$

$$A_{12} = (-1)^{1+2} \begin{vmatrix} 0 & 2 \\ -10 & 3 \end{vmatrix} = -20$$

$$A_{13} = (-1)^{1+3} \begin{vmatrix} 0 & 5 \\ -10 & 4 \end{vmatrix} = 50$$

$$A_{21} = (-1)^{2+1} \begin{vmatrix} 1 & 0 \\ 4 & 3 \end{vmatrix} = -3$$

$$A_{22} = (-1)^{2+2} \begin{vmatrix} 3 & 0 \\ -10 & 3 \end{vmatrix} = 9$$

$$A_{23} = (-1)^{2+3} \begin{vmatrix} 3 & 1 \\ -10 & 4 \end{vmatrix} = -22$$

$$A_{31} = (-1)^{3+1} \begin{vmatrix} 1 & 0 \\ 5 & 2 \end{vmatrix} = 2$$

$$A_{32} = (-1)^{3+2} \begin{vmatrix} 3 & 0 \\ 0 & 2 \end{vmatrix} = -6$$

$$A_{33} = (-1)^{3+3} \begin{vmatrix} 3 & 1 \\ 0 & 5 \end{vmatrix} = 15$$

Now adj A = transpose of co-factor matrix.

$$= \text{transpose of} \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

$$= \text{transpose of} \begin{bmatrix} 7 & -20 & 50 \\ -3 & 9 & -22 \\ 2 & -6 & 15 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & -3 & 2 \\ -20 & 9 & -6 \\ 50 & -22 & 15 \end{bmatrix}.$$

Illustration 15 : Find the adjoint of the matrix, $\begin{bmatrix} 2 & -1 \\ -4 & -3 \end{bmatrix}$

Solution : Let $A = \begin{bmatrix} 2 & -1 \\ -4 & -3 \end{bmatrix}$, then

adj A = transpose of $\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ where A_{11}, A_{12}, A_{21} and A_{22} are co-factors of the element of the matrix A.

Now $A_{11} = (-1)^{1+1}(-3) = -3$

$A_{12} = (-1)^{1+2}(-4) = 4$

$A_{21} = (-1)^{2+1}(-1) = 1$

$A_{22} = (-1)^{2+2}(2) = 2$

$\therefore$ adj A = transpose of $\begin{bmatrix} -3 & 4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ 4 & 2 \end{bmatrix}$.

Illustration 16 : Show that $(AB)^1 = B^1 \cdot A^1$ with the help of the following matrices.

$$A = \begin{bmatrix} 8 & 3 \\ 7 & 2 \\ 23 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 5 \end{bmatrix}$$

Solution : $$AB = \begin{bmatrix} 8 & 3 \\ 7 & 2 \\ 23 & 4 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 5 \end{bmatrix} = \begin{bmatrix} 16 & 6 & 23 \\ 14 & 4 & 17 \\ 46 & 8 & 43 \end{bmatrix}$$

$$(AB)^1 = \begin{bmatrix} 16 & 14 & 46 \\ 6 & 4 & 8 \\ 23 & 17 & 43 \end{bmatrix} \quad \ldots\ldots\ldots\ldots \quad \text{(i)}$$

$$B^1A^1 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \\ 1 & 5 \end{bmatrix} \cdot \begin{bmatrix} 8 & 7 & 23 \\ 3 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 16 & 14 & 46 \\ 6 & 4 & 8 \\ 23 & 17 & 43 \end{bmatrix} \quad \ldots\ldots\ldots\ldots \quad \text{(ii)}$$

2.6 INVERSE OF MATRIX :

A square matrix is said to be inversible if there exists another square matrix B such that AB=BA=I (where I is the identity matrix).

The matrix B is called the inverse of A and is denoted by A^{-1}. In other words, $A \cdot A^{-1} = A^{-1} \cdot A = I \cdot$

If A is a non-singular square matrix then inverse of A is given by $A^{-1} = \frac{1}{|A|} \cdot \text{adj } A \cdot$

Proof : We know that :

$A \cdot (\text{adj}A) = (\text{Adj}A)A = |A| \cdot I$ where $|A| \neq 0$

$$\Rightarrow A\left(\frac{1}{|A|} \text{ adj } A\right) = \left(\frac{1}{|A|} \text{ adj } A\right) = 1$$

$$\Rightarrow A^{-1} \cdot = \frac{1}{|A|} \cdot \text{adj } A.$$

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ and adj A = transpose of $\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$

$$\therefore \quad A^{-1} = \frac{1}{|A|}\begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}.$$

Illustration 17 : If $A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$, show that $A^{-1} = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

Solution : Assuming $B = A^{-1} = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

we have $AB = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}\begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 7-3-3 & -3+3+0 & -3+0+3 \\ 7-4-3 & -3+4+0 & -3+0+3 \\ 7-3-4 & -3+3+0 & -3+0+4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 1 \text{ (identity matrix of order 3).}$$

Now $BA = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$

$$= \begin{bmatrix} 7-3-3 & 21-12-9 & 21-9-12 \\ -1+1+0 & -3+4+0 & -3+3+0 \\ -1+0+1 & -3+0+3 & -3+0+4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$= I$

$\therefore \quad AB = I = BA$

Thus $\quad A^{-1}B = B = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}.$

Illustration 18 : If $A = \begin{bmatrix} 5 & 7 \\ 3 & 4 \end{bmatrix}$, prove that $A^{-1} = \begin{bmatrix} -4 & 7 \\ 3 & -5 \end{bmatrix}$

Solution : Let $B = \begin{bmatrix} -4 & 7 \\ 3 & -5 \end{bmatrix}$

$\therefore \quad AB = \begin{bmatrix} 5 & 7 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} -4 & 7 \\ 3 & -5 \end{bmatrix}$

$$= \begin{bmatrix} -20+21 & 35-35 \\ -12+12 & 21-20 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

As $AB = I = BA$, B is the inverse of A.

i.e., $A^{-1} = B = \begin{bmatrix} -4 & 7 \\ 3 & -5 \end{bmatrix}.$

Illustration 19 : Find the inverse of the matrix, $A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & -2 \\ -2 & 0 & 1 \end{bmatrix}$.

Solution :

$$\det \quad A = |A| = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & -2 \\ -2 & 0 & 1 \end{bmatrix}$$

$= 1.(3-0) - (-1)(2-4)+0(0+6)$
$= 3-2$
$= 1$ .... (i)

The co-factors of the elements of the Matrix A are

$$A_{11} = (-1)^{1+1}\begin{vmatrix} 3 & -2 \\ 0 & 1 \end{vmatrix} = 3$$

$$A_{12} = (-1)^{1+2}\begin{vmatrix} 2 & -2 \\ -2 & 1 \end{vmatrix} = 2$$

$$A_{13} = (-1)^{1+3}\begin{vmatrix} 2 & 3 \\ -2 & 0 \end{vmatrix} = 6$$

$$A_{21} = (-1)^{2+1}\begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

$$A_{22} = (-1)^{2+2}\begin{vmatrix} 1 & 0 \\ -2 & 1 \end{vmatrix} = 1$$

$$A_{23} = (-1)^{2+3}\begin{vmatrix} 1 & -1 \\ -2 & 0 \end{vmatrix} = 2$$

$$A_{31} = (-1)^{3+1}\begin{vmatrix} -1 & 0 \\ 3 & -2 \end{vmatrix} = 2$$

$$A_{32} = (-1)^{3+2}\begin{vmatrix} 1 & 0 \\ 2 & -2 \end{vmatrix} = 2$$

$$A_{33} = (-1)^{3+3}\begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} = 5$$

$$\text{adj A} = \text{transpose of} \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 1 & 2 \\ 2 & 1 & 2 \\ 6 & 2 & 5 \end{bmatrix} \quad \text{...} \quad \text{(ii)}$$

Therefore $\quad A^{-1} = \frac{1}{|A|} \text{ adj A}$

$$= \frac{1}{1}\begin{bmatrix} 3 & 1 & 2 \\ 2 & 1 & 2 \\ 6 & 2 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 1 & 2 \\ 2 & 1 & 2 \\ 6 & 2 & 5 \end{bmatrix}.$$

2.7 SOLUTION OF SYSTEM OF LINEAR EQUATIONS USING MATRIX :

In the following system of equations

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3.$$

Let $\quad A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$

Then this system of equations can be expressed

$$\text{as}\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

this is $AX = B$

$\Rightarrow X = A^{-1}B$. (where A is non-singular matrix.)

Then equating the corresponding elements of the above matrix equation, the values of the variable can be found out.

Illustration 20 : Solve the system of equation using matrix method.

$$3x + 2y = 7$$
$$4x - y = 2$$

Solution : Expressing the equations in matrix form, we have

$$\begin{bmatrix} 3 & 2 \\ 4 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \end{bmatrix}$$

It can also be written as $AX = B$

where $A = \begin{bmatrix} 3 & 2 \\ 4 & -1 \end{bmatrix} \cdot X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} 7 \\ 2 \end{bmatrix}$

$\det A = -3 - 8 = -11$ .... (i)

As $|A| = -11$, it is non-singular and its inverse exists. The co-factors of the matrx A are

$A_{11} = -1$ $\quad A_{21} = -2$

$A_{12} = -4$ $\quad A_{22} = 3$

Adjoint A = transpose of $\begin{bmatrix} -1 & -4 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ -4 & 3 \end{bmatrix}$ .... (ii)

$\therefore\ A^{-1} = \dfrac{1}{|A|} \cdot \text{adj } A = \dfrac{1}{-11}\begin{bmatrix} -1 & -2 \\ -4 & 3 \end{bmatrix}$ .... (iii)

again $X = A^{-1}B = \dfrac{1}{-11}\begin{bmatrix} -1 & -2 \\ -4 & 3 \end{bmatrix}\begin{bmatrix} 7 \\ 2 \end{bmatrix}$

$$= \frac{1}{-11}\begin{bmatrix} -7-4 \\ -28+6 \end{bmatrix}$$

$$= \frac{1}{-11}\begin{bmatrix}-11\\-22\end{bmatrix} = \begin{bmatrix}1\\2\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}1\\2\end{bmatrix}$$

Equating the corresponding elements on both sides, we have x = 1 and y = 2.

Illustration 21 : The following shown the monthly salary of an undertaking for the month of May 2016.

Month	No. of employees			Total monthly Salary (in₹)
	Clerks	Peons	Labourers	
May	1	3	1	2,075
June	2	3	1	2,625
July	2	4	2	3,350

Compute the monthly salary of each type of employees.

Solution : The matrix expression of the table data will be :

$$\begin{bmatrix}1 & 3 & 1\\2 & 3 & 1\\2 & 4 & 2\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}2075\\2625\\3350\end{bmatrix}$$

$$\text{Let } A = \begin{bmatrix}1 & 3 & 1\\2 & 3 & 1\\2 & 4 & 2\end{bmatrix},\ X = \begin{bmatrix}x\\y\\z\end{bmatrix} \text{ and } B = \begin{bmatrix}2075\\2625\\3350\end{bmatrix}$$

$$|A| = \begin{bmatrix}1 & 3 & 1\\2 & 3 & 1\\2 & 4 & 2\end{bmatrix} = -2.$$

Co-factors of the elements of the matrix A are :

$$A_{11} = (-1)^{1+1}\begin{vmatrix}3 & 1\\4 & 2\end{vmatrix} = 2$$

$$A_{12} = (-1)^{1+2}\begin{vmatrix} 2 & 1 \\ 2 & 2 \end{vmatrix} = -2$$

$$A_{13} = (-1)^{1+3}\begin{vmatrix} 2 & 3 \\ 2 & 4 \end{vmatrix} = 2$$

$$A_{21} = (-1)^{2+1}\begin{vmatrix} 3 & 1 \\ 4 & 2 \end{vmatrix} = -2$$

$$A_{22} = (-1)^{2+2}\begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} = 0$$

$$A_{23} = (-1)^{2+3}\begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} = 2$$

$$A_{31} = (-1)^{3+1}\begin{vmatrix} 3 & 1 \\ 3 & 1 \end{vmatrix} = 0$$

$$A_{32} = (-1)^{3+2}\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = 1$$

$$A_{33} = (-1)^{3+3}\begin{vmatrix} 1 & 3 \\ 2 & 3 \end{vmatrix} = -3$$

$$\text{adj}\,A = \text{transpose}\begin{bmatrix} 2 & -2 & 2 \\ -2 & 0 & 2 \\ 0 & 1 & -3 \end{bmatrix} = \begin{bmatrix} 2 & -2 & 0 \\ -2 & 0 & 1 \\ 2 & 2 & -3 \end{bmatrix} \quad \ldots\ldots\ldots\ldots \quad \text{(ii)}$$

$$A^{-1} = \frac{1}{|A|}\,\text{adj}A = \frac{1}{-2}\begin{bmatrix} 2 & -2 & 0 \\ -2 & 0 & 1 \\ 2 & 2 & -3 \end{bmatrix} \quad \ldots\ldots\ldots\ldots \quad \text{(iii)}$$

Now $AX = B$

$\Rightarrow \quad X = A^{-1}B$

$$= \frac{1}{-2}\begin{bmatrix} 2 & -2 & 0 \\ -2 & 0 & 1 \\ 2 & 2 & -3 \end{bmatrix}\begin{bmatrix} 2075 \\ 2625 \\ 3350 \end{bmatrix} = \begin{bmatrix} -2075 + 2625 + 0 \\ 2075 + 0 - \frac{3350}{2} \\ -2075 - 2625 + \frac{3}{2}.3350 \end{bmatrix}$$

$$= \begin{bmatrix} 550 \\ 2075 - 1675 \\ -4700 + 5025 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 550 \\ 400 \\ 325 \end{bmatrix}$$

$x = 550$, $y = 400$ and $z = 325$

The monthly salary of each clerk, peon and labourer are ₹550, ₹400 and ₹325 respectively.

2.5 QUESTIONS :

1. Choose the correct answer from the alternatives given below :

(i) The inverse of the matrix $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ can not be found as A is a :

(a) singular matrix (b) non-singular matrix
(c) square matrix (d) Identity matrix

(ii) A matrix with single column with and any number of rows is known as :

(a) Singular matrix (b) Row matrix (c) Column matrix (d) Diagonal matrix

(iii) A row matrix has only :

(a) one element (b) one row with one or more columns
(c) one column with one or more rows (d) one row and one column

(iv) The co-factor of 3 in $\begin{bmatrix} 1 & 3 & 1 \\ 2 & 9 & 7 \\ -1 & 0 & 2 \end{bmatrix}$ is :

(a) 11 (b) –11 (c) 22 (d) –22

(v) If $A = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$ then A^2 is equal to :

(a) $\begin{bmatrix} 9 & 7 \\ 4 & 11 \end{bmatrix}$ (b) $\begin{bmatrix} 18 & 7 \\ 14 & 11 \end{bmatrix}$ (c) $\begin{bmatrix} 19 & 17 \\ 14 & 11 \end{bmatrix}$ (d) $\begin{bmatrix} 17 & 19 \\ 11 & 14 \end{bmatrix}$

(vi) The solutions of the following equations, if solved simultaneously are :

$$2x + 3y = 1$$
$$3x + 4y = 1$$

(a) –1, –1 (b) 1, 1 (c) 1, –1 (d) –1, 1.

(vii) The matrix expression of the following system of linear equation is :

$$7x - 3y - z = 3$$
$$4x + y - 2z = 3$$
$$5x + y + z = 7$$

(a) $\begin{bmatrix} 7 & -3 & -1 \\ 4 & 1 & 2 \\ 5 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 7 \end{bmatrix}$ (b) $\begin{bmatrix} 7 & -3 & -1 \\ 4 & 1 & -2 \\ 5 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 7 \end{bmatrix}$

(c) $\begin{bmatrix} 7 & -3 & -1 \\ 4 & 1 & 2 \\ 5 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ -7 \end{bmatrix}$ (d) $\begin{bmatrix} 7 & -3 & -1 \\ 4 & 1 & 2 \\ 5 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 7 \end{bmatrix}$

(viii) If $x = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $A^{-1}B = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $x = A^{-1}B$, the value of x, y and z respectively are :

(i) 1, 2, 3 (b) 3, 2, 1 (c) 2, 1, 3 (d) 2, 3, 1

(ix) The inverse of $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is :

(a) $\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(x) If the matrix A is both symmetric and skew symmetric, then
(a) A is a diagonal matrix. (b) A is a zero matrix.
(c) A is a square matrix. (c) Identity matrix.

(xi) If A is square matrix such that $A^2 = A$, then $(I+A)^3 - 7A$ is equal to :
(a) A (b) I–A (c) I (d) 3A

2. Answer the following in one word/ term each :

(i) A matrix that comprises one element only.
(ii) A matrix that consists of zero elements.
(iii) A square matrix in which all the principal diagonal elements are non-zeroes and all other elements are zeroes.
(iv) A diagonal matrix in which all the leading diagonal elements are equal.
(v) A square matrix in which all the leading diagonal elements are unity or one, and other elements are zeroes.
(vi) A square matrix in which all the elements above or below the principal diagonal are zeros.
(vii) A matrix that appears with equal number of rows and columns.

3. Correct the underlined portion of the following sentences :

(i) If the transpose of a matrix is equal to the matrix itself, then the matrix is called a <u>skew-symmetric</u> matrix.
(ii) If A and B are two matrices then $A \times B$ is <u>equal</u> to $B \times A$,
(iii) If $AX = B$, then $X = B^{-1}A$.
(iv) A matrix whose determinant value is zero is called a <u>non-singular</u> matrix.
(v) A matrix $A = [a_{ij}]_{m \times n}$ is said to be a square matrix if <u>m > n</u>.
(vi) A matrix obtained by deleting the rows or columns or both of a matrix is called its <u>super-matrix</u>.

4. Fill in the blanks :

(i) A square matrix, which when multiplied by its transpose amounts to an identity matrix is called a/an __________ matrix.
(ii) A symmetric matrix that reproduces itself is termed as a/an _________ matrix.
(iii) A square matrix A is said to be symmetric if $A^1 =$ __________.
(iv) A square matrix A is said to be skew-symmetric, if $A^1 =$ __________.
(v) A square matrix with 1's on the main diagonal and 0's elsewhere, is called a _______ matrix.

5. Answer the following questions in one sentences each.

(i) What is a matrix ?
(ii) Give a numerical example of a matrix.

(iii) Define a null matrix.
(iv) What do you mean by a square matrix ?
(v) What is meant by a singleton matrix ?
(vi) What do you mean by transpose of a matrix ?

6. Answer the following questions in within 30 words.

(i) What do you mean by Equal matrices ? Explain with examples.
(ii) Explain orthogonal matrix with examples.
(iii) What is skew symmetric matrix ? Explain with examples.
(iv) Explain and illustrate triangular matrices.
(v) What is adjoint matrix ? Explain with example.
(vi) Explain the condition necessary for addition of matrices.

7. Answer the following questions within 50 words.

(i) What is equal matrix ? How it is different from equivalent matrix, explain with example.
(ii) What is subtraction of matrices state the conditions necessary for subtraction and explain the procedure of subtraction.
(iii) What is multiplication of matrices ? Explain the condition and procedure necessary for multiplication.
(iv) What is transpose of a matrix ? Prove that transpose of a transposed matrix reproduces the original matrix.
(v) Explain the procedure of finding the Adjoint of a matrix with an example.
(vi) What is singular and non-singular matrix ? Explain with example.

8. If $A = \begin{bmatrix} -2 & 4 \\ -3 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 7 \\ 5 & 3 \end{bmatrix}$ find

(a) $A + B$ (b) $B - A$ (c) $-2A$ (d) $3B$

9. Find the transpose of (a) $\begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & -2 & 5 \\ 7 & 2 & 8 \\ 1 & 1 & 0 \end{bmatrix}$

10. Find the sum of the matrices given below :

(a) $\begin{bmatrix} 4 & 2 \\ 5 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ -2 & -3 \end{bmatrix}$ (b) $\begin{bmatrix} 17 \\ 12 \end{bmatrix} + \begin{bmatrix} -3 \\ 8 \end{bmatrix}$

(c) $\begin{bmatrix} \frac{1}{5} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{4} \end{bmatrix} + \begin{bmatrix} \frac{1}{3} & \frac{4}{5} \\ \frac{1}{2} & \frac{3}{4} \end{bmatrix}$ (d) $\begin{bmatrix} x & y \\ 2x & 3y \end{bmatrix} + \begin{bmatrix} 4 & 2 \\ 5 & 0 \end{bmatrix}$

11. Evaluate : $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} + \begin{bmatrix} -1 & -2 & 0 \\ -2 & -2 & 5 \\ -4 & -5 & -7 \end{bmatrix}$

12. Solve the following matrix equations :

(a) $\begin{bmatrix} 2x+y & x \\ z & p \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 3 & 4 \end{bmatrix}$

(b) $\begin{bmatrix} x & 2 & y \\ 4 & z & 9 \end{bmatrix} + \begin{bmatrix} y & 4 & -x \\ 4 & 2 & p \end{bmatrix} = \begin{bmatrix} 10 & 6 & 6 \\ 8 & 10 & 7 \end{bmatrix}$

(c) $\begin{bmatrix} x+9 & y \\ z & 4 \end{bmatrix} + 2\begin{bmatrix} x & 1 \\ y & p \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 2x+1 & z \\ z-3 & 3y \end{bmatrix} = \begin{bmatrix} 5 & 7 \\ 4 & 9 \end{bmatrix}$

13. If $A = \begin{bmatrix} 2 & 3 \\ -4 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 6 \\ 2 & 9 \end{bmatrix}$

find (a) $(A')'$ (b) $(A+A)'$ (c) $3A' - B$
(d) $2A + 4B$ (e) $(2A+B)'$ (f) $4A + 9B + A'$

14. Find the product of the following matrices :

(a) $\begin{pmatrix} 2 & 4 & 9 & 8 \end{pmatrix} \begin{pmatrix} 8 \\ 4 \\ 9 \\ 7 \end{pmatrix}$ (b) $\begin{pmatrix} 4 & 2 \\ 6 & 3 \end{pmatrix} \begin{pmatrix} 3 \\ -6 \end{pmatrix}$

(c) $\begin{pmatrix} 2 & 4 & 9 \\ 2 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 4 & 0 \\ 2 & 3 & 0 \end{pmatrix}$ (d) $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 4 & 2 & 0 \\ 5 & 0 & 1 \\ 1 & 0 & 7 \end{pmatrix}$

15. If $A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & -1 & 0 \\ 3 & 7 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 5 & 6 \\ 1 & 1 & 2 \\ -2 & 0 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 6 & 7 & 8 \\ 0 & 1 & 1 \\ -3 & 1 & 2 \end{bmatrix}$

Show that :
(a) $A + (B + C) = (A + B) + C$
(b) $A(B + C) = AB + AC$

16. Prove that the matrix $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & 4 & 0 \end{bmatrix}$ is a root of the polynomial $f(x) = x^3 - x^2 - 8x + 8$

(Hints : Prove $f(A) = A^3 - A^2 - 8A + 8.1 = 0$)

17. Verify that $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ satisfies the equation $x^2 - (a + d)x + (ad - bc) = 0$

where I and 0 are 2 × 2 matrices.

18. If $\begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ find the value of x and y.

19. If $\begin{bmatrix} x+y & x-z \\ 2x-y & 0 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 1 & 0 \end{bmatrix}$ find the value of x, y, z.

20. If $A = \begin{bmatrix} 1 & 0 & -2 \\ 2 & 3 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 4 & -1 & 3 \\ 0 & 2 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 2 & -3 & 0 \\ 1 & 4 & 5 \end{bmatrix}$ then find $A - 3B + 2C$.

21. Find the matrix which when added to $\begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix}$ gives $\begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}$.

22. If $A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 3 \\ -3 & 1 \end{bmatrix}$, then show that $(A + B)^2 \neq A^2 + 2AB + B^2$.

23. Determine the value of k for which the matrix $\begin{bmatrix} k & k \\ 2 & k \end{bmatrix}$ has no inverse.

24. Determine the value of k for which the matrix $\begin{bmatrix} 2 & 3 & 4 \\ 2 & 4-k & 4 \\ k & 3 & 4 \end{bmatrix}$ has no inverse.

25. Give one example of identity matrix and one example of null null matrix each of the order 3.

26. If $\begin{bmatrix} 1 & 1 & 1 \\ -3 & 4 & 1 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 11 \end{bmatrix}$ find the value of x, y and z.

27. Find the value of x and y if $\begin{bmatrix} 3 & 2 \\ 7 & x \end{bmatrix} \begin{bmatrix} 5 & -2 \\ -7 & y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

28. If $A = \begin{bmatrix} 1 & -2 & 2 \\ 3 & 1 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 4 \\ 1 & 2 \\ 3 & -2 \end{bmatrix}$ show that $(AB)' = B' - A'$.

29. Find A if $A^2 = \begin{bmatrix} 10 & 6 \\ 6 & 10 \end{bmatrix}$.

30. Find the adjoint of the matrices given below :

(a) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 1 & 0 \\ 3 & 4 & -1 \\ 1 & 5 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} -1 & -2 & -3 \\ -4 & -5 & -6 \\ 0 & -7 & -8 \end{bmatrix}$

31. Find the inverse of the following matrices :

(a) $\begin{bmatrix} 3 & 2 \\ 4 & 5 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 2 \\ -1 & -3 \end{bmatrix}$

32. Find the inverse of the matrices given below :

(a) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 1 & 0 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$

33. Find the determinant, adjoint and inverse of the matrix.

$$A = \begin{bmatrix} 3 & 2 & -1 \\ 2 & 1 & -5 \\ 1 & 1 & -2 \end{bmatrix}$$

34. If $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}$, find $(AB)^{-1}$.

35. If $A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 5 \\ 6 & 7 \end{bmatrix}$, show that $(AB)^{-1} = B^{-1}A^{-1}$.

36. If $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$, prove that $A^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -4 & -3 & -1 \\ \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \end{bmatrix}$.

37. If $A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & -2 \\ -2 & 0 & 1 \end{bmatrix}$, show that $AA^{-1} = I$ where I is the identity matrix of order 3.

38. Matrix x and y are such that $3x + 4y = I$ and $x - 2y = 21$ where I denotes

the identity $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. (Hints : Solve the two equations.)

Find x and y.

Solve the following system of equations by matrix method :

39. $3x - 2y = 12$
    $5x - 3y = 1$
40. $x + 2y = 7$
    $3x + 2y = 13$
41. $x + y = 10$
    $2x + 5y - 29 = 0$
42. $5x + 7y = 12$
    $11y + 3x = 14$
43. $x + 3y - 2z = 1$
    $3x + 4y + z = 5$
    $5x + 5y - 3Z = 5$
44. $3x - 2y + z = 1$
    $2x + y - 5z = 2$
    $x - y - 2z = 3$
45. $x + y - 3z = -6$
    $2x + y + z = 7$
    $x - y + 2z = 5$
46. If $A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 3 & 4 & 7 \end{pmatrix}$, find A^{-1} and using A^{-1} solve the following system of linear equations :
    $x + y + z = 5$
    $2x + y + z = 7$
    $3x + 4y + 7z = 21.$
47. If the cost of 5 cutlets and 7 chops is ₹12 and cost of 3 cutlets and 11 chops is ₹14. Find the cost of each chops and cutlets by using matrices.
48. A man purchases a chair and a table for ₹13,000, another man purchases at the same price 4 chairs and a table for ₹28,000. What is the cost of each chair and each table ? Use matrix to get the answer.
49. There are skilled, semi-skilled labours in 3 factories. Their remuneration is stated below ?

	No. of labourers			
Factory	Skilled	Semi-skilled	Unskilled	Total remuneration (₹)
A	2	4	10	1780
B	2	5	10	1900
C	1	3	5	1010

Find the salary of each skilled, semi-skilled and un-skilled labourer.

50. If $A=\begin{pmatrix}1 & 3 & 3\\ 1 & 4 & 3\\ 1 & 3 & 4\end{pmatrix}$ and $B=\begin{pmatrix}7 & -3 & -3\\ -1 & 1 & 0\\ -1 & 0 & 1\end{pmatrix}$

show that AB = I, where I is the identity matrix of order 3. Also show that $B = A^{-1}$.

❖

ANSWER

1. (i) a (ii) c (iii) b (iv) b (v) b (vi) d (vii) b (viii) a (ix) c (x) b (xi) c.
2. (i) singe-ton matrix, (ii) zero or null matrix (iii) Diagonal matrix (iv) scalar matrix, (v) unity or identity matrix (vi) Triangular matrix, (vii) Square matrix.
3. (i) symmetric, (ii) Not equal, (iii) $A^{-1}B$, (iv) singular (v) m = n (vi) Sub-matrix
4. (i) orthogonal (ii) indempotent (iii) A (iv) –A (v) unity / Identity.
5. (i) Matrix is an ordered array of elements in rows and columns.

(ii) $\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}$

(iii) When all the elements contained by a matrix are zeros, is called a null matrix.

(iv) When all the number of elements of rows and number of elements of columns are equal, the matrix is termed as a squarle matrix.

(v) A matrix having one element is called a single ton matrix.

(vi) When the row of a matrix are changed to columns or vice-verse, such resultant matrix is called transpose of the matrix.

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CHAPTER - 3

SET THEORY

Structure

3.1 Introduction

In everyday life, we usually speak of "a pack of cards" 'a bunch of keys', 'a herd of cattle', 'a crowd at a cricket match' and collection of articles etc. to donote group of different objects. In mathematics, we use the word set to denote such collections.

Thus, a set is a collection of well-defined objects. By "well-defined", we mean that, it must be possible to tell beyond doubt, whether or not a given object belongs to the collection that we are considering. If we are asked to write the set of all intelligent students of a class, it is not possible to do so. No two persons will have the common list. Thus the collection of intelligent students in the class is not a set. It is not necessary that a set contains only the same type of objects. If we have a book, a football, a glass & a table, then these objects also form a set although, they are different from eachother. These objects are called elements or members of the set.

In mathematics, we have set of numbers of various kinds, sets of points on a line, sets of lines in a plane etc.

A few examples of the sets are given below:-

(i) The set of pupils in a school

(ii) The set of teachers in a school

(iii) The set of natural numbers between 2 and 20

(iv) The set of vowels in English alphabet

(v) The set of odd numbers between 4 and 16

(vi) The set of a pen, a pencil, a notebook, a tennis ball and a racket.

3.2 Finite and Infinite sets

A set is called a finite set, if we can count the number of elements of the set.

A set in which the number of elements cannot be counted is called an infinite set.

Set of students in a school, set of people living in Delhi, a set of natural numbers from 3 to 19 are some of the examples of finite sets.

Set of all natural numbers, set of points on a line & set of circles with a fixed centre are some of the examples of infinite sets.

3.3 Set Notation

We use capital letters A, B, C, D,...... to describe a set. The elements of a set are denoted by small alphabets a, b, c, x, y etc.

A set can be expresed in the following two ways :

(i) Tabulation or Roster Method.

(ii) Rule method or set Builder form.

(i) Roster method : In this method, the elements of the set are listed together & are placed within braces { }. For example, set of natural numbers less than 5 is written as {1, 2, 3, 4} whereas, the set of vowels in English alphabet is {a, e, i, o, u,}. In the case of Infinite set, as we cannot list all the elements, we write some elements & then put 3 or 4 dots before putting the braces. Thus, set of all natural numbers is written as {1, 2, 3, 4.....}.

The set of whole members is written as : {0, 1, 2, 3, ...}

The set of prime members is written as : {2, 3, 5, 7, 11, ...}

(ii) Rule Method : In this method, we write a set by some special property that has to be satisfied by the elements to become the member of that set. Thus, the set {1, 2, 3, 4, ...} can be rewritten as {All natural numbers less than 5}.

Set of integers {....., -3, -2, -1, 0, 1, 2, 3, ... }can be written as {set of integers}.

We also write the set by rule method in the form {x; x is a natural number <5} and {x:x is and integer}. Here x denotes all the elements satisfying the given property.

Thus, set P= { x:x is prime} is read as "P is the set of numbers x such that x is a prime number:

Sometimes, we also use symbol. 'E' in expressing a set. Thus, set of even numbers is written as $\{ 2x : x \in N\}$

Set of natural numbers between 5 and 20.

$= \{ x : x \in N \text{ and } 5 < x < 20\}$

Example 1: Use the roster method to describe each of the following sets:

(i) Set of integers greater than 10 and less than 15.

(ii) Set of positive integers divisible by 3 and less the 18.

Solution :

(i) Let the given set be denoted by A. then A = {11, 12, 13, 14}.

(ii) Let the given set be denoted by B.

Then B = {3, 6, 9, 12, 15}.

Examples 2 : Use the rule method to describe each of the following.

(i) {1, 3, 5, 7, 9, 11, ...}

(ii) {5, 10, 15, 20, ...}

Solution :

(i) {1, 3, 5, 7, 9, 11, ...} = set of all odd numbers.

= { x : x is an odd number}

This set can slso be written as : $\{2x - 1 : x \in N\}$

(ii) { 5, 10, 15, 20, ...} = set of multiples of 5.

$= \{5x : x \in N\}$

Example 3 : Use roster method to express the set,

$$A = \{x : x = 5y\text{-}1, y \in N, y > 3\}$$

Solution : As y >3, we start substituting y = 4, 5, 6, ... and we get,

A = { 19, 24, 29, 34 ....}

Example 4 : Write the set containing all days of the week beginning with S.

Solution : let S be the required set.

Then S = {Sunday, Saturday}.

3.4 Single Member set

A set which has only one member is known as a single member set. For example, a set of natural numbers between 7 and 9 is a single member set having only one element 8. i.e., {8} it is also called a 'singleton'.

3.5 Empty or Null set

Can you think of a natural number less than 1? Certainly, there is no natural number less than 1. Thus, set of all natural numbers less than 1, will not contain any element. Hence, a set which contains no element is called an empty or null set. It is also called a void set. In Listing elements of an empty set, we have braces { } and having nothing within the braces. Null set is usually denoted by the Greek letter 'ϕ' (phi). Clearly the set {0} is not an empty set as it is a set which has one element 'zero' belonging to it. The set of triangles with two obtuse angles is { } & the set of natural numbers between 5 and 6 is also ϕ.

3.6 Disjoint Sets

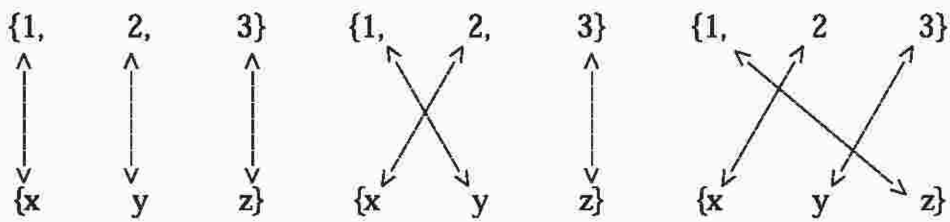
If no element of A is in B and no element of B is in A, then A and B are called disjoint sets. For example, sets {1, 3, 5} and {2, 4, 6} are disjoint sets as there is no common element in them. In other words, two sets which have no common members are called disjoint sets.

3.7 One-to-one correspondence

Two sets are said to be in one to-one correspondence, if they can be matched in such a way, that each element of one set is associated with a single element of the other.

Considers the sets X = {1, 2, 3} and Y = {x, y, z}

We can pair the elements of one set with the elements of the other in different ways.



and so on.

Thus, we observe that, each member of one set is matched with one and only one member of the other set. so, X and Y have one-to-one correspondence between them.

Let A = {a,b,c} and B = {p,q,r,s}, then it is clear that A and B are not in one-to-one correspondence, If the elements of A and B are matched, then one element of B remains unmatched. Thus, we say that A has fewer elements than B or B has more elements than A. In other words, set b is larger than set A.

3.8 Equivalent sets

Two sets X and Y are said to be equivalent sets, if the number of elements in X is equal to the numbers of elements in Y i.e , there is one-to-one correspondence between the elements of X and Y. For example, {1,2,3} and {x,y,z} are equivalent sets. The symbol "~" is used to denote equivalence. Thus A~B is read as 'A is equivalent to B'.

3.9 Equal sets

Two sets A and B are called equal sets if every element of A is also an element of B & every element of B is also an element of A. For example {r,s,t} and {t, r, s} are equal sets.

Obviously, if two sets are equal, they are equivalent too, but if two sets are equivalent they may not be equal. Corresponding to a set, there are infinite number of equivalent sets.

Example : Given A = {2,4,6,8} and B = {2,8, 4,6} we see that every element of A is a member of B and each element of B is also a member of A.

Therefore, set A = set B

Further, if X is a set of letters in the word 'ronak' and y is a set of letters in the word 'kanor' then x= {ronak} and y = {kanor}

$\therefore$ x = y.

From the above, we observe that the order of elements and repetition of elements do not change a set.

3.10 Subsets

Let A and B be-two sets, such that :

A = {all months of the year}

B = { all months of the year starting with j}

We see that each element of B is also a member of A. Thus, B is Included in the set A. B is called subset of A . Thus, we define a subset as :

If every element of a set A is also an element of another set B, then A is called a subset of B. and we write it as $A \subset B$ or $B \supset A$. Symbol $\subset$ is used to denote " is a subset of" or " is contained in" and symbol $\supset$ is used to denote "is a superset of" or "contains". Thus, $A \subset B$ is read as A is contained in B and $B \supset A$ is read as B contains A.

For A to be a subset of B, every element of A must be in B, but every element of B may or may not be in A. If every element of B is also in A, then $B \subset A$, too. This shows that A and Bare same sets.

Thus, $A \subset B$ and $B \subset A \Leftrightarrow A = B$.

In case every element of A is in B, but every element of B is not in A i.e., if $A \subset B$ but $B \not\subset A$, then A is said to be a proper subset of B. For example, the set of odd natural numbers is a proper subset of the set of natural numbers.

Clearly, every set is a subset of itself and null set or ϕ is a subset of every set.

Example : Consider a set A = {1}. It has two possible subsets {1} and ϕ.

Again, in set B = {1,2}, the possible subsets are ϕ, {1}, {2}, {1,2}

Further the set {1,2, 3} has ϕ, {1}, {2}, {3}, {1,2}, {1,3}, {2,3} {1,2,3} as its sub sets.

Thus, we observe that :

A set with 1 element has 2 sub sets.

A set with 2 elements has 2^2 sutsets.

A set with 3 elements has 2^3 subsets.

A set with n elements has 2^n subsets.

Proceeding in the same way, it can easily be verified that the number of proper subsets of a set with 'n' elementsa is 2^n-1.

3.11 Power Set

Elements of a set can also be some sets. Such sets are called sets of sets. For example, {ϕ, {1}, {2}, {1,2} is a set whose elements are four sets ϕ, {1}, {2}, {1,2}. The set of all subsets or a given set X is called the power set of X and is denoted by P(X).

For example, if X = {x,y,z}, then,

P(X) = {ϕ, {x}, {y}, {z}, {x, y}, {x, z}, {y, z}, {x, y, z}.

3.12 Universal Set

Every set is likely to be a subset of another set. A set which is such that all the sets under consideration are its subset is called the universal set or universe. It is denoted by U. Set of natural numbers is a universal set for all the finite set of natural number. The set of students of Class X is the universal set for the set of students of Class XA, the set of students of Class X B, the set of students of Class X C, etc. For plane geometry, the universal set is the set of all points of the plane and so on.

3.13 Complement of a set

Complement of a set A is the set which contains all those elements of the universal set which are not in A. It is denoted by A' or $\overline{A}$.

Thus, $A' = \{x : x \in U, x \notin A\}$

For example, if $U = \{1, 2, 3, 4, 5, 6, 7\}$ and $A = \{2, 4, 3\}$

then $A' = \{1, 5, 6, 7\}$

Clearly, complement of the complement of a set A is set itself, i.e. $(A')' = A$

SET OPERATIONS

3.14 Venn Diagrams

The idea of sets can also be represented diagrammatically. These representation of sets are called Venn Diagrams.

In Venn diagrams, universal set U is usually shown by a rectangular region and the subsets of U by cicrular regions inside it. The elements of U are represented as points inside the rectangle, whereas the points inside the circle denote the elements of the subsets. Note that the size of the region has nothing to do with the number of elements of set.

In Fig. 3.1, U represents the set of natural number < 15 and A= {odd numbers<15}

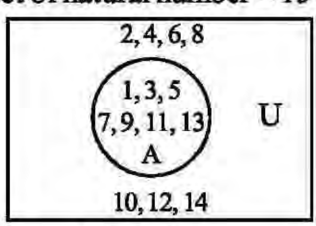


Fig. 3.1

3.15 : Union of Sets

Let A and B be two given sets. A set in which elements of A or B or both are included is called union of A and B and is denoted by $A \cup B$ i.e. $A \cup B = \{x : x \in A \text{ or } x \in B\}$.

In other words, union is described as an 'either' 'or' idea i.e. $A \cup B$ contains all those elements which are either in A or in B. For example, if $A=\{1, 2, 3, 4\}$ and $B=\{5, 6, 7\}$ then,

$A \cup B = \{1, 2, 3, 4, 5, 6, 7\}$

and if $X = \{2, 4, 6\}$ and $Y = \{1, 4, 6\}$

then, $X \cup Y = \{1, 2, 4, 6\}$]

Here, one element is written only once.

Diagrammatically, $A \cup B$ is represented by the shaded portion in the Venn diagram given below :

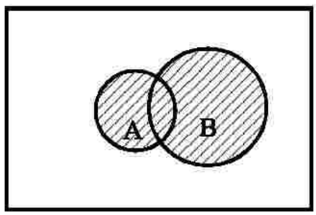


Fig. 3.2

3.16 : Intersection of Sets

Let A and B be two given sets. The set of those elements which are common to both A and B is called intersection of A and B and is denoted by $A \cap B$.

Thus, $A \cap B = \{x : x \in A \text{ or } x \in B\}$.

In other words, intersection is described as an 'and' idea i.e. $A \cap B$ contains those elements which are in A as well as in B. For example, if $A=\{1, 2, 3, 4\}$ and $B=\{2, 4, 5, 6, 9\}$ then,

$A \cap B = \{2, 4\}$

Diagrammatically, $A \cap B$ is represented by the shaded portion in the Venn diagram given below :

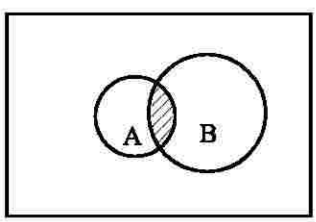


Fig. 3.3

3.17 : Properties of Set operations

(i) Commutative Property :

(a) Union is commutative i.e. if A and B are two sets, then $A \cup B = B \cup A$.

(b) Intersection is commutative i.e. If A and B are two sets, then : $A \cap B = B \cap A$.

For example, if A = {a, b, c, d} and B = {p, q, b, c,}, then

(a) $A \cup B$ = {a, b, c, d, p, q} and $B \cup A$ = {p, q, a, c, b, d} clearly $A \cup B = B \cup A$.

(b) $A \cap B$ = {b, c} and $B \cap A$ = {b, c}

$\therefore A \cap B = B \cap A$

(ii) Associative Property :

(a) Union is associative i.e. if A, B, C are three sets, then

$(A \cup B) \cup C = A \cup (B \cup C)$.

(b) Intersection is associative, i.e., if A, B, C are three sets, then

$\therefore (A \cap B) \cap C = A \cap (B \cap C)$.

For example, if A = {1, 2, 3, 4} and B = (2, 4, 6, 8} and C {1, 2, 4, 5}, then,

(a) $(A \cup B) \cup C$ = { 1, 2, 3, 4, 6, 8}$\cup$ {1, 2, 4, 5} = {1, 2, 3, 4, 5, 6, 8} and

$A \cup (B \cup C)$ = {1, 2, 3, 4} $\cup$ {1, 2, 4, 5, 6, 8} = {1, 2, 3, 4, 5, 6, 8}

$\therefore (A \cup B) \cup C = A \cup (B \cup C)$

(b) $(A \cap B) \cap C$ = { 2, 4}$\cap$ {1, 2, 4, 5} = {2, 4} and

$A \cap (B \cap C)$ = {1, 2, 3, 4} $\cap$ {2, 4} = {2, 4}

Thus, $(A \cap B) \cap C = A \cap (B \cap C)$.

(iii) Distributive Properties :

If A, B, C are three sets, then :

(a) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ i.e. intersection is distributive over union.

(b) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ i.e. union is distributive over intersection.

For example, if A = {1, 2, 3, 4, 5}, B = (2, 3, 5, 7, 9} and C = {3, 4, 6, 8, 10} then,

(a) $A \cap (B \cup C)$ = (1, 2, 3, 4, 5}$\cap$ ({2, 3, 5, 7, 9} $\cup$ {3, 4, 6, 8, 10})

$= \{1, 2, 3, 4, 5\} \cap \{2, 3, 4, 5, 6, 7, 8, 9, 10\}$

$= \{2, 3, 4, 5\}$

and $(A \cap B) \cup (A \cap C) = (\{1, 2, 3, 4, 5\} \cap \{2, 3, 5, 7, 9\}$

$\cup (\{1, 2, 3, 4, 5,\} \cap \{3, 4, 6, 8, 10\})$

$= \{2, 3, 5\} \cup \{3, 4\} = \{2, 3,4, 5\}$

$\therefore A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(b) $A \cup (B \cap C) = (1, 2, 3, 4, 5\} \cup (\{1, 3, 5, 7, 9\} \cap \{3, 4, 6, 8, 10\})$

$= \{1, 2, 3, 4, 5\} \cup \{3\} = \{1, 2, 3, 4, 5\}$

and $(A \cup B) \cap (A \cup C) = (\{1, 2, 3, 4, 5\} \cup \{2, 3, 5, 7, 9\})$

$\cap \{1, 2, 3, 4, 5,\} \cup \{3, 4, 6, 8, 10\})$

$= \{1, 2, 3, 4, 5, 7, 9\} \cap \{1, 2, 3, 4, 5, 6, 8, 10\}$

$= \{1, 2, 3, 4, 5\}$

Thus, $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(iv) (a) $A \cup \phi = A$ (b) $A \cap \phi = \phi$

Let $A = \{p, q, r, s, t\}$, then,

(a) $A \cup \phi = \{p, q, r, s, t\} = \cup \phi = \{p, q, r, s, t\} = A$

(b) $A \cap \phi = \{p, q, r, s, t\} \cap \phi = \phi$.

(v) If U is a universal set, then

(a) $A \cup U = U$ (b) $A \cap U = A$

Let $U = \{a, b, c, d, e, f\}$ and $A = (b, d, f\}$, then

(a) $A \cup U = (b, d, f) \cup \{a, b, c, d, e, f\}$

$= \{a, b, c, d, e, f\} = U$

(b) $A \cap U = \{b, d, f) \cap \{a, b, c, d, e, f\} = \{b, d, f\} = A$

(vi) $A \cup A = A$

(vii) $A \subset A \cup B, B \subset A \cup B$

(viii) $A \cap B \subset A$ and $A \cap B \subset B$ [$A \cap B$ contains only those elements which are in A as well as in B]

(ix) If $A \cup B = \phi$, then $A = \phi = A$.

(x) If A and B are disjoint sets, then

$$A \cap B = \phi$$

(xi) $A \cap A' = \phi$ i.e., A and A' are disjoint sets. For example Let U = {1, 2, 3, 4, 5, 6, 7, 8} and A = {1, 3, 5, 7}, then A' = {2, 4, 6, 8}

$\therefore\ A \cap A' = \{1, 3, 5, 7\} \cap \{2, 4, 6, 8\} = \phi$.

(xii) $A \cup A' = U$

Let A = {1, 2, 3, 8} and U = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}

then, A' = {4, 5, 6, 7, 9, 10}

$A \cup A' = \{1, 2, 3, 8\} \cup \{4, 5, 6, 7, 9, 10\}$

$= \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

$= U$.

(xiii) $(A')' = A$ i.e., the complement of the complement of a set is the set itself.

Let U = {a, b, c, d, e}

and A = {a, c, e}

then, A' = {b, d}

and $(A')' = \{a, c, e\} = A$

(xiv) If A = U, then, $A' = \phi$.

(xv) Demorgan's Laws :

(i) $(A \cup B)' = A' \cap B'$

(ii) $(A \cap B)' = A' \cup B'$.

i.e., the complement of union is the intersection of complements and complement of an intersection is the union of the complements.

Example 1 : Let U = {1, 2, 3, 4, 5, 6, 7}, A = {2, 4, 6}

B = {1, 3, 5, 7} and C = {1, 2, 4, 7}

find : (i) $A \cup B$ (ii) $B \cup C$ (iii) $B \cap A$ (iv) $A \cap C$ (v) A' (vi) $A' \cap B'$

Solution :

(i) $A \cup B = \{2, 4, 6\} \cup \{1, 3, 5, 7\}$

$= \{1, 2, 3, 4, 5, 6, 7\}$

(ii) $B \cup C = \{1, 3, 5, 7\} \cup \{1, 2, 4, 7\}$

$= \{1, 2, 3, 4, 5, 7\}$

(iii) $B \cap A = \{1, 3, 5, 7\} \cap \{2, 4, 6\}$

$= \{ \ \}$.

(iv) $A \cap C = \{2, 4, 6\} \cap \{1, 2, 4, 7\} = \{2, 4\}$

(v) $A' = \{1, 3, 5, 7\}$

(vi) $A' \cap B' = \{1, 3, 5, 7\} \cap \{2, 4, 6\}$

$= \{ \ \}$

Example 2 : If $A = \{2, 6, 8\}$, $B = \{1, 3, 7, 8\}$, $C = \{3, 6, 8, 9\}$ and $U = \{1, 2, 3, 6, 7, 8, 9\}$, then verify that :

(i) $(A \cup B\}' = A' \cap B'$

(ii) $(A \cap B)' = A' \cup B'$

Solution : (i) $A \cup B = \{1, 2, 3, 6, 7, 8\}$

$\therefore (A \cup B)' = \{9\}$

Also, $A' = \{1, 3, 7, 9\}$ and $B' = \{2, 6, 9\}$.

$\therefore (A' \cap B') = \{9\}$.

Thus, $(A \cup B)' = A' \cap B'$.

(ii) $(A \cap B) = \{8\}$

$\therefore (A \cap B)' = \{1, 2, 3, 6, 7, 9\}$

$A' \cup B' = \{1,3,7,9\} \cup \{2,6,9\} = \{1,2,3,6,7,9\}$

$(A \cap B)' = A' \cup B'$

Example 3 : If $A = \{2x; x \in N\}$, $B = \{3x; x \in N\}$ and $C = \{5x; x \in N\}$, then find

(i) $A \cap B$ (ii) $A \cap C$ (iii) $(A \cap B) \cap C$.

Solution : Here A = {2, 4, 6, 8, 10, ...}

B = {3, 6, 9, 12, 15, ...}

C = {5, 10, 15, 20, 25, ...}

and

(i) $A \cap B = \{6, 12, 18, ...\}$

(ii) $A \cap C = \{10, 20, 30, ...\}$

(iii) $(A \cap B) \cap C = \{6, 12, 18, 24, 30, ...\} \cap \{5, 10, 15, 20, 25, 30, ...\}$

$= (30, 60, 90, ...\}$

3.18. Applications of Sets

If A is a finite set, then the number of elements of A can be counted & is denoted by n(A). We state below two results which will be used in solving the problems on applications of sets.

(i) If A and B are two sets, then :

$n(A \cup B) = n(A) + n(B) - n(A \cap B)$

For example, if A = {1, 2, 3, 5, 6} and B = {2, 3, 6, 8, 9, 10, 12},

then, $A \cup B = \{1, 2, 3, 5, 6\} \cup \{2,3, 6, 8, 9, 10, 12\}$

$= \{1, 2, 3, 5, 6, 8, 9, 10, 12\}$

and $A \cap B = \{1, 2, 3, 5, 6\} \cap \{2, 3, 6, 8, 9, 10, 12\}$

$= \{2, 3, 6\}$

Now, n(A) = 5, n(B) = 7, $n(A \cup B) = 9$ and $n(A \cap B) = 3$.

$\therefore n(A) + n(B) - n(A \cap B) = 5 + 7 - 3 = 9 = n(A \cup B)$

$\therefore n(A \cup B) = n(A) + n(B) - n(A \cap B)$

(ii) If A and B are two disjoint sets, then $n(A \cup B\} = n(A) + n(B)$.

If A and B are disjoint sets, then :

$n(A \cap B) = 0$

$\therefore n(A \cup B) = n(A) + n(B)\ 0 = n(A) + n(B)$.

Now, let us consider some examples.

Example 4 : The population of a town is 6000. Out of these 3400 persons read Hindustan Times and 2700 persons read Times of India. There are 700 persons, who read both the papers. Find the number of persons who do not read either of the two papers.

Solution : Let A be the set of persons who read Hindustan Times and B, the set of persons who read Times of India. Clearly, we have to find $n(A \cup B)'$.

We are given that $n(A) = 3400$

$n(B) = 2700$

and $n(A \cap B) = 700$

$\therefore\ n(A \cup B) = n(A) + n(B) - n(AB)$

$= 3400 + 2700 - 700$

$= 5400.$

$\therefore\ n(A \cup B)' = 6000 - 5400 = 600$

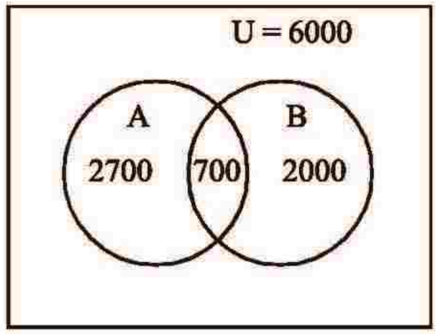


Fig. 3.4

So, the number of persons who do not read either of the two papers = 600.

Example 5 : In an examination 80 students secured first class marks in English or Mathematics. Out of these 50 students, obtained first class marks in Mathematics & 10 students in English & Mathematics both. How many students secured first class marks in English only ?

Solution : Let E be the set of students who secured first class marks in English and M the set of students who secured first class marks in Mathematics.

The given information may be written as :

$n(E \cup M) = 80,\ n(M) = 50,\ n(E \cap M) = 10$

We know that,

$n(E \cup M) = n(E) + n(M) - n(E \cap M).$

Substituting the values of $n(E \cup M)$, $n(M)$ and $n(E \cap M)$, we get,

$80 = n(E) + 50 - 10$

$\Rightarrow n(E) = 80 - 40 = 40$

$\therefore\ n(E) = 40.$

Thus the number of students who secured first class marks in English only

$= n(E) - n(E \cap M)$

$= 40 - 10 = 30.$

❖❖❖

3.19 Questions

1. Choose the correct answer from the alternatives given in each bit :

(i) Which of the following is a set ?

(a) A set of all rivers in India.

(b) A group of 10 best sportsmen in B.J.B College.

(c) A Group of eleven best cricket players in the world.

(d) A set of all beautiful flowers.

(ii) Which of the following in a well defined set ?

(a) All healthy students of a class

(b) All good girls of a class.

(c) All the excellent lecturers of a college

(d) All natural numbers less than 20

(III) Which of the following is a void set ?

(a) Set of even numbers between 10 and 19

(b) Set of squares with five sides

(c) All days of the week beginning with S.

(d) All prime numbers less than 9

IV) Which of the following pair of sets are equal sets ?

(a) A = {m,n,p,q} and B= {p, n, m, q, o}

(b) A = {m, n, p, q} and B = {1, 2, 3, 4}

(c) A = {m, n, p, q} and B = {p, n, m, q, n}

(d) A = {0} and B = {ϕ}

V. Which of the following pair of sets are equivalent ?

(a) A = {a, b, c, d} and B = {a, b, c, d, e}

(b) A= {x, y, z} and B = {1,2,3,4}

(c) A = {0} and B = {ϕ}

(d) A = {January, June, July) and B = {Sunday, Monday, Friday, Saturday}

VI) If {p, i, a, e,o}={x : x is a vowel in English Alphabet) then p is equal to :

a) b

b) c

c) d

d) u

VII) A is a set such that $\{0, 1, 3\} \subset A \subset \{0,3,2,1,4\}$
then set A is :

(a) $A = \{0,1,2,3\}$

(b) $A = \{0,1,3,4\}$

(c) both (a) and (b)

(d) $A = \{0, 3, 2, 5\}$

(viii) $A \cap (B \cup C) = (A \cap C) \cup (A \cap B)$ indicates the following property of set operations :

(a) Commutative property

(b) Associative Property

(c) Distributive Property

(d) Idem potence property

IX) A is a set of the factors of 6, therefore :

(a) $A = \{1,2,3\}$

(b) $A = \{0,1,2,3\}$

(c) $A = \{2,3\}$

(d) $A = \{1,2,3,6\}$

X) The total number of subsets of a finite set containing n elements is :

a) n^2

b) 2^n

c) $2n$

d) $n+n^2$

2. Answer the following quertions in one word or term :

(i) A set having only one element.

(ii) Complement of set A

(iii) A set containing no element.

(iv) The set of sets

(v) Name the set, whose number of elements can be counted.

(vi) Name the sets, if and only if each one is a subset of the other.

3. Fill in the blanks :

(i) $A \cup \phi$ = _________ .

(ii) $A \cap \phi$= __________ .

(iii) $(A \cup B)' = A' - B'$

(iv) A single ton set is a set containing ______ element.

(v) An empty set is denoted by ________.

(vi) According to De Morgan's Law $(A \cap B)'$ = _________ .

4. Correct the underlined portions of the following sentences :

(i) A set having only one zero as its element is called <u>zero</u> set.

(ii) If set A = {All girls} and B = {All boys}, then $A \cap B$ is an <u>Equivalent set.</u>

(iii) The notation $\supset$ is read as <u>subset.</u>

(iv) For only two sets A and B; $(A \cap B)' = \underline{A' \cap B'}$

(v) Number of male students in a wo-men's college will constitute <u>single ton</u> set.

(vi) Intersection of set A and set $\cup$ is equal to <u>$\cup$</u>

5. Answer the following questions in one senence each.

(i) Write the tabular form of $\{x : x \text{ is an odd number and } 0 \leq x \leq 10\}$

(ii) Write the set builder form of A = {a,e,i, o, u}

(iii) Define set.

(iv) What is a disjoint set ?

(v) What is a subset ?

(vi) What is an empty set ?

6. Answer the following questions within 30 words.

(i) If $A = \{x : x^2 - 4x+3=0\}$ and $B = \{0,1,2,3\}$ find $A \cap B$

(ii) Define the following terms and give two example of each :

(a) Empty set (b) Subset

(iii) What is a power set ? Find all subsets of the set A, where $A = \{3,5,m,n\}$

(iv) Explain universal set with examples.

(v) If $A = \{3,5,a,b\}$ and $B = \{1,5,b,d\}$ verify that.

(a) $A \cap B = B \cap A$

(b) $A \cup B = B \cup A$

7. Answer the following question within 50 words.

(i) What is a venn-diagram ? Does venn-diagram constitute a proof ? Draw a venn-diagram to show a subset of set.

(ii) What is complement law ? Explain this law with an example. Also prove that $A \cap A' = \phi$

(iii) What do you mean by De-Morgan's Law ?
Prove that $(A \cap B)'' = A' \cap B'$ and $(A \cap B)' = A \cup 'B'$

(iv) Define intersection of two sets with examples.

(v) What is union of two sets ? Draw a venn-diagram to show the union with numerial figure.

(vi) If $A = \{1,3,5,7,9\}$, $B = \{2,4,6,8,10\}$ and

$C = \{3,4,7,8,11,12\}$

Show that : (a) $(A \cup B) \cup C = A \cup (B \cup C)$

(b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

8. If A = {1,5,7,10}, B = {5,10,12,16} and C = {1,10,12,18,21} Draw venn -diagram to show.

(a) $A \cup B$

(b) $B \cup C$

(c) $A \cap C$

(d) $(A \cup C) \cap B$

(e) $(A \cap B) \cap C$

9 If A {p, q, r}, B =(x: x is a vowel" find the cartesian product of set A and B and count the number of elements in it.

Ans : A x B = {(P,a), (P,e), (P,i), (P,o), (P,u), (q,a), (q,a), (q,e), (q,i), (q,o), (q,u), (r,a), (r,e), (r,i), r,o), (r,u)} and n(AxB) = 15]

10.

(a) Identify the elements of A if

B = {1,2,3} and A x B = {(4.1), (4.2), (4.3), (5.1), (5.2), (5.3) (6.1), (6.2) (6.3)}

[Ans. {4,5,6}]

(b) If A = {x:x^2-4x+3}and B = {0,1,2,3}, find A x B.

[Ans. AB-{(3.0), (3.1), (3.2), (3.3), (1,0), (1.1)(1.2), (1,3)}]

11. Prove that $A-B = A \cap B$ and hence show that :

(a) $B \cap (A-B) = \phi$ and $B \cup (A-B) = A \cup B$

12. Out of total 150 students who appeared +2 Arts examination from a College, 45 failed in English, 50 failed in Economics and 30 in mathematics. Those who failed both in English and Economics were 30 those who failed both in Economics & Mathematics were 32 and those who failed both in English and mathmatics were 35. The students who failed in all the three subjects were 25. Find the number of students who failed at least in any one of the subjects.

[Ans. $n(A)+n(B)+n(c) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C) = 53$

Where A=English, B = Economics, C= Mathematics.

13. In a test there are 500 candidates. Out of them 360 candidates can write Hindi and 130 condidates can write English only. How many of them cand write both English and Hindi ? [Ans-10]

14. In a survey of 500 people, it was found that 280 read Times of India and 230 read The Samaj while 40 read both the papers.

(a) Howmaney read at least one paper ?

(b) How many read neither times of India nor the Samaj ?

(c) How many read exactly one paper ?

[Ans. (a) 470 (b) 30 (c) 190

15. A,B,C be any there sets, then prove that : $A \times (B \cup C) = (A \times B) \cup (A \times C)$

16. In a school, 60% students contributed towards National Defence Fund and 70% Students towards the National Savings Scheme. What percentage of students contributed towards both the schemes ? Assume that all the students have contributed to one or both the schemes. [Ans - 30%]

17. A survey was conducted on the T.V programmes watched by 120 students of a school hostel. It was revealed that 70 students watched 'sports' programme and 56 students watched 'Youth Forum' where, as 24 students watched both the programmes. Find the number of students who did not watch T.V on that day. [Ans. 18]

Answer

1. (i) a (ii) d (iii) b (iv) c (v) c (vi) d (viii) c (viii) c (ix) d (x) b

2. (i) Singleton (ii) A' (iii) Null / Void / empty (iv) Power set (v) Finite Set (vi) Equale Set

3. (i) A (ii) ϕ (iii) $\cap$ (iv) One (v) ϕ (vi) $A' \cup B'$

4. (i) Single-ton set (ii) Disjoint Set (iii) Power Set (iv) $A' \cup B'$ (v) Null Set (vi) A

5. (i) {1,3,5,7,9}

   (ii) {x : x is set of vowel in English alphabet }

   (iii) A set is a collection of well-defined objects.

   (iv) If no element of set A is in set B and no element of set B is in set A then set A and B are called disjoint sets.

   (v) If every element of a set A is also an element of another set B then A is called a subset of B.

   (vi) A set which contains no element is called an empty or null set.

■■■

CHAPTER - 4

FUNCTIONS

Structure

4.1 Introduction

One of the most important concepts in mathematics is that of a function. The word 'function' is derived from the Latin word meaning operation. It we square a real number x, we get another real number x^2.

Let us take another example. If we consider the marks obtained by different students, we can form pair of numbers to express this. In the example of squaring of numbers we take the pair (x, x^2). In the example of students and marks, we can have pairs (Madan, 21), (Mohan, 13), (Arun, 37). etc..... Thus we see that we can associate the members of one set with the members of second set by taking ordered pairs.

4.2 Relation

A relation is a subset of cartesian product set which also contains ordered pair of elements within it. For example, a polygon has several vertices, a number can have several factors, but a real number has only one number as its square, a circle has one centre so on. If a given element in an ordered pair, is associated with exactly one element, it is a special type of relation which is called a function. Thus, a relation from a set A to B is denoted by:

$$R = \{ (a, b) \mid a \in A \text{ and } b \in B \text{ and } aRb\}$$

For example : $R = \{ (1, 2), (2, 2), (1, 6), (2, 6), (3, 6), (6, 6)\}$

Here, the first element is a factor of the second element in each pair.

It cannot be accepted as a function as the first elements in the pairs are associated with more than one second element of the pair.

4.3 Meaning of function :

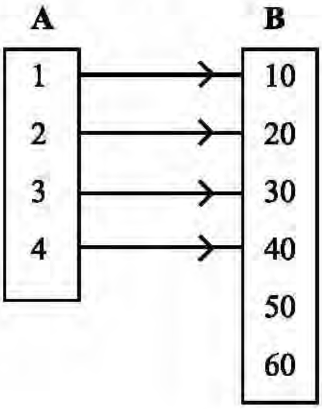
A function from A to B is a set of ordered pair of elements, in which every element of A is associated with exactly one element of B. If $f : A \to B$,

$$f = \{ (a, b)\ a \in A \text{ and } b \in B \text{ and has unique assignment with } b\}$$

For example : $f = \{ (1, 10), (2, 20), (3, 30), (4, 40) \}$, which are taken from the sets.

$A = \{ 1, 2, 3, 4\}$ and $B = \{10, 20, 30, 40, 50, 60\}$.

Diagrammatically,



In the above diagram, the elements of set A i.e., 1, 2, 3, 4 assciated with only one element of set B i.e., 10, 20, 30, 40. The set of first elements of i.e., set A is called domain of the function, whereas the set of second elements i.e., set B is called the co-domain and the set of second elements which are associated with the elements of the first is called range. In the above example,

Domain = {1, 2, 3, 4}

co-domain = {10, 20, 30, 40, 50, 60}

Range = {10, 20, 30, 40}

In the above function in a given pair. (1, 10), 10 is called image of 1 or 1 is called pre-image of 10.

A function may be expressed in several ways. Take, for example, a function where (x,y) are ordered pair and y = 3x +1.

Now, it can be written as :

(a) $f = \{ (x,y) \mid y = 3x + 1\}$

(b) $f : x \rightarrow y$, where $y = 3x + 1$

(c) $f = \{ (x, 3x + 1) \mid x \in R\}$

(d) $f(x) = 3x + 1$

(e) $y = 4x - 7$

(f) $f : R \rightarrow R$, where $f(x) = 3x + 1$

(i) Equal functions :

Two functions F : A → B and g : A → B are said to be equal, if f (a) = g (a), for all a ∈ A.

Example : A = { 1, 2, 3} and B = {1, 4, 9} and f = { (1, 1), (2, 4), (3,9)}

Similarly, g: A → B , where ' b ' is the square of ' a '.

f = { (1,1), (2, 4), (3, 9) }

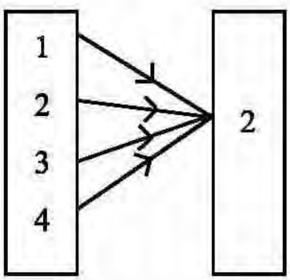
In both the functions : $f(1) = 1, f(2) = 4$ and $f(3) = 9$.

Hence, f = g or both functions are equal.

(ii) Constant function :

A function : f : R→ R is defined by f (x) = c, for all x ∈ R.

Example : f = { (1, 2), (2, 2), (3, 2), (4, 2)}



(iii) Identify function :

A function f : A → A is defined by f (a) = a for all a ∈ A is called the identity function of A.

Example : f = { (1, 1), (2, 2), (3, 3) } is an identity function of the set A = { 1, 2, 3}.

(iv) Real function :

A function f : A → B is called a real function, if A and B are subsets of the set of real numbers.

(v) Polynomial function :

Let ' n ' be a positive integer and $a_0, a_1, a_2, a_3, ..., a_n$ be any real numbers except that $a_n \neq 0$

A function f : R → R is defined by the formula : $f(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_0$ is called a polynomial function of degree *n*.

Example : $f(x) = 4x + 1, f(x) = x^2 + 5, f(x) = 2x^3 + 7$ are polynomial functions of degree 1, 2, 3 respectively & are called linear, quadratic & cubic functions respectively. Symbolically, $f = \{x, f(x) \mid f(x)$ is a polynomial function$\}$.

(vi) Rational function :-

A rational function is defined the quotient of two polynomial functions as :-

$f(x) = \dfrac{g(x)}{h(x)}$, where g (x) and h (x) are two polynomial functions and $h(x) \neq 0$

Example : $f(x) = \dfrac{x^2 + 3x + 5}{x^4 + 7}$

(vii) Even and odd function :-

A real function is said to be an even function, if $f(-x) = f(x)$.

Example : $\therefore f(x) = x^2$

$\therefore f(-x) = (-x)^2 = x^2 = f(x)$

Hence, $f(x) = x^2$ is an even function.

A real function $f(x)$ is said to be odd function if $f(-x) = -f(x)$

Example : If $f(x) = x^3$

Then $f(-x) = (-x)^3 = -x^3 = -f(x)$

Hence, $f(x) = x^3$ is an odd function.

(viii) Absolute value function :

A function $f : R \rightarrow R$ defined by :

$$f(x) = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$

is called absolute value function. It is also known as modulus function & written as $y = f(x) = |x|$.

(ix) Signum function :

A function f (x) is said to be signum function, if :

$$Y = f(x) = \begin{cases} 1, & \text{for } x > 0 \\ 0, & \text{for } x = 0 \\ -1, & \text{for } x < 0 \end{cases}$$

(x) Greatest Integer function :-

A function $f(x)$ is said to be greatest integer function, if $f(x) = [x]$, which implies $[x]$ is the greatest integer less than or equal to x.

Example : f = { (3.2, 3), (4.7, 4), (-3.4, -4), (-5,-5)}

<u>4.4 CLASSIFICATION OF FUNCTIONS :</u>

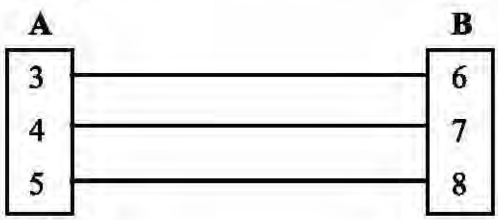
(i) One-One function :

A function $f: A \rightarrow B$ is said to be one-one function if distinct elements of A have distinct images. It is also called injective function.

Example : A = { 3, 4, 5} and B = {6, 7, 8} and f = {(3, 6), (4, 7), (5, 8)}

Hence each element of domain have different and distinct image in the co-domain.

Diagrammatically,



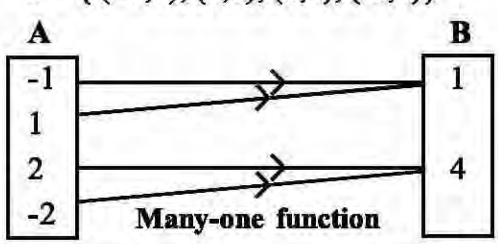
One-one function

(ii) Many-one function:

A function f : A → B is said to be many-one function if more than one element of A have the same image in B.

Example : Let A = { -1, 1, 2, -2} and B = { 1, 4} and

F = { (-1,1), (1,1), (2,4), (-2,4)}

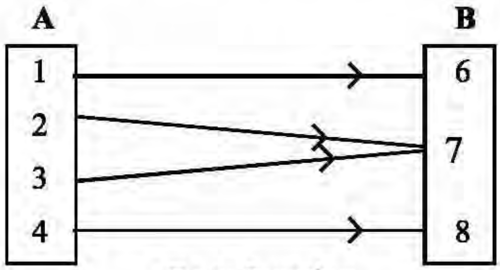


Many-one function

(iii) Onto function :

A function f: A → B is said to onto if every element of B is an image of at least one element of A : Onto function is also called subjective function.

Example : Let A = {1, 2, 3, 4} and B {6, 7, 8} and f : A → B is such that f (1) = 6, f (2) = 7, f (3) = 7 and f (4) = 8. Then f is an onto function. All the elements of B are images of elements of A.



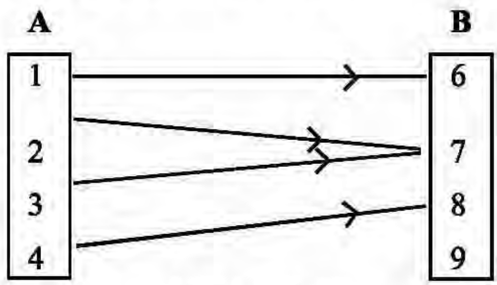
Onto function

(iv) Into function :

A function f: A → B is called into if there is at least one element in B which is not an image of elements of A.

Example : Let A = {1, 2, 3, 4} and B = { 6, 7, 8, 9} and f : A → B is such that f (1) = 6, f (2)= 7, f (3) = 7 and f (4) = 8, but 9 is an element of B which is not the image of the elements of A.

Then, the function f is an into function.

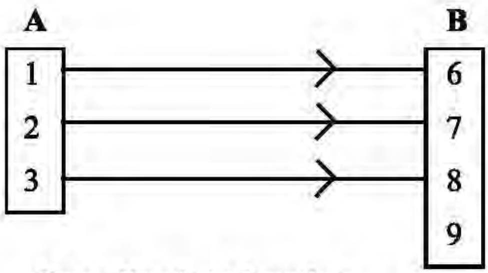


Into function

(v) One - One into function : -

A function which is one - one & into is called one-one into function.

Example :

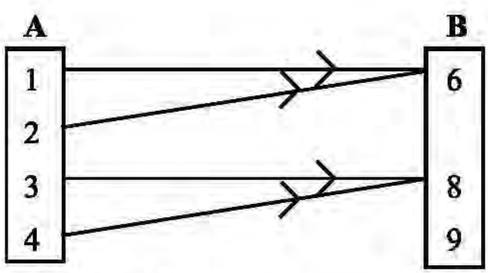


One - One into function

(vi) Many - one into function :-

A function which is many-one, as well as, into is called many-one into function.

Example :

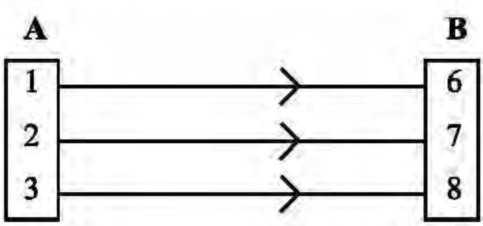


Many - one into function

(vii) One-One onto function :

A function which one-one as well as onto is called one-one onto function.

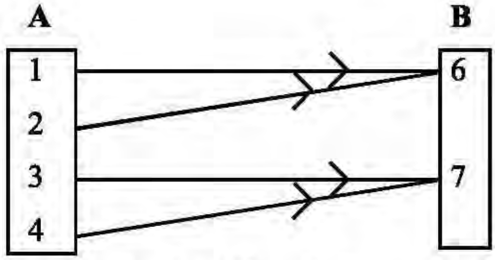
Example :



(viii) Many - one onto function :

A function which is many-one as well as onto is called many-one onto function.

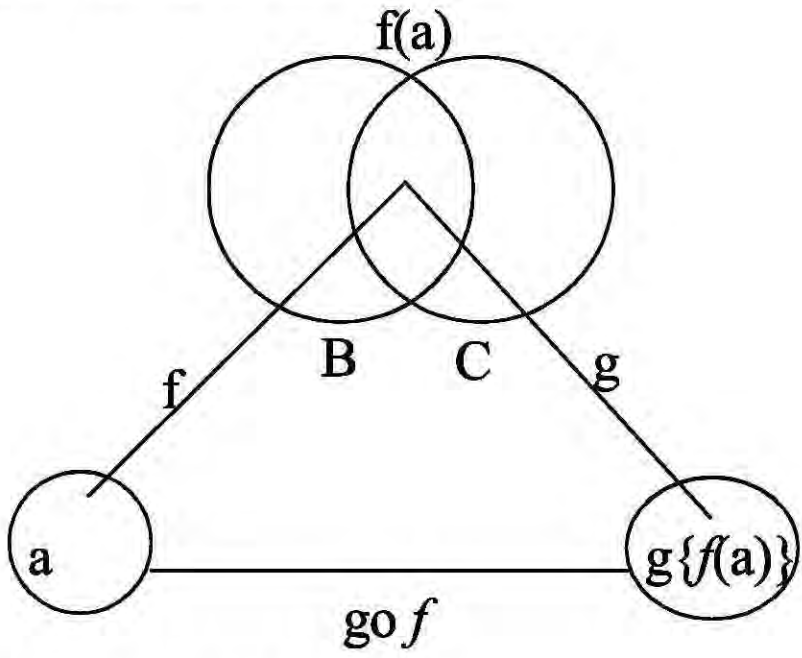
Example :



Many- one onto function.

4.5 COMPOSITE FUNCTION

Let f : A → B and g : C → D be two functions, such that the range of F ⊂ C. The set of ordered pairs {a, d) | a ∈ A and d ∈ g (*f* (a))} is a function from the set A into set D. This function is called composite of *f* and g and is denoted by go *f*.



Similarly, fog can be obtained

Example : Let $f(x) = x^2 + 5$ and $g(x) = x+2$

Then, go $f = g[f(x)]$

$= g[x^2+5]$
$= x^2 + 5 + 2 \qquad (\because g(x) = x+2)$
$= x^2 + 7$

and $fog = f[g(x)]$
$= f[x+2]$
$= (x + 2)^2 + 5 \qquad (\because f(x) = x^2 + 5)$
$= x^2 + 4 + 4x + 5$
$= x^2 + 4x + 9$

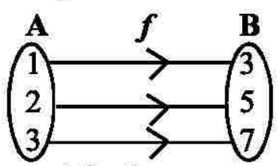
4.6 INVERSE FUNCTION :-

If $f : A \rightarrow B$ is one-one onto function, then a function $g : B \rightarrow A$, which associates each element of B with a unique element of A, such that, $b = f(a)$, is called an inverse function. Inverse function is denoted by f-1.

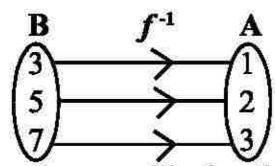
$\therefore f^{-1} = \{ (b,a) \mid b = f(a) \}$
$= \{ (b,a) \mid (a,b) \in f \}$

Example : If $y = f(x) = 2x + 1$, then $x = \dfrac{y-1}{2}$

Thus, $x = \dfrac{y-1}{2}$ can be written as $f^{-1}(x)$



A finction



Inverse of the function.

Thus, $f = \{ (x,y) \mid y = 2x + 1 \}$ and $f^{-1} = \{ (y,x) \mid x = \dfrac{y-1}{2} \}$

4.7 ALGEBRA OF FUNCIONS :-

Addition, subtraction, multiplication and division on function can be defined as below :
Let f and g be two function of x, such that,
$y = f(x)$ and $z = g(x)$

Then, the basic operations of addition, subtration, multiplication and division can be stated as follows.

(a) $(f + g) : x \rightarrow y + z$ or $y + z = f(x) + g(x)$
(b) $(f - g) : x \rightarrow y - z$ or $y - z = f(x) - g(x)$
(c) $(fg : x \rightarrow yz$ or $yz = f(x).\, g(x)$

(d) $\left(\frac{f}{g}\right) : x \rightarrow \frac{y}{z}$ or $\frac{y}{z} = \frac{f(x)}{g(x)}$, g (x) $\neq$ 0.

Example : If $f(x) = x + 2$ and $g(x) = x^2 + 1$, then find $f + g$, $f - g$, fg and $\frac{f}{g}$.

Solution : Let $f(x) = x + z = y$, $g(x) = x^2 + 1 = z$
(i) $f + g = y + z = x + z + x^2 + 1 = x^2 + x + 3$
(ii) $f - g = y - z = x + 2 - (x^2 + 1) = x - x^2 + 1$
(iii) $fg = y \times z = (x + 2)(x^2 + 1) = x^3 + 2x^2 + x + 2$
(iv) $\frac{f}{g} = \frac{y}{z} = \frac{x + 2}{x^2 + 1}$

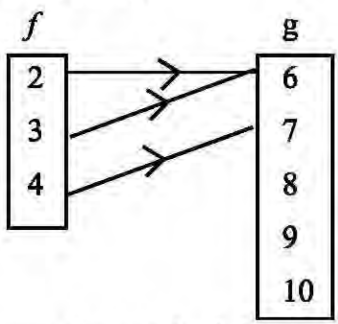
4.8 APPLICATIONS :-

Example 1 = If A = {2, 3, 4} and B = { 6, 7, 8, 9, 10}, verify whether the following are functions or not.

(a) f = { (2, 6)}, (3, 6), (4, 7) }

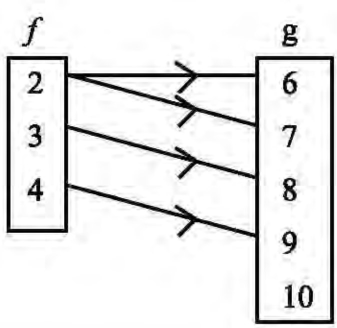
(b) g = { (2, 6), (2, 7), (3, 8), (4, 9) }

Solution : (a)



In the above diagram, every element 2 of f is associated with only one element of g, hence, f is a function.

(b)



In the above diagram, the element 2 of f is associated with 6 and 7 (i.e) more than one element of (g), hence g is not a function.

Example 2 : find the value of the function, if $f(x) = \frac{x+2}{x^2-1}$

(a) $f(2) + f(3)$ (b) $f(0) \times f(4)$

Solution : $f(x) = \frac{x+2}{x^2-1}$

$$\therefore f(2) = \frac{2+2}{4-1} = \frac{4}{3}, f(0) = \frac{0+2}{0-1} = -2$$

$$f(4) + \frac{4+2}{16-1} = \frac{6}{15} = \frac{2}{5} \text{ and } f(3) = \frac{3+2}{9-1} = \frac{5}{8}$$

$$\therefore \text{(a)} f(2) + f(3) = \frac{4}{3} + \frac{5}{8} = \frac{32+15}{24} = \frac{47}{24}$$

$$\text{(b)} f(0) \times f(4) = -2 \times \frac{2}{5} = -\frac{4}{5}$$

Example 3 : If $f(x) = x + 2$ and $g(x) = x^2$, find fog and got for the functions.

Solution :- $f(x) = x + 2$

$g(x) = x^2$

Then, $fog(x) + f[g(x)] = f[x^2] = x^2 + 2$.

Similarly, $gof(x) = g[f(x)] = g[x + 2] = (x+2)^2 = x^2 + 4x + 4$

Example 4 : Identify the even and odd function from the following:-

(a) $x^6 + 2x^4 + 7x^2 + 4$

(b) $x^7 - 5x^5 + 2x^3 - x$

Solution : (a) Let $f(x) = x^6 + 2x^4 + 7x^2 + 4$

Then, $f(-x) = (-x)^6 + 2(-x)^4 + 7(-x)^2 + 4$

$= x^6 + 2x^4 + 7x^2 + 4$

$= f(x)$

As f(–x) = f(x), it is an even function.

(b) Let $f(x) = x^7 - 5x^5 + 2x^3 - x$

Then $f(-x) = (-x)^7 - 5(-x)^5 + 2(-x)^3 - (-x)$

$= -x^7 + 5x^5 - 2x^3 + x$

$= -(x^7 - 5x^5 + 2x^3 - x)$

$= -f(x)$

As $f(-x) = -f(x)$, it is an odd function.

Example 5 : Find the inverse of the function $f(x) = 5x + 1$

Solution : The function $f(x) = 5x + 1$ is a linear function and it is 1-1 function.

Let $y = 5x + 1$

$\Rightarrow 5x = y - 1$

$\Rightarrow x = \frac{y-1}{5}$

Now, if, $f = \{ (x,y) \mid y = 5x + 1) \}$

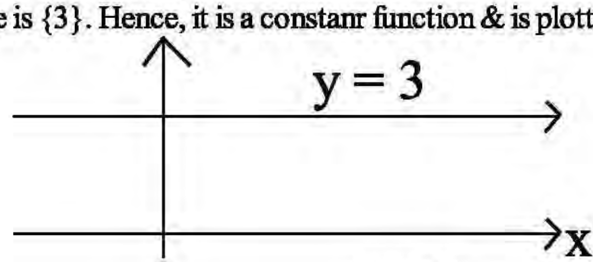
Then $f^{-1} = \left\{ (y,x) \mid x = \frac{y-1}{5} \right\}$

Example 6 : Draw a graph of the function $f(x) = 3$.

Solution : Let as take several sets of ordered pairs, then given function $f(x) = y = x^2$

x	0	1	2	3	-3	-2	-1
y	3	3	3	3	3	3	3

From the obove table, we find that the domain of the function is a set of real numbers and the range is {3}. Hence, it is a constanr function & is plotted in the graph paper as below.



4.9. Questions :-

1. From the alternatives given below, choose and write the correct answer along with its serial number.

(i) A is a function if :

(a) A = {(a,1), (a,2), (c,1, (d,4)}

(b) A = {a,1), (a,2), (b,3), (c,d)}

(c) A = {(a,2), (b,2), (c,2), (d,3)}

(d) A = {(a-1), (b,2), (b,3), (c,4)}

(ii) A is a constant function if :

(a) A = {a,2), (a,3), (b,3), (b,4)}

(b) A = {(a,3),(b,3), (c,3), (d,3)}

(c) A = {(a,2), (b,3), (c,4), (d,4)}

(iii) If A = {1,2,3}, B = {4,5} and
f = (1,4), (2,4), (3,5) is a function

(a) 1-1 into

(b) 1-1 onto

(c) Many-one into

(d) Many-one onto

(iv) If A = {a, b,c}, B = {1,2,3} and f = {(a,1), (b,2), (c,3)}
is a function :

(a) Constant function

(b) Onto function

(c) Into function

(d) Manyone into function

(v) $Y^2 = x^2+2xy+y$ is a/an

(a) Implicit function

(b) Explicit function

(c) Linear function

(d) Even fundtion

(vi) $f(x) = x^5 - 1$ is

(a) an even funcetion

(b) an odd funcetion

(c) both odd and even

(d) Neither even nor odd

(vii) If $f(x) = \frac{2}{x^2-4}$ and $g(x) = x+2$, then $\frac{2}{x-2}$ is

(a) f+g

(b) f-g

(c) fg

(d) $\frac{f}{g}$

(viii) If f (x) = 3, then the function is :

(a) Identity function

(b) Constant function

(c) Odd function

(d) Even function

(ix) If $f(x) = x^3$ and $g(x) = 2x+1$, then fog (x) would be :

(a) $(2x+1)^2$

(b) $(2x+1)^3$

(c) $(2x+1)^4$

(d) $(2x+1)^5$

(x) If the selling price per unit and cost per unit are ₹ x and ₹ y respectively and 5 units of the product are sold, then the profit function P is :

(a) $P = 5x + y$

(b) $P = 5x - y$

(c) $P = 5x - 5y$

(d) $P = 5x + 5y$

2. Answer the following questions in one word or term.

(i) What is the name of a function, when at least one element of the codomain which doesnot correspond to any element of the domain ?

(ii) What would be the name of a function, when its range is equal to its codomain ?

(iii) Name the function, when all elements of A are associated with the same elements of B.

(iv) What is the name of the function, when $f(-x) = f(x)$

(v) Name the function, when $f(x) = 1$, for $x > 0$

$f(x) = 0$, for $x = 0$

$f(x) = -1$, for $x < 0$.

3. Fill in the Blanks :

(i) If $f(-x) = -f(x)$, then $f(x)$ is an _______ function

(ii) The function $f(x) = Ka^x$, where $k \neq 0$

(iii) The function $f(x) = x$ is called ______ function

(iv) The domain of the function $f = \{(1,2), (2,3), (3,4)\}$ is ______

(v) The range of the function $f = \{(2,1), (3,2), (3,4)\}$ is ________

(vi) if $f(x) = 3x + 2$, $g(x) = x^2$ then fog = ________.

4. Correct the underlined portion of the following sentences .

(i) The function $f(x) = 2^x$ is called a <u>logarithmic</u> function.

(ii) The function $f(x) = 3x^4 - 3x^2 + 1$ is an <u>odd</u> function.

(iii) If f and g are two functions and defined as $f(x) = 3x + 1$ and $g(x) = x^2 + 2$, then f-g is equal to <u>$x^2 - 3x + 1$</u>

(iv) If in a function, there in at least one extra element in set B which is not associated with any element of set A, the function is called <u>onto</u>.

(v) If all the elements of set A are associated with one element of Set B, it is called <u>identity</u> function.

5. Answer the following questions in one sentence each.

(i) What is a function ?

(ii) Find the domain of the function, f= {(1,2), (2,4), (3,9)}

(iii) What is an odd function ?

(iv) What is an even function.

(v) Find the range of the function f (x)=2

(vi) What do you mean by inverse function ?

(vii) What is a composite function

6. If a function f is defined by $f(x) = 9x^2+1$, find f (o), $f\left(\frac{1}{2}\right)$, + $f\left(\sqrt{3}\right)$, + f (-2) and $f\left(-\frac{2}{3}\right)$.

7. Given A= {2,4,6,8} and B = {6,8,10,12} clarify the following ordered pair set as function or not a function.

(i) C_1 = {(2,6), (4,8), (6,10), (8,12)}

(ii) C_2 = {(2,6), (4,8), (6,10)}

(iii) C_3 = {(2,6), (4,6), (6,8), (8,8)}

(iv) C_4 = {(2,10), (4,10), (4,10), (6,10), (8,10)}

(v) C_5 = {(2,6), (2,8), (4,10), (6,12), (8,12)}

8. Find the domain and range of the following functions.

(i) f= {(1,2), (2,4), (3,6), (4,8)}

(ii) f= {x, x+2} : x ∈ {1,2,3}

(iii) $f(x) = \frac{x-2}{x-2}$

(iv) f (x) = 1

9. From the following, findout which are even and which are odd functions.

(i) $f(x) = x^2+5$

(ii) $f(x) = x^3-x$

(iii) $f(x) = 19x^4 - 7x^2 + 1$

(iv) $f(x) = 3x+5x^7$

10. If $f(x) = 3x+2$, $g(x) = x^2$, $h(x) = 2x^3+1$

Find (i) fog (ii) foh (iii) fof (iv) hof (v) gof (vi)gog

11. Find the inverse of the following functions :

(i) $f(x) = 5x+2$

(ii) $f(x) = \frac{2x-1}{x-1}$ and domain of f is $\{x : x>1\}$

(iii) $f = \{(1,2), (2,3), (4,6), (5,7), (8,10)\}$

(iv) $f(x) = \frac{x+1}{x-1}, x \neq 1$

12. If $g(x) = \frac{1}{x}$ and $x \neq 0$, prove that $gog^{-1} = g^{-1}og$

13. Draw the graph for the following functions :

(i) $f(x) = 5$

(ii) $f(x) = x+1$

(iii) $f(x) = x^2$

(iv) $f(x) = I x I$

(v) $f(x) = [x]$

14. Show that $e^x + e^{-x}$ is an even function.

15. Explain the meaning of 1-1 into and 1-1 onto function.

16. Show diagramatically many-one into and many one onto function.

17. Classify the following functions as one-one onto, one-one into, many-one onto, many-one into.

(i) $f : R \rightarrow R, f(x) = 5x^2+8$

(ii) $g ; R \rightarrow R, g(x) = 5x + 3$

(iii) $f = \{(1,2), (2,3), (3,3)\}$

(iv) $f : R \rightarrow R, f(x) = 1 x 1$

(v) $f: R \to R, f(x) = [x]$

18. Find the inverse of the function $f(x) = 2x + 1$ and draw a graph for the inverse function.

19. Find the quotient of an identify function and modulus function.

{Hints : $f(x) = x$ and $g(x) = |x|$ }

20. Find f+g, f-g, f.g, $\frac{1}{f}, \frac{f}{g}$, if $f(x) = x^3+1$

21. Show that $\lim_{x \to 0} \frac{e^{2+x} - e^2}{x} = e^2$

22. Show that $\lim_{x \to 0} \frac{3^{2+x} - 9}{x} = 9 \text{Log } e^3$

23. Draw a graph of the function $f(x) = x^2$

24. Find the values of functions

(i) $f(x) = \text{Log}_e x$, for $x = e$

(ii) $f(x) = \text{Log}_{10} x$, for $x = 10$

25. If $f(x) = \text{Log}\left(\frac{1-x}{1+x}\right)$, show that $f(b) + f(c) - f\left(\frac{b+c}{1+bc}\right)$.

26. Explain into and onto function with help of example.

Answer

1. (i) c, (ii) b (iii) d (iv) b (v) a (vi) d (vii) c (viii) b, (ix) b (x) c

2. (i) into function (ii) Onto function (iii) constant function
(iv) Even function (v) Signum function.

3. (i) Odd (ii) exponential (iii) Identity (vi) {1,2,3}
(v) {1,2,4} (vi) $3x^2 + 2$

(4) (i) Exponential (ii) Even (iii) $-x^2 + 3x - 1$
(iv) into (v) Contant.

5. (i) A function f form A to B written as $f : A \rightarrow B$ is a relation in which to every element of A corresponds a unique element of B.

(ii) Domain {1,2,3}

(iii) A function is said to be odd, when f (-x) = -f (x)

(iv) A function is said to be even, when (f (-x) = f (x)

(v) Range = {2}

(vi) A composite function may be called as a function of a function.

■■■

CHAPTER - 5

Limit and Continuity

Structure

5.1 Limit of a function

5.2 Some theorems of limit

5.3 Evaluation of limits

5.4 Some standard limits

5.5 Continuity of a function

5.6 Questions.

5.1 Limit of a function

In the last chapter the meaning of a function is clearly explained. This chapter will explain the limits of such functions. Let us take an example, **f(x) = 2x –1**. In this function the values of the independent variable, x constitute domain of the function and the corresponding values of the dependent variable, y or f(x) constitute the range of the function. The function assumes different values for the different values of x. That is, f(x)=1 as x = 1, f(x) = 3 as x = 2, so on. Sometimes it becomes difficult to determine the values of the range (y) of the function for given values of the domain(x). In the function, $f(x)=\frac{x^2-1}{x-1}$ the values of $f(x)=\frac{0}{0}$ when x = 1, similarly in $f(x)=\frac{x+2}{x-2}$, the vlaue of $f(x)=\frac{4}{0}$ when x = 2. Such indeterminate forms of the function cause a great difficulty. In order to avoid such indeterminate values of the function, the use of limit has been introduced.

Let us try to understand the concept of limit. Observe the graph of a function f(x) = x or y = x. Let A(x, y) be any point of the straight line representing f(x) = x. Draw a perpendicular from the point A(x,y) on x-axis. Now AB = x and OB = x as y = x.

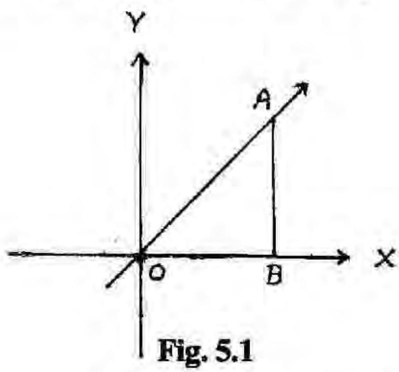


Fig. 5.1

As the point, 'A' approaches towards the origin, 'O' the length of AB decreases. Ultimately the length of AB becomes zero as the point reaches the origin, O. In this approach the function f(x) or y or the perpendicular, AB approaches towards zero as the value of x in A(x,y) or OB approaches towards zero. We write as $x \to 0$, $f(x) \to 0$.

Let us take a numerical example to understand the meaning of 'approach'. Let us consider a function f(x) = 4x + 1 and observe the values of the function as x approaches either from lower or higher side.

Case I. (When the value of x is **less than one** and then approaches 1).

Table 5.1

Value of x	Value of f(x) = 4x + 1
.95	4.8
.96	4.84
.97	4.88
.98	4.92
.99	4.96

Case II. (When the value of x is **more than one** and then approaches 1).

Table 5.2

Value of x	Value of f(x) = 4x+1
1.05	5.2
1.005	5.02
1.0005	5.002
1.00005	5.0002
1.000005	5.00002

In case I we observed that as 'x' approaches from .95 to 1, the function f(x) = 4x + 1 comes closer to 5. Thus we say, the limit of the function f(x) = 4x + 1 is 5 as x approaches 1. It is also known as **left hand limit.** We write,

$$\lim_{x\to 1^-} 4x+1 \text{ or, } \lim_{x\to 1} 4x+1=5.$$

In case II, we observed that as 'x' approaches from 1.05 to 1, the function f(x) = 4x + 1 comes closer to 5. Thus we say, the limit of the function f(x) = 4x + 1 is 5 as x approaches 1. It is also known as **right hand limit.** We write,

$\lim_{x \to 1^+} 4x + 1$ or, $\lim_{x \to 1} 4x + 1 = 5$

Example 1. Find $\lim_{x \to 1} f(x)$ and f(1), where $f(x) = \frac{x^2 - 1}{x - 1}$.

Solution : Let us consider the behaviour of the function, $f(x) = \frac{x^2 - 1}{x - 1}$ in the graph as well as in the numerical tables as x approaches 1.

The function $\frac{x^2 - 1}{x - 1}$ does not exist as x approaches 1 since the division by zero (x–1) is not defined. But $\lim_{x \to 1} \frac{x^2 - 1}{x - 1}$ exists and is equal to 2 as shown below :

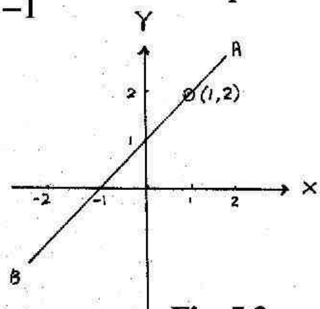


Fig. 5.2

On the above graph as x approaches 1 from lower side (left hand side), the value of f(x) comes closer to 2. Similarly as x approaches from higher side (right hand side), the function also comes closer to 2. Hence,

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2$$

Table 5.3 **Table 5.4**

x	$f(x) = \frac{x^2 - 1}{x - 1}$	x	$f(x) = \frac{x^2 - 1}{x - 1}$
.6	1.6	1.5	2.5
.7	1.7	1.2	2.2
.8	1.8	1.1	2.1
.9	1.9	1.01	2.01
.99	1.99	1.001	2.001

In Table 5.3 x approaches 1 from left i.e. from a lower value.

In Table 5.4 x approaches 1 from right i.e. from a higher value.

In both the tables we find that as x approaches 1, the function $\frac{x^2-1}{x-1}$ approaches 2.

Hence,

$$\lim_{x\to 1^-} = \frac{x^2-1}{x-1} = 2 \text{ and } \lim_{x\to 1^+} = \frac{x^2-1}{x-1} = 2.$$

5.2 Some theorems on limit :

Some theorems on the operation of addition, subtraction, multiplication, division and multiplication by a constant are given below :

Let f(x) and g(x) be two functions of x and both f(x) and g(x) exist.

(I) $\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$

(Limit of sum = Sum of limits)

(II) $\lim_{x\to a}[f(x)-g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$

(Limit of difference = Difference of limits)

(III) $\lim_{x\to a}[f(x)\cdot g(x)] = \left[\lim_{x\to a} f(x)\right]\cdot\left[\lim_{x\to a} g(x)\right]$

(Limit of product = Product of limits)

(IV) $\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} \left(\lim_{x\to a} g(x) \neq 0\right)$

(Limit of quotient = Quotient of limits)

(V) $\lim_{x\to a} k\cdot f(x) = k\cdot \lim_{x\to a} f(x)$ (where k is a constant)

(VI) $\lim_{x\to a} k = k$ (where k is a constant)

(VII) $\lim_{x\to a}\sqrt{f(x)} = \sqrt{\lim_{x\to a} f(x)}$

5.3. Evaluation of limits :

While determining the limiting value of a function with respect to a particular value of the independent variable we may come across indeterminant forms of the function such as $\frac{0}{0}, \frac{a}{0}, \frac{\infty}{\infty}$.

To overcome such situations we should make different approaches to find the limit of a function. We may **directly substitute** the value of the independent variable in the function or we may simplify the function by **cancelling the common factor** of the numerator and demominator of the rational function or we may **rationalise** the numerator or denominator of function. Let us examine such situations through various examples.

Example 2. Evaluate :

(i) $\lim_{x\to 2}(x+5)$, (ii) $\lim_{x\to 2} x(x+5)$, (iii) $\lim_{x\to 2} x(x-1)$

Solution : (i) The limit of $\lim_{x\to 2}(x+5)$ as $x \to 2$ is determined by :

$$\lim_{x\to 2}(x+5) = 2+5 = 7.$$

(ii) $\lim_{x\to 2} x(x+5) = \lim_{x\to 2}(x^2+5x) = (2)^2 + 5\times 2 = 14$

(iii) $\lim_{x\to 2} x(x-1) = \lim_{x\to 2}(x^2-x) = (2)^2 - 2 = 4-2 = 2$

Example 3. Evaluate :

(i) $\lim_{x\to 5} = \frac{x^2-1}{x+1}$, (ii) $\lim_{x\to 3} = \frac{x^2-5x}{x-2}$

Solution : (i) $\lim_{x\to 5} = \frac{x^2-1}{x+1} = \frac{(5)^2-1}{5+1} = \frac{25-1}{6} = \frac{24}{6} = 4$

(ii) $\lim_{x\to 3} = \frac{x^2-5x}{x-2} = \frac{(3)^2-5\times 3}{3-2} = \frac{9-15}{1} = -6$

Example 4. Evaluate :

(i) $\lim_{x\to 1}\frac{x^2-3x+2}{x^2-1}$, (ii) $\lim_{x\to 3} = \frac{x^2-5x+6}{x^2-9}$

Solution : (i) The given function $\lim_{x\to 1}\frac{x^2-3x+2}{x^2-1}$ at $x = 1$ takes the form of $\frac{a}{0}$ which is an indeterminate quantity. After factorising the numerator and denominator the common factor (x–1) can be cancelled from numerator and denominator to arrive at the limiting value.

$$\therefore \lim_{x\to 1} = \frac{x^2-3x+2}{x^2-1} = \lim_{x\to 1} = \frac{x^2-2x-x+2}{x^2-1}$$

$$= \lim_{x\to 1}\frac{(x-1)(x-2)}{(x-1)(x+1)}$$

$$= \lim_{x\to 1}\frac{x-2}{x+1}$$

$$= -\frac{1}{2}$$

(ii) The given function $\lim_{x\to 3}\frac{x^2-5x+6}{x^2-9}$ at x = 3 takes the form of $\frac{0}{0}$ which is an indeterminate quantity. So by cancelling a common factor from numerator and denominator, the limit can be easily found out.

$$\lim_{x\to 3}\frac{x^2-5x+6}{x^2-9} = \lim_{x\to 3}\frac{x^2-3x-2x+6}{x^2-9}$$

$$= \lim_{x\to 3}\frac{(x-3)(x-2)}{(x+3)(x-3)}$$

$$= \lim_{x\to 3}\frac{x-2}{x+3}$$

$$= \frac{1}{6}.$$

Example 5. Evaluate :

(i) $\lim_{x\to 0}\frac{\sqrt{1+x}-\sqrt{1-x}}{x}$, (ii) $\lim_{x\to 1}\frac{(2x-3)(\sqrt{x}-1)}{2x^2+x-3}$, (iii) $\lim_{x\to\infty}\left[\sqrt{x^2+x+1}-x\right]$

Solution : (i) As the limit takes indeterminate form $\frac{0}{0}$, we rationalise the numerator.

$$\lim_{x\to 0}\frac{\sqrt{1+x}-\sqrt{1-x}}{x} = \lim_{x\to 0}\left[\frac{\sqrt{1+x}-\sqrt{1-x}}{x}\times\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}-\sqrt{1-x}}\right]$$

$$= \lim_{x \to 0}\left[\frac{(1+x)-(1-x)}{x\left(\sqrt{1+x}+\sqrt{1-x}\right)}\right]$$

$$= \lim_{x \to 0}\left[\frac{2x}{x\left(\sqrt{1+x}+\sqrt{1-x}\right)}\right]$$

$$= \lim_{x \to 0}\frac{2}{\sqrt{1+x}+\sqrt{1-x}}$$

$$= \frac{2}{2}$$

$$=1.$$

(ii) $$\lim_{x \to 1}\left[\frac{(2x-3)\left(\sqrt{x}-1\right)}{2x^2+x-3}\right] = \lim_{x \to 1}\left[\frac{(2x-3)\left(\sqrt{x}-1\right)}{2x^2+3x-x-3}\right]$$

$$= \lim_{x \to 1}\left[\frac{(2x-3)\left(\sqrt{x}-1\right)}{(2x+3)(x-1)}\right]$$

$$= \lim_{x \to 1}\left[\frac{(2x-3)\left(\sqrt{x}-1\right)}{(2x+3)\left(\sqrt{x}+1\right)\left(\sqrt{x}-1\right)}\right]$$

$$= \lim_{x \to 1}\left[\frac{(2x-3)}{(2x+3)\left(\sqrt{x}+1\right)}\right]$$

$$= \frac{-1}{5 \times 2}$$

$$= -\frac{1}{10}.$$

(iii) $$\lim_{x \to \infty}\left[\sqrt{x^2+x+1}-x\right]$$

$$= \lim_{x \to \infty}\left[\frac{\left(\sqrt{x^2+x+1}-x\right)\left(\sqrt{x^2+x+1}+x\right)}{\left(\sqrt{x^2+x+1}+x\right)}\right]$$ (Rationalising the denominator)

$$= \lim_{x\to\infty} \frac{\left(\sqrt{x^2+x+1}\right)^2 - x^2}{\left(\sqrt{x^2+x+1}+x\right)}$$

$$= \lim_{x\to\infty} \frac{x^2+x+1-x^2}{\sqrt{x^2+x+1}+x}$$

$$= \lim_{x\to\infty} \frac{x^2+x+1-x^2}{\sqrt{x^2+x+1}+x}$$

$$= \lim_{x\to\infty} \frac{1+\frac{1}{x}}{\sqrt{1+\frac{1}{x}+\frac{1}{x^2}}+1} \text{ (Dividing both numerator and denominator by x)}$$

$$= \frac{1+0}{\sqrt{1+0+0}+1}\left(\text{As } x\to\infty, \frac{1}{x} \text{ or } \frac{1}{x^2}\to 0\right)$$

$$= \frac{1}{1+1}$$

$$= \frac{1}{2}.$$

Example 6. Evaluate :

(i) $\lim\limits_{x\to\infty} \frac{ax^3 + bx^2 + cx}{dx^3 + e}$ (where a, b, c, d, e are constants)

(ii) $\lim\limits_{x\to 2} \frac{x^3 - 3x^2 + 4}{x^3 - 4x + 4}$

(iii) $\lim\limits_{x\to 0} \frac{\sqrt{1-x} - 1}{x}$

Solution : (i) On direct substitution of the value of x, the function assumes the indeterminate form of $\frac{\infty}{\infty}$. Now, divide both numerator and denominator by x^3 and then find the limit.

$$\lim_{x\to\infty} \frac{ax^3 + bx^2 + cx}{dx^3 + e} = \lim_{x\to\infty} \frac{a + \frac{b}{x} + \frac{c}{x^2}}{d + \frac{e}{x^3}}$$ (dividing the numerator and denominator by x^3)

$$= \frac{a + 0 + 0}{d + 0} \left(\text{As } x \to \infty, \frac{1}{x} \text{ or } \frac{1}{x^2} \text{ or } \frac{1}{x^3} \to 0 \right)$$

$$= \frac{a}{d}.$$

(ii) On direct substitution of the values of x, the function assumes the form of $\frac{0}{0}$. So, that indeterminate situation may be avoided by factorising both the numerator and denominator.

$$\lim_{x\to 2} \frac{x^3 - 3x^2 + 4}{x^2 - 4x + 4}$$

$$= \lim_{x\to 2} \frac{x^3 + x^2 - 4x^2 + 4}{x^2 - 4x + 4}$$

$$= \lim_{x\to 2} \frac{x^2(x+1) - 4(x^2 - 1)}{x^2 - 4x + 4}$$

$$= \lim_{x \to 2} \frac{(x+1)\{x^2 - 4(x-1)\}}{x^2 - 4x + 4}$$

$$= \lim_{x \to 2} \frac{x^2(x+1) - 4\{(x+1)(x-1)\}}{x^2 - 4x + 4}$$

$$= \lim_{x \to 2} \frac{(x+1)\{x^2 - 4(x-1)\}}{x^2 - 4x + 4}$$

$$= \lim_{x \to 2} \frac{(x+1)\{x^2 - 4x + 4\}}{x^2 - 4x + 4}$$

$$= \lim_{x \to 2}(x+1)$$

= 2+1 (Putting the value of x)

= 3.

(iii) On direct substitution of the value of x, the function assumes the indeterminate form of $\frac{0}{0}$. Hence, in order to avoid such situation, the function should be rationalised by multiplying $\left(\sqrt{1+x}+1\right)$ both to the numerator and denominator.

$$\lim_{x \to 0} \frac{\sqrt{1+x}-1}{x} = \lim_{x \to 0} \frac{\left(\sqrt{1+x}-1\right)\left(\sqrt{1+x}+1\right)}{x\left(\sqrt{1+x}+1\right)}$$

$$\lim_{x \to 0} \frac{\left(\sqrt{1+x}-1\right)}{x} = \lim_{x \to 0} \frac{\left(\sqrt{1+x}-1\right)\left(\sqrt{1+x}+1\right)}{x\left(\sqrt{1+x}-1\right)}$$

$$= \lim_{x \to 0} \frac{\left(\sqrt{1+x}\right)^2 - (1)^2}{x\left(\sqrt{1+x}+1\right)}$$

$$= \lim_{x \to 0} \frac{1+x-1}{x\left(\sqrt{1+x}+1\right)}$$

$$= \lim_{x \to 0} \frac{x}{x\left(\sqrt{1+x}+1\right)}$$

$$= \frac{1}{\sqrt{1+0}+1} \text{ (Putting the value of x)}$$

$$= \frac{1}{2}$$

Example 7. Evaluate :

(i) $\lim\limits_{x \to 1} \frac{x^3+1}{x+1}$

(ii) $\lim\limits_{x \to 2} \frac{x^3+8}{x+2}$

Solution : (i) On direct substitution the function becomes $\frac{0}{0}$. Hence after factorisation of the function, the limit may be found out.

$$\lim_{x \to -1} \frac{x^3+1}{x+1} = \lim_{x \to -1} \frac{(x+1)(x^2-x+1)}{x+1} \quad (\text{As } a^3+b^3=(a+b)(a^2-ab+b^2))$$

$$= (-1)^2 - (-1) + 1$$

$$= 1+1+1$$

$$= 3.$$

(ii) On direct substitution the function becomes $\frac{0}{0}$. Hence the limit of the function may be found out after due factorisation.

$$\lim_{x \to 2} \frac{x^3+8}{x+2} = \lim_{x \to 2} \frac{(x+2)(x^2-2x+4)}{x+2} \quad (\text{As } a^3+b^3=(a+b)(a^2-ab+b^2))$$

$$= \lim_{x \to 2} (x^2-2x+4)$$

$$= (-2)^2 - 2(-2) + 4$$

$$= 4+4+4$$

$$= 12.$$

5.4 Some standard limits :

(I) If n be any positive integer, then

$$\lim_{x \to a} \frac{x^n - a^n}{x - a} = n\, a^{n-1} \text{ (where } x > 0)$$

(II) $\lim_{x \to 0} \frac{e^x - 1}{x} = 1$

(III) $\lim_{x \to 0} \frac{a^x - 1}{x} = \log_e a$

(IV) $\lim_{x \to 0} \frac{\log(1 + x)}{x} = 1$

Let us apply the standard limits in the following problems.

Example 8. Evaluate : $\lim_{x \to 1} \frac{x^3 - 1}{x - 1}$

Solution : $\lim_{x \to 1} \frac{x^3 - 1}{x - 1} = \lim_{x \to 1} \frac{(x)^3 - (1)^3}{x - 1}$

$$= 3(1)^{3-1} \left(\because \lim_{x \to a} \frac{x^n - a^n}{x - a} = n\, a^{n-1} \right)$$

$$= 3.$$

Example 9. Evaluate : $\lim_{x \to 2} \frac{x^3 - 8}{x^2 - 4}$

Solution : $\lim_{x \to 2} \frac{x^3 - 8}{x^2 - 4} = \lim_{x \to 2} \left[\frac{x^3 - 8}{x - 2} \div \frac{x^2 - 4}{x - 2} \right]$

$$= \lim_{x \to 2} \frac{x^3 - 8}{x^2 - 2} \div \lim_{x \to 2} \frac{x^2 - 4}{x^2 - 2}$$

$$= 3(2)^{3-1} \div 2(2)^{2-1} \left(\because \lim_{x \to a} \frac{x^n - a^n}{x - a} = n\, a^{n-1} \right)$$

$$= 12 \div 4$$

$$= 3.$$

Example 10. Evaluate : $\lim_{x\to a}\frac{(x+2)^{5/3}-(a+2)^{5/3}}{x-a}$

Solution : $\lim_{x\to a}\frac{(x+2)^{5/3}-(a+2)^{5/3}}{x-a}$

$=\lim_{x+2\to a+2}\frac{(x+2)^{5/3}-(a+2)^{5/3}}{(x+2)-(a+2)}$ (when $x\to a$, $x+2\to a+2$)

$=\frac{5}{3}(a+2)^{5/3-1}\left(\because \lim_{x\to a}\frac{x^n-a^n}{x-a}=n\,a^{n-1}\right)$

$=\frac{5}{3}(a+2)^{2/3}$.

Example 11. Find the value of the positive integer n, so that $\lim_{x\to 2}\frac{x^n-(2)^n}{x-2}=32$

Solution : $\lim_{x\to 2}\frac{x^n-(2)^n}{x-2}=32$

$\Rightarrow n(2)^{n-1}=32$

$\Rightarrow n(2)^{n-1}=4\times 2^{4-1}$

$\Rightarrow n=4$

$\therefore$ The required positive integer n is 4.

Example 12. Evaluate : $\lim_{x\to 0}\frac{e^{2x}-1}{x}$

Solution : $\lim_{x\to 0}\frac{e^{2x}-1}{x}=\lim_{x\to 0}\frac{e^{2x}-1}{2x}\times 2$

$=2\left[\lim_{2x\to 0}\frac{e^{2x}-1}{2x}\right]$

$=2\times 1\ \left(\because \lim_{x\to 0}\frac{e^x-1}{x}=1\right)$

$=2.$

Example 13. Find the value of $\lim_{x\to0}\frac{5^x-2^x}{x}$.

Solution : $\lim_{x\to0}\frac{5^x-2^x}{x}=\lim_{x\to0}\frac{(5^x-1)-(2^x-1)}{x}$

$$=\log_e 5-\log_e 2 \quad \left(\because \lim_{x\to0}\frac{a^x-1}{x}=\log_e a\right)$$

$$=\log_e 5/2.$$

Example 14. Show that $\lim_{x\to0}\frac{\log(1+x)}{x}=1$.

Solution : Let $y=\lim_{x\to0}\frac{\log(1+x)}{x}$

$$\Rightarrow xy=\log_e(1+x)$$

$$\Rightarrow e^{xy}=(1+x)$$

$$\Rightarrow e^{xy}-1=x$$

$$\Rightarrow \frac{e^{xy}-1}{x}=1$$

$$\Rightarrow \frac{e^{xy}-1}{xy}\cdot y=1$$

$$\Rightarrow \lim_{xy\to0}\frac{e^{xy}-1}{xy}\cdot y=1$$

$$\Rightarrow \lim_{xy\to0}\frac{e^{xy}-1}{xy}\cdot\lim_{xy\to0}y=1$$

$$\Rightarrow 1\cdot\lim_{x\to0}y=1 \quad (\text{As } x\to0 \text{ gives } xy\to0)\ \&\ \lim_{x\to0}\frac{e^x-1}{x}=1$$

$$\Rightarrow \lim_{x\to0}\frac{\log(1+x)}{x}=1 \quad (\text{Substituting the value of } y)$$

5.5 CONTINUITY OF A FUNCTION

A function f(x) is said to be continuous at x = a if and only if

(i) f(x) is defined at x = a

(ii) $\lim\limits_{x \to a^-} f(x) = \lim\limits_{x \to a^+} f(x)$

(iii) $\lim\limits_{x \to a} f(x) = f(a)$

(iv) $\lim\limits_{h \to 0} f(a-h) = f(a) = \lim\limits_{h \to 0} f(a+h)$

When any of the above conditions is not satisfied by a function, it is discontinuous at x=a.

Properties of continuous functions

Let f(x) and g(x) be two real functions, continuous at a. Let c be a real number, then,

(i) f(x) + g(x) is continuous at x = a.

(ii) f(x) – g(x) is continuous at x = a.

(iii) f(x).g(x) is continuous at x = a.

(iv) is continuous at x = a.

(v) c · f(x) is continuous at x = a.

(vi) $\frac{1}{f(x)}$ is continuous at x = a, provided $f(a) \neq 0$.

The test of continuity of a function can also be made graphically. Let us take two functions :

(a) f(x) = 2x

(b) $f(x) = \begin{cases} 3 & x > 0 \\ 0 & x \leq 0 \end{cases}$

(a) Graphic representation of the function y = 2x where y = f(x)

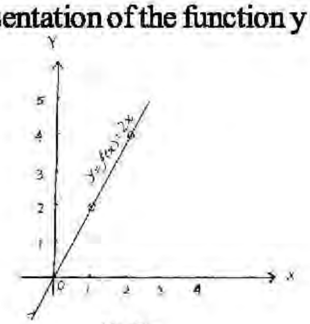


Fig 5.3

The graphic line of the function is continuous.

(b) Graphic representation of the function $y = \begin{cases} 3 & x > 0 \\ 0 & x \le 0 \end{cases}$

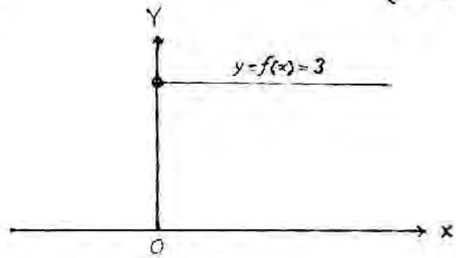


Fig 5.4

There is a jump in the graph x = 0 to x = 1, so the function is discontinuous at x = 0.

Example 15. Examine the continuity of the following function at x = 0, 1.

$$f(x) = \begin{cases} -x^2 & \text{if } x \le 0 \\ 4x - 3 & \text{if } 0 < x \le 1 \\ 4x^2 - 3^2 & \text{if } 1 < x \end{cases}$$

Solution :

(i) At x = 0

$f(0) = -(0)^2 = 0$

$\text{L.H.L.} = \lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^-}(-x^2) = 0$

$\text{R.H.L.} = \lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^+}(4x - 3) = -3$

As LHL $\neq$ RHL $\lim\limits_{x \to 0} f(x)$ does not exist.

Hence the function is discontinuous.

(ii) At x = 1

$f(1) = 4 \times 1 - 3 = 4 - 3 = 1$

$\text{LHL} = \lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^-}(4x - 3) = 4 \times 1 - 3 = 4 - 3 = 1$

$\text{RHL} = \lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^+}(4x^2 - 3x) = 4 \cdot 1^2 - 3 \times 1 = 4 - 3 = 1$

As $\lim\limits_{x \to 1} f(x) = 1 = f(1)$, the function is continuous at x = 1.

Example 16. A function is defined by

$$f(x) = \begin{cases} x-1 & when\ x \le 0 \\ x^2 & when\ x > 0 \end{cases}$$

show that the function f(x) is discontinuous at x = 0.

Solution :

Left Hand Limit (L.H.L.) $= \lim_{x \to 0^-} f(x)$

$$\lim_{x \to 0^-} (x-1)$$

$$= \lim_{h \to 0} (0-h) - 1$$

$$= -1.$$

Right Hand Limit (R.H.L.) $= \lim_{x \to 0^+} f(x)$

$$= \lim_{x \to 0^+} (x^3)$$

$$= \lim_{h \to 0} (0+h)^3$$

$$= 0$$

As L.H.L $\neq$ R.H.L the function f(x) is discontinuous at x = 0.

❖

5.6 QUESTIONS

1. From the given alternatives choose and write the correct answer against each bit along with its serial number :

(i) $\lim_{x\to 1}(3x+2)$ is equal to

(a) 4 (b) 5

(c) 6 (d) 0

(ii) $\lim_{x\to 2}\left(\frac{1}{x^2}+7\right)$ is

(a) ∞ (b) 0

(c) 6 (d) 7

(iii) $\lim_{x\to 2}\frac{x^5-32}{x^3-8}$ is equal to

(a) $\frac{20}{3}$ (b) $\frac{3}{8}$

(c) 1 (d) $\frac{3}{16}$

(iv) $\lim_{x\to a}\frac{x^3-a^3}{x^2-a^2}$ is equal to

(a) $\frac{3}{2}$ (b) $\frac{3t}{2}$

(c) $\frac{2}{3t}$ (d) $\frac{t}{3}$

(v) $\lim_{x\to -4}\frac{x^2-16}{x+4}$ is equal to

(a) 4 (b) 8

(c) –8 (d) 16

(vi) $\lim_{x \to 0} \frac{\sqrt{1+x} - \sqrt{1-x}}{x}$ is equal to

(a) 2 (b) 3

(c) 0 (d) 1

(vii) $\lim_{x \to -2} \frac{x^3 + 8}{x + 2}$ is equal to

(a) 10 (b) 12

(c) 16 (d) 8

(viii) $\lim_{x \to \infty} \frac{ax^2 + bx + c}{cx^2 + d}$ is equal to

(a) $\frac{b}{c}$ (b) $\frac{a}{c}$

(c) $\frac{c}{d}$ (d) $\frac{a}{d}$

(ix) $\lim_{x \to \infty} \left(\frac{x-1}{x+1} \right)$ is equal to

(a) 1 (b) 2

(c) $\frac{1}{2}$ (d) 0

(x) $\lim_{x \to \infty} \frac{9x^4 + 8x^3 + 7x^2 + 6x + 5}{4x^4 + 3x^3 + 2x^2 + x + 1}$ is equal to

(a) 5 (b) 6

(c) $\frac{7}{2}$ (d) $\frac{9}{4}$

(xi) If a function is defined by $f(x)=\begin{cases}5x-4 & 0<x\le 1\\ 4x^3-3 & 1<x\le 2\end{cases}$

then

(a) the function is continuous at $x=1$ (b) the function is not continuous at $x=1$

(c) the function is continuous at $x=3$ (d) the function is continuous at $x=4$

(xii) The function $f(x)=\begin{cases}x-1 & x<2\\ 2x-3 & x\ge 2\end{cases}$ is continuous

(a) at $x=2$ but not continuous at other points

(b) at $x=-2$ but not continuous at other points

(c) at $x=3$ but not continuous at other points

(d) at every point

(xiii) $f(x)=\dfrac{x^2+2x-3}{x+1}$, $x\neq 1$, $f(-1)=k$ and f(x) is continuous at $x=-1$, then the value of k is

(a) 1 (b) –1

(c) –2 (d) –4

(xiv) $f(x)=\begin{cases}x-1 & \text{when } x>0\\ \dfrac{1}{2} & \text{when } x+0\\ x+1 & \text{when } x<0\end{cases}$, then f(x) is

(a) discontinuous at $x=0$ (b) continuous at $x=0$

(c) undefined at $x=0$ (d) all of these.

(xv) A function f(x) is said to be continuous at $x=a$ if and only if

(a) f(x) is defined at $x=a$ (b) $\lim\limits_{x\to a^-} f(x)=\lim\limits_{x\to a^+} f(x)$

(c) $\lim\limits_{x\to a} f(x)=f(a)$ (d) all of the above are satisfied.

2. Express each of the following in one word /one term :

(i) $1+\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}$....................

(ii) $\lim_{x\to a^+} f(x)$

(iii) $\lim_{x\to 1}[(x+2^2-6]$

(iv) Is $f(x)=\begin{cases} x-1 & x\le 2 \\ 2x-3 & x>0 \end{cases}$ continuous at $x=2$.

(v) $\lim_{x\to 3}\frac{3-3}{\sqrt{x-2-\sqrt{4-x}}}$

3. Answer each of the following questions in one sentence :

(i) Evaluate $\lim_{x\to 1}\frac{x^3-1}{x-1}$

(ii) Evaluate $\lim_{x\to 5}\frac{x^2-9x+20}{x^2-6x+5}$

(iii) Evaluate $\lim_{x\to\infty}\left(\frac{5}{x^2}+7\right)$

(iv) Evaluate $\lim_{x\to\infty} e^{x^2+2x+1}$

(v) Write any one condition when a function f(x) will be said to be continuous at $x=1$.

4. Rectify the underlined portions of the following sentences :

(i) In $\lim_{x\to a} f(x)=l$, the function f(x) is said to tend to a limit $\underline{a}$.

(ii) In $\lim_{x\to a} f(x)=l$, l is called the $\underline{\text{left}}$ hand limit.

(iii) $\lim_{x\to 4}\sqrt[3]{6x+3}=\underline{8}$.

(iv) $\lim_{x \to a} \frac{(x+2)^{5/2} - (a+2)^{5/2}}{x-a} = \underline{\frac{5}{4}(a+2)^{3/2}}$

(v) In a continuous function $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = \underline{f(x)}$.

5. Fill in the blanks :

(i) $\lim_{x \to \infty} \frac{x^2 - 1}{x - 1} =$ ....

(ii) $\lim_{x \to 1} \frac{x^2 - 3x + 2}{x^2 - 1} =$............

(iii) If $\lim_{x \to 2} \frac{x^n - (2)^n}{x - 2} = 32$, then n is equal to ....

(iv) A function is said to be continuous at $x = a$ if and only if .... is defined at $x = a$

(v) If $f(x) = \begin{cases} x - 1 & when\ x \le 0 \\ x^2 & when\ x > 0 \end{cases}$, then

the function is .... at $x = 0$

6. Answer the following questions within 30 words each :

(a) Evaluate : $\lim_{x \to 2}(x+3)$.

(b) Evaluate : $\lim_{x \to 5}(x^2 - 3x)$.

(c) Evaluate : $\lim_{x \to 3} \frac{x^3 - 4x}{x - 2}$.

(d) Find the value of $\lim_{x \to 2} x(x^2 - 1)$.

(e) Evaluate : $\lim_{x \to 2} \frac{x^3 - 2^3}{x - 2}$

(f) Evaluate : $\lim_{x\to1} \frac{(2x-3)(\sqrt{x}-1)}{3x^2+3x-6}$

(g) Evaluate : $\lim_{x\to a} \frac{x^8-a^8}{x^3-a^3}$ $\left(\text{Hints}: \frac{x^8-a^8}{x^3-a^3} = \frac{\frac{x^8-a^8}{x-a}}{\frac{x^3-a^3}{x-a}}\right)$

(h) Evaluate : $\lim_{x\to3} \frac{x^4-81}{x^3-27}$

(i) Evaluate : $\lim_{x\to3} \frac{x-3}{\sqrt{x-2}-\sqrt{4-x}}$

(j) Evaluate : $\lim_{x\to\infty} \sqrt{4x^2+x+1}-2x$

(k) If $\lim_{x\to a} \frac{x^2-a^2}{x-a} = 12$, then find the value of a.

(l) Find the values of 'a' if $\lim_{x\to a} \frac{x^5-a^5}{x-a} = \lim_{x\to2} x+3$

(m) Find the value of $\lim_{x\to0} \frac{(1+x)^m-1}{x}$ (Hint : take y = 1+x or x = y–1)

(n) Show that the following function is discontinuous at x = 2

$$f(x) = \begin{cases} x-1 & \text{when } x \le 2 \\ x^2 & \text{when } x > 0 \end{cases}$$

7. Answer each of the following questions within fifty words :

(a) Test the continuity of the following function at x = 0

$$f(x) = \begin{cases} 4 & \text{when } x > 2 \\ 1 & \text{when } x \le 0 \end{cases}$$

(b) Evaluate $\lim_{x \to \infty} \frac{5x^2 + 4x + 1}{x^2 + 1}$

(c) Evaluate $\lim_{x \to 0} 8$

(d) Evaluate $\lim_{x \to -1} \frac{x^3 - 4x + 1}{x - 1}$

(e) Evaluate $\lim_{x \to 1} \frac{x^3 - 1}{x - 1}$

(f) Evaluate $\lim_{x \to 3} \frac{x^4 - 81}{x - 3}$

(g) Evaluate $\lim_{x \to 3} \frac{e^4 - e^3}{x - 3}$

(h) Find the value of n, if $\lim_{x \to 2} \frac{x^n - 2^n}{x - 2} = 32$

Answer the following long questions.

A. Evaluate the following limits :

8. $\lim_{x \to 2} \frac{x + 1}{x - 1}$

9. $\lim_{x \to 0} \left[3 - (x + 1)^2\right]$

10. $\lim_{x \to -1} \frac{x^2 - 7x + 10}{x^2 - 5x + 6}$

11. $\lim_{x \to 0} \frac{4x}{\sqrt{2 + x} - \sqrt{2 - x}}$

12. $\lim_{x \to \infty} \frac{x^3 + x^2 + x + 2}{x^3 - 1}$

13. $\lim_{x \to 2} 5$

14. $\lim_{x \to -3} \frac{x^3 + 27}{x + 3}$

15. $\lim_{x \to a} \left[\frac{x^m - a^m}{x^n - a^n} \right]$

16. $\lim_{x \to 0} \frac{1 - \sqrt{1 - x^2}}{x}$

17. $\lim_{x \to 2} \frac{x^2 \sqrt{x} - 4\sqrt{2}}{x - 2}$

18. $\lim_{x \to a} \frac{\sqrt{a + 2x} - \sqrt{3x}}{\sqrt{3a + x} - 2\sqrt{x}}$

19. $\lim_{h \to 0} \frac{3h^3 + 2h - h}{h^2 - h}$

20. $\lim_{x \to \infty} \frac{5x^2 - 3x + 7}{\sqrt{x^4 + 2}}$

21. $\lim_{x \to a} \frac{(x+2)^{7/3} - (a+2)^{7/3}}{x - a}$

22. $\lim_{x \to 0} \frac{x}{\sqrt{2 + x} - \sqrt{2 - 3x}}$

23. $\lim_{x \to 1} \frac{x^3 - x^2 - 2x + 2}{x - 1}$

B. Test the continuity of each of the following functions :

24. $f(x) = 2x + 3$

25. $f(x) = \begin{cases} \frac{1}{2}, & x \neq 0 \\ 2 & x > 0 \end{cases}$ at $x = 0$

26. $f(x)=\begin{cases}2x-3 & x<2\\ x-1 & x\geq 2\end{cases}$ at $x=2$

27. $f(x)=\begin{cases}x, & \text{when } x>0\\ 0, & \text{when } x=0\\ -x, & \text{when } x<0\end{cases}$ at $x=0$

28. $f(x)=\begin{cases}x^2, & \text{when x is greater than zero}\\ 2, & \text{when x is equal to zero}\\ x-1, & \text{when x is less than zero}\end{cases}$

29. $f(x)=\begin{cases}\dfrac{x^2-4}{x-2}, & \text{when x is not equal to 2}\\ 16 & \text{when x is equal to 2}\end{cases}$

C. Answer the following questions :

30. Given, $f(x)=\begin{cases}2x-1, & \text{when } x\leq 5\\ 3c, & \text{when } x>5\end{cases}$

If the function, f(x) is continuous at c = 5, find the value of c.

31. Show that the following function is continuous at x = 1.

$$f(x)=\begin{cases}3x-2, & \text{if } 0<x\leq 1\\ 4x^2-3x, & \text{if } 1<x\leq 2\end{cases}$$

32. From the following function find,

(a) Left-hand limit at x = 1

(b) f(x) = 2

(c) Right-hand limit at x = 1

$$f(x)=\begin{cases}4x+1, & \text{when } x>1\\ 3x+2, & \text{when } x=1\\ 5, & \text{when } x<1\end{cases}$$

Also test the continuity of the function at x = 1.

33. Show that the function f(x) = |x| is a continuous function.

ANSWERS

1. (i) b (ii) d (iii) a (iv) b (v) c (vi) d (vii) b (viii) b (ix) a
   (x) d (xi) a (xii) d (xiii) d (xiv) a (xv) d
2. (i) e (ii) RHL (iii) 3 (iv) Continuous (v) 1
3. (i) 3 (ii) ¼ (iii) 7 (iv) e^9 (v) $\lim_{x \to a^-} f(x) = f(a) = \lim_{x \to a^+} f(x)$
4. (i) e (ii) Right (iii) 3 (iv) $\frac{5}{2}(a+2)$ (v) f(a)
5. (i) 2 (ii) $-\frac{1}{2}$ (iii) 4 (iv) f(x) (v) Discontinuous
6. (a) 5 (b) 10 (c) 15 (d) 6 (e) 12 (f) $-\frac{1}{18}$ (g) $\frac{8}{3}a^5$ (h) 4 (i) 1
   (j) $\frac{1}{4}$ (k) 6 (l) ± 1 (m) m
7. (a) discontinuous (b) 5 (c) 8 (d) –3 (e) 3 (f) 108 (g) e^3 (h) 4

8. 3 9. 2 10. 8 11. 8 12. 1 13. 5 14. 13 15. $\frac{m}{n}a^{m-n}$

16. $\frac{1}{2}$ 17. $\frac{5}{2}$ 18. $\frac{2}{3\sqrt{3}}$ 19. 1 20. 5 21. $\frac{7}{3}(a+2)^{4/3}$

22. $\frac{1}{\sqrt{2}}$ 23. –1 24. continuous at every point

25. discontinuous 26. continuous 27. continuous 28. Discontinuous

29. Discontinuous at $x = 2$ 30. $c = 3$.

❖❖❖

CHAPTER - 6

DIFFERENTIATION

STRUCTURE

6.1 Meaning of differential coefficient

6.2 Derivatives of some important functions

6.3 Derivative of a function of a function

6.4 Derivative of implicit function

6.5 Derivative of functions in parametric forms.

6.6 Derivative of higher order

6.7 Miscellaneous illustrations

6.8 Questions

6.1 Meaning of differential coefficient

The differential coefficient of a function is the rate of change of the function for a particular value of the variable. If y or f (x) is a function of x and h be any small increment to x which corresponds to f (a+h)-f(a) increase in y or f (x) at a point a, then the ratio of the two increments $\frac{f(a+h)-f(a)}{h}$ represents the difference coefficient. It represents the average rate of change of f(x) in an interval of (a, a+h).

The limiting value of the ratio as h tends to zero represents the instantaneous rate of change of f (x) at 'a'. This instantaneous rate of change is known as differential coefficient of the function with respect to the independent variable at a particular point i.e. $\lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$ or $\frac{dy}{dx}$ is called the differential coefficient or derivative of y with respect to x. It is also denoted by f '(x) . Now

$$\frac{dy}{dx} = f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$$

Example 1.

If f (x) = x^2+4, find f ' (4)

Solution : Now f (x) = x^2+4

$$\therefore\ f'(4) = \lim_{h \to 0} \frac{f(4+h)-f(4)}{h}$$

$$= \lim_{h \to 0} \frac{[(4+h)^2+4]-[4^2+4]}{h}$$

$$= \lim_{h\to 0} \frac{16+h^2+8h+4-16-4}{h}$$

$$= \lim_{h\to 0} \frac{h^2+8h}{h}$$

$$= \lim_{h\to 0} \frac{h(h+8)}{h}$$

$$= \lim_{h\to 0}(h+8)$$

$$= 8$$

6.2 Derivatives of some important functions

The above mothod of finding out derivative is known as "differentiation from first principle or ab initio." To make computation more rapid and easier we may use some standard results of differentiation.

(i) Derivative of x^n, where n is any integer.

$$\frac{d(x^n)}{dx} = n.x^{n-1}$$

(ii) Derivative of a constant (c)

$$\frac{d(c)}{dx} = 0$$

(iii) Derivative of a multiple of a function

$$\frac{d(k.u)}{dx} = k.\frac{du}{dx}$$

(iv) Derivative of exponential function

$$\frac{d(e^x)}{dx} = e^x$$

(v) Derivative of the logarithmic function

$$\frac{d(\log_e x)}{dx} = \frac{1}{x}$$

(vi) Derivative of $\log_a x$ (x>0)

$$\frac{d(\log_a x)}{dx} = \frac{1}{x} \log_a e$$

(vii) Derivative of sum or difference two or more functions.

$$\frac{d(u \pm v)}{dx} = \frac{du}{dx} \pm \frac{dv}{dx} \quad \text{(Where u and v are functions of x)}$$

(viii) Derivative of product of two functions

$$\frac{d(u.v)}{dx} = v.\frac{du}{dx} + u.\frac{dv}{dx} \quad \text{(Where u and v are functions of x)}$$

(ix) Derivative of the quotient of two functions

$$\frac{d\left(\frac{u}{v}\right)}{dx} = \frac{v.\frac{du}{dx} - u.\frac{dv}{dx}}{v^2} \quad \text{(Where u and v are functions of x)}$$

Example 2.

Find the differential coefficient of x^2 from the first principle.

Solution : Let $y = x^2$

Now, $y + \Delta y = (x+\Delta x)^2$

$\Rightarrow \quad \Delta y = (x + \Delta x)^2 - y$

$\Rightarrow \quad \Delta y = x^2 + (\Delta x)^2 + 2.x.\Delta x - x^2$

$\Rightarrow \quad \Delta y = \Delta x\,(\Delta x + 2x)$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta x(\Delta x + 2x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} (\Delta x + 2x)$$

$$= 2x$$

Example 3.

Find the derivative of x^7 w.r.t.x

Solution : Let $y = x^7$

$$\therefore \frac{dy}{dx} = \frac{d(x^7)}{dx} = 7x^{7-1} = 7x^6 \quad \left(\because \frac{d(x^n)}{dx} = nx^{n-1}\right)$$

Example 4.

Find the derivative of 8 w.r.t.x.

Solution : Let y = 8

$$\therefore \frac{dy}{dx} = \frac{d(8)}{dx} = 0 \quad (\because \frac{d(c)}{dx} = 0)$$

Example 5.

Find the derivative of $7x^2$ w.r.t.x

Solution : Let $y = 7x^2$

$$\therefore \frac{dy}{dx} = \frac{d(7x^2)}{dx}$$

$$= 7.\frac{d(x^2)}{dx}$$

$$= 7.\ 2x$$

$$= 14x.$$

Example 6.

Find the derivative of $7x^2$ w.r.t.x

Solution : Let $y = 2x^2$

$$\therefore \frac{dy}{dx} = \frac{d(2e^x)}{dx}$$

$$= 2.\frac{d(e^x)}{dx}$$

$$= 2\ e^x$$

Example 7.

Find the derivative of 5 log x w.r.t.x

Solution : Let y = 5 log x

$$\therefore \frac{dy}{dx} = \frac{d(5.\log_e x)}{dx}$$

$$= 5.\frac{d(\log_e x)}{dx}$$

$$= 5.\frac{1}{x}$$

$$= \frac{5}{x}$$

Example 8.

Find the derivative of $\log_5 x$ w.r.t.x

Solution : Let $y = \log_5 x$

$$\therefore \frac{dy}{dx} = \frac{d(\log_5 x)}{dx} = \frac{1}{x}\log_5 e$$

Example 9.

Find the derivative of x^3+2x^2+7 w.r.t.x

Solution : Let $y = x^3 + 2x^2+7$

$$\therefore \frac{dy}{dx} = \frac{d(x^3 + 2x^2 + 7)}{dx}$$

$$= \frac{d(x^3)}{dx} + \frac{d(2x^2)}{dx} + \frac{d(7)}{dx}$$

$$= 3x^2 + 2.2x + 0$$

$$= 3x^2 + 4x$$

Example 10.

Find the derivative of $x^4 - 3x^2 - 8$ w.r.t.x

Solution : Let $y = x^4 - 3x^2 - 8$

$$\therefore \frac{dy}{dx} = \frac{d(x^4 - 3x^2 - 8)}{dx}$$

$$= \frac{d(x^4)}{dx} - \frac{d(3x^2)}{dx} - \frac{d(8)}{dx}$$

$$= 4x^3 - 3.2x - 0$$

$$= 4x^3 - 6x$$

Example 11.

Find the derivative of $(x^2+3)(x^3-7)$ w.r.t.x

Solution : Let $y = (x^2+3)(x^3-7)$

$$\therefore \frac{dy}{dx} = (x^3 - 7)\frac{d(x^2 + 3)}{dx} + (x^2 + 3)\frac{d(x^3 - 7)}{dx}$$

$$= (x^3 - 7)\left\{\frac{d(x^2)}{dx} + \frac{d(3)}{dx}\right\} + (x^2 + 3)\left\{\frac{d(x^3)}{dx} - \frac{d(7)}{dx}\right\}$$

$$= (x^3-7).2x + (x^2+3).3x^2$$

$$= 2x(x^3-7) + 3x^2(x^2+3)$$

$$= 2x^4-14x+3x^4+9x^2$$

$$= 5x^4+9x^2-14x$$

Example 12.

Find the derivative of $\frac{x^2 + 9}{x + 2}$ w.r.t.x.

Solution : Let $y = \frac{x^2 + 9}{x + 2}$

$$\therefore \frac{dy}{dx} = \frac{d\left(\frac{x^2+9}{x+2}\right)}{dx}$$

$$= \frac{(x+2)\frac{d(x^2+9)}{dx} - (x^2+9).\frac{d(x+2)}{dx}}{(x+2)^2}$$

$$= \frac{(x+2)\left\{\frac{d(x^2)}{dx} + \frac{d(9)}{dx}\right\} - (x^2+9)\left\{\frac{d(x)}{dx} + \frac{d(2)}{dx}\right\}}{(x+2)^2}$$

$$= \frac{(x+2).2x - (x^2+9).1}{(x+2)^2}$$

$$= \frac{2x^2+4x-x^2-9}{(x+2)^2}$$

$$= \frac{x^2+4x-9}{(x-2)^2}$$

6.3 Derivative of a function of a function

In order to differentiate a composite of two or more functions we apply Chain Rule. The technique to find out the derivative of a composite function is explained below.

Let y be function of t and t be the function of x. Now y = f (t) and u=f(x). Then by Chain Rule, we have, $\frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx}$

The Chain Rule may also be extended to more than two functions.

Example 13.

Find the derivative of $(x^3+8)^7$ w.r.t.x

Solution :

Let $t = x^3+8$ and y $(x^3+8)^7 = t^7$

Now $\frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx}$

$$= \frac{d(t^7)}{dt} \cdot \frac{d(x^3+8)}{dx}$$

$$= 7t^6 \left\{ \frac{d(x^3)}{dx} + \frac{d(8)}{dx} \right\}$$

$$= 7(x^3+8)^6(3x^2)$$

$$= 21\ x^2\ (x^3+8)^6$$

6.4. Derivative of implicit function

In equation $x^2+y^2+4xy = 16$, y is not explicitly defined. Hence it is an implicit function. The method to differentiate implicit is illustrated below. Let us take an implicit function :

$$x^2+y^2-4xy = 16$$

Now differentiating both sides with respect to x, we have

$$\frac{d(x^2+y^2-4xy)}{dx} = \frac{d(16)}{dx}$$

$$\Rightarrow \frac{d(x^2)}{dx} + \frac{d(y^2)}{dx} - \frac{d(4xy)}{dx} = 0$$

$$\Rightarrow 2x + \frac{d(y^2)}{dy} \cdot \frac{dy}{dx} - 4\left\{ y.\frac{dx}{dx} + x\frac{dy}{dx} \right\} = 0$$

$$\Rightarrow 2x + 2y.\frac{dy}{dx} - 4y - 4x\frac{dy}{dx} = 0$$

$$\Rightarrow 2y.\frac{dy}{dx} - 4x\frac{dy}{dx} = 4y - 2x$$

$$\Rightarrow \frac{dy}{dx}(2y-4x) = 4y - 2x$$

$$\Rightarrow \frac{dy}{dx} = \frac{4y-2x}{2y-4x} = \frac{2(2y-x)}{2(y-2x)}$$

$$\therefore \frac{dy}{dx} = \frac{2y-x}{y-2x}$$

6.5. Derivative of functions in parametric forms.

Sometimes two variables may be related to be a third variable which show separate relation between the first two variables. The third variable in called the parameter. If y=f(t) and x=g (t), then 't' is known as the parameter. To differentiate functions of parametric form we apply the following Chain Rule.

$$\frac{dy}{dx}=\frac{dy}{dt}\div\frac{dx}{dt}$$

$$\text{or}\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

Example 14 If $y = at^2$ and $x = 5bt$, find $\frac{dy}{dx}$

Solution :

$y = at^2$ and $x = 5bt$

Differentiating both the functions separately w.r.t.x we have,

$$\frac{dy}{dt}=\frac{d(at^2)}{dt}=a\frac{d(t^2)}{dt}=a.2t=2at \text{ .... (i)}$$

$$\text{Similarly, } \frac{dx}{dt}=\frac{d(5bt)}{dt}=5b.\frac{d(t)}{dt}=5b \text{ .... (ii)}$$

$$\text{Now, } \frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{2at}{5b}$$

6.6. Derivative of higher order.

The differential Coefficient of a function is denoted $\frac{dy}{dx}$ or f ' (x) and is called the first derivative. If this first derivative $\frac{dy}{dx}$ or f ' (x) is again differentiated w.r.t.x, the resultant derivative is denoted by $\frac{d^2y}{dx^2}$ or f " (x) and is called the second derivative. The successive derivatives are denoted by

$$\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3} .... \frac{d^ny}{dx^n},$$

or f'(x), f"(x), f'''(x) ....f^n(x).

Example 15.

Find the second order derivative of x^3 with respect to x.

Solution :

Let $y = x^3$

$$\therefore \frac{dy}{dx} = \frac{d(x^3)}{dx}$$

$= 3x^2$ .... (First order derivative)

Again $$\frac{d^2y}{dx^2} = \frac{d\left(\frac{dy}{dx}\right)}{dx}$$

$$= \frac{d(3x^2)}{dx}$$

$$= 3\frac{d(x^2)}{dx}$$

$= 3.2x$

$= 6x$ .... (Second order derivative)

6. Miscellaneous Illustrations :

Example 16

Find the differential coefficient of the following functions w.r.t.x from the first principle.

(a) x (b) $\sqrt{x+5}$ (c) $\frac{1}{2x+3}$

Solution :

(a) Let y=x

Then $y + \Delta y = x + \Delta x$

$\Rightarrow \quad \Delta y = x + \Delta x - y$

$$\Rightarrow \frac{\Delta y}{\Delta x} = \frac{x + \Delta x - x}{\Delta x} = 1 \; (\because \; y = x)$$

$$\text{Now} \frac{dy}{dx} = \lim_{\Delta_x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} 1 = 1$$

$$\therefore \frac{dy}{dx} = 1$$

(b) Let $y = \sqrt{x+5}$

Then $y + \Delta y = \sqrt{(x + \Delta x) + 5}$

$$\Rightarrow \Delta y = \sqrt{(x + \Delta_x) + 5} - \sqrt{x+5} \qquad (\because y = \sqrt{x+5})$$

$$\Rightarrow x_y = \frac{\sqrt{(x + \Delta_x) + 5} - \sqrt{x+5}}{1} \times \frac{\sqrt{(x + \Delta_x) + 5} + \sqrt{x+5}}{\sqrt{(x + \Delta x) + 5} + \sqrt{x+5}}$$

$$\Rightarrow \Delta y = \frac{x + \Delta x + 5 - x - 5}{\sqrt{(x + \Delta_x + 5} + \sqrt{x+5}}$$

$$\Rightarrow \frac{\Delta y}{\Delta x} = \frac{\Delta x}{\sqrt{x + \Delta_x + 5} + \sqrt{x+5}} \times \frac{1}{\Delta x}$$

$$\Rightarrow \lim_{\Delta_x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta_x \to 0} \frac{1}{\sqrt{x + \Delta_x + 5} + \sqrt{x+5}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2\sqrt{x+5}}$$

(c) Let $y = \frac{1}{2x+3}$

Then $y + \Delta y = \frac{1}{2(x+\Delta_x)+3}$

$$\Rightarrow \Delta y = \frac{1}{2(x+\Delta x)+3} - \frac{1}{2x+3}$$

$$\Rightarrow \Delta y = \frac{2x+3-2x-2\Delta x-3}{[2(x+\Delta x)+3][2x+3]}$$

$$\Rightarrow \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{-2.\Delta x}{[2(x+\Delta x)+3][2x+3]} \times \frac{1}{\Delta x}$$

$$\Rightarrow \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{-2}{[2(x+\Delta x)+3][2x+3]}$$

$$= \frac{-2}{[2x+3][2x+3]}$$

$$= \frac{-2}{(2x+3)^2}$$

Example 17.

Find the derivative of $e^x + 2\log_e x + \sqrt{x}$ w.r.t.x.

Solution :

Let $y = e^x + 2\log_e x + \sqrt{x}$

Now $\frac{dy}{dx} = \frac{d(e^x + 2\log_e x + \sqrt{x}}{dx}$

$$= \frac{d(e^x)}{dx} + \frac{d(2\log_e x)}{dx} + \frac{d(x)^{1/2}}{dx}$$

$$= e^x + 2.\frac{d(\log_e x)}{dx} + \frac{1}{2}.x^{\frac{1}{2}-1}$$

$$= e^x + 2.\frac{1}{x} + \frac{1}{2}x^{-\frac{1}{2}}$$

$$= e^x + \frac{2}{x} + \frac{1}{2\sqrt{x}}$$

Example 18.

If $y = 7x^3.e^x+3$, then find $\frac{dy}{dx}$.

Solution :

Now $y = 7x^3.e^x + 3$

$$\therefore \frac{dy}{dx} = \frac{d(7x^3.e^x + 3)}{dx}$$

$$= \frac{d(7x^3.e^x)}{dx} + \frac{d(3)}{dx}$$

$$= 7.\frac{d(x^2.e^x)}{dx}$$

$$= 7.\left[e^x \frac{d(x^3)}{dx} + x^3 \frac{d(e^x)}{dx}\right]$$

$$= 7\left[e^x.3x^2 + x^3e^x\right]$$

$$= 7x^2e^x(3 + x)$$

Example 19.

Find the derivative of $\frac{x^2 + 2x - 7}{x + 2}$ w.r.t.x.

Solution :

Now $y = \frac{x^2 + 2x - 7}{x + 2}$

$$\therefore \frac{dy}{dx} = \frac{d\left(\frac{x^2 + 2x - 7}{x + 2}\right)}{dx}$$

$$= \frac{(x+2).\frac{d(x^2+2x-7)}{dx} - (x^2+2x-7).\frac{d(x+2)}{dx}}{(x+2)^2}$$

$$= \frac{(x+2)\left\{\frac{d(x^2)}{dx} + \frac{d(2x)}{dx} - \frac{d(7)}{dx}\right\} - (x^2+2x-7)\left\{\frac{d(x)}{dx} + \frac{d(2)}{dx}\right\}}{(x+2)^2}$$

$$= \frac{(x+2)(2x+2) - (x^2+2x-7)}{(x+2)^2}$$

$$= \frac{2x^2+2x+4x+4-x^2-2x+7}{(x+2)^2}$$

$$= \frac{x^2+4x+11}{(x-2)^2}$$

Example 20.

Find the differential coefficient of the following functions :

(a) $\frac{1}{(2x+7)^5}$ (b) $e^{\log x + x}$

Solution :

(a) Let u =(2x+7) and y = $\frac{1}{(2x+7)^5}$ = $(2x+7)^{-5}$

Now y = $(2x+7)^{-5} = u^{-5}$

By Chain Rule, $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

$$= \frac{d(u^{-5})}{du} \times \frac{d(2x+7)}{dx}$$

$$= -5 \cdot u^{-5-1} \cdot \left\{\frac{d(2x)}{dx} + \frac{d(7)}{dx}\right\}$$

$$= -5\frac{1}{u^6}\cdot 2$$

$$= -\frac{10}{(2x+7)^6}$$

(b) Let u =logx+x and y = $e^{\log x + x}$

Now y = $e^{\log x + x}$

$\Rightarrow$ y =e^u

By Chain Rule, $\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$

$$= \frac{d(e^u)}{du} \times \frac{d(\log x + x)}{dx}$$

$$= e^u \cdot \left\{\frac{d(\log x)}{dx} + \frac{d(x)}{dx}\right\}$$

$$= e^u \cdot \left\{\frac{1}{x} + 1\right\}$$

$$= e^{\log x + x} \cdot \left\{\frac{1+x}{x}\right\}.$$

Example 21.

If $x^2 + 4y^2 = 12$, find $\frac{dy}{dx}$.

Solution :

(a) The function $x^2 + 4y^2 = 12$ is an implicit function. So differentiating both sides of the function w.r.t.x, we have,

$$\frac{d(x^2 + 4y^2)}{dx} = \frac{d(12)}{dx}$$

$$\Rightarrow \frac{d(x^2)}{dx}+\frac{d(4y^2)}{dx}=1$$

$$\Rightarrow 2x+\frac{d(4y^2)}{dy}\cdot\frac{dy}{dx}=1$$

$$\Rightarrow 2x+4\cdot 2y\cdot\frac{dy}{dx}=1$$

$$\Rightarrow 8y\cdot\frac{dy}{dx}=1-2x$$

$$\therefore \frac{dy}{dx}=\frac{1-2x}{8y}.$$

Example 22.

If $y=\sqrt{x+\sqrt{x+\sqrt{x+....\infty}}}$, then find $\frac{dy}{dx}$.

Solution :

$$y=\sqrt{x+\sqrt{x+\sqrt{x+....\infty}}}$$

Squaring both sides of the function, we have,

$$\Rightarrow y^2=x+\sqrt{x+\sqrt{x+....\infty}}$$

$$\Rightarrow y^2=x+y$$

Differentiating both sides with respect to x, we have,

$$\frac{d(y^2)}{dx}=\frac{d(x+y)}{dx}$$

$$\Rightarrow \frac{d(y^2)}{dy}\cdot\frac{dy}{dx}=\frac{d(x)}{dx}+\frac{dy}{dx}$$

$$\Rightarrow 2y.\frac{dy}{dx}-\frac{dy}{dx}=1$$

$$\Rightarrow \frac{dy}{dx}(2y-1)=1$$

$$\Rightarrow \frac{dy}{dx}=\frac{1}{2y-1}.$$

Example 23.

If $y = at^2$ and $x = 2at$, find $\frac{dy}{dx}$.

Solution :

Given that $y = a{\cdot}t^2$ and $x = 2a{\cdot}t$

So, $\frac{dy}{dt}=\frac{d(at^2)}{dt}=2a\cdot t$ and

$$\frac{dy}{dt}=\frac{d(2a\cdot t)}{dt}=2a$$

Now, $\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{2at}{2a}=t.$

Example 24.

If $x=a\left(t+\frac{1}{t}\right)$ and $y=a\left(t-\frac{1}{t}\right)$, then show that $\frac{dy}{dx}=\frac{t^2+1}{t^2-1}$.

Solution :

Given that $x=a\left(t+\frac{1}{t}\right)$

$$\Rightarrow \frac{dx}{dt}=\frac{d\left\{a\left(t+\frac{1}{t}\right)\right\}}{dt}$$

$$= a\left\{\frac{d(t)}{dt} + d\left(\frac{\frac{1}{t}}{dt}\right)\right\}$$

$$= a\left(1 - \frac{1}{t^2}\right)$$

Similarly, $y = a\left(t - \frac{1}{t}\right)$

$$\Rightarrow \frac{dy}{dx} = \frac{da\left(t - \frac{1}{t}\right)}{dx}$$

$$= a\left\{\frac{d(t)}{dt} - \frac{d\left(\frac{1}{t}\right)}{dt}\right\}$$

$$= a\left(1 + \frac{1}{t^2}\right)$$

Now, $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{a\left(1 + \frac{1}{t^2}\right)}{a\left(1 - \frac{1}{t^2}\right)}$

$$= \frac{\frac{t^2 + 1}{t^2}}{\frac{t^2 - 1}{t^2}}$$

$$= \frac{t^2 + 1}{t^2 - 1}.$$

Example 25.

If $f(x) = x^2 + 5x - 2$, then find $f'(x)$ and $f'(20)$.

Solution :

Given that $f(x) = x^2 + 5x - 2$

$$\Rightarrow f'(x) = \frac{d(x^2 + 5x - 2)}{dx}$$

$$= \frac{d(x^2)}{dx} + \frac{d(5x)}{dx} - \frac{d(2)}{dx}$$

$$= 2x + 5$$

Now $f'(2) = 2(2) + 5 = 4 + 5 = 9$

and $f'(20) = 2(20) + 5 = 40 + 5 = 45$.

Example 26.

If $f(x) = ax^2 + bx + 8$, $f'(2)=11$ and $f'(7) = 31$, then find the values of a and b.

Solution :

Given that $f(x) = ax^2 + bx + 8$

$$\Rightarrow f'(x) = \frac{d(ax^2 + bx + 8)}{dx}$$

$$= \frac{d(ax^2)}{dx} + \frac{d(bx)}{dx} + \frac{d(8)}{dx}$$

$$= 2ax + b. \qquad (\because \text{ a and b are constants})$$

As $f'(2) = 11$

$\Rightarrow 2a(2) + b = 11$

$\Rightarrow 4a + b = 11$ .... (i)

$f'(7) = 31$

$\Rightarrow 2a(7) + b = 31$

$\Rightarrow 14a + b = 31$ .... (ii)

Now $14a + b = 31$ (Solving the two equations simultaneously)

$$\begin{array}{r} -4a \pm b = -11 \\ \hline 10a = 20 \end{array}$$

$a = 2$

Putting a = 2 in equation (i), we have,

$4a + b = 11$

$\Rightarrow \quad 4 \times 2 + b = 11$

$\Rightarrow \quad b = 11 - 8$

$\Rightarrow \quad b = 3$

$\therefore \quad a = 2$ and $b = 3$.

Example 27.

Find the second order derivatives of the following functions :

(a) x^5, (b) e^x (c) $\log_e x$

Solution :

(a) Let $y = x^5$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{d(x^5)}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx} = 5x^4$$

$$f''(x) = \frac{d^2y}{dx^2} = \frac{d(5x^4)}{dx}$$

$= 5 \times 4x^3$

$= 20x^3$.

(b) Let $y = e^x$

$$\text{Now} \quad f'(x) = \frac{dy}{dx} = \frac{d(e^x)}{dx}$$

$= e^x$.

$$\therefore \quad f''(x) = \frac{d^2y}{dx^2} = \frac{d(e^x)}{dx}$$

$= e^x$.

(c) Let $y = \log_e x$

Now $f'(x) = \frac{dy}{dx} = \frac{d(\log_e x)}{dx} = \frac{1}{x}$

and $f''(x) = \frac{d^2y}{dx^2} = \frac{d\left(\frac{1}{x}\right)}{dx} = \frac{d(x^{-1})}{dx}$

$$= -1 \cdot x^{-1-1} = -x^{-2} = -\frac{1}{x^2}.$$

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6.8. Questions

1. Choose and write the correct answer from the alternatives given in each bit.

(i) $\frac{dy}{dx}$ is equal to :

(a) $\lim\limits_{\Delta x \to 0}\left[\frac{f(x+\Delta x)-f(x)}{\Delta x}\right]$ (b) $\lim\limits_{\Delta x \to 1}\left[\frac{f(x+\Delta x)-f(x)}{\Delta x}\right]$

(c) $\lim\limits_{\Delta x \to \infty}\left[\frac{f(x+\Delta x)-f(x)}{\Delta x}\right]$ (d) $\lim\limits_{x \to 0}\left[\frac{f(x+\Delta x)-f(x)}{\Delta x}\right]$

(ii) f ' (x) is equal to

(a) $\frac{1}{\frac{dy}{dx}}$ (b) $\frac{d[f(x)]}{dx}$ (c) $\frac{1}{f(x)}$ (d) $\sqrt{f(x)}$

(iii) If f (x) is equal to $\left(2x+\frac{1}{2}\right)^2$, then f ' (2) is :

(a) 16 (b) 32 (c) 18 (d) 36

(iv) Under Chair Rule $\frac{dy}{dx}$ is equal to

(a) $\frac{dy}{du}\times\frac{dx}{du}$ (b) $\frac{dy}{dx}\times\frac{dx}{du}$ (c) $\frac{dy}{du}\times\frac{du}{dx}$

(d) $\frac{dx}{dy}\times\frac{dy}{dx}$

(v) Derivative of a^x is equal to :

(a) $a^x \log_e a$ (b) $a^x \log_a e$ (c) $a^x.\log_a e$ (d) $a^x.\log e^a$

(vi) Derivative of e^x is equal to :

a) $a^x.\log_e a$ (b) $e^x\log_e e$ (c) $\frac{1}{x.\log_e a}$ (d) $e^x.\log_e a$

(vii) If $f(x) = \left(x + \frac{1}{x}\right)^2$, then f' is equal to :

(a) 0 (b) 1 (c) 2 (d) -1

(viii) It is an implicit function if :

(a) $X^2 + 2x + 7 = y$ (b) X3+y3+7xy = 12

(c) $y = 2x + 7$ (d) $y = X^2 + 2x$

(ix) The differential coefficient of a constant $\overline{K}$ is :

(a) K (b) $\frac{1}{K}$ (c) K^2 (d) 0

(x) The second order derivative of x^2 w.r.t. x is :

(a) 2x (b) x (c) $\frac{1}{x}$ (d) 2

2. Express / Answer each of the following in one word / term

(i) The alternative name of differential coefficient.

(ii) The rationalised value of the numerator of $\frac{\sqrt{a^2 + x^2} - \sqrt{a^2 - x^2}}{\sqrt{a^2 + x^2} + \sqrt{a^2 - x^2}}$

(iii) The Rule by which $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

(iv) The value of $\frac{dy}{dx}$ in $x^2 + y^2 = 16$

(v) The approximate value of e in natural logarithm.

3. Answer each of the following in one sentence :

(i) Differentiate (x-2) (x+3) w.r.t.x.

(ii) Differentiate x^3 +logx + 5

(iii) Find f '(2) if f (x) is equal to $x^3 + 2x^2+3x+7$.

(iv) Find the differential coefficient of e^{3x}

(v) Differentiate x^4 w.r.t.x^2.

4. Rectify the underlined portions of the following sentences :

(i) The differential coefficient of the sum of two functions is equal to the difference of their differential co-efficients.

(ii) It $y=\frac{1}{x}$, then $\frac{dy}{dx}.\frac{dx}{dy}=\underline{0}$

(iii) $x^2 - 5xy + xy^2 - 10 = 0$ is an explicit function.

(iv) $\frac{d(\log_a x)}{dx}=\frac{1}{x.\underline{\log e}}$

(v) $n.\underline{x}^{n+1} = x^n$

5. Fill in the blanks :

(i) $\frac{d(x^5)}{dx}=5$______

(ii) $\frac{d\left(\frac{1}{x^5}\right)}{dx}=-5.\frac{1}{----}$

(iii) In derivative of a function of a function,

$$\frac{dy}{dx}=\frac{dy}{\_\_\_\_}\times\frac{du}{dx}$$

(iv) The second derirative of x^4 is -------- .

(v) If $y = \sqrt{1+x^4}$, then $y.\frac{dy}{dx}$ is equal to --------.

6. Answer each of the following questions in 30 words :

(i) Find the derivative of $x^4 + 3x^3 + \log x + e^x - \frac{1}{x} + 7$

(ii) Find the differential coefficient of x^2 w.r.t.x from first principle.

(iii) State the formula for finding the derivative of quotient of two functions.

(iv) If $p = 4q^2 - 2q$, find $\frac{dp}{dq}$.

(v) Find $\frac{dy}{dx}$ for the function $x^3+y^3+3xy=7$.

(vi) Find the derivative of 5^x

(vii) Find the derivative of e^{3x}

(viii) If $x = 3t^2$ and $y=4t$ find $\frac{dy}{dx}$ $\left[\text{Hints: } \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right]$

(ix) Find the differential coefficient of log (log x) w.r.t.x.

(x) If $y = 2x^3+3x^2+4x+5$, find $\frac{dy}{dx}$ at $x = 2$

7. Answer each of the following questions within 50 wards.

(i) Find $\frac{dy}{dx}$ if, $y = (x^2 - 5)(4x^3 - 3x^2 - 7)$.

(ii) Find $\frac{dy}{dx}$ if $y = \frac{x^2-1}{x^2+1}$.

(iii) Find $\frac{dy}{dx}$ if $y = \sqrt{x}.\log x^2$.

(iv) Differentiate $\frac{x^2 + e^x}{\log x + 10}$ w.r.t.x.

(v) Explain derivative of a function of a function with an example.

(vi) If $y = at^4$ and $x = 2at^2$, find $\frac{dy}{dx}$.

(vii) Find the second order derivative of x^2+5x+1.

(viii) If $x^2-3xy-4y^2 = 0$, find $\frac{dy}{dx}$.

(ix) Differentiate $(3x+8)^7$ w.r.t.x.

(x) If $y = \left(\frac{2x}{x^2+1}\right)^3$, find $\frac{dy}{dx}$.

8. Answer the following long questions :

(A) Differentiate the following functions with respect to x :

(i) $5x^4 + 4x^3+3x^2+2x+1$

(ii) $x^2. e^{5x}$

(iii) x^x

(iv) $5^x + \log_5 x$

(v) $\frac{e^{2x} + e^{-2x}}{e^{2x} - e^{-2x}}$

(vi) $\frac{2x^2 + x + 1}{\sqrt{x}}$

(vii) $\left(3x^2+\frac{2}{x}+5\right)^6$ (Hints : Apply chain rule)

(viii) $\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)^2$

(ix) $\frac{x^3-2x^2+3}{x^4+x^2-5}$

(x) $x^2+y^2=16$

(xi) $x^2y - xy^2 = 4$

(xii) $x+y=\frac{xy}{200}$

(B) (i) Differentiate x^2 with respect to x from first principle.

(ii) Differentiate e^{2x} from first princeple.

(iii) If $f(x) = ax^2+bx+10$, $f'(2) = 14$ and $f'(3) = 20$, then find the values of a and b.

(iv) If $y = x^3e^x + 18$, then find $\frac{dy}{dx}$

(v) If $y = at^2$ and $x = 2bt$, then show that $\frac{dy}{dx}=\frac{at}{b}$.

(vi) If $y = e^{5\log x+2x}$, then find $\frac{dy}{dx}$

(vii) If $x^2+y^2=9$, then find $\frac{dy}{dx}$

(viii) If $y=\sqrt{\frac{(x-5)(x^2+3)}{(2x^2+4x+1)}}$, then find $\frac{dy}{dx}$, (Hints : take log on both sides)

(ix) If $x = 2at^2$, $y = at^3$, then find $\frac{dy}{dx}$.

(x) If $y = \frac{e^x - 1}{e^x + 1}$, then show that $\frac{dy}{dx} = \frac{2e^x}{(e^x + 1)^2}$

Answers

1. (i) a (ii) b (iii) c (iv) c (v)a (vi) b (vii) a (viii) b (ix) d (x) d

2. (i) Derivative (ii) $2x^2$ (iii) Chain rule (iv) $-\frac{x}{y}$ (v) 2.7183

3. (i)$2x+1$ (ii) $3x^2 + \frac{1}{2}$ (iii) 20 (iv) $3e^{3x}$ (v)$2x^2$

4. (i) Sum (ii) 1 (iii) Implict (iv)log a (v)x^{n-1}

5 (i) x^4 (ii) x^6 (iii) du (iv) $12x^2$ (v) $2x^3$

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CHAPTER - 7

INTEGRATION

STRUCTURE

7.1 Introduction

7.2 Definition and meaning of integration

7.3 Properties of indefinite integrals.

7.4 Some standard results of indefinite integrals.

7.5 Integration by substitution

7.6 Comparison between differentiation and integration.

7.7 Questions.

7.1 Introduction :

In the 17th century major advance in integration came with the independent discovery of the fundamental theorem of calculus by Issac Newton and Gottfried Leibniz. In differentiation we find the rate of change between two or more variables from a given functional relationship existing between them. But in integration we find the functional relationship between two or more variables from a given rate of change between them. We observe that integration is a reverse process of differentation. So the result of integration is termed as antiderivative. To better understand the technique of integration, let us define the term integration.

7.2 Definition and meaning of Integration

If f(x) is any differentiable function of x such that, $\frac{d[f(x)]}{dx} = f'(x)$ then f(x) is called the antiderivative of f'(x). That is, if we integrate f'(x) we get f(x) and if we differentiate f(x), we get f'(x) Symbolically, we write,

$$\int f(x)dx = f'(x) + C$$

where C is called the constant of integration, f'(x)+C represents a set of functions called antiderivatives. The anti derivative is also called integral or primitive of f(x).

Let us take an example : $\frac{d\left(\frac{x^2}{2}\right)}{dx} = x$

In this example x is the derivative of $\frac{x^2}{2}$. $\frac{x^2}{2}$ is the antiderivative of x. Hence the the integral of x is $\frac{x^2}{2}$. We write, $\int (x)\,dx = \frac{x^2}{2}$

But the integral or antiderivative of x may be many other functions other than $\frac{x^2}{2}$. The possible antiderivatives may be many because,

$$\frac{d\left(\frac{x^2}{2}+5)\right)}{dx}=x$$

Also, $$\frac{d\left(\frac{x^2}{2}+9)\right)}{dx}=x..........\text{so on.}$$

That is, $$\frac{d\left(\frac{x^2}{2}+C)\right)}{dx}=x, \text{ where C is a constant}$$

As there are indefinite number of functions which differ by the constant, but give the same derivative, the integration may provide an indefinite number of antiderivatives. That is why such type of integrals are called indefinite integrals. We define integration by,

$$\int f(x)dx = f'(x) + C$$

where the notations are as under :

$\int$ = Integral sign

x = Variable of integration

f(x) = Integrand

$f'(x) \rightarrow$ Integral of f(x) w.r.t. x

C $\rightarrow$ Constant of integration

7.3 Properties of Indefinite Integrals

(i) The process of differentiation and integration are opposite to each other. That is,

$$\frac{d[f(x)]}{dx} = f'(x) \text{ and} \int f'(x)dx = f(x) + c$$

(ii) Two indefinite integrals with the same derivative are equivalent, That is,

$\int f(x)dx = \int g(x)dx$, even though they differ by parameters of constants.

(iii) $\int [f(x) + g(x)] = \int f(x)dx + \int g(x)dx$

(iv) For any real number k,

$$\int kf(x)dx = k\int f(x)dx$$

7.4 Some standard results of indefinite integrals

We have listed below derivatives of some important functions and the corresponding integrals:

	Derivatives	**Integrals or Antiderivatives**
1.	$\frac{d\left(\frac{x^{n+1}}{n+1}\right)}{dx} = x^n$	$\int x^n dx = \frac{x^{n+1}}{n+1} + c$
2.	$\frac{d(\log x)}{dx} = \frac{1}{x}$	$\int \left(\frac{1}{x}\right) dx = \log x + c$
3.	$\frac{d(e^x)}{dx} = e^x$	$\int e^x.dx = e^x + c$
4.	$\frac{d(x)}{dx} = 1$	$\int 1 dx = x + c$
5.	$\frac{d\left(\frac{a^x}{\log_e a}\right)}{dx} = a^x$	$\int a^x dx = \frac{a^x}{\log_e a} + c$

6. $\frac{d\left(\frac{e^{ax}}{a}\right)}{dx} = e^{ax}$ $\quad\quad$ $\int e^{ax}dx = \frac{e^{ax}}{a} + c$

7. $\frac{d\left(\frac{a^{mx}}{m\log_e a}\right)}{dx} = a^{mx}$ $\quad\quad$ $\int a^{mx}dx = \frac{a^{mx}}{m.\log_e a} + c$

On the basis of the concept of integration and above -mentioned standard results of important indefinite integrals let us work out some practical problems.

Example 1. Write the antiderivative for each of the following functions :

(i) x^4 $\quad\quad$ (ii) x^{r-1}

Solution :

(i) We have to find out a function whose derivative is x^4

Now, $\frac{d(x^5)}{dx} = 5x^{5-1}$

$\Rightarrow \frac{d(x^5)}{dx} = 5x^4$

$\Rightarrow x^4 = \frac{1}{5}\frac{d(x^5)}{dx}$

$\therefore$ An antiderivative of $x^4 = \frac{1}{5}x^5$

(ii) Similarly, $\frac{d(x^r)}{dx} = r.x^{r-1}$

$\Rightarrow x^{r-1} = \frac{1}{r}\frac{d(x^r)}{dx}$

$$\Rightarrow x^{r-1} = \frac{d(\frac{1}{r}.x^r)}{dx}$$

$\therefore$ An antiderivative of $x^{r-1} = \frac{1}{r}.x^r$

Example 2.

Find the integrals of the following functions :

(i) $5x^4$ (ii) $n.x^{n-1}$

Solution :

(i) $\int 5x^4 dx = 5\int x^4 dx$

$$= 5x\frac{x^{4+1}}{4+1} + c \qquad (\because \int x^n dx = \frac{x^{n+1}}{n+1} + c)$$

$$= 5 \times \frac{x^5}{5} + c$$

$$= x^5 + c$$

(ii) $\int nx^{n-1} dx = n\int x^{n-1} dx$

$$= n\frac{x^{n-1+1}}{n-1+1} + c$$

$$= n\frac{x^n}{n} + c$$

$$= x^n + c$$

Example 3. Integrate the following functions w.r.t.x

(i) $\int dx$ (ii) $\int \frac{5}{x^2} dx$ (iii) $\int 3\sqrt{x} dx$ (iv) $\int \frac{e^x}{5} dx$

Solution :

(i) $\int dx = \int 1.dx = \int x^0.dx = \frac{x^{0+1}}{0+1} + c = x + c$

(ii) $\int \frac{5}{x^2}dx = 5\int x^{-2}dx = 5\frac{x^{-2+1}}{-2+1} + c = 5\frac{x^{-1}}{-1} + c = -5\frac{1}{x} + c$

(iii) $\int 3\sqrt{x}dx = 3\int x^{\frac{1}{2}}dx = 3\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c = 3\frac{x^{3/2}}{3/2} + c = 3 \times \frac{2}{3} \times x^{3/2} + c$

$= 2x^{\frac{3}{2}} + c$

(iv) $\int \frac{e^x}{5}dx = \frac{1}{5}\int e^x.dx = \frac{1}{5}e^x + c$

Example 4. Find the integrals of the following functions :

(i) $x^2 + 5x + \frac{1}{x} - 5$ (ii) $(x - \frac{1}{x})^2$

Solution :

$(i)\int (x^2 + 5x + \frac{1}{x} - 5)dx$

$= \int x^2.dx + \int 5x.dx + \int \frac{1}{x}.dx - \int 5.dx$

$= \frac{x^{2+1}}{2+1} + 5\frac{x^{1+1}}{1+1} + \log x - 5\frac{x}{0+1} + c$

$= \frac{x^3}{3} + \frac{5x^2}{2} + \log x - 5x + c$

(ii) $\int (x - \frac{1}{x})^2 dx = \int (x^2 + \frac{1}{x^2} - 2.x.\frac{1}{x})dx$

$$= \int x^2 dx + \int \frac{1}{x^2}.dx - \int 2.dx$$

$$= \frac{x^{2+1}}{2+1} + \frac{x^{-2+1}}{-2+1} - 2.\frac{x^{0+1}}{0+1} + c$$

$$= \frac{x^3}{3} - \frac{1}{x} - 2x + c$$

Example 5. Evaluate the following integrals :

(i) $\int (3e^{3x} + 1)dx$ (ii) $\int x^3(1 - \frac{1}{x^3})dx$

Solution : (i) Let us find a function whose derivative is $3e^{3x} + 1$.

Now, $\frac{d(e^{3x})}{dx} = 3.e^{3x}$ and $\frac{d(x)}{dx} = 1$

$$\therefore \frac{d(e^{3x} + x)}{dx} = 3e^{x} + 1$$

As antiderivative of $3e^{x}+1$ is $e^{3x}+x$,

$$\int (3e^{3x} + 1)dx = e^{3x} + x + C$$

(ii) $\int x^3(1 - \frac{1}{x^3})dx = \int (x^3 - 1)dx$

$$= \int x^3 dx - \int 1.dx$$

$$= \frac{x^{3+1}}{3+1} - \frac{x^{0+1}}{0+1} + c$$

$$= \frac{x^4}{4} - x + c$$

Example 6 . Evaluate :

(i) $\int 5^x dx$ (ii) $\int e^{7x} dx$ (iii) $\int 5^{2x} dx$ (iv) $\int 3e^{2t} dt$

Solution :

(i) $\int 5^x dx = \frac{5^x}{\log 5} + c \quad (\because \int a^x dx = \frac{a^x}{\log_e a} + c)$

(ii) $\int e^{7x} dx = \frac{e^{7x}}{7} + c \quad (\because \int e^{ax} dx = \frac{e^{ax}}{a} + c)$

(iii) $\int 5^{2x} dx = \frac{5^{2x}}{2\log 5} + c \quad (\because \int a^{mx} dx = \frac{a^{mx}}{m\log a} + c)$

(iv) $\int 3e^{2t} dt = 3\int e^{2t} dt \quad = 3\left\{\frac{e^{2t}}{2} + c\right\}$

Example 7 Evaluate :

(i) $\int \left(e^{3x} + 3^{7x} + x^{-3/2} - e^{x/2}\right) dx$

(ii) $\int \frac{ax^4 + bx^3 + cx^2 + dx + e}{x^2} dx$, where a, b, c, d, e are constants

Solution :

(i) $\int (e^{3x} + 3^{7x} + x^{-3/2} - e^{x/2}) dx$

$= \int e^{3x} dx + \int 3^{7x} dx + \int x^{-3/2} dx - \int e^{x/2} dx$

$$=\frac{e^{3x}}{3}+\frac{3^{7x}}{7\log 3}+\frac{x^{-3/2+1}}{-\frac{3}{2}+1}-\frac{e^{x/2}}{\frac{1}{2}}+c$$

$$=\frac{e^{3x}}{3}+\frac{3^{7x}}{7\log 3}+\frac{x^{-\frac{1}{2}}}{-\frac{1}{2}}-2e^{x/2}+c$$

$$=\frac{e^{3x}}{3}+\frac{3^{7x}}{7\log 3}-2\frac{1}{\sqrt{x}}-2e^{x/2}+c$$

(ii) $\int\frac{ax^4+bx^3+cx^2+dx+e}{x^2}dx,$

$$=\int\frac{ax^4}{x^2}dx+\int\frac{bx^3}{x^2}.dx+\int\frac{cx^2}{x^2}+\int\frac{dx}{x^2}.dx+\int\frac{e}{x^2}dx$$

$$=\int ax^2.dx+\int b.x.dx+\int c.dx+\int d\frac{1}{x}dx+e\int x^{-2}dx$$

$$=a\frac{x^{2+1}}{2+1}+b\frac{x^{1+1}}{1+1}+c.\frac{x^{0+1}}{0-1}+d.\log x+e\frac{x^{-2+1}}{-2+1}+c$$

$$=a\frac{x^3}{3}+b\frac{x^2}{2}+c.x+d.\log x+e\frac{x^{-1}}{-1}+c$$

$$=\frac{a}{3}x^3+\frac{b}{2}x^2+c.x+d\ \log x-e\frac{1}{x}+c$$

Solution :

L.H.S $= \int e^{x\log a}.e^{x}dx$

$= \int e^{x\log a + x}dx$

$= \int e^{x(\log a+1)}dx$

$= \int e^{x(\log_e a+\log_e e)}.dx$

$= \int e^{x.\log_e ae}.dx$

$= \frac{e^{x\log ae}}{\log ae} + c$

$= \frac{\log_e (ae)^x}{\log_e ae} + c$

Example 9. If $\frac{d[f(x)]}{dx} = 3x^2 - 2x$ and $f(2) = 0$, then find f(x).

Solution :

Given, $\frac{d[f(x)]}{dx} = 3x^2 - 2x$

$\Rightarrow f(x) = \int (3x^2 - 2x)dx$

$\Rightarrow f(x) = 3\frac{x^{2+1}}{2+1} - 2\frac{x^{1+1}}{1+1} + c$

$\Rightarrow f(x) = x^3 - x^2 + c$

Putting x = 2, we have

$f(2) = 2^3 - 2^2 + c$

$\Rightarrow \quad 0 = 8 - 4 + c \qquad \text{(Given } f(2) = 0)$

$\Rightarrow \quad c = -4$

Putting the value of c in f(x), we have,

$f(x) = x^3 - x^2 - 4.$

Example 10. Evaluate $\int \frac{1}{\sqrt{x+a} - \sqrt{x-b}}\,dx$

Solution :

$$\frac{1}{\sqrt{x+a}-\sqrt{x-b}} = \frac{\sqrt{x+a}+\sqrt{x-b}}{\left(\sqrt{x+a}-\sqrt{x-b}\right)\left(\sqrt{x+a}+\sqrt{x-b}\right)}$$

$$= \frac{\sqrt{x+a}+\sqrt{x-b}}{(x+a)-(x-b)}$$

$$= \frac{\sqrt{x+a}+\sqrt{x-b}}{a+b}$$

$$= \frac{1}{a+b}\left\{(x+a)^{\frac{1}{2}} + (x-b)^{\frac{1}{2}}\right\}$$

$$\therefore \quad \int \frac{1}{\sqrt{x+a}-\sqrt{x-b}}\,dx = \int \frac{1}{a+b}\left\{(x+a)^{\frac{1}{2}} + (x-b)^{\frac{1}{2}}\right\}dx$$

$$= \frac{1}{a+b}\left\{\int (x+a)^{\frac{1}{2}}dx + \int (x-b)^{\frac{1}{2}}dx\right\}$$

$$= \frac{1}{a+b}\left\{\frac{2}{3}(x+a)^{3/2} + \frac{2}{3}(x-b)^{3/2}\right\} + C$$

$$\left\{\because \int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{(n+1)a} + C\right\}$$

7.5 Integration by Substitution

There are three important methods for finding integrals by reducing the functions into standard forms. They are :

1. Integration by substitution
2. Integration by parts
3. Integration by partial fractions.

We shall study the method of integration by substitution in this chapter. This method has no generalised rule to select any substitution. Under this method the integral $\int f(x)dx$ is transformed into another form by changing the independent variable x to t by substituting x=g(t). By this transformation, the integral can be easily evaluated.

Let us take an integral I which is equal to $\int f(x)dx$.

$\therefore I = \int f(x)dx$

Putting, $x = g(t)$ .... (i)

or, $\frac{dx}{dt} = g'(t)$

or, $dx = g'(t)dt$ .... (ii)

Now, $\therefore I = \int f(x)dx = \int f\{g(t)\}g'(t)dt$

Under this substitution, the variable 'x' has changed to 't' which will enable in to evaluate the integral easily. Let us take examples and standard forms to understand the method clearly :

Some important forms of Integrals :

(i) $\int f'(x)dx = \log f(x) + c$, where $f'(x)$ is the derivative of f(x).

(ii) $\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{(n+1)a} + c$, where a, b are constants and $n \neq -1$.

(iii) $\int \frac{1}{x+1}.dx = \log|x+1| + c$

(iv) $\int \frac{1}{ax+b}.dx = \frac{1}{a}\log|ax+b| + c$

Proof :

(i) $$\int \frac{f'(x)}{f(x)} dx = \log\{f(x)\} + C$$

Let $t = f(x)$

$$\Rightarrow \frac{dt}{dx} = f'(x)$$

$$\Rightarrow dt = f'(x).dx$$

Substituting the above in $\int \frac{f'(x)}{f(x)} dx$, we have

$$\int \frac{f'(x)}{f(x)} dx = \int \frac{1}{t}.dt \quad (\text{As } f'(x)dx = dt \text{ and } f(x) = t)$$

$$= \log t + c$$

$$= \log\{f(x)\} + c$$

Using this form of (i), other forms of integrals may be derived.

Example 11. Evaluate $\int (2x+5)^7 dx$

Solution :

Let $2x+5 = t$

$$\Rightarrow \frac{d(2x+5)}{dx} = \frac{dt}{dx}$$

$$\Rightarrow \frac{d(2x)}{dx} + \frac{d(5)}{dx} = \frac{dt}{dx}$$

$$\Rightarrow 2 = \frac{dt}{dx}$$

$$\Rightarrow dx = \frac{1}{2} \cdot dt$$

Now, $\int (2x+5)^7 dx = \int t^7 \cdot \frac{1}{2} dt$

$$= \frac{1}{2} \cdot \frac{t^{7+1}}{7+1} + C$$

$$= \frac{1}{2} \cdot \frac{(2x+5)^{7+1}}{7+1} + C$$

$$= \frac{(2x+5)^8}{8 \times 2} + C$$

$$= \frac{(2x+5)^8}{16} + C.$$

Example : 12 Evaulate $\int \frac{dx}{(5-3x)^4}$

Solution : Let $I = \int \frac{dx}{(5-3x)^4}$ and $5\text{-}3x = t.$

$5 - 3x = t.$

$$\Rightarrow \frac{d(5-3x)}{dx} = \frac{dt}{dx} \quad \text{(Differentiating both sides)}$$

$$\Rightarrow \frac{d(5)}{dx} - \frac{d(3x)}{dx} = \frac{dt}{dx}$$

$$\Rightarrow -\frac{d(3x)}{dx} = \frac{dt}{dx}$$

$$\Rightarrow -3.1 = \frac{dt}{dx}$$

$$\Rightarrow dx = -\frac{1}{3}dt.$$

Now, $$I = \int \frac{dx}{(5-3x)^4}$$

$$= \int \frac{-\frac{1}{3}dt}{t^4}$$

$$= \int -\frac{1}{3}t^{-4} \cdot dt$$

$$= -\frac{1}{3}\int t^{-4} \cdot dt$$

$$= -\frac{1}{3}\frac{t^{-4+1}}{(-4+1)} + C$$

$$= -\frac{1}{3}\frac{t^{-3}}{(-3)} + C$$

$$= \frac{1}{9}\frac{1}{t^3} + C$$

$$= \frac{1}{9(5-3x)^3} + C.$$

Example 13. Integrate $(x^3+9).3x^2$ with respect to x.

Solution :

Let $(x^3+9) = t$

$$\Rightarrow \frac{d(x^3+9)}{dx} = \frac{dt}{dx}$$

$$\Rightarrow \frac{d(x^3)}{dx} + \frac{d(9)}{dx} = \frac{dt}{dx}$$

$$\Rightarrow 3x^2 = \frac{dt}{dx}$$

$$\Rightarrow dx = \frac{1}{3x^2} \cdot dt$$

Now, $\int (x^3 + 9) 3x^2 .dx$

$= \int t.3x^2 \cdot \frac{1}{3x^2} \cdot dt$ (Substituting the values of dx and $x^3 + 9$)

$$= \int t \cdot dt$$

$$= \int \frac{t^{1+1}}{1+1} + C$$

$$= \frac{(x^3+9)^2}{2} + C.$$

Example 14. Find the integrals of the following.

(i) 5^{2x} (ii) e^{5x}

Solution : (i) Let $2x = t$

$$\Rightarrow 2 = \frac{dt}{dx}$$

$$\Rightarrow dx = \frac{1}{2}.dt$$

Now, $\int 5^{2x}.dx = \int 5^t . \frac{1}{2}.dt$

$$= \frac{1}{2} \int 5^t.dt$$

$$=\frac{1}{2}\frac{5^t}{\log_e 5}+c$$

$$=\frac{1}{2}\frac{5^{2x}}{\log_e 5}+c$$

(ii) Let $5x = t$

$$\Rightarrow \frac{d(5x)}{dx}=\frac{dt}{dx}$$

$$\Rightarrow 5=\frac{dt}{dx}$$

$$\Rightarrow dx=\frac{1}{5}dt$$

Now, $\int e^{5x}dx=\int e^t.\frac{1}{5}dt$

$$=\frac{1}{5}\int e^t.dt$$

$$=\frac{1}{5}e^t$$

$$=\frac{1}{5}e^{5x}$$

Example 15. Evaluate : $\int \frac{(5+\log x)^3}{x}dx$

Solution : Let $5+\log x = t$

$$\Rightarrow \frac{d(5+\log x)}{dx}=\frac{dt}{dx}$$

$$\Rightarrow \frac{d(5)}{dx} + \frac{d(\log x)}{dx} = \frac{dt}{dx}$$

$$\Rightarrow \frac{1}{x} = \frac{dt}{dx}$$

$$\Rightarrow dx = x.dt$$

Now $\int \frac{(5+\log x)^3}{x} dx$

$$= \int \frac{(t)^3}{x} \cdot x \, dt$$

$$= \int t^3 \cdot dt$$

$$= \frac{t^{3+1}}{3+1} + c$$

$$= \frac{t^4}{4} + c$$

$$= \frac{(5+\log x)^4}{4} + c$$

Example 16. Evaluate $\int \frac{dx}{x+\sqrt{x}}$

Solution :

$$\int \frac{dx}{x+\sqrt{x}} = \int \frac{dx}{\sqrt{x}\left(\sqrt{x}+1\right)}$$

Let $\sqrt{x} + 1 = t$

$$\Rightarrow \frac{d(\sqrt{x}+1)}{dx} = \frac{dt}{dx}$$

$$\Rightarrow \frac{d(\sqrt{x})}{dx} + \frac{d(1)}{dx} = \frac{dt}{dx}$$

$$\Rightarrow \frac{d(x^{\frac{1}{2}})}{dx} = \frac{dt}{dx}$$

$$\Rightarrow \frac{1}{2}x^{\frac{1}{2}-1} = \frac{dt}{dx}$$

$$\Rightarrow \frac{x^{-\frac{1}{2}}}{2} = \frac{dt}{dx}$$

$$\Rightarrow \frac{1}{2\sqrt{x}} = \frac{dt}{dx}$$

$$\Rightarrow dx = 2\sqrt{x}dt$$

Now $\int \frac{dx}{x+\sqrt{x}} = \int \frac{dx}{\sqrt{x}(\sqrt{x}+1)}$

$$= \int \frac{2\sqrt{x}.dt}{\sqrt{x}.t}$$

$$= \int \frac{2}{t}dt$$

$$= 2\int \frac{1}{t}dt$$

$$= 2\log t + c$$

$$= 2\log(\sqrt{x}+1) + c$$

Example 17. Evaluate $\int(x^3+2x^2+4x-5)^2(3x^2+4x+4)dx$

Solution : Let $x^3+2x^2+4x-5=t$

$$\Rightarrow \frac{d(x^3+2x^2+4x-5)}{dx}=\frac{dt}{dx}$$

$$\Rightarrow 3x^2+4x+4=\frac{dt}{dx}$$

$$\Rightarrow dx=\frac{1}{3x^2+4x+4}dt$$

Now, $\int(x^3+2x^2+4x-5)^2(3x^2+4x+4)dx$

$$=\int t^2(3x^2+4x+4)\frac{1}{3x^2+4x+4}dt$$

$$=\int t^2.dt$$

$$=\frac{t^{2+1}}{2+1}+c$$

$$=\frac{1}{3}(x^3+2x^2+4x-5)^3+c$$

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7.7 QUESTIONS

1. From the given alternatives in each bit, choose and write the correct answer against each bit :

(i) In $\int f(x)dx$, f(x) is

(a) Integral

(b) Integrand

(c) Integration

(d) Constant of integration

(ii) Integration is the opposite process of

(a) Logarithm

(b) Determinant

(c) Differentiation

(d) Matrix multiplication

(iii) Constant of integration appears in

(a) Indefinite Integrals

(b) Definite Integrals

(c) Derivatives

(d) All types of integrals.

(iv) $\int \log x \, dx = ?$

(a) $\frac{1}{x} + c$

(b) $x(\log x + 1)$

(c) $(\log x)^2 + c$

(d) $\frac{1}{2}(\log x)^2 + c$

(v) $\int \log_{10}x \; dx = ?$

(a) $\frac{1}{x} + c$

(b) $\frac{1}{x}\log_e x + c$

(c) $x(\log x - 1)\log_{10} e + c$

(d) $\frac{1}{x}\log_e 10 + c$

(vi) The integral of $\int 2x(x^2+1)dx$ is

(a) $(x^2+1)^2 + c$

(b) $\frac{(x^2+1)^2}{2} + c$

(c) $\frac{x^4}{4} + x^2 + c$

(d) $x^4 + 2x^2 + c$

(vii) $\int 6x^2(1+x^3)dx = ?$

(a) $(1+x^3)^2 + C$

(b) $\frac{(1+x^3)^2}{2} + C$

(c) $2x^3 + 6x^6 + C$

(d) $12x^2 + 30x^4 + C$

(viii) $\int (2x+5)^6 dx = ?$

(a) $\frac{1}{14}(2x+5)^7 + C$

(b) $(2x+5)^7 + C$

(c) $x^2 + 5x + C$

(d) $\frac{1}{28}(2x+5)^7 + C$

(ix) $\int 5e^x \cdot dx = ?$

(a) $5\, e^{2x} + C$

(b) $e^x + C$

(c) $\frac{e^x}{2} + C$

(d) $5\, e^x + C$

(x) $\int (3x^2 + 4x^3 + 5x^4 + 6)dx = ?$

(a) $6x^3 + 12x^4 + 20\, x^5 + C$

(b) $x^5 + x^4 + x^3 + C$

(c) $\frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + C$

(d) Zero.

2. **Express / answer the following in one word/term :**

(i) A technique which is just opposite of the technique of differentiation.

(ii) A value by which two functions with same derivative differ.

(iii) The process by which an integral is found.

(iv) Evaluate $\int dx$.

(v) What is f(x) in $\int f(x)\cdot dx$?

3. Answer each of the following in one sentence :

(i) Show the derivative and antiderivative of x^{n+1} with respect to x.

(ii) Show that : $\int (2x^2 + e^x)dx = \frac{2}{3}x^3 + e^x + C_1 + C_2$

(iii) Find the value of constant of integration if $\frac{df(x)}{dx} = 2x$.

(iv) Integrate $(x+1)^3$ with respect to x.

(v) What do you understand by integration by substitution ?

4. Rectify the underlined portion of the following :

(i) $\int k.d(x) = \underline{k} + C$ (where k is an integer)

(ii) $\int e^{ax}dx = \underline{e}^{ax} + C$

(iii) $\int a^x dx = \underline{a}^x + C$

(iv) The integral of the function e^{2x+3} is $\underline{(e^{2x+3} + C)}$.

(v) $\int 4x(x^2+1)dx = 4(x^2+1)^2 + C$

5. Fill in the blanks :

(i) $\int \frac{1}{x}dx = ...+ C$

(ii) If $\frac{d[f(x)]}{dx} = f'(x)$, then $\int f'(x)dx = ...+ C$.

(iii) $\int \{f(x) + g(x)\}dx = \int f(x)dx + ...$

(iv) $\int K \cdot f(x)dx = Kx$.........................

(v) The process of differentiation and integration are .... to each other.

6. Answer each of the following questions within 30 words :

(i) Evaluate : $\int x^5 dx$

(ii) Evaluate : $\int \left(e^x + \frac{1}{e^x}\right)^2 dx$

(iii) Evaluate : $\int \left(x^2 + 3\right)^2 dx$

(iv) Evaluate : $\int \frac{x^2}{x-1} dx$ (Hints : $x^2 = x^2 - 1 + 1$)

(v) Show that : $\int \frac{dx}{x \cdot \log x} dx = \log|\log x| + C$ (Hints : Take $\log x = t$)

(vi) Find the value of $\int 2x\left(1 + x^2\right)^5 dx$

(vii) If $\int f'(x) = 4x^3 + 3x^2 + 2x + 7$, find f(x).

(viii) Evaluate : $\int 3^x dx$

(ix) Evaluate : $\int 10^{2x} dx$

(x) Find the primitive of $3 . \sqrt{x}$

7. Answer each of the following question within 50 words :

(i) Evaluate : $\int \left(x + \frac{1}{x}\right)^2 .dx$

(ii) Evaluate : $\int \left(x^3 + 2x^2 - 7x + 1\right)dx$

(iii) Evaluate : $\int \frac{x^3}{x-1} dx$

(iv) Evaluate : $\int \frac{1}{\sqrt{x+3}-\sqrt{x-2}} dx$

(v) Find the integral of $\left(2^x - 3^x\right)^2$ with respect to x.

(vi) Find $\int \frac{dx}{2x-1}$

(vii) Evaluate : $\int \frac{x^3}{\left(1+x^2\right)^2} dx$

(viii) Evaluate : $\int x \cdot e^{x^2} \cdot dx$

(ix) Evaluate : $\int \frac{dx}{\sqrt{x}+x}$

(x) Integrate $(x^4 + 3x^2 + 6x - 10)(4x^3 + 6x + 6)$ with respect to x.

8. (a) Evaluate :

(i) $\int \left(2e^{3x} + 1\right) dx$

(ii) $\int \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2 dx$

(iii) $\int \frac{x^3 + 2x^2 - 1}{x^2} dx$

(iv) $\int \frac{x^2 + x + 1}{\sqrt{x}} dx$

(v) $\int \sqrt{x}\left(4x^2 + 3x - 9\right) dx$

(vi) $\int \frac{x^3 - 2x^2 + x - 2}{x-2} dx$

(vii) $\int (ax^2 + bx + c)dx$ (where a, b, and c are constants)

(viii) $\int (5x^4 + 4x^3 + 3x^2 + 2x + 1)dx$

(ix) $\int (2^x + 3^x)^2 dx$

(x) $\int (x-2)(x^2 - 4x + 3)dx$

(xi) $\int (x-2)\sqrt{(x^2 - 4x + 5)}dx$

(xii) $\int x \cdot e^{x^2} dx$

(xiii) $\int \frac{(\log x)^2}{2} dx$

(xiv) $\int x\sqrt{2x^2 - 1}\, dx$

(xv) $\int \frac{x}{9 - 4x^2} dx$

(b) (i) If $\frac{d[f(x)]}{dx} = 4x^3 - 5x^4$ and f(2) = 0, then find f(x).

(ii) If $\frac{d[f(x)]}{dx} = 3x^2 - 2x$ and f(2) = 0, then the value of constant of integration.

(iii) Find a function whose derivative is $(1-x)\sqrt{x}$.

(iv) Compare differentiation with integration.

ANSWERS

1. (i) a (ii) c (iii) a (iv) a (v) c (vi) b (viii) a (viii) a (ix) d (x) c

2. (i) Integration (ii) Constant of integration (iii) Integration
(iv) $x + C$ (v) Integrand.

4. (i) kx (ii) $\frac{e^{ax}}{a}$ (iii) $\frac{a^x}{\log_e a}$ (iv) $\frac{1}{2}e^{2x+3} + C$ (v) $(x^2+1)^2$

5. (i) log x (ii) f(x) (iii) $\int g(x)dx$ (iv) $\int f(x)dx$ (v) Opposite.

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CHAPTER - 8

MEASURE OF CENTRAL TENDENCY-MATHEMATICAL AVERAGES

STRUCTURE

8.1 Introduction

8.2 Meaning and Definition of Central Tendency

8.3 Objective

8.4 Requisites of a good measure of Central Tendency

8.5 Different measures of Central Tendency.

8.6 Mathematical Averages.

- 8.6.1 Arithmetic Mean - its meaning, merits, demerits, uses, properties, computation and practical problems.
- 8.6.2 Geometric Mean - its meaning. computation, mathematical properties, uses, merits. demerits and practical problems.
- 8.6.3 Harmonic Mean - its meaning, computation, mathematical properties uses, merits, demerits, practical problems, relationship between A.M., G.M. and H.M.

8.7 Questions.

8.1. Introduction

Collection, classification and tabulation of data are not the ends in themselves. They are only means to an end. Compact data tables in their face speak very little. To bring to the fore, what they all have to say, the statistician must analyse them vigourously by using different statistical tools and techniques available in his kit. Measures of central tendency, measures of dispersion, skewness, correlation, regression, interpolation etc are some common techniques used for the purpose of analysis by the statisticians. Selection and use of a technique depends upon the purpose of study, nature of data and degree of accuracy needed.

In this chapter, an attempt has been made to introduce the students with measures of central tendency which is a very popular and commonly used statistical technique for the purpose of analysis and interpretation of data.

8.2. Meaning and Definition of Central Tendency

Measures of central tendency refers to a group of statistical methods those are being used to find out the central value or the average value or the indicator of a frequency distribution. An array of figures or a crammed data table usually confuse us. Measures of central tendecy aim at finding a single value from among the mass data that is likely to posses the features of the mass. It helps us to have a better understanding of a situation and does away with confusion. Let us take one example. A boy whose B.Com final results are just declared went to his home and told his mother the marks he has secured in all the 18 subjects. Surely it will be difficult for his mother to remember all the marks and gauge his overall performance immediately. But instead of telling all the marks if the son would have told the average score or percentage of marks, then it would have been more meaningful and convincing to the mother. This avarage score or percentage of marks is called the central value. Different methods used for finding such central values are called measures of central tendency.

The 'single value' that is found out from the mass is called 'central value' because it is most likely to be at the centre of the distribution, around which most values are clustered or distributed.

It is also called the average value because it is neither the smallest nor the largest value, but a value in between.

Defination of 'Avarage' or 'Central Value'

The term avarage or central value has been defined by different authors differently. Some important among them may be presented as follows :

Croxton & Cowden : "An average value is a single value within the range of the data that is used to represent all the values in the series. Since the average is somewhere within the range of the data it is also called a measure of central value"

A.E. Wough : "An average is a single value selected from a group of values to represent them in some way-a value which is supposed to stand for the whole group, of which it is a part, as typical of all the values in the group."

Clark : "Average is an attempt to find one single figure to describe the whole of figures".

Leabo : "The average is sometimes described as a number which is typical of the whole group."

Features :

Thus we can say that a measure of central tendency is the measure of a single typical value which is the best representative of the whole group. From the detinitions, it can be concluded that an average or central value has the following features :

(i) It is a single value expressed in quantitative form as typical value.

(ii) As a typical value it represents a group of values.

(iii) It depicts the charecterstics of the whole group by giving a central idea.

(iv) It lies somewhere in between the smallest and largest value in the distribution;

(v) Most of the values in the distribution are clustered or distributed around it and

(vi) It describes the charateristics of the entire group for the purpose of analysis and comparison either at a point of time or over a period of time.

8.3. Objectives

The important objectives of an 'average' or a measure of central tendency can be stated as follows :

(i) To describe a distribution or series in a precise manner by determining the central value.

(ii) To facilitate comparison between different disiributions by reducing mass data to one single value.

(iii) To help in computation of other statistical measures like dispersion, skewness, kurtosis etc.

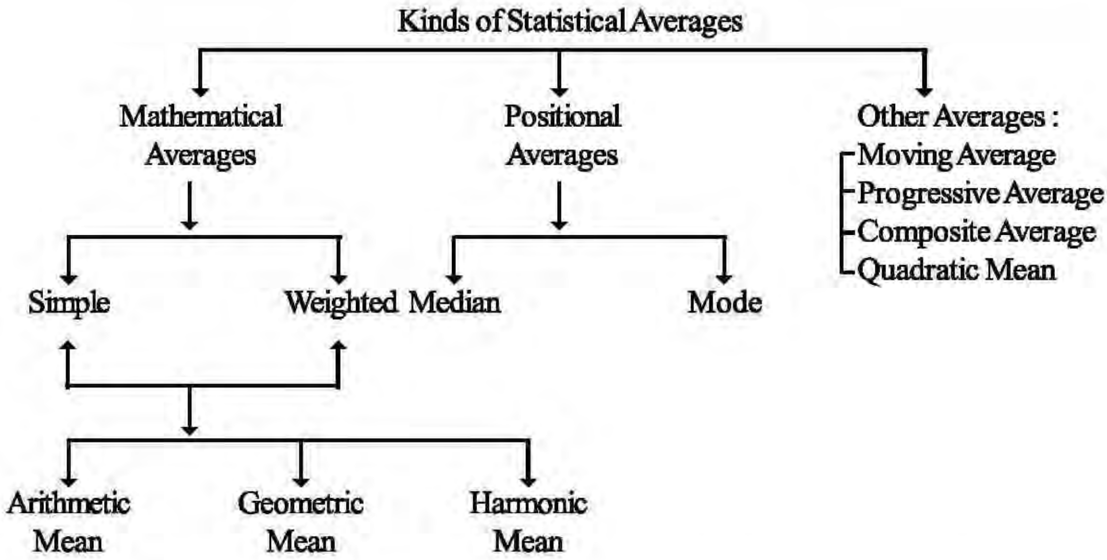
8.4. Requisites of a good measure of cental tendency.

An ideal average must have the following charecterstic features :

- It should be rigidly defined to avoid ambiguity. It must have one and only one interpretation by different persons.
- It should be easy to understand and calculate.
- It should be based an all the observations or the whole data.
- It should be suitable for further mathematical treatment, which means the average should have some important and interesting mathematical properties and thus can be used in further statistical theory and analysis.
- It should be affected as little as possible by fluctuations in sampling or must have sampling stability.
- It should not be affected much by extreme values in the observation.

8.5. Different measures of Central Tendency

The various measures of central tendency or kinds of staistical averages can be presented as follows :



In the following section we shall deal with two kinds of averages i.e mathematical and positional averages in detail. Other averages like moving, progressive, composite etc are not discussed here as they are of very limited use and beyond our syllabus.

8.6 Mathematical Averages

These are averages whose values are mathematically derived. They are subject to further mathematical analysis. Arithmetic Mean, Geometric Mean and Harmonic Mean are popular and very commonly used averages under this category.

8.6.1. ARITHMETIC MEAN : ITS MEANING

Mean or Arithmetic Mean is the most popular measure of central tendency and used very frequently. Arithmetic Mean of a given set of observations is their sum divided by the number of observations. For example the Arithmetic Mean of 10,12,18,20,25, 30 and 32 is equal to

$$\frac{10+12+18+20+25+30+32}{7}=\frac{147}{7}=21$$

If $X_1, X_2 X_3 \ldots X_n$ are given 'n' observations, then their Arithmetic Mean usually denoted by $\overline{X}$ is given by the formula.

$$\overline{X} = \frac{X_1 +_2 + X_3 + \ldots\ldots\ldots\ldots + Xn}{n} = \frac{\sum x}{n}$$

Where $\sum X$ is the sum of the observations and 'n' is the number of Observations .

Merits, Demerits and uses of Arithmetic Mean

Merits :

1. It is easy to calculate.
2. It is very simple to understand.
3. It is rigidly defined and expressed in a mathematical formula.
4. Its calculation is based on all the values in the observation.
5.` It is least affected by sampling fluctuations.
6. It is capable of further algebraic treatment.
7. It is the best measure to compare two or many series.

Demerits :

1. It is not possible to calulate Arithmetic Mean if all the values are not given.
2. Presence of extreme values in the series has greater effect on the Mean.
3. It may not be represented in the actual data and hence, is theoritical. For example : average number of children in the families.
4. It cannot be determined by inspection or be located graphically.
5. In the absence of original observations, Arithmetic Mean may lead to wrong conclusions.
6. It can not be determined in case of qualitative data like honesty, beauty, love, emotion etc.

Uses of Arithmetic Mean :

1. It is the most popular measure of central tendency and therefore, is extensively used in practical statistics. When all the values in the series are considered as equally important and when the distribution does not have very large and very small items, it is the most appropriate average.

2. In every field, government or private, economic or non-economic, estimates are always obtained by using Arithmetic Mean.
3. For the purpose of comparison, for calculating profit per unit, output per man and per machine, average income, average expenditure etc, businessmen always use the Arithmetic Mean.
4. It is used when the distribution does not have open end classes.

   Arithmetic Mean has numerous uses and it is not possible to list them all. When ever some body says about average, usually people mean 'Arithmetic Mean. But it should not be used when ratios and percentages are being studied.

Properties of Arithmetic Mean

Property I : The algebraic sum of deviations from Mean is zero symbolically if the Mean of 'n' observations $X_1, X_2, X_3 ....X_n$ is $\overline{X}$ then

$$(X_1 - \overline{X}) + (X_2 - \overline{X}) + ...(X_n - \overline{X}) = 0$$

Or

$$\sum_{1=1}^{n} (X - \overline{X}) = 0$$

Property II : If each observation is increased by 'a' then the Mean is also increased by 'a'. If the Mean of 'n' observations $X_1, X_2, X_3 ....Xn$ is $\overline{X}$, then the Mean of the observations

$$(X_1 + \alpha), (X_2 + \alpha)..........(X_n + \alpha) \text{ is } \overline{X} + \alpha$$

Property III : If each observation is decreased by 'a' then the new Mean is also decreased by 'a'. If Mean of n observations $X_1, X_2, X_3 ....Xn$ is $\overline{X}$ then the Mean of the observations.

$$(X_1 - \alpha), (X_2 - \alpha), (X_3 - \alpha)..........(X_n - \alpha) \text{ is } \overline{X} - \alpha$$

Property IV : If each observation is multiplied by 'P' $(P \neq O)$, then the new Mean is also multiplied by 'P'

If Mean of 'n' observations $X_1, X_2, X_3.........Xn$ is $\overline{X}$, then the Mean of

$PX_1, PX_2, PX_3.........PX_n$ is $P\overline{X}$

Property V : If each observation is divided by 'P' $(P \neq O)$ then the Mean of new observations is also devided by 'P', i.e new $\overline{X} = \frac{\overline{X}}{P}$

Property VI : The sum of the squares of deviatons of the items from the Mean of a series is the lowest of the sum of the square of deviatons taken from any other actual or assumed value.

Sybolically $\sum(X-\overline{X})^2 < \sum(X-A)^2$

Or $\sum d^2 < \sum dx^2$

Property VII : If Mean and number of items of different series are given the combined Mean can be obtained by $\overline{X_{1.2}} = \frac{n_1\overline{X_1} + n_2\overline{X}2 + ...}{n_1 + n_2}$

Property VIII : The Mean of first 'n' natural numbers can be abtained by the model

$\overline{X} = \frac{n+1}{2}$

Computation of Arithmetic Mean :

Arithmetic Mean may be (i) Simple Arithmetic Mean or (ii) Weghted Arithmetic Mean. In simple Arithmetic Mean, all the items of observations are given equal importance, where as, in weighted Arithmetic Mean different observations are given different weights keeping in view their relative importance.

Simple Arithmetic Mean of un-grouped and grouped data can be found out by using three different methods : They are (i) Direct Method (ii) Short-cut Method and (iii) Step-Deviation Method. Formula used for calulating Mean under different methods may be presented in a table

	Method	Un-grouped Data (Individual Series)	Grouped Data (Discrete / Continuous Series)
1.	Direct Method	$\overline{X} = \frac{\Sigma X}{N}$	$\overline{X} = \frac{\Sigma mf}{N}$
2.	Short-cut Method	$\overline{X} = A + \frac{\Sigma d}{N}$	$\overline{X} = A + \frac{\Sigma fd}{N}$
3.	Step Deviation Method	$\overline{X} = A + \frac{\Sigma d_1}{N} \times C$	$\overline{X} = A + \frac{\Sigma fd_1}{N} \times C$

as follows :

Where $\overline{X}$ = Arithmetic Mean

A = Assumed Mean

N = Total number of observation i.e. Σf

m = Mid value or middle point of each class interval

f = Frequency of the class.

C = Common factor.

d = Deviation from the assumed mean A i.e X- A

d_1 = Deviation devided by the common factor i.e $\frac{X - A}{C}$ or d/c.

Σ = The Greek letter sigma is read as 'Sum of' or 'Summation of'.

Individual Observations :

In a series of individual observations, where frequencies are not given, Arithmetic Mean can be calculated by using different methods as illustrated in the following example :

Illustration I

Calculate Arithmetic Mean from the following data by :

(i) Direct method

(ii) Short-cut method

(iii) Step - deviation method

Roll No. :	1	2	3	4	5	6	7
Marks :	70	60	30	50	40	80	20

Solution

Direct Method :

Marks (×)
70
60
30
50
40
80
20
ΣX 350
N = 7

$$\bar{X} = \frac{\Sigma X}{N} = \frac{350}{7} = 50 \text{ marks}$$

Short-Cut method :

Marks ×	X - A (d)
70	+ 30
60	+ 20
30	– 10
50	+ 10
40	0
80	+ 40
20	– 20
	$\Sigma d = 70$

Assumed mean A = 40, N = 7

$$\bar{X} = A + \frac{\Sigma d}{N}$$

$$= 40 + \frac{70}{7}$$

$$= 40 + 10$$

$$= 50 \text{ marks}$$

Step-Deviation method

Marks	X-A	d/c
X	d	d_1
70	30	3
60	20	2
30	- 10	- 1
50	10	1
40	0	0
80	40	4
20	- 20	- 2
		$\Sigma d_1 = 7$

Here Assumed Mean 'A' = 40 and

Common factor 'C' = 10

$$\overline{X} = A + \frac{\Sigma d_1}{N} X C$$

$$= 40 + \frac{7}{7} \times 10$$

$$= 40 + 10$$

$$= 50 \text{ Marks}$$

Grouped Data

Discrete Frequency Distribution

In a discrete frequency distribution where, along with the value of 'X', their frequencies are given, the Arithmetic Mean can be determined by adopting different methods as follows :

Illustration 2

Calculate Arithmetic Mean of the following data by using

(i) Direct method (ii) Short-cut method and

(iii) Step-deviation method.

Monthly (₹) Income	10,000	12,000	15,000	18,000	20,000	22,000
Number of People	5	10	15	7	8	5

Solution

Direct Method

Income X	Number of People f	fx
10,000	5	50,00
12,000	10	1,20,000
15,000	15	2,25,000
18,000	7	1,26,000
20,000	8	1,60,000
22,000	5	1,10,000
	Σ f = 50	Σ fx 91,000

$$\overline{X} = \frac{\sum fx}{N} \text{ or } \frac{\sum fx}{\sum f}$$

$$= \frac{7,91,000}{50} = 15,820$$

Short-cut method

Income (X)	Number of People (f)	Deviation (d) (X-A)	fd
10,000	5	- 5000	- 25000
12,000	10	- 3000	- 30000
15,000	15	0	0

18,000	7	3000	21000
20,000	8	5000	40000
22,000	5	7000	35000
	Σ f = 50		Σ fd = 41,000

Let the Assumed mean A = 15,000

$$\overline{X} = A + \frac{\Sigma fd}{N}$$

$$= 15000 + \frac{41.000}{50}$$

$$= 15000 + 820$$

$$= ₹\ 15,820$$

Step Deviation method

Income (₹)	Number of People	Deviation (d)	d/c	fd_1
X	f	X-A	d_1	
10,000	5	- 5000	-5	- 25
12000	10	-3000	-3	-30
15000	15	0	0	0
18,000	7	3000	3	21
20,000	8	5000	5	40
22000	5	7,000	7	35
	Σ f = 50			$\Sigma fd_1 = 41$

$$\overline{X} = A + \frac{\Sigma fd_1}{\Sigma f} \times C$$

Where A = 15,000 C = 1000

$$\overline{X} = 15,000 + \frac{41}{50} \times 1000$$

$= 15,000 + 820$

$= 15,820$

Continuous frequency Distribution

When we calculate the central tendency of a continuous series we assume that the frequencies of the class interval are concentrated at the centre of the class - interval.

Thus, where class intervals are given along with frequencies, Arithmetic Mean can be calculated by using all the three methods as illustrated below.

Before that, the continuous series is first converted into a discrete series by assigning mid-points or mid-values in place of each-individual class interval.

Illustration 3

From the following data calculate average income under all the methods

Income in thousand (₹)	0-10	10-20	20-30	30-40	40- 50	50-60
Number of Persons	6	14	16	27	22	15

Solution

Direct Method

Income (x)	Number of Person (f)	Mid Points (m)	(fm)
0 - 10	6	5	30
10 - 20	14	15	210
20 - 30	16	25	400
30 - 40	27	35	945
40 - 50	22	45	990
50 - 60	15	55	825
	$\Sigma f = 100$		$\Sigma fm = 3400$

$$\bar{X} = \frac{\Sigma fm}{\Sigma f}$$

$$\frac{3400}{100} = 34$$

∴ The mean income is ₹ 34,000

Short-cut Method

Income in 000 ₹ (x)	Number of Person (f)	Mid Points (m)	Deviation from Assumed mean (d)	(f d)
0 - 10	6	5	- 30	- 180
10 - 20	14	15	- 20	- 280
20 - 30	16	25	- 10	- 160
30 - 40	27	35	0	0
40 - 50	22	45	10	220
50 - 60	15	55	20	300
	Σ f = 100			Σ fd = -100

$$X = A + \frac{\Sigma fd}{\Sigma f} = 35 + \frac{-100}{100}$$

$= 35 - 1$

$= 34$

∴ The Mean income is ₹ 34,000

Step-Diviation method

Income in 000 Rs	Number of Person (f)	Mid Value (m)	Deviation from Assumed mean (d) (m - A)	d/c d_1	fd_1
0 - 10	6	5	- 30	- 3	- 18
10 - 20	14	15	- 20	- 2	- 28
20 - 30	16	25	- 10	- 1	- 16

30 - 40	27	35	0	0	0
40 - 50	22	45	10	1	22
50 - 60	15	55	20	2	30
	$\Sigma f = 100$				$\Sigma fd_1 = -10$

$$\overline{X} = A + \frac{\Sigma fd_1}{\Sigma f} \times C$$

$$= 35 + \frac{10}{100} \times 10$$

$$= 35 - 1 = 34$$

While calculating Arithmetic Mean in case of grouped data, if the seris is found to be an inclusive series, then it must be converted to an exclusive one. This can be illustrated as follows:

Illustration 4.

From the following data calculate Arithmetic Mean by using step-deviation method :

X	10-19	20-29	30 - 39	40 - 49	50 - 59	60 - 69	70 - 79	80 -89
f	6	8	7	10	17	38	9	3

Solution

As the data is presented in the form of an inclusive series, we have to transform it into an exclusive series before computation of Arithmetic Mean.

The difference between the lowest limit of the second class interval and the upper limit of the first class interval, is 1 (20-19) We will subtract 1/2 (half) of this difference i.e 0.5 from the lower limit and add 0.5 to the upper limit of each class interval. Then the new classes will be formed as follows :

X	f	Mid Value	m-A	d/c	fd_1
9.5 - 19.5	6	14.5	- 30	- 3	- 18
19.5 - 29.5	8	24.5	- 20	- 2	- 16
29.5 - 39.5	7	34.5	- 10	- 1	- 7
39.5 - 49.5	10	44.5	0	0	0
49.5 - 59.5	17	34.5	+10	1	17

59.5 - 69.5	38	64.5	+20	2	76
69.5 - 79.5	9	74.5	+30	3	27
79.5 - 89.5	3	84.5	+40	4	12
	$\Sigma f = 100$				$\Sigma fd_1 = -10$

$$\bar{X} = A + \frac{\Sigma fd_1}{\Sigma f} \times C$$

$$= 44.5 + \frac{91}{98} \times 10$$

$$= 44.5 + \frac{910}{98}$$

$$= 44.5 + 9.29$$

$$= 53.79$$

Missing Frequency / Incorrect values :

Sometimes due to oversight, a wrong value might have been included for calculating Mean or a question may be asked to find out a missing frequency. In such cases the problem can be solved simply by using the mathematical propoties of Mean.

Illustration 5

The average weekly wage of a group of 30 workers in a factory was calculated to be 350. But later it was discoverd that one figure was misread as 450 instead of 480. Calculate the correct average.

Solution :

$\sum X = N\bar{X} = 30 \times 350$ = 10,500 (Incorrect Total)

Less the in-correct figure 450

10,050 (Correct Total)

Add the correct figure 480

10,530

$\therefore$ Correct $\Sigma x = 10,530$

Correct $\overline{X} = \frac{10,530}{30} = 351$

Illustration 6

Find the value of P for the following distribution whose Mean is 23.33 :

X	10	15	P	25	30
f	2	4	6	8	10

Solution :

X	f	fx
10	2	20
15	4	60
P	6	6P
25	8	200
30	10	300
	$\sum f=30$	$\sum fx=580+6P$

$$\overline{X} = \frac{\sum fx}{N}$$

$$= \frac{580+6P}{30}$$

$$\therefore \frac{580+6P}{30} = 23.33$$

$$6P = (23.33 \times 30) - 580$$
$$= 700 - 580$$
$$= 120$$

$$P = \frac{120}{6}$$
$$= 20.$$

Weighted Arithmetic Mean

In calculation of simple Arithmetic Mean, all values in the series are given equal importance or equal weights. But sometimes it is necessary to assign different weights to different values in the series, for the purpose of better representation. Hence, the necessity of calculating weighted Arithmetic Mean arises.

If W_1, W_2, W_3 .... Wn are the weights assigned to value X_1, X_2, X_3 .... X_n respectively then the weighted Arithmetic Mean is difined as :

$$\text{Weighted Arithmetic Mean } \frac{W_1X_1 + W_2X_2 + W_3X_3 \ldots\ldots\ldots + W_nX_n}{W_1 + W_2 + W_3 + W_n}$$

$$= \frac{\Sigma WX}{\Sigma W}$$

Illustration 7

A candidate obtained following percentage of marks in an examination.

English 70 Statistics 60, Mathematrics 80, Accounting 75 and Economics 58. Find weighted Arithmetic Mean if weights 2,1, 3, 3 & 1 are alloted to the subjects respectively.

Solution :

Subjects	Marks (x)	Weights (w)	Wx
English	70	2	140
Statisties	60	1	60
Mathematic	80	3	240
Accounting	75	3	225
Economics	58	1	58
		$\Sigma W = 10$	$\Sigma WX = 723$

Weighted A.M

$$= \frac{\Sigma WX}{\Sigma W}$$

$$= \frac{723}{10}$$

$$= 72.3$$

Where weighted Arithmetic Mean must be used : (i) if it is disired to find the averages of ratios percentages or rates.(ii) When it is required to calculate the average of series from the average of its component parts i.e it is useful to compute the Mean of the Means (iii) while computing the standard birth and death rates and (iv) in the construction of index numbers, weighted arithmetic average should be used.

Combined Arithmetic Mean

If the Means of different samples from a universe or different components of a group along with their sizes are known, then their combined Mean can be found out by using the following formula :

$$\text{Combined Mean} = \frac{N_1\overline{X_1} + N_2\overline{X_2}......+.....N_n\overline{X_n}}{N_1 + N_2 + ...+ N_n}$$

Where $N_1, N_2 .... N_n$ are size of different samples or components and $\overline{X}_1, \overline{X}_2 ...... \overline{X}_n$ and are their respective Means.

Illustration 8

The average score of girls in class X examination of of a school is 67 and that of boys is 63. The average score of the whole class is 64.5. Find the percentage of boys and girls in the class.

Solution :

Given - Mean score of girls $\overline{X_1} = 67$

,, of boys $\overline{X_2} = 63$

$\overline{X}_n$

combined Mean $\overline{X} = 64.5$

Let the number of Girls in the class be N_1

and number of Boys in the class be N_2

Then $\overline{\overline{X}} = \frac{N_1\overline{X_1} + N_2\overline{X_2}}{N_1 + N_2}$

$$64.5 = \frac{67N_1 + 63N_2}{N_1 + N_2}$$

$64.5\ N_1 + 64.5\ N_2 = 67N_1 + 63N_2$

$2.5N_1 = 1.5\ N_2$

$25N_1 = 15\ N_2 : \ ^{N_1}\!/_{N_2} = {}^{15}\!/_{25}$

Hence the ratio of girls to boys in the class is :3:5 then % of girls = $\frac{100}{8} \times 3 = 37.5\%$ and

Boys = $\frac{100}{8} \times 5 = 62.5\%$.

Practical Problems :

Illustration 9

The average score of boys in a competitive examination is 70 and that of the girls is 75. The average score of the whole number of chandidates appearing the exam is 72. Find the percentage of boys and girls appearing the exmination.

Solution

Let N_1 and N_2 respectively be the number of boys and girls.

$\overline{X_1}$ - Average score of boys = 70

$\overline{X_2}$ - Average score of girls = 75

$\overline{X}$ - Average score of all condidates 72

$$\overline{\overline{X}} = \frac{N_1\overline{x_1} + N_2\overline{x_2}}{N_1 + N_2} = \frac{70N_1 + 75N_2}{N_1 + N_2} = 72$$

$= 70N_1 + 75N_2 = 72N_1 + 72N_2$

$\Rightarrow -2N_1 = -3N_2$

$\Rightarrow 2N_1 = 3N_2$

$\therefore \frac{N_1}{N_2} = \frac{3}{2}$

The ratio of boys to girls is 3:2.

Percentage of boys = $\frac{3}{5} \times 100 = 60\%$

Percentage of girls $= \frac{2}{5} \times 100 = 40\%$

Illustration 10

Find the missing frequency (p) for the following distribution whose Mean is 7.68 :

X	3	5	7	9	11	13
f	6	8	15	P	8	4

Solution

X	f	fx
3	6	18
5	8	40
7	15	105
9	P	9P
11	8	88
13	4	52
	$\Sigma f = 41 + P$	$\Sigma fx = 303 + 9P$

$$\bar{X} = \frac{\Sigma fx}{\Sigma f} = 7.68$$

$$= \frac{303 + 9P}{41 + P} = 7.68$$

$$P = \frac{11.88}{1.32}$$

$$P = 9$$

Illustration 11

Find the missing frequencies (P_1, P_2) for the following distribution whose Mean is 2.9 and N = 10 :

X	1	2	3	4	5
f	2	1	P_1	P_2	1

Solution

X	f	fx
1	2	2
2	1	2
3	P_1	$3P_1$
4	P_2	$4P_2$
5	1	5

$\Sigma f = 4+P_1+P_2$, $\Sigma fx = 9+3P_1+4P_2$

$\Sigma f = N = 10$

$\therefore\ 4 + P_1 + P_2 = 10$

$P_1 + P_2 = 10\text{-}4 = 6$ (i)

$$\overline{X} = \frac{\Sigma \int X}{N} = \frac{9+3P_1+4P_2}{10}$$

$$\frac{9+3P_1+4P_2}{10} = 2.9$$

$\therefore\ 3P_1 + 4P_2 = 29–9 = 20$

Now $P_1+P_2 = 6$ (i)

$3P_1+4P_2=20$ (ii)

Multiplying the 1st equatation by 3 and subtracting the 2nd from it.

$$\begin{array}{r} 3P_1 + 3P_2 = 18 \\ -\ 3P_1 + 4P_2 = 20 \\ \hline -\ P_2 = -\ 2 \end{array}$$

or $P_2 = 2$ $\therefore$ $P_1 = 6–2 = 4$

$\therefore$ The missing frequencies P_1 and P_2 are 4 and 2 respectively.

Illustration 12

From the following data find out the Arithmetic Mean.

X (Less than)	5	10	15	20	25	30
Frequency	5	15	30	38	45	50

Solution

For finding out the Mean the open ended classification be converted to close ended classofication first

X	f	mid value (m)	A=12.5 m–A	d_1 d/c	fd_1
0-5	5	2.5	-10	-2	-10
5-10	10	7.5	- 5	- 1	- 10
10-15	15	12.5	0	0	0
15-20	8	17.5	5	1	8
20-25	7	22.5	10	2	14
25-30	5	27-5	15	3	15
	$\Sigma f = 50$				$\Sigma f\ d_1 = 17$

$$\overline{X} = A + \frac{\Sigma fd_1}{\Sigma f} \times C$$

Where A = 12.5, N or $\Sigma f = 50$, C = 5

Putting the values $\overline{X} = 12.5 + \frac{17}{50} \times 5$

$= 12.5 + 1.7 = 14.2.$

8.6.2 GEOMETRIC MEAN : ITS MEANING :

Like Arithmetic Mean (A.M), Geometric Mean (GM) is a mathematical measure of central tendency or average. It is defined as the Nth root of the product of N items. Thus, Geometric Mean of a series of items X_1, X_2, X_3 .... Xn is Nth root of the product of all X values.

Symbolically $G.M = \sqrt[n]{X_1 X_2 X_3 \ldots\ldots\ldots X_n}$

For simplifying calculations of G.M logorithm is used as follows :

$$GM = (X_1.X_2.X_3\ldots\ldots\ldots X_n)$$

$$\log G.M = \frac{1}{N}(\log X_1 + \log X_2 + \log X_3 + \ldots\ldots \log X_n)$$

$$\log GM = \frac{\Sigma \log X}{N}$$

$$GM = \text{Antilog}\left[\frac{\Sigma \log X}{N}\right]$$

Computation of Geometric Mean :

Individual Observations :

Formula

Without use of logarithm

$$GM = \sqrt[n]{X_1.X_2.X_3\ldots\ldots\ldots X_n}$$

This formula is suitable when there are only 2 or 3 values of X.

By using logarithm formula :

$$GM = \text{Antilog } \frac{\Sigma \log X}{N}$$

Steps for Calculation.

- Find the logarithm of the given values i.e log X
- Find the sum total of logs i.e Σ log X.
- Divide Σ log X. by the numbers of

  items (N) i.e $\frac{\Sigma \log X}{N}$
- Find the antilog of $\frac{\Sigma \log X}{N}$

Illustration 13

Calculate the Geometric Mean of

10 15 20 25 30

Solution :

X	log of X
10	1.000
15	1.1761
20	1.3010
25	1.3979
30	1.4771
	$\Sigma \log X = 6.3521$

$$G.M = Al \frac{\sum \log X}{N}$$

$\Sigma \log X = 6.3521$

$N = 5$

$$\therefore G.M = \text{Antilog } \frac{6.3521}{5}$$

$= \text{Antilog } 1.27042$

$= 18.64$

Discrete Series :

The formula for calculating Geometric Mean in case of discrete series is

$$GM = \text{Antilog} \left[\frac{\sum f \log X}{N} \right]$$

<u>Steps for Calculation</u>

- Find the logarithm of the given values i.e log X
- Multiply the frequency of each item with the log X values and obtain f log X.
- Find the total of the products i.e $\Sigma f \log X$
- Find the antilog of $\frac{\sum f \log X}{N}$

Illustration : 14

Calculate Geometric Mean from the following data

Marks	10	15	20	25	30
No. of Students	2	4	6	8	10

Solution

X	f	log X	f log X
10	2	1.0000	2.0000
15	4	1.1761	4.7044
20	6	1.3010	7.8060
25	8	1.3979	11.1832
30	16	1.4771	14.7710
	$\Sigma f = 30$		$\Sigma f\log x = 40.4646$

$$G.M = \text{Antilog}\frac{\Sigma f\log X}{N}$$

$$= \text{Antilog}\frac{40.4646}{30}$$

$= \text{Antilog } 1.3488$

$= 22.32$

Continuous Series :

The formula used for calculating Geometrie Mean in continuous series is

$$G.M = \text{Antilog}\left[\frac{\Sigma f \log m}{N}\right]$$

Where m stands for mid values of clases and 'N' for total frequency.

Step for Calculation

- Find the mid-values of different classes i.e., 'm'.
- Find the Logarithms of mid values i.e., 'log m'.
- Multiply the respective frequencies wIth log m values and find out '*f* Log m'.
- Find the sum of f log m values i.e., 'Σ f log m'.

- Divide Σ f log m by total number of items i.e., 'N' or 'Σ f and find $\frac{\sum \int \log m}{N}$
- Find antilog of $\frac{\sum \int \log m}{N}$

Illustration : 15

Calculate Geometric Mean from the following data :

Marks	0-10	10 - 20	20 - 30	30 - 40	40 - 50
No. of Students	5	10	12	8	5

Solution :

X	m.v	*f*	log m	*f* log m
0 - 10	5	5	0.6990	3.4950
10 - 20	15	10	1.1761	11.7610
20 - 30	25	12	1.3979	167748
30 - 40	35	8	1.5441	12 3528
40 - 50	45	5	1.6532	8.2660
		$\Sigma f = 40$		ΣfLog m=52.6496

$$G.M = \text{Antilog} \left[\frac{\sum \int \log m}{N}\right]$$

$$= \text{Antilog} \ \frac{52.6496}{40}$$

$$= \text{Antilog} \ 1.3162$$

$$= 20.71$$

Weighted Geometric Mean

When different values X_1, X_2 .... X_n of the variable are not of equal importance and are assigned different weights such as W_1, W_2 ...... Wn respectively according to their importance, then weighted Geometric Mean is calculated. The formula for calculation of Weghted Geometric Mean is

$$G.M (W) = \text{Antilog} \left[\frac{1}{\Sigma W} \Sigma W\log X\right]$$

$$= \text{Antilog} \left[\frac{\Sigma W\log X}{\Sigma W}\right]$$

Where G.M (W) - Weighted Geometric Mean

W = Weights assigned to Values of Variable

X = Different Values of the variable

N/Σw =Total number of observations.

Illustration 16

From the following data calculate the Geometric Mean :

X (Variable)	8	25	19	28
W (Weights)	3	5	7	15

Solution

X	W	log X	W log X
8	3	0.9031	2.7093
25	5	1.3970	6.9850
19	7	1.2788	8.9516
28	15	1.4472	21.7080
	Σ w = 30		Σ w logX=40.3539

$$G.M(w) = \text{Antilog}\left[\frac{\sum W\log X}{\sum W}\right]$$

$$= \text{Antilog}\ \frac{40.3539}{30}$$

$$= \text{Antilog}\ 1.3451$$

$$= 22.14$$

Combined Geometric Mean :

Like combined Arithmetic Mean, combined Geometric Mean of two or more serieses can be found out by using the mathematical properties of G.M. The relevant formula for calculating Combined Geometric Mean is :

$$G.M_{1.2} = \text{Antilog}\left[\frac{N_1\log G_1 + N_2\log G_2 .... + N_n\log G_n}{N_1 + N_2 .... + N_n}\right]$$

Where $G_1, G_2 ..... G_n$: Geometric mean of different series

$N_1, N_2 ...... N_n$ Number of observations in each series.

Illustration 17

Find the combined Geometric Mean of the following 2 series from the given information :

Series	G.M.	No. of Observations
I	5	3
II	15	5

Solution :

$$G.M_{1.2} = \text{Antilog}\left[\frac{N_1\log G_1 + N_2\log G_2}{N_1 + N_2}\right]$$

Substituting the respective values in the above formula

$$G.M_{1.2} = \text{Antilog}\left[\frac{3\log 5 + 5\log 15}{3+5}\right]$$

$$= \text{Antilog}\left[\frac{3\times 0.6990 + 5\times 1.1761}{8}\right]$$

$$= \text{Antilog}\left[\frac{2.0970+5.8805}{8}\right]$$

$$= \text{Antilog}\left[\frac{7.9775}{8}\right]$$

$$= \text{Antilog } 0.9972$$

$$= 9.9357$$

Thus the G.M of the two series taken together is 9.9357

Mathematical Properties of G.M

1. If any value of a series is zero the value of Geometric Mean becomes infinity, hence impracticable.
2. It cannot be calculated if there are odd numbers of negative values in the series, as it is not possible to find the root of a negative value.
3. Since G.M is the n th root of the products of the observations, n th powers of G.M gives the product of the observations.

   $G.M = \sqrt[n]{X_1.X_2.X_3.......X_n}$

   $\therefore (G.M)^n = X, X_2 X_3 ..... X_n$
4. Any series of numbers having same N and same products will have the some Geometric Mean.

Illustration - 18

	X_1	X_2	X_3	X_4
	4	16	32	64
	4	2	2	2
	4	8	4	2
	4	1	1	1
Product	256	256	256	256

$GM_1 = \sqrt[4]{4\times4\times4\times4} = 4$

$GM_2 = \sqrt[4]{32\times2\times4\times1} = 4$

$GM_3 = \sqrt[4]{32\times2\times4\times1} = 4$

$GM_4 = \sqrt[4]{64\times2\times2\times1} = 4$

5. For any series of positive values the Geometric Mean is smaller than the Arithmetic Mean and higher than the Harmonic Mean. Symbolically

   H.M > G.M < A.M

6. The sum of deviations of the logarithms of the original values above and below the logarithm of the G.M. are equal.

Illustration - 19

X	log X	Deviation From Log of G.M
5	0.6990	– 0.3010
10	1.0000	0
20	1.3010	+ 0.3010

$$\text{GM} = \sqrt[3]{5\times10\times20}$$

$$= \sqrt[3]{1000}$$

$$= 10$$

Thus deviation of logarithm obove Geometric Mean (0.3010) is equal to below the Geometric Mean i.e. .3010.

Uses of Geometric Mean

Specific fields in which use of Geometric Mean is more popular and appropriate are mentioned below.

(i) In computing average of ratios and percentages, and in determining rates of increase or decrease.

(ii) In construction of index number

(iii) In social and economic situations where more weights are given to smaller items.

(iv) If two series have the same Mean, Geometric Mean can be considered for their comparison

Merits and Demerits of Geometric Mean

The important merits of G.M are :

1. It is the only average which can be used for averaging percentage of growth / declime over a period of time.

2. It is simple to calculate and is capable of further mathematical treatment.
3. It is useful in construction of index numbers.
4. It is based on all the values of observations.
5. It is little affected by sampling fluetuations.
6. It gives less weightage to large items and more weightage to small items.

Demerits of Geometric Mean.

The main demerits of Geometric Mean are :

1. It is difficult to understand and calculate.
2. It cannot be calculated when a value is either zero or negative
3. It cannot be found out by inspection.
4. If some of values in a series are not given, it cannot be calculated.

Practical Problems on G.M

Illustration 20

In 3 decades the population of New Delhi increased by 10%, 20% and 30% respectively find the average population growth per decade.

Solution :

The appropriate measure of central tendency here is G.M. not A.M.

Decade	Percent Growth	% population on prev decade	log X
1st	10	110	2.0414
2nd	20	120	2.0792
3rd	30	130	2.1139
			$\Sigma \log X = 6.2345$

$$GM = \text{Antilog} \frac{\sum \log X}{N}$$

$$\text{Putting the values, } GM = \text{Antilog} \frac{6.2345}{3}$$

$$= \text{Antilog } 2.0782$$

$$= 119.8$$

The average rate of increase in population is

$$119.8 - 100 = 19.8\%$$

Illustration - 21

A machine was purchased for 1 lakh rupees in 2010. Depreciation on the deminishig balance was charged @30% in the 1st year 25% in the second year, 20% in 3rd year and 10% in the next two years. What is the average rate of depreciation during the whole period ?

Solution

Year	Deminishing Value taking 100 as base	log X
2010	100-30 = 70	1.8451
2011	100 - 25 = 75	1.8751
2012	100 - 20 = 80	1.9031
2013	100 - 10 = 90	1.9542
2014	100 - 10 = 90	1.9542
		Σ Log X = 9.5317

$$G.M = \text{Antilog}\ \frac{\sum \log X}{N}$$

Substituting the values

$$G.M = \frac{9.5317}{5}$$

= Antilog 1.9063

= 80.60

Since the diminishing value is 80.60, the depreciation will be

100-80.60 = 19.40%

$\therefore$ The average rate of depreciation charged during the whole period is 19.4%.

Illustration 22

The Geometric Mean of 6 observations was calculated as 8.16. It was later discovered that one of the observations was wrongly recorded as 12, in fact it was 22. Calculate the correct G.M.

Solution

$$G.M = (X_1, X_2, X_3 \ldots\ldots X_n)^{\frac{1}{n}}$$

$\therefore (G.M)^n$ = Product of the 'x' values

Thus, the product of the numbers = $(8.16)^6$

The corrected product will be $\frac{(8.16)^6 \times 22}{12}$

Hence the corrected value of GM will be $\left(\frac{(8.16)^6 \times 22}{12}\right)^{1/6}$

log GM = 1/6 {log $(8.16)^6$ + log22 - log12}

log GM = 1/6 {6 log 8.16 + log 22 - log 12}

= 1/6 {6 × 0.9117 + 1.3424 - 1.0792}

= 1/6 {5.4702 + 1.3424 - 1.0792}

= 1/6 {6.8126 - 1.0792}

= 1/6 {5.7334}

= 0.9556

GM = Antilog 0.9556

= 9.03

Illustration 23

The weighted GM of 5 numbers : 150, 260, 350, 250 and 175 is 208. If the weights assigned to 1st 4 numbers are 10,6,4 and 2 respectively, find out the weight of the 5th number.

Solution

Let the weight of the 5th number be 'W' given GM is 208

Hence $GM_{(w)} = \text{Antilog } \frac{\sum w\log x}{\sum w}$

$$\log GM = \frac{\sum w\log X}{\sum w}$$

$$\log 208 = \frac{\sum w\log X}{\sum w}$$

X	W	log X	Wlog X
150	10	2.1761	21.7610
260	6	2.4150	14.4990
350	4	2.5441	10.1764
250	2	2.3979	4.7958
175	W	2.2430	2.2430 W

$\Sigma W = 22 + W$

$\Sigma W \log X = 51.2322 + 2.2430 W$

Hence $\log 208 = \dfrac{51.2322 + 2.2430w}{22 + W}$

$$2.3181 = \frac{51.2322 + 2.2430W}{22 + W}$$

$$50.9982 + 2.3181W = 51.2322 + 2.2430 W$$

$$2.3181W - 2.2430 W = 51.2322 - 50.9982$$

$$0.0751 W = 0.234$$

$$W = \frac{0.234}{0.0751}$$

= 3.1 or 3 approximately.

$\therefore$ The weight assigned to the 5th number is 3.

Illustration - 24

The Arithmetic Mean of two numbers is 15 and their Geometric Mean is 14.142 find the numbers.

Solution

Let the two numbers be 'a' and 'b'

Then $A.M = \dfrac{a+b}{2} = 15$

,, $a + b = 30$

$$G.M = \sqrt[2]{ab} = 14.142$$

$$ab = (14.142)^2 = 199.9396 = 200$$

$$(a + b)^2 - (a - b)^2 = 4ab$$

$$(30)^2 - (a - b)^2 = 4 \times 200$$

$$900 - 800 = (a - b)^2$$

$$100 = (a - b)^2$$

$$a - b = 10$$

$$\therefore a = \frac{30 + 10}{2} = 20 \quad \therefore b = 10$$

8.6.3 HARMONIC MEAN ITS MEANING

Harmonic Mean is another mathematical measure of central tendency or average. It is defined as the reciprocal of the Arithmetic Mean of the reciprocal of the values of the variables. Thus symbolically Harmonic Mean is

$$H.M = \text{Reciprocal } \frac{\Sigma \frac{1}{x}}{N} \text{ or } \frac{N}{\Sigma \frac{1}{x}}$$

The term reciprocal with reference to a number means one divided by that number If p.q and r are numbers then 1/p, 1/q and 1/r are their reciprocal respectively . Thus it is the inverse of the Mean of the receprocals of the given values.

Computation of Harmonic Mean

Individual Series

In individual series or un-grouped data the following formula is used to calculate Harmonic Mean :

$$H.M = \frac{N}{\Sigma \frac{1}{X}}$$

Where N = Number of observations

X = Values of the variable

Steps for Calculating H.M

- Find the reciprocal of values i.e 1/x
- Find the total of the reciprocals i.e $\Sigma \frac{1}{x}$
- Find the A.M of the reciprocals i.e $\frac{\Sigma \frac{1}{x}}{N}$
- Find the reciprocal of the A.M. of reciprocals i.e $\frac{N}{\Sigma \frac{1}{x}}$

Note : For receiprocal you may refer to 'Reciprocal Table'.

Illustraction 25

Find the Harmonic Mean of the following values

2 5 10 20 25 50

Solution

X	2	5	10	20	25	50

Reciprocal

1/x	1/2	1/5	1/10	1/20	1/25	1/50

$$\Sigma \frac{1}{X} = \frac{1}{2} + \frac{1}{5} + \frac{1}{10} + \frac{1}{20} + \frac{1}{25} + \frac{1}{50}$$

$$= \frac{50 + 20 + 10 + 5 + 4 + 2}{100}$$

$$= \frac{91}{100}$$

N = 6

$$\text{H.M} = \frac{6}{\frac{91}{100}} = \frac{600}{91} = 6.59$$

GROUPED DATA

Discrete Series :

In case of discrete series where frequencies are given Harmonic Mean is calculated by using the following formula $HM = \frac{N}{\Sigma f \frac{1}{X}}$ or $\frac{\Sigma f}{\Sigma f \frac{1}{X}}$

Where f stands for frequencies.

N (= Σf) stands for number of observations and X stands for values of the variable

Steps for calculating H.M.

- Find the reciprocal of all 'X' Values i.e 1/x
- Multiply the reciprocals with respective frequencies i.e $f \frac{1}{x}$
- Find the sum of the products i.e. $\Sigma f \frac{1}{x}$
- Divide 'N' by $\Sigma f \frac{1}{x}$

Illustration 26

Calculate Harmonic Mean from the following data :

X	1	2	3	4	5	6	7
f	5	4	6	2	5	12	14

Solution

X	f	1/x	f(1/x)
1	5	1/1	5
2	4	1/2	2
3	6	1/3	2
4	12	1/4	3
5	15	1/5	3
6	12	1/6	2
7	14	1/7	2
	Σf = 68		$\Sigma f \frac{1}{x} = 19$

$$H.M = \frac{N}{\Sigma(f \frac{1}{x})}$$

Where N =68, $\Sigma f \frac{1}{x} = 19$

$$H.M = \frac{68}{19}$$

$$= 3.58$$

Continuous Series

In continuous series where class along with frequencies are given, the mid-value of each class is considered as the values of the variable, i.e., X and the following formula is used to calculate Harmonic Mean.

$$H.M = \frac{N}{\Sigma f \frac{1}{m}}$$

Where N stands for number of observations and 'm' stands for mid-values of each class and 'f' stands for frequency of each class.

Steps for calculation of H.M

- Find the mid-value of each class i.e., m
- Find out the reciprocal of mid-values i.e., 1/m
- Multiply the respective frequencies with the reciprocals i.e., $f \frac{1}{m}$
- Find the sum of the products i.e., $\sum f \frac{1}{m}$
- Divide N by the sum of the product i.e., $\frac{N}{\sum\left(f \times \frac{1}{m}\right)}$

Illustration 27

From the following data calculate the Harmonic Mean :

Mark	0-10	10-20	20-30	30-40	40-50
No. of Students	15	20	30	10	05

Solution :

Mark	No. of Students	Mid-Value	Reciprocal	
X	f	m	1/m	$f\,\frac{1}{m}$
0-10	15	5	1/5	3
10-20	20	15	1/15	4/3
20-30	30	25	1/25	6/5
30-40	10	35	1/35	2/7
40-50	05	45	1/45	1/9
	$\Sigma f = 80$			$\Sigma\left(f\,\frac{1}{m}\right) = \frac{1868}{315} = 5.93$

$$H.M = \frac{N}{\Sigma\left(f\,\frac{1}{m}\right)}$$

Substituting the values.

$$H.M = \frac{80}{5.93}$$

$$= 13.49$$

Weighted Harmonic Mean

When different weights are assigned to different values of the variable according to their importance, in that case, instead of calculating simple H.M we have to calculate weighted Harmonic Mean.

The formula used for calculation of weighted Harmonic Mean is given as under

$$H.M\,(w) = \frac{N}{\sum(W \cdot \frac{1}{X})} \;or\; \frac{\sum f}{\sum(w \cdot \frac{1}{x})}$$

Where W : Weights assigned to different values of X

N/ Σf: No of observations.

Illustration 28

From the following data compute weighted Harmonic Mean.

X	5	12	15	20	24
W	5	7	8	6	4

Solution

X	W	1/x	$W.\frac{1}{X}$
5	5	1/5	1
12	7	1/12	7/12
15	8	1/15	8/15
20	6	1/20	6/20
24	4	1/24	4/24

$\Sigma W = 30$ $\qquad$ $\Sigma w\frac{1}{X} = 1+\frac{7}{12}+\frac{8}{15}+\frac{6}{20}+\frac{4}{24}$

$$H.M\,(w) = \frac{\Sigma W}{\Sigma(w\frac{1}{x})}$$

$$\Sigma w \frac{1}{X} = 1 + \frac{7}{12} + \frac{8}{15} + \frac{6}{20} + \frac{4}{24}$$

$$= \frac{120+70+64+36+20}{120}$$

$$= \frac{310}{120} = \frac{31}{12} = 2.58$$

Substituting the values.

$$H.M\,(w) = \frac{30}{2.58} = 11.63$$

Combined Harmonic Mean

Combined Harmonic Mean of two or more series can be calculated by using the algebraic properties of Harmonic Mean. The formula for calculation of combined Harmonic Mean is given below :

$$H.M_{1.2} = \text{Reciprocal of } \frac{(N_1 recH_1 + N_2 recH_2 ....)}{N_1 + N_2 ....}$$

Where	$H.M_{1.2}$:	Combined H.M of series 1 and 2
	N_1, N_2	Denote number of observation in series 1 and 2 respectively.
	H_1, H_2	Harmonic Mean of series 1 and 2.

Illustration 29

Calculate the combined H.M of the following series :

Series	H.M	N
I	II	30
II	15	40

Solution :

$$H.M_{1.2} = \text{Reciprocal of } \frac{N_1 \cdot \frac{1}{H_1} + N_2 \cdot \frac{1}{H_2}}{N_1 + N_2}$$

$$= \frac{N_1 + N_2}{\frac{N_1}{H_1} + \frac{N_2}{H_2}}$$

Given $H_1 = 11$; $H_2 = 15$

$N_1 = 30$; $N_2 = 40$

Substituting the values.

$$H.M_{12} = \frac{30 + 40}{\frac{30}{11} + \frac{40}{15}} = \frac{70}{\frac{450 + 440}{165}}$$

$$= \frac{70 \times 165}{890} = 12.98$$

Mathematical Properties of Harmonic Mean :

1. Harmonic Mean cannot be computed from a series if any of its values is zero.
2. Harmonic Mean can be calculated for a series with any number of negative values.
3. For any series in which all the values are not equal the value of H.M is less than the G.M and A.M. .
4. If Harmonic Mean of two or more series are given, the combined Harmonic Mean for all the series can be found out as explained under combined Harmonic Mean.
5. If any two of the 3 factors i.e H.M, number of observations 'N' and sum of the reciprocal of X values i.e $\Sigma 1/x$ are given the value of the 3rd factor can be found out.

Uses of Harmonic Mean

Harmonic Mean is considered to be a better average in the following fields :

1. In computation of average rate of speed, average rate of price, average rate of income etc. where relationship between two types of units that are capable of being expressed as reciprocals.
2. For graphic presentation of data Harmonir Mean is considered as a better average as it makes use of ratios.

Merit : When distance covered or speed remain same for each travel Harmonic Mean is more suitable but when time remains same for each travel Arithmetic Mean is considered as a better average.

Merits and Demerits of H.M

Following are the merits of H.M

1. It is rigidly defined.
2. It is capable of further algebraic manipulation.
3. It gives more weightage to smaller items and less weghtage to larger items.
4. It is the most useful average where we deal with average of rates and times.
5. It is based on all the values of the series.

Demerits

The main demerits of Harmonic Mean are :

1. It is dificult to understand and comparatively difficult to calculate.
2. When items of variables have both negative and positive values or if one of the item is 'zero' then it is not possible to calculate Harmonic Mean.
3. It unnecessarily gives more weightage to small items which is some times not desirable at all. Giving more weights to smaller items is not the real characteristic of good measure of central tendency.

Practical Problems :

Illustration 30

In an Office there are 4 typists A.B.C. and D. A can type a letter in 5 minutes. B.C and D can do the same work in 4, 6 and 8 minutes respectively. Find out the average time taken for typing a letter in the Office,

Solution

In the problem the variable is time and the constant is 'letter'

The appropriate mean here is H.M

N = 4 (4 persons A.B.C. & D)

Values of 'X' = 5,4,6 and 8

$$H.M = \frac{N}{\Sigma(1/x)}$$

$$= \frac{4}{1/5 + 1/4 + 1/6 + 1/8} = \frac{4}{\frac{24+30+20+15}{120}}$$

$$= \frac{4}{\frac{89}{120}} = \frac{4 \times 120}{89} = 5.39$$

$\therefore$ The average time taken for typing a letter is 5.39 minutes.

Illustration 31

A vender sells 3 grades of lemon at 4 for one rupee, 3 for one rupee and 2 for one rupee. Calculate the average price per lemon.

Solution

It is a question based on rates. The variable is number of lemons and the constant is one rupee.

$$H.M = \frac{N}{\Sigma(1/x)}$$

Here N = 3 (Three varities)

X values are 4.3 and 2

$$H.M = \frac{3}{\frac{1}{4} + 1/3 + 1/2} = \frac{3}{\frac{3+4+6}{12}}$$

$$= \frac{3}{\frac{13}{12}} = \frac{3 \times 12}{13} = \frac{36}{13} = 2.78$$

Average rate of lemom sold per rupee is 2.78

$$\therefore \text{Average price per lemon} = \frac{1}{2.78}$$

$$= 0.36$$

Illustration 32

A cyclist covers successive quarters of a mile at the speed of 12, 10, 8, 7 km/hr respectively. Find the average speed.

Solution

In the question the variable is speed in kilometres and the constant is hour. The appropriate average is H.M

$$H.M = \frac{N}{\Sigma(1/x)}$$

Given N = 4 (Four quarters)

X values are 12,10,8 and 7.

$$H.M = \frac{4}{\frac{1}{12}+\frac{1}{10}+\frac{1}{8}+\frac{1}{7}} = \frac{4}{\frac{70+84+105+120}{840}}$$

$$= \frac{4}{\frac{379}{840}} = \frac{4\times 840}{379} = 8.87$$

$\therefore$ The average apeed is 8.87km per hour.

Illustration 33

A person buys kerosene at ₹ 0.80, ₹ 01.20, ₹ 1.80 and ₹ 2.80 per litre for 4 succissive years. What is the average cost of the oil if he spends ₹ 1000 every year ?

Solution

Price per Ltr.	Money Spent ₹
0.80	1000
1.20	1000
1.80	1000
2.80	1000

As equal sum of money is spent on every year. It will be appropriate to calculate simple Harmonic Mean.

Number = 4

X = 0.80,1.20,1.80 and 2.80

$$H.M = \frac{4}{\frac{1}{0.8}+\frac{1}{1.2}+\frac{1}{1.8}+\frac{1}{2.8}}$$

$$= \frac{4}{1.25 + 0.83 + 0.56 + 0.36}$$

$$= \frac{4}{3} = 1.33$$

∴ The average cost of kerosene per ltr. is ₹ 1.33 during the period.

Illustration 34

A train runs 25 kms at an average speed of 30 km/hr. another 50 kms at a speed of 40 km. per hour. Then due to repair of the track travels for 6 minutes at a speed of 10 km/hr. and finally covers the remaining distance of 24 kms at a speed of 24 km/hr. what is the average speed in km/hr. ?

Solution

In this case we have to use weighted Harmonic Mean. Distance covered is used as weight.

For 6 minutes the train traveled at a speed of 10km/hr. Hence the distance covered is 1km.

Speed km./hr. (X)	**Distance Covered** (W)	(W/X)
30	25	25/30
40	50	50/40
10	1	1/10
24	24	24/24
	Σw = 100	

$$H.M = \frac{\Sigma W}{\Sigma(W/X)} = \frac{100}{\frac{25}{30} + \frac{50}{40} + \frac{1}{10} + 1}$$

$$= \frac{100}{\frac{100 + 150 + 12 + 120}{120}} = \frac{100}{\frac{382}{120}}$$

$$= \frac{100 \times 120}{382} = 31.41$$

∴ Average speed of the train is 31.41 kms/hr.

Relationship between A.M, G.M and H.M

(i) For any two positive numbers

$G.M = \sqrt{A.M \times H.M}$

Let 'a' and 'b' are two positive numbers. Then

$$A.M = \frac{a+b}{2}$$

$$G.M = (ab)^{1/2} = \sqrt{ab}$$

$$H.M = \frac{2}{\frac{1}{a}+\frac{1}{b}} = \frac{2ab}{a+b}$$

$$A.M \times H.M = \frac{a+b}{2} \times \frac{2ab}{a+b}$$

$= ab$

$(G.M)^2$

$\therefore \sqrt{A.M \times H.H} = G.M$

(ii) When all values of the series differ in size A.M > G.M and G.M > H.M.

or AM > GM > H.M

Let a and b are two positive numbers such that $a \neq b$

Then $A.M = \frac{a+b}{2}$ $G.M = \sqrt{ab}$

$$H.M = \frac{2ab}{a+b}$$

Let us 1st prove that A.M > G.M.

$$\therefore \frac{a+b}{2} > \sqrt{ab}$$

$$a + b > 2\sqrt{ab}$$

$$a + b - 2\sqrt{ab} > 0$$

$$(\sqrt{a} - \sqrt{b})^2 > 0$$

As the square of any real quantity is positive, $(\sqrt{a} - \sqrt{b})^2$ will be positive.

hence $\frac{a+b}{2} > \sqrt{ab}$

Let us prove that G.M > H.M

hence $\because \sqrt{ab} > \frac{2ab}{a+b}$

$(a+b)\sqrt{ab} > 2ab$

$(a+b) > \frac{2ab}{\sqrt{ab}}$

$(a+b) > 2\sqrt{ab}$

This has already been proved.

Hence G.M > H.M

Since A.M > G.M and G.M > H.M.

it is automatically proved that

AM > GM > H.M.

(iii) If all values of a series are equal then AM = G.M = H.M

❖❖❖

8.7 QUESTIONS

1. Choose the the correct answer from the given alternatives :

a. The calculation of Arithmatic Mean is based on :

(i) All values; (ii) Extreme values; (iii) Middle value; (iv) Few values.

b. The sum of square of deviations taken from Arithmatic Mean is :

(i) Maximum; (iii) Two ;

(ii) Minimum; (iv) One.

c. The sum of deviations taken from the Mean is always equal to :

(i) One; (ii) Zero; (iii) Assumed Mean; (iv) Median

d. The most common measure of central tendency is

(i) Median (ii) Arithmatic Mean (iii) Mode (iv) Geometric Mean.

e. When speed is the variable and the distance covered is different for different speeds, the most suitable average is :

(i) Werghted mean; (iii) Weryhted Harmonic mean

(ii) Weighted Geometric Mean; (iv) Harmonic Mean

f. If the Geometric Mean of 5 items is 3, then the product of these items is :

(i) 9; (ii) 27; (iii) 243: (iv) 143.

g. Harmonic Mean gives greater weightage to :

(i) Smaller items; (iii) Average values ;

(ii) Larger items; (iv) Middle values.

h. In calculation of simple Arithmatic Mean, all the values are given :

(i) Equal importance; (iii) In equal importance;

(ii) Relative importance; (iv) Subjective importance.

i. When all the values in a series differ in size the relationship between A.M., G.M. and H.M. is :

(i) AM > GM > HM; (iii) HM > GM > AM;

(ii) GM > AM > HM; (iv) AM > HM > GM;

j. In case of changes in the rates from year to year, the average that is most suitable is :

(i) A.M.; (ii) G.M. (iii) H.M. (iv Medrar.

k. Geometric Mean of 2, 4, and 8 is :

(i) 6; (ii) 8; (iii) 4; (iv) 16.

l. If the Mean of 5, 7, 6 and x is 5, then the value of x is :

(i) 2; (ii) 3; (iii) 4; (iv) 5.

m. If the algebraic sum of deviations of 20 observations from 30 is 20, then the Mean of the observations is :

(i) 24; (ii) 21; (iii) 31; (iv) 30.

n. If Arithmetic Mean of 2 numbers is 8 and the Harmonic Mean is 2, then their Geometric Mean will be :

(i) 6; (ii) 4; (iii) 16; (iv) 4.5.

o. When all the values of a series are equal, then AM will be :

(i) Equal to G.M and H.M (iii) Lesser than G.M.

(ii) Greater than G.M and H.M. (iv) AM ≠ GM ≠ HM.

2. Correct the underlined portions of the following sentences :

a. The sum of the square of deviations taken from Mean is the highest than the sum of square of deviations taken from any other value.

b. While calculating weighted Arithmatic Mean all values are given equal weights.

c. Mean is a positional average.

d. Harmonic Mean gives greater weightage to bigger items.

e. Median is affected by the presence of extreme values in the series.

f. When all the values in a series are equal their A.M, G.M and H.M are unequal.

g. A.M, G.M and H.M are positional measures of central tendency.

h. When all the values in a series are not equal A.M is smaller than G.M and H.M.

i. The sum of deviations of the items from actual Mean is always lowest.

j. The number of items in a series is equal to the quotient of the sum of values of the series and their Median.

3. Fill in the blanks :

a. ________ is more stable measure of central tendency as it is least affected by sampling fluctuation.

b. Harmonic Mean gives ________ weightage to smaller values.

c. Simple Arithmetic Mean gives ________ werghtage to values of the variable.

d. Mean is affected by presence of ________ values in the series.

e. Reciprocal of the ________ of a number is the number itself.

f. If Mean and number of items of different series are given, the ________ Mean can be obtained.

g. If all the items of a series are divided by p, the value of the Mean is also ________ by p.

h. Harmonic Mean can not be computed when any of the value in the series is ________ .

i. When all the values of series are equal then the A.M, G.M and H.M of the series are ________ .

j. If any value in a series is zero, the value of Geometric Mean becomes ________ .

4. Answer the following in one word/term :

a. A single value that represents a series of values.

b. The number of times, a value is repeated in a series.

c. An average that gives unequal weightage to different values.

d. A variable that takes all possible values in the series.

e. A variable that varies only by finite value.

f. A table in which the data grouped into classes and the number of cases which fall in each class are recorded.

g. The difference between the upper limit and lower limit of a class.

h. A continuous classificatin that does not include the upper limit of the class interval.

i. A continuous classification that includes the upper limit of the class interval in the same class.

j. Nth root of the product of 'n' number of variables.

5. Answer the following in one sentence each.

(i) What is an average ?

(ii) What is an weighted average ?

(iii) What do you mean by raciprocal of a number ?

(iv) What is a continuous variable ?

(v) What is class interval ?

(vi) Define Geometric Mean ?

(vii) Define Harmonic Mean ?

Answer :

1. (a) i (b) ii (c) ii (d) ii (e) iii (f) iii (g) i (h) i (i) i (j) ii (k) iii (l) i (m) iii (n) ii (o) i

2. (a) Least (b) Unequal (c) Mathematical (d) Smaller (e) Arithmetic Mean (f) Equal (g) Mathmatical (h) Greater (i) Zero (j) Mean.

3. (a) Mean (b) More (c) Equal (d) Extreme (e) Raciprocal (f) Combined (g) Divided (h) Zero (i) Equal (j) Zero

4. (a) Central Value/ Average (b) Frequency (c) Weighted Average (d) Continuous Variable (e) Discrete variable (f) Frequency Distribution Task (g) Class Interval (h) Exclusive (i) Inclusive (j) Geometric Mean.

5. (i) An average is a single value that represents a group of values.

(ii) An average that gives different weights to different values in the series is called an weighted average.

(iii) Reciprocal of a number means one divided by that number.

(iv) A variable that is capable of manifesting every conceivable fractional value within a range of possibilities is called a continuous varible.

(v) The difference between upper and lower limit of a class is called as the class - interval of that class.

(vi) Geometric Mean is defined as the Nth root of the product of N items or values.

(vii) Harmonic Mean is defined as the raciprocal of the Arithmetic Mean of raciprocal values in an observation.

No. 6 Answer the following in 30 words :

a. What is a measure of central tendency ?

b. Write any three features of an ideal measure of central tendency.

c. What is weighted Arithmatic Mean ?

d. Write any three demerits of Arithmatic Mean.

e. Write any two mathematical properties of Arithmetic Mean.

f. Define Geometric Mean.

g. Write any three advantages of Arithmetic Mean.

h. Write the uses of Geometric Mean.

i. Write the uses of Harmonic Mean.

j. What are the demerits of Harmonic Mean ?

No. 7. Answer the following with 50 words :

a. Write any five mathmatical properties of Arithmetic Mean.

b. Write the main limitations of Arithmetic Mean.

c. What is a combined average ? How is it calculated. ?

d. Explain the specific uses of Geometric Mean.

e. Write the mathematical properties of Geometric Mean.

f. Write any five features of an ideal measure of central tendency.

g. What is a central value ? What are its uses ?

h. Why is Mean considered as an ideal measure of central tendency ?

i. Write the specific uses of Harmonic Mean.

j. What is weighted Arithmetic Mean ? What are its advantages over simple Arithmetic Mean ?

Long Answer type questions.

8. What is a measure of central tendency ? Explain the features of an ideal measure of central tendency.

9. Explain the merits and demerits of Arithmetic Mean.

10. Explain the mathematical properties of Arithmetic Mean.

11. Explain the relationship between Arithmetic Mean, Geometric Mean and Harmonic Mean.

12. What is meant by central tendency ? State the important measures of central tendency and point aout the limitations of Mode.

Problems

13. Calculate Arithmetic Mean from the following data :

Marks	0-10	10-20	20-30	30-40	40-50	50-60
Number of Students	6	14	16	27	22	15

14. Calculate Mean wages from the following data :

Wages in	1-7	8-14	15-20	22-28	29-35
Number of firms	3	17	12	11	7

15. Calculate Arithmetic Mean from the following data :

Marks Above	0	10	20	30	40	50
Number of Students	40	37	30	20	7	3

16. Calculate the simple and weighted Arithmetic Mean price per tonne of coal purchased by a firm during 6 months from the following data :

Months	Jan	Feb	Mar	Apr	May	June
Price per tonne	42.50	51.25	59.00	53.00	47.50	52.00
Tonnes purchased	25	30	40	50	10	45

17. Find the missing frequency from the following data :

Marks	0-10	10-20	20-30	30-40	40-50	50-60
No. of Students	2	18	?	20	17	6

18.(a) In a class of 50 students 10 have failed and their average score is 1.8 the total marks secured by the entire class in 288. Find the average marks of the students passed.

(b) The average marks of 100 student swere calculated as 50. It was later discovered that one entry was misread as 93 instead of 63. Find the correct average.

19. The average daily wages of all the workers in a factory is ₹444. If the average wages paid to male and female workers are ₹480 and ₹360 respectively find the percentage of male and female workers employed by the factory.

20. Calculate the combined Arithmetic Mean from the following data :

	Class A	Class B
No. of Students	40	50
Average Mark	61	64

21. Find out the missing frequencies if the Mean of the distributon is 50 and total number of observation is 120.

Class	0-20	20-40	40-60	60-80	80-100
Frequency	17	?	32	?	19

22. A candidate obtained the following percentage of marks in different subjects :

English	Statistics	Cost A/c	Economics	Income Tax
46%	67%	72%	58%	53%

It is agreed to give double weights to marks in English and Economics as compared to other subjects. Calculate simple and weighted Arithmetic Mean.

Geometric Mean :

23. If the rate of growth in population of Bhubaneswar in the last three decades is 10%, 20% and 30%, find the average rate of growth.

24. A series comprises of two numbers and its Mean is 10 and Geometric Mean in 8 Find out the numbers.

25. Find the GM from the following data

X	4	12	18	26
f	2	4	3	1

26. The population of India increased from 100 crores in 2001 to 120 crorers in 2011. Find the annual rate of growth of the population.

27. A machinery depreciate by 40% in the first year 25% in the 2nd year and 10% per annum during the next 3 years. What is the average rate of depreciation during the whole period ?

Harmonic Mean

28. A cyclist pedals from his house to his office at a speed of 10 km. p.h. and back from his office to his house at 15 k.m. p.h. Find the average speed.

29. An investor buys ₹2000 worth of shares in a company each month. During the 1st 4 months he bought the shares at a price of ₹10, ₹12, ₹15 and ₹18 per share. Find out the average price per share.

30. Mr Pal travels 8 kms at 4 km. p.h, 6kms at 3 km. p.h, and 4 kms at 2 km. p.h. Find out his average speed per hour.

31. Interest paid on each of the three different sums of money yeilding 10%, 12% and 15% simple interest per annum fetches the same. What is the average yeild percent on the sum invested ?

32. Calculate the H.M. of the following :

X	0-10	10-20	20-30	30-40	40-50
f	4	7	28	12	9

◆◆◆

CHAPTER - 9

Positional Averages

Structure

9.1 Positional Average

9.2 Median - Its meaning and definition, methods of determination, Properties, Merits, Demerits and uses.

9.3 Partition Values - Quartiles, Deciles and Percentiles.

9.4 Mode - its meaning and definition, methods of determination, Merits, Demerits and uses of Mode.

9.5 Choice of Suitable Average

9.6 Comparison among Mean, Median and Mode.

9.7 Questions.

9.1 Positional Averages :

These are a group of averages whose values are found out simply by location i.e by locating their position in the distribution. Like mathematical averages they are not 'derived' values, they are existing values, picked out only by identifying their position. Median Mode and other partition values like Quartiles, Deciles and Percentiles are some commonly used positional measures of central tendency. They have no mathematical properties like the mathematical averages and they also do not need all the values of the observation for their determination.

9.2 MEDIAN

–Meaning & Definition

Median is the value that separates the higher half of a distribution from the lower half. In simple terms it is the value that lies in the middle of the distribution, when the data is arranged either in ascending or descending order of magnitude. As the value of the Median is identified with a particular 'position' it is called a positional average. Its value is found out not by calculation but simply by location.

Important definitions of Median by some prominent writers are quoted as follows.

Port L.S. Conner

"The Median is that value of the variable which divides the group into two equal parts, one part comprising all values greater and the other, all values smaller than the Median".

J.R. Stockton and C.T. Clark

"Median is the value of the middle item in an array"

Yele and Kendal

"The Median may be defined as the middle most of the central value of the variable, when the values are arranged in order of magnitude or as the value is such, that greater and smaller values occur with equal frequency".

Charecterstics of Median

- It is the middle value of the data series or frequency distribution.
- It divides the distribution into two equal halves.
- The division is made in such a manner that 50% of values are higher than the Median while other 50% are lower than the Median.
- Both ends of the distribution upper & lower remain equidistant from the Median.

Methods of Determination of Median

As has been told, the value of the Median is only located, not calculated from the series like Arithmetic Mean or Geometric Mean. Two methods are being used for the poorpose. They are

(i) Tabular method

(ii) Graphic method

Tabular method

Under this method the data are presented in a table and Median is determined by using techniques as mentioned below.

Ungrouped Data

The formula used for finding Median in ungrouped data or individual series is :

$$M = \text{Size of} \left(\frac{N+1}{2}\right) \text{th item.}$$

where, M = Median N = Number of items in the series.

Steps for Calculation

- Arrange the data in ascending / descending order of magnitude.
- Locate the Median item by using the formula $\frac{N+1}{2}$
- The items such located is the value of Median.

When 'N' is odd number $\frac{N+1}{2}$ th term of the series is the Median and finding its value poses little problem. But when 'N' is even number, for finding the value of the Median, we have to modify the formula and find out the average of two middle terms. Thus

$$\text{Median} = \frac{N/2\text{th term} + \{N/2+1\}\text{ th term}}{2}$$

Example 1.

Calculate Median from the following data

12	30	20	15	25	10	2	40	4	8	32

Solution

The data are 1st arranged in ascending order of their magnitude.

Serial Number	Items (X)
1	2
2	4
3	8
4	10
5	12
6	15
7	20
8	25
9	30
10	32
11	40

$M = \text{Size of } \frac{N+1}{2}$ th items.

Here N = 11

$$M = \text{Size of } \frac{11+1}{2} \text{th item}$$

= Size of 6th item

Size of 6th item is 15

Hence the value of Median is 15

Example 2

Find out the Median from the following data :

15 20 20 23 23 25 26 27 35 40

Solution

The data are already in ascending order therefore we can directly go for calculation of Median.

Sl. No.	Item(X)
1	15
2	20
3	20
4	23
5	24
6	25
7	26
8	27
9	35
10	40

As 'N' is even number for finding Median we can use the formula :

$$\text{Madian} = \frac{(N/2)\text{th term} + (N/2+1)\text{th term}}{2}$$

$$= \frac{(10/2)\text{th term} + (10/2+1)\text{th term}}{2}$$

$$= \frac{5\text{th term} + 6\text{th term}}{2}$$

$$= \frac{23+25}{2}$$

$\therefore$ The Value of Median is 24.

Grouped Data

Discrete Series

The formula used for calculating Median is

$$M = \text{Size of}\left(\frac{N+1}{2}\right)\text{th item}$$

Steps for Calculation

- Arrange the data in ascending / deasending order of magnitude.
- Find the cumulative frequencies.
- Find the size of $\frac{N+1}{2}$ th item
- Locate the item from the cumulative frequency and corresponding value of X
- The corresponding value of 'X' is the Median value.

Example 3

Calculate the Median from the following data

X	15	20	25	30	35	40
f	10	15	25	5	5	20

Solution

X	f	Cf
15	10	10
20	15	25
25	25	50
30	5	55
35	5	60
40	20	80
	$\Sigma f = 80$	

$$M = \text{Size of} \left(\frac{N+1}{2} \right) \text{th item}$$

$$= \text{Size of} \left(\frac{80+1}{2} \right) \text{th item}$$

= Size of 40.5th item

The value of 40.5 th item lies within the 50 whose Value is 25

Hence Median = 25

Continuous Series

In case of continuous series Median is found out in two stages :

- First the Median class is found out by using the formula N/2.
- In the second stage by using a method of interpolation the value of the Median is found out. The formula used for interpolation is given below :

  When the series is arranged in ascending order.

$$M = L_1 + \frac{L_2 - L_1}{f_1}(m - c)$$

When the series is arranged in descending order

$$M = L_2 - \frac{L_2 - L_1}{f_1}(m - c)$$

Where M = Median

L_1 = Lower limit of the median class

L_2 = Upper limit of the median class

f_1 = Frequency of the median class

m = N/2

C = Cumulative frequency of the class preceding median class

Steps for Calculation

- Arrange the series either in ascending or descending order
- Calculate cumulative frequencies.
- Find out the Median class by using the formula N/2
- Interpolate the value of Median from the Median class by using the appropriate formula.

Example 4 :

Find out Median from the following table

Marks (X)	0-10	10-20	20-30	30-40	40-50	50-60	60-70	70-80	80-90	90-100
No. of Candidates(*f*)	5	20	40	70	85	65	50	35	20	10

Solution

Marks X	No. of Candidates f	C.F
0-10	5	5
10-20	20	25
20-30	40	65
30-40	70	135

40-50	85	320
50-60	65	285
60-70	50	335
70-80	35	370
80-90	20	390
90-100	10	400
	$\Sigma f = 400$	

Median = The size of N/2th item

N = 400

The size of N/2th item

The 200th item lies in

Class 40-50

Now applying the formula

$$M = L_1 + \frac{L_2 - L_1}{f_1}(m - c)$$

Where $L_1 = 40$ $L_2 = 50$ $f_1 = 85$

$m = 200$ $c = 135$

Median $= 40 + \frac{50-40}{85}(200-135)$

$= 40 + \frac{10}{85} \times 65$

$= 40 + \frac{650}{85}$

$= 40 + 7.65$

$= 47.65$

Example 5 : (Inclusive series)

Calculate Median from the following data

X	10-14	15-19	20-24	25-29	30-34	35-39
f	5	10	15	20	10	5

Solution

Since the series is an inclusive one, to make it continuous it must be made exclusive by deducting. 0.5 from the lower limit and adding 0.5 to the upper limit of all the classes.

Marks	No. of Candidates	
X	*f*	C.F
9.5-14.5	5	5
14.5-19.5	10	15
19.5-24.5	15	30
24.5-29.5	20	50
29.50-34.5	10	60
34-5 - 39-5	5	65

Median = Size of N/2th item

=65/2 = 32.5th item

From the cumulative frequency Column it is observed that Median lies in the class 24.5 - 29.5

Now applying the formula

$$M = L_1 + \frac{L_2 - L_1}{f_1}(m - c)$$

$$= 24.5 + \frac{29.5 - 24.5}{20}(32.5 - 30)$$

$$= 24.5 + \frac{5}{20} \times 2.5$$

$$= 24.5 + 0.63$$

$$= 25.13$$

Example 6 : (Open ended class interval)

Marks obtained by 65 students in an exmination are given below. Find the Median marks

Marks More than	70%	60%	50%	40%	30%	20%
No. of Students	7	18	40	40	63	65

Solution

As cumulative frequencies are given, firstly we have to find out simple frequencies.

Mark(x)	f	C.*f*
20-30	2	2
30-40	23	25
40-50	0	25
50-60	22	47
60-70	11	58
70-80	7	65

Median = Size of N/2th item

= Size of 65/2 = 32.5th item

From the cumulative frequency it is observed that Median lies in the class 50-60.

Applying the formula

$$\text{Median} = L_1 + \frac{L_2 - L_1}{f_1}(m - c)$$

Where L_1 =50, L_2=60, f_1=22 m=32.5, C=25

Substituting the values

$$\text{Median} = 50 + \frac{60-50}{22} \times (32.5 - 25)$$

$$= 50 + \frac{10}{22} \times 7.5$$

$$= 50 + \frac{75}{22}$$

$$= 53.41$$

Example 7 (Mid Value series)

Following is the distribution of marks obtained by 32 students. Calculate the Median Mark.

Mid Value	5	15	25	35	45	55
Frequency	4	6	10	7	3	2

Solution

Since we are given the mid values, we should first find out the upper and lower limits of the various classes. As the mid values are 5,15,25 ..... the class interval is 10. For determining the limits we will use following method and find out different classes.

L_1 = Mid value - 1/2 of class interval

L_2 = Mid value + 1/2 of class interval

Marks	No. of Student	C.f
0-10	4	4
10-20	6	10
20-30	10	20
30-40	7	27
40-50	3	30
50-60	2	32

Median = Size of N/2th item

= size of item 32/2=16th item

It can be observed from the cumulative frequency column that Median lies in the class 20-30.

By applying the formula

$$\text{Median} = L_1 + \frac{L_2 - L_1}{f_1}(m - c)$$

Where L_1 = 20, L_2 = 30, f_1=10, m= 16 and C= 10

Substituting the values

$$\text{Median} = 20 + \frac{30-20}{10}(16-10)$$

$= 20 + 6 = 26.$

Graphic Method

Median, unlike Mean can be located graphically with the help of ogives or cummulative frequency curves. The steps involved under the method are as follows.

- Draw an ogive curve - either less than ogive or more than ogive.
- Find out the value of N/2 and locate it on y axis.
- Draw a horizontal line from this point so that it intersect the curve.
- From the point of intersection, draw a perpendicular to 'X' axis.
- The point of intersection on X axis gives fives the value of the Median.

Median also can be located by using both the ogives (more than & less than) together under such circumstance, we need not have to calculate N/2. From the point of intersection of both the ogives when a perpendicular is drawn to 'X' axis, it gives the value of Median.

Example 8

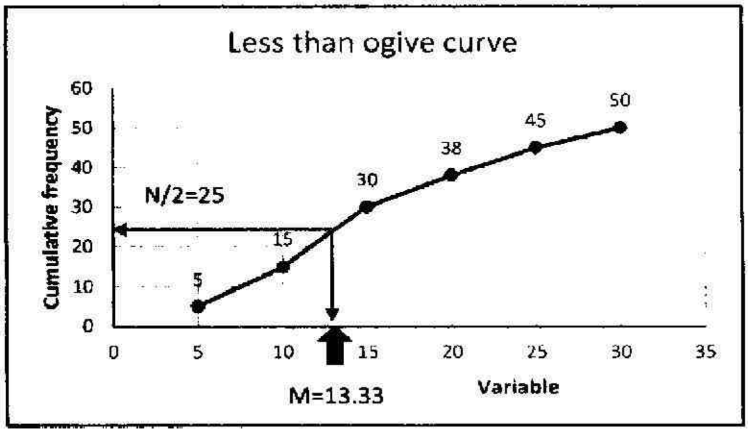
From the following data find out the value of Median by graphical method.

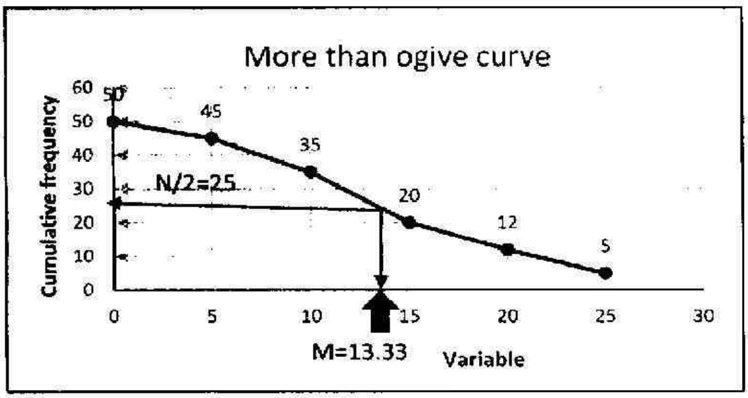
X	0-5	5-10	10-15	15-20	20-25	25-30
(f)	5	10	15	8	7	5

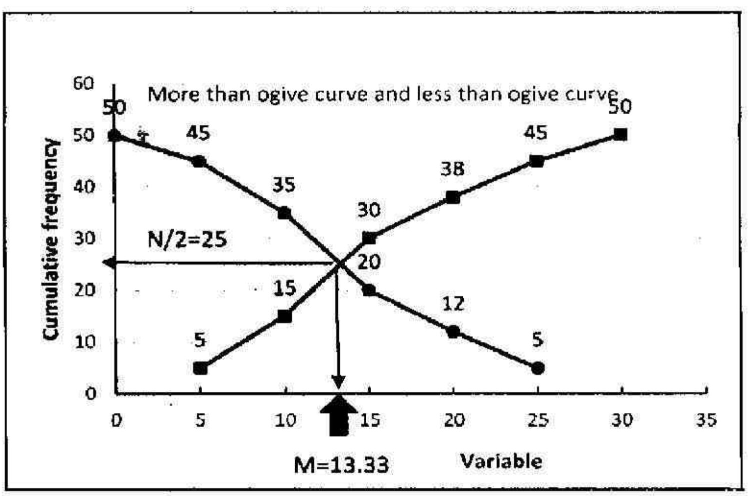
Solution

Less Than method		c.f	More than method		c.f
less then	5	5	More than	0	50
,,	10	15	,,	5	45
,,	15	30	,,	10	35
,,	20	38	,,	15	20
,,	25	45	,,	20	12
,,	30	50	,,	25	5

Using the 'less than ogive' and 'more than ogive' the Median is computed graphically as follows :







Properties of Median

The important properties of Median are :

(i) The sum of the absolute deviations of the items from Median will be the least of all other deviations taken from any other value or average.

(ii) If Median of a series is known the missing frequency of a class can be found out provided the total number of observation (N) is given.

Merits of Median

Median, as a measure of central tendency has the following merits :

1. It is easy to understand and simple to calculate.
2. It does not require all the items of observation
3. It is not affected by extreme values.
4. It can also graphically located
5. It can be easily located in open ended series where the lowest or highest class limits are not known.
6. It is considered as the most appropriate average for calculation of Mean deviation as the sum of absolute deviation taken from Median is the least.
7. Median represents value that exists in the series.

Demerits of Median

1. It is not based on all the observations of a series.
2. It is not capable of further algebaraic treatment like Mean. Geometric Mean and Harmonic Mean.
3. It requires arranging of data in ascending / descending order.
4. It is very much affected by fluctuation in sampling.
5. It cannot be computed exactly where the numbers of item in a series is even.
6. Sometimes it gives a value that is never found in the series.
7. When numbers of observations is very few, it gives erroneous results.

Uses of Median

- It is useful in those cases where all the observations are not available or the classifications are not available or the classifications are open ended.
- It is useful in cases where numerical measurements are not possible like skill, honesty. intelligence, etc, which are called descriptive statistics. Here the values are not counted or measured but scored.

9.3 Partition Values

Quartiles, Deciles and Percentiles are partition values that divide the distribution or series of data into 4, 10 and 100 parts respectively. When the distribution is divided into two equal parts, we have only one partition value i.e Median. But when it is divided into 4 equal parts there are 3 partition values, 3 quartiles (Q1, Q2 and Q3). When it is divided into ten equal parts there are 9 partition values or 9 Deciles (d_1, d_2, d_3, d_4, d_5, d_6, d_7, d_8 and d_9) Similarly for its division into 100 parts there are 99 partition values or 99 percentiles (P_1, P_2, P_3 .... P_{99})

The division of a series into many parts and different partition values throw more light on the nature of the distribution and helps in analysis of data. More specifically partion values are used in productivity, ratings and ranking of test scored in psychological and educational statistics.

Quartiles

As has been told, quartiles divide the series of data into 4 equal parts. The first quartile (Q_1), which is called as the lower quartile, covers the first quarter or 25% of items of the series. It is the partition value of the first half of the series. The second (Q_2) is the Median itself and the third qualite (Q_3) divides the upper half of the series into two equal halves - This is also called the upper quartile and it covers the first 75% of the items of the series.

Computation of Quartiles

Quartiles are determined in the same manner as Median. Methods used for finding out quartiles for ungrouped and grouped data are discussed as below.

Ungrouped Data

Individual Observation

Like Median for finding out quartiles, the observations must be arranged either in ascending or descending order of magnitude and then the following formula be used to determine the values :

Q_1 = First Quartile = Value of $\frac{N+1}{4}$ th item and

Q_3 = Third Quartile = Value of $\frac{3(N+1)}{4}$ th item.

Example 9

From the following data, calculate Q_1 and Q_3

Serial Number	1	2	3	4	5	6	7	8	9	10
Marks	12	30	20	15	25	10	2	40	4	8

Solution

The data are first arranged in ascending order.

2 4 8 10 12 15 20 25 30 40

Q_1 = The size of $\frac{N+1}{4}$ th item

= The size of $\frac{10+1}{4} = \frac{11}{4} = 2.75$th item

= The value of 2nd item + 0.75 (Value of 3rd item – Value of 2nd item)

= 4 + {0.75(8-4)}

= 4+3 = 7

Q_3 = The size of $\frac{3(N+1)}{4}$ th item

= The size of $\frac{3(10+1)}{4}$ th item

= The size of 8.25 the item

= The value of 8th item + 0.25 (9th item - 8th item)

= 25 + 0.25 (30-25)

= 25 + 1.25

= 26.25

Grouped Data

Discrete Series

In case of discrete variable distribution, the formula used for finding out. Quartiles is the same as in individual observations.

Thus the values Q_1 = The size of $\frac{N+1}{4}$ th item,

Q_2 = The size of $\frac{N+1}{2}$ th item (Median)

Q_3 = The size of $\frac{3(N+1)}{4}$ th item

Before determining quartiles, the data must be arranged in order of their magnitude and a cumulative frequency column be added to the data table.

Example 10

Calculate lower and upper quartiles from the following data

X	58	59	60	61	62	63	64	65	66
f	2	3	6	15	10	5	4	3	1

Solution

X	*f*	c*f*
58	2	2
59	3	5
60	6	11
61	15	26
62	10	36
63	5	41
64	4	45
65	3	48
66	1	49

Lower Quartile Q_1 = Size of $\frac{N+1}{4}$ th item.

Size of $\frac{49+1}{4}$ th item

= 12.5th item.

12.5 lies in the row where c.f is 26

Hence $Q_1 = 61$

Q_3 the upper Quartile

Q_3 = The size of $\frac{3(N+1)}{4}$ th item

= The size of $\frac{3(49+1)}{4}$ th item

= The size of 37.5th item

37.5 lies in the row where

$C.f = 41$

Hence $Q_3 = 63$

Continuous Series

Computation of quartiles in a continuous distribution is done in two stages, First, the size of the quartile class is found out in the same manner as in case of discrete series, but in place of (N+1) only 'N' is taken for the purpose. For first quartile, the size of (N/4)th item is taken into account and for the 3rd quartile 3(N/4)th item is found out. In the second stage the values of the quartile is interpolatedd by using a formula of interpolation as in case of Median. The interpolation formula used for compution of different quartiles are given below.

$$Q_1 = L_1 + \frac{L_2 - L_1}{f}(N/4 - C)$$

$$Q_2 = L_1 + \frac{L_2 - L_1}{f}(N/2 - C)$$

$$Q_3 = L_1 + \frac{L_2 - L_1}{f}(3N/4 - C)$$

L_1 = Lower limit of the quartile class.

L_2 = Upper limit of the quartile class.

f = Frequency of the quartile class.

N = Number of observations, and

C = Cumulative frequency of the class preceding the quartile class.

Example 11

Find out Q_1, Q_2 and Q_3 from the following data

Marks	0-7	7-14	14-21	21-28	28-35	35-42	42-49
No. of Students	3	4	7	11	2	14	9

Solution

Marks X	No. of Students f	C.f
0-7	3	3
7-14	4	7
14-21	7	14
21-28	11	25
28-35	2	27
35-42	14	41
42-49	9	5

Q_1 = Size of N/4th item

= Size of $\frac{50}{4} = 12.5$ item

= Q_1 lies in the class 14-21

Where $L_1 = 14$, $L_2 = 21$, $f = 7$ and $C = 7$

$$14+\frac{21\text{-}14}{7}(12.5-7)$$

$= 14 + 5.5$

$= 19.5$

Q_2 = Size of N/2th item

= Size of 50/2th item

Q_2 or Median lies in the class 21-28

Where $L_1 = 21$, $L_2 = 28$, $f = 11$ and $C=14$

$$\therefore\ Q_2 = 21+\frac{28-21}{11}(25-14)$$

$= 21+7$

$= 28$

Q_3 = The size of 3 N/4th item

= The size $\frac{3\times 50}{4}$ th 37.5th item

37.5 item lies in the class 35-42

Where $L_1 = 35$, $L_2=42$, f-14 and $C=27$

$$Q_3 = 35+\frac{42-35}{14}(37.5-27)$$

$= 35+5.25$

$= 40.25$

Deciles

Deciles are those values which divide a distribution into ten equal parts. There are nine deciles denoted by D_1, D_2, D_3..... D_9 respectively. The value of the 5th Decile (D_5) is the same as the value of Median as it stands in the middle of the distribution i.e in the place of the Median.

Computation of Deciles

Deciles are calculated by using the same method as in case of Median and Ouartiles. For the purpose, the data are first arranged in order of magnitude i.e either in ascending or descending order and then appropriate formula is used to locate them.

Ungroupd Data

Individual Series / Observations :

First Decile = D_1 = Value of $\frac{N+1}{10}$ th item

Second Decile = D_2 = Value of $\frac{2(N+1)}{10}$ th item.

Third Decile = D_3 = Value of $\frac{3(N+1)}{10}$ th item.

Nineth Decile = D_9 = Value of $\frac{9(N+1)}{10}$ th item

Example 12

Find out the 3rd and 8th Decile from the following data :

13, 14, 7, 12, 9, 17, 8, 10, 6, 15, 18, 21, 20

Solution

Arranging the given data in ascending order we get.

6, 7, 8, 9, 10, 12, 13, 14, 15, 17, 18, 20, 21

N = 13

Value of D_3 = $\frac{3(N+1)}{10}$ th item

= $\frac{3(13+1)}{10}$ th

= 4.2th item

= Value of 4th item +0.2 (5th term - 4th term)

= 9 + 0.2 (10-9) = 9.2

$$\text{Value of } D_8 = \frac{8(N+1)}{10} \text{ th item}$$

$$= \frac{8(13+1)}{10} \text{ th item}$$

= 11.2th item

= 11th item + 0.2. (12th term - 11th term)

= 18 + 0.2 (20-18)

= 18 + 0.4

= 18.4

Grouped Data

Discrete Series

In case of discrete frequency distribution the same formula, as in case of individual series, is used. But here, after arranging the data in ascending or discending order, a cumulative frequency column is added. Thus

$$\text{First Decile } D_1 = \text{Value of } \frac{N+1}{10} \text{ th item.}$$

$$\text{Second Decile } D_2 = \text{Value of } \frac{2(N+1)}{10} \text{ th item.}$$

$$\text{Symbolically } D_k = \frac{K(N+1)}{10} \text{ th item}$$

Where the value of 'K' is 1, 2, 3............9 i.e., the number of the Decile.

Example 13

Calculate D_3, D_5 and D_7 from the following data :

Marks	10	20	30	40	50	63
No. of Students	4	7	15	48	7	2

Solution

Marks (X)	No. of Students (f)	Comulative Frequency (c.f)
10	4	4
20	7	11
30	15	26
40	18	44
50	7	51
60	2	53
	$\Sigma f = 53$	

$D_3 = \text{Size of } \frac{3(N+1)}{10} \text{th term}$

$= \text{Size of } \frac{3(53+1)}{10} \text{th term}$

= Size of 16.2th term

Cumulative frequency, greater than 16.2 is 26 and the corresponding variable is 30

Hence $D_3 = 30$

$D_5 = \text{Size of } \frac{5(53+1)}{10} \text{th term}$

= Size of 27th term

Cumulative frequency just greater than 27 is 44. The corresponding variable is 40.

Hence $D_5 = 40$

$D_7 = \text{Size of } \frac{7(53+1)}{10} \text{th item}$

= Size of 37.8th item

Cumulative frequency just greater than 37.8 is 44. The corresponding variable is 40

Hence $D_7 = 40$.

Continuous Series

In a continuous frequency distribution where class frequencies are there, first, the Decile class is found out by using the following formula :

$$Dk = \text{size of } \frac{Kn}{10} \text{ th item.}$$

Where K stands for the number of the Decile i.e 1,2,3 9

The decile class is identified by refering to the cumulative frequency column as in case of discrete series.

After identification of the decile class the value of the decile is interpolated by using the following formula:

$$Dk = L_1 + \frac{L_2 - L_1}{f}\left(\frac{KN}{C} - C\right)$$

Where
- K = Number of the Decile
- L_1 = Lower limit of Decile class,
- L_2 = Upper limit of Decile class,
- f = Frequency of the Decile class,
- N = Number of observations,
- C = Cumulative frequency of the class preceding the Decile Class

Example 14

From the following distribution calculate the 6th and 9th Decile :

X	1-5	6-10	11-15	16-20	21-25	26-30	31-35	36-40
f	3	10	20	30	20	9	5	3

Solution

This is an inclusive distribution. Hence we shall make it an exclusive one by deducting 0.5 and adding 0.5 in lower and upper limit of all the classes :

X	f	c.f
0.5 - 5.5	3	3
5.5 - 10.5	10	13
10.5 - 15.5	20	33
15.5 - 20.5	30	63
20.5 - 25.5	20	83
25.5 - 30.5	9	92
30.5 - 35.5	5	97
35.5 - 40.5	3	100

The value of D_6 = Size of $\frac{6N}{10}$ th item

= size of $\frac{6 \text{ x} 100}{10}$ th item

= 60th item

Cumulative frequency just greater than 60 is 63.

The corresponding Decile class is 15.5 - 20.5

Hence $D_6 = 15.5 + \frac{20.5 - 15.5}{30}(60 - 33)$

$= 15.5 + \frac{5}{30} \times 27$

$= 15.5 + 4.5 = 20$

D_9 = The size of $\frac{9N}{10}$ th item

= The size of $\frac{9 \times 100}{10}$ th item

= The size of 90th item

Cumulative frequency just higher than 90 is 92.

The corresponding decile class is 25.5 - 30.5

$$\therefore D_9 = 25.5 + \frac{30.5 - 25.5}{9}(90 - 83)$$

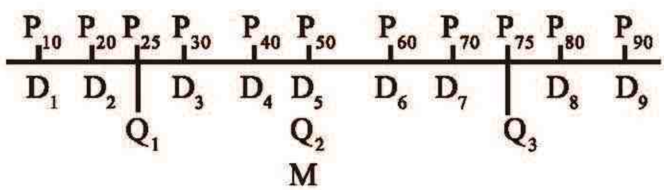
$$= 25.5 + \frac{5}{9} \times 7$$

$$= 25.5 + 3.89$$

$$= 29.39$$

Percentiles

Percentiles are partition values which divide the series into 100 equal parts. There are 99 percentiles which are denoted by P_1, P_2, P_3 .... P_{99}. The value of 50th percentile in same as the value of Median. The relationship among Median, quartiles, deciles and percentiles can be understood well from the following line dragram.



(Frequency Distribution & Relation of Partition Values)

Computation of Percentile
Ungrouped Data / Individual Series

In case of individual series, percentile is calculated by using the following formula :

$$Pk = \text{value of } \frac{k(N+1)}{100} \text{ the term}$$

where $K = 1,2,3$ 99

N = Number of items

For finding percentile, the values of the variable must be arranged either in ascending or descending order.

Example 15

From the following data calculate P_{25}, P_{78} and P_{90}

Sl. No	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
Marks	18	20	25	17	9	11	23	37	38	42	36	35	8	10	11	21	20	41	35

Solution

The series is 1st arranged in ascending order.

Sl. No.	Marks
1	8
2	9
3	10
4	11
5	11
6	17
7	18
8	20
9	20
10	21
11	23
12	25
13	35
14	35
15	36
16	37
17	38
18	41
19	42

P_{25} =Value of $\frac{25(19+1)}{100}$ the term

=Value of $\frac{25(20)}{100}$ th term

=Value of 5th term of i.e 11

P_{78}=Value of $\frac{78(19+1)}{100}$ th term

= Value of 15.6th term

=Value of 15th term + 0.6 (Value of 16th term – Value of 15th term)

= 36.0.6 (37-36)

= 36 + 0.6 = 36.6

P_{90} =Value of $\frac{90(19+1)}{100}$ th term

= value of 18th term i.e 41

Grouped Data

Discrete Series

For computing percentile in a discrete distribution, the same formula which is used in case of Individual series in used. Thus

Pk = The value of $\frac{k(N+1)}{100}$ th item

Where K = Number of percentile i.e 1,2,3....99

N = Number of observations.

Example 16

From the following data calculate P_{35} and P_{76}

x	20	25	30	35	40	45	50	55
f	1	3	5	9	14	10	6	4

Solution

As the values of X are arranged in ascending order, no further arrangement is required However, cumulative frequencies are to be calculated,

X	f	C.f
20	1	1
25	3	4
30	5	9
35	9	18
40	14	32
45	10	42
50	6	48
55	4	52
	$\Sigma f = 52$	

P_{35} = Value of $35\frac{52+1}{100}$ th term

= Value of 18.55 th term

= Value of 18th term + .55 (Value of 19th term - value of 18th term)

= 35 + .55 (40-35)

= $35 + .55 \times 5 = 37.75$

P_{76} = Value of $\frac{76(52+1)}{100}$ th term

= Value of $76\times\frac{53}{100}$ th term

= Value of 40.28th term i.e., 45.

Continuous Series

For calculating percentiles in case of continuous frequency distribution where class intervals are given, following steps are to be followed :

- Compute the cumulative frequencies
- Calculate $\frac{K \times N}{100}$ for kth percentile

  where k = 1,2,3 99
- Find the cumulative frequency just greater than the value $\frac{K \times N}{100}$ and the corresponding class, which is called the percentile class.
- Interpolate the value of the percentile by using the formula.

$$P_k = L_1 + \frac{L_2 - L_1}{f}\left(\frac{K \times N}{100} - C\right)$$

Where L_1 = Lower limit of percentile class

L_2 = Upper limit of percentite class

f = Frequency of percentile class

K = Percentile Number

C = Cumulative frequency of the class preceeding the percentile class.

Example 17

Calculate P_{20} and P_{45} from the following data :

Marks More than	0	10	20	30	40	50	60	70
Number of Students	100	92	77	57	40	25	15	5

Solution

Since we are given cumulative frequencies. We have to find out simple frequencies first.

Marks	No. of Students	C.f
0-10	8	8
10-20	15	23
20-30	20	43
30-40	17	60
40-50	15	75
50-60	10	85
60-70	10	95
70-80	5	100
	$\Sigma f = 100$	

P_{20} = Value of $\frac{20 \times N}{100}$ th term

=Value of $\frac{20 \times 100}{100}$ th term

=Value of 20th term

Just greater than 20 in cumulative frequency columns is 23 and corresponding percentile class is 10-20

$$\text{Now } P_{20} = 10 + \frac{20-10}{15}(20-8)$$

$$= 10 + \frac{10}{15} \times 12$$

$$= 10 + 8 = 18$$

P_{45} = Value of $\frac{45 \times 100}{100}$ th term

= 45th term

Just greater than 45 in cumulative frequency colum is 60. Hence the percentile class is the corresponding class i.e 30-40.

$$P_{45} = L_1 + \frac{L_2 - L_1}{f}(\frac{45 \times N}{100} - C)$$

$$= 30 + \frac{40-30}{17}(\frac{45 \times 100}{100} - 43)$$

$$= 30 + \frac{10}{17}(45-43)$$

$$= 30 + \frac{20}{17}$$

$$= 30 + 1.18$$

$$= 31.18$$

9.4 'MODE' - Its meaning and definition :

Like Median, Mode is an average of position. It is the most frequently occuring value in a distribution. In other words it is a value that has the greatest frequency in a series. For example the Mode of the distribution 5,6,5,9,8,5,9,10,4,5 is '5' as it appears most frequently in the series. Mode is also otherwise known as 'Norm'.

Mode has been defined by different authors in different ways. Some of the definitions may be quoted as follows :

Croxton and Cowden

"The Mode of a distribution is the value at the point around which the items tend to be most heavily concentrated"

Stockton and Clark "The value of the variable that occurs maximum time or most offen".

A.M Tuttle "Mode is the value which has the greatest frequency density"

Kenny and Keeping "The value of the variable which accurs most frequently in a distribution is called Mode."

When in a frequency distribution two or more values have equal maximum frequency, then the value of Mode is ill defined. In a series of observation if one value has the highest frequency, and the value of Mode is unique, it is called a uni-modal series. Similarly if in a series of observations two, three or more values are having the equal maximum frequency the series is called a bi-modal, tri-modal or multi-modal series accordingly.

Methods of Determining Mode

Mode of a series is determined by using the following methods :

- Method of inspection
- Method of grouping
- Method of graph and
- Method of empirical relation

However before determination of Mode under any method, the following points must be kept in mind.

- The series must be arranged in ascending order.
- If the class intervals are unequal, they should be made equal.
- If the series is an inclusive one it must be converted to exclusive series If the class limits of the modal class only is converted to exclusive that will solve the purpose.
- It is not necessary to complete the open end class intervals by estimating their lower and upper class limits.

An appropriate method must be chosen from the above, keeping in view the nature of observations. All the methods are discussed below in detail :

Method of inspection

This method is very simple. The value of Mode or modal class is found out just by inspecting the distribution and locating the value against which maximum frequency occurs.

But while using this method, one must see that the difference between the maximum frequency and the next maximum frequency is not very small. In that case, the method of grouping is considered more appropriate than the method of inspection.

Method of grouping

Under this method, two tables are prepared to indentify the modal value. The first table prepared is called the 'grouping table' and the second table is called "analysis table"

Grouping Table

A grouping table normally has six columns. The contents of the columns are given below :

Column I : It contains the orginal frequencies. The maximum frequecy in the coloumn is under-lined or circled

Column II : In this column the frequencies in Column-I are combined in twos i.e 1 and 2, 3 and 4, 5 and 6 and so on. The maximum frequency so combined is under-lined or circled.

Column III : Here the 1st frequency of Column-I is left and others are combined in twos as in Column-II, again the highest value in this column is marked.

Column IV : In this column the frequencies of Column-I are combined in threes i.e. 1,2 and 3, 4,5 and 6; 7,8 and 9 and so on and again the highest value is marked.

Column V : The first frequency of Column-I is left and the remaining frequencies are grouped in 3s as in Column-IV. The highest value in the column is marked, as in other columns.

Column VI : In this column first two frequencies of Column-I are left and the other frequencies are grouped in 3s (threes) i.e 3, 4 and 5, 6, 7 and 8, 9, 10 and II and so on. Again the highest value in the column is identified.

It can be noted here that when the number of items are more, frequencies may be grouped in fours and fives in the same manner as in twos and threes.

Analysis Table

After preparation of the grouping table, an analysis table is prepared. In this table, column numbers appear as row headings and probable value of Mode i.e X values appear as column headings. The table is prepared in the following format :

ANALYSIS TABLE

X → / Col No. ↓	X_1	X_2	X_3	Xn
Col No.I				
Col No.II				
Col No.III				
Col No.IV				
Col No.V				
Col No.VI				
Total				

The values against which frequences are marked 'maximum', in grouping table in each column, are identified by putting a vertical bar mark in the relevant box of the analysis table. The total row records the number of vertical bars against each value of 'X'. The value of X representing highest number of vertical bars in the total column, is considered as modal value or modal class.

The procedure for preparing grouping and analysis table can be better understood from the following practical example :

Example 18

Find out Mode from the following distribution :

Size of item	2	3	4	5	6	7	8	9	10	11	12	13
Frequency	3	8	10	12	16	14	10	8	17	5	4	1

Solution

<u>**GROUPING TABLE**</u>

Size	Col I	Col II	Col III	Col IV	Col V	Col VI
2	3	11		21		
3	8		18		30	
4	10	22				38
5	12		<u>28</u>	42		
6	16	<u>30</u>			<u>40</u>	
7	14		24			32
8	10	18		35		
9	8		25		30	
10	<u>17</u>	22				26
11	5		9	10		
12	4	5				
13	1					

x→ / Col No.↓	2	3	4	5	6	7	8	9	10	11	12	13
Col No.I									1			
Col No.II					1	1						
Col No.III				1	1							
Col No.IV				1	1	1						
Col No.V					1	1	1					
Col No.VI			1	1	1							
Total			1	3	5	3	1		1			

In analysis table the total corresponding to value of variable (x) = 5 is highest. Hence 6 is the modal value.

In case of continuous frequency distribution, by inspection method or by grouping method only the modal class is found out.

For finding the value of Mode from the Modal class, the following interpolation formula. must be used :

$$Z = L_1 + \frac{f_1 - f_0}{2f_1 - fo - f_2}(L_2 - L_1) \qquad \text{or } Z = L_1 + \frac{\Delta_1}{\Delta_1 + \Delta_2} x\, i$$

Where Z = Mode

L_1 = Lower limit of modol class

L_2 = Upper limit of modal class

f_1 = Frequency of modal class

f_o = Frequency of the class preceding the modal class

f_2 = Frequency of the class succeeding the modal class.

Δ_1 = $(f_1 - f_0)$ $\Delta_2 = (f_1 - f_2)$ $i = (L_2 - L_1)$

Note-I : If the 1st class is the modal class then *fo* is taken as 'o'. Similarly if the last class is modal class f_2 is taken as zero.

Note-II : If the modal value lies outside the model class, the following formula is used to calculate Mode.

$$Z = L_1 + \frac{f_2}{f_0 + f_2} \times (L_2 - L_1)$$

Example 19

Find the Mode from the following data.

Marks	0-10	10-20	20-30	30-40	40-50	50-60	60-70	70-80	80-90	90-100
Number of Students	3	5	7	10	12	15	12	6	2	8

Solution

Since the series is regular and there is significant difference between the maximum and next maximum frequencies we may not prepare a grouping table. By inspection we can say Mode lies in the class 50-60

Thus, $Z = L_1 + \frac{f_1 - f_0}{2f_1 - fo - f_2}(L_2 - L_1)$

Where $L_1 = 50$, $L_2 = 60$, $f_1 = 15$, $f_0 = 12$, $f_2 = 12$

Putting the values

$$Z = 50 + \frac{15 - 12}{2 \times 15 - 12 - 12} \times (60 - 50)$$

$$= 50 + \frac{3}{6} \times 10$$

$$= 50 + 5$$

$$= 55$$

Graphic Method

Graphic method of determing mode is suitable for a continuous frequency distribution. The basic condition for using this method is that the class intervals must be equal.

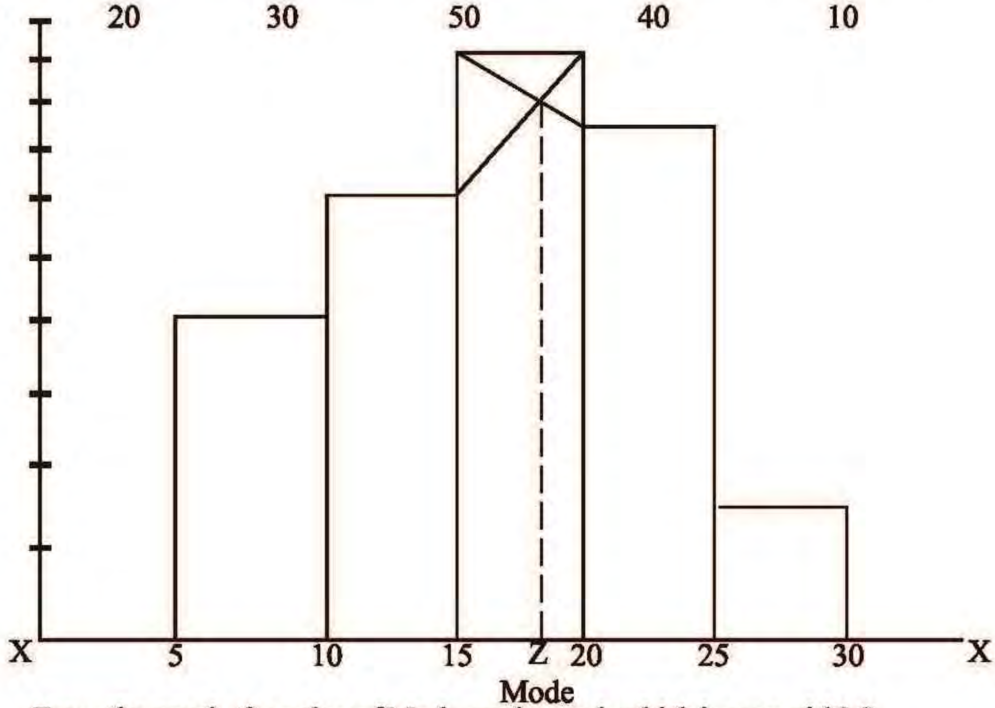
Steps for locating Mode

- Draw a histogram of the given data.
- Select the rectangle representing the highest frequency.
- Draw two lines diagonally from the upper corners of the bar to the upper corners of the two adjacent bars on either side of the rectangle.
- From the point of intersection of the two diagonals, draw a perpendicular to x-axis representing the variable X.
- Read the value of the series.

Example 20

Determine Mode from following data using graphical method :

X	5-10	10-15	15-20	20-25	25-30
Fy	20	30	50	40	10



From the graph, the value of Mode can be read, which is around 18.3.

Method of empirical relation.

When it is not possible to determine Mode, by all other methods discussed above, then only this method of empirical relation shall be used to find out the value of Mode. Normally when a distribution is a bi-modal, tri-modal or multi-modal one, in otherwords Mode is all defined, then this method is used to get a definite value of Mode. According to Karl Pearson when a series is moderately asymmetrical, then its Mean. Median and Mode bears a mathematical relation as follows.

Mode = 3 Median - 2 Mean. or

$$\text{Median} = \frac{\text{Mode} + 2\,\text{Mean}}{3}$$

$$\text{Mean} = \frac{3\text{Median} - \text{Mode}}{2}$$

Thus, from the value of Median and Mean the value of Mode can be determined by using this relationship.

Merit, Demerits and Uses of Mode

Merits

Some of the merits of Mode can be mentioned as follows:

1. It is very easy to understand.
2. In some cases it can be located by mere inspection
3. Graphical location of Mode is possible.
4. It is not affected by extreme values.
5. For calculation of Mode in case of open ended classification, it is not required to estimate both the extreme class limits.
6. It gives the most representative value from within the series.

Demerits

The main demerits of Mode are :

1. It is not rigidly defined. There are different formula for its calculations, which some times give different results.

2. It is not based on all the observations.
3. It is not capable of further mathematical treatment.
4. It is significantly affected by fluctuations in sampling.
5. When the class intervals of a series are un-equal, it is not possible to find out its Mode.

Uses of Mode

Mode as a measure of central tendencey, has the following specific uses :

1. Un-like other averages, it is capable of studying qualitative data as its determination is based on the frequencies and not on the values of the variable. Therefore it is used to study qualitative phenomenon.
2. It is very useful for businessmen as it helps them in studying trend in fashion, and deciding the quantities of different goods to be produced..

Practical Problems

Example 21

Determine the missing frequencies when Mode is 36 and total frequency is 30 :

X	10-20	20-30	30-40	40-50	50-60
f	-	5	12	-	2

Solution

The value of Mode is 36. Hence the modal class is 30-40

$$Z = L_1 + \frac{f_1 - f_0}{2f_1 - f_0 - f_2} \times (L_1 - L_2)$$

Given $L_1 = 30$ $L_2 = 40$ f_1- 12 f_0=5

Let $f_2 = x$

Substituting the values

$$36 = 30 + \frac{12-5}{2\times12-5-x}\times(40-30)$$

$$36 = 30 + \frac{7}{19 - x}\times10$$

$$6 = \frac{70}{19 - x}$$

6 (19-x) = 70

114 - 6x = 70

6x = 44

x = 44/6 = 7.33, hence f_2 = 7.33

As N=30 the other missing frequency is

30- (5+12+733+2) = 3.67

Example 22

Calculate Mode from the following data.

More than	0	10	20	30	40	50	60	70	80
frequencies	100	96	94	76	54	33	14	4	1

Solution

As the frequencies are given in comulative form, they must be decumulated first, as follows :

X	f	cf
0-10	04	04
10-20	02	06
20-30	18	24
30-40	22	46
40-50	21	67
50-60	19	86
60-70	10	96
70-80	03	99
80-90	01	100

As the highest frequency 22 is very close to the next highest frequency, it is necessary to find out modal class by using grouping and analysis table in stead of simple inspection method.

GROUPING TABLE

	Col I	Col II	Col III	Col IV	Col V	Col VI
0-10	4	6		24		
10-20	2		20		42	
20-30	18	40				61
30-40	22		43	62		
40-50	21	40			50	
50-60	19		29			32
60-70	10	13		14		
70-80	3		4			
80-90	1					

Analysis Table

× → / Col No.1	0-10	10-20	20-30	30-40	40-50	50-60	70-80	80-90
Col No.I				1				
Col No.II			1	1	1	1		
Col No.III				1	1			
Col No.IV				1	1	1		
Col No.V					1	1	1	
Col No.VI			1	1	1			
Total			2	5	5	3	1	

From the analysis table it has been found out that there are two classes i.e. 30-40 and 40-50 are having equal totals of 5 each and thus mode can not be found out by the grouping method. The empirical relation method should be used to find out Mode. Mean and Median of the distribution should be found out first.

X	M	f	(x-A) d	$\left(\frac{M-A}{10}\right)$ d_1	fd_1	c.f
0-10	5	4	-40	-4	-16	4
10-20	15	2	-30	-3	-6	6
20-30	25	18	-20	-2	-36	24
30-40	35	22	-10	-1	-22	46
40-50	45	21	0	0	0	67
50-60	55	19	10	1	19	86
60-70	65	10	20	2	20	96
70-80	75	3	30	3	9	99
80-90	85	1	40	4	4	100
					$\Sigma fd_1 = -28$	

$$\overline{X} = A + \frac{\Sigma fd_1}{N} xC$$

Where $A = 45$, $N = 100$ $\Sigma fd_1 = -28$ $C = 10$

Putting the values

$$\overline{X} = 45 + \frac{-28}{100} x10$$

$= 45-2.8$

$= 42.2$

Median = The size of $\frac{N}{2}$ th item

$= \frac{100}{2} = 50\text{th item}$

∴ From the cumulative frequency it can be found out that 50th item lies in the class 40-50. Hence it is the Median class

$$\text{Median} = L_1 + \frac{L_2 - L_1}{f}(N/2 - C)$$

Where $L_1 = 40$, $L_2 = 50$, $f_1 = 21$ $N/2 = 50$ $C = 46$

$$\text{Median} = 40 + \frac{50-40}{21} \times (50-46)$$

$$= 40 \ \frac{10}{21} \times 4$$

$= 40 + 1.9$

$= 41.9$

∴ Mode $= 3 \text{ Median} - 2 \text{ Mean}$

$= 3 \times 41.9 - 2 \times 42.2$

$= 125.7 - 84.4$

$= 41.3$

Example 23

Calculate Mode from the following data.

(X) Mid values	-	150	250	350	450	550
Frequencies	=	27	9	7	3	2

Solution

Since we are given mid values, first we have to determine the lower and upper limits of the classes.

Lower limits of a class is

Mid value – 1/2 (Difference between two mid values)

Upper limit – Mid values + 1/2 (Difference between two mid values)

Here the difference between two mid values are 100 (250 – 150)

Hence the class intervals are as follows :

X	frequencies
100 - 200	27
200 - 300	9
300 - 400	7
400 - 500	3
500 - 600	2

By inspection the modal class is 100 - 200

$$\text{Mode} = L_1 + \frac{f_1 - f_0}{2f_1 - f_0 - f_2} \times i$$

$$= 100 + \frac{27-0}{2\times 27 - 0 - 9} \times 100 \quad (\text{Here } f_0 = 0)$$

$$= 100 + \frac{27}{54-9} \times 100$$

$$= 100 + \frac{27}{45} \times 100$$

$$= 100 + \frac{2700}{45}$$

$= 100 + 60$

$= 160$

Example 24

The Median and Mode of the following wage distribution of 230 persons are known to be Rs. 335 and Rs. 340 respectively. Three frequency values from the table are missing. Find out the missing frequencies.

Wages (Rs.)	0-100	100-200	200-300	300-400	400-500	500-600	600-700
No. of Persons	4	16	60	–	–	–	4

Solution

Let the missing frequencies be x, y and z.

x	f	c.f.
0-100	4	4
100-200	16	20
200-300	60	80
300-400	x	80+x
400-500	y	80+x+y
500-600	z	80+x+y+z
600-700	4	84+x+y+z

Now N $= 84 + x + y + z$

$230 = 84 + x + y + z$

$\therefore x + y + z = 230 - 84 = 146.$

Since the Median and Mode of the series are 330 and 335 the Median as well as the Modal class is 300 – 400.

Median $=$ Size of $N/2$ th item

$=$ Size of $230/2$ th i.e. 115th item.

$$\text{Median} = L_1 + \frac{L_2 - L_1}{f}(N/2 - C)$$

Putting the values

$$\text{Median} = 300 + \frac{400-300}{x}(115-80)$$

$$= 300 + \frac{100}{x} \times 35$$

$$= 300 + \frac{3500}{x}$$

$$\text{or,} \quad 335 = 300 + \frac{3500}{x}$$

or, $35x = 3500$

or, $x = 100$

$\therefore$ The frequency of Modal class is 100.

$$\text{Mode} = L_1 + \frac{f_1 - f_0}{2f_1 - f_0 - f_2} \times (L_2 - L_1)$$

Where $L_1 = 300$, $L_2 = 400$ $f_1=100$, $f_0=60$, $f_2=y$

Now putting the values

$$340 = 300 + \frac{100-60}{2\times100-60-y} \times (400-300)$$

$$= 300 + \frac{40}{140-y} \times 100$$

$$340\text{-}300 = \frac{4000}{140-y}$$

$$40(140-y) = 4000$$

$$5600-40y = 4000$$

$$40y = 1600$$

$$y = \frac{1600}{40}$$

$$= 40$$

$x+y+z = 146$

where $x = 100$, $y = 40$, hence z

$$= 146 - (100 + 40) = 6$$

$\therefore$ The three missing frequencies are respectively 100, 40 and 6.

Example 25

Fiom the data given below, find the Mean, Median and Mode.

Variable	10-13	13-16	16-19	19-22	22-25	25-28	28-31	31-34	34-37	37-40
Frequence	8	15	27	51	50	54	36	18	9	7

Solution

Variable X	Frequency f	M.V m	d	d_1	fd_1	c.f
10-13	8	11.5	-12	-4	-32	8
13-116	15	14.5	- 9	-3	-45	23
16-19	27	17.5	-6	-2	-54	50
19-22	51	20.5	-3	-1	-51	101
22-25	50	23.5	0	0	0	151
25-28	54	26.5	+3	1	54	205
28-31	36	29.5	+6	2	72	241
31-34	18	32.5	+9	3	54	259
34-37	9	35.5	+12	4	36	268
37-40	7	38.5	+15	5	35	275
	ΣfN=275				**$\Sigma fd_1 = 69$**	

$$\overline{X} = A + \frac{\sum fd_1}{N} X\ i$$

Where $A = 23.5, N=275$ $C=3$ $\Sigma fd_1 = 69$

Putting the values

$$\overline{X} = 23.5 + \frac{69}{275} \times 3$$

$$= 23.5 + \frac{207}{275} = 23.5 + 0.75 = 24.25$$

Median = Size of N/2 th item i.e $\frac{275}{2} = 137.5$th item

∴ Median class is 22-25.

$$\text{Median} = L_1 + \frac{L_2 - L_1}{f}\left(N/_2 - C\right)$$ (where $L_1 = 22$, $L_2 = 25$, $f = 50$, $C = 101$, $N/2 = 137.5$)

$$= 22 + \frac{25-22}{50}(137.5 - 101)$$

$$= 22 + \frac{3}{50} \times 36.5 = 22 + 2.19 = 24.19\,.$$

Mode

Since the difference between the highest and next highest frequencies are very small to find out the modal class grouping and analysis table be prepared as follows.

Grouping Table

	Col I	Col II	Col III	Col IV	Col V	Col VI
10-13	8	23		50		
13-16	15		42		93	
16-19	27	78				128
19-22	51		101	155		
22-25	50	104			140	
25-28	54		90			108
28-31	36	54		63		
31-34	18		27		34	
34-37	9	16				
37-40	7					

Analysis Table

x→ / Col No.↓	10-13	13-16	16-19	19-22	22-25	25-28	28-31	31-34	34-37	37-40
Col No.I						1				
Col No.II					1	1				
Col No.III				1	1					
Col No.IV				1	1	1				
Col No.V					1	1	1			
Col No.VI			1	1	1					
Total			1	3	5	4	1			

As the maximum frequency occurs in the class 22-25 as per the table it is the modal class

Applying the formula

$$\text{Mode} = L_1 + \frac{f_1 - fo}{2f_1 - fo - f_2} \times i$$

Where $L_1 = 22$ $f_1 = 50$ $f_o = 51$, $f_2 = 54$, $i = 3$

$$= 22 + \frac{50-51}{2\times 50 - 51 - 54} \times 3$$

$$= 22 + \frac{-1}{-5} \times 3 = 22 + .6 = 22.6$$

$\therefore$ Mean = 24.25 Median = 24.19 Mode = 22.60

In an asymmetrical distribution as above Mean is greater than that of Median, which is greater that of Mode.

9.5 Choice of a suitable average :

As has been discussed in the previous chapters, all the measures of central tendency i.e. Mean, Median, Mode, Geometric Mean, Harmonic Mean aim at finding a central value or representative value from the data. But all these measures cannot be used indiscriminately. For best possible results, an appropriate choice of average must be made by the analyst. Appropriateness of an average can be judged from the following factors :

* Level of measurement of data
* Shape of the distribution
* Stability of the measure

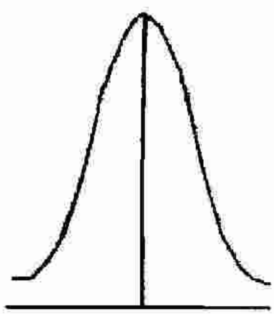
Level of measurement of data

There are four levels of measurement of data : They are **nominal, ordinal, ratio** and **interval.** At nominal level the observations can be distinguished but cannot be arranged in any order. For example different brands of a product or different colours of automobiles, different racial, social or religious groups etc. At ordinal level the observations can be arranged in ascending or descending order but no arithmetic operations are possible. For example while describing the quality of ice-cream, one may say it is very good, satisfactory, bad or very bad. Similarly when the performance is ranked like excellent, very good, good, satisfactory, bad and very bad, then such data are ordinal in nature. Such data can be ranked or graded but can not be mathematically manipulated. At interval level it is assumed that a given interval or scale measures the same amount of difference irrespective of where the interval appears. For example temperature difference between 10°C to 20°C is same as difference between 40°C and 50°C. At ratio level the data maintain mathematical proportion. For example 10 kg. is just the double of 5 kg and there is an absolute value '0'. But we cannot say that temperature at 10°C is just double of the temperature at 5°C and 0°C means there is no temperature at all.

Basing on the data type discussed above now we can say that in case of nominal data only we can use Mode, for ordinal data, Median and Mode can be used and for ratio and interval data, all the measures can be used.

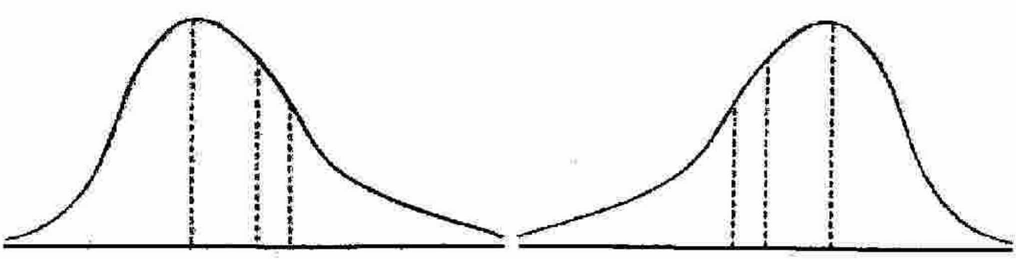
Shape of the distribution

In case of symmetric distribution the Mean, Median and Mode are equal. But in case of a skewed or asymmetric distribution all the measures of central value are different. For positively skewed distribution Mode is the smallest and Median lies between Mode and Mean. When the distribution is negatively skewed Mode is the highest and Median lies between Mean and Mode. Thus under both the circumstances, Median lies in the centre of the distribution and hence is the average of choice for a skewed distribution.



$M_o = M_e = \overline{X}$

(a)



(b) (c)

Stability of the measure :

Mean is generally more stable than Median and Mode. When different samples are collected from a population and their Mean. Median and Mode are calculated the Means of the samples will be more in agreement than Medians and Modes. Therefore, Mean is considered as a more reliable measure than others and is the most popular average of choice.

9.6 Comparison among Mean, Median and Mode

A comparison may be made among Mean, Median and Mode taking into consideration certain bases as follows :

	Basis	Mean	Median	Mode
1.	Average type	It is a mathematical average.	It is a positional average.	It is a positional average.
2.	Calculation	It is based on all the observation	It is the value lying in the middle of the distribution.	It is the most frequently occuring value.
3.	Mathematical treatment	It is capable of further mathematical treatment.	It is not capable of mathematical treatment.	It is not capable of mathematical treatment.
4.	Items	It takes into consideration of all the items.	It does not consider all the items.	Not all the items are required.
5.	Method of calculation	It does not require arrangement of values	Arranging of the values is essential.	Arranging of the values is essential.
6.	Extreme values	It is affected by presence of extreme values.	It is not affected by extreme values.	It is not affected by extreme values.
7.	Reliability	It is stable, more reliable and most popular average	Only used in appropriate cases.	Only used in appropriate cases and not popular.
8.	Result	There is only one mean.	There is only one median.	There may be one or more than one mode in a series.

9.7 QUESTIONS

No.1 Choose and write the correct answer from the alternatives given below :

(a) Sum of deviations of the values of variables ignoring '+' and '–' sign is minimum when taken from :

(i) Mean (iii) Harmonic Mean;
(ii) Median (iv) Mode

(b) A frequency distribution having an unique modal value is a :

(i) Bimodal series (iii) Multimodal series
(ii) Unimodal series (iv) Trimodal series

(c) Median is a

(i) Mathematical Average (iii) Positional Average
(ii) Arithmetic Average (iv) Moving Average

(d) A frequency distribution has

(i) 10 Deciles (iii) 9 Deciles
(ii) 100 Deciles (iv) 99 Deciles

(e) Quartile divides a series into

(i) 3 equal parts (iii) 10 equal parts
(ii) 4 equal parts (iv) 9 equal parts

(f) The second quartile is equal to

(i) Median (iii) Mode
(ii) Mean (iv) G.M.

(g) In a moderately asymmetric distribution, Mode is equal to

(i) 3 Median – 2 Mean (iii) 2 Mean – 3 Median
(ii) 3 Mean – 2 Median (iv) 2 Median – 2 Mean

(h) In a symmetric distribution the 1st and 3rd quartiles are equidistant from

(i) Mean (iii) Mode
(ii) Median (iv) H.M.

(i) The 5th Decile is same as

(i) 1st quartile (iii) 50th percentile

(ii) 3rd quartile (iv) 10th percentile

(j) A series should be arranged either in ascending or descnding order for calculation of :

(i) Mean (iii) Geometric Mean

(ii) Median (iv) H.M

No.2 Correct the underlined portions of the following sentences :

(i) Mode is a mathematical average.

(ii) Second quartile and Mean of a series are equal.

(iii) Mode is the value that has the smallest frequency.

(iv) Median and Mean are not affected by the presence of extreme values in the series.

(v) Deciles divide the series into 100 equal parts.

(vi) The 50th percentile and Mode of a frequency distribution are same.

(vii) The value of Mode and Mean can be determined graphically.

(viii) Median is a derived measure of central tendency.

(ix) For calculation of Mean the values in the variable must be arranged according to their magnitude.

(x) In a symmetric distribution the1st quartlile and 3rd quartile are equidistant from the Mode.

No.3 Fill in the blanks :

(i) Median is a ________________ average.

(ii) Mode is the value that has the greatest __________.

(iii) Percentiles divide the series into ________ equal parts.

(iv) In a moderately skewed distribution if the value of Mean and Median are 8 and 9 the value of Mode will be __________.

(v) ________ is the most ill defined average.

(vi) When Mean, Median and Mode of a series are equal the series is called ______ series.

(vii) In a positively skewed distribution Mode is ______ than Mean.

(viii) Median is more suitable for _________ clsssification.

(ix) A series having 2 modal values is called _______ series.

(x) Second quartile and __________ decile of a series are equal.

4. Express the following in one word / term each :

(i) Series having an unique mode

(ii) A series whose Mean, Median and Mode are equal

(iii) A value that divides a series into some equal parts.

(iv) An average whose value is simply determined by location.

(v) Values that divide a frequency distribution into ten equal parts.

(vi) A series having more than one modal value.

(vii) A series whose Mean, Median and Mode are not equal.

(viii) A partition value that divides the series into two equal parts.

(ix) Value that most frequently occurs in a series.

(x) Method used to locate the value of Median / Mode from a class interval.

5. Answer the following in one sentence each :-

(i) What is a symmetric distribution ?

(ii) What do you mean by a bi-modal series ?

(iii) What is a partition value ?

(iv) What is a percentile ?

(v) What is a positional average ?

(vi) Define Mode.

(vii) What is Quartile ?

ANSWER

No.1. (a) ii, (b) ii, (c) iii, (d) iii, (e) ii, (f) i, (g) i, (h) iii, (i) iii, (j) ii.

No.2 (i) positional (ii) Median (iii) Highest (iv) Mode (v) Percentile (vi) Median

(vii) Median (viii) Positional (ix) Mdeian (x) Median.

No.3 (i) positional, (ii) Frequency (iii) Hundred (iv) ii (v) Mode (vi) Symmetric, (vii) More, (viii) Open ended (ix) Bimdal (x) 5th decile

No.4 (i) Unimodal (ii) Symmetric (iii) Partition value (iv) Positional (v) Deciles (vi) multimodal (vii) Assymmetrical (viii) Mdeian (ix) Mode (x) Interpolation.

No. 5

(i) A frequency distribution whose Mean Median and Mode are equal, is called a symmetric distribution.

(ii) A series is said to be bimodal if it has two Modes.

(iii) A value thai divides a series into some equal parts like two four, ten e fc is called a partiton value.

(iv) Percentile is a partition value that divides a series into hundred equal parts.

(v) An average that is located simply by identifying its position is called a positional average.

(vi) Mode or modal value is that value in a series of observations which occurs with the greatest frequency.

(vi) Quartile is a partition value that divides a series into 4 equal parts.

No. 6. Answer the following in 30 word.

a) What is a partition value ?

b) What do you mean by a positional measure of central tendency ?

c) State the difference between mathematical and positional measure of central tendency.

d) State the specific uses of Median .

e) Define Mode

f) Write any two uses of Mode.

g) What Points should be remembered while calculating Mode ?

i) Write any two limitations of Median.

j) Write any two important merits of Mode.

No. 7. Answer the following within 50 words.

a) State the difference between Mean and Median.

b) Write various uses of Mode.

c) Explain the empirical relationship between Mean, Median and Mode.

d) Explain the limitations of Mode.

e) Explain the steps for finding out Mode.

f) Write the uses of partition values.

g) State the demerits of Median.

h) Note down the factors to be considered for chosing an appropriate measure of central tendency.

i) List the advantages of Median.

Long Answer Type.

8) Define Median. Explain its merits, demerits and uses.

9) Explain the difference between Mean, Median and Mode.

10) What is Mode ? state its merits, demerits and uses.

11) What is a partition value ? Briefly explain different partition values and their uses.

12) Write explainatory notes on the following.

(i) Quartiles

(ii) Deciles

(iii) Percentiles

13. Explain the empirical relationship among Mean. Median and Mode with suitable example.

14. Explain how you will choose an average under different situations.

15. Define Mode. Explain the areas of its applicability and discuss its relative advantages and disadvantages.

16. Explain different methods of locating Mode.

Problems :

17. Find the Mean and Median from the data given below.

Marks	0-10	10-20	20-30	30-40	40-50	50-60
Number of students	12	18	27	20	15	8

18. From the following data find out the value of Median and upper Quartile :

Marks More than	70	60	50	40	30	20
Number of students	7	28	40	40	63	75

19. Findout the missing frequency from the following data. When Median = 50 and number of items is 100.

X	0-20	20-40	40-60	60-80	80-100
f	14	?	27	?	15

20. One hundred and twenty students appeared for certain test and the following marks distributions are obtained.

Marks	0-20	20-40	40-60	60-80	80-100
Students Find	10	30	36	30	14

(i) The percentage of students who failed, if 35 marks are required for passing.

21. Calculate Median from the following data.

Weight in gms.	410-419	420-429	430-439	440-449	450-459	460-469	470-479
No. of apples	14	20	42	54	45	18	7

22. From the following data calculate the 1st Quartile the Median and the 3rd Quartile.

Height (cms)	141-150	151-160	161-170	171-180	181-190
Frequency	5	16	56	19	4

23. Find the 45th and 60the percentile from the following data:

X	20-23	25-30	30-35	35-40	40-45	45-50
f	10	20	20	15	15	20

24. Find the upper Quartile, 7th Decile and 65th percentile from the following data.

X	30-40	40-50	50-60	60-70	70-80	80-90	90-100
f	1	3	11	21	43	32	9

25. Following is the distribution of Marks in statistics obtained by 50 students :

More than :	0	10	20	30	40	50
Number of Students :	50	46	40	20	10	3

If 60% of the students passed the test, find the minimum marks obtained by a pass student.

26. Find D_2, P_3 and P_{90} from the following data.

X	10	10-20	20-40	40-60	60-80	80 and above
f	8	10	222	25	10	5

27. Given the following distribution of Income :

Income (in lakh)	0-1	1-20	2-3	3-4	4-5	5-6	6-7	7-8
No. of Filiies	4	6	10	15	8	5	4	2

Find out (i) The Highest income among the poorest 25%.

(ii) The lowest income among the richest 30%.

28. From the following data calculate the percentage of workers getting wages :

(i) More than 44

(ii) Between 22 and 58

Wages in ₹	0-10	10-20	20-30	30-40	40-50	50-60	60-70	70-80
Number of works	20	45	85	160	70	55	35	30

29. From the following data obtain the value of Mode.

X	0-10	10-20	20-30	30-40	40-50
f	8	15	15	7	2

30. Determine the missing frequencies from the following information when Median and Mode are known to be 33.5 and 34 respectively and number of observations 100.

X	0-10	10-20	20-30	30-40	40-50	50-40	60-70
f	4	16	?	?	?	6	4

31. From the following data determine the value of Mode :

Mid value	5	15	25	35	45	55	75	85
Frequency	12	35	105	148	141	125	136	98

32. Calculate Mode from the following data.

Marks below	10	20	30	40	50	60	70	80	90
No of Students	4	6	14	46	67	86	96	99	100

33. The Mean, Median and Mode of a set of 75 items were ascertained to be 27,29 and 34 respectively. Afterwards it was noticed that an item was taken as 44 in stead of 53. Find the correct value of Mean, Median and Mode.

34. Calculate Median and Mode by grouping method from the follwing series :

Mid value

X	5	10	15	20	25	30	35	40	45
frequency	7	13	19	24	32	28	17	8	6

35. Calculate Median and Mode from the following :

Size	6-10	11-15	16-20	21-25	26-30
Frequency	20	30	50	40	10

❖❖❖

CHAPTER-10

MEASURES OF DISPERSION-Positional Dispersion

STRUCTURE

10.0 INTRODUCTION

The various measures of central tendency explain about the concentration of data on one point. They give only one figure to represent the entire series of data. The entire data in the series concentrate/cluster around a central value, called average. But the average alone is not adequate to describe or represent the given series of data. The averages like Mean, Median, Mode, Harmonic Mean, Geometric Mean do not reveal how the values of observations are dispersed or scattered on both sides of the central value. The composition of the series is not fully explained by a value of central tendency only. An average is not the fully representative of a given series of data. Sometimes the individual values of a series have significant variations in relation to the average. The study of central tendency is to be supplemented by the study of dispersion or variation for giving an idea about the spread of values which is explained below.

Daily wages earned by three workers in one factory in a week:

Worker	MON (₹)	TUE (₹)	WED (₹)	THU (₹)	FRY (₹)	SAT (₹)	SUN (₹)	Total (₹)
A	500	500	500	500	500	500	500	3,500
B	520	480	500	570	490	475	465	3,500
C	100	300	400	500	1.000	650	550	3,500

Worker A gets avarage daily wage of ₹ 500 and there is no sign of any variation. The wage data are not scattered at all. Worker B also gets an average daily wage of ₹ 500 and the daily wage scatters modetately. The degree of variation of daily wage is very less for worker B. Worker C though gets an average daily wage of ₹ 500, his daily wage has a significant variation. The average daily wage of worker C does not represent the daily wage in the particular week.

Thus we observe that although all the three workers have the same average, yet they widely differ from one another in terms of the composition of weekly wages. When the extent of variation/scatterness of individual values from the central value in a series of data is large or significant, then the measure of central tendency or average cannot be representative of the distribution. The mean wage may not be a so impotant characteristic of income distribution, if the variations of daily wage from the mean wage is wide. For the students of social sciences, variation in the income is more important than that of the mean income.

10.1 MEANINGAND DEFINITION

The variation of observations / items in a series of distribution of data from its central value, is called dispersion. It is the variability in the size of items in a series. Some important definitions of Dispersion are given below.

1. "Dispersion is the measure of the variations of the items."

   A.L.Bowley

2. " The degree to which numerical data tend to spread about an average value is called the variation or dispersion of the data." - Spiegel.

3. " Dispersion or spread is the degree of the scatter or variation of the variable about a central value."

   D.C. Brooks and W.F.L. Dick

4. "Measures of variability are usually used to indicate how tightly bunched the sample values are, around the mean."

   Dyckman and Thomas.

5. " Dispersion is a measure of the extent to which the individual items vary."

   Prof. L.R. Corner

6. " The measurement of the scatterness of the mass of figures in a series about an average is called measure of variation or dispersion."

   Simpson & Kafka.

It is clear from the above discussions and definitions that dispersion is the scatter or spread or variation of data from some central value. When the diviations of the items from their average (i.e., Mean, Median or Mode) are found out, the average of the deviations represents the series. For this, measures of dispersion are known as the averages of the second order and Mean, Median or Mode etc. are called the averages of the first order.

10.2 OBJECTIVES OF DISPERSION

The objectives of the measures of dispersion are as follows:

(i) Determining the Reliability of an average:

Measures of dispersion determine the reliability of an average. It tests whether an average represents the series or not. When dispersion is small, the average closely represents the series and it is reliable. The average is a typical representative of the universe. But when the dispersion is large, the average does not represent the series and is not a typical representative. Such an average may not be reliable.

(ii) Determining the nature and causes of variation:

Another objective of measuring dispersion is to determine the nature and causes of variation of the mass data. Then only it becomes easy to control the variability and take remedial measures. For example, in the fields of social science, to control the unequal distribution of wealth and income, it is important to study the variation or scatter in wealth and income of the society.

(iii) Comparing the variability of two or more series of data:

Measures of dispersion enable a comparison of two or more series with the help of relative measures dispersion. The higher the degree of variability, the lesser is consistency or uniformity in the values of the variables. Conversely the lower the degree of variability, the higher is the degree of consistency of data.

Comparative study of variability is very fruitful and useful in the fields like measuring profitability of companies, fluctuation in the value of shares, studies relating to demand and supply, price and demand etc.

(iv) Facilitating the use of other statistical measures:

Measures of dispersion facilitate further statistical analysis. It is the basis to study the pros and cons of series of data or universe with the help of many powerful analytical tools of statistics. Such tools or parameters of statistics are regression analysis, correlation analysis, testing of hypothesis, analysis of variance, statistical quality control etc.

10.3 CHARACTERISTICS OF A GOOD MEASURE OF DISPERSION

The characteristics of a good measure of dispersion are the same as those of the characteristics of a good measure of central tendency. These are:

(i) **Simple to understand:** A good measure of dispersion should be designed to simplify the complexity of data. It should be readily understood by all, even by a layman.

(ii) **Easy to compute:** A good measure of dispersion should not only be simple to understand but also be easy to compute. The ease of computation should not be at the cost of other characteristics.

(iii) **Based on all observations:** Any good measure of dispersion should be based on all the observations / items of universe / sample under study. If any of the items is dropped, then the value of the dispersion will be changed and findings will be misleading.

(iv) **Not affected by extreme observations:** Although each item of the series influences the value of dispersion, none of the items should have undue or uneven influence. When one of the items in the universe is very small or very large, it will affect the dispersion unduly. In other words, extreme items may distort the value of dispersion and reduce its utility.

(v) **Rigidly defined :** A good measure of dispersion should be rigidly defined. It should have one and only one interpretation / explanation. It should be preferably defined by an algebraic formula. The dispersion should be free from any personal prejudice and bias of the investigator; otherwise, the result will be futile and ambiguous.

(vi) <u>Capable of further algebraic treatment and analysis:</u>

An ideal measure of dispersion should be capable of further algebraic treatment and analysis. One should prefer to have a dispersion that could be used for advance statistcal analysis. For example, combined mean deviation, combined standard deviation will enhance the utility of simple measures of dispersion.

(vii) <u>Sampling stability:</u> Last but not the least, the sample taken for study should give a stable value of dispersion. A good measure of dispersion should not be affected by sampling fluctuations. For example, if we take a number of samples from the students of a college, we should expect the same value. It dose not mean that there can be no difference in the values of dispersion of different samples. There will be difference, but the difference should be very small or insignificant.

10.4 MEASURES OF DISPERSION (ABSOLUTE AND RELATIVE) :

The dispersion of a series of distribution can be studied by different measures such as range, interquartile range, quartile deviation, mean deviation, standard deviation, coefficient of range, coefficient of quartile deviation, coefficient of mean deviation and coefficient of standard deviation (variation). These above measures of dispersion are classified on the basis of : (i) **Unit of measurement** as absolute and relative, and on the basis of (ii) **Nature of calculation** as positional and mathematical.

MEASURES OF DISPERSION

(On the basis of **Unit of measurement**)

Absolute Dispersion	Relative Dispersion
↓	↓
(Range,Inter-Quartile range Quartile Deviation, Mean Deviation, Standard Deviation, Lorenz curve.)	(Coefficient of Range, Coefficient of Quartile Deviation, Coefficient of Mean Deviation, Cofficient of Standard Deviation.)

Absolute Measures of Dispersion :- These are expressed in the same statistical unit in which the original data are given. For example, income and expenditure are measured in terms of rupee, dollar, pound etc; distance and length are measured in terms of metre, kilometre etc; weight is measured in terms of kilogram, ton etc; liquid is measured in terms of litre, kilolitre, gallon ctc. The absolute measures may be used to compare the variations in two series provided the data are in same units of measurement and have almost the same average value.

Relative measures of Dispersion :- These are expressed in terms of the ratio or percentage but not in terms of the units of original data. It is the ratio of an absolute measure of dispersion to an appropriate average. The relative measure is the coefficient of any absolute measure of dispersion. In a nutshell, a relative measure is known as the coefficient of variation. A relative measure is a prime number. It is free from any unit of measurement, whatever may be the unit of measurement of a series of distribution. The word coefficient means a number, that is independent of any unit of measurement. It should be kept in mind that while calculating a relative measure of dispersion, the average value is used as the base (denominator). Relative measures help in the comparison of data of two or more series with different units of measurements. For example, comparison of height and weight of students in a class, distance covered and fuel consumed while travelling by a motorcar etc.

Measures of Dispersion
(On the Basis of **Nature of calculation**)

Positional Dispersion
↓
(Range,Inter Quartile Range,
Quartile Deviation,
Decile, Percentile)

Mathematical Dispersion
↓
(Mean Deviation
Standard Deviation
Coefficient of variation)

10.5 POSITIONAL DISPERSION (Range, Inter Qrartile Range, and Quartile Deviation)

A measure of dispersion ascertained from a distribution on the basis of its nature of calculation is of two types; such as (i) positional dispersion and (ii) mathematical dispersion. A positional dispersion is computed by taking into account the physical spread / position of the variables in a series of distribution. Positional dispersions are the measures that describe the spread or scatter among values of variables. Range, Inter Quartile Range and Quartile Deviation are Positional measures of dispersion as these are based on the position of certain values of a variable in the series.

10.5.1 Range:

Range is the difference between the largest and the smallest observations of a distribution. It is the simplest method of studying dispersion. It is based on the physical location / position of the two extreme values of the series.

Range = The Largest value of a distribution – The Smallest value of a distribution

= L–S.

The range of a distribution depends only on the largest and the smallest observations of the series. The frequencies and class interval of various ungrouped or grouped distribution have no role in determining the range. Thus the range of grouped frequency distribution is the difference between the upper limit of the highest class interval and lower limit of the lowest class interval.

Merits :

1) It is easy to calculate and simple to understand.
2) It is not necessary to calculate any measure of central tendency to determine range.
3) Range is popularly used in our day to day life to know the variation in daily temperature, expenditure, rainfall etc.

Demerits :

1) It is not based on all the observations of a distribution. Only two extreme items are taken into account.
2) It is adversely affected by sampling fluctuations.
3) It is not possible to calculate Range in open-end series.
4) Frequencies are ignored in discrete and continuous series.
5) Range fails to compare the distributions in different units.

Coefficient of Range :

The relative measure of range is called the coefficient of range. Sometimes for the purpose of comparison, coefficient of range is calculated. It is obtained by dividing the range by the sum of extreme items. The coefficient of range is also called 'The Ratio of the Range' or 'The coefficient of the scatter.' Symbollically, cofficient

of range = $\frac{L-S}{L+S}$.

The following example would explain the use of the above formula :

Illustration 1.

From the weight of seven students of class x, find the range and coefficient of range of weights in a series of individual observation :

Number of students :	1	2	3	4	5	6	7
Weight (kg) :	45	40	42	47	50	52	55

Solution :

Here, Largest value = 55

Smallest value = 40.

$\therefore$ Range = L-S = 55kg - 40kg = 15 kg.

Coefficient of Range = $\frac{L-S}{L+S} = \frac{55-40}{55+40} = \frac{15}{95} = 0.158$

Illustration 2.

From the following data calcuate range and its coefficient in a discrete series.

Weight (in kg.)	40	42	45	47	50	52	55
Number of students	5	7	6	4	8	3	2

Solution :

Here, L = 55kg, S = 40kg

$\therefore$ Range = L-S = 55kg-40kg = 15 kg.

$$\text{Coefficient of Range} = \frac{L-S}{L+S} = \frac{55-40}{55+40} = \frac{15}{95} = 0.158.$$

When range and coefficient of range are computed in a disrete series, the frequencies of the variable are not taken into account.

Illustration 3

Calculate range and coefficient of range from the following data in a continuous series :

Marks :	0-10	10-20	20-30	30-40	40-50
Numbers of students :	5	7	10	8	20

Solution :

Here, Largest value (L) = 50

Smallest value (S) = 0

$\therefore$ Range = L-S = 50-0 = 50

$$\text{Coefficient of range} = \frac{L-S}{L+S} = \frac{50-0}{50+0} = \frac{50}{50} = 1.$$

When range and coefficient of range are computed in a continuous series, both the frequencies and class intervals of the variables are ignored.

Utility of Range :

In spite of all the serious limitations of range, it has a lot of utility in our day to day life. The uses and applications of range are described below :

(i) Analysis of fluctuations in stock and commodity markets - Range is helpful in analysing the change in the prices of stocks, shares, gold, exchange rates etc. In fact range is used when the variations are not much.

(ii) Forecasting weather conditions : The meterologist makes use of range to forecast the weather conditions like minimum and maximum temperature, rainfall, humidity etc.

(iii) Quality control : Range is used in the quality control of manufacturing products. Control charts are prepared under the method of statistical quality control (SQC) by using Range.

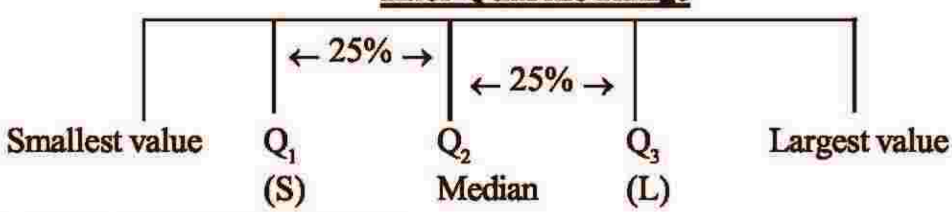
(iv) Useful in day to day life : Range is the most frequently used measure of dispersion in our every day life. Difference in temperature from day to day, difference in the daily pocket expenditure of collegiates, difference in the salary of employees of Government of Odisha etc. are the examples of range.

10.5.2 INTER-QUARTILE RANGE

Range as a measure of positional dispersion, is subject to certain limitations. The scatter within the largest and smallest item is not taken into account by range. It is suggested to discard the extreme items and in its place, a limited range is to be established. This is possible by taking the range of the quartilies. It includes only the middle fifty percent of the distribution. It is called the inter quartile range. It excludes one quartar (twenty-five percent) of the observations at the lower end and another quarter at the upper end of the distribution. In a nutshell, interquartile range is the difference between the third quartile and the first quartile.

Inter Quartile Range (IQR) = Q_3-Q_1

Inter Quartile Range



10.5.3 QUARTILE DEVIATION

Half of the difference between the first quartile and the third quartile is called semi-interquartile range or quartile deviation.

Quartile Deviation (QD) = $\frac{Q_3 - Q_1}{2}$

Quartile deviation represents the average difference of two extreme quartiles from the middle quartile (median / second quartile). In a symmetrical distribution, the two extreme quartiles. (Q_3 and Q_1) are equi-distant from the median (Median – Q_1 = Q_3 – Median). Thus the difference between Q_3 and Median or Median and Q_1, is taken as another measure of dispersion. The median $\pm$Q.D. covers exactly 50% of the distributions.

In real life, one seldom finds a series of perfectly symmetrical distribution of data in the field of social sciences. All the distributions of data under study are asymmetrical. As a result of which an asymmetrical distribution includes approximately fifty percent of the observations.

When quartile deviation is small, it denotes a high degree of uniformity and insignificent variation in the middle fifty percent of observations. But a large quartile deviation indicates the variation among the central items is significant.

Quartile deviation is an absolute measure of dispersion. For further and better statistical analysis of data, coefficient of quartile deviation is to be taken into consideration.

$$\text{Coefficient of Quartile Deviation} = \frac{\frac{Q_3 - Q_1}{2}}{\frac{Q_3 + Q_1}{2}} = \frac{Q_3 - Q_1}{Q_3 + Q_1}.$$

In case of series with frequencies, cumulative frequencies are to be calculated first before applying this formula.

Advantages :

Quartile deviation is superior to range in some respects, along with all the advantagesof range. The other advantages are

(i) It is capable of computing disersion in open-end distributions.

(ii) It is not affected by extreme values.

(iii) It is effective in erratic or highly skewed distributions.

Disadvantages :

(i) It is not based on all observations of the siries. It takes only middle 50% of items ignoring the first 25% and last 25% of data.

(ii) It is not capable of further mathematical treatment.

(iii) It is highly affected by fluctuations in sampling.

(iv) It cannot be called a measure of dispersion as it does not show the scatter around an average, rather it shows a distance on a scale.

(v) It gives only a rough idea about the degree of variability in a series.

Illustration 4.

Find the value of quartile deviation and its coefficient from the following data :

Wages : (₹)	200	340	610	750	820	940	1010
Workers :	1	2	3	4	5	6	7

Solution

Here wages of seven workers are given in the distribution.

$$Q_1 = \text{value of} \left(\frac{N+1}{4}\right) \text{th item.}$$

$$= \text{value of} \left(\frac{7+1}{4}\right) \text{th item.}$$

$$= \text{value of 2nd item.}$$

Here, The value of 2nd item is ₹ 340.

$\therefore Q_1 = ₹ 340.$

$$Q_3 = \text{value of } 3 \left(\frac{N+1}{4}\right) \text{th item.}$$

$$= \text{value of } 3 \left(\frac{7+1}{4}\right) \text{th item.}$$

= value of 6th item.

Here, the value of 6th item = ₹ 940.

$\therefore Q_3$ = ₹ 940.

Inter Quartile Range (IQR.) = ₹ 940 – ₹ 340 = ₹ 600.

Semi Interquartile Range or Quartile Deviation

$$= \frac{Q_3 - Q_1}{2} = \frac{\text{Rs.}940 - \text{ Rs. } 340}{2} = \text{Rs.}\frac{600}{2} = \text{Rs.}300$$

$$\text{Coefficient of QD} = \frac{Q_3 - Q_1}{Q_3 + Q_1} = \frac{940 - 340}{940 + 340} = \frac{600}{1280} = 0.469$$

Illustration 5.

Find out the Interquartile Range, Quartile Deviation and its coefficient of following series :

Wages :(₹)	200	325	612	740	818	920
Number of workers :	3	5	8	6	7	2

Solution :

Calculation of I.QR, QD and its coefficient :

Wages	Number of workers	cumulative frequency
(₹)	(f)	(cf)
200	3	3
325	5	8
612	8	16
740	6	22
818	7	29
920	2	31

$N = cf = 31.$

$Q_1 = \text{value of } \left(\frac{N+1}{4}\right) \text{th item}$

$= \text{value of} \left(\frac{31+1}{4}\right) \text{th item.}$

= value of 8 th item

value of 8th is ₹ 325

$\therefore Q_1 =$ ₹ 325.

$Q_3 = \text{value of } 3\left(\frac{N+1}{4}\right) \text{th item.}$

$= \text{value of } 3\left(\frac{31+1}{4}\right) \text{th item.}$

= value of 24th item.

value of 24th item is ₹ 818.

Inter Quartile Range (IQR) = $Q_3 - Q_1$ = ₹ 818 – ₹ 325= ₹ 493.

$$\text{Quartile Deviation (Q.D.)} = \frac{Q_3 - Q_1}{2} = \frac{₹818 - ₹325}{2} = \frac{₹493}{2} = ₹\ 246.5.$$

$$\text{Coefficient of Q.D.} = \frac{Q_3 - Q_1}{Q_3 + Q_1} = \frac{₹\ (818 - 325)}{₹\ (818 + 325)} = \frac{₹493}{₹1143} = 0.431.$$

Illustration 6.

Find the Inter Quartile Range, Quartile Deviation and its coefficient from the following data:

Marks:	0-10	10-20	20-30	30-40	40-50	50-60	60-70	70-80	80-90	90-100
Number of students :	4	8	7	15	22	8	10	9	6	11

Solution :

Calculation of IQR, Q.D and its coefficient :

Marks (X)	Number of students (f)	Cumulative frequency (cf)
0-10	4	4
10-20	8	12
20-30	7	19
30-40	15	34
40-50	22	56
50-60	8	64
60-70	10	74
70-80	9	83
80-90	6	89
90-100	11	100

N = cf = 100

Q_1 = Value of $\left(\frac{N}{4}\right)$th item

= Value of $\left(\frac{100}{4}\right)$th item

= Value of 25th item

Q_1 lies in the Class 30-40

Q_3 = Value of $3\left(\frac{N}{4}\right)$ th item

= Value of $\left(\frac{3x100}{4}\right)$ th item

= Value of 75th item

Q_3 lies in the class 70-80

$$Q_1 = L + \frac{N/4 - cf}{f} \text{ x } i$$

L=30, N/4 = 25, cf=19, f=15, i = 10

$$Q_1 = 30 + \frac{25-19}{15} \text{x}10$$

$$= 30 + \frac{6\text{x}10}{15}$$

=30+4

=34

$$L + \frac{3N/4 - cf}{f} \text{ x } i$$

L=70, 3N/4=75, cf=74, f=9, i=10

$$Q_3 = 70 + \frac{75-74}{9} \text{x}10$$

$$= 70 + \frac{1}{9}\text{x}10 = 70 + 1.10 = 71.1$$

Inter Quartile Range = $Q_3 - Q_1$ = 71.1-34=37.1

$$\text{Quartile Deviation (QD)} = \frac{Q_3 - Q_1}{Q_3 + Q_1} = \frac{71.1-34}{71.1+34} = \frac{37.1}{105.1} = 0.353.$$

Illustration 7.

Calculate the Inter Quartile Range, Quartile Deviation and its coefficient from the following data :

Marks :	10	20	30	40	50	60
No of Students :	8	15	22	20	10	5

Solution : Calculation of Inter Quartile Range, Quartile Deviation and its coefficient:

Marks	No. of Students (Frequency) = f	c.f	Marks	No. of Student frequency (f)	cf
10	8	8	40	20	65
20	15	23	50	10	75
30	22	45	60	5	80

$Q_1 = \text{Value of} \left(\frac{N+1}{4}\right) \text{th item} = \left(\frac{81}{4}\right) \text{th item}$

$= 20.25\text{th item}$

Value of 20.25th item is 20

$\therefore\ Q_1 = 20$

$Q3 = \text{Value of } 3\left(\frac{N+1}{4}\right) \text{th item} = 3\left(\frac{81}{4}\right) \text{th item} = \text{value of } 60.75\text{th item} = 40$

$\therefore\ Q_3 = 40$

(i) Inter Quartile Range = $Q_3\text{-}Q_1 = 40–20=20.$

(ii) Quartile Deviation = $\frac{Q_3 - Q_1}{2} = \frac{40-20}{2} = 10$

(iii) Cofficient of Quartile Deviation $= \frac{Q_3 - Q_1}{Q_3 + Q_1}$

$$= \frac{40-20}{40+20} = \frac{20}{60} = \frac{1}{3} = 0.333$$

Illustration 8.

Find the value of (i) Inter Quartile Range (ii) Quartile Deviation (iii) Coefficient of Quartile Deviation.

Wages (₹)	No. of Workers
More than 70	7
More than 60	18
More than 50	40
More than 40	40
More than 30	63
More than 20	65

Solution : The series is to be converted into a simple frequency distribution and data are arranged from the bottom.

Calculation of IQR, QD and its coefficient.

Wages (₹)	No. of Workers (f)	Cumulative Frequency (cf)
20-30	2	2
30-40	23	25
40-50	0	25
50-60	22	47
60-70	11	58
70-80	7	65

Q_1 = Value of $\left(\frac{N}{4}\right)^{th}$ item = Value of $\left(\frac{65}{4}\right)$th item

= Value of 16.25 the item which lies in the class inter value 30-40.

$$\therefore Q_1 = L + \frac{N/4 - cf}{f} \times i$$

Where L=30, N/4 = 16.25, cf=2, f = 23, i=10

$$Q_1 = 30 + \left(\frac{16.25 - 2}{23}\right) \times 10 = ₹36.1$$

$$Q_3 = \text{Value of} \left(\frac{3N}{4}\right)\text{th item} = \text{Value of} \left(\frac{3 \times 65}{4}\right)\text{th item}$$

= value of 48.75th item which lies in 60-70 class itnervals.

$$\therefore \quad Q_3 = L + \frac{\frac{3N}{4} - cf}{f} \times i = L + \left(\frac{\frac{3N}{4} - cf}{f}\right) \times i$$

where $L = 60$, $\frac{3N}{4} = 48.75$, $cf = 47$, $f = 11$, $i = 10$.

$$\therefore Q_3 = 60 + \frac{(48.75 - 47)}{11} \times 10 = 60 + \left(\frac{17.5}{11}\right) = 60 + 1.6 = 61.6.$$

(i) Inter Quartile Range = $Q_3 - Q_1 = 61.6 - 36.1 = 25.5$

(ii) Semi IQR / Quartile Deviation $= \frac{Q_3 - Q_1}{2} = \frac{61.6 - 36.1}{2} = \frac{25.5}{2} = 12.25$

(iii) Coefficient of Quartile Deviation $= \frac{Q_3 - Q_1}{Q_3 + Q_1} = \frac{61.6 - 36.1}{61.6 + 36.1} = \frac{25.5}{97.7} = 0.261$

Illustration 9.

Calculate Inter Quartile Range, Quartile Deviation and its relative measure from following series.

Hourly Wage (₹)	No. of Workers	Hourly Wage (₹)	No. of Workers
20-29	306	50-59	96
30-39	182	60-69	42
40-49	144	70-79	34

Solution : Calculation of IQR, QD and coefficient of QD.

Hourly Wages (₹)	No. of Workers (f)	Cumulative Frequency (cf)
20-29	306	306
30-39	182	488
40-49	144	632
50-59	96	728
60-69	42	770
70-79	34	804

$Q_1 = \text{Value of} \left(\frac{N}{4}\right) \text{th item} = \text{Value of} \left(\frac{804}{4}\right) \text{th item}$

= value of 201st item which lies in 20-29 class itnervals which is an inclusive series = 19.5 – 29.5.

$\therefore \quad L = 19.5, \frac{N}{4} = 201, cf = 0, f = 306, i = 10.$

$$\therefore \quad Q_1 = L + \frac{\frac{N}{4} - cf}{f} \times i \cdot$$

$$\therefore \quad Q_1 = 19.5 + \left(\frac{201-0}{306} \times 10\right) = 19.5 + \left(\frac{201 \times 10}{306}\right) = 19.5 + 6.57 = 26.07$$

$\therefore \quad Q_3 = \text{Value of } \left(\frac{3N}{4}\right)\text{th item} = \text{Value of } \left(\frac{3 \times 804}{4}\right)\text{th item} = \text{value of 603rd item}$ which lies in 40-49 class interval which is inclusive series = 39.5-49.5.

$$\therefore \quad Q_3 = L + \frac{\frac{3N}{4} - cf}{f} \times i, \text{ where } L = 39.5, \frac{3N}{4} = 603, cf = 488, f = 144, i = 10.$$

$$\therefore \quad Q_3 = 39.5 + \left(\frac{603-488}{144} \times 10\right) = 39.5 + \frac{115 \times 10}{144} = 39.5 + 7.99 = 47.49$$

(i) Inter Quartile Range = $Q_3 - Q_1$ = ₹47.49 – ₹26.07 = ₹21.42

(ii) Quartile Deviation $= \frac{Q_3 - Q_1}{2} = \frac{₹47.49 - ₹26.07}{2} = \frac{₹21.42}{2} = ₹10.71$

(iii) Coefficient of Quartile Deviation $= \frac{Q_3 - Q_1}{Q_3 + Q_1} = \frac{47.49 - 26.07}{47.49 + 26.07} = \frac{21.42}{73.56} = 0.261$

10.6 QUESTIONS

1 Multiple Choice Questions.

Select the correct answers from the alternatives given below in each bit :

(a) The simplest measure of dispersion is :

(i) Inter Quartile range (ii) Quartile Deviation (iii) Range (iv) Mean Deviation

(b) A relative measure of dispersion is :

(i) Range (ii) Inter Quartile range (iii) Coefficient of QD, (iv) Quartile Deviation.

(c) An absolute measure of dispersion is :

(i) Coefficient of range (ii) Coefficient I.Q.R (iii) Standard Deviation (iv) Coefficient of Mean Deviation

(d) The measure of dispersion not suitable for open-end class is :

(i) Quartile Deviation (iii) Mean Deviation

(ii) Range (iv) Inter Quartile Range

(e) An absolute measure of dispersion is expressed in :

(i) Ratio (ii) Percentage (iii) Decimal (iv) the same unit of original data.

(f) Relative measure of dispersion can be expressed in :

(i) Ratio/percentage/decimal (ii) the same unit of original data (iii) in different units original data (iv) rates.

(g) A relative measure of dispersion is always

(i) equal to unity (ii) more than unity (iii) Less than or equal to unity

(iv) less than zero

Answer (a) iii, (b) iii, (c) iii, (d) ii (e) iv (f) i (g) iii

2. Express the following in one word/term each :

(i) The other name of Quartile Deviation

(ii) Difference between extreme values of a series divided by sum of extreme values.

(iii) The formula of Inter Quartile Range.

Answer : Semi Inter-quartile range ; (ii) Co-efficient of Range; (iii) $Q_3 - Q_1$.

3. Answer the following within one sentence each :

(i) Give Spiegel's definition on a measure of dispersion.

(ii) What purpose is served by a relative measure of dispersion ?

(iii) How is Range calculated in a continuous series ?

(iv) What is the formula of coefficient of Quartile Deviation ?

Ans : (i) "The degree to which numerical data tend to spread about an average value is called the variation or dispersion of data; (ii) They serve the purpose of comparison; (iii) It is calculated by deducting the lower limit of first class interval from the upper limit of the last class interval; (iv) Coefficient of Quartile Deviation = $\frac{Q_3 - Q_1}{Q_3 + Q_1}$.

4. Fill in the gaps

(a) _______ is the simplest measure of dispersion.

(b) Coefficient of range is called coefficient of _______.

(c) Range, I QR and QD are _______ dispersions.

(d) Positional dispersions are _______ of positional averages.

(e) Mathematical dispersion is _______ of further mathematical analysis.

(f) Range _______ be used in open-end distributions.

(g) Range is _______ by fluctuations of sampling.

(h) Median lies half way on the same scale from Q_1 to Q_3 in _______ series.

(i) $Q_1 + QD \neq Q_3 - QD$ in _______ series of distribution

(j) QD includes atleast _______ percent in asymmetrical series.

Answer : (a) Range (b) Scatter (c) positional (d) counterparts (e) capable (f) cannot (g) affected (h) symmetrical (i) asymmetrical (j) fifty.

5. Correct the underlined portion of the following sentences.

(a) Quartile deviation is a/an <u>relative</u> measure of dispersion.

(b) Coefficient of standard deviation is a/an <u>absolute</u> measure of dispersion.

(c) Relative measure of dispersions are expressed in <u>original data.</u>

(d) Quartile Deviation and Semi I.Q.R. are <u>different</u>.

(e) Median ± QD covers <u>more than</u> 50% of the distribution.

(f) Coefficient of Q.D. is an <u>absolute</u> measure dispersion.

(g) QD is <u>affected</u> by extreme values.

(h) QD takes only <u>extreme</u> 50% items.

Ans.: (a) Absolute (b) Relative (c) Ratio/percentage (d) Same (e) exactly (f) Relative (g) not affected (h) middle.

6. Answer the following sentences within 30 words each :

(a) State the merits of range

(b) State the merits of Inter-quartile range

(c) State the merits of quartile deviation.

(d) Mention any three properties of a good measure of dispersion

(e) What are the positional dispersions ?

(f) State the objectives of dispersion.

7. Answer the following sentences within 50 words each.

(a) State the objectives of measuring dispersion.

(b) Express the utility of relative measures of dispersion.

(c) Why is QD calculated ?

(d) State the usefulness of range.

(e) Mention the demerits of QD.

(f) Give any three definitions of measure of dispersion.

8. What do you understand by dispersion ? Explain briefly the various methods used for measuring dispersion.

9. What is Quartile Deviation ? Explain its advantages and limitations.

10. Define range. Describe the merits, demerits and usefulness of range.

11. Calculate range and its coefficient from the following data. Price of mustard Oil per kilogram from Monday to Saturday.

Monday	Tuesday	Wednesday	Thursday	Friday	Saturday
₹ 160	158	170	142	176	187

Ans : Range = ₹45, Coefficient of Range = 0.137

12. The following are the prices of shares of X.Ltd in a week.

Days	Prices ₹	Days	Prices ₹	Days	Prices ₹
Mon	200	Wed	208	Fri	220
Tue	210	Thu	160	Sat	250

Calculate Range and its coefficient.

Ans : Range = 90 , Coefficient of Range = 0.714

13. Marks obtained in an OPSC examination by 25 candidates are as follows :

Marks :	5-9	10-14	15-19	20-24	25-29	30-34
Number of Candidates	1	3	8	5	4	2

Find (i) Coefficient of range (ii) Inter quartile range

Ans : (i) 0.795 (ii) 10.78

14. Calculate coefficient of Quartile Deviation from the following data :

Size :	4-8	8-12	12-16	16-20	20-24	24-28	28-32	32-36	36-40
Frequencies :	6	10	18	30	15	12	10	6	2

Ans : Coefficient of QD = 0.521

15. Find the Quartile Deviation and its coefficient from the following distribution :

Size :	0-10	10-20	20-30	30-40	40-50	50-60	60-70	70-80	80-90	90-100
Frequencies	4	8	10	18	16	14	12	8	6	4

Ans : QD = 16.25, Coefficient of QD = 0.339

16. Calculate coefficient of Quartile Deviation of the following

Age :	16-20	20-24	24-28	28-32	32-36	36-40
Number of Men :	200	250	400	300	250	100

Ans : Coefficient of QD = 0.16

17. Calculate QD and its coefficient from the following data.

Marks :	0-10	10-20	20-30	30-40	40-50	50-60	60-70
Number of Students :	5	6	10	15	8	4	2

Ans : QD = 10.875, Coefficient of QD = 0.32

●●●

CHAPTER-11

MATHEMATICAL DISPERSION - I MEAN DEVIATION

STRUCTURE

11.0 INTRODUCTION TO MATHEMATICAL DISPERSION

The positional dispersions described in the previous chapter namely Range, Inter-Quartile Range, Quartile Deviation are not real measures of dispersion in a strict sense. They suffer from a common defect that only two extreme values of a series are considered for their calculation. They do not show the spread or scatterness of the distribution from their central value. More over, they are not based on all the observations of a series.

Therefore, it is always better to take such a measure of dispersion which takes into account all the observations of a series and it is calculated from a measure of central value. The central value of a series of distribution may be Mean, Median or Mode. The method of calculating deviations from Mean, Median or Mode can throw some light on the formation of the series and dispersal/spread of items around a central value. This method of dispersion is called the "Method of Averaging Deviations". In this method, the deviations of items from any measure of central tendency are averaged for the purpose of studying the dispersion of the series.

The other two measures of dispersion, i.e., Mean Deviation and Standard Deviation overcome the limitations of positional dispersions. Mean Deviation and Standard Deviation are rightly called the mathematical dispersion. They involve some algebraic and arithmetical operations in their computations. The mathematical operations are addition, subtraction, multiplication, division and square root etc.

11.1 INTRODUCTION TO MEAN DEVIATION

The Mean Deviation is popularly known as Average Deviation. It is the average of the deviations of individual items in a distribution from their Mean, Median or Mode. Mean Deviation is defined as the arithmetic mean of the absolute deviations of items in a series taken from its central value. The absolute deviation implies that while taking the deviations from a measure of central value, the plus and minus signs are ignored. Mean Deviation is also known as 'Mean absolute deviation' or 'first moment of dispersion'.

Theoretically, deviations can be taken from any of the three measures of central tendency as mentioned earlier. But in actual practice, mean deviation is calculated either from Mean or from Median. Usually, Mode is not considered for this purpose because its value is indetermined at times. Of the other two central values, Median is better as the sum of the deviations from Median is always less than the sum of the deviations taken from mean. So the value of the Mean Deviation from Median is always less than the value of the Mean Deviation from Mean. The 'plus' and 'minus' signs are ignored in aggregating the deviations. When the algebraic signs are not ignored the deviations taken from arithmetic mean will always be zero and deviations taken from Median will also be nearing zero in a moderately asymmetrical series. It is therefore, futile to study mean deviation by taking into consideration the algebraical signs. However, Arithmetic Mean is more commonly used in calculating the value of average deviations. This is the reason why the average deviation is called Men Deviation. As the purpose of a measure of dispersion is to study the variation of items from a central value, it is immaterial and insignificant if plus and minus signs are ignored.

11.2 Calculation of Mean Deviation — Procedure :

If X_1 X_2, X_3...... X_N are N number of observations, $\overline{X}$ is the arithmetic mean, M is the Median and Z is Mode of the series. Symbolically,

(i) $\delta\overline{X} = \frac{\sum|D|}{N}$, where

$\delta\overline{X}$ stands for the Mean Deviation from Mean, |D| for the deviations of the values of variable taken from Mean, i.e., $|(X-\overline{X})|$, read as the modulus value or absolute value of the deviations where minus sign is ignored, and N for the number of items.

(ii) $\delta M = \frac{\sum|X-M|}{N} = \frac{\sum|D|}{N}$, where

δM stands for the Mean Deviation from Median,

$|D|$ for the deviations of the values of variable taken from Median, i.e. $|X - M|$ and N for the number of items.

(iii) $\delta Z = \frac{\sum|X-Z|}{N} = \frac{\sum|D|}{N}$, where

δZ stands for the Mean Deviation from Mode,

$|D|$ for the deviations of the values of variable taken from Mode, i.e., $|X - Z|$ and N for the number of items.

11.3 Calculation of coefficient of Mean Deviation

Mean Deviation is an absolute measure of dispersion. It is expressed in the same units of measurement in which the original data are expressed. The coefficient of Mean Deviation is the relative measure of dispersion and is obtained by dividing the Mean Deviation by the particular average from which it has been computed. It is expressed in terms of ratio, percentage and not in the terms of the unit of measurement of original data. Coefficient of Mean Deviation is also known as the Mean Coefficient of Dispersion.

Coefficient of Mean Deviation from Mean, Median and Mode would be symbolically as follows :

Coefficient of Mean Deviation	Deviations taken from :		
	Mean ($\overline{X}$)	Median(M)	Mode (Z)
	$\frac{MD}{\overline{X}} = \frac{\delta\overline{X}}{\overline{X}}$	$\frac{MD}{M} = \frac{\delta M}{M}$	$\frac{MD}{Z} = \frac{\delta Z}{Z}$

Below are given the computation of Mean Deviation under different series :

11.4 Calculation of Mean Deviation in Individual series (Direct Method) :

$$MD = \frac{\sum |D|}{N}$$

Illustration 1:

The daily wages of 7 labourers are as follows :

Serial Number	1	2	3	4	5	6	7
Wages (₹)	350	300	325	250	400	450	445

Find the Mean Deviation and its coefficient from (i) Mean and (ii) Median.

Solution :

(i) Calculation of Mean Deviation and its coefficient (from Mean)

Wages(X) (₹)	$\lvert D \rvert = \lvert X - \overline{X} \rvert$
350	10
300	60
325	35
250	110
400	40
450	90
445	85
$\Sigma X = 2520$	$\Sigma \lvert D \rvert = 430$

$\overline{X} = \frac{\Sigma X}{N} = \frac{2520}{7} = 360$, Mean Deviation = MD = $\frac{\Sigma|D|}{N}$

Mean Deviation (MD) = $\delta\overline{X} = \frac{\Sigma|D|}{N} = \frac{430}{7} =$ ₹61.43

Cofficient of Mean Deviation = $\frac{MD}{\overline{X}} = \frac{\delta\overline{X}}{\overline{X}} = \frac{61.43}{360} = 0.171$

(ii) Calculation of Mean Deviation (from Median)

By arranging the observations in ascending order we get, 250, 300, 325, 350, 400, 445,450 Here N=7

Sl. No.	Wages(X)	$\|D\| = \|X - M\|$	
	(₹)	(₹)	
1.	250	100	Median = M= Value of $\left(\frac{N+1}{2}\right)$th
2.	300	50	item = Value of $\left(\frac{7+1}{2}\right)$th
3.	325	25	item = Value of 4th item
4.	350	0	$\therefore$ Median = M = 350
5.	400	50	
6.	445	95	
7.	450	100	
	N=7	$\Sigma\|D\| = 420$	

Mean Deviation = $\delta M = \frac{\Sigma|D|}{N} = \frac{₹420}{7} =$ ₹60

Coefficient of Mean Deviation = $\frac{\delta M}{M} = \frac{MD}{M} = \frac{₹60}{350} = 0.171$

11.5 Calculation of Mean Deviation in individual series (Shortcut method) :

In this method first mean or median is calculated. Then the total of the values of observations below the Mean or Median and above it are found out. Then the total of values of the observation below the Mean or Median is subtracted from total of values of observation above it. The difference of the two values is then divided by the number of items to get Mean Deviation. For calculating the Mean Deviation from Mean in shortcut method, the formula is : MD = $\frac{\overline{X}y - \overline{X}x}{N}$

Where MD= Mean Deviation

N = Number of Observations

$\overline{X}y$ = Total of the values of observations above the Arithmatic mean.

$\overline{X}x$ = Total of the values of observations below the Arithmetic mean.

Similarly, when Mean Deviation is calculated from Median, the formula is :

$$MD = \frac{(My - Mx)}{N}, \text{where}$$

MD = Mean Deviation

N = Number of Observations,

My = Total of the observations above the Median.

Mx = Total of the observations below the Median.

Illustration 2. Calculate Mean Deviation and its coefficient from the wages of a group of employees given below :

Sl No.	1	2	3	4	5	6	7	8	9
Wages (₹)	3700,	3000	4200	3400	5800	3800	3200	4600	4300

Solution :

(i) Calculation of Mean Deviation and its Coefficient from Mean :

Employees	Salary (X) (₹)
1	3,700
2	3000
3.	4200
4.	3400
5.	5,800
6.	3,800
7.	3,200
8.	4,600
9.	4.300
N=9	$\Sigma X = 36,000$

$$\text{Mean} = \frac{\sum X}{N} = \frac{3600}{9} = 4000$$

Total of the values of items above mean $\overline{X}y = (4200 + 4600 + 4300 + 5800) = 18,900$

Total of the values of items below mean

$\overline{X}x = (3,700 + 3000 + 3,400 + 3,800 + 3200) = 17,100$

$$MD = \frac{(\overline{X}y - \overline{X}x)}{N} = \left(\frac{18,900 - 17,100}{9}\right)$$

$$= \frac{1800}{9} = ₹200$$

Cofficient of Mean Deviation =

$$= \frac{MD}{\overline{X}} = \frac{₹200}{₹4,000} = 0.05$$

ii) Calculation of mean Deviation from Median :

(In Ascending Order)	
Employees	Salary (₹)
1	3,000
2	3,200
3.	3,400
4	3,700
5	3,800
6	4,200
7.	4,300
8	4,600
9	5,800
N=9	

Median = Value of $(\frac{N+1}{2})$ th item = Value of $\left(\frac{9+1}{2}\right)$ th item = value of 5th item.

$\therefore$ Median = 3,800

My = (4,200 +4,300+4,600 +5,800)= ₹ 18,900 = Total value of items above median.

Mx = (3000+3,200+3,400+3700) = ₹ 13,300 = Total of the values of items below median.

$$MD = \frac{(My - Mx)}{N}$$

$$= \frac{₹(18900 - 13,300)}{9}$$

$$= \frac{₹\ 5,600}{9} = ₹\ 622.2$$

$$\text{Coefficient of Mean Deviation} = \frac{MD}{M}$$

$$= \frac{622.2}{3,800} = 0.164$$

11.6 Calculation of Mean Deviation in Discrete Series.

In discrete series, the formula for calculating Mean Deviation are :

(i) Mean Deviation from Mean = $MD = \frac{\sum f|d\bar{x}|}{N}$, where

MD = Mean Deviation,

$|d\bar{x}|$ = Deviation from Mean ignoring minus sign,

$\sum f|d\bar{x}|$ = Sum of the product of deviations and their respective frequencies,

N = Number of items.

(ii) Mean Deviation from Median = $MD = \frac{\sum f|dm|}{N}$, where

$|dm|$ = deviation from Median ignoring minus sign,

$\sum f|dm|$ = Sum of the products of deviations and their respective frequencies.

(iii) Mean Deviation from Mode = $MD = \frac{\sum f|dz|}{N}$, where

$|dz|$ = deviation from Mode ignoring minus sign,

$\sum f|dz|$ = Sum of the product of deviations and their respective frequencies.

Illustration 3 : Find the Mean Deviation and its coefficient of the following series from (i) Mean, (ii) Median and (iii) Mode :

Size (x) :	2	3	4	5	6	8	9
Frequency (f) :	6	8	5	2	3	4	6

Solution :

(i) Calculation of Mean Deviation and its coefficient from Mean :

Size (x) (1)	Frequency (f) (2)	Product (Col 1 & 2) (fx) (3)	Deviations from Mean $\|d\overline{x}\|$ (4)	Product (Col.2&4) $f\|d\overline{x}\|$ (5)
2	6	12	3	18
3	8	24	2	16
4	5	20	1	5
5	2	10	0	0
6	3	18	1	3
8	4	32	3	12
9	6	54	4	24
	$\sum f = N = 34$	$\sum fx = 170$		$\sum f\|d\overline{x}\| = 78$

$$\text{Mean} = \overline{X} = \frac{\sum fx}{N} = \frac{170}{34} = 5$$

$$\text{Mean Deviation} = \text{MD} = \frac{\sum f|d\overline{x}|}{N} = \frac{78}{34} = 2.294$$

$$\text{Coefficient of MD} = \frac{\text{MD}}{\text{Mean}} = \frac{2.294}{5} = 0.459$$

(ii) Calculation of Mean Deviation and its coefficient from Median

Size (x) (x)	Frequency (f) (f)	Cummulative Frequencies (cf) (cf)	Deviation from |dm| Median |dm|	Product of functions f|dm|
2	6	6	2	12
3	8	14	1	8
4	5	19	0	0
5	2	21	1	2
6	3	24	2	6
8	4	28	4	16
9	6	34	5	30
	$\sum f = N = 34$			$\sum f\|dm\| = 74$

$$\text{Median} = M = \text{Value of}\left(\frac{N+1}{2}\right)\text{th item} = \text{Value of}\left(\frac{34+1}{2}\right)\text{th item}$$

$$= \text{Value of } 17.5\text{th item} = 4.$$

$$\text{Mean Deviation} = MD = \frac{\sum f|dm|}{N} = \frac{74}{34} = 2.176.$$

$$\text{Coefficient of Mean Deviation} = \frac{\text{Mean Deviation}}{\text{Median}} = \frac{2.176}{4} = 0.544.$$

(iii) Calculation of Mean Deviation and its coefficient from Mode.

Size 1 (x)	Frequency (f) 2 (f)	Deviation from Mode 3 (dz)	Product of Deviation and Frequency (col. 2 × col.3) 4 f|dz|
2	6	1	6
3	8	0	0
4	5	1	5
5	2	2	4
6	3	3	9
8	4	5	20
9	6	6	36
	$\sum f = N = 34$		$\sum f\|dz\| = 80$

By visual inspection Mode (z) = 3 as it has the maximum frequency 8.

$$\text{Mean Deviation} = \text{MD} = \frac{\sum f|dz|}{N} = \frac{80}{34} = 2.353.$$

$$\text{Coefficient of Mean Deviation} = \frac{\text{MD}}{2} = \frac{2.353}{3} = 0.784$$

11.7 Calculation of Mean Deviation in Continuous Series

Calculation of Mean Deviation in continuous series is the same as that of calculation of Mean Deviation in discrete series. The only difference is that one is to find out the mid-point of the different class intervals and take the deviations of these points from any measure of central tendency i.e. Mean, Median or Mode.

Illustration 4.

Calculate the Mean Deviation and its co-efficients from Mean from the following data :

Class Interval :	0-4	4-8	8-12	12-16	16-20
Frequency :	4	6	7	5	3

Solution : Calculation of Mean Deviation and its coefficient

Class Interval	Mid-Value (x)	Frequency (f)	fx	$\|d\bar{x}\| = \|x - \bar{x}\|$	$f\|d\bar{x}\|$
0-4	2	4	8	7.52	30.08
4-8	6	6	36	3.52	21.12
8-12	10	7	70	0.48	3.36
12-16	14	5	70	4.48	22.40
16-20	18	3	54	8.48	25.44
		$\sum f = 25$	$\sum fx = 238$		$\sum f\|d\bar{x}\| = 102.4$

$$\text{Mean} = \frac{\sum fx}{\sum f} = \frac{\sum fx}{N} = \frac{238}{25} = 9.52$$

$$\text{Mean Deviation} = \text{MD} = \frac{\sum f|dx|}{N} = \frac{102.4}{25} = 4.096.$$

$$\text{Coefficient of Mean Deviation} = \frac{\text{Mean Deviation}}{\text{Mean}} = \frac{4.096}{9.52} = 0.43.$$

Illustration 5

Calculate the Mean Deviation and its co-efficient from Median from the following data :

Class Interval :	10-20	20-30	30-40	40-50	50-60
Frequency :	5	7	9	7	6

Solution : Calculation of Mean Deviation and its coefficient from Median.

Class Interval	Frequency (f)	Mid-Point (x)	Cumulative Frequency (cf)	Deviations |dm| = |x–m|=|x–35.55|	f|dm|
10-20	5	15	5	20.55	102.75
20-30	7	25	12	10.55	73.85
30-40	9	35	21	0.55	4.95
40-50	7	45	28	9.45	66.15
50-60	6	55	34	19.45	116.70
	$\sum f = N = 34$				$\sum f\|dm\| = 364.40$

$$\text{Median} = \text{Value of} \left(\frac{N}{2}\right) \text{th item} = \text{value of} \left(\frac{34}{2}\right) \text{th item} = \text{Value of 17th item.}$$

From the visual inspection of cummulative frequencies column, Median lies in (30-40) class interval

$$\therefore \text{ Median} = M = L + \frac{\frac{N}{2} - cf}{f} \times i, \text{ where,}$$

$$L = \text{Lower limit of median class} = 30, \ \frac{N}{2} = 17,$$

c.f. = Cummulative frequency of the class proceeding median class = 12, f = frequency of Median class = 9, i = Range of class Interval = 10

$$\therefore \text{ Median} = 30 + \left(\frac{17-12}{9} \times 10\right) = 30 + \frac{50}{9} = 30 + 5.55 = 35.55$$

$$\text{Mean Deviation} = \frac{\sum f|dm|}{N} = \frac{364.40}{34} = 10.72$$

$$\text{Coefficient of Mean Deviation} = \frac{\text{Mean Deviation}}{\text{Median}} = \frac{10.72}{35.55} = 0.302$$

11.8 Advantages of Mean Deviation :

1. Mean Deviation is simple to understand and easy to calculate. It is readily understood by the general public in socio-economic studies.
2. It is based on all the obsevations of the distribution.
3. It is rigidly defined.
4. It is less affected by extreme items than that of Standard Deviation.
5. It is better for comparison of deviations of different series.
6. It is useful in forecasting business cycle.

11.9 Limitations of Mean Deviation :

1. It is illogical to ignore the '±' signs in computing Mean Deviation.
2. It is not capable of further algebraic treatment.
3. It is rarely used in sociological studies to draw inferences.
4. If the Mean, Median or Mode is a fraction, then it is difficult to compute.
5. If Mean Deviation is calculated from Mode, it is not reliable as Mode does not properly represent the series.
6. In case of open-end series, Mean Deviation does not give accurate result.

11.10 Application / Usefulness of Mean Deviation :

The drawbacks of Mean Deviation can not prevent one from using it in different studies bacause of its simplicity in meaning and computation.

(i) Mean Deviation is meaningful in studying small samples.

(ii) It is very useful in studying business cycles.

(iii) It is better applicable to economic surveys made by National Bureau of Economic Research

(iv) Mean Deviation is especially effective in presenting reports to the general public those who have some fundamental knowledge on statistical methods.

Illustration 6 .

Calculate the Mean Deviation from the following data relating to the weights (to the nearest kilogram) of 85 children :

Weight (Kgs) :	20	25	28	30	35	38	40	50	55
Number of children :	2	5	8	10	12	18	15	9	6

Solution : Calculation of Mean Deviation from Mean.

Weight kg (x)	No. of Children (f)	fx	Deviations from Mean $d\overline{x}$	$f\lvert d\overline{x}\rvert$
20	2	40	17.33	34.66
25	5	125	12.33	61.65
28	8	224	9.33	74.64
30	10	300	7.33	73.30
35	12	420	2.33	27.96
38	18	684	0.67	12.06
40	15	600	2.67	40.05
50	9	450	12.67	114.03
55	6	330	17.67	106.02
	$\sum f = N = 85$	$\sum fx = 3173$		$\sum f\lvert d\overline{x}\rvert = 544.37$

$$\text{Mean} = \frac{\sum fx}{N} = \frac{3173}{85} = 37.33 \text{ Kgs}$$

$$\text{Mean Deviation} = \frac{\sum f\lvert d\overline{x}\rvert}{N} = \frac{544.37}{85} = 6.404 \text{ Kgs}$$

$$\text{Coefficient of Mean Deviation} = \frac{MD}{\text{Mean}} = \frac{6.404}{37.33} = 0.172$$

Illustration 7.

Calculate the value of Mean Deviation and its co-efficient from Median of the following data :

Marks :	20-30	30-40	40-50	50-60	60-70
Number of Students :	6	12	18	25	20

Solution :

Calculation of Mean Deviation an its coefficient from Median.

Marks	Mid Value	Number of students (f)	Cumulative frequency (cf)	Deviation of Median |dm|	f|dm|
20-30	25	6	6	26.8	160.80
30-40	35	12	18	16.8	201.60
40-50	45	18	36	6.8	122.40
50-60	55	25	61	3.2	80.00
60-70	65	20	81	13.2	264.00
		$\sum f = N = 81$			$\sum f\|dm\| = 828.80$

Median = The value of $\frac{N}{2}$ th item or $\left(\frac{81}{2}\right)$ th item.= value of 40.5th item, which lies in 50-60 class.

$$\therefore \text{Median} = L + \frac{N/2 - cf}{f} \times i, \text{ where } \frac{N}{2} = 40.5, cf = 36, f = 25, i = 10$$

$$= 50 + \left(\frac{40.5 - 36}{25}\right) \times 10$$

$$= 50 + \left(\frac{10}{25} \times 4.5\right) = 50 + \frac{45}{25} = 50 + 1.8 = 51.8$$

$$\text{Mean Deviation} = \frac{\sum f|dm|}{N} = \frac{828.80}{81} = 10.232$$

$$\text{Coefficient of Mean Deviation} = \frac{MD}{Median} = \frac{10.232}{51.80} = 0.198.$$

Illustration 8.

Calculate Median and Mean Deviation for the following data :

Age (Years) :	1-5	6-10	11-15	16-20	21-25	26-30	31-35
Number of persons :	7	10	16	32	24	18	10

Solution : Calculation of Median and Mean Deviation :

Age (Years)	Mid Value (x)	Number of Persons (f)	Cumulative frequency (cf)	Deviations $\|(x-m)\|=\|dm\|$	$f\|dm\|$
0.5–5.5	3	7	7	16.48	115.36
5.5–10.5	8	10	17	11.48	114.80
10.5–15.5	13	16	33	6.48	103.68
15.5–20.5	18	32	65	1.48	47.36
20.5–25.5	23	24	89	3.52	84.48
25.5–30.5	28	18	107	8.52	153.36
30.5–35.5	33	10	117	13.52	135.20
		$\Sigma f = 117$			$\Sigma f\|dm\| = 754.24$

Median = Value of $\frac{N}{2}$ th item = Value of $\left(\frac{117}{2}\right)$ th item = Value of 58.5th item which lies in 15.5 – 20.5 class interval.

$$\text{Median} = L + \left(\frac{N/2 - cf}{f} \times i\right), \text{ where } L = 15.5,\ \frac{N}{2} = 58.5,\ cf = 33,\ f = 32,\ i = 5.$$

$$\therefore \text{ Median} = 15.5 + \left(\frac{58.5 - 33}{32} \times 5\right) = 15.5 + 3.98 = 19.48$$

$$\text{Mean Deviation} = \frac{\sum f|dm|}{N} = \frac{754.24}{117} = 6.45$$

$$\text{Coefficient of Mean Deviation} = \frac{\text{Mean Deviation}}{\text{Median}} = \frac{6.45}{19.48} = 0.331.$$

Shortcut Method :

When value of any average (Mean/Median/Mode) is in fraction, shortcut method is applied.

Mean Deviation from Mean, Median or Mode

$$= \frac{\sum xf_1 - \sum xf_2 - \left[\sum f_1 - \sum f_2\right](\overline{x}/M/Z)}{N}, \text{ where}$$

Σxf_1 = Sum of the product of items and frequency corresponding to those items/midvalues which are greater than the Mean/Median/Mode.

Σxf_2 = Sum of the product of items and frequency corresponding to those items/mid values which are lesser than the Mean/Median/Mode.

Σf_1 = Sum of the frequencies corresponding to mid values / items greater than the Mean/Median/Mode.

Σf_2 = Sum of the frequencies corresponding to mid values / items lesser than the Mean/Median/Mode.

N = Number of Items.

$\overline{x}$ = Mean, M = Median, z = Mode.

Illustration 9.

Calculate Median Deviation from Median from the following data in (i) Direct method and (ii) Shortcut method.

Class Interval :	0-10	10-20	20-30	30-40	40-50	50-60
Frequency :	15	36	53	42	17	2

Solution : (i) Direct Method.

Class Interval	Mid Value (x)	Frequency (f)		Cumulative frequency (c f)	Deviations from Median |(x–m)| = |dm|	Deviations xf f|dm|	(x f)	
0-10	5	15	f_2	15	20.94	314.10	75	xf_2
10-20	15	36		51	10.94	393.84	540	
20-30	25	53		104	0.94	49.82	1320	
30-40	35	42	f_1	146	9.06	380.52	1470	xf_1
40-50	45	17		163	19.06	324.02	765	
50-60	55	2		165	29.06	58.12	110	
		Σf =N=165				Σf|dm| = 1520.42		

Median = Value of $\frac{N}{2}$ th item = Value of $\left(\frac{165}{2}\right)$ th item = Value of 82.5th item.

By visual inspection Median lies in 20-30 class interval.

$\therefore$ Median = $L + \left(\frac{N/2 - cf}{f} \times i\right)$, where L = 20, $\frac{N}{2}$ =82.5, cf = 51, f = 53, i = 10.

Putting the above values to the formula of interpolation,

$$\text{Median} = M = 20 + \left(\frac{82.5 - 51}{53} \times 10\right) = 20 + \left(\frac{31.5 \times 10}{53}\right)$$

$$= 20 + \frac{315}{53} = 20 + 5.94 = 25.94$$

$$\text{Mean Deviation} = \frac{\sum f|dm|}{N} = \frac{1520.42}{165} = 9.215.$$

(ii) Short cut Method.

$$\text{Mean Deviation from Median} = \frac{\sum xf_1 - \sum xf_2 - \left[\sum f_1 - \sum f_2\right](M)}{N}$$

Σxf_1 = 1470+765+110=2345

Σxf_2 = 75+540+1325 = 1940

Σf_1 = 42+17+2 = 61

Σf_2 = 15 + 36 + 53 = 104

M = Median

N = Number of items.

$$\therefore \text{Mean Deviation} = \frac{2345 - 1940 - (61 - 104)(25.94)}{165}$$

$$= \frac{405 - (-43)(25.94)}{165} = \frac{405 + 1115.42}{165} = \frac{1520.42}{165} = 9.215$$

Illustration 10.

Calculate Median Deviation from Mean in Shortcut method.

Hourly Wage (Less than (₹) :	10	20	30	40	50	60
Number of Family :	3	8	16	26	30	32

Solution :

Hourly Wage (₹)	Mid Value (x)	Number of family (f)		Product of x and f (xf)
0-10	5	3	Σf_2	15
10-20	15	5		75
20-30	25	8		200
30-40	35	10	Σf_1	350
40-50	45	4		180
50-60	55	2		110
		$\Sigma f = N = 32$		$\Sigma xf = 930$

$$\text{Mean} = \bar{x} = \frac{\sum xf}{N} = \frac{₹930}{32} = ₹29.06$$

Mean Deviation from Mean (under shortcut method)

$$= \frac{\sum xf_1 - \sum xf_2 - \left[\sum f_1 - \sum f_2\right](\bar{x})}{N}, \text{ where}$$

$\Sigma xf_1 = 350 + 180 + 110 = 640$

$\Sigma xf_2 = 15 + 75 + 200 = 290$

$\Sigma f_1 = 10 + 4 + 2 = 16$

$\Sigma f_2 = 3 + 5 + 8 = 16$

$\bar{x} = 29.06$

$N = 32$

∴ Substituting the above values to the formula,

$$\text{Mean Deviation} = \frac{640 - 290 - [16 - 16]29.06}{32}$$

$$= \frac{350 - (0 \times 29.06)}{32} = \frac{350}{32} = 10.938$$

11.11 QUESTIONS :

1. From the alternatives given below in each bit, choose and write the correct answer along with its serial number :

(a) The dispersion which shows the scatterness of various items from its central value is :
(i) Range (ii) Interquartile range (iii) Quartile deviation (iv) Mean deviation

(b) Mean deviation is not calculated from :
(i) Geometric Mean (ii) Arithmetic Mean (iii) Median (iv) Mode

(c) The most preferred measure of central tendency used to calculate Mean Deviation is :
(i) Mean (ii) Median (iii) Mode (iv) Harmonic Mean

(d) The unitless dispersion is :
(i) Mean Deviation (ii) Quartile Deviation
(iii) Range (iv) Coefficient of Mean Deviation

(e) Mean deviation does not give accurate result in case of :
(i) closed-end series (ii) Exclusive series
(iii) Open-end series (iv) Inclusive series

(f) The dispersion which is based on all the observations of a series :
(i) Range (ii) Inter quartile range (iii) Quartile deviation (iv) Mean deviation

Answers : (a) (iv) Mean deviation; (b) (i) Geometric Mean; (c) (ii) Median; (d) (iv) coefficient of Mean Deviation; (e) (iii) open-end series, (f) (iv) Mean Deviation.

2. Fill in the gaps.

(a) Mean deviation is ________ defined.
(b) Mean deviation ________ affected by extreme items.
(c) Mean deviation is useful in ______ business cycles.
(d) Mean deviation is ________ capable of further algebraic treatment.
(e) Mean deviation is difficult to compute if Mean, Median and Mode is in ________.
(f) It is ________ to igonre '±' signs in computing Mean Deviation.
(g) Mean Deviation is meaningful in studying ________ samples.
(h) Mean deviation is rarely used in ______ studies / surveys.
(i) ________ is based on all the observations of the series.
(j) Mean deviation is the ________ mean of deviations taken from a central value.
(k) Mean deviation is also known as mean ________ deviation.
(l) Mean deviation is also known as first ________ of dispersion.

Ans : (a) Rigidly (b) less (c) forecasting (d) Not (e) Fractions (f) Illogical (g) small (h) sociological (i) Mean deviation (j) Arithmetic (k) Absolute (l) Moment.

3. Answer the following questions in one sentence each :

(a) Define mean deviation.

(b) What is the other name of mean deviation ?

(c) What is the formula of computing mean deviation from mode in discrete series.

(d) What do you mean by coefficient of mean deviation.

(e) Which measure of dispersion is comparison friendly ?

(f) Why is Median preferable to Mean in calculating Mean Deviation ?

4. Express the following in one word / term each :

(i) The measure of dispersion which is computed from either Mean, Median and Mode.

(ii) The other name of Mean Deviation.

(iii) The relative measure of Mean Deviation.

(iv) The average which is considered most suitable for computing Mean Deviation.

Answers : (i) Mean Deviation, (ii) Average Deviation, (iii) Coefficient of Mean Deviation (iv) Median.

5. Correct the underlined portions of the following sentences :

(a) Mean deviation is <u>more</u> popular than standard deviation.

(b) Mean deviation <u>considers</u> the '±' signs in taking deviation.

(c) The definition of mean deviation is <u>flexible</u>.

(d) Coefficient of Mean deviation is an <u>absolute</u> measure of dispersion.

(e) The calculation of mean deviation in discrete series and continuous series is <u>different</u>.

(f) Mean deviation is based on <u>middle fifty percent</u> of the observations.

(g) Mean deviation is useful in studying <u>large</u> samples.

(h) Mean Deviation is <u>frequently</u> used in sociological studies.

(i) It is <u>logical</u> to ignore '±' signs in computing Mean Deviation.

Answer : (a) less, (b) Ignores, (c) Rigid, (d) Relative, (e) Same, (f) all, (g) small, (h) rarely, (i) illogical.

6. Answer the following questions within 30 words each :

(a) State the formula for computing mean deviation from mean in shortcut method and explain the symbols used therein.

(b) State the formula for computing mean deviation from median in shortcut method and explain the symbols used therein.

(c) Explain the empirical relationship between Quartile Deviation and Mean Deviation.

(d) Give any three advantages of Mean Deviation.

(e) Give any three uses / applications of Mean Deviation.

Q.7. Answer the following within 50 words each :

(a) State the merits of Mean Deviation.

(b) State the demerits of Mean Deviation.

(c) State the usefulness of Mean Deviation.

(d) Why is Mean Deviation preferred to Range or Quartile Deviation ?

(e) Explain the procedure of computing Mean Deviation in continuous series.

LONG QUESTIONS

Q.8. Define Mean Deviations. Explain its merits, demerits and applications.

Q.9. State the different formulae for calculating Mean Deviation with examples.

Q.10. With Median as the base, calculate the Mean Deviation and the variability of the two series X and Y:

Series X	Series Y	Series X	Series Y
3484	487	4124	620
4572	508	3682	382
5624	408	3680	186
4388	266	4308	218

Ans : Coefficient of mean deviation :- X = 11.6%, Y = 30.7%

Q.11. Calculate the Mean Deviation from the following data relating hourly wage of 100 workers.

Wages (₹)	60	61	62	63	64	65	66	67	68
Number of workers :	2	0	15	29	25	12	10	04	3

Ans : M.D = 1.239. Mean = $\overline{X}$ = 63.89

12. Find the Mean Deviation from the Mean of the following distribution of dividend payment of companies during the year ending 2016-17.

Wages (₹)	0-3	3-6	6-9	9-12	12-15	15-18	18-21
Number of workers :	2	7	10	12	9	6	4

Ans : MD= 3.823, Mean = $\overline{X}$ = 10.68

13. Find the Mean Deviation from median from data given below relating to the annual turnover of vegetable shops of an urban market :

Turnover (₹'000) :	1-3	3-5	5-7	7-9	9-11	11-13	13-15	15-17
Number of Shops :	6	53	85	56	21	26	4	4

Ans : Mean Deviation = 2.252, Median = M=6.612

Q.14. Calculate Median, Mean Deviation and coefficient of Mean Deviation from the following data.

Age in years :	1-5	6-10	11-15	16-20	21-25	26-30	31-35	36-40	41-45
Number of persons :	7	10	16	32	24	18	10	5	1

Ans : Median = M=19.95, Mean Deviation = 7.09, Coefficient of Mean Deviation = 0.355

15. Compute Mean Deviation from Mean, Median an Mode from the following data :

Class:	0-10	10-20	20-30	30-40	40-50	50-60	60-70	70-80	80-90
Frequency :	3	8	15	20	25	10	9	6	4

Ans : Mean deviation (i) from Mean = 14.99, (ii) from Median=15.07, from Mode = 15

16. Calculate Mean Deviation and its coefficient from Mean, Median for the following data :

Mark less than :	10	20	30	40	50	60	70	80
Number of Students :	3	8	16	26	37	50	56	60

Ans : Median = M = 15.18, Coefficient of Mean Deviation = 0.347

17. Calculate Mean Deviation and its coefficient from Mean for the following data :

X:	10	11	12	13	14
F :	3	12	18	12	3

Ans : Mean = $\overline{X}$ =12, Mean Deviation = 0.75, Cofficient of Mean Deviation = 0.0625

18. Calculate Mean Deviation from the Mean for the following data. Also find the coefficient of Mean Deviation .

Size :	0-4	4-8	8-12	12-16	16-20
Frequency :	4	6	8	5	2

Ans : Mean = $\overline{X}$ = 9.2 MD = 3.84

Coefficient of MD = 0.42.

❖❖❖

CHAPTER-12

MATHEMATICAL DISPERSION - II STANDARD DEVIATION

STRUCTURE

12. INTRODUCTION

Standard Deviation is an important absolute measure of dispersion. The concept of 'Standard Deviation was first introduced by Karl Pearson in 1823. It is taken as the most widely used measure of dispersion. It is popular because it is free from any defects suffered by the other measures of dispersion. It satisfies most of the properties of an ideal measure of dispersion. Standard Deviation is also called 'root mean square Deviation, because it is the square root of the Mean of the squared deviation from the Arithmetic Mean. Standard Deviation is denoted by the small Greek letter σ (read as sigma). It measures the absolute dispersion of a distribution. The greater is the magnitude of standard deviation the greater is the amount of dispersion. It indicates that the magnitude of dispersion is more from the value of their Arithmetic Mean. A lower standard deviation indicates a low degree of dispersion/variability from the mean and there is higher degree of uniformity and homogeneity, of the items in the series.

When we compare two or more distributions with the same mean, it is better to take the Standard Deviations into consideration. Higher is the Standard deviation of a series, it has a least representative mean and conversely lower is the Standard Deviation of a series it has a most representative mean. So the standard Deviation is very useful in determining the representativeness of the mean of a series.

12.1 Definition.

Standard Deviation is defined "as the square root of the Arithmetic Mean of the squares of deviations of the given observations from their Arithmetic mean of a series."

12.2. Difference between Mean Deviation and Standard Deviation.

		Mean Deviation	Standard Deviation
1.	Base	Mean Deviation can be calculated from Arithmatic Mean, Median or Mode.	Standard Deviation can be calculated from Arithmetic Mean only.
2.	Algebraic Signs.	In calculating Mean Deviation '$\pm$' signs are ignored.	In calculating standard Deviation '$\pm$'signs are not ignored.
3.	Further Algebraic Treatment.	Mean Deviation is not capable of further algebraic treatment.	Standard Deviation is capable of further algebraic treatment.

12.3. Calculation of Standard Deviation - Individual Series

(a) Direct Method : For calculating the Standard Deviation of a series of individual observations when the Arithmetic mean is a round figure (whole number), we can apply the direct method. The formula for calculating Standard Deviation is :

$SD = \sigma = \frac{\sqrt{\Sigma x^2}}{N}$, Where SD= σ = Standard Deviation,

$x = X - \overline{X}$ = Deviation of the items from the Arithmetic Mean

Σx^2 = Sum of the squares of deviations, N=Number of items / Observations.

(b) Indirect Method : When the actual Arithmetic Mean becomes a fraction, say 26.784, it becomes cumbersome to find deviations and square the deviations. In such a case deviations are taken from an asumed mean and necessary adjustment is made in the formula of Standard Deviation. The formula for Standard Deviation in indirect method is.

$$SD = \sigma = \sqrt{\frac{\Sigma d^2}{N} - \left(\frac{\Sigma d}{N}\right)^2}$$

Σd^2 = Sum of the squares of deviations,

Σd = Sum of the deviations.

N = Number of items.

Illustration 1

Calculate the Standard Deviation of the following series of individual observations :

Hourly Wages (₹) 100, 110, 120, 115, 125, 140,132, 158, 167, 173

Solution : Calculation of S.D

Sl. No	**Hourly wage (X)**	**Diviation From Actual Mean**	**Squares of Deviations Actual mean (x^2)**
		$x = (x - \overline{X})$	
1.	100	-34	1,156
2.	110	-24	576
3.	120	-14	196
4.	115	-19	361
5.	125	-9	81
6.	140	6	36
7.	132	-2	4
8.	158	24	576
9.	167	33	1,089
10.	173	39	1521
N=10	**ΣX = 1340**	**ΣX = 0**	**Σx2=5, 596**

Arithmetic Mean $\overline{X} = \frac{\Sigma X}{N} = \frac{1340}{10} = 134$

Standard Deviation $= \sigma = \sqrt{\frac{\Sigma x^2}{N}} = \sqrt{\frac{5,596}{10}} = \sqrt{559.6} = 23.65$

Illustration 2.

Calculate Standard Deviation by shortcut / indirect method from the detail given in previous illustration :

Solution : Calculation of Standard Deviation

Hourly wages() (X)	**Deviation from Assumed Mean (135) (X-135)=d**	**Squares of Deviations (d^2)**
100	-35	1225
110	-25	625
120	-15	225
115	-20	400
125	-10	100
140	5	25
132	-3	9
158	23	529
167	32	1024
173	38	1,444
N=10	**$\Sigma d = -10$**	**$\Sigma d^2 = 5,606$**

Standard Deviation $= \sigma = \sqrt{\frac{\Sigma d^2}{N} - \left(\frac{\Sigma d}{N}\right)^2} = \sqrt{\left(\frac{5606}{10}\right) - \left(\frac{-10}{10}\right)^2}$

$= \sqrt{560.6 - 1} = \sqrt{559.6} = 23.65$

Illustration 3.

Calculate Standard Deviation by using original values of the variable from the following data.

Variable (X) 2 3 4 5 7 8 10 12

Solution : Calculation of Standard Deviation

Variable (X)	Squence of Variable (X^2)
2	4
3	9
4	16
5	25
7	49
8	64
10	100
12	144
ΣX= 51	**ΣX² = 411**

Standard Deviation $(s)=\sqrt{\frac{\Sigma X^2}{N}-\left(\frac{\Sigma X}{N}\right)^2}$

$$=\sqrt{\frac{411}{8}-\left(\frac{51}{8}\right)^2}=\sqrt{51.375-40.641}=\sqrt{10.734}$$

$$=\sqrt{10.734}=3.27$$

12.4 Calculation Standard Deviation - Discrete Series.

For discrete series, Standard Deviation is calculated in three different methods.

i) Actual Arithmetic mean method $\sigma=\sqrt{\frac{\Sigma fx^2}{N}}$ where x= $X-\overline{X}$

ii) Assumed mean methiod $s=\sqrt{\frac{\Sigma fd^2}{N}-\left(\frac{\Sigma fd}{N}\right)^2}$, Where d = (X-A)

iii) Step Deviation method : $s=\sqrt{\frac{\Sigma fd^2}{N}-\left(\frac{\Sigma fd^2}{N}\right)^2}\times i$

Where $d = \left(\frac{X-A}{i}\right)$ and i = class interval

Illustration 4.

Compute the Standard Deviation of the following distribution from actual mean.

(X)	2	3	4	5	6	8	9
(f)	6	8	5	2	3	4	6

Solution : Computation of Standard Deviation from actual mean

Variable Frequency

(X)	(f)	(fX)	$x=(X-\overline{X})$	x^2	fx^2
2	6	12	3	9	54
3	8	24	2	4	32
4	5	20	1	1	5
5	2	10	0	0	0
6	3	18	1	1	3
8	4	32	3	9	36
9	6	54	4	16	96
	Σf=N=34	**Σfx=170**			**Σfx²=226**

$$\text{Mean} = \overline{X} = \frac{\sum fX}{N} = \frac{170}{34} = 5$$

Standard Deviation $\sigma = \sqrt{\frac{\sum fx^2}{N}} = \sqrt{\frac{226}{34}}$

$= \sqrt{6.64706} = 2.577$

Illustration 5.

Calculate Standard Deviation in shortcut method from the following data:

X :	4	5	6	8	9	12
f :	5	3	7	10	3	2

Solution :

Calculation of Standard Deviation

X	f	d=X-A =x-8	d^2	fd	fd^2
4	5	-4	16	-20	80
5	3	-3	9	-9	27
6	7	-2	4	-14	28
8	10	0	0	0	0
9	3	1	1	3	3
12	2	4	16	8	32
Σf=N=30			**Σd^2=46**	**Σfd=-33**	**fd^2=170**

Let the Assumed Mean be 8

$$S.D=\sigma=\sqrt{\frac{\Sigma fd^2}{N}-\left(\frac{\Sigma fd}{N}\right)^2}=\sqrt{\frac{170}{30}-\left(\frac{-33}{30}\right)^2}=\sqrt{5.667-1.210}=\sqrt{4.457}=2.112$$

$$\text{Arithmetic mean}=\overline{X}=\text{Assumed mean}+\frac{\Sigma fd}{N}=8+\left(\frac{-33}{30}\right)=8-1.1=7.9$$

Illustration 6

Calculate Standard Deviation from following data :

X:	4	6	8	10	12	14	16
f:	2	3	6	8	5	3	2

Solution :

As the variable X increases by equal intervals, step deviation method should be followed. Let the Assumed Mean be 10.

Calculation of Standard Deviation

X	f	d=(X-A)/2 = (X-10)/2	d^2	fd	fd^2
4	2	- 6/2 = -3	9	- 6	18
6	3	- 4/2 = -2	4	- 6	12
8	6	- 2/2 = -1	1	- 6	6
10	8	0/2 = 0	0	5	0
12	5	2/2 = 1	1	6	5
14	3	4/2 = 2	4	6	12
16	2	6/2 = 3	9	6	18
	Σf=N=29			**Σfd=--1**	**Σfd^2=71**

$$S.D.(\sigma)=\sqrt{\frac{\sum fd}{N}-\left(\frac{\sum fd}{N}\right)^2}Xi=\sqrt{\frac{71}{29}-\left(\frac{-1}{29}\right)^2}X2$$

$$=\sqrt{2.448-0.001}\times 2$$

$$=\sqrt{2.447}\times 2=1.56\times 2=3.12.$$

$$\text{Arithmetic Mean}=\text{Assumed Mean}+\frac{\sum fd}{N}\times i=10+\frac{-1}{29}\times 2=10\frac{-2}{99}=10-0.069$$

$$= 9.931$$

12.5 Continuous Series

The method of computation of Standard Deviation in continuous series is similar to the method adopted for discrete series. But in actual practice step deviation method is used in most of the studies, The formula is :

$$\text{Standard Deviation}(\sigma)=\sqrt{\frac{\sum fd^2}{N}-\left(\frac{\sum fd}{N}\right)^2}\text{ x i, where}$$

d = Deviations from assumed mean = $\left(\frac{X-A}{i}\right)$

A = Assumed Mean

i = Magnitude of the class interval.

Illustration - 7

The monthly wages of labourers in a factory are given below. Find out the Mean and Standard Deviation of their wages.

Wages (in '000 rupees) :	10-20	20-30	30-40	40-50	50-60	60-70
Number of workers :	12	14	10	8	7	5

Solution : Calculation of Mean and Standard Deviation.

Wages 000)	No. of Workers (f)	Mid value (X)	Deviations from Assumed mean (X-A) (d)	Squares of Deviations (d^2)	Product of frequency & Deviation (fd)	Product of frequency & squares of deviations (fd^2)
10-20	12	15	$-30/10 = -3$	9	-36	108
20-30	14	25	$-20/10 = -2$	4	-28	56
30-40	10	35	$-10/10 = -1$	1	-10	10
40-50	8	45	$0/0 = 0$	0	0	0
50-60	7	55	$10/10 = 1$	1	7	7
60-70	5	65	$20/10 = 2$	4	10	20
	$\Sigma f = N = 56$			$\Sigma fd = -57$		$\Sigma fd^2 = 201$

Arithmetic Mean ($\bar{x}$) $= \text{Assumed Mean} + \frac{\sum fd}{N} \times i$

$$= 45 + \left(\frac{-57}{56} \times 10\right)$$

$$= 45 - 10.179$$

$$= 34.821.$$

Standard Deviation (σ) $= \sqrt{\frac{\sum fd^2}{N} - \left(\frac{\sum fd}{N}\right)^2} \times i$

$$= \sqrt{\frac{201}{56} - \left(\frac{-57}{56}\right)^2} \times 10$$

$$= \sqrt{3.589 - 1.036} \times 10$$

$$= \sqrt{2.553} \times 10$$

$$= 1.598 \times 10$$

$$= 15.98.$$

12.6 Mathematical Properties of Standard Deviation

Standard Deviation is the best among all the measures of dispersion. It is capable of further algebraic treatment. It satisfies all the features of an ideal measure of dispersion. It has some very important mathematical properties for enhancing the utility of various statistical studies. These are :

(a) Combined Standard Deviation : As one computes the combined Arithmetic Mean of two or more groups, one can also compute the combined Standard Deviation of two or more groups.

Symbolically,

Combined Standard Deviation (σ_{12}) = $\sqrt{\dfrac{n_1\left(\sigma_1^2+d_1^2\right)+n_2\left(\sigma_2^2+d_2^2\right)}{n_1+n_2}}$

where, n_1 = Number of observations in 1st group,

n_2 = Number of Observations in 2nd group,

σ_1^2 = Square of SD of 1st group,

σ_2^2 = Square of SD of 2nd group,

$d_1 = \bar{x}_1 - \bar{x}_{12}$ (Difference between the Mean of 1st group and their Combined Mean)

$d_2 = \bar{x}_2 - \bar{x}_{12}$ (Difference between the Mean of 2nd group and their Combined Mean)

Illustration-8

Given below the marks of students of SCS (Junior) College, Puri and BJB (Junior) College, Bhubaneswar :

Particulars	BJB (Jr.) College, BBSR	SCS (Jr.)College, Puri
Numbers of students	755	690
Average marks in (Business Mathematics & Statistics)	98	90
Standard Deviation	3	4

Calculate : (i) Combined Mean and (ii) Combined Standard Deviation.

Solution :

Calculation of Combined Mean and Combined Standard Deviation.

Given :

$\bar{x}_1 = 98,\ \bar{x}_2 = 90$

$\sigma_1 = 3,\ \sigma_2 = 4$

$n_1 = 755,\ n_2 = 690$

(i) Combined Mean = $\bar{x}_{12} = \frac{\bar{x}_1 n_1 + \bar{x}_2 n_2}{n_1 + n_2}$

$$= \frac{(98 \times 755) + (90 \times 690)}{755 + 690} = \frac{73990 + 62100}{1445} = \frac{136090}{1445} = 94.18$$

(ii) Combined Standard Deviation

Combined Standard Deviation = $\sigma_{12} = \sqrt{\frac{n_1(\sigma^2 + d_1^2) + n_2(\sigma_2^2 + d_2^2)}{n_1 + n_2}}$

$d_1 = \bar{X}_1 - \bar{X}_{12} = 98 - 94.18 = 5.82$

$d_2 = \bar{X}_2 - \bar{X}_{12} = 90 - 94.18 = -4.18$

$$\sigma_{12} = \sqrt{\frac{755(3^2 + 5.82^2) + 690(4^2 + (-4.18)^2)}{755 + 690}} = \sqrt{\frac{32368.662 + 23,095.956}{1455}}$$

$$= \sqrt{\frac{55464.618}{1445}}$$

$$= \sqrt{38.384} = 6.195$$

(b) **Standard Deviation of first n natural numbers** : It can be computed by the formula :

$$\sigma = \sqrt{\frac{n^2 - 1}{12}} .$$

Illustration 9

Find the Standard Deviation of the first 50 (i.e., 1 to 50) natural numbers.

Solution :

$$\sigma = \sqrt{\frac{50^2 - 1}{12}} = \sqrt{\frac{2500 - 1}{12}} = \sqrt{\frac{2499}{12}} = \sqrt{208.25} = 14.431$$

(c) **Standard Deviation is independent of change of origin :**

Standard Deviation of a series is not affected by the change of origin. Change of origin means each item of a series is either increased or decreased by a constant number. Here the resultant change of origin will not change the value of Standard Deviation.

Illustration 10 :

Calculate the Standard Deviation of original group and new group if each item of the original group is increased by 3.

Original Group (x_1) =	3	5	7
New Group (x_2) =	6	8	10

Solution :

Group 1 (Original Group) (x_1)	Deviation from Mean (d_1)	Squares of Deviations $\left(d_1^2\right)$
3	–2	4
5	0	0
7	+2	4
$\sum x_1 = 15$		$\sum d_1^2 = 8$
$n_1 = 3$		

$$\text{Mean of Group 1} = \frac{\sum x_1}{n_1} = \frac{15}{3} = 5$$

$$\sigma_1 = \sqrt{\frac{\sum d_1^2}{n_1}} = \sqrt{\frac{8}{3}} = 1.633$$

Again,

Group 2 (New Group (x_2)	Deviation from Mean (d_2)	Squares of Deviations $\left(d_2^2\right)$
6	-2	4
8	0	0
10	2	4
$\sum x_2 = 24$		$\sum d_2^2 = 8$

Mean of Group 2 = $\frac{\sum x_2}{n_2} = \frac{24}{3} = 8$

Standard Deviation of Group 2 = $\sigma_2 = \sqrt{\frac{\sum d_2^2}{n_2}} = \sqrt{\frac{8}{3}} = 1.633$

Thus the Standard Deviation of both the groups are the same.

(d) **Standard Deviation is dependent upon the change in scale.** If each item of a group of data is either multiplied on divided by a constant number, then the value of Standard Deviation will either increase or decrease proportionately by the same constant.

Illustration 11.

Calculate and compute the Standard Deviation of the original series and new series if each item is decreased by 3 times in new series, from the following data:

Series I	Series II
15	5
21	7
27	9

Solution :

(x_1)	(d_1)	(d_1^2)	(x_2)	(d_2)	(d_2^2)
15	-6	36	5	-2	4
21	0	0	7	0	0
27	6	36	9	2	4
$\sum x_1 = 63$		$\sum d_1^2 = 72$	$\sum x_2 = 21$		$\sum d_2^2 = 8$

$\bar{x}_1 = \frac{\sum x_1}{n} = \frac{63}{3} = 21$

$\sigma_1 = \sqrt{\frac{\sum d_1^2}{n}} = \sqrt{\frac{72}{3}} = \sqrt{24} = 4.899$

$\bar{x}_2 = \frac{\sum x_2}{n} = \frac{21}{3} = 7$

$\sigma_2 = \sqrt{\frac{\sum d_2^2}{n_2}} = \sqrt{\frac{8}{3}} = 1.633$

$\therefore \sigma_2$ is also decreased by one-third.

(e) **There is a fixed relationship between Quartile Deviation, Mean Deviation and Standard Deviation.**

In a symmetrical or moderately asymmetrical series of a distribution, there is an empirical relationship between standard deviation and other measures of dispersion. The following properties exist among them :

(i) Quartile Deviation = $\frac{2}{3}$ of Standard Deviation

Hence QD < SD.

(ii) Mean Deviation = $\frac{4}{5}$ of Standard Deviation.

Hence MD < SD and also QD < MD.

The Quartile Deviation is less than Mean Deviation and Mean Deviation is less than Standard Deviation. Their overall relationship is like this : QD < MD < SD.

Illustration 12

If Standard Deviation of series is 15, determine the value of Quartile Deviation and Mean Deviation to prove QD < MD < SD.

Solution :

$$SD = 15$$

$$\therefore QD = \frac{2}{3} \times 15 = 10$$

$$MD = \frac{4}{5} \times 15 = 12$$

Now we get QD=10, MD=12 for SD =15.

$$\therefore QD(10) < MD(12) < SD(15).$$

(f) The sum of the squares of deviations of items from its Arithmetic Mean in a series is the lowest compared to the sum of squares of deviations from any other averages i.e., Median and Mode.

(g) **Variance can be calculated when standard deviation is known.** R.A. Fisher introduced variance for the first time.

Variance (v) = Square of Standard Deviation = σ^2.

$v = \sigma^2$., $\therefore \sigma = \sqrt{v}$.

(h) **Probable Error / Standard Error** can be Calculated from Standard Deviation.

SEσ / PEσ = 0.6745σ.

12.7 Merits and limitations of Standard Deviation :

Standard Deviation is the most ideal and largely used measure of dispersion. Its merits are :

(i) It is based on all the observations / items of the series.

(ii) It is rigidly defined.

(iii) It is amendable to further algebraic treatment.

(iv) It is the best measure of dispersion as the sum of the squares of deviations from Arithmetic Mean is the lowest compared to the squares of deviations taken from any other average.

(v) It is not affected by fluctuations of sampling.

(vi) It is useful in further statistical analysis for calculating Skewness, correlation analysis, regression analysis etc.

In spite of its popularity, wide acceptability and higher degree of accuracy, it is not free from flaws and limitations. The limitations of Standard Deviation are :

(i) It is difficult to compute.

(ii) It cannot be computed in open end series.

(iii) It gives more weightage to extreme items and less weightage to those items which are nearer to mean.

(iv) It is unable to compare two or more sets of data expressed in different unit of measurement.

12.8 : Coefficient of Standard Deviation and Coefficient of Variation :

Standard Deviation is an absolute measure of dispersion like the formerly discussed dispersions such as Range, Quartile Deviation and Mean Deviation. Standard Deviation as an absolute measure is expressed in terms of the unit of measurement of the original items. For example, for expressing the Standard Deviation of height and income or expenditure, centimeters/meters and rupees are used respectively.

But to make the utility of Standard Deviation more meaningful, its relative measures are used. The relative measures of Standard Deviation are used mostly in comparative studies. These are :

(i) Coefficient of Standard Deviation

(ii) Coefficient of Variation.

Coefficient of Standard Deviation = $\frac{\sigma}{\bar{x}}$, where

σ = Standard Deviation and

$\bar{x}$ = Arithmetic Average of the series.

Coefficient of Variation is expressed as a percentage of the coefficient of Standard Deviation It was developed by Karl Pearson :

(i) For comparing two or more groups of data, expressed in different units of measurement;

(ii) For comparing data groups in the same unit of measurement, but where their mean values differ widely.

(iii) For two or more sets of data in same units of measurement having no difference or insignificant difference in their mean values.

Coefficient of variation is computed as follows :

$$\text{Coefficient of Variation (CV)} = \frac{\sigma}{\bar{x}} \times 100$$

$$= \text{Coefficient of standard deviation} \times 100.$$

While comparing the variability between different distributions, the distribution giving the minimum coefficient of variation is proved as less variable, more consistent,, more uniform, more stable and more homogeneous and vice versa.

Coefficient of variation can also be applicable to compare the price trends in stock and commodity markets.

12.9 QUESTIONS :

1. From the alternatives given below in each bit, choose and write the correct answer alongwith its serial numbers.

Multiple choice questions :

(a) Standard deviation of 3 and 4 is :

(i) 2 (ii) 3.5 (iii) 1.5 (iv) 5

(b) The concept of Standard Deviation was developed by :

(i) Croxton and Cowden (ii) Karl Pearson

(iii) Lord Bowley (iv) Clark

(c) Coefficient of Standard Deviation is the :

(i) Absolute measure of dispersion (ii) Relative measure of dispersion

(iii) Square root of variance (iv) Root-mean square deviation

(d) Coefficient of variation is :

(i) $\frac{\sigma}{\bar{x}}$ (ii) σ^2 (iii) $\frac{\sigma}{100}\times 100$ (iv) $\frac{\sigma}{\text{Median}}$

(e) One of the following, which is not required to compute Combined Standard Deviation is :

(i) Median (ii) Mean

(iii) Standard Deviation (iv) Number of items

(f) The measure of dispersion suitable for studying fluctuations in share prices, is :

(i) Quartile Deviation (ii) Mean Deviation

(iii) Range (iv) Standard Deviation

(g) If each of the observations of a series is divided by 3, the Standard Deviation of the new observations is :

(i) $\frac{1}{3}$rd of the S.D. of the original observations

(ii) 3 times of the S.D. of the original observations

(iii) 6 times of the S.D. of the original observations

(iv) Not changed.

(h) If each observation of a series is decreased by 5, then the standard deviation of the new observations is :

(i) Decreased by 5 (ii) increased by 5

(iii) becomes 5 times of the S.D. of original observations (iv) Not changed

(i) The relationship between Mean Deviation and Standard Deviation is :

(i) 3 MD = 2 SD (ii) 6 MD = 5 SD

(iii) 5 MD = 4 SD (iv) MD = SD

(j) Standard error calculated from SD is equal to :

(i) 0.6745σ (ii) 0.7654σ (iii) 0.5476σ (iv) 0.06745σ

Answer : (a) (iii); (b) (ii); (c) (ii); (d) (iii);
(e) (i); (f) (iv); (g) (i); (h) (iv);
(i) (iii); (j) (i).

2. Fill in the gaps :

(a) _______ is the best measure of dispersion.

(b) Standard Deviation is ____________ of the change of origin.

(c) SD is _________ upon change in scale.

(d) Coefficient of variation is the _______ of coefficient of Standard Deviation.

(e) The Standard Deviation of first 'n' natural numbers is _________.

(f) Standard Deviation can only be calculated from ________.

(g) SD is free from any defects suffered by __________ measures of dispersion.

(h) S.D. is capable of __________ algebraic treatment.

(i) When mean is 12, variance is 9, the coefficient of variation is ________.

(j) If in a series the coefficient of variations is 55.42%, and Mean is 46, the Standard Deviation shall be _______.

(k) If the coefficient of variation of a series is 48.86 and its Standard Deviation is 24.43, the Arithmetic Mean shall be _______.

l. The higher the degree of variability, the ________ is the consistency in the value of variables.

Answer : (a) Standard Deviation, (b) independent, (c) dependent, (d) percentage

(e) $\sqrt{\frac{n^2-1}{12}}$

(f) Mean, (g) other, (h) further, (i) 25%, (j) 25.495, (k) 50, (l) lesser

3. Answer the following questions in one sentence each :

(a) Define standard deviation.

(b) Which measure of dispersion is known as the second moment of dispersion ?

(c) What is coefficient of variation ?

(d) Why do we use coefficient of variation ?

(e) Which method of dispersion is considered as the best and why ?

(f) Give the mathematical formula for calculating combined standard deviation of two groups.

(g) Define variance.

Answer :

(a) Standard Deviation is the square of the arithmetic mean of the squares of deviations measured from Mean.

(b) Standard Deviation

(c) Coefficient of variation is a relative measure of standard deviation where it is equal to $\frac{\sigma}{\bar{x}} \times 100$ (expressed as percentage)

(d) Coefficient of variation is used for comparing two or more series.

(e) Standard deviation as a measure of dispersion is the best because it covers all the items in the series and it takes into account the ± signs of the deviations from mean.

(f) Combined standard deviation = (σ_{12})

$$= \sqrt{\frac{N_1\sigma_1^2 + N_1(\bar{x}_1 - \bar{x}_{12})^2 + N_2\sigma_2 + N_2(\bar{x}_2 - \bar{x}_{12})^2}{N_1 + N_2}}$$

(g) Variance is the square of standard deviation (v) = σ^2.

4. Express the following in one word/ term each :

(i) The square of the standard deviation

(ii) Relative measure of standard deviation

(iii) The measure of dispersion always computed from Mean as the average.

(iv) The usually accepted measure of dispersion for studying relative effect.

Answer : (i) variance; (ii) coefficient of standard deviation; (iii) Standard deviation; (iv) coefficient of Standard Deviation or coefficient of variation.

5. Correct the underlined portion of the following sentences :

(a) Mean Deviation is the best measure of dispersion.

(b) Standard deviation can be calculated from any average.

(c) The higher the coefficient of variation, the greater would be the consistency.

(d) The lower the coefficient variation the lesser would be the consistency.

(e) Variance can be calculated from mean deviation.

(f) S.D. is affected by fluctuations of sampling.

(g) SD can be applicable to measure pricing trend in stock markets.

(h) Karl Person first introduced variance.

(i) R.A. Fisher first used SD.

(j) SD gives comparatively lesser importance to extreme items.

Answer : (a) Standard Deviation, (b) Arithmetic, (c) lesser (d) greater (e) standard (f) not affected (g) C.V.,(h) S.D. (i) variance (j) greater.

6. **Answer the following questions within 30 words each :**

(a) Name the methods of measuring dispersion.

(b) State any two features of Standard Deviation.

(c) Name the methods of relative measures of dispersion, and give their formulae.

(d) What are coefficient of variation and variance ?

(e) State the relationship between MD, SD and QD.

(f) Find out the S.D of natural numbers from 1 to 50.

(g) State any three merits of Standard Deviation.

(h) State the differences between MD and SD.

(i) Give any three demerits of Standard Deviation.

7. **Answer the following questions within 50 words each :**

(a) Distinguish between MD and SD.

(b) Give any three merits of standard deviation.

(c) What do you mean by probable error ?

(d) Why is standard deviation considered as the best measure of dispersion ?

(e) State and explain the formula to compute combined Standard Deviation.

Group - C

Long Answers

8. Define dispersion. Explain the features of a good measure of dispersion.

9. What is Standard Deviation ? Discuss its merits and demerits.

10. Define Standard Deviation. Explain the properties of Standard Deviation.

11. The following table gives the weekly pocket expenditure of 100 students selected at random. Calculate the Standard Deviation of their expenditure.

Expenditure (₹)	60-62	62-64	64-66	66-68	68-70	70-72
Number of Students :	5	18	42	20	8	7

Answer : SD = ₹ 2.42.

12. Calculate the Mean and Standard Deviation and Coefficient of SD from the following distribution :

Age (Years) :	20-25	25-30	30-35	35-40	40-45	45-50
Number of Persons :	170	110	80	45	40	35

Answer : Mean = 30.21, SD = 7.94, Coefficient of SD = 0.263.

13. Calculate the Standard Deviation and Coefficient of variation from the following data :

Value (₹) :	90-99	80-89	70-79	60-69	50-59	40-49	30-39
Number of Persons :	2	12	22	20	14	4	1

Answer : Mean = 68.10, SD = 12.51, Coefficient of variation = 18.37%.

14. A consignment of 180 articles is classified according to the size of the articles as under. Find the Standard Deviation and its Coefficient :

Size (More than) :	90	80	70	60	50	40	30	20	10	0
Number of Articles :	0	5	14	34	65	110	150	170	176	180

Answer : Mean = 45.222, SD = 16.83, Coefficient of SD = 0.372.

15. From the following data, calculate the Standard Deviation and its Coefficient :

Size :	44-46	46-48	48-50	50-52	52-54
Frequency (f) :	3	24	27	21	5

Answer : Mean = 49.025, SD = 1.962, Coefficient of SD = 0.04.

16. Calculate SD and CV for the following distribution :

Class :	4-8	8-12	12-16	16-20	20-24	24-28	28-32	32-36	36-40
Frequency :	11	13	16	14	14	9	17	6	4

Answer : SD = 9.287, CV = 46.62%.

17. A factory produces two types of electric lamps P and Q. In an experiment relating to their life, the following results were obtained :

Length of life (in Hours)	Number of Lamps	
	Type - P	Type - Q
500-700	5	4
700-900	11	30
900-1100	26	12
1100-1300	10	8
1300-1500	8	6
Total	60	60

Compute the variability of the two varieties of lamps using coefficient of variation.

Answer : CV (P) = 21.6%, CV (Q) = 23.4%

18. The followings are the scores of two batsmen for 8 matches.

Rahul :	12	115	76	42	7	19	49	80
Kuldeep :	47	12	76	73	24	51	63	54

Who, among the two, is the more consistent batsman ?

Answer : CV (R) = 71.14%, CV (K) = 41.96%.

So, Kuleeep is the more consistent batsman.

❖❖❖

APPENDIX

Appendix 1—Numerical Tables

TABLE 1 : LOGARITHMS

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
10	.0000	0043	0086	0120	0170	0212	0253	0294	0334	0374	4	8	12	17	21	25	29	33	37
11	.0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	4	8	11	15	19	23	26	30	34
12	.0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	3	7	10	14	17	21	24	28	31
13	.1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	3	6	10	13	16	19	23	26	29
14	.1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	3	6	9	12	15	18	21	24	27
15	.1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	3	6	8	11	14	17	20	22	25
16	.2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	3	5	8	11	13	16	18	21	24
17	.2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	2	5	7	10	12	15	17	20	22
18	.2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	2	5	7	9	12	14	16	19	21
19	.2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	2	4	7	9	11	13	16	18	20
20	.3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2	4	6	8	11	13	15	17	19
21	.3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	16	18
22	.3424	3444	3464	3483	3502	3522	3541	3562	3579	3598	2	4	6	8	10	12	14	15	17
23	.3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	6	7	9	11	13	15	17
24	.3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9	11	12	14	16
25	.3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	3	5	7	9	10	12	14	15
26	.4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2	3	5	7	8	10	11	13	15
27	.4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2	3	5	6	8	9	11	13	14
28	.4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2	3	5	6	8	9	11	12	14
29	.4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1	3	4	6	7	9	10	12	13
30	.4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1	3	4	6	7	9	10	11	13
31	.4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1	3	4	6	7	8	10	11	12
32	.5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1	3	4	5	7	8	9	11	12
33	.5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1	3	4	5	6	8	9	10	12
34	.5315	5315	5340	5353	5366	5378	5391	5403	5416	5428	1	3	4	5	6	8	9	10	11
35	.5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1	2	4	5	6	7	9	10	11
36	.5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1	2	4	5	6	7	8	10	11
37	.5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1	2	3	5	6	7	8	9	10
38	.5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1	2	3	5	6	7	8	9	10
39	.5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1	2	3	4	5	7	8	9	10
40	.6021	6031	6042	6053	6065	6075	6085	6096	6107	6117	1	2	3	4	5	6	8	9	10
41	.6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1	2	3	4	5	6	7	8	9
42	.6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1	2	3	4	5	6	7	8	9
43	.6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1	2	3	4	5	6	7	8	9
44	.6435	6444	6454	6465	6474	6484	6493	6503	6513	6522	1	2	3	4	5	6	7	8	9
45	.6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	1	2	3	4	5	6	7	8	9
46	.6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	1	2	3	4	5	6	7	7	8
47	.6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1	2	3	4	5	5	6	7	8
48	.6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1	2	3	4	4	5	6	7	8
49	.6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1	2	3	4	4	5	6	7	8
50	.6990	6993	7007	7016	7024	7033	7042	7050	7059	7067	1	2	3	3	4	5	6	7	8
51	.7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1	2	3	3	4	5	6	7	8
52	.7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1	2	2	3	4	5	6	7	7
53	.7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1	2	2	3	4	5	6	6	7

TABLE 1 : LOGARITHMS

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
54	.7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1	2	2	3	4	5	6	6	7
55	.7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1	2	2	3	4	5	5	6	7
56	.7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1	2	2	3	4	5	5	6	7
57	.7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1	2	2	3	4	5	5	6	7
58	.7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1	1	2	3	4	4	5	6	7
59	.7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1	1	2	3	4	4	5	6	7
60	.7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	1	1	2	3	4	4	5	6	6
61	.7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	1	1	2	3	4	4	5	6	6
62	.7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1	1	2	3	3	4	5	6	6
63	.7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1	1	2	3	3	4	5	5	6
64	.8062	8069	8075	8082	8989	8096	8102	8109	8116	8122	1	1	2	3	3	4	5	5	6
65	.8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1	1	2	3	3	4	5	5	6
66	.8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	1	1	2	3	3	4	5	5	6
67	.8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1	1	2	3	3	4	5	5	6
68	.8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1	1	2	3	3	4	4	5	6
69	.8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1	1	2	2	3	4	4	5	6
70	.8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	1	1	2	2	3	4	4	5	6
71	.8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1	1	2	2	3	4	4	5	5
72	.8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	1	1	2	2	3	4	4	5	5
73	.8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	1	1	2	2	3	4	4	5	5
74	.8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	1	1	2	2	3	4	4	5	5
75	.8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	1	1	2	2	3	3	4	5	5
76	.8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1	1	2	2	3	3	4	5	5
77	.8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1	1	2	2	3	3	4	4	5
78	.8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	1	1	2	2	3	3	4	4	5
79	.8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	1	1	2	2	3	3	4	4	5
80	.9031	9036	9042	9047	9053	9058	6063	9069	9074	9079	1	1	2	2	3	3	4	4	5
81	.9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	1	1	2	2	3	3	4	4	5
82	.9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	1	1	2	2	3	3	4	4	5
83	.9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1	1	2	2	3	3	4	4	5
84	.9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1	1	2	2	3	3	4	4	5
85	.9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1	1	2	2	3	3	4	4	5
86	.9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1	1	2	2	3	3	4	4	5
87	.9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	0	1	1	2	2	3	3	4	4
88	.9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	0	1	1	2	2	3	3	4	4
89	.9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	0	1	1	2	2	3	3	4	4
90	.9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	0	1	1	2	2	3	3	4	4
91	.9590	9595	9600	9605	9609	9614	9619	9624	9628	9933	0	1	1	2	2	3	3	4	4
92	.9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	0	1	1	2	2	3	3	4	4
93	.9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	0	1	1	2	2	3	3	4	4
94	.9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	0	1	1	2	2	3	3	4	4
95	.9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	0	1	1	2	2	3	3	4	4
96	.9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	0	1	1	2	2	3	3	4	4
97	.9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	0	1	1	2	2	3	3	4	4
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	0	1	1	2	2	3	3	4	4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0	1	1	2	2	3	3	3	4

TABLE II : ANTILOGARITHMS

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
.00	1000	1002	1005	1007	1009	1012	1014	1016	1019	1021	0	0	1	1	1	1	2	2	2
.01	1023	1026	1028	1030	1033	1035	1038	1040	1042	1045	0	0	1	1	1	1	2	2	2
.02	1047	1050	1052	1054	1057	1059	1062	1064	1067	1069	0	0	1	1	1	1	2	2	2
.03	1072	1074	1076	1079	1081	1084	1086	1089	1091	1094	0	0	1	1	1	1	2	2	2
.04	1096	1099	1102	1104	1107	1109	1112	1114	1117	1119	0	1	1	1	1	2	2	2	2
.05	1122	1125	1127	1130	1132	1135	1138	1140	1143	1146	0	1	1	1	1	2	2	2	2
.06	1148	1151	1153	1156	1159	1161	1164	1167	1169	1172	0	1	1	1	1	2	2	2	2
.07	1175	1178	1180	1183	1186	1189	1191	1194	1197	1199	0	1	1	1	1	2	2	2	2
.08	1202	1205	1208	1211	1213	1216	1219	1222	1225	1227	0	1	1	1	1	2	2	2	3
.09	1230	1233	1236	1239	1242	1245	1247	1250	1253	1256	0	1	1	1	1	2	2	2	3
.10	1259	1262	1265	1268	1271	1274	1276	1279	1282	1285	0	1	1	1	1	2	2	2	3
.11	1288	1291	1294	1297	1300	1303	1306	1309	1312	1315	0	1	1	1	2	2	2	2	3
.12	1318	1321	1324	1327	1330	1334	1337	1340	1343	1346	0	1	1	1	2	2	2	2	3
.13	1349	1352	1355	1358	1361	1365	1368	1371	1374	1377	0	1	1	1	2	2	2	3	3
.14	1380	1384	1387	1390	1393	1396	1400	1403	1406	1409	0	1	1	1	2	2	2	3	3
.15	1413	1416	1419	1422	1426	1429	1432	1435	1439	1442	0	1	1	1	2	2	2	3	3
.16	1445	1449	1452	1455	1459	1462	1466	1469	1472	1476	0	1	1	1	2	2	2	3	3
.17	1479	1483	1486	1489	1493	1496	1500	1503	1507	1510	0	1	1	1	2	2	2	3	3
.18	1514	1517	1521	1524	1528	1531	1535	1538	1542	1545	0	1	1	1	2	2	2	3	3
.19	1549	1552	1556	1560	1563	1567	1570	1574	1578	1581	0	1	1	1	2	2	3	3	3
.20	1585	1589	1592	1596	1600	1603	1607	1611	1614	1618	0	1	1	1	2	2	3	3	3
.21	1622	1626	1629	1633	1637	1641	1644	1648	1652	1656	0	1	1	2	2	2	3	3	3
.22	1660	1663	1667	1671	1675	1679	1683	1687	1690	1694	0	1	1	2	2	2	3	3	3
.23	1698	1702	1706	1710	1714	1718	1722	1726	1730	1734	0	1	1	2	2	2	3	3	4
.24	1738	1742	1746	1750	1754	1758	1762	1766	1770	1774	0	1	1	2	2	2	3	3	4
.25	1778	1782	1786	1791	1795	1799	1803	1807	1811	1816	0	1	1	2	2	2	3	3	4
.26	1820	1824	1828	1832	1837	1841	1845	1849	1854	1858	0	1	1	2	2	3	3	3	4
.27	1862	1866	1871	1875	1879	1884	1888	1892	1897	1901	0	1	1	2	2	3	3	3	4
.28	1905	1910	1914	1919	1923	1928	1932	1936	1941	1945	0	1	1	2	2	3	3	4	4
.29	1950	1954	1959	1963	1968	1972	1977	1982	1986	1991	0	1	1	2	2	3	3	4	4
.30	1995	2000	2004	2009	2014	2018	2023	2028	2032	2037	0	1	1	2	2	3	3	4	4
.31	2042	2046	2051	2056	2061	2065	2070	2075	2080	2084	0	1	1	2	2	3	3	4	4
.32	2089	2094	2099	2104	2109	2113	2118	2123	2128	2133	0	1	1	2	2	3	3	4	4
.33	2138	2143	2148	2153	2158	2163	2168	2173	2178	2183	0	1	1	2	2	3	3	4	4
.34	2188	2193	2198	2203	2208	2213	2218	2223	2228	2234	1	1	2	2	3	3	4	4	5
.35	2239	2244	2249	2254	2259	2265	2270	2275	2280	2286	1	1	2	2	3	3	4	4	5
.36	2291	2296	2301	2307	2312	2317	2323	2328	2333	2339	1	1	2	2	3	3	4	4	5
.37	2344	2350	2355	2360	2366	2371	2377	2382	2388	2393	1	1	2	2	3	3	4	4	5
.38	2399	2404	2410	2415	2421	2427	2432	2438	2443	2449	1	1	2	2	3	3	4	4	5
.39	2455	2460	2466	2472	2477	2483	2489	2495	2500	2506	1	1	2	2	3	3	4	5	5
.40	2513	2518	2523	2529	2535	2541	2547	2553	2559	2564	1	1	2	2	3	4	4	5	5
.41	2570	2576	2582	2588	2594	2600	2606	2612	2618	2624	1	1	2	2	3	4	4	5	5
.42	2630	2636	2642	2649	2655	2661	2667	2673	2679	2685	1	1	2	2	3	4	4	5	6
.43	2692	2698	2704	2710	2716	2723	2729	2735	2742	2748	1	1	2	3	3	4	4	5	6
.44	2754	2761	2767	2773	2780	2786	2793	2799	2805	2812	1	1	2	3	3	4	4	5	6
.45	2818	2825	2831	2838	2844	2851	2858	2864	2871	2877	1	1	2	3	3	4	5	5	6
.46	2884	2891	2897	2904	2911	2917	2924	2931	2938	2944	1	1	2	3	3	4	5	5	6
.47	2951	2958	2965	2972	2979	2985	2992	2999	3006	3013	1	1	2	3	3	4	5	5	6
.48	3020	3027	3034	3041	3048	3055	3062	3069	3076	3083	1	1	2	3	4	4	5	6	6
.49	3090	3097	3105	3112	3119	3126	3133	3141	3148	3155	1	1	2	3	4	4	5	6	6

TABLE II : ANTILOGARITHMS

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
.50	3162	3170	3177	3184	3192	3199	3206	3214	3221	3228	1	1	2	3	4	4	5	6	7
.51	3236	3243	3251	3258	3266	3273	3281	3289	3296	3304	1	2	2	3	4	5	5	6	7
.52	3311	3319	3327	3334	3342	3350	3357	3365	3373	3381	1	2	2	3	4	5	5	6	7
.53	3388	3396	3404	3412	3420	3428	3436	3443	3451	3459	1	2	2	3	4	5	6	6	7
.54	3467	3475	3483	3491	3499	3508	3516	3524	3532	3540	1	2	2	3	4	5	6	6	7
.55	3548	3556	3565	3573	3581	3589	3597	3606	3614	3622	1	2	2	3	4	5	6	7	7
.56	3631	3639	3648	3656	3664	3673	3681	3690	3698	3707	1	2	3	3	4	5	6	7	8
.57	3715	3724	3733	3741	3750	3758	3767	3776	3784	3793	1	2	3	3	4	5	6	7	8
.58	3802	3811	3819	3828	3837	3846	3855	3864	3873	3882	1	2	3	4	4	5	6	7	8
.59	3890	3899	3908	3917	3926	3936	3945	3954	3963	3972	1	2	3	4	5	5	6	7	8
.60	3981	3990	3999	4009	4018	4027	4036	4046	4055	4064	1	2	3	4	5	6	6	7	8
.61	4074	4083	4093	4102	4111	4121	4130	4140	4150	4159	1	2	3	4	5	6	7	8	9
.62	4169	4178	4188	4198	4207	4217	4227	4236	4246	4256	1	2	3	4	5	6	7	8	9
.63	4266	4276	4285	4295	4305	4315	4325	4335	4345	4355	1	2	3	4	5	6	7	8	9
.64	4365	4375	4385	4395	4406	4416	4426	4436	4446	4457	1	2	3	4	5	6	7	8	9
.65	4467	4477	4487	4498	4508	4519	4529	4539	4550	4560	1	2	3	4	5	6	7	8	9
.66	4571	4581	4592	4603	4613	4624	4634	4645	4656	4667	1	2	3	4	5	6	7	9	10
.67	4677	4688	4699	4710	4721	4732	4742	4753	4764	4775	1	2	3	4	5	7	8	9	10
.68	4786	4797	4808	4819	4831	4842	4853	4864	4875	4887	1	2	3	4	6	7	8	9	10
.69	4898	4909	4920	4932	4943	4955	4966	4977	4989	5000	1	2	3	5	6	7	8	9	10
.70	5012	5023	5035	5047	5058	5070	5082	5093	5105	5117	1	2	4	5	6	7	8	9	11
.71	5129	5140	5152	5164	5176	5188	5200	5212	5224	5236	1	2	4	5	6	7	8	10	11
.72	5248	5260	5272	5284	5297	5309	5321	5333	5346	5358	1	2	4	5	6	7	9	10	11
.73	5370	5383	5395	5408	5420	5433	5445	5458	5470	5483	1	3	4	5	6	8	9	10	11
.74	5495	5508	5521	5534	5546	5559	5572	5585	5598	5610	1	3	4	5	6	8	9	10	12
.75	5623	5636	5649	5662	5675	5689	5702	5715	5728	5741	1	3	4	5	7	8	9	10	12
.76	5754	5768	5781	5794	5808	5821	5834	5848	5861	5875	1	3	4	5	7	8	9	11	12
.77	5888	5902	5916	5929	5943	5957	5970	5984	5998	6012	1	3	4	5	7	8	10	11	12
.78	6026	6039	6053	6067	6081	6095	6109	6124	6138	6152	1	3	4	6	7	8	10	11	13
.79	6166	6180	6194	6209	6223	6237	6252	6266	6281	6295	1	3	4	6	7	9	10	11	13
.80	6310	6324	6339	6353	6368	6383	6397	6412	6427	6442	1	3	4	6	7	9	10	12	13
.81	6457	6471	6486	6501	6516	6531	6546	6561	6577	6592	2	3	5	6	8	9	11	12	14
.82	6607	6622	6637	6653	6668	6683	6699	6714	6730	6745	2	3	5	6	8	9	11	12	14
.83	6761	6776	6792	6808	6823	6839	6855	6871	6887	6902	2	3	5	6	8	9	11	13	14
.84	6918	6934	6950	6966	6982	6998	7015	7031	7047	7063	2	3	5	6	8	10	11	13	15
.85	7079	7096	7112	7129	7145	7161	7178	7194	7211	7228	2	3	5	7	8	10	12	13	15
.86	7244	7261	7278	7295	7311	7328	7345	7362	7379	7396	2	3	5	7	8	10	12	13	15
.87	7413	7430	7447	7464	7482	7499	7516	7534	7551	7568	2	3	5	7	9	10	12	14	16
.88	7586	7603	7621	7638	7656	7674	7691	7709	7727	7745	2	4	5	7	9	11	12	14	16
.89	7762	7780	7798	7816	7834	7852	7870	7889	7907	7925	2	4	5	7	9	11	13	14	16
.90	7943	7962	7980	7998	8017	8035	8054	8072	8091	8110	2	4	6	7	9	11	13	15	17
.91	8128	8147	8166	8185	8204	8222	8241	8260	8279	8299	2	4	6	8	9	11	13	15	17
.92	8318	8337	8356	8375	8395	8414	8433	8453	8472	8492	2	4	6	8	10	12	14	15	17
.93	8511	8531	8551	8570	8590	8610	8630	8650	8670	8690	2	4	6	8	10	12	14	16	18
.94	8710	8730	8750	8770	8790	8810	8831	8851	8872	8892	2	4	6	8	10	12	14	16	18
.95	8913	8933	8954	8974	8995	9016	9036	9057	9078	9099	2	4	6	8	10	12	15	17	19
.96	9120	9141	9162	9183	9204	9226	9247	9268	9290	9311	2	4	6	8	11	13	15	17	19
.97	9333	9354	9376	9397	9419	9441	9462	9484	9506	9528	2	4	7	9	11	13	15	17	20
.98	9550	9572	9594	9616	9638	9661	9683	9705	9227	9750	2	4	7	9	11	13	16	18	20
.99	9772	9795	9817	9840	9863	9886	9908	9931	9954	9977	2	5	7	9	11	14	16	18	20

III. POWERS, ROOTS AND RECIPROCALS

n	n^2	n^3	$\sqrt{n}$	$\sqrt[3]{n}$	$\sqrt{10n}$	$\sqrt[3]{10n}$	$\sqrt[3]{100n}$	$\frac{1}{n}$
1	1	1	1	1	3.162	2.154	4.642	1
2	4	8	1.414	1.260	4.472	2.714	5.848	.5000
3	9	27	1.732	1.442	5.477	3.107	6.694	.3333
4	16	64	2	1.587	6.325	3.420	7.638	.2500
5	25	125	2.236	1.710	7.701	3.684	7.937	.2000
6	36	216	2.449	1.817	7.746	3.915	8.434	.1667
7	49	343	2.646	1.913	8.361	4.121	8.879	.1429
8	64	512	2.828	2.000	8.944	4.309	9.283	.1250
9	81	729	3.000	2.080	9.487	4.481	9.655	.1111
10	100	1000	3.162	2.154	10.0	4.642	10.000	.1000
11	121	1331	3.317	2.224	10.488	4.791	10.323	.09091
12	144	1728	3.464	2.289	10.954	4.932	10.627	.08333
13	169	2197	3.606	2.351	11.402	5.066	10.914	.07692
14	196	2744	3.742	2.410	11.832	5.192	11.187	.07143
15	225	3375	3.873	2.466	12.247	5.313	11.447	.06667
16	256	4096	4.000	2.520	12.649	5.429	11.696	.06250
17	289	5913	4.123	2.571	13.038	5.540	11.935	.05882
18	324	5832	4.243	2.621	13.416	5.646	12.164	.05556
19	361	6859	4.359	2.668	13.784	5.749	12.386	.05263
20	400	8000	4.472	2.714	14.142	5.848	12.599	.0500
21	441	9261	4.583	2.759	14.491	5.944	12.806	1.04762
22	484	10648	4.690	2.802	14.832	6.037	13.006	.04545
23	529	12167	4.796	2.844	15.166	6.127	13.200	.04348
24	576	13824	4.899	2.884	15.492	6.214	13.389	.04167
25	625	15625	5.000	2.924	15.811	6.300	13.572	.0400
26	676	17576	5.099	2.962	16.125	6.383	13.751	.03846
27	729	19683	5.196	3.000	16.432	6.463	13.925	.03704
28	784	21952	5.292	3.037	16.733	6.542	14.095	.03571
29	841	24389	5.385	3.072	17.029	6.619	14.260	.03448
30	900	27000	5.477	3.107	17.321	6.694	14.422	.03333
31	961	29791	5.568	3.141	17.607	6.768	14.581	.03226
32	1024	32768	5.657	3.175	17.889	6.840	14.736	.03125
33	1089	35937	5.745	3.208	18.166	6.910	14.888	.03030
34	1156	39304	5.831	3.240	18.439	6.980	15.037	.02941
35	1225	42875	5.916	3.271	18.708	7.047	15.183	.02857
36	1296	46656	6.000	3.302	18.974	7.114	15.326	.02778
37	1369	50653	6.083	3.332	19.235	7.179	15.467	.02703
38	1444	54872	6.164	3.362	19.494	7.243	15.605	.02632
39	1521	59319	6.245	3.391	19.748	7.306	15.741	.02564
40	1600	64000	6.325	3.420	20.00	7.368	15.874	.0250
41	1681	68921	6.403	3.448	20.248	7.429	16.005	.02439
42	1764	74088	6.481	3.476	20.494	7.489	16.134	.02381
43	1849	79507	6.557	3.503	20.736	7.548	16.261	.02326
44	1936	85184	6.633	3.530	20.976	7.606	16.386	.02273
45	2025	91125	6.708	3.557	21.213	7.663	16.510	.02222
46	2116	97336	6.782	3.583	21.448	7.719	16.631	.02174
47	2209	103823	6.856	3.609	21.679	7.775	16.751	.02128
48	2304	110592	6.928	3.634	21.909	7.830	16.869	.02083
49	2401	117649	7.000	3.659	22.136	7.884	16.985	.02041
50	2500	125000	7.071	3.684	22.361	7.937	17.100	.020

III. POWERS, ROOTS AND RECIPROCALS

n	n^2	n^3	$\sqrt{n}$	$\sqrt[3]{n}$	$\sqrt{10n}$	$\sqrt[3]{10n}$	$\sqrt[3]{100n}$	$\frac{1}{n}$
51	2601	132651	7.141	3.708	22.583	7.990	17.213	.01961
52	2704	140608	7.211	3.733	22.804	8.041	17.325	.01923
53	2809	148877	7.280	3.756	23.022	8.093	17.435	.01887
54	2916	157464	7.348	3.780	23.238	8.143	17.544	.01852
55	3025	166375	7.416	3.803	23.452	8.193	17.652	.01818
56	3136	175616	7.483	3.826	23.664	8.243	17.758	.01786
57	3249	185193	7.550	3.849	23.875	8.291	17.863	.01754
58	3364	195112	7.616	3.871	24.083	8.340	17.967	.01724
59	3481	205379	7.681	3.893	24.290	8.387	18.070	.01695
60	3600	216000	7.746	3.915	24.495	8.434	18.171	.01667
61	3721	226981	7.810	3.936	24.698	8.481	18.272	.01639
62	3844	238328	7.874	3.958	24.900	8.527	18.371	.01613
63	3969	250047	7.937	3.979	25.100	8.573	18.469	.01587
64	4096	262144	8.000	4.000	25.298	8.618	18.566	.01562
65	4225	274625	8.062	4.021	25.495	8.662	18.663	.01538
66	4356	287496	8.124	4.041	25.690	8.707	18.758	.01515
67	4489	300763	8.185	4.062	25.884	8.750	18.852	.01493
68	4624	314432	8.246	4.082	26.077	8.794	18.945	.01471
69	4761	328509	8.307	4.102	26.268	8.837	19.038	.01449
70	4900	343000	8.367	4.121	26.458	8.879	19.129	.01429
71	5041	357011	8.426	4.141	26.646	8.921	19.220	.01408
72	5184	373248	8.485	4.160	26.833	8.963	19.310	.01389
73	5329	389017	8.544	4.179	27.019	8.004	19.399	.01370
74	5476	405224	8.602	4.198	27.203	9.045	19.487	.01351
75	5625	421875	8.660	4.217	27.386	9.086	19.574	.01333
76	5776	438976	8.718	4.236	27.568	9.126	19.661	.01316
77	5929	456533	8.775	4.254	27.740	9.166	19.747	.01299
78	6084	474552	8.832	4.273	27.928	9.205	19.832	.01282
79	6241	493039	8.883	4.291	28.107	9.244	19.916	.01266
80	6400	512000	8.944	4.309	28.284	9.283	20.000	.01250
81	6561	531441	9.000	4.327	28.460	9.322	20.083	.01235
82	6724	551368	9.055	4.344	28.636	9.360	20.165	.01220
83	6889	571787	9.110	4.362	28.810	9.398	20.247	.01205
84	7056	592704	9.165	4.380	28.983	9.435	20.328	.01190
85	7225	614125	9.220	4.397	29.155	9.473	20.408	.01176
86	7396	636056	9.274	4.414	29.326	9.510	20.488	.01163
87	7569	658503	9.327	4.431	29.496	9.546	20.567	.01149
88	7744	681472	9.381	4.448	29.665	9.583	20.646	.01136
89	7921	704969	9.434	4.465	29.833	9.619	20.724	.01124
90	8100	729000	9.487	4.481	30.000	9.655	20.801	.01111
91	8281	753571	9.538	4.498	30.166	9.691	20.878	.01099
92	8464	778688	9.592	4.514	30.332	9.726	20.954	.01087
93	8649	804357	9.644	4.531	30.496	9.761	21.029	.01075
94	8836	830584	9.695	4.547	30.659	9.796	21.105	.01064
95	9025	857375	9.747	4.563	30.822	9.830	21.179	.01053
96	9216	884736	9.798	4.579	30.984	9.865	21.253	.01042
97	9409	912673	9.849	4.595	31.145	9.899	21.327	.01031
98	9604	941192	9.899	4.610	31.305	9.933	21.400	.01020
99	9801	970299	9.909	4.626	31.464	9.967	21.472	.01010
100	10000	1000000	10.000	4.642	31.623	10.000	21.544	.0100