APPLICATION OF DERIVATIVES

To live is the rarest thing in the world. Most people exist, that is all.

- Oscar Wilde

8.0 Introduction

Derivatives and related concepts have found application in almost every branch of scientific study. It is too difficult to enlist them here. Out of many applications we shall study in this section some of the simple ones, particularly rectilinear motion, behaviour of functions, tangents and normal to plane curves and extremization of functions. We shall further see how the method of differential calculus saves much of the labour in computing certain limits of functions.

8.1 Velocity and Acceleration in Rectilinear Motion

In physical problems where motion of any kind is studied, time, symbolised by the letter t is taken as the independent variable. The simplest case in this type of study is rectilinear motion i.e. when the motion is in a straight line. If the moving object travels equal distance in the same direction in equal intervals of time, the motion is called **uniform**. In this case the distance travelled in a given direction per unit time, called the velocity of the object, is constant. However, in nature, most of the motions are non-uniform. We study below non uniform rectilinear motion.

We co-ordinatise the line (L) of motion by specifying

(i) a unit of measurement of distance

(ii) a positive direction

and (iii) a fixed point *0* called the origin of distance from which all measurements are made.

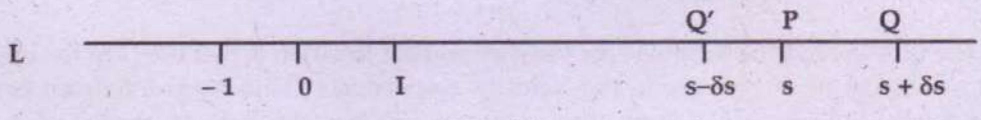


Fig - 8.1

Let a moving object be at P with coordinate s at time t. Clearly s is a functions of t which we denote by

$$s = f\ (t).$$

When the positive direction of motion is from left to right, let Q be the point where the object is at time $t + \delta t$. The distance travelled in time interval δt is $(s + \delta s) - s = \delta s$. The average velocity

$$= \frac{\text{distance}}{\text{time}}$$

$$= \frac{\delta s}{\delta t} = \frac{f\ (t + \delta t) - f\ (t)}{\delta t}.$$

Hence on taking limit as $\delta t \to 0$, whenever it exists, instantaneous velocity at time t is given by

$$v = \lim_{\delta t \to 0} \frac{\delta s}{\delta t} = \frac{ds}{dt} = f'\ (t).$$

When the motion is from right to left, i.e. when s is decreasing as time passes, δs becomes negative. So the distance travelled in time δt is $-\delta s$ and the instantaneous velocity would then be given by

$$\lim_{\delta t \to 0} \frac{\delta s}{\delta t} = -\frac{ds}{dt}.$$

Thus the sign of $\frac{ds}{dt}$ gives the direction of motion and the absolute value of $\frac{ds}{dt}$ gives the **speed** of the object at time t.

Acceleration of an object moving with a velocity v at any time t is defined to be the instantaneous rate of change of velocity i.e.

$$a = \frac{dv}{dt} = \frac{d^2s}{dt^2}.$$

Example: 1

A ball is thrown vertically upwards with initial speed of $98m/sec$. If the motion of the ball satisfies the equation $s = v_0t - \frac{1}{2}gt^2$ where v_0 is the initial velocity and g $(= 9.8m/sec^2)$ is the acceleration due to gravity, find the height to which the ball will rise.

Solution :

Here $s = v_0t - \frac{1}{2}gt^2$

$$\Rightarrow \quad v = \frac{ds}{dt} = v_0 - gt. \qquad \text{.........} (1)$$

When the ball rises its velocity gradually decreases and ultimately becomes zero. Then the ball begins to fall. So putting $v = 0$ in (1) we get

$$0 = v_0 - gt$$

$$\Rightarrow \quad t = \frac{v_0}{g} = \frac{98}{9.8} \text{ secs} = 10 \text{ secs}.$$

Thus the ball will begin to fall after 10 secs and by that time the ball will rise to a height of

$$s = 98 \times 10 - \frac{1}{2} \times 9.8 \times 10^2 = 490\text{m}.$$

The idea behind the formulae for velocity and acceleration is that they are the instantaneous rates of change of displacement and velocity respectively. Thus as pointed out earlier in the introduction, the concept of derivative is useful to determine rate of change of quantities in various fields provided of course that the quantity undergoes continuous change.

Example : 2

At a certain instant, the side of a square is increasing at the rate of .3cm/sec. and at the same time the area is increasing at the rate of 30sq.cm/sec. Find the length of the side of the square at this instant.

Solution :

Let the length of the side of the square at the given instant be x cm.

The rate at which it is increasing = $\frac{dx}{dt}$ = .3cm/sec.

The area of the square = x^2sq. cm.

The rate at which the area is increasing = $\frac{dx^2}{dt} = 2x\frac{dx}{dt} = 2x \times 0.3$sq.cm/sec.

According to the question,

$2x \times 0.3 = 30$

$\Rightarrow \quad x = \dfrac{30}{2 \times 0.3} = 50$

Hence the length of the side of the square at the given instant is 50 cm.

EXERCISE 8(a)

1. Find the velocity and acceleration at the end of 2 seconds of the particle moving according to the following rules.

   (i) $s = 2t^2 + 3t + 1$ (ii) $s = \sqrt{t} + 1$ (iii) $s = \dfrac{3}{2t + 1}$

   (iv) $s = t^3 - 6t^2 + 15t + 12$.

2. The sides of an equilateral triangle are increasing at the rate of $\sqrt{3}$ cm./sec. Find the rate at which the area of the triangle is increasing when the side is 4 cm. long.

3. Find the rate at which the volume of a spherical balloon will increase when its radius is 2 metres if the rate of increase of its radius is 0.3m/min.

4. The surface area of a cube is decreasing at the rate of 15sq.cm./sec. Find the rate at which its edge is decreasing when the length of the edge is 5cm.

8.2 Tangent and Normal to plane curves.

Let $\overleftrightarrow{PT}$ be the tangent to a curve $y = f(x)$ at the point P $(c, f(c))$ of the curve. As we have seen in 12.3, the slope of the line $\overleftrightarrow{PT}$ is given by

$$\tan\psi = f'(c)$$

provided that $\overleftrightarrow{PT}$ is not vertical. Hence the equation to the tangent line through $(c, f(c))$ is

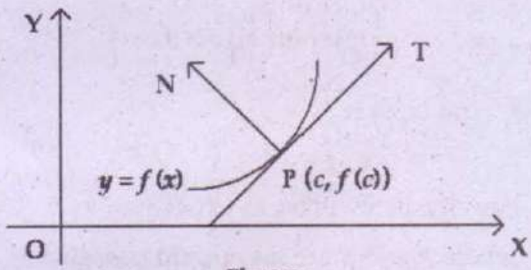


Fig 8.2

$$y - f(c) = f'(c)\ (x - c) \qquad \text{......... (1)}$$

provided the tangent line is not parallel to y-axis in which case, the equation is simply

$$x = c. \qquad \text{......... (2)}$$

A line $\overleftrightarrow{PN}$ perpendicular to the tangent $\overleftrightarrow{PT}$ is called a **normal** to the curve at P. When the tangent through $(c, f(c))$ is not parallel to the y-axis i.e. when $f'(c)$ is finite, the slope of the normal to the curve at $(c, f(c))$ is $-1/f'(c)$ and the equation to the normal is

$$y - f(c) = -\frac{1}{f'(c)}\ (x - c)$$

i.e. $$(x - c) + f'(c)\,[\,y - f(c)\,] = 0. \qquad \text{......... (3)}$$

When the tangent is parallel to the y-axis, the normal is parallel to the x-axis. Since it passes through the point $(c, f(c))$, the equation to the normal in this case becomes

$$y = f(c). \qquad \text{......... (4)}$$

Example 3 :

Find the equation to the tangent and normal to the curve $3x^2 + 5y^2 = 23$ at $(-1, 2)$.

Solution :

Differentiating the given equation implicitly we have

$$6x + 10y\,\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{3x}{5y}.$$

$$\therefore \quad \left.\frac{dy}{dx}\right]_{(-1,\,2)} = -\frac{3\,(-1)}{5.2} = \frac{3}{10} = \text{slope of the tangent.}$$

Hence the required equation of the tangent is

$$y - 2 = \frac{3}{10}\,(x - (-1))$$

$$\Rightarrow \quad 3x - 10y + 23 = 0.$$

The equation of the normal is

$$x - (-1) + \frac{3}{10}\,(y - 2) = 0$$

$$\Rightarrow \quad 10x + 3y + 4 = 0.$$

Example 4 :

Prove that the equation to a tangent to the curve

$$\left(\frac{x}{a}\right)^{\frac{2}{3}} + \left(\frac{y}{b}\right)^{\frac{2}{3}} = 1$$

at (x, y) is

$$\frac{X}{x}\left(\frac{x}{a}\right)^{2/3} + \frac{Y}{y}\left(\frac{y}{b}\right)^{2/3} = 1,$$

where X and Y are the current coordinates of a point on the tangent.

Solution :

Differentiating the given equation of the curve implicitly we have

$$\frac{2}{3}\left(\frac{x}{a}\right)^{-\frac{1}{3}} \cdot \frac{1}{\mathrm{a}} + \frac{2}{3}\left(\frac{y}{b}\right)^{-\frac{1}{3}}\frac{1}{b}\cdot\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{\left(\frac{y}{b}\right)^{1/3} b}{\left(\frac{x}{a}\right)^{1/3} a}$$

If (X, Y) is a point on the tangent then the equation to the tangent is given by

$$\mathrm{Y} - y = -\frac{\left(\frac{y}{b}\right)^{1/3} b}{\left(\frac{x}{a}\right)^{1/3} a}\,(\mathrm{X} - x)$$

$$= -\frac{y}{x}\,\frac{\left(\frac{x}{a}\right)^{2/3}}{\left(\frac{y}{b}\right)^{2/3}}\;(X - x)$$

$$\Rightarrow \quad (Y - y)\left(\frac{y}{b}\right)^{2/3}\cdot\frac{1}{y} = -\left(\frac{x}{a}\right)^{2/3}\cdot\frac{1}{x}\;(X - x)$$

$$\Rightarrow \frac{Y}{y}\left(\frac{y}{b}\right)^{2/3} + \frac{X}{x}\left(\frac{x}{a}\right)^{2/3} = \left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} = 1. \qquad \text{(since } (x, y) \text{ is on the curve)}$$

EXERCISE 8(b)

1. Find the equations to the tangents and normals to the following curves at the indicated points.
   (i) $y = 2x^2 + 3$ at $x = -1$
   (ii) $y = x^3 - x$ at $x = 2$
   (iii) $y = \sqrt{x} + 2x + 6$ at $x = 4$
   (iv) $y = \sqrt{3}\,\sin x + \cos x$ at $x = \frac{\pi}{3}$
   (v) $y = (\log x)^2$ at $x = \frac{1}{e}$
   (vi) $y = \frac{1}{\log x}$ at $x = 2$
   (vii) $y = x\,e^{-x}$ at $x = 0$
   (viii) $y = a\,(\theta - \sin\theta),\ y = a\,(1 - \cos\theta)$ at $\theta = \frac{\pi}{2}$
   (ix) $\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} = 1$ at $(a\cos^3\theta,\ b\sin^3\theta)$.

2. Find the point on the curve

   $y^2 - x^2 + 2x - 1 = 0$

   where the tangent is parallel to the x-axis.

3. Find the point (s) on the curve

   $x = \frac{3at}{1 + t^2}, \qquad y = \frac{3at^2}{1 + t^2}$

   where the tangent is perpendicular to the line $4x + 3y + 5 = 0$.

4. Find the point on the curve

   $x^2 + y^2 - 4xy + 2 = 0$

   where the normal is parallel to the x-axis.

5. Show that the line $y = mx + c$ touches the parabola $y^2 = 4ax$ if $c = \frac{a}{m}$.

6. Show that the line y = mx + c touches the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ if $c^2 = a^2m^2 + b^2$.

[**Hints :** Find equation to tangent at a point (x', y') of the curve and compare it with $y = mx + c$.]

7. Show that the sum of the intercepts on the coordinate axes of any tangent to the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$ is constant.

8. Show that the curves $y = 2^x$ and $y = 5^x$ intersect at an angle

$$\tan^{-1}\left|\frac{ln\left(\frac{5}{2}\right)}{1 + ln\,2\,ln 5}\right|$$

(**Note :** Angle between two curves is the angle between their tangents at the point of intersection)

9. Show that the curves $ax^2 + by^2 = 1$ and $a'x^2 + b'y^2 = 1$ intersect at right angles if

$$\frac{1}{a} - \frac{1}{b} = \frac{1}{a'} - \frac{1}{b'}.$$

10. Find the equation of the tangents drawn from the point (1, 2) to the curve

$$y^2 - 2x^3 - 4y + 8 = 0.$$

11. Show that the equation to the normal to

$$x^{2/3} + y^{2/3} = a^{2/3}$$

is $y \cos\theta - x \sin\theta = a \cos 2\theta$ where θ is the inclination of the normal to x-axis.

12. Show that the length of the portion of the tangent to $x^{2/3} + y^{2/3} = a^{2/3}$ intercepted between the axes is constant.

13. Find the tangent to the curve

$$y = \cos(x + y), \quad 0 \le x \le 2\pi$$

which is parallel to the line $x + 2y = 0$.

14. If tangents are drawn from the origin to the curve $y = \sin x$ then show that the locus of the points of contact is $x^2y^2 = x^2 - y^2$.

15. Find the equation of the normal to the curve given by

$$x = 3\cos\theta - \cos^3\theta$$

$$y = 3\sin\theta - \sin^3\theta \qquad \text{at } \theta = \frac{\pi}{4}.$$

16. If $x\cos\alpha + y\sin\alpha = p$ is a tangent to the curve $\left(\frac{x}{a}\right)^{\frac{n}{n-1}} + \left(\frac{y}{b}\right)^{\frac{n}{n-1}} = 1$ then show that $(a\cos\alpha)^n + (b\sin\alpha)^n = p^n$.

17. Show that the tangent to the curve $x = a(t - \sin t)$, $y = at(1 + \cos t)$ at $t = \frac{\pi}{2}$ has slope $\left(1 - \frac{\pi}{2}\right)$.

8.3 Increasing and Decreasing Functions

Definition :

A function f is said to be **increasing** in an interval I if $f(x_1) \le f(x_2)$ whenever $x_1 < x_2$ and $x_1, x_2 \in$ I. If $f(x_1) < f(x_2)$ whenever $x_1 < x_2$ and $x_1, x_2 \in$ I then f is said to be **strictly increasing** in I.

A function f is said to be **decreasing** in I if $f(x_1) \ge f(x_2)$ whenever $x_1 < x_2$ and $x_1, x_2 \in$ I. A **strictly decreasing** function is defined with obvious modifications.

A function is called **monotonic** if it is either increasing or decreasing.

For graphical representation of such functions see fig.8.3.

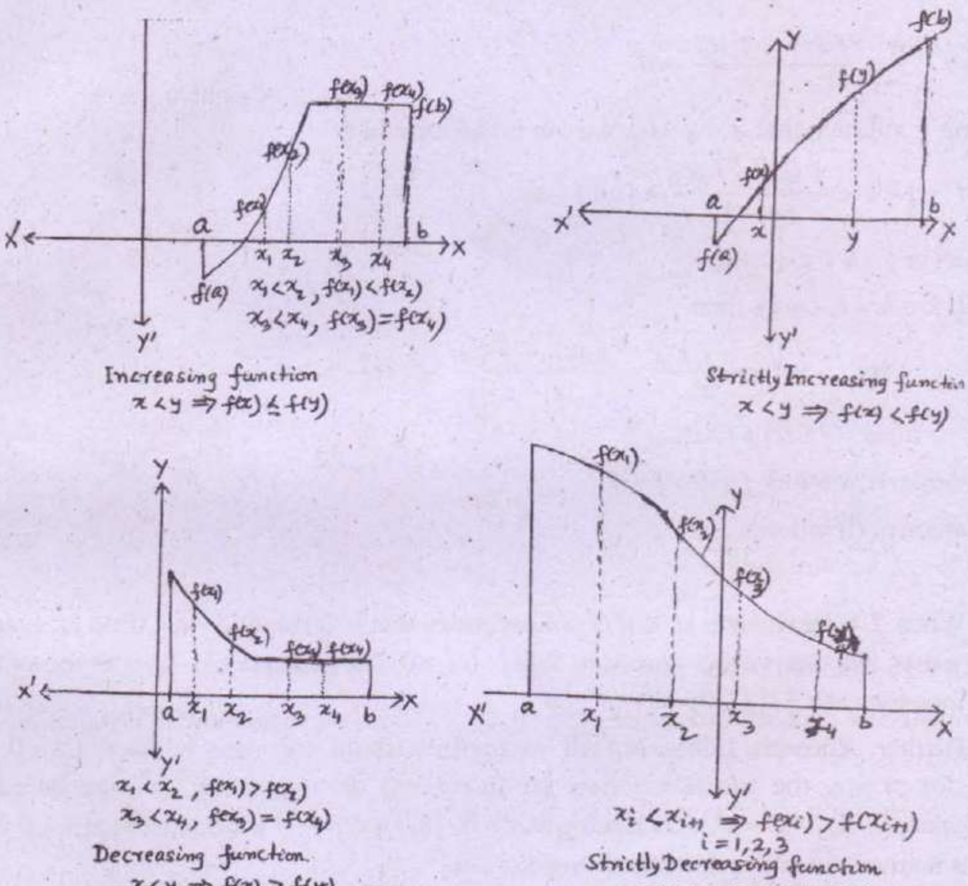


Fig. 8.3

Definition :

A function f is said to be increasing **at a point** c if there is an open interval I containing c such that for each $x \in$ I,

$$x < c \Rightarrow f(x) < f(c) \text{ and } x > c \Rightarrow f(x) > f(c).$$

The definition of f decreasing **at a point** c is similar.

We observe that if a function is increasing (decreasing) in an open interval I, then it is increasing (decreasing) at each point of I.

The next theorem gives a useful criterion to decide whether a differentiable function is increasing or decreasing at point.

Theorem 1 :

Let f be a function defined in an open interval containing a point c. Then

(i) $f'(c) > 0 \Rightarrow f$ is increasing at c.

(ii) $f'(c) < 0 \Rightarrow f$ is decreasing at c.

Proof :

We prove the first statement.

The proof of the second is similar.

We have

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} > 0.$$

Taking $\in > 0$ such that $\in < f'(c)$, we can find δ for which

$$f'(c) - \in < \frac{f(x) - f(c)}{x - c} < f'(c) + \in$$

whenever $c - \delta < x < c + \delta$,

If $x \in (c - \delta, c + \delta)$, then

$$x < c \quad \Rightarrow \quad \frac{f(c) - f(x)}{c - x} = \frac{f(x) - f(c)}{x - c} > f'(c) - \in > 0.$$

$$\Rightarrow \quad f(c) > f(x),$$

and similarly, $x > c \Rightarrow f(x) > f(c)$.

So statement (i) follows.

Remarks :

When f is increasing at c, it is not necessary that $f'(c)$ should exist (find an example), but if $f'(c)$ exists, we can only conclude that $f'(c) \geq 0$. The example of $f(x) = x^3$ shows that though $f(x)$ increases at $x = 0, f'(0) = 0$.

Further, Theorem 1 does not tell us anything about the case when $f'(c) = 0$. If $f'(c) = 0$, then for $x = c$, the function f may be increasing or decreasing or may be neither of it. For example (i) $f(x) = x^3$ is increasing at $x = 0$, (ii) $f(x) = -x^3$ is decreasing at $x = 0$ and (iii) $f(x) = x^2$ is neither increasing nor decreasing at $x = 0$.

In such cases we have to look for sign of $f'(x)$ in $(c - \delta, c)$ and $(c, c + \delta)$ for small values of δ. If $f'(x) > 0$ in both $(c - \delta, c)$ and $(c, c + \delta)$ then f is increasing at c and decreasing in case $f'(x) < 0$.

Example 5 :

Test whether the function $f(x) = x^3 - 27x + 6$ is increasing or decreasing

Solution :

$$f(x) = x^3 - 27x + 6$$

$$\Rightarrow \quad f'(x) = 3x^2 - 27 = 3(x^2 - 9) = 3(x + 3)(x - 3).$$

Now $f'(x) > 0 \quad \Rightarrow \quad (x + 3(x - 3) > 0$

$\Rightarrow \quad (x + 3)$ and $(x - 3)$ have the same sign

$\Rightarrow \qquad x < -3 \text{ or } x > 3.$

Hence if $|x| > 3$ then f is increasing. Further $|x| < 3 \Rightarrow f'(x) < 0$ and hence f is decreasing for $x \in (-3, 3)$.

EXERCISE 8(c)

1. Find the intervals where the following functions are (a) increasing and (b) decreasing.

   (i) $y = \sin x,\ x \in [0, 2\pi]$

   (ii) $y = \ln x,\ x \in R_+$

   (iii) $y = a^x, a > 0,\ x \in R$

   (iv) $y = \sin x + \cos x,\ x \in [0, 2\pi]$

   (v) $y = 2x^3 + 3x^2 - 36x - 7$

   (vi) $y = \dfrac{1}{x-1},\ x \neq 1$

   (vii) $y = \begin{cases} x^2 + 1, & x \le -3 \\ x^3 - 8x + 13, & x > -3 \end{cases}$

   (viii) $y = 4x^2 + \dfrac{1}{x}$

   (ix) $y = (x-1)^2\ (x+2)$

   (x) $y = \dfrac{\ln x}{x},\ x > 0$

   (xi) $y = \tan x - 4(x-2),\ x \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$

   (xii) $y = \sin 2x - \cos 2x,\ x \in [0, 2\pi]$.

2. Give a rough sketch of the functions given in question 1.

3. Show that the function $\dfrac{e^x}{x^p}$ is strictly increasing for $x > p > 0$.

4. Show that $2\sin x + \tan x \geq 3x$ for all $x \in \left(0, \dfrac{\pi}{2}\right)$.

8.4 Maxima and Minima

Definition 1 :

A function f is said to have a **maximum (local maximum or relative maximum)** at a point $x = c$ if $f(c) > f(c+h)$ for $0 < |h| < \delta$ for some $\delta > 0$.

Definition 2 :

The value $f(c)$ is said to be an **absolute maximum** if $f(c) \geq f(x)$ for all values of x where f is defined.

Definition 3 :

The function f is said to have a **minimum (local minimum or relative minimum)** at a point $x = c$ if $f(c) < f(c+h)$ for $0 < |h| < \delta$ for some $\delta > 0$.

Definition 4 :

The value $f(c)$ is said to be an **absolute minimum** if $f(c) \leq f(x)$ for all values of x for which f is defined.

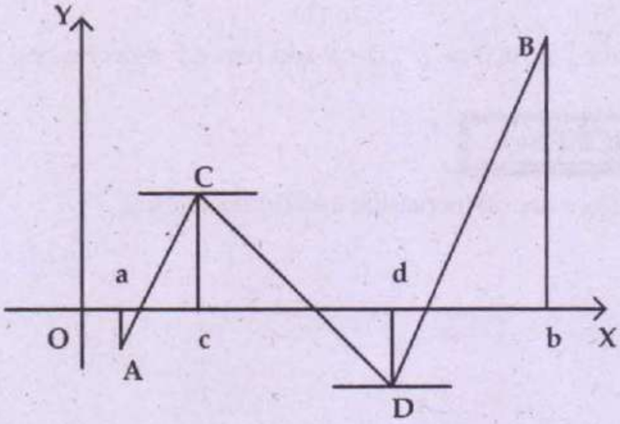


Fig 8.4

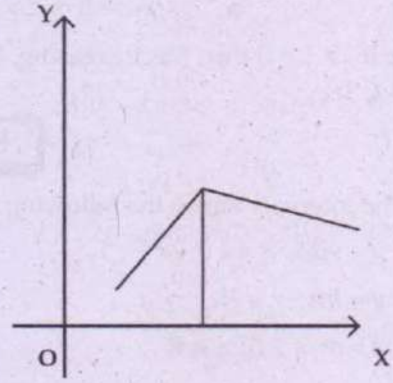
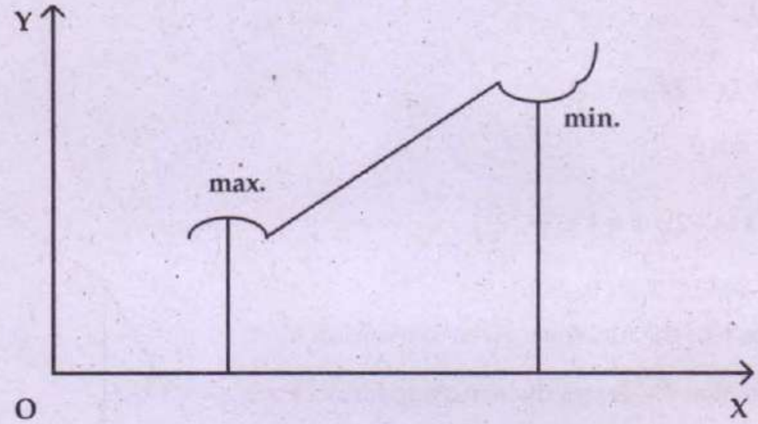


Fig. 8.5

A maximum or minimum of a function is a local property in the sense that it occurs in a 'near' locality of a point. A maximum value in some locality may even be less than a minimum value in another locality (See Q. 1 (ix) in Exercise 8 (d)).



a maximum value < *a* minimum value

Fig 8.6

By an **extremum** (plural extrema) we mean either a maximum or a minimum.

Necessary Condition for an extremum

Theorem 2

If f has an extremum at a point $x = c$, then $f'(c) = 0$ or $f'(c)$ does not exist.

Proof :

We have either of the possibilities, f′(c) exist or does not exist.

Let us suppose that f′ (c) exists.

(i) Suppose that f has a maximum at $x = c$. Then for $0 < |h| < \delta$ for some $\delta > 0$.

$f(c) > f(c + h)$.

If we chose $h > 0$, then

$f(c) > f(c + h)$ and $f(c) > f(c - h)$ ....(1)

Inequalities in (1) are true even though f is not differentiable at $x = c$.

Now suppose $f'(c)$ exists, Then from (1), we have,

$$\frac{f(c+h)-f(c)}{h} < 0 \text{ and } \frac{f(c-h)-f(c)}{-h} > 0.$$

Thus on one hand we obtain,

$$f'(c+) = \lim_{h \to 0+} \frac{f(c+h)-f(c)}{h} \le 0.$$

and on the other hand,

$$f'(c-) = \lim_{h \to 0+} \frac{f(c-h)-f(c)}{-h} \ge 0.$$

Since $f'(c)$ exists, $f'(c+) = f'(c-)$ whence $f'(c) = 0$.

(ii) If we suppose that f has a relative minimum at $x = c$ then by definition

$$f(c) < f(c+h)$$

for $0 < |h| < \delta$ for some $\delta > 0$. If $f'(c)$ exists then we would obtain $f'(c) = 0$ just in the same way as (i).

Definition 5 :

A point $x = c$ is called a **critical point of f if either $f'(c) = 0$ or $f'(c)$ does not exist.**

The condition in the above theorem is not sufficient for a function to have an extremum, i.e. the derivative at a point $x = c$ may vanish, i.e. becomes zero or may not exist without the function having an extremum at that point.

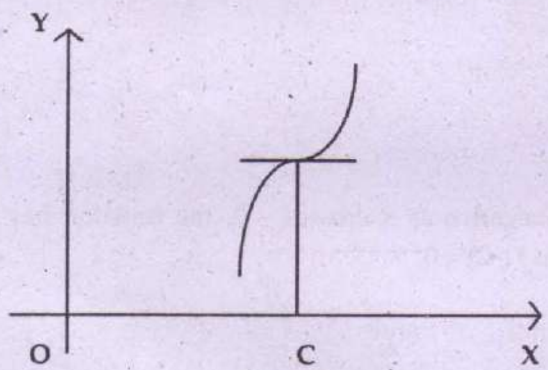


Fig 8.7

(f' (c) exists and is zero but f(c) is not an extremum)

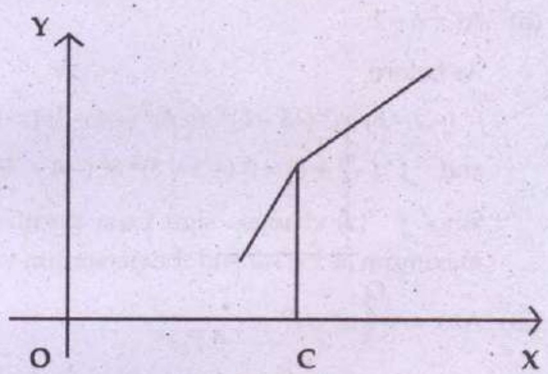


Fig 8.8

(f' (c) does not exist and f(c) is not an extremum)

We follow the following working rule :

Sign Test : Let c be a critical point of f where f is continuous. Then

(i) If there exists a $\delta > 0$ such that $f'(x) > 0$ for all $x \in (c-\delta, c)$ and $f'(x) < 0$ for all $x \in (c, c+\delta)$, then $f(c)$ is a maximum value of f. Here f(x) changes sign from positive to negative as x passes through c.

(ii) If there exists a $\delta > 0$ such that $f'(x) < 0$ for all $x \in (c-\delta, c)$ and $f'(x) > 0$ for all $x \in (c, c+\delta)$, then $f(c)$ is a minimum value of f. Here f(x) changes sign from negative to positive as x passes through c.

Example 6 :

Test the function $y = (x-1)^5 (x+2)^4$ for extreme values.

Solution :

Here $y = f(x) = (x-1)^5 (x+2)^4$

$\Rightarrow f'(x) = 5(x-1)^4 (x+2)^4 + (x-1)^5 . 4(x+2)^3$

$= 3(x-1)^4 (x+2)^3 (3x+2).$ .... (1)

on factorisation.

Now $f'(x) = 0$ for $x = 1, -2$ and $-\frac{2}{3}$ which are the possible extremum points. The function is continuous at all these points.

(i) At $x = 1$: If x is less than 1, then putting $x = 1 - \delta$, where $\delta > 0$ is sufficiently small, we get from (1)

$f'(1-\delta) = 3(-\delta)^4 (3-\delta)^3 (5-3\delta) > 0$

and when $x = 1 + \delta$ we get

$f'(1+\delta) = 3\delta^4 (3+\delta)^3 (5+3\delta) > 0.$

Since $f'(x)$ does not change sign as x crosses 1 the function has no extremum at $x = 1$.

(ii) At $x = -2$:

As before

$f'(-2-\delta) = 3(-3-\delta)^4 (-\delta)^3 (-4-3\delta) > 0$

and $f'(-2+\delta) = 3(-3+\delta)^4 \delta^3 (-4+3\delta) < 0.$

Since $f'(x)$ changes sign from positive to negative as x crosses -2, the function has a maximum at $x = -2$ and the maximum value is $f(-2) = 0$.

(iii) At $x = -\frac{2}{3}$:

$f'\left(-\frac{2}{3} - \delta\right) = 3\left(-\frac{5}{3} - \delta\right)^4 \left(\frac{4}{3} - \delta\right)^3 (-3\delta) < 0$

and $f'\left(-\frac{2}{3} + \delta\right) = 3\left(-\frac{5}{3} + \delta\right)^4 \left(\frac{4}{3} + \delta\right)^3 (3\delta) > 0.$

So f has a minimum at $x = -\frac{2}{3}$ and the minimum value is $f\left(-\frac{2}{3}\right) = -\left(\frac{5}{3}\right)^5 \cdot \left(\frac{4}{3}\right)^4$.

We have yet another method to test a function for extreme points.

Theorem 3 : (Second Derivative Test)

A function f has a maximum at $x = c$ if $f'(c) = 0$ and $f''(c) < 0$ and a minimum at $x = c$ if $f'(c) = 0$ and $f''(c) > 0$.

Proof :

Let $f'(c) = 0$ and $f''(c) < 0$.

Since $f''(c) = \lim_{h \to 0} \frac{f'(c+h) - f'(c)}{h} < 0$

it follows that a certain neighbourhood $(c-\delta, c+\delta)$ of c,

$$\frac{f'(c+h)-f'(c)}{h} < 0$$

and $$\frac{f'(c-h)-f'(c)}{-h} < 0$$

where $0 < h < \delta$. Hence we obtain the inequalities $f'(c+h) < f'(c) < f'(c-h)$ and since $f'(c) = 0$, we have

$$f'(c+h) < 0 < f'(c-h)$$

where $0 < h < \delta$. Thus $f'(x)$ changes sign from positive to negative as x passes through c.

Hence f has a maximum at $x = c$.

Similarly the case for $f''(c) > 0$ can be proved.

Comment :

What will happen if $f'(c) = 0$ and also $f''(c) = 0$?

Consider $f(x) = x^3$ at $x=0$. Here $f'(0) = f''(0) = 0$. In cases as such we need further discussion involving derivatives of order higher than the second.

In this context we present the following theorem without proof which justifies that higher order derivatives beyond the second are essential to decide extreme points of functions on many occasions.

Theorem

If $c \in (a, b)$ and

1. $f'(c) = f''(c) = f'''(c) = \ldots = f^{(n-1)}(c) = 0$
2. $f^{(n)}(c) \neq 0$,
3. **then if n is even and**

$f^{(n)}(c) > 0 \Rightarrow x = c$ is a point of local minimum,

$f^{(n)}(c) < 0 \Rightarrow x = c$ is a point of local maximum,

but if n is odd then $x = c$ gives neither maximum nor minimum. It is a point of inflexion on the curve $y = f(x)$.

As an application of this result, take two examples :

Example (i) Test the function $y = x^6+5$ for extreme values.

Solution :

Here $f'(x) = 6x^5 = 0$ for $x = 0$ therefore $x = 0$ is a critical point.

Here $f''(0) = f'''(0) = f^{(4)}(0) = f^{(5)}(0) = 0$

But $f^{(6)}(0) = 720 > 0$ (6 is even)

$\Rightarrow$ the function attains a local minimum at $x = 0$

Example (ii)

Test the function $f(x) = x^3$ for extreme values.

Solution :

$f'(x) = 3x^2 = 0$ *for* $x = 0$, *so* $x = 0$ is a critical point.

also $f''(0) = 0$, but $f'''(0) = 6 > 0$

since the third derivative at $x = 0$ is different from 0 and 3 is odd, by the above theorem $x = 0$ gives neither local maximum nor local minimum. It is a point of inflexion on the graph of the function. (Refer Example 12)

You can see from the graph of $y = x^3$ how the tangent to the graph of $y = x^3$ crosses it at the origin.

Example 7

Find the maxima and minima of $f(x) = \sin x + \cos x, x \in [0, 2\pi]$.

Solution :

$$f(x) = \sin x + \cos x, x \in [0, 2\pi]$$

$$\Rightarrow f'(x) = \cos x - \sin x$$

and the roots of the equation $f'(x) = 0$ are given by

$$\tan x = 1 = \tan \frac{\pi}{4}$$

$$\Rightarrow \quad x = \frac{\pi}{4}, \pi + \frac{\pi}{4}.$$

Now $f''(x) = -\sin x - \cos x$

$$f''\left(\frac{\pi}{4}\right) = -\sin \frac{\pi}{4} - \cos \frac{\pi}{4} = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} < 0.$$

So $x = \frac{\pi}{4}$ is a local maximum and the maximum value is $f\left(\frac{\pi}{4}\right) = \sqrt{2}$.

$$f''\left(\pi + \frac{\pi}{4}\right) = -\sin\left(\pi + \frac{\pi}{4}\right) - \cos\left(\pi + \frac{\pi}{4}\right)$$

$$= -\left(-\frac{1}{\sqrt{2}}\right) - \left(-\frac{1}{\sqrt{2}}\right) > 0.$$

So $x = \pi + \frac{\pi}{4}$ is a local minimum and the minimum value is $f\left(\pi + \frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -\sqrt{2}$.

To find the absolute maximum or absolute minimum of a function over a closed interval we have to consider the values of the function at the end points also besides the extreme points. This is because the absolute maximum or absolute minimum may occur at the end points (See fig 8.4) which need not be critical points.

Example 8

Find the greatest and least values of $x^4 - 2x^2 + 3$ in $[-2, 2]$.

Solution :

The extreme points are the roots of the equation,

$$f'(x) = 0$$

i.e. $\quad 4x^3 - 4x = 4x(x+1)(x-1) = 0.$

So the extreme points are $x = 0, -1$ and 1 all of which are in $[-2, 2]$.

Now $\quad f''(x) = 12x^2 - 4 = 4(3x^2 - 1)$

$\Rightarrow \quad f''(0) = -4 < 0$

$f''(-1) = 4(3-1) > 0$

$f''(1) = 4(3-1) > 0$

Hence $x = 0$ is a local maximum and $x = -1$ and $x = 1$ are points of minima,

$f(0) = 3$ (maximum)

$f(-1) = 2$ (minimum)

$f(1) = 2$ (minimum)

Further $f(-2) = 11 = f(2)$

Since max. $\{3, 2, 11\} = 11$

and min. $\{3, 2, 11\} = 2$

the absolute maximum value is 11 and the absolute minimum value is 2.

Note that the absolute maximum value is attained at one end point and absolute minimum value is attained at two interior points of the interval $[-2, 2]$.

Example 9

Find the point on the curve $4x^2 + a^2y^2 = 4a^2,\ 4 < a^2 < 8$, that is farthest from the point $(0, -2)$.

Solution :

The equation of the curve can be written as

$$\frac{x^2}{a^2} + \frac{y^2}{4} = 1,\ 4 < a^2 < 8.$$

Let P $(a\cos\phi, 2\sin\phi)$ be a point on the curve. Then the distance of P from $(0, -2)$ is

$$l = \sqrt{\left[\, a^2 \cos^2\phi + (2\sin\phi + 2)^2 \right]}$$

$$\Rightarrow \quad l^2 = a^2\cos^2\phi + 4(\sin\phi + 1)^2 = u \text{ (say).}$$

Now for l to be maximum, u must be maximum. Hence $\frac{du}{d\phi}$ must vanish.

Now $\dfrac{du}{d\phi} = -2a^2\cos\phi\sin\phi + 8(\sin\phi + 1)\cos\phi$

$= 2\cos\phi\,[\,(4 - a^2)\sin\phi + 4\,],$

$\dfrac{d^2u}{d\phi^2} = -2\sin\phi\,[\,(4-a^2)\sin\phi + 4\,] + 2\cos\phi\,[\,(4-a^2)\cos\phi\,]$

$= 2(4-a^2)(\cos^2\phi - \sin^2\phi) - 8\sin\phi$

$$= 2\,(4-a^2)\cos 2\phi - 8\sin\phi.$$

$$\therefore \quad \frac{du}{d\phi} = 0 \Rightarrow \cos\phi = 0 \text{ or } (4-a^2)\sin\phi + 4 = 0.$$

$$(4-a^2)\sin\phi + 4 = 0 \Rightarrow \sin\phi = \frac{4}{a^2-4} > 1$$

since $a^2 - 4 < 4$.

Hence discarding this possibility we have $\cos\phi = 0 \Rightarrow \phi = \frac{\pi}{2}$.

($\phi = \frac{3\pi}{2}$ gives the minimum value of l) .

Further $\left.\frac{d^2u}{d\phi^2}\right]_{\phi=\frac{\pi}{2}} = 2\,(4-a^2)\,(-1) - 8 = 2\,(a^2-8) < 0.$

Hence the maximum value of l is given by $\phi = \frac{\pi}{2}$ for which P (0, 2) and $l_{max} = 4$.

Example 10

Find maximum value of the following function on $[-2, 2]$

$$f(x) = \begin{cases} 3x+2, & x \le 0 \\ 2-3x, & x > 0. \end{cases}$$

Solution :

Here $f'(x) = \begin{cases} 3 & \text{if } x < 0 \\ -3 & \text{if } x > 0. \end{cases}$

The function is not differentiable at $x = 0$ because $f'(0+) = -3$ and $f'(0-) = 3$. Possible extremum point on $[-2, 2]$ is $x = 0$. Now for $x < 0, f'(x) > 0$ and hence f is increasing in $(-\delta, 0)$ and similarly f is decreasing in $(0, \delta)$. Hence $x = 0$ is a maximum point . The maximum value is $f(0) = 2$.

Example 11

Show that the rectangle of maximum area that can be inscribed in a given circle is a square.

Solution :

Let A B C D be a rectangle inscribed in a circle with centre at 0 and radius $= |OC| = a$. By symmetry O is on the diagonal AC.

Let OE $\perp$ BC.

Suppose $|OE| = x = \frac{1}{2}\,|AB|$

and $|EC| = y = \frac{1}{2}\,|BC|$.

A = Area of ABCD $= |AB|\,.\,|BC|$

$= 4\,xy$

$= 4x\,(a^2-x^2)^{\frac{1}{2}}$.

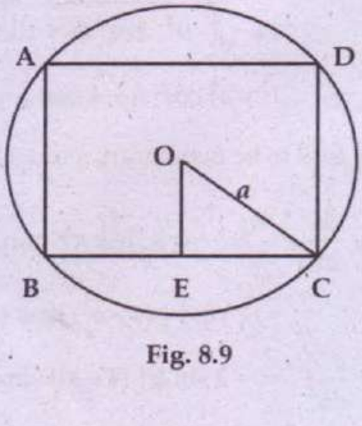


Fig. 8.9

$$\therefore \quad \frac{dA}{dx} = 4\,(a^2-x^2)^{\frac{1}{2}} + 4x\,.\,\frac{1}{2}\,(a^2-x^2)^{-\frac{1}{2}}\,(-2x)$$

$$= 4\,(a^2 - x^2)^{\frac{1}{2}} - 4x^2\,(a^2 - x^2)^{-\frac{1}{2}}.$$

Since A is to be maximised we put $\frac{dA}{dx} = 0$.

$\Rightarrow \quad 4\,(a^2 - x^2)^{\frac{1}{2}} = 4x^2\,(a^2 - x^2)^{-\frac{1}{2}}$

$\Rightarrow \quad a^2 - x^2 = x^2$

$\Rightarrow \quad x = \frac{a}{\sqrt{2}}$, discarding the negative value.

Hence $y = \left(a^2 - \frac{a^2}{2}\right)^{1/2} = \frac{a}{\sqrt{2}}$.

Thus $\frac{1}{2}\ |AB| = \frac{1}{2}\ |BC| \Rightarrow |AB| = |BC| \Rightarrow$ ABCD is a square.

Note that the same result would have been obtained if we had wanted to obtain a rectangle of **minimum** area. But to obtain a minimum A which could be as near to zero as we please we should have either x or y sufficiently near zero (since $A = 4xy$).. But then the result $y = x$ as implied by $\frac{dA}{dx} = 0$ would certainly not produce a rectangle inscribed in the circle. Hence it is only possible that the relative extremum produces a rectangle of maximum area.

Note : Concavity, convexity and point of inflexion.

We have used the second derivative of a function to determine the nature of an extremum. Now we shall see how it gives important information about the nature of the graph of a function.

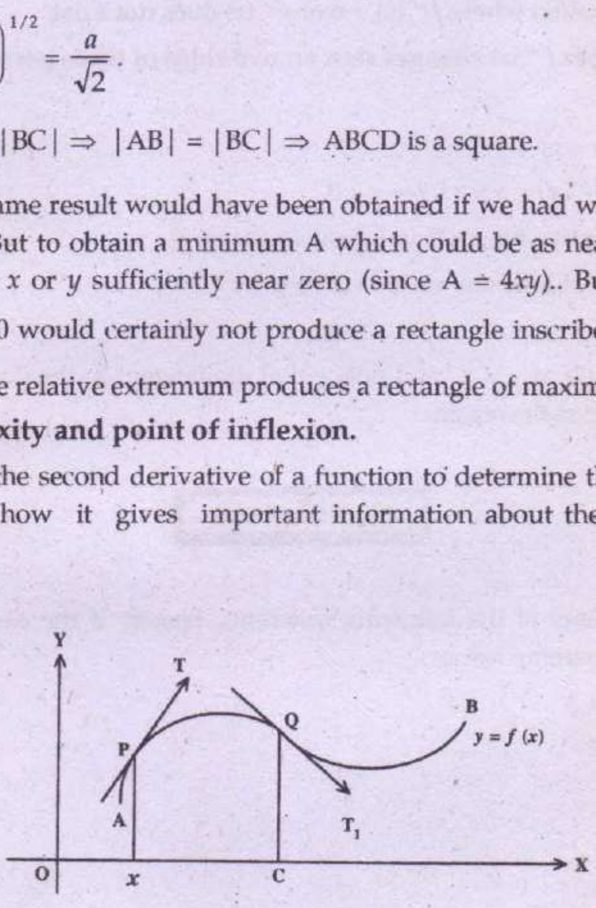


Fig. 8.10

As we move from the point P (see fig 8.10) to the point Q along the curve $y = f(x)$ the tangent line to the curve gradually turns clockwise, its slope $f'(x)$ decreases and becomes zero at the maximum point. Then it assumes negative value and decreases further till we reach Q after which the tangent starts turning anticlockwise and $f'(x)$ gradually increases to become zero at the point of minimum and continues to increase as we move along the curve towards B. The part of the curve AQ where $f'(x)$ is a decreasing function i.e. where $f''(x) < 0$ is said to be **convex w.r.t. the x-axis (or convex upwards)** and the part QB of the curve where $f'(x)$ is an increasing function ($f''(x) > 0$) is said to be concave w.r.t. x-axis (or concave upwards). The point Q where the tangent changes its direction of rotation i.e. where $f'(x)$ attains an extreme value is called a **point of inflexion**. We have the following definition.

Definition :

A point Q on a curve is said to be a **point of inflexion** if the curve is concave on one side and convex on the other side of Q w.r.t. x-axis.

It is necessary that at a point of inflexion $f''(x) = 0$. But the condition is not sufficient. The point $x = 0$ is not a point of inflexion for the curve $y = x^4$ because $f''(x)$ does not change sign as x passes through the origin. A point on the curve may be an inflexion point even when the second derivative does not exist there, e.g. consider $y = x^{1/3}$. Thus **in order to find points of inflexion we must have to**

(i) **find the points where $f''(x) = o$ or $f''(x)$ does not exist.**

(ii) **test whether $f''(x)$ changes sign on two sides of these points.**

Example 12

Take the function $f(x) = x^3$

We have $f'(x) = 3x^2$, $f''(x) = 6x = 0$ for $x = 0$.

Also $f''(x) = 6x$ is negative for $x < 0$ and positive for $x > 0$.

Hence $x = 0$ is a point of inflexion on the curve $y = x^3$

Do yourself

Draw a freehand graph of $y = x^3$ and notice that the tangent to the curve at $x = 0$, which is the x axis, crosses the curve at the origin.

EXERCISE 8(d)

1. Find the extreme points of the following functions. Specify if the extremum is a maximum or minimum. Find the extreme values.

(i) $y = x^2 + 2x + 3$

(ii) $y = 5x^2 - 2x^5$

(iii) $y = \dfrac{3x}{x^2 + 1}$

(iv) $y = x^2\sqrt{1 - x^2}$

(v) $y = 2x^3 - 15x^2 - 36x + 18$

(vi) $y = 60/(x^4 - x^2 + 25)$

(vii) $y = (x - 1)^3$

(viii) $y = (x - 2)^3 (x + 3)^4$

(ix) $y = x + \dfrac{1}{x}$

(x) $y = 4\cos 2x - 3\sin 2x, x \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$

(xi) $y = \sin x \cos x, x \in \left(\dfrac{\pi}{8}, \dfrac{\pi}{2}\right)$

(xii) $y = \cos x (1 + \sin x), x \in [0, 2\pi]$

(xiii) $y = \sin^p x \cos^q x, p, q > 0, x \in \left[0, \frac{\pi}{2}\right]$

(xiv) $y = x e^{-x}, x \in (-2, 2)$.

2. Show that the following functions do not possess maximum or minimum.

   (i) x^3 (ii) x^5 (iii) $3x^3 - 12x^2 + 16x - 5$

   (iv) $4 - 3x + 3x^2 - x^3$ (v) $ln\ |x|, x \neq 0$.

3. Use the function $f(x) = x^{1/x}, x > 0$ to show that $e^{\pi} > \pi^e$.

4. Prove the inequality $x^2 e^{-x^2} \leq e^{-1}, x \in R$.

5. If $f(x) = a\ ln\ x + bx^2 + x$ has extreme values at $x = -1$ and $x = 2$ then find a and b.

6. Show that $\frac{x}{1 + x \tan x}, x \in \left(0, \frac{\pi}{2}\right)$ is maximum when $x = \cos x$.

7. Determine the absolute maximum and absolute minimum of the following function on $[-1, 1]$.

$$f(x) = \begin{cases} (x+1)^2, & x \leq 0 \\ (x-1)^2, & x > 0. \end{cases}$$

8. Find extreme values of

$$f(x) = \begin{cases} \frac{x}{1-x^2}, & -1 < x < 0 \\ x^3 - x, & 0 \leq x < 2 \end{cases} \quad \text{on } (-1, 2).$$

9. Find two numbers x and y whose sum is 15 such that xy^2 is maximum.

10. If the sum of two positive numbers is constant then show that their product is maximum when they are equal.

11. Determine a rectangle of area 25 sq. units which has minimum perimeter.

12. Find the altitude of a right circular cylinder of maximum volume inscribed in a sphere of radius r.

13. Show that the radius of the right circular cylinder of greatest curved surface that can be inscribed in a given cone is half the radius of the base of the cone.

14. Show that the semivertical angle of a cone of given slant height is $\tan^{-1}\sqrt{2}$ when its volume is maximum.

15. A cylindrical open water tank with a circular base is to be made out of 30 sq. metres of metal sheet. Find the dimensions so that it can hold maximum water. (Neglect thickness of sheet)

16. A cylindrical vessel of capacity 500 cubic metres open at the top is to be constructed. Find the dimensions of the vessel if the material used is minimum given that the thickness of the materialused is 2 cm.

17. Find the coordinates of the point on the curve $x^2y - x + y = 0$ where the slope of the tangent is maximum.

18. Find the points on the curve $y = x^2 + 1$ which are nearest to the point (0, 2).

19. Show that the minimum distance of a point on the curve $\frac{a^2}{x^2} + \frac{b^2}{y^2} = 1$ from the origin is $a + b$.

20. Show that the vertical angle of a right circular cone of minimum curved surface that circumscribes a given sphere is $2 \sin^{-1}\left(\sqrt{2} - 1\right)$.

21. Show that the semi-vertical angle of a right circular cone of minimum volume that circumscribes a given sphere is $\sin^{-1}\left(\frac{1}{3}\right)$.

22. Show that the shortest distance of the point (0, 8*a*) from the curve $ax^2 = y^3$ is $2a\sqrt{11}$.

23. Show that the triangle of greatest area that can be inscribed in a circle is equilateral.

(**Hints :** It BC is any chord then for the Δ ABC to have maximum area, the point A must be on the perpendicular bisector of BC so as to have the largest height AD. Let $m\angle BAD = \alpha$. Let BO = OA = *r*. Then area of $\Delta ABC = \Delta = \frac{1}{2}\ |BC|\ |AD| = \frac{1}{2} \cdot 2r \sin 2\alpha\ (r + r\cos 2\alpha) = r^2 \sin 2\alpha$ $(1 + \cos 2\alpha)$. Then maximise Δ to obtain $2\alpha = \frac{\pi}{3}$).

8.5 DIFFERENTIALS AND CALCULATION OF ERROR.

Let a function f be differentiable on an interval (a, b). Using the notation $y = f(x)$ we have by definition of derivative

$$\frac{dy}{dx} = f'(x) = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} \qquad \ldots\ldots\ldots (1)$$

where δx and δy are increments in x and y respectively.

From (1) we can write

$$\lim_{\delta x \to 0} \left[\frac{\delta y}{\delta x} - f'(x)\right] = 0$$

or $$\delta y = \delta x\, f'(x) + \delta x \cdot \alpha \qquad \ldots\ldots\ldots (2)$$

where α is a function of δx such that $\lim \alpha = 0$ as $\delta x \to 0$ and $f'(x)$ is independent of δx. The first term on the r.h.s. of (2) is the principal part of δy. It is denoted by dy or df and is called the differential of y relative to increment δx.

Thus $dy = \delta x f'(x)$.

Taking $y = f(x) = x$ we see that $dx = \delta x$ .... (3)

since $f'(x) = 1$. Hence $dy = f'(x)\, dx$ .... (4)

It is thus apparent that the derivative $f'(x)$ of a function f is in fact the quotient of the differentials df and dx. So $f'(x)$ is also called the differential coefficient. Since $\delta x \cdot \alpha$ is an infinitesimal of higher order than δx we have, from (2),

$$\delta y \approx dy \qquad \text{......... (5)}$$

when δx is small.

Equations (4) and (5) can be used for the calculation of error or approximate values as illustrated through examples. But first we have the following

Theorem 4

Let u and v be differentiable functions on an interval I. Then

(i) $d(u \pm v) = du \pm dv$

(ii) $d(uv) = vdu + udv$

(iii) $d\left(\dfrac{u}{v}\right) = \dfrac{vdu - udv}{v^2}$, provided $v \neq 0$

(iv) $du = 0$ iff u is a constant.

The proofs are straightforward from the definition of differentials. As an illustration we only prove (iii).

Proof :

(iii) : Let $y = \dfrac{u}{v}$ where u and v are functions of x. Then

$$dy = \left[\frac{d}{dx}\left(\frac{u}{v}\right)\right] dx \quad \text{from (4)}$$

$$= \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}\, dx$$

$$= \left[v\frac{du}{dx}dx - u\frac{dv}{dx}dx\right] / v^2$$

$$= (v\, du - udv) / v^2.$$

Calculation of Errors :

If $y = f(x)$ and δx is an error in x, then the error in y i.e. $\delta y = f(x + \delta x) - f(x)$. The **relative error** is $\dfrac{\delta y}{y}$ and **percentage of error** is $\dfrac{\delta y}{y} \times 100$.

We use $\delta y \approx df = f'(x)\, dx$ to calculate δy.

Example 13

If $y = \sqrt{x + 1}$, find δy and dy when $x = 8$ and $dx = 0.02$.

Solution :

Here $y = (x+1)^{\frac{1}{2}}$

$\Rightarrow \delta y = (x+\delta x+1)^{\frac{1}{2}} - (x+1)^{\frac{1}{2}}$

$= (9.02)^{\frac{1}{2}} - 3$ (putting values of x and δx)

$= 3.003331 - 3$

$= .003331$ (upto 4 significant ditgits)

$dy = f'(x)\,dx$

$= \frac{1}{2}(x+1)^{-\frac{1}{2}}\,dx$

$= \frac{1}{2}(8+1)^{-\frac{1}{2}} \times 0.02$ $(\because dx = \delta x)$

$= \frac{1}{2} \times \frac{1}{3} \times 0.02$

$= 0.003333.$

Example 14

The radius of a wire as measured by a screw gauge is found to be 1.26 mm. If the correct radius is 1.25 find the approximate error, relative error and percentage error in calculation of area of its cross section.

Solution :

The correct radius = r = 1.25mm. Error in measuring radius = δr = 0.01mm. Area of cross section of the wire = A = πr^2.

Approximate error in calculation of A is δA

$\approx dA$

$= \pi \times 2r dr$

$= 3.141 \times 2 \times 1.25 \times 0.01$ sq.mm.

$= 0.0785$ sq. mm.

Relative error $= \frac{\delta A}{A} = \frac{2\pi r\,dr}{\pi r^2}$

$= 2\,\frac{dr}{r}$

$= \frac{0.02}{1.25} = 0.016.$

$\therefore$ Percentage error $= \frac{\delta A}{A} \times 100 = 1.6.$

Note that actual error in calculation of A $= \pi\{(1.26)^2 - (1.25)^2\} = 0.0788$ sq.mm.

Example 15

Find approximate value of $(80.8)^{\frac{1}{4}}$.

Solution :

Put $y = x^{\frac{1}{4}}$

Then $\delta y = (x + \delta x)^{\frac{1}{4}} - x^{\frac{1}{4}}$

$$\Rightarrow \quad (x + \delta x)^{\frac{1}{4}} = x^{\frac{1}{4}} + \delta y \approx x^{\frac{1}{4}} + dy$$

But $dy = \frac{1}{4}\, x^{-\frac{3}{4}}\, dx$

$$\Rightarrow \quad (x + \delta x)^{\frac{1}{4}} \approx x^{\frac{1}{4}} + \frac{1}{4}\, x^{-\frac{3}{4}}\, dx$$

Now substituting $x = 81$, $\delta x = dx = -0.2$

we get

$$(81 - 0.2)^{\frac{1}{4}} \approx (81)^{\frac{1}{4}} + \frac{1}{4}\,(81)^{-\frac{3}{4}} \times (-0.2)$$

$$\Rightarrow (80.8)^{\frac{1}{4}} \approx 3 - 0.0018 = 2.9982.$$

EXERCISE 8(e)

1. Determine the differentials in each of the following cases.

   (i) $y = x^3 - 1$ (ii) $y = \sin^2 x$ (iii) $y = \dfrac{1+\sqrt{x}}{1-\sqrt{x}}$

   (iv) $z = \cos 2t - 2\cot t$ (v) $r = \dfrac{4}{1+\sin\theta}$ (vi) $x^2 y = 2$

   (vii) $xy^2 + yx^2 = 1$.

2. Find δf and df when

   (i) $f(x) = 2x^2 - 1$, $x = 1$, $\delta x = 0.02$.

   (ii) $f(x) = \sqrt{x}$, $x = 16$, $\delta x = 0.3$.

   (iii) $f(x) = (x + 1)^3$, $x = 8$, $\delta x = 0.04$.

   (iv) $f(x) = ln\,(1 + x)$, $x = 1$, $\delta x = 0.04$.

3. Find approximate values of the following.

   (i) $\sqrt[3]{28}$ (ii) $\sqrt[6]{63}$ (iii) $\sqrt{48.96}$

   (iv) $(1.99)^7$ (v) $2^{3.02}$ (vi) $\sin 59^0$

4. Find the percentage of error in calculation of the surface area of a spherical balloon of diameter 14.02 m. if the true diameter is 14m.

5. Find approximately the difference between the volumes of two cubes of sides 3 cm. and 3.04 cm.

6. The height of a regular cone is 3 times the radius of its base. The radius of the base was wrongly measured to be 5 cm. where as its true radius is 4.88 cm. Find the relative error in measuring the curved surface area of the cone.

*8.6 INDETERMINATE FORMS

(Additional topic for interested students, not for examination)

The quotient of two functions $f(x)/g(x)$ is said to be in indeterminate form $\frac{0}{0}$ as x tends to a certain point c if

$$\lim_{x \to c} f(x) = 0 = \lim_{x \to c} g(x).$$

We have earlier learnt that

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)},$$ provided the limits on the right hand side exist and $\lim_{x \to c} g(x) \neq 0$.

So we have to adopt some other methods to find the limit of $f(x) / g(x)$ when $\lim_{x \to c} g(x) = 0$.

(i) If $\lim_{x \to c} f(x)$ exists and is not equal to zero and $\lim_{x \to c} g(x) = 0$ then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = +\infty \text{ or } -\infty.$$

(ii) If $\lim_{x \to c} f(x) = 0$ and $\lim_{x \to c} g(x) = 0$ then we make use of the following theorem.

Theorem (L' Hospital's Rule) (Pronounced as Lo-pital's rule)

Let f and g be two functions differentiable on some open interval containing c such that $g'(x) \neq 0$ when $x \neq c$. Further let $f(c) = g(c) = 0$.

Then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$$

provided the latter limit exists.

Proof :

Consider the interval $[c, c + h]$ for some $h > 0$. The functions f and g satisfy the conditions of Cauchy's Mean Value Theorem on this interval.

Hence we have an $y \in (c, c + h)$ such that

$$\frac{f(c+h)-f(c)}{g(c+h)-g(c)} = \frac{f'(y)}{g'(y)}.$$

i.e. $$\frac{f(c+h)}{g(c+h)} = \frac{f'(y)}{g'(y)} \qquad (\because f(c) = 0 = g(c))$$

Further $c < y < c + h$ implies that as $c + h \to c$ y also tends to c. Hence from the above equality we obtain by putting $x = c + h$ and letting $h \to 0$

$$\lim_{x \to c+} \frac{f(x)}{g(x)} = \lim_{y \to c+} \frac{f'(y)}{g'(y)}$$

$$= \lim_{x \to c+} \frac{f'(x)}{g'(x)} \qquad \text{(note)}$$

provided the latter limit exists.

Similar considerations applied to the interval $[c-h, c]$ would result in

$$\lim_{x \to c-} \frac{f(x)}{g(x)} = \lim_{x \to c-} \frac{f'(x)}{g'(x)}.$$

Combining the two results the theorem follows.

This rule can be applied repeatedly as long as successive derivatives of f and g continue to satisfy the conditions of the rule and their quotients are in the indeterminate form $\frac{0}{0}$.

Note (1) The same rule is also applicable for finding the limit of $\frac{f(x)}{g(x)}$ when $f(x) \to +\infty$ or $-\infty$ and $g(x) \to +\infty$ or $-\infty$ as $x \to c$. In such a case the function is said to be in the indeterminate form $\frac{\infty}{\infty}$ as $x \to c$.

(2) The rule is also applicable for the indeterminate forms $\frac{0}{0}$ and $\frac{\infty}{\infty}$ when $x \to \infty$ or $-\infty$.

(3) We define in an analogous way other indeterminate forms viz. $0 \times \infty$, $\infty - \infty$, 0^0, 1^∞ and ∞^0. For example, we define the function $[f(x)]^{g(x)}$ to be in the indeterminate form ∞^0 as $x \to c$ if $\lim_{x \to c} f(x) = \infty$ and $\lim_{x \to c} g(x) = 0$. All such forms can be reduced to $\frac{0}{0}$ or $\frac{\infty}{\infty}$ form by taking logarithms or other simplification procedures (See examples).

We shall not discuss the proofs of all these cases given in the notes as they are beyond the scope of the book.

Warning : When we apply the L' Hospital's rule we take the derivative of $f(x)$ and $g(x)$ separately and do not take derivative of the quotient $\frac{f(x)}{g(x)}$.

Example 16

Evaluate the following limits.

(i) $\lim_{x \to 0} \frac{\tan x}{x}$ (ii) $\lim_{x \to 0} \left(\frac{1}{x} - \cot x\right)$

(ii) $\lim_{x \to \infty} x^{\frac{1}{x}}$ (iv) $\lim_{x \to \infty} \frac{x^n}{e^x}$

Solution :

(i) $\lim_{x \to 0} \frac{\tan x}{x}$ $\left(\frac{0}{0} \text{ form}\right)$

$= \lim_{x \to 0} \frac{\sec^2 x}{1}$ (by L' Hospital's rule)

$= 1.$

(ii) $\lim_{x \to 0} \left(\frac{1}{x} - \cot x\right)$

$= \lim_{x \to 0} \left(\frac{\sin x - x\cos x}{x \sin x}\right)$ $\left(\frac{0}{0} \text{ form}\right)$

$= \lim_{x \to 0} \frac{\cos x - (\cos x - x\sin x)}{\sin x + x\cos x}$

$$= \lim_{x \to 0} \frac{x \sin x}{\sin x + x \cos x} \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x \to 0} \frac{\sin x + x \cos x}{\cos x + \cos x - x \sin x} = \frac{0}{2} = 0.$$

(iii) Let $y = x^{\frac{1}{x}}$ (∞^0 form)

$$\Rightarrow \quad lny = \frac{1}{x}\, lnx$$

$$\therefore \quad \lim_{x \to \infty} lny = \lim_{x \to \infty} \frac{\ln x}{x} \quad \left(\frac{\infty}{\infty} \text{ form}\right)$$

$$= \lim_{x \to \infty} \frac{(1/x)}{1} = 0.$$

Hence $\ln \lim_{x \to \infty} y = \lim_{x \to \infty} lny = 0.$

$$\Rightarrow \quad \lim_{x \to \infty} y = e^0 = 1$$

$$\Rightarrow \quad \lim_{x \to \infty} x^{\frac{1}{x}} = 1.$$

(iv) $\lim_{x \to \infty} \frac{x^n}{e^x} \quad \left(\frac{\infty}{\infty} \text{ form}\right)$

$$= \lim_{x \to \infty} \frac{nx^{n-1}}{e^x} \quad \left(\frac{\infty}{\infty} \text{ form}\right)$$

$= \ldots\ldots\ldots$

$$= \lim_{x \to \infty} \frac{n!}{e^x} \qquad \text{(Applying L' Hospital's rule successively)}$$

$= 0.$

EXERCISE 8 (t)

(Not for examination)

Find the following limits.

1. $\lim_{x \to 0} \frac{\tan ax}{x}$
2. $\lim_{x \to 0} \frac{\sin ax}{\sin bx}$
3. $\lim_{x \to 1} \frac{\ln x}{x^2 - 1}$
4. $\lim_{x \to \pi/2} \frac{1 - \sin x}{\cos x}$
5. $\lim_{x \to 0} \frac{x - \sin x}{\tan^3 x}$
6. $\lim_{x \to 2} \frac{x^3 - 12x + 16}{3x^3 - 8x^2 - 4x + 16}$
7. $\lim_{x \to 1} \frac{\ln(2-x)}{1 - x^2}$
8. $\lim_{x \to 0+} \frac{\sqrt{1-x} - \sqrt{1+x}}{\sqrt{x}}$
9. $\lim_{x \to 1} \frac{2\sqrt{x} - 3\sqrt[3]{x} + 1}{(x-1)^2}$
10. $\lim_{x \to \infty} \frac{x^3 - 3x + 1}{2x^3 - 7x^2 + 5}$

11. $\lim_{x \to 2} \frac{4^x - 2^{3+x} + 16}{(x-2)^2}$

12. $\lim_{x \to 0+} \frac{\ln \tan x}{\ln \sin 2x}$

13. $\lim_{x \to 0} \frac{e^x - e^{-x} - 2\sin x}{x \sin x}$

14. $\lim_{x \to \infty} \frac{\ln (1/x)}{xe^x}$

15. $\lim_{x \to 0+} \frac{e^{-1/x}}{x}$

16. $\lim_{x \to \infty} \frac{x^n + x^{-n}}{(x+2)^n}$

17. $\lim_{x \to 0} \frac{e^x - e^{-x}}{x \cos x}$

18. $\lim_{x \to \infty} \frac{\sin^{-1} x}{x}$

19. $\lim_{x \to 0+} \log\tan x \tan 2x$

20. $\lim_{x \to \pi/2} (\tan x)^{\cos x}$

21. $\lim_{x \to 1} x^{\frac{1}{1-x}}$

22. $\lim_{x \to 0+} x^{\sin x}$

23. $\lim_{x \to 1} \left(\frac{1}{x-1} - \frac{1}{\ln x}\right)$

24. $\lim_{x \to \infty} \left(x - \sqrt{x^2 - 1}\right)$

25. $\lim_{x \to \infty} \left(x - \sqrt{x^2 + 2x}\right)$

26. $\lim_{x \to 0} \left(\frac{1}{\sin^2 x} - \frac{1}{x^2}\right)$

27. $\lim_{x \to \pi/2} (\tan 3x - \tan x)$

28. $\lim_{x \to 0+} \frac{\sqrt{a^x - b^x}}{\sqrt{x}}, a > b$

29. $\lim_{x \to \infty} \left(1 + \frac{3}{x}\right)^x$

30. $\lim_{x \to 0+} \left(e^{\frac{1}{x}}\right)^{\ln(1+x)}$

31. $\lim_{x \to 0} \left(\frac{\sin x}{x}\right)^{1/x}$

32. $\lim_{x \to 0} (1 + x^2)^{1/x}$

33. $\lim_{x \to 0} (\cot x - \operatorname{cosec} x)$

34. $\lim_{x \to 1} (2 - x)^{\operatorname{cosec} \pi x}$

35. $\lim_{x \to 0} \left(\frac{x^2 + 2x - 1}{x^2 - 1}\right)^{\frac{1}{x}}$

36. $\lim_{x \to 0} \frac{(1+x)^{\frac{1}{x}} - e}{x}$

37. $\lim_{x \to 0+} (\tan x)^{1/\ln x}$

38. $\lim_{x \to 0} \frac{x^3 \sin \frac{1}{x}}{\tan x}$

39. $\lim_{x \to 0} \frac{2^x - 1}{\sqrt{(1+x)} - 1}$

40. $\lim_{x \to 1} (1 - x) \tan \frac{\pi x}{2}$

41. $\lim_{x \to 0} \frac{x \cos x - \sin x}{x^2 \sin x}$

42. $\lim_{x \to 0} \frac{e^{-\frac{1}{x^2}}}{x^2}$.

43. $\lim_{x \to 0} \left(\frac{\sin x}{x}\right)^{\frac{1}{x^2}}$.

CHAPTER 9

Integration

I fear the day that technology will surpass our human interaction. The world will have a generation of idiots.

- Albert Einstein

9.0 Introduction

In the previous chapter, we have studied the derivative of functions and its applications. If a function is differentiable then we can find its derivative. In this chapter, we shall deal with how to find a function if its derivative is given. The process of finding the function when its derivative is given is called integration and the function found is known as the **integral, primitive** or **antiderivative** of the given function.

9.1 ANTIDERIVATIVE (PRIMITIVE)

If $g(x)$ is the derivative of $f(x)$, then $f(x)$ is said to be an **anti-derivative**. (or **integral**) of $g(x)$. For example, as $\cos x$ is the derivative of $\sin x$, $\sin x$ is an anti-derivative of $\cos x$. This fact is symbolically written as $\int \cos x \, dx = \sin x$. The symbol $\int$ (an elongated S) is used to denote the operation of integration and called the integral sign. The function (here $\cos x$) to be integrated is called the **integrand**, 'dx' denotes the fact that the integration is to be performed with respect to x (i.e. x is the **variable of integration**).

Observe that each one of $\sin x+31$, $\sin x-7\sqrt{2}$, $\sin x+5.7-11\sqrt{3}$ has the same derivative, $\cos x$. So each one of them is an anti-derivative of $\cos x$. That means, anti-derivative of a function is not unique. In fact, the anti-derivative of $\cos x$ is $\sin x+\text{K}$ where K is any constant, not necessarily the same at each occurrence.

If $g(x)$ is the derivative of $f(x)$ (and so of $f(x) + \text{K}$), then $f(x) + \text{K}$ denotes the family of all anti-derivatives of $g(x)$. Here K is an indefinite constant (though can be given any particular value depending on physical circumstances). Therefore, $f(x) + \text{K}$ is called the **indefinite integral** of $g(x)$. We write $\int g(x)\,dx = f(x) + \text{K}$, K being the constant of integration which is arbitrary. One may use any other symbol for constant like A, B, C, ... , a, b, c, λ, μ, etc. for the constant of integration.

If two functions $\text{F}(x)$, $f(x)$ represent the integrals of a given function $g(x)$, then they must differ by a constant.

Because $\text{F}'(x) = g(x)$ and $f'(x) = g(x)$,

$$\frac{d}{dx}\left[\text{F}(x) - f(x)\right] = \text{F}'(x) - f'(x) = g(x) - g(x) = 0.$$

Hence, $\text{F}(x) - f(x)$ must be a constant.

9.2 SIMPLE INTEGRATION FORMULAE

The following formulae for integrals are directly obtained from the corresponding formulae for derivatives.

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + K \quad (n \neq -1)$$

$$\int \frac{1}{x}\,dx = \ln|x| + K$$

$$\int \cos x\,dx = \sin x + K$$

$$\int \sin x\,dx = -\cos x + K$$

$$\int \sec^2 x\,dx = \tan x + K$$

$$\int \operatorname{cosec}^2 x\,dx = -\cot x + K$$

$$\int \sec x \tan x\,dx = \sec x + K$$

$$\int \operatorname{cosec} x \cot x\,dx = -\operatorname{cosec} x + K$$

$$\int e^x\,dx = e^x + K$$

$$\int a^x\,dx = \frac{a^x}{\ln a} + K$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}x + K \text{ or } -\cos^{-1}x + c$$

$$\int \frac{1}{1+x^2}\,dx = \tan^{-1}x + K \text{ or } -\cot^{-1}x + c$$

$$\int \frac{1}{x\sqrt{x^2-1}}\,dx = \sec^{-1}x + K \text{ or } -\operatorname{cosec}^{-1}x + c$$

9.3 ALGEBRA OF INTEGRALS

(i) $\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$

(ii) $\int \lambda f(x)\,dx = \lambda f(x)\,dx$, for a constant λ.

Proof :

(i) Let $f(x)\,dx = F(x) + K_1$ and $g(x)\,dx = G(x) + K_2$
So that $F'(x) = f(x)$ and $G'(x) = g(x)$.

$\therefore \quad \int f(x)\,dx \pm \int g(x)\,dx = (F(x) + K_1) \pm (G(x) + K_2)$

$= F(x) \pm G(x) + K$, where $K = K_1 \pm K_2$. .... (1)

But $\frac{d}{dx}[F(x) \pm G(x) + K] = F'(x) \pm G'(x) = f(x) \pm g(x)$.

Then $\int [f(x) \pm g(x)]\,dx = F(x) \pm G(x) + K$. .... (2)

Form (1) and (2) $\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$.

(ii) $\frac{d}{dx}[\lambda\int f(x)\,dx] = \lambda\,\frac{d}{dx}\left(\int f(x)\,dx\right) = \lambda f(x).$

$\therefore \quad \int \lambda f(x)\,dx = \lambda \int f(x)\,dx$.

Combining the above two rules it can be established that

$\int[\lambda_1 f_1(x) + \lambda_2 f_2(x) + \dots + \lambda_n f_n(x)]\,dx$

$= \lambda_1\int f_1(x)\,dx + \lambda_2\int f_2(x)\,dx + \dots + \lambda_n\int f_n(x)\,dx.$

Example 1 : Integrate the following

(i) $\int (x^6 + x^2 + x + 1)\,dx$ (ii) $\int \left(\sqrt{y}+\frac{1}{\sqrt{y}}+\frac{1}{y^2}+\frac{1}{y^3}\right)dy$

(iii) $\int \left(4\cos x - 3e^x + \frac{2}{\sqrt{1-x^2}}\right)dx$ (iv) $\int 6x^3(x+5)^2\,dx$

(v) $\int 5\tan^2 x\,dx$ (vi) $\int \frac{x^4}{x^2+1}\,dx$ (vii) $\int e^{3x}\,dx$

Solutions :

(i) $\int (x^6 + x^2 + x + 1)\,dx = \int x^6\,dx + \int x^2\,dx + \int x\,dx + \int 1.dx$

$= \frac{x^7}{7} + \frac{x^3}{3} + \frac{x^2}{2} + x + k$ [The constant of integration 'k' is mentioned in the final answer]

(ii) $\int \left(\sqrt{y}+\frac{1}{\sqrt{y}}+\frac{1}{y^2}+\frac{1}{y^3}\right)dy = \int\left(y^{\frac{1}{2}}+y^{-\frac{1}{2}}+y^{-2}+y^{-3}\right)dy$

$= \frac{y^{\frac{1}{2}+1}}{\frac{1}{2}+1} + \frac{y^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + \frac{y^{-2+1}}{-2+1} + \frac{y^{-3+1}}{-3+1} + K = \frac{2}{3}y^{\frac{3}{2}} + 2y^{\frac{1}{2}} - y^{-1} - \frac{1}{2}y^{-2} + K$

$= \frac{2}{3}\,y\sqrt{y} + 2\sqrt{y} - \frac{1}{y} - \frac{1}{2y^2} + K.$

Remark: The integral is expressed in that variable in which the variable of infegration in given.

(iii) $\int \left(4\cos x - 3e^x + \frac{2}{\sqrt{1-x^2}}\right)dx = 4\int \cos x\,dx - 3\int e^x\,dx + 2\int \frac{dx}{\sqrt{1-x^2}}$

$= 4\sin x - 3e^x + 2\sin^{-1}x + K$

(iv) $\int 6x^3(x+5)^2\,dx = \int 6x^3(x^2+10x+25)\,dx = \int (6x^5+60x^4+150x^3)dx$

$= 6\int x^5 dx + 60\int x^4\,dx + 150\int x^3 dx$

$= 6\frac{x^6}{6} + 60\frac{x^6}{5} + \frac{150}{4}x^4 + K = x^6 + 12x^5 + \frac{75}{2}x^4 + K.$

(v) Here a trigonometric formula helps to convert the given integrand to standard form.

$\int 5\tan^2 x\, dx = \int 5(\sec^2 x - 1)\, dx = 5\int \sec^2 x\, dx - 5\int 1.dx$

$= 5\tan x - 5x + K.$

(vi) When the highest power of the variable in the numerator is equal to or higher than that in the denominator in a rational fraction, a division will help.

$\frac{x^4}{x^2+1} = x^2 - 1 + \frac{1}{x^2+1}$.

$\therefore \int \frac{x^4}{x^2+1}dx = \int x^2 dx - \int 1.dx + \int \frac{1}{x^2+1}dx = \frac{x^3}{3} - x + \tan^{-1}x + K.$

(vii) $\int e^{3x}\, dx = \int (e^3)^x\, dx = \frac{(e^3)^x}{\ln e^3} + K = \frac{e^{3x}}{3} + K$

Points to note :

(i) Like differentiation, integration is an operation on functions.

(ii) Differentiation and integration are processes

such that (a) $\frac{d}{dx}\left[\int f(x)dx\right] = f(x)$

and (b) $\int \left[\frac{d}{dx}f(x)\right]dx = f(x) + c$

(iii) The integral of a function is not unique. Two integrals of a function differ by a constant.

(iv) If the degree of a polynominal is n, its integral is again a polynomial of degree n+1. But its derivative is a polynomial of degree n-1.

(v) There exist functions which have no antiderivatinc, eg. $f(x) = 0$, when x is rational and $f(x)=1$ when x is irrational. In either words integrals of all functions cannot be found.

(vi) Geometrically indefinite integral of a function represents a family of curves placed parallel to each other having parallel tangents at the points of intersection of the curves of the family with the lines perpendicular (orthogonal) to the axis representing the variable of integration. For illustration, draw the graphs of the family $F(x)=x^2+c$, $c \in R$ which is the integral of $f(x)=x$ and consider their intersections with $x=k$.

EXERCISE- 9(a)

Integrate the following :

1. (i) $\int 2\, dx$ (ii) $\int 3x^2\, dx$ (iii) $\int 4x^3\, dx$

(iv) $\int x^5 dx$ (v) $\int x^{31}\, dx$ (vi) $\int \left(2\sqrt{x} + \frac{3}{\sqrt{x}}\right)dx$

(vii) $\int \frac{1}{x\sqrt{x}}\, dx$ (viii) $\int \left(x^{4/7} + \frac{1}{x^{1/3}}\right)dx$ (ix) $\int \frac{4}{x}dx$

(x) $\int \left(\frac{3}{x^2}+\frac{1}{x^{12}}\right)dx$ (xi) $\int \left(x^2+\sqrt{x}\right)^2 dx$ (xii) $\int (x+3)(2-x)\,dx$

(xiii) $\int \frac{\left(\sqrt{x}+2\right)^2}{x^4}dx$ (xiv) $\int \frac{\left(x+\sqrt{x}\right)(2x+1)}{x^2}dx.$

2. (i) $\int \cos x\,dx$ (ii) $\int \frac{dx}{\cos^2 x}$

(iii) $\int \frac{dx}{1-\cos^2 x}$ (iv) $\int \frac{\sin x}{\cos^2 x}dx$

(v) $\int \frac{2\cos x}{1-\cos^2 x}dx$ (vi) $\int \frac{1-\sin^3 x}{\sin^2 x}dx$

(vii) $\int \frac{\sin^2 x}{1+\cos x}dx$ (viii) $\int \frac{\cos 2x}{\cos x+\sin x}dx$

(ix) $\int \frac{\cos^4 x-\sin^4 x}{\cos x-\sin x}dx$ (x) $\int \frac{\cos 2x}{\sin^2 x\cos^2 x}dx$

(xi) $\int \frac{a^2\sin^2 x+b^2\cos^2 x}{\sin^2 2x}dx$ (xii) $\int (\tan x+\cot x)^2\,dx$

(xiii) $\int \frac{1-\cos 2x}{1+\cos 2x}dx$ (xiv) $\int \sec^2 x\,.\,\mathrm{cosec}^2 x\,dx$

(xv) $\int \frac{a\sin^3 x+b\cos^3 x}{\sin^2 x\cos^2 x}dx$ (xvi) $\int \sqrt{1+\sin 2x}\,dx$

(xvii) $\int \sqrt{1-\cos 2x}\,dx$ (xviii) $\int \sqrt{1+\cos 2x}\,dx$

(xix) $\int \frac{\cos 3x\cos 2x+\sin 3x\sin 2x}{1-\cos^2 x}dx$ (xx) $\int (a\cot x+b\tan x)^2\,dx.$

3. (i) $\int (e^x+2)\,dx$ (ii) $\int 3^x\,dx$

(iii) $\int a^{x+2}\,dx$ (iv) $\int a^{3x}\,dx$ (v) $\int \frac{e^{2x}+1}{e^x}dx$

4. (i) $\int \left(\frac{5}{\sqrt{1-x^2}}+\frac{7}{1+x^2}\right)dx$ (ii) $\int \frac{3x^2}{x^2+1}dx$ (iii) $\int \frac{x^6}{x^2+1}dx$

(iv) $\int \frac{x^4+x^3+x^2+x+2}{x^2+1}dx$ (v) $\int \left(\sqrt{1-x^2}+\frac{x^2}{\sqrt{1-x^2}}\right)dx$ (vi) $\int \frac{x^2+\sqrt{x^2-1}}{x^3\sqrt{x^2-1}}dx$

5. Find the unique antiderivative F(x) of $f(x)=2x^2+1$, whese F(o) = -2.

9.4 INTEGRATION BY SUBSTITUTION.

When the integrand is not in a standard form, it can sometimes be transformed to integrable form by a suitable substitution. The integral

$\int f\{g(x)\}\, g'(x)\, dx$ can be converted to

$\int f(\theta)\, d\theta$ by substituting $g(x)$ by θ, so that if

$\int f(\theta)\, d\theta = \mathrm{F}(\theta) + \mathrm{K}$, then

$\int f\{g(x)\}\, g'(x)\, dx = \mathrm{F}\{g(x)\} + \mathrm{K}$.

This is a direct consequence of chain rule, for

$$\frac{d}{dx}\left[\mathrm{F}\{g(x)\} + \mathrm{K}\right] = \frac{d}{d\theta}\left[\mathrm{F}(\theta) + \mathrm{K}\right] . \frac{d\theta}{dx} = f(\theta)\frac{d\theta}{dx} = f\{g(x)\}\ g'(x).$$

There is no definite formula for substitution. Keen observation of the form of the integrand will help in choosing the function for which substitution is to be made. However, one must be sure that the derivative of the function so chosen must be present along with dx as in the above case. Occasionally, mere adjustment of a constant may be necessary.

Any symbol for variable viz. s, t, u, v, w, x, y, z may be chosen for substitution other than the variable of the given integral. However, after the integration is over, the original variable should be put back .

Example 2 :

(i) $\int (ax+b)^n\, dx,\ n \neq -1.$

Put $ax + b = \theta$, so that $\dfrac{d\theta}{dx} = a$ or $d\theta = a\,dx$.

Hence, $\int (ax+b)^n\, dx = \dfrac{1}{a}\int (ax+b)^n\, a\,dx = \dfrac{1}{a}\int \theta^n\, d\theta$

$$= \frac{1}{a}\,\frac{\theta^{n+1}}{n+1} + \mathrm{C} = \frac{1}{a}\,\frac{(ax+b)^{n+1}}{n+1} + \mathrm{C} \text{ putting back for } \theta.$$

(ii) $\int \cos(ax+b)\, dx = \dfrac{1}{a}\int \cos(ax+b) \,.\, a\,dx$

$= \dfrac{1}{a}\int \cos\theta\, d\theta$, putting $ax + b = \theta$ and $a\,dx = d\theta$

$= \dfrac{1}{a}\sin\theta + \mathrm{C}$, where C is an arbitrary constant.

$= \dfrac{1}{a}\sin(ax+b) + \mathrm{C}$, putting back for θ .

Similarly,

$$\int \sin(ax+b)\, dx = -\frac{1}{a}\cos(ax+b) + \mathrm{C}$$

$$\int \sec^2(ax+b)\, dx = \frac{1}{a}\tan(ax+b) + \mathrm{C}$$

$$\int \operatorname{cosec}^2(ax+b)\, dx = -\frac{1}{a}\cot(ax+b) + \mathrm{C}$$

$$\int \sec(ax+b)\tan(ax+b)\,dx = \frac{1}{a}\sec(ax+b) + C$$

$$\int \operatorname{cosec}(ax+b)\cot(ax+b)\,dx = -\frac{1}{a}\operatorname{cosec}(ax+b) + C$$

$$\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b} + K$$

$$\int \frac{dx}{\sqrt{1-(ax+b)^2}} = \frac{1}{a}\sin^{-1}(ax+b) + K,$$

$$\int \frac{dx}{1+(ax+b)^2} = \frac{1}{a}\tan^{-1}(ax+b) + K.$$

(iii) $\int \frac{g'(x)}{g(x)}\,dx = \int \frac{d\theta}{\theta}$, putting $g(x) = \theta$ so that $\frac{d\theta}{dx} = g'(x) \Rightarrow g'(x)\,dx = d\theta$.

$= ln\,|\theta| + C = ln\,|g(x)| + C.$

Taking different functions for $g(x)$, we get

$$\int \frac{dx}{ax+b} = \frac{1}{a}\int \frac{a\,dx}{ax+b} = \frac{1}{a}\,ln\,|ax+b| + C.$$

$$\int \cot x\,dx = \int \frac{\cos x}{\sin x}\,dx = \int \frac{d(\sin x)}{\sin x} = ln\,|\sin x| + C$$

$$\int \tan x\,dx = \int \frac{\sec x\tan x}{\sec x}\,dx = \int \frac{d(\sec x)}{\sec x} = ln\,|\sec x| + C$$

$$\int \sec x\,dx = \int \frac{\sec x(\sec x+\tan x)}{\sec x+\tan x}\,dx = \int \frac{\sec x\tan x+\sec^2 x}{\sec x+\tan x}\,dx$$

$$= \int \frac{d(\sec x+\tan x)}{\sec x+\tan x} = ln\,|\sec x+\tan x| + K = ln\left|\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right| + K.$$

$$\int \operatorname{cosec} x\,dx = \int \frac{\operatorname{cosec} x(\operatorname{cosec} x-\cot x)}{\operatorname{cosec} x-\cot x}\,dx = \int \frac{-\operatorname{cosec} x\cot x+\operatorname{cosec}^2 x}{\operatorname{cosec} x-\cot x}\,dx$$

$$= \int \frac{d(\operatorname{cosec} x-\cot x)}{\operatorname{cosec} x-\cot x}\,dx = ln\,|\operatorname{cosec} x-\cot x| + K = ln\left|\tan\frac{x}{2}\right| + K.$$

(iv) $\int \frac{x^4+4x^3}{x^5+5x^4+7}\,dx = \frac{1}{5}\int \frac{5x^4+20x^3}{x^5+5x^4+7}\,dx$

$= \frac{1}{5}\int \frac{d\theta}{\theta}$, taking $\theta = x^5+5x^4+7$ so that $d\theta = (5x^4+20x^3)\,dx = 5(x^4+4x^3)\,dx$

$= \frac{1}{5}\,ln\,|\theta| + K$

$= \frac{1}{5}\,ln\,|x^5+5x^4+7| + K.$

(v) $\int \sin^7 x\cos x\,dx = \int \theta^7\,d\theta$, putting $\sin x = \theta$ so that $\cos x\,dx = d\theta$.

$= \frac{\theta^8}{8} + C = \frac{1}{8}\sin^8 x + C.$

(vi) $\int 2\,e^{\tan^2 x}\tan x\sec^2 x\,dx = \int e^{\theta}\,d\theta$, putting $\tan^2 x = \theta$ so that $2\tan x \,.\, \sec^2 x\,dx = d\theta$

$= e^{\theta} + C = e^{\tan^2 x} + C.$

(vii) $\int \frac{(\tan^{-1} x)^3}{1+x^2}\, dx = \int z^3\, dz$, taking $\tan^{-1} x = z$ so that $\frac{1}{1+x^2}\, dx = dz$

$= \frac{z^4}{4} + C = \frac{1}{4}(\tan^{-1}x)^4 + C$.

(viii) $\int \frac{3(\ln x)^2}{x}\, dx = \int 3z^2\, dz$, putting $\ln x = z$ so that $\frac{1}{x}\, dx = dz$.

$= z^3 + C = (\ln x)^3 + C$.

EXERCISE 9 (b)

Integrate the following : (In some cases suggestions have been given for substitution)

1. (i) $\int \sin 3x\, dx$ (ii) $\int \cos ax\, dx$

(iii) $\int \cos(2-7x)\, dx$ (iv) $\int \sin \frac{x}{2}\, dx$

(v) $\int \sec^2 4x\, dx$ (vi) $\int \text{cosec}^2 \frac{x}{3}\, dx$

(vii) $\int \sec(x+2)\tan(x+2)\, dx$ (viii) $\int \text{cosec}\left(x+\frac{\pi}{4}\right)\cot\left(x+\frac{\pi}{4}\right) dx$ $\left(x+\frac{\pi}{4}=z\right)$

(ix) $\int x^2 \cos x^3\, dx$ $(x^3 = z)$ (x) $\int e^x \sec e^x \tan e^x\, dx$ $(e^x = z)$

(xi) $\int \frac{\sec^2\sqrt{x}}{\sqrt{x}}\, dx$ $\left(\sqrt{x} = z\right)$.

2. (i) $\int \sin x \cos x\, dx$ $(\sin x = 9)$ (ii) $\int \tan^3 x \sec^2 x\, dx$ $(\tan x = 9)$

(iii) $\int \frac{\text{cosec}^2 x}{1+\cot x}\, dx$ (iv) $\int \frac{\sin x}{\cos^3 x}\, dx$

(v) $\int \frac{\cos x}{\sin^5 x}\, dx$ (vi) $\int \frac{\text{cosec}^2(\ln x)}{x}\, dx$ $(\ln x = z)$

(vii) $\int \sqrt{1-\sin x}\, \cos x\, dx$.

3. (i) $\int x\sqrt{x^2+3}\, dx$ $(x^2+3 = 9)$ (ii) $\int \frac{7dx}{2-3x}$

(iii) $\int \frac{x}{\sqrt{x^2-a^2}}\, dx$ (iv) $\int \frac{x^2+1}{(x^3+3x+7)^3}\, dx$ $(x^3+3x+7 = 9)$

(v) $\int (x^4-3x^2+1)^4(2x^3-3x)\, dx$, $(x^4-3x^2+1 = 9)$

4. (i) $\int e^{3x}\, dx$ (ii) $\int e^{2x+7}\, dx$

(iii) $\int e^{\frac{x}{3}}\, dx$ (iv) $\int e^{x^3} x^2\, dx$, $(x^3 = 9)$

(v) $\int a^{2x}\, dx$ (vi) $\int 2a^{x^2} x\, dx$, $(x^2 = 9)$

(vii) $\int e^{2\tan x}\sec^2 x\,dx$ $(2\tan x = 9)$ (viii) $\int \frac{e^x}{(e^x-2)^2}\,dx$, $(e^x - 2 = 9)$

(ix) $\int e^{\cos^2 x}\sin 2x\,dx$.

5. (i) $\int \frac{\sin^{-1}x}{\sqrt{1-x^2}}\,dx$ (ii) $\int \frac{dx}{\sqrt{1-(x-1)^2}}$

(iii) $\int \frac{(\sec^{-1}x)^2}{x\sqrt{x^2-1}}\,dx$ (iv) $\int \frac{dx}{x\left[1+(\ln x)^2\right]}$

(v) $\int \frac{dx}{x^2+2x+2}$ $\left(\text{Integrate} \int \frac{dx}{(x+1)^2+1}\right)$

6. (i) $\int \tan 3x\,dx$ (ii) $\int \cot\frac{x}{3}\,dx$

(iii) $\int \sec(2x+1)\,dx$ (iv) $\int \operatorname{cosec} 7x\,dx$

(v) $\int 2x\cot(x^2+3)\,dx$, $(x^2+3 = z)$ (vi) $\int e^x \tan e^x\,dx$

(vii) $\int (\sec 2x - 3)^2\,dx$.

7. (i) $\int \frac{e^x+e^{-x}}{e^x-e^{-x}}\,dx$ (ii) $\int 3^x e^{2x}\,dx$

(iii) $\int \frac{(x+1)\,ln(x^2+2x+2)}{x^2+2x+2}\,dx$

8. (i) $\int \frac{\sin x}{\sin(x+\alpha)}\,dx$, $(x+\alpha = z)$ (ii) $\int \frac{\sin x}{\cos(x-\alpha)}\,dx$

(iii) $\int \frac{\tan x+\tan\alpha}{\tan x-\tan\alpha}\,dx$ $(x-\alpha = z)$.

9.5 INTEGRATION OF SOME TRIGONOMETRIC FUNCTIONS :

If the integrand is of the form $\sin mx\cos nx$, $\sin mx\sin nx$ or $\cos mx\cos nx$, a trigonometric transformation will help to reduce it to the sum of sines or cosines of multiple angles which can be easily integrated.

$$\sin mx\cos nx = \frac{1}{2}\,.\,2\sin mx\cos nx = \frac{1}{2}\left[\sin(m+n)x+\sin(m-n)x\right]$$

$$\sin mx\sin nx = \frac{1}{2}\left[\cos(m-n)x-\cos(m+n)x\right]$$

$$\cos mx\cos nx = \frac{1}{2}\left[\cos(m-n)x+\cos(m+n)x\right]$$

In case where there are more than two factors, successive transformations will help.

For example, $\sin mx\cos nx\cos kx = \frac{1}{2}\sin mx\left[\cos(n+k)x+\cos(n-k)x\right]$

$$= \frac{1}{2}\left[\sin mx\cos(n+k)x+\sin mx\cos(n-k)x\right]$$

$$= \frac{1}{4}\left[\sin(m+n+k)x+\sin(m-n-k)x+\sin(m+n-k)x+\sin(m-n+k)x\right].$$

Example 3 :

Evaluate (i) $\int \sin 3x \cos 2x \, dx$ (ii) $\int \sin 2x \sin x \, dx$

(iii) $\int \cos 4x \cos 3x \, dx$ (iv) $\int \sin 3x \, . \, \sin 2x \cos 4x \, dx$.

Solution :

(i) $\int \sin 3x \cos 2x \, dx = \frac{1}{2} \int 2 \, . \sin 3x \cos 2x \, dx$

$= \frac{1}{2} \int (\sin 5x + \sin x) \, dx = \frac{1}{2} \left(-\frac{1}{5} \cos 5x - \cos x \right) + C = -\frac{1}{10} (\cos 5x + 5 \cos x) + C.$

(ii) $\int \sin 2x \, . \sin x \, dx = \frac{1}{2} \int 2 \sin 2x \sin x \, dx$

$= \frac{1}{2} \int (\cos x - \cos 3x) \, dx = \frac{1}{2} (\sin x - \frac{1}{3} \sin 3x) + C$

$= \frac{1}{6} (3 \sin x - \sin 3x) + C .$

[Also $\int \sin 2x \sin x \, dx = \int 2 \sin^2 x \cos x \, dx = \int 2z^2 dz$, putting $\sin x = z$

$= \frac{2}{3} z^3 + K = \frac{2}{3} \sin^3 x + K.$

(Verify if the results obtained in the two processes are consistent.)

(iii) $\int \cos 4x \cos 3x \, dx = \frac{1}{2} \int (\cos 7x + \cos x) \, dx = \frac{1}{2} \left(\frac{1}{7} \sin 7x + \sin x \right) + C$

$= \frac{1}{14} (\sin 7x + 7 \sin x) + C .$

(iv) $\sin 3x \sin 2x \, . \cos 4x = \frac{1}{2} (\cos x - \cos 5x) \cos 4x$

$= \frac{1}{2} (\cos x \cos 4x - \cos 5x \cos 4x) = \frac{1}{4} (2 \cos 4x \cos x - 2 \cos 5x \cos 4x)$

$= \frac{1}{4} (\cos 5x + \cos 3x - \cos 9x - \cos x)$

So $\int \sin 3x \, . \sin 2x \cos 4x \, dx = \frac{1}{4} \int (\cos 5x + \cos 3x - \cos 9x - \cos x) \, dx$

$= \frac{1}{4} \left[\frac{1}{5} \sin 5x + \frac{1}{3} \sin 3x - \frac{1}{9} \sin 9x - \sin x \right] + C$

$= \frac{1}{180} \left[9 \sin 5x + 15 \sin 3x - 5 \sin 9x - 45 \sin x \right] + C.$

Higher powers of sine and cosine in the integrand ($\sin^m x \cos^n x$, m, n nonnegative integers) may be treated in the same way or better may be simplified to sum of sines and cosines of multiple angles by using multiple angle formula.

For example,

$\sin^2 x = \sin x \, . \sin x = \frac{1}{2} (2 \sin x \, . \sin x) = \frac{1}{2} (\cos 0 - \cos 2x)$

$= \frac{1}{2} (1 - \cos 2x)$ as has been done above.

But $\sin^2 x = \frac{1}{2} \cdot 2\sin^2 x = \frac{1}{2}(1-\cos 2x)$ is a simpler process using the formula of $\cos 2x$.

Similarly, $\cos^2 x = \frac{1}{2}(1+\cos 2x)$

$$\cos^3 x = \frac{1}{4} \cdot 4\cos^3 x = \frac{1}{4}(3\cos x + \cos 3x)$$

$$\sin^3 x = \frac{1}{4} \cdot 4\sin^3 x = \frac{1}{4}(3\sin x - \sin 3x)$$

$$\cos^4 x = \left(\frac{2\cos^2 x}{2}\right)^2 = \left(\frac{1+\cos 2x}{2}\right)^2 = \frac{1}{4}(1 + 2\cos 2x + \cos^2 2x)$$

$$= \frac{1}{8}(2 + 4\cos 2x + 2\cos^2 2x) = \frac{1}{8}(3 + 4\cos 2x + \cos 4x)$$

$$\sin^4 x = \left(\frac{2\sin^2 x}{2}\right)^2 = \left(\frac{1-\cos 2x}{2}\right)^2 = \frac{1}{8}(3 - 4\cos 2x + \cos 4x)$$

$$\sin^2 x\cos^2 x = \frac{1}{4}(4\sin^2 x\cos^2 x) = \frac{1}{4}\sin^2 2x = \frac{1}{8}(1-\cos 4x)$$

$$\sin^3 x\cos^2 x = \sin x \cdot \frac{1}{4}\sin^2 2x = \frac{1}{8}\sin x(1-\cos 4x)$$

$$= \frac{1}{16}(2\sin x - 2\cos 4x\sin x) = \frac{1}{16}(2\sin x - \sin 5x + \sin 3x).$$

Example 4 :

Integrate (i) $\int \sin^2 x\,dx$ (ii) $\int \cos^3 x\,dx$

(iii) $\int \sin^4 x\,dx$ (iv) $\int \sin^3 x\cos^3 x\,dx$.

Solutions :

(i) $\int \sin^2 x\,dx = \int \frac{1}{2}(1-\cos 2x)\,dx = \frac{1}{2}\left(x - \frac{1}{2}\sin 2x\right) + C$

$$= \frac{1}{4}(2x - \sin 2x) + C.$$

(ii) $\int \cos^3 x\,dx = \frac{1}{4}\int (3\cos x + \cos 3x)\,dx$

$$= \frac{1}{4}(3\sin x + \frac{1}{3}\sin 3x) + C = \frac{1}{12}(9\sin x + \sin 3x) + C.$$

(iii) $\int \sin^4 x\,dx = \int \left(\frac{2\sin^2 x}{2}\right)^2 dx = \int \left(\frac{1-\cos 2x}{2}\right)^2 dx$

$$= \frac{1}{4}\int (1 - 2\cos 2x + \cos^2 2x)\,dx = \frac{1}{8}\int (3 - 4\cos 2x + \cos 4x)\,dx$$

$$= \frac{1}{8}\left[3x - \frac{4}{2}\sin 2x + \frac{1}{4}\sin 4x\right] + C$$

$$= \frac{1}{32}[12x - 8\sin 2x + \sin 4x] + C.$$

(iv) $\int \sin^3x \cos^3x\, dx = \frac{1}{8} \int (2 \sin x \cos x)^3\, dx = \frac{1}{8} \int \sin^3 2x\, dx$

$$= \frac{1}{32} \int 4 \sin^3 2x\, dx = \frac{1}{32} \int (3 \sin 2x - \sin 6x)\, dx$$

$$= \frac{1}{32} \left[-\frac{3}{2} \cos 2x + \frac{1}{6} \cos 6x \right] + C = \frac{1}{192} (\cos 6x - 9 \cos 2x) + C .$$

When one (or both) of the powers in $\sin^m x \cos^n x$ is an odd positive integer, there is a more convenient way for integration. If n is odd, keeping one cosine factor (to be the derivative of $\sin x$), the remaining even power of cosx can be converted to sine function. Then we get a polynomial (or rational fraction) in $\sin x$ with the derivative ($\cos x$) of $\sin x$. A substitution for $\sin x$ will help to integrate the function. Similarly, if m is odd, keeping one sine factor, the remaining even power of $\sin x$ can be converted to a polynomial in $\cos x$ and a substitution for $\cos x$ may be made.

Example 5 :

Evaluate,

(i) $\int \sin^3x\, dx$ (ii) $\int \cos^5x\, dx$

(iii) $\int \sin^4x \cos^3x\, dx$ (iv) $\int \sin^3x \cos^5x\, dx$

(v) $\int \frac{\cos^3 x}{\sin^4 x}\, dx$. (vi) $\int \tan^6\theta\, d\theta$

(vii) $\int \frac{\sin 6x + \sin 4x}{\cos 6x + \cos 4x}$.

Solutions :

(i) $\int \sin^3x\, dx = \int (1 - \cos^2x) \sin x\, dx$.

$$= \int -(1 - z^2)\, dz \text{ putting } \cos x = z$$

$$= -z + \frac{z^3}{3} + c = -\cos x + \frac{1}{3} \cos^3x + C.$$

(ii) $\cos^5x\, dx = \int (1 - \sin^2x)^2 \cos x\, dx$

$$= \int (1 - \theta^2)^2\, d\theta \text{ putting } \sin x = \theta$$

$$= \int (1 - 2\theta^2 + \theta^4)\, d\theta = \theta - \frac{2}{3} \theta^3 + \frac{1}{5} \theta^5 + C$$

$$= \sin x - \frac{2}{3} \sin^3x + \frac{1}{5} \sin^5x + C .$$

(iii) $\int \sin^4x \cos^3x\, dx = \int \sin^4x (1 - \sin^2x) \cos x\, dx$

$$= \int \theta^4 (1 - \theta^2)\, d\theta \quad \text{putting } \sin x = \theta$$

$$= \int (\theta^4 - \theta^6)\, d\theta = \frac{1}{5} \theta^5 - \frac{1}{7} \theta^7 + C$$

$$= \frac{1}{5} \sin^5x - \frac{1}{7} \sin^7x + C .$$

(iv) $\int \sin^3x \cos^5x\, dx = \int \sin^3x \cos^4x . \cos x\, dx = \int \sin^3x\,(1-\sin^2x)^2 \cos x\, dx$

$= \int \theta^3 (1-\theta^2)^2\, d\theta$ putting $\sin x = \theta$

$= \int (\theta^3 - 2\theta^5 + \theta^7)\, d\theta = \frac{\theta^4}{4} - \frac{2}{6}\,\theta^6 + \frac{1}{8}\,\theta^8 + C$

$= \frac{1}{4}\sin^4x - \frac{1}{3}\sin^6x + \frac{1}{8}\sin^8x + C.$

Alternatively, $\int \sin^3x \cos^5x\, dx = \int \cos^5x\ (1-\cos^2x) \sin x\, dx$

$= \int z^5 (1-z^2)(-dz)$ putting $\cos x = z$

$= \int (-z^5 + z^7)\, dz = \frac{1}{8}z^8 - \frac{1}{6}z^6 + K$

$= \frac{1}{8}\cos^8x - \frac{1}{6}\cos^6x + K.$

(v) $\int \frac{\cos^3 x}{\sin^4 x}\, dx = \int \frac{1-\sin^2 x}{\sin^4 x}\cos x\, dx = \int \frac{1-\theta^2}{\theta^4}\, d\theta$, putting $\sin x = \theta$

$= \int (\theta^{-4} - \theta^{-2})\, d\theta = \frac{\theta^{-3}}{-3} - \frac{\theta^{-1}}{-1} + C$

$= -\frac{1}{3\theta^3} + \frac{1}{\theta} + C = \frac{1}{\sin x} - \frac{1}{3\sin^3 x} + C.$

$= \operatorname{cosec}x - \frac{1}{3}\operatorname{cosec}^3x + C.$

Alternatively, $\int \frac{\cos^3 x}{\sin^4 x}\, dx = \int \cot^3x \operatorname{cosec}x\, dx = \int \cot^2x . \cot x \operatorname{cosec}x\, dx$

$= \int (\operatorname{cosec}^2x - 1) \cot x \operatorname{cosec}x\, dx = \int (z^2-1)(-dz)$ putting $\operatorname{cosec}x = z$

$= \int (1-z^2)\, dz = z - \frac{z^3}{3} + C$

$= \operatorname{cosec}x - \frac{1}{3}\operatorname{cosec}^3x + C.$

(vi) $\int \tan^6\theta\, d\theta = \int \tan^4\theta . \tan^2\theta\, d\theta = \int \tan^4\theta\, (\sec^2\theta - 1)\, d\theta$

$= \int \tan^4\theta . \sec^2\theta\, d\theta - \int \tan^4\theta . d\theta = \frac{1}{5}\tan^5\theta - \int \tan^2\theta\,(\sec^2\theta - 1)\, d\theta$

$= \frac{1}{5}\tan^5\theta - \int \tan^2\theta \sec^2\theta\, d\theta + \int \tan^2\theta\, d\theta = \frac{1}{5}\tan^5\theta - \frac{1}{3}\tan^3\theta + \int (\sec^2\theta - 1)\, d\theta$

$= \frac{1}{5}\tan^5\theta - \frac{1}{3}\tan^3\theta + \tan\theta - \theta + C.$

(vii) $\int \frac{\sin 6x + \sin 4x}{\cos 6x + \cos 4x}\, dx = \int \frac{2 . \sin 5x . \cos x}{2 . \cos 5x . \cos x}\, dx = \int \frac{\sin 5x}{\cos 5x}\, dx$

$= -\frac{1}{5}\, ln\, |\cos 5x| + C = \frac{1}{5}\, ln\, |\sec 5x| + C.$

EXERCISE 9 (c)

Integrate the following :

1. (i) $\int \sin 4x \cos 3x \, dx$ (ii) $\int \cos 5x \cos 2x \, dx$

(iii) $\int \sin x \cos 4x \, dx$ (iv) $\int \sin 6x \sin 3x \, dx$

(v) $\int \cos 4x \cos 5x \sin 2x \, dx$ (vi) $\int \sin \frac{3x}{4} \cos \frac{x}{2} \, dx$

(vii) $\int \cos 2x \cos \frac{x}{2} . dx$ (viii) $\int \sin \frac{x}{2} \sin \frac{x}{3} \cos \frac{x}{4} \, dx$.

2. (i) $\int \cos^2 x \, dx$ (ii) $\int \sin^3 x \, dx$

(iii) $\int \cos^4 x \, dx$ (iv) $\int \sin^5 x \, dx$

(v) $\int \cos^7 x \, dx$ (vi) $\int \sin^6 x \, dx$

(vii) $\int \cos^5 x \sin^3 x \, dx$ (viii) $\int \sin^{20} x \cos^3 x \, dx$

(ix) $\int \frac{\sin^3 x}{\cos^6 x} \, dx$ (x) $\int \cot^3 x \operatorname{cosec}^{16} x \, dx$

(xi) $\int \sec^{30} x \tan x \, dx$ (xii) $\int \sin^3 x \sec^{14} x \, dx$.

3. (i) $\int \sin^4 x \cos^4 x \, dx$ (ii) $\int \sin 3x \cos^2 x \, dx$

(iii) $\int \cos 2x \sin^3 x \, dx$ (iv) $\int \sin^4 x \cos^2 x \, dx$.

4. (i) $\int \tan^5\theta \sec^4\theta \, d\theta$ (ii) $\int \cot^4\theta \operatorname{cosec}^4\theta \, d\theta$

(iii) $\int \sec^{11}\theta \tan\theta \, d\theta$ (iv) $\int \cot\theta \operatorname{cosec}^7\theta \, d\theta$

(v) $\int \tan^3\theta \, d\theta$ (vi) $\int \cot^4\theta \, d\theta$

(vii) $\int \tan^5\theta \, d\theta$ (viii) $\int \cot^6\theta \, d\theta$.

5. (i) $\int \frac{\sin ax - \sin bx}{\cos ax - \cos bx} \, dx$ (ii) $\int \frac{\cos px + \cos qx}{\sin px + \sin qx} \, dx$

(iii) $\int \frac{\sin 4x - \sin 2x}{\cos x} \, dx$ (iv) $\int \frac{\sin 2x}{a \cos^2 x + b \sin^2 x + c} \, dx$.

9.6 INTEGRATION BY TRIGONOMETRIC SUBSTITUTION :

The following trigonometric identities can be utilized to simplify certain forms of functions in the integrand with trigonometric substitutions.

$$1-\sin^2\theta = \cos^2\theta \text{ (or } 1-\cos^2\theta = \sin^2\theta)$$

$$\tan^2\theta + 1 = \sec^2\theta \text{ (also } \cot^2\theta + 1 = \text{cosec}^2\theta)$$

$$\sec^2\theta - 1 = \tan^2\theta \text{ (also cosec}^2\theta - 1 = \cot^2\theta).$$

The irrational forms $\sqrt{a^2-x^2}$, $\sqrt{x^2+a^2}$, $\sqrt{x^2-a^2}$ can be simplified to radical free functions by putting $x = a\sin\theta$, $x = a\tan\theta$, $x = a\sec\theta$ respectively (or $x = a\cos\theta$, $x = a\cot\theta$, $x = a\,\text{cosec}\theta$ respectively). The substitution $x = a\tan\theta$ (or $x = a\cot\theta$) can be useful in case of presence of $x^2 + a^2$ in the integrand, particularly when it is present in the denominator.

Example 6 :

Integrate, (i) $\int \frac{dx}{\sqrt{a^2-x^2}}$ (ii) $\int \frac{dx}{x^2+a^2}$ (iii) $\int \frac{dx}{\sqrt{x^2+a^2}}$

(iv) $\int \frac{dx}{\sqrt{x^2-a^2}}$ (v) $\int \frac{dx}{x\sqrt{x^2-a^2}}$.

Solutions :

(i) Let $x = a\sin\theta$, so that $dx = a\cos\theta\, d\theta$ and $\theta = \sin^{-1}\frac{x}{a}$

$$\because \int \frac{dx}{\sqrt{a^2-x^2}} = \int \frac{a\cos\theta\, d\theta}{\sqrt{a^2-a^2\sin^2\theta}} = \int \frac{a\cos\theta}{a\cos\theta}\, d\theta = \int d\theta$$

$$= \theta + C = \sin^{-1}\frac{x}{a} + C.$$

(ii) Let $x = a\tan\theta$, so that $dx = a\sec^2\theta\, d\theta$ and $\theta = \tan^{-1}\frac{x}{a}$.

$$\because \int \frac{dx}{x^2+a^2} = \int \frac{a\sec^2\theta}{a^2\tan^2\theta + a^2}\, d\theta = \int \frac{1}{a}\, d\theta = \frac{1}{a}\theta + C = \frac{1}{a}\tan^{-1}\frac{x}{a} + C.$$

(iii) Let $x = a\tan\theta$, so that $dx = a\sec^2\theta\, d\theta$

$$\because \int \frac{dx}{\sqrt{x^2+a^2}} = \int \frac{a\sec^2\theta}{\sqrt{a^2\tan^2\theta + a^2}}\, d\theta = \int \frac{a\sec^2\theta}{a\sec\theta}\, d\theta = \int \sec\theta\, d\theta$$

$$= ln\,|\sec\theta + \tan\theta| + C = ln\left|\sqrt{\frac{x^2}{a^2}+1} + \frac{x}{a}\right| + C$$

$$= ln\left|\frac{x+\sqrt{x^2+a^2}}{a}\right| + C = ln\left(x+\sqrt{x^2+a^2}\right) + K. \quad \text{(where } K = C - ln\,|a|)$$

(iv) Let $x = a\sec\theta$, so that $dx = a\sec\theta\tan\theta\, d\theta$

$$\because \int \frac{dx}{\sqrt{x^2-a^2}} = \int \frac{a\sec\theta\tan\theta}{\sqrt{a^2\sec^2\theta - a^2}}\, d\theta = \int \frac{a\sec\theta\tan\theta}{a\tan\theta}\, d\theta$$

$$= \int \sec\theta\, d\theta = ln\,|\sec\theta + \tan\theta| + C = \log\left|\frac{x}{a} + \sqrt{\frac{x^2}{a^2}-1}\right| + C$$

$$= ln \left| \frac{x+\sqrt{x^2-a^2}}{a} \right| + C = ln \left| x+\sqrt{x^2-a^2} \right| + K. \qquad \text{(where } K = C - ln\ |a|.$$

(v) Let $x = a\sec\theta$, then $\theta = \sec^{-1}\frac{x}{a}$.

$$\because \quad \int \frac{dx}{x\sqrt{x^2-a^2}} = \int \frac{a\sec\theta\tan\theta}{a\sec\theta\sqrt{a^2\sec^2\theta - a^2}}\,d\theta = \int \frac{a\sec\theta\tan\theta}{a\sec\theta \,.\, a\tan\theta}\,d\theta$$

$$= \frac{1}{a}\int d\theta = \frac{1}{a}\theta + C = \frac{1}{a}\sec^{-1}\frac{x}{a} + C.$$

The results obtained above can be used as standard formulae.

1. $\displaystyle\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + C.$

2. $\displaystyle\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C.$

3. $\displaystyle\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\frac{x}{a} + C.$

4. $\displaystyle\int \frac{dx}{\sqrt{x^2+a^2}} = ln\left|x+\sqrt{x^2+a^2}\right| + A$

5. $\displaystyle\int \frac{dx}{\sqrt{x^2-a^2}} = ln\left|x+\sqrt{x^2-a^2}\right| + A$

Example 7 :

Integrate

(i) $\displaystyle\int \frac{dx}{\sqrt{25-16x^2}}$ (ii) $\displaystyle\int \frac{e^x}{e^{2x}+9}\,dx$

(iii) $\displaystyle\int \frac{dx}{x\sqrt{x^8-4}}$ (iv) $\displaystyle\int \frac{\cos\theta\,d\theta}{\sqrt{4\sin^2\theta+1}}$

(v) $\displaystyle\int \frac{x+5}{\sqrt{x^2+6x-7}}\,dx.$

Solutions :

(i) $\displaystyle\int \frac{dx}{\sqrt{25-16x^2}} = \frac{1}{4}\int \frac{dx}{\sqrt{\left(\frac{5}{4}\right)^2 - x^2}} = \frac{1}{4}\sin^{-1}\frac{x}{5/4} + C$

$= \dfrac{1}{4}\sin^{-1}\dfrac{4x}{5} + C$, using formula.

(ii) $\displaystyle\int \frac{e^x}{e^{2x}+9}\,dx = \int \frac{dz}{z^2+3^2}$ (where $z = e^x$)

$$= \frac{1}{3}\tan^{-1}\frac{z}{3} + C = \frac{1}{3}\tan^{-1}\left(\frac{e^x}{3}\right) + C.$$

(iii) $\int \frac{dx}{x\sqrt{x^8-4}} = \frac{1}{4}\int \frac{4x^3}{x^4\sqrt{x^8-4}}\,dx$

$= \frac{1}{4}\int \frac{dz}{z\sqrt{z^2-2^2}}$, putting $x^4 = z$, $4x^3\,dx = dz$.

$= \frac{1}{4}\,\frac{1}{2}\,.\sec^{-1}\frac{z}{2} + C$ using formula

$= \frac{1}{8}\sec^{-1}\left(\frac{x^4}{2}\right) + C.$

(iv) $\int \frac{\cos\theta\, d\theta}{\sqrt{4\sin^2\theta+1}}\,d\theta = \int \frac{dz}{\sqrt{4z^2+1}}$, putting $\sin\theta = z$, $\cos\theta\, d\theta = dz$.

$= \frac{1}{2}\int \frac{dz}{\sqrt{z^2+\left(\frac{1}{2}\right)^2}} = \frac{1}{2}\,.\,ln\left|z+\sqrt{z^2+\frac{1}{4}}\right| + C$

$= \frac{1}{2}\,ln\left|\sin\theta+\sqrt{\sin^2\theta+\frac{1}{4}}\right| + C.$

(v) $\int \frac{x+5}{\sqrt{x^2+6x-7}}\,dx = \int \frac{x+3+2}{\sqrt{(x+3)^2-16}}\,dx = \int \frac{z+2}{\sqrt{z^2-16}}\,dz$, putting $x+3 = z$.

$= \int \frac{z}{\sqrt{z^2-16}}\,dz + 2\int \frac{dz}{\sqrt{z^2-16}}$

$= \frac{1}{2}\int \frac{d(z^2-16)}{\sqrt{z^2-16}} + 2\int \frac{dz}{\sqrt{z^2-16}}$

$= \frac{1}{2}\int \frac{du}{\sqrt{u}} + 2\int \frac{dz}{\sqrt{z^2-16}}$ (where $z^2 - 16 = u$ in the first integration)

$= \sqrt{u} + 2\,ln\left|z+\sqrt{z^2-16}\right| + c$

$= \sqrt{z^2-16} + 2\,ln\left|z+\sqrt{z^2-16}\right| + c$

$= \sqrt{(x+3)^2-16} + 2\,ln\left|x+3+\sqrt{(x+3)^2-16}\right| + c$

$= \sqrt{x^2+6x-7} + 2\,ln\left|x+3+\sqrt{x^2+6x-7}\right| + c.$

EXERCISE 9 (d)

Integrate the following :

1. (i) $\int \frac{dx}{\sqrt{11-4x^2}}$ (ii) $\int \frac{e^{3x}}{\sqrt{4-e^{6x}}}\,dx$ $(e^{3x} = z)$

(iii) $\int \frac{dx}{x\sqrt{25-(\ln x)^2}}$ (iv) $\int \frac{\cos\theta}{\sqrt{4-\sin^2\theta}}\,d\theta$

(v) $\int \frac{x^2}{\sqrt{36-x^6}}\,dx$ $(x^3 = z)$ (vi) $\int \frac{x+3}{\sqrt{9-x^2}}\,dx$

(vii) $\int \frac{dx}{\sqrt{5-x^2-4x}}$ $(x+2=z)$ (viii) $\int \frac{x+3}{\sqrt{5-x^2-4x}}\,dx.$

2. (i) $\int \frac{dx}{3x^2+7}$ (ii) $\int \frac{e^{4x}}{e^{8x}+4}\,dx$ $(e^{4x}=z)$

(iii) $\int \frac{dx}{x\left\{(ln\,x)^2+25\right\}}$ (iv) $\int \frac{\sec\theta\,\tan\theta}{\sec^2\theta+4}\,d\theta$

(v) $\int \frac{x^9}{x^{20}+4}\,dx$ $(x^{10}=z)$ (vi) $\int \frac{3x+4}{x^2+4}\,dx$

(vii) $\int \frac{dx}{x^2+6x+13}$ $(x+3=z)$ (viii) $\int \frac{x+5}{x^2+6x+13}\,dx.$

3. (i) $\int \frac{dx}{x\sqrt{4x^2-9}}$ (ii) $\int \frac{dx}{\sqrt{e^{4x}-5}}$ $(e^{2x}=z)$

(iii) $\int \frac{dx}{x\,ln\,x\sqrt{(ln\,x)^2-4}}$ (iv) $\int \frac{\sec\theta\,d\theta}{\sin\theta\sqrt{3\tan^2\theta-1}}$

(v) $\int \frac{dx}{x\sqrt{x^{14}-b^2}}$ $(x^7=z)$ (vi) $\int \frac{x^2+3}{x\sqrt{x^2-4}}\,dx$

(vii) $\int \frac{dx}{(x+1)\sqrt{x^2+2x-3}}$ $(x+1=z)$ (viii) $\int \frac{x^2+2x+4}{(x+1)\sqrt{x^2+2x-3}}\,dx.$

4. (i) $\int \frac{dx}{\sqrt{3x^2+4}}$ (ii) $\int \frac{4e^x}{\sqrt{3e^{2x}+4}}\,dx$

(iii) $\int \frac{dx}{x\sqrt{(ln\,x)^2+8}}$ (iv) $\int \frac{d\theta}{\sin^2\theta\sqrt{\cot^2\theta+2}}$

(v) $\int \frac{x^2}{\sqrt{x^6+a^6}}\,dx$ (vi) $\int \frac{3x+4}{\sqrt{5x^2+8}}\,dx$

(vii) $\int \frac{e^x\cos e^x}{\sqrt{\sin^2 e^x+9}}\,dx$ (viii) $\int \frac{2x+11}{\sqrt{x^2+10x+29}}\,dx.$ $(x+5=z)$

5. (i) $\int \frac{dx}{\sqrt{4x^2-6}}$ (ii) $\int \frac{e^{5x}}{\sqrt{e^{10x}-4}}\,dx$

(iii) $\int \frac{dx}{x\sqrt{(ln\,x)^2-4}}$; $x>e^2$ (iv) $\int \frac{\cos\theta\,d\theta}{\sin^2\theta\sqrt{\operatorname{cosec}^2\theta-4}}$

(v) $\int \frac{dx}{\sqrt{x}\sqrt{x-a^2}}$ $\left(\sqrt{x}=z\right)$ (vi) $\int \frac{x-2}{\sqrt{3x^2-8}}\,dx$

(vii) $\int \frac{3x+4}{x\sqrt{2x^2-5}}\,dx$ (viii) $\int \frac{x^2+2x+2}{x\sqrt{x^2-4}}\,dx$

(ix) $\int \frac{dx}{\sqrt{x^2+8x}}$ $(x+4=z)$ (x) $\int \frac{x+7}{\sqrt{x^2+8x}}\,dx.$

9.7 INTEGRATION BY PARTS :

If ϑ and ω are differentiable functions of x, then

$$\frac{d}{dx}(\vartheta\omega) = \vartheta\frac{d\omega}{dx} + \omega\frac{d\vartheta}{dx}$$

or $$\vartheta\frac{d\omega}{dx} = \frac{d}{dx}(\vartheta\omega) - \omega\frac{d\vartheta}{dx}.$$

Integrating both sides,

$$\int \vartheta\frac{d\omega}{dx}\,dx = \int\frac{d}{dx}(\vartheta\omega)\,dx - \int\omega\frac{d\vartheta}{dx}\,dx$$

$$= \omega\vartheta - \int\omega\frac{d\vartheta}{dx}\,dx.$$

Setting $\frac{d\omega}{dx} = u$, $\omega = \int u dx$, the result can be written as

$$\int u\,\vartheta\,dx = \left(\int u dx\right)\times\vartheta - \int\left(\int u dx\right)\times\frac{d\vartheta}{dx}\,dx. \qquad \text{.........(A)}$$

This rule is called the rule of 'integration by parts' and is used to integrate the product of two functions.

In words, Integral of the product of two functions

= (Integral of first function) × second function

– Integral of (integral of first × derivative of the second).

Briefly, **Int. of product = (Int. first) × second – $\int$ (Int. first) (der. second) dx.**

Application of this rule does not end the work of integration, but leaves another integral . If this remaining integral is easier or simpler than the original one (integral of product of two functions) the rule has been properly applied. The success depends on the proper choice of the first function (function to be integrated in both terms on the right of (A)) and the second function (function to be differentiated in the second term on the right) .

In fact, since $u\vartheta = \vartheta u$, 'first' and 'second' have no significance of order of writing; these two terms are used merely to fix the idea that one function (first) is to be chosen to be integrated in both terms and the other (second) to be differentiated in the second term on the right of (A) .

For example, in integrating $x\,e^x$ as $\int x\,e^x\,dx = \frac{x^2}{2}\,e^x - \int\frac{x^2}{2}e^x\,dx$, the rule has been applied without proper choice of 'first' and 'second' function and the resulting new integral is more difficult than the original one. In this case, e^x should be chosen as the 'first function'.

Then, $\int e^x\,x\,dx = \left(\int e^x\,dx\right)x - \int\left(\int e^x\,dx\right).\frac{dx}{dx}.dx = e^x.x - \int e^x.dx$

so that the resulting integral is easier than the original one and can be easily integrated.

The following table gives a proper choice of 'first' and 'second' functions in certain cases. Here m ∈ N, n may be zero or any positive integer.

Function to be integrated	First function	Second function
$x^n e^x$	e^x	x^n
$x^n \cos x$	$\cos x$	x^n
$x^n \sin x$	$\sin x$	x^n

$x^n (ln\, x)^m$	x^n	$(ln\, x)^m$
$x^n \sin^{-1}x$	x^n	$\sin^{-1}x$
$x^n \cos^{-1}x$	x^n	$\cos^{-1}x$
$x^n \tan^{-1}x$	x^n	$\tan^{-1}x$

Usually from among exponential function (E), trigonometric function (T), algebraic function (A), logarithmic function (L) and inverse trigonometric function (I), the choice of 'first' and 'second' function is made in the order ETALI.

Example 8 :

Integrate (i) $\int x \cos x\, dx$ (ii) $\int x^2 e^x\, dx$

(ii) $\int \tan^{-1} x\, dx$ (iii) $\int (ln\, x)^2\, dx$.

Solutions :

(i) $\int x \,.\, \cos dx = \left(\int \cos x\, dx\right) . x - \int \left(\int \cos x\, dx\right) \times \frac{dx}{dx} . dx$.

$= x \sin x - \int \sin x \;.\; 1.\, dx = x \sin x + \cos x + C$.

(ii) $\int x^2\, e^x\, dx = \left(\int e^x dx\right) . x^2 - \int \left(\int e^x\, dx\right) \frac{d}{dx}\, (x^2) . dx$

$= x^2 e^x - 2\int x\, e^x\, dx = x^2 e^x - 2 \left[\left(\int e^x\, dx\right) . x - \int \left(\int e^x\, dx\right) . 1 . dx \right]$

$= x^2 e^x - 2x\, e^x + 2\, e^x + C$

$= (x^2 - 2x + 2)\, e^x + C$.

When the integrand is $x^n e^x$, the rule of integration by parts has to be applied n times in succession followed by an ordinary integration. Similar steps are to be taken when the integrand is either $x^n \cos ax$ or $x^n \sin bx$.

(iii) $\int \tan^{-1} x\, dx = \int 1. \tan^{-1} x\, dx$. (here '1' is the first function)

$= x \,.\, \tan^{-1} x - \int x \,.\, \frac{1}{1+x^2}\, dx$

$= x \tan^{-1} x - \frac{1}{2} \int \frac{2x}{1+x^2}\, dx$

$= x \tan^{-1} x - \frac{1}{2}\, ln\, (1 + x^2) + C$.

When $\sin^{-1}x$, $\cot^{-1}x$, $\tan^{-1}x$ etc. or $\log x$ is present alone in the integrand, $1 = x^0$ has to be taken as the first function.

(iv) $\int (ln\, x)^2\, dx = \int 1. (ln\, x)^2\, dx = x \,.\, (ln\, x)^2 - \int x \,.\, \frac{2\,(ln\, x)}{x}\, dx$

$= x\, (ln\, x)^2 - 2 \int 1.\; ln\, x\, dx = x\, (ln\, x)^2 - 2 \left[x \,.\, ln\, x - \int x \,.\, \frac{1}{x}\, dx \right]$

$= x\, (ln\, x)^2 - 2x\, ln\, x + 2 \int dx$

$= x\, (ln\, x)^2 - 2x\, ln\, x + 2x + C$

$= x\, [\, (ln\, x)^2 - 2\, (ln\, x) + 2\,] + C$.

When the integrand is of the form $e^x\{f(x)+f'(x)\}$, the integral is $e^x f(x)$ which can be verified by differentiating the later. Using integration by parts

$$\int e^x f(x)\,dx = e^x . f(x) - \int e^x . f'(x)\,dx + C$$

So $$\int e^x\{f(x)+f'(x)\}\,dx = e^x f(x) + C.$$

Any function $e^x g(x)$ can be integrated in this way if $g(x)$ can be expressed as the sum of a function and its derivative.

Example 9 :

$$\int e^x \frac{1+\sin x}{1+\cos x}\,dx = \int e^x \frac{\sin x}{1+\cos x}\,dx + \int \frac{e^x}{1+\cos x}\,dx$$

$$\left(\text{observe that } \frac{d}{dx}\left(\frac{\sin x}{1+\cos x}\right) = \frac{1}{1+\cos x}\right)$$

$$= e^x . \frac{\sin x}{1+\cos x} - \int e^x \frac{1}{1+\cos x}\,dx + C + \int \frac{e^x}{1+\cos x}\,dx$$

$$= e^x \frac{\sin x}{1+\cos x} + C.$$

In some cases, integrating by parts we get a multiple of the original integral on the right side, which can be transferred and added to the given integral on the left which then can be evaluated.

Example 10 :

Evaluate (i) $\int e^{ax}\cos bx\,dx$ (ii) $\int e^{ax}\sin bx\,dx$

(iii) $\int \sqrt{a^2-x^2}\,dx$ (iv) $\int \sqrt{x^2+a^2}\,dx$

(v) $\int \sqrt{x^2-a^2}\,dx.$

Solution :

(i) $$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a}.\cos bx - \int \frac{e^{ax}}{a}(-b\sin bx)\,dx$$

$$= \frac{e^{ax}}{a}\cos bx + \frac{b}{a}\int e^{ax}\sin bx\,dx$$

$$= \frac{e^{ax}}{a}\cos bx + \frac{b}{a}\left[\frac{e^{ax}}{a}\sin bx - \int \frac{e^{ax}}{a}(b\cos bx)\,dx\right]$$

$$= \frac{e^{ax}}{a}\cos bx + \frac{b}{a^2}e^{ax}\sin bx - \frac{b^2}{a^2}\int e^{ax}\cos bx\,.\,dx + C$$

$$\Rightarrow \left(1+\frac{b^2}{a^2}\right)\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2}(a\cos bx + b\sin bx) + C$$

$$\therefore \int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx) + K, \text{ where } K = \frac{ca^2}{a^2+b^2}. \quad \text{...... (1)}$$

(ii) Similarly $$\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) + K. \quad \text{......... (2)}$$

(iii) $\int\sqrt{a^2-x^2}\,dx \quad = \int 1.\sqrt{a^2-x^2}\,dx = x\sqrt{a^2-x^2} - \int x\left(\dfrac{-2x}{2\sqrt{a^2-x^2}}\right)dx$

$$= x\sqrt{a^2-x^2} + \int \frac{x^2}{\sqrt{a^2-x^2}}\,dx$$

$$= x\sqrt{a^2-x^2} + \int \frac{a^2-(a^2-x^2)}{\sqrt{a^2-x^2}}\,dx.$$

$$= x\sqrt{a^2-x^2} + a^2\int \frac{dx}{\sqrt{a^2-x^2}} - \int\sqrt{a^2-x^2}\,dx$$

$$\therefore 2\int\sqrt{a^2-x^2}\,dx = x\sqrt{a^2-x^2} + a^2\int\frac{dx}{\sqrt{a^2-x^2}} = x\sqrt{a^2-x^2} + a^2\sin^{-1}\frac{x}{a} + 2C.$$

$$\Rightarrow \int\sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C. \quad \text{...... (3)}$$

Similarly it can be proved that,

$$\int\sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\,ln\left|x+\sqrt{x^2+a^2}\right| + C \quad \text{....... (4)}$$

$$\int\sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\,ln\left|x+\sqrt{x^2-a^2}\right| + C. \quad \text{....... (5)}$$

The results established above in (1), (2), (3), (4), (5) may be used as formulae.

EXERCISE 9 (e)

Evaluate the following :

1. (i) $\int(1+x)\,e^x\,dx$ (ii) $\int x^3 e^x\,dx$

   (iii) $\int x^2 e^{ax}\,dx$ (iv) $\int(3x+2)^2\,e^{2x}\,dx$

2. (i) $\int x\,\sin x\,dx$ (ii) $\int x^2\,\cos x\,dx$

   (iii) $\int x^2\,\sin ax\,dx$ (iv) $\int x\,\cos^2 x\,dx$

   (v) $\int x\,\sin^3 x\,dx$ (vi) $\int 2x\,\sin 2x\cos x\,dx$

   (vii) $\int 2x\,\cos 3x\cos 2x\,dx$ (viii) $\int 2x^3\,\cos x^2\,dx$

   (ix) $\int x\,\mathrm{cosec}^2 x\,dx$ (x) $\int x\,\tan^2 x\,dx$.

3. (i) $\int x\,ln\,(1+x)\,dx$ (ii) $\int x^7\,ln\,x\,dx$

   (iii) $\int(ln\,x)^3\,dx$ (iv) $\int ln\,(x^2+1)\,dx$

(v) $\int \frac{lnx}{x^5}\,dx$ (vi) $\int ln\,(x^2+x+2)\,dx$

(vii) $\int ln\left(x+\sqrt{x^2+a^2}\right)\,dx$ (viii) $\int ln\left(x+\sqrt{x^2-a^2}\right)\,dx$.

4. (i) $\int \sin^{-1}x\,dx$ (ii) $\int x\,\sin^{-1}x\,dx$

(iii) $\int \cos^{-1}x\,dx$ (iv) $\int x\,\tan^{-1}x\,dx$

(v) $\int x^2\,\tan^{-1}x\,dx$ (vi) $\int \sec^{-1}x\,dx$

(vii) $\int x\,\text{cosec}^{-1}x\,dx$.

5. (i) $\int e^{3x}\,\cos 2x\,dx$ (ii) $\int e^{2x}\,\sin x\,dx$

(iii) $\int e^{x}\,\cos^2 x\,dx$ (iv) $\int x\,e^{x^2}\,\sin x^2\,dx$

(v) $\int e^{ax}\,\sin(bx+c)\,dx$. (vi) $\int (2x^2+1)e^{x^2}\,dx$.

6. (i) $\int \sqrt{9-x^2}\,dx$ (ii) $\int \sqrt{5-4x^2}\,dx$

(iii) $\int \sqrt{1-x^2-2x}\,dx\ (x+1=z)$ (iv) $\int e^{z}\sqrt{4-e^{2z}}\,dx$

(v) $\int \cos\theta\sqrt{5-\sin^2\theta}\,d\theta$.

7. (i) $\int \sqrt{x^2+4}\,dx$ (ii) $\int \sqrt{7x^2+2}\,dx$

(iii) $\int \sqrt{4x^2+12x+13}\,dx\ (2x+3=z)$ (iv) $\int e^{2z}\sqrt{e^{4z}+6}\,dz$

(v) $\int \sec^2\theta\sqrt{\sec^2\theta+3}\,d\theta$ (vi) $\int (2x^2+1)e^{x^2}\,dx$.

8. (i) $\int \sqrt{x^2-8}\,dx$ (ii) $\int \sqrt{3x^2-2}\,dx$

(iii) $\int \sqrt{x^2-4x+2}\,dx\ (x-2=z)$ (iv) $\int a^{z}\sqrt{a^{2z}-4}\,dz$

(v) $\int \sec\theta\tan\theta\sqrt{\tan^2\theta-3}\,d\theta$.

9. (i) $\int e^{x}\,(\tan x+ln\sec x)\,dx$ (ii) $\int e^{x}\,(\cot x+ln\sin x)\,dx$

(iii) $\int \frac{e^x}{x}\,(1+x\,ln\,x)\,dx$ (iv) $\int \frac{xe^x}{(1+x)^2}\,dx$.

10. (i) $\int \left[\frac{1}{ln\,x}-\frac{1}{(ln\,x)^2}\right]dx$ (ii) $\int \sin(ln\,x)\,dx$

(iii) $\int \sin x\,ln\,(\text{cosec}x-\cot x)\,dx$.

9.8 PARTIAL FRACTIONS AND INTEGRATION OF RATIONAL FUNCTIONS.

A function of the form $\frac{P(x)}{Q(x)}$, where P (x) and Q (x) are polynomials, is called a rational function. It is a proper algebraic fraction if the degree of P (x) is less than that of Q (x); otherwise it is an improper algebraic fraction. In the latter case, a division can be performed and the rational function can be written as $\frac{P(x)}{Q(x)} = \theta(x) + \frac{R(x)}{Q(x)}$, where θ (x) is a polynomial and $\frac{R(x)}{Q(x)}$ is a proper fraction (i.e. degree of R (x) is less than that of Q (x)). Since integration of a polynomial is quite easy, discussion of the method of integration of proper algebraic fractions alone will be sufficient in the context of integration of any rational function.

A proper fraction $\frac{R(x)}{Q(x)}$ can be decomposed into simpler fractions, called partial fractions, and each simpler fraction can be integrated separately by the methods outlined earlier.

Four different cases arise depending on the factors of the denominator Q (x).

(i) The partial fraction (p.f.) corresponding to every non-repeated linear fractor $ax + b$ of Q (x) is of the form $\frac{A}{ax+b}$, where A is a constant.

If $Q(x) = (a_1x + b_1)(a_2x + b_2)(a_3x + b_3)$

$$\frac{R(x)}{Q(x)} = \frac{A_1}{a_1x+b_1} + \frac{A_2}{a_2x+b_2} + \frac{A_3}{a_3x+b_3}.$$

(ii) To a repeated linear factor of the form $(ax + b)^n$, there corresponds n partial fractions of the form $\frac{A_r}{(ax+b)^r}$ $(r = 1, 2, 3, \ldots\ldots\ldots\ldots n)$.

If $Q(x) = (ax + b)^3 (cx + d)$

$$\frac{R(x)}{Q(x)} = \frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \frac{A_3}{(ax+b)^3} + \frac{A_4}{cx+d}.$$

(iii) To a non-repeated quadratic factor $lx^2 + px + q$, there corresponds a p.f. of the form $\frac{A_1x+B_1}{lx^2+px+q}$.

If $Q(x) = (lx^2 + px + q)\ (ax + b)$

$$\frac{R(x)}{Q(x)} = \frac{A_1x+B_1}{lx^2+px+q} + \frac{A_2}{ax+b}.$$

(iv) To a repeated quadratic factor $(lx^2 + px + q)^n$, there correspond n *p.f..s* of the type $\frac{A_rx+B_r}{(lx^2+px+q)^r}$ $(r = 1, 2, \ldots\ldots\ldots\ldots n)$.

If $Q(x) = (lx^2 + px + q)^2\ (ax + b)$

$$\frac{R(x)}{Q(x)} = \frac{A_1x+B_1}{lx^2+px+q} + \frac{A_2x+B_2}{(lx^2+px+q)^2} + \frac{A_3}{ax+b}.$$

When the given proper algebraic fraction $\frac{R(x)}{Q(x)}$ **is equated to the sum of all its p.f.s., we get an identity. To find the values of the constants,** both sides are multiplied by Q (x), thereby clearing the fractions, and the coefficient of like powers of x on both sides are equated. This results in as many equations as the number of constants to be determined which can be solved to determine the constants.

Alternatively, any suitable particular value can be given to x on both sides to get an equation involving the constants. This can be repeated by taking different values of x to get as many equations as necessary to determine the constants.

It is very easy to find the constants in the p.f.s corresponding to non-repeated linear factors of Q (x) and also in the p.f. corresponding to the highest power of a repeated linear factor by putting such value for x as to make each such factor zero in turn after clearing the fractions.

Example 11 :

Integrate (i) $\int \frac{4x+5}{x^2+x-2}\,dx$ (ii) $\int \frac{x^2}{(x+1)^2(x-2)}\,dx$

(ii) $\int \frac{2x^2+x+3}{(x^2+2)(x-1)}\,dx$ (iv) $\int \frac{4x^2}{(x-3)(x+1)}\,dx$

(v) $\int \frac{dx}{x^2-a^2}\,dx;$ (vi) $\int \frac{dx}{(x+a)(x+b)}$

Solution :

(i) $x^2+x-2 = (x+2)(x-1)$

$$\because \quad \int \frac{4x+5}{x^2+x-2} = \frac{A}{x+2} + \frac{B}{x-1} \qquad \text{(say)}$$

clearing the fractions,

$$4x+5 = A(x-1) + B(x+2). \qquad \text{.........(1)}$$

Method 1 :

Equating the coefficients of x on both sides of (1)

$$4 = A + B.$$

Equating the constant terms $5 = 2B - A$.

Solving these two equations we get, $A = 1$, $B = 3$.

Method 2 :

Alternatively, setting $x = 0$, $x = 2$ in succession in (1) we get

we get $5 = 2B - A$

and $13 = A + 4B$

which can be solved to yield $A = 1$, $B = 3$.

Method 3 :

Any two arbitrary values for x will serve the purpose, (as we observe in method 2). But the most convenient values for x which will yield the solutions readily are those which make the factors $x-1$, $x+2$ zero in turn.

Setting $x = 1$ in (1), one gets $9 = 3B \Rightarrow B = 3$

Setting $x = -2$, $-3 = -3A \Rightarrow A = 1$.

This method, called the method of suppression, yields the value of the corresponding constant at once by suppressing the other constants.

Thus by using any of the methods, the constants associated with the p.f.s. can be determined.

Here $\dfrac{4x+5}{x^2+x-2} = \dfrac{1}{x+2} + \dfrac{3}{x-1}$

$$\therefore \quad \int \frac{4x+5}{x^2+x-2}\,dx = \int \frac{dx}{x+2} + 3\int \frac{dx}{x-1} = ln\,|x+2| + 3\,ln\,|x-1| + C.$$

(ii) $\displaystyle\int \frac{x^2}{(x+1)^2(x-2)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x-2}$ (say).

Clearing the fractions, $x^2 = A(x+1)(x-2) + B(x-2) + C(x+1)^2$. .... (2)

Putting $x = -1$ in (2), $1 = -3B \Rightarrow B = -\dfrac{1}{3}$.

Putting $x = 2$, $4 = 9C \Rightarrow C = \dfrac{4}{9}$.

A cannot be found by method of suppression.

Equating the coeficients of x^2 on both sides of (2)

$$1 = A + C \Rightarrow A = \frac{5}{9}.$$

$$\therefore \quad \frac{x^2}{(x+1)^2(x-2)} = \frac{\frac{5}{9}}{x+1} - \frac{\frac{1}{3}}{(x+1)^2} + \frac{\frac{4}{9}}{x-2}$$

$$\Rightarrow \quad \int \frac{x^2}{(x+1)^2(x-2)}\,dx = \frac{5}{9}\int\frac{dx}{x+1} - \frac{1}{3}\int\frac{dx}{(x+1)^2} + \frac{4}{9}\int\frac{dx}{x-2}$$

$$= \frac{5}{9}\,ln\,|x+1| + \frac{1}{3}\,\frac{1}{x+1} + \frac{4}{9}\,ln\,|x-2| + C.$$

(iii) Let $\displaystyle\int \frac{2x^2+x+3}{(x^2+2)(x-1)} = \frac{Ax+B}{x^2+2} + \frac{C}{x-1}$.

$\because \quad 2x^2 + x + 3 = (Ax+B)(x-1) + C(x^2+2)$ .... (3)

setting $x = 1$ in (3), $6 = 3C \Rightarrow C = 2$.

Equating the coefficients of x^2, $2 = A + C \Rightarrow A = 0$.

Equating the constant terms, $3 = 2C - B \Rightarrow B = 1$.

$$\therefore \quad \frac{2x^2+x+3}{(x^2+2)(x-1)} = \frac{1}{x^2+2} + \frac{2}{x-1}$$

$$\Rightarrow \quad \int \frac{2x^2+x+3}{(x^2+2)(x-1)}\,dx = \int\frac{dx}{x^2+2} + 2\int\frac{dx}{x-1} = \frac{1}{\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}} + 2\,ln\,|x-1| + C.$$

(iv) $\dfrac{4x^2}{(x-3)(x+1)}$ is not a proper fraction. Since the coefficient of x^2 in numerator is 4 and that in denominator is 1, the quotient will be 4 and the remaining part is a proper fraction.

Let $\frac{4x^2}{(x-3)(x+1)} = 4 + \frac{A}{x-3} + \frac{B}{x+1}$.

$\therefore \quad 4x^2 = 4(x-3)(x+1) + A(x+1) + B(x-3)$. .... (4)

Setting $x = 3$ in (4), $36 = 4A \Rightarrow A = 9$. Setting $x = -1$, $4 = -4B \Rightarrow B = -1$.

$$\text{Then } \frac{4x^2}{(x-3)(x+1)} = 4 + \frac{9}{x-3} - \frac{1}{x+1}$$

$$\Rightarrow \int \frac{4x^2}{(x-3)(x+1)}\,dx = \int 4\,dx + \int \frac{9}{x-3}\,dx - \int \frac{dx}{x+1}$$

$$= 4x + 9\ln|x-3| - \ln|x+1| + C.$$

(v) $$\frac{1}{x^2-a^2} = \frac{1}{(x-a)(x+a)} = \frac{1}{2a}\left(\frac{1}{x-a} - \frac{1}{x+a}\right)$$

$$\therefore \int \frac{dx}{x^2-a^2} = \frac{1}{2a}\int\left(\frac{1}{x-a} - \frac{1}{x+a}\right)dx = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C.$$

(vi) $$\frac{1}{(x+a)(x+b)} = \frac{1}{a-b}\left(\frac{1}{x+b} - \frac{1}{x+a}\right)$$

$$\therefore \int \frac{dx}{(x+a)(x+b)} = \frac{1}{a-b}\int\left(\frac{1}{x+b} - \frac{1}{x-a}\right)dx = \frac{1}{a-b}\ln\left|\frac{x+b}{x-a}\right| + C.$$

The results of (v) follows from (vi) by taking b= $-a$. These two results may be used as standard formula.

(1) $$\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C$$

(2) $$\int \frac{dx}{(x+a)(x+b)} = \frac{1}{a-b}\ln\left|\frac{x+b}{x+a}\right| + C.$$

N.B. (*i*) Sometimes **special techniques** can be used after careful observation. When the integrand contains only even powers of x in both numerator and denominator we divide the numerator and denominator by x^2. For example

$$\int \frac{x^2}{x^4+x^2+1}\,dx = \frac{1}{2}\int \frac{(x^2+1)+(x^2-1)}{x^4+x^2+1}\,dx$$

$$= \frac{1}{2}\int \frac{1+\frac{1}{x^2}}{x^2+1+\frac{1}{x^2}}\,dx + \frac{1}{2}\int \frac{1-\frac{1}{x^2}}{x^2+1+\frac{1}{x^2}}\,dx$$

$$= \frac{1}{2}\int \frac{\left(1+\frac{1}{x^2}\right)dx}{\left(x-\frac{1}{x}\right)^2+3} + \frac{1}{2}\int \frac{\left(1-\frac{1}{x^2}\right)dx}{\left(x+\frac{1}{x}\right)^2-1}$$

$$= \frac{1}{2}\int \frac{du}{u^2+3} + \frac{1}{2}\int \frac{dv}{v^2-1}, \text{ putting } u = x - \frac{1}{x},\quad v = x + \frac{1}{x}.$$

$$= \frac{1}{2}\cdot\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{u}{\sqrt{3}}\right) + \frac{1}{2}\cdot\frac{1}{2}\log\left|\frac{v-1}{v+1}\right| + C$$

$$= \frac{1}{2\sqrt{3}}\tan^{-1}\left(\frac{x^2-1}{\sqrt{3}x}\right) + \frac{1}{4}\log\left|\frac{x^2-x+1}{x^2+x+1}\right| + C.$$

(*ii*) To integrate $\int \frac{dx}{x^4+1}$ we write

$$\int \frac{dx}{x^4+1} = \frac{1}{2}\int \frac{(x^2+1)-(x^2-1)}{x^4+1}\,dx$$

$$= \frac{1}{2}\int \frac{x^2+1}{x^4+1}\,dx - \frac{1}{2}\int \frac{x^2-1}{x^4+1}\,dx$$

$$= \frac{1}{2}\int \frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}}\,dx - \frac{1}{2}\int \frac{1-\frac{1}{x^2}}{x^2+\frac{1}{x^2}}\,dx$$

$$= \frac{1}{2}\int \frac{1+\frac{1}{x^2}}{\left(x-\frac{1}{x}\right)^2+2}\,dx - \frac{1}{2}\int \frac{1-\frac{1}{x^2}}{\left(x+\frac{1}{x}\right)^2-2}\,dx$$

$$= \frac{1}{2}\int \frac{du}{u^2+2} - \frac{1}{2}\int \frac{dv}{v^2-2} \quad \text{putting } u = x-\frac{1}{x} \text{ and } v = x+\frac{1}{x}$$

$$= \frac{1}{2}\cdot\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{u}{\sqrt{2}}\right) - \frac{1}{2}\cdot\frac{1}{2\sqrt{2}}\log\left|\frac{v-\sqrt{2}}{v+\sqrt{2}}\right| + C.$$

$$= \frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{x^2-1}{\sqrt{2}x}\right) - \frac{1}{4\sqrt{2}}\log\left|\frac{x^2-\sqrt{2}x+1}{x^2+\sqrt{2}x+1}\right| \, C.$$

EXERCISE 9 (f)

Evaluate the following integrals.

1. (i) $\int \frac{4x-9}{x^2-5x+6}\,dx$ (ii) $\int \frac{3x}{(x-4)(x+2)}\,dx$

(iii) $\int \frac{5x-12}{(2x-3)(x-6)}\,dx$ (iv) $\int \frac{20x+3}{6x^2-x-2}\,dx$

(v) $\int \frac{2x^2}{(x-1)(x-2)(x-3)}\,dx$ (vi) $\int \frac{12x^4-2x^3-4x^2+x-3}{6x^2-x-2}\,dx.$

2. (i) $\int \frac{2x+9}{(x+3)^2}\,dx$ (ii) $\int \frac{5x^2+4x+4}{(x-2)(x+2)^2}\,dx$

(iii) $\int \frac{x^2+7x+4}{x^3+x^2-x-1}\,dx$ (iv) $\int \frac{x^4+3x^3+x^2-1}{x^3+x^2-x-1}\,dx.$

3. (i) $\int \frac{4x^2 - x + 3}{(x^2+1)(x-1)} dx$ (ii) $\int \frac{5x}{(x^2-2x+2)(x+1)} dx$

(iii) $\int \frac{3}{x^3-1} dx$ (iv) $\int \frac{x^5+x^4+x^3+x^2+4x+1}{x^3+1} dx.$

4. (i) $\int \frac{dx}{x^2-5}$ (ii) $\int \frac{dx}{2x^2+8x+7}$

(iii) $\int \frac{x+3}{2x^2+8x+7} dx$ (iv) $\int \frac{4x^2+20x+25}{2x^2+8x+7} dx$

(v) $\int \frac{e^x}{e^{2x}+3e^x+1} dx$ (vi) $\int \frac{\tan^2\theta+1}{\tan^2\theta-1} d\theta.$

5. (i) $\int \frac{dx}{3-x^2}$ (ii) $\int \frac{dx}{7-x^2+6x}$

(iii) $\int \frac{x-5}{7-x^2+6x} dx$ (iv) $\int \frac{\cos\theta}{3-\sin^2\theta} d\theta.$

(v) $\int \frac{x^2 dx}{(x^2+3)(x^2+2)}$ (vi) $\int \frac{x^3 dx}{x^4+3x^2+2}$ (put $x^2 = t$)

(vii) $\int \frac{dx}{\sin x(3+2\cos x)}$ (put $\cos x = z$).

9.9 INTEGRATION (Continued)

Some functions which are not rational functions can be converted to rational form by suitable substitution and then can be integrated by standard formulae.

Example 12 :

(i) $\int \frac{\sqrt{x+3}}{x+1} dx = \int \frac{t}{t^2-2} \cdot 2\,t dt$ setting $x+3=t^2$

$$= 2\int \frac{t^2}{t^2-2}\, dt = 2\int\left(1+\frac{2}{t^2-2}\right) dt$$

$$= 2\left[t + \frac{2}{2\sqrt{2}}\, ln \left|\frac{t-\sqrt{2}}{t+\sqrt{2}}\right|\right] + c$$

$$= 2\left[\sqrt{x+3} + \frac{1}{\sqrt{2}}\, ln \left|\frac{\sqrt{x+3}-\sqrt{2}}{\sqrt{x+3}+\sqrt{2}}\right|\right] + c.$$

(ii) $\int \frac{x}{(x-1)^{\frac{1}{4}}} dx = \int \frac{v^4+1}{v} 4v^3\, dv$, putting $x-1 = v^4$

$$= 4\int (v^6+v^2)\, dv = 4\left(\frac{v^7}{7}+\frac{v^3}{3}\right) + c$$

$$= \frac{4}{21} v^3(3v^4+7) + c = \frac{4}{21}(x-1)^{\frac{3}{4}}(3x+4) + c.$$

Integrands of the type $\frac{ax+b}{\sqrt{px+q}}$, $(ax^2+bx+c)\sqrt{px+q}$, $\frac{ax^2+bx+c}{\sqrt{px+q}}$ can be integrated in terms of powers of $px+q$.

Since $ax + b = \frac{a}{p}(px + q) + \frac{bp - aq}{p}$.

$$\int \frac{ax+b}{\sqrt{px+q}}\,dx = \frac{a}{p}\int \sqrt{px+q}\,dx + \frac{bp-aq}{p}\int \frac{dx}{\sqrt{px+q}}$$

$$= \frac{2a}{3p^2}(px+q)^{\frac{3}{2}} + \frac{2(bp-aq)}{p^2}(px+q)^{\frac{1}{2}} + c.$$

Similarly $ax^2 + bx + c = \frac{a}{p^2}(px+q)^2 + L(px+q) + m$

where $L = \frac{bp - 2aq}{p^2}$, $m = \frac{cp^2 + aq^2 - bpq}{p^2}$.

$\therefore \quad (ax^2 + bx + c)\sqrt{px+q} = \frac{a}{p^2}(px+q)^{\frac{5}{2}} + L(px+q)^{\frac{3}{2}} + m(px+q)^{\frac{1}{2}}$

and $\frac{ax^2+bx+c}{\sqrt{px+q}} = \frac{a}{p^2}(px+q)^{\frac{3}{2}} + L(px+q)^{\frac{1}{2}} + m(px+q)^{-\frac{1}{2}}$

which can be easily integrated.

Since $(ax^2 + bx + c) = a\left[\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}\right]$ setting $x + \frac{b}{2a} = v$ and $\frac{4ac-b^2}{4a^2} = k^2$ or $-k^2$ depending on its sign, $\sqrt{ax^2+bx+c}$ can be put in one of the forms $\sqrt{v^2+k^2}$, $\sqrt{v^2-k^2}$ or $\sqrt{k^2-v^2}$. Then $\frac{1}{\sqrt{ax^2+bx+c}}$ as well as $\sqrt{ax^2+bx+c}$ can be integrated using standard formulae. Problems of integration of such forms have been dealt with earlier.

The integral $\int \frac{dx}{(px+q)\sqrt{ax^2+bx+c}}$ can be converted to the form $k\int \frac{dt}{\sqrt{lt^2+mt+n}}$ by the substitution $px + q = \frac{1}{t}$ and can be evaluated .

Example 13 :

$$\int \frac{dx}{(x+1)\sqrt{x^2+1}} = \int \frac{-\frac{1}{t^2}\,dt}{\frac{1}{t}\sqrt{\left(\frac{1}{t}-1\right)^2+1}} \qquad \left(\text{putting } x + 1 = \frac{1}{t}\right)$$

$$= -\int \frac{dt}{\sqrt{2t^2-2t+1}} = -\frac{1}{\sqrt{2}}\int \frac{dt}{\sqrt{\left(t-\frac{1}{2}\right)^2+\frac{1}{4}}}$$

$$= -\frac{1}{\sqrt{2}}\, ln \left| t - \frac{1}{2} + \sqrt{\left(t-\frac{1}{2}\right)^2+\frac{1}{4}}\right| + c$$

$$= -\frac{1}{\sqrt{2}}\, ln \left|\frac{1 - x + \sqrt{2(x^2+1)}}{x+1}\right| + k.$$

EXERCISE 9 (g)

Evaluate the following integrals.

1. (i) $\int \frac{\sqrt{2x+3}}{x}\, dx$ (ii) $\int \frac{\sqrt{x^2-7}}{x}\, dx$

(iii) $\int \frac{\sqrt{x}}{x+2}\, dx$ (iv) $\int x\,(3x+2)^{\frac{1}{3}}\, dx$

(v) $\int \frac{x+2}{(2x-1)^{\frac{1}{3}}}\, dx$ (vi) $\int (x+2)\,(x+1)^{\frac{1}{4}}\, dx$

(vii) $\int \frac{x-1}{(x+2)^{\frac{3}{4}}}\, dx$ (viii) $\int \frac{dx}{\sqrt{x}-\sqrt[3]{x}}$ $(x = t^6)$.

2. (i) $\int \frac{3x+4}{\sqrt{2x-3}}\, dx$ (ii) $\int (7x+4)\sqrt{3x+2}\, dx$

(iii) $\int (3x+1)\,(x-2)^{\frac{9}{2}}\, dx$ (iv) $\int \frac{2x+5}{(x+2)^{\frac{7}{2}}}\, dx$

(v) $\int \frac{x^2+2x+1}{\sqrt{x+4}}\, dx$ (vi) $\int (x^2+2x+7)\sqrt{x+1}\, dx$.

3. (i) $\int \frac{dx}{\sqrt{4x^2-4x+5}}$ (ii) $\int \sqrt{4x^2-4x+5}\, dx$

(iii) $\int \frac{dx}{\sqrt{x^2-6x+5}}$ (iv) $\int \sqrt{x^2-6x+5}\, dx$

(v) $\int \frac{dx}{\sqrt{1+2x-x^2}}$ (vi) $\int \sqrt{1+2x-x^2}\, dx$.

4. (i) $\int \frac{dx}{(1+x)\sqrt{1-x^2}}$ $\left(1+x=\frac{1}{t}\right)$ (ii) $\int \frac{dx}{(2-x)\sqrt{5-4x+x^2}}$

5. (i) $\int \frac{dx}{(2x+5)\sqrt{x+2}}$ $(x+2=t^2)$ (ii) $\int \frac{1+x^2}{x\sqrt{x^4+1}}\, dx$ $\left(x-\frac{1}{x}=v\right)$

(iii) $\int \sqrt{\frac{x-1}{2x+1}}\, dx$ $(2x+1=v^2)$ (iv) $\int \frac{x}{\sqrt{(a^2-x^2)(x^2-b^2)}}\, dx$ $(x^2-b^2=v^2)$.

9.10 INTEGRATION OF SOME MORE TRIGONOMETRIC FUNCTIONS

The integral $\int \frac{dx}{a + b\cos x + c\sin x}$ can be evaluated by converting cos*x* and sin*x* to $\tan\frac{x}{2}$ $(= t)$.

$$\int \frac{dx}{a + b\cos x + c\sin x} = \int \frac{dx}{a + b\frac{1-t^2}{1+t^2} + c\frac{2t}{1+t^2}},$$

$$= \int \frac{1+t^2}{a(1+t^2) + b(1-t^2) + 2ct}\,dx = \int \frac{\sec^2\frac{x}{2}}{(a-b)t^2 + 2ct + (a+b)}\,dx$$

$= \frac{2}{a-b}\int \frac{dt}{t^2 + \frac{2c}{a-b}t + \frac{a+b}{a-b}}$, which is a standard integral, the denominator being a quadratic in t. It can be written in the form

$$\frac{2}{a-b}\int \frac{dt}{\left(t + \frac{c}{a-b}\right)^2 + k^2} = \frac{2}{a-b}\int \frac{dv}{v^2 + k^2}$$

or $\frac{2}{a-b}\int \frac{dt}{\left(t + \frac{c}{a-b}\right)^2 - k^2} = \frac{2}{a-b}\int \frac{dv}{v^2 - k^2}$ and can be integrated.

Integrals of the type $\int \frac{dx}{a + b\cos x}$, $\int \frac{dx}{a + b\sin x}$

or $\int \frac{dx}{a\cos x + b\sin x}$ are particular cases of the general form discussed above.

Example 14 :

(i) $$\int \frac{dx}{2 + \sin x} = \int \frac{dx}{2 + \frac{2\tan(x/2)}{1 + \tan^2(x/2)}}$$

$$= \frac{1}{2}\int \frac{1 + \tan^2(x/2)}{\tan^2(x/2) + 1 + \tan(x/2)}\,dx$$

$$= \frac{1}{2}\int \frac{\sec^2(x/2)}{\tan^2(x/2) + \tan(x/2) + 1}\,dx$$

$$= \int \frac{dt}{t^2 + t + 1} \qquad \text{setting } \tan\frac{x}{2} = t$$

$$= \int \frac{dt}{\left(t + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \frac{1}{\frac{\sqrt{3}}{2}}\tan^{-1}\frac{t + \frac{1}{2}}{\frac{\sqrt{3}}{2}} + c.$$

$$= \frac{2}{\sqrt{3}}\tan^{-1}\frac{2\tan(x/2) + 1}{\sqrt{3}} + c.$$

(ii) $\int \frac{dx}{1+2\cos x} = \int \frac{dx}{1+2\frac{1-\tan^2(x/2)}{1+\tan^2(x/2)}} = \int \frac{1+\tan^2(x/2)}{3-\tan^2(x/2)}\,dx$

$= \int \frac{\sec^2(x/2)}{3-\tan^2(x/2)}\,dx = 2\int \frac{dt}{3-t^2}$ putting $\tan\frac{x}{2} = t$

$= 2\,\frac{1}{2\sqrt{3}}\,ln\left|\frac{\sqrt{3}+t}{\sqrt{3}-t}\right| + c = \frac{1}{\sqrt{3}}\,ln\left|\frac{\sqrt{3}+\tan x/2}{\sqrt{3}-\tan x/2}\right| + c.$

If the integrand is $\frac{p\cos x + q\sin x + r}{a\cos x + b\sin x + c}$ the numerator can be written in the form $\lambda(a\cos x + b\sin x + c) + \mu(-a\sin x + b\cos x) + \Upsilon$, so that $\int \frac{p\cos x + q\sin x + r}{a\cos x + b\sin x + c}\,dx =$ $\int\left(\lambda + \frac{\mu\, d(a\cos x + b\sin x + c)}{a\cos x + b\sin x + c} + \frac{\gamma}{a\cos x + b\sin x + c}\right)dx = \lambda x + \mu\, ln\,|a\cos x + b\sin x + c| +$ $\Upsilon\int \frac{dx}{a\cos x + b\sin x + c}$ when the last integral can be evaluated by the method outlined above.

Example 15 :

To evaluate $\int \frac{2\sin x + 3\cos x}{3\sin x + 4\cos x}\,dx$, it can be seen that

$2\sin x + 3\cos x = \lambda(3\sin x + 4\cos x) + \mu(3\cos x - 4\sin x)$ implies $3\lambda - 4\mu = 2$ and $4\lambda + 3\mu = 3$, giving

$$\lambda = \frac{18}{25} \quad \mu = \frac{1}{25}.$$

$$\therefore \int \frac{2\sin x + 3\cos x}{3\sin x + 4\cos x}\,dx = \int \frac{\frac{18}{25}(3\sin x + 4\cos x) + \frac{1}{25}(3\cos x - 4\sin x)}{3\sin x + 4\cos x}\,dx$$

$$= \frac{18}{25}\int dx + \frac{1}{25}\int \frac{d(3\sin x + 4\cos x)}{3\sin x + 4\cos x}$$

$$= \frac{18}{25}x + \frac{1}{25}\,ln\,|3\sin x + 4\cos x| + c.$$

Integral of the type $\int \frac{dx}{a + b\cos^2 x + c\sin^2 x}$ can be evaluated by multiplying the numerator and denominator by $\sec^2 x$ (or $\mathrm{cosec}^2 x$), thereby converting the denominator to a quadratic of $\tan x$ ($\cot x$).

Example 16 :

$$\int \frac{dx}{3\sin^2 x + 2\cos^2 x} = \int \frac{\sec^2 x}{3\tan^2 x + 2}\,dx$$

$= \frac{1}{3}\int \frac{dt}{t^2 + \frac{2}{3}}$ putting $\tan x = t$

$$= \frac{1}{3}\,\frac{1}{\sqrt{\frac{2}{3}}}\tan^{-1}\frac{\sqrt{3}t}{\sqrt{2}} + c = \frac{1}{\sqrt{6}}\tan^{-1}\left(\frac{\sqrt{3}}{\sqrt{2}}\tan x\right) + c.$$

EXERCISE 9 (h)

Evaluate the following integrals.

1. (i) $\int \frac{dx}{4+5\cos x}$ (ii) $\int \frac{dx}{3+\cos x}$

(iii) $\int \frac{dx}{3+\sin x}$ (iv) $\int \frac{dx}{1+2\sin x}$

(v) $\int \frac{dx}{2\sin x+3\cos x}$ (vi) $\int \frac{dx}{1+\cos x+\sin x}$

2. (i) $\int \frac{3\sin x+28\cos x}{5\sin x+6\cos x}\,dx$ (ii) $\int \frac{12\sin x-2\cos x+3}{\sin x+\cos x}\,dx$

(iii) $\int \frac{5\sin x}{3-2\sin x}\,dx$ (iv) $\int \frac{2\cos x+7}{4-\sin x}\,dx.$

3. (i) $\int \frac{dx}{2\cos^2 x+3\cos x}$ (ii) $\int \frac{dx}{4\sin^2 x-\sin x}$

(iii) $\int \frac{\sin x\cos x}{\sin^2 x-2\sin x+3}\,dx$ (iv) $\int \frac{dx}{\cos x-\cos 3x}.$

4. (i) $\int \frac{d\theta}{4+3\sin^2\theta}$ (ii) $\int \frac{d\theta}{2-3\cos^2\theta}$

(iii) $\int \frac{d\theta}{4\cos^2\theta+9\sin^2\theta}$ (iv) $\int \frac{d\theta}{2+3\cos^2\theta-4\sin^2\theta}.$

5. (i) $\int \frac{\sin 3x}{\cos 7x\cos 4x}\,dx$ (ii) $\int \frac{\cos 2x}{\sin 7x\cos 5x}\,dx.$

6. (i) $\int \frac{dx}{\cos x\,(5+3\cos x)}$ (ii) $\int \frac{dx}{\cos x\,(1+2\sin x)}.$

9.11 DEFINITE INTEGRAL

It was stated earlier that integration can be considered as a process of summation. In such a case the integral is called definite integral.

Let us consider a function $f(x)$ continuous in the interval $[a, b]$. Let the interval be divided into n sub-intervals of lengths $h_1, h_2, \ldots\ldots\ldots h_n$ by the points $x_1, x_2, \ldots\ldots x_{n-1}$ such that

$$a = x_0 < x_1 < x_2 < x_3 \ldots\ldots\ldots\ldots\ldots < x_{n-1} < x_n = b.$$

Then $h_r = x_r - x_{r-1}$. $(r = 1, 2, \ldots\ldots\ldots n)$

Let ϑ_r be any point in $[x_{r-1}, x_r]$. $(r = 1, 2, \ldots\ldots\ldots\ldots n)$

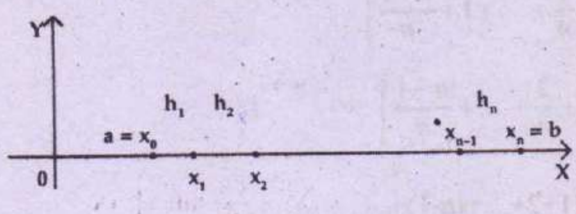


Fig. 9.1

Taking $S = h_1 f(v_1) + h_2 f(v_2) + \ldots\ldots\ldots + h_n f(v_n)$, if the limit of S exists as $n \to \infty$ and maximum of $h_r \to 0$, then this limit is called the definite integral of $f(x)$ from a to b. It is denoted by

$$\int_a^b f(x)\, dx.$$

So $\int_a^b f(x)\, dx = \text{Lt}\,[\, h_1 f(v_1) + h_2 f(v_2) + \ldots\ldots\ldots\ldots\ldots + h_n f(v_n)\,]$ as $n \to 0$.

[Here a is called the **lower limit** and b, the **upper limit** of integration. Here the term 'limit' has different implication from limit of a function.]

In this discussion, which is quite general, the subintervals are taken with arbitrary length (but with the condition that the maximum of these tends to zero). It is often convenient to take equal subintervals each of length $h = \dfrac{b-a}{n}$ so that $h \to 0$ as $n \to \infty$. Then $x_r = a + rh$ $(r = 0, 1, 2, \ldots\ldots n)$. It is also convenient to take v_r as one of the end points of the corresponding subinterval. Then, if the limit exists,

$$\int_a^b f(x)\,dx = \underset{n\to\infty}{lt}\; h\,[\, f(a) + f(a+h) + \ldots\ldots\ldots\ldots\ldots + f(a+\overline{n-1}\,h)\,]$$

$$= \underset{n\to\infty}{lt}\; h \sum_{r=0}^{n-1} f(a+rh)$$

Also $\int_a^b f(x)\, dx = \underset{n\to\infty}{lt}\; h\,[\, f(a+h) + f(a+2h) + \ldots\ldots\ldots\ldots + f(a+nh)\,]$

$$= \underset{n\to\infty}{lt}\; h \sum_{r=1}^{n} f(a+rh).$$

Example 17

Find $\int_1^2 x\, dx$ as a limit of sum.

Solution: $\int_a^b f(x)\text{dx} = \lim\limits_{n\to\infty} h\,[f(a)+f(a+h)+\ldots.+f(a+(n-1)h]$ where $h=\dfrac{b-a}{n}$.

Here, $a=1$, $b=2$, $h=\dfrac{1}{n}$ and $f(x)=x$.

$$\therefore \int_1^2 x\, dx = \lim_{n\to\infty}\frac{1}{n}\left[f(1) + f\left(1+\frac{1}{n}\right) + \ldots. + f(i)\left(1+\frac{n-1}{n}\right)\right]$$

$$= \lim_{n\to\infty}\frac{1}{n}\left[1+1+\frac{1}{n}+\ldots.+1+\frac{n-1}{n}\right]$$

$$= \lim_{n\to\infty}\frac{1}{n}\left[n+\frac{1}{n}+\frac{2}{n}+\ldots.+\frac{n-1}{n}\right]$$

$$= \lim_{n\to\infty}\frac{1}{n}\left[n+\frac{1}{n}(1+2+\ldots.+(n-1)\right]$$

$$= \lim_{n\to\infty} \frac{1}{n}\left[n + \frac{1}{n}\frac{n\,(n-1)}{2} \right]$$

$$= \lim_{n\to\infty} \frac{1}{n}\left(\frac{3n-1}{2} \right)$$

$$= \lim_{n\to\infty} \left[\frac{3}{2} - \frac{1}{2n} \right]$$

$$= \frac{3}{2}.$$

EXERCISE 9 (i)

Evalnate the following as limit of sum.

1. $\int_1^3 5x^2 dx$, 2. $\int_{-1}^1 e^x dx$, 3. $\int_0^6 x\, dx$, 4. $\int_0^4 \left(x^2 + 2x + 4\right) dx$

9.12 FUNDAMENTAL THEOREM OF INTEGRAL CALCULUS :

This theorem establishes a relation between definite integral and indefinite integral. Here the theorem will be given without proof.

Statement :

If $f(x)$ is continuous in the interval $[a, b]$ and $F(x)$ is an anti-derivative of $f(x)$, then

$$\int_a^b f(x)\, dx = F(b) - F(a).$$

From indefinite integration, it is known that $\int f(x)\ dx = F(x) + K$ where K is an arbitrary constant. For any K, the difference of the values of $F(x) + K$ at $x = b$ and at $x = a$ is $[F(x) + K]_{x=b} - [F(x) + K]_{x=a} = F(b) + K - F(a) - K = F(b) - F(a)$ and this is equal to the definite integral $\int_a^b f(x)\ dx$.

So we can write $\int_a^b f(x)\ dx = |F(x) + K|_a^b = F(b) - F(a)$. As K is eliminated in the process, we can also write $\int_a^b f(x)\ dx = |F(x)|_a^b = F(b) - F(a)$.

In case of definite integrals, the results

(i) $\int_a^b [g(x) + h(x)]\, dx = \int_a^b g(x)\ dx + \int_a^b h(x)\, dx$ and

(ii) $\int_a^b \lambda\, g(x)\, dx = \lambda \int_a^b g(x)\ dx$ follow from the corresponding results of indefinite integrals.

Example 18

(i) $\int_1^2 x^3\, dx = \left|\frac{x^4}{4}\right|_1^2 = \left(\frac{x^4}{4}\right)_{x=2} - \left(\frac{x^4}{4}\right)_{x=1} = \frac{2^4}{4} - \frac{1^4}{4} = \frac{15}{4}.$

(ii) $\int_0^{\frac{\pi}{2}} \sin x\, dx = |-\cos x|_0^{\frac{\pi}{2}} = \left(-\cos\frac{\pi}{2}\right) - (-\cos 0) = 0 - (-1) = 1.$

(iii) $\int_0^1 \frac{dx}{1+x^2} = |\tan^{-1} x|_0^1 = \tan^{-1}1 - \tan^{-1}0 = \frac{\pi}{4} - 0 = \frac{\pi}{4}.$

(iv) $\int_0^{\frac{\pi}{2}} x \cos x\, dx$

We first find the indefinite integral

$\int x \cos x\, dx = \sin x \,.\, x - \int \sin x \,.\, 1\, dx = x \sin x + \cos x + K.$

$\therefore \quad \int_0^{\frac{\pi}{2}} x \cos x\, dx = |x \sin x + \cos x|_0^{\pi/2} = \left(\frac{\pi}{2}\sin\frac{\pi}{2} + \cos\frac{\pi}{2}\right) - (0\,.\,\sin 0 + \cos 0) = \frac{\pi}{2} - 1.$

(v) $\int_0^1 \frac{dx}{(x+1)(x+2)}$

The indefinite integral is determined first.

As $\frac{1}{(x+1)(x+2)} = \frac{1}{(x+1)} - \frac{1}{(x+2)},$

$\int \frac{dx}{(x+1)(x+2)} = \int \frac{dx}{x+1} - \int \frac{dx}{x+2} = ln \left|\frac{x+1}{x+2}\right| + K.$

$\therefore \quad \int_0^1 \frac{dx}{(x+1)(x+2)} = \left[ln \left|\frac{x+1}{x+2}\right|\right]_0^1 = ln\,\frac{2}{3} - ln\,\frac{1}{2} = ln\,\frac{4}{3}.$

(vi) $\int_0^{\frac{\pi}{2}} (3x^2 + 2x + \cos x)\, dx = \int_0^{\frac{\pi}{2}} 3x^2\, dx + \int_0^{\frac{\pi}{2}} 2x\, dx + \int_0^{\frac{\pi}{2}} \cos x\, dx$

$= |x^3|_0^{\frac{\pi}{2}} + |x^2|_0^{\frac{\pi}{2}} + |\sin x|_0^{\frac{\pi}{2}} = \frac{\pi^3}{8} - \frac{\pi^2}{4} + 1.$

In case of substitution it is often convenient to carry the limits of integration with the substituted variable, so that conversion back to the original variable after the integration will not be necessary as is done in case of indefinite integrals.

(vii) $\int_2^3 2x\, e^{x^2}\, dx = \int_4^9 e^z\, dz,$ putting $x^2 = z$, so that $z = 4$ when $x = 2$ and $z = 9$ when $x = 3$.

Then the integral can be evaluated as $\int_4^9 e^z\, dz = |e^z|_4^9 = e^9 - e^4.$

(viii) $\int_0^{\frac{\pi}{4}} \sin^5 x \cos x\, dx = \int_0^{\frac{1}{\sqrt{2}}} z^5\, dz$ (putting $\sin x = z$)

$$= \left| \frac{z^6}{6} \right|_0^{\frac{1}{\sqrt{2}}} = \frac{1}{8.6} - 0 = \frac{1}{48}.$$

9.13 ELEMENTARY PROPERTIES OF DEFINITE INTEGRALS.

(i) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

(ii) $\int_a^b f(x)\,dx = \int_a^b f(y)\,dy = \int_a^b f(z)\,dz$

i.e. definite integral is independent of the symbol for variable of integration.

(iii) $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx, \quad a < c < b.$

Proof :

Let $F'(x) = f(x)$.

(i) $\int_a^b f(x)\,dx = |F(x)|_a^b = F(b) - F(a)$

$\int_b^a f(x)\,dx = |F(x)|_b^a = F(a) - F(b)$

So $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx.$

(ii) $\int_a^b f(x)\,dx = |F(x)|_a^b = F(b) - F(a)$

$\int_a^b f(y)\,dy = |F(y)|_a^b = F(b) - F(a)$

$\int_a^b f(z)\,dz = |F(z)|_a^b = F(b) - F(a)$

So $\int_a^b f(x)\,dx = \int_a^b f(y)\,dy = \int_a^b f(z)\,dz.$

(iii) $\int_a^c f(x) + \int_c^b f(x)\,dx = |F(x)|_a^c + |F(x)|_c^b$

$= F(c) - F(a) + F(b) - F(c) = F(b) - F(a) = |F(x)|_a^b$

$= \int_a^b f(x)\,dx$, by the Fundamental Theorem.

When the left limit and right limit at a point of discontinuity of a function differ by a finite number the discontinuity is called finite discontinuity. When the integrand is not continuous in the interval of integration but has a finite number of finite discontinuities, the integral can be evaluated by dividing the interval into subintervals by the points of discontinuity within the interval. The finite discontinuities, if at all, at the two ends of the interval of integration should be ignored.

Example 19 :

(i) $\int_1^4 [x]\, dx = \int_1^2 [x]\, dx + \int_2^3 [x]\, dx + \int_3^4 [x]\, dx$

$= \int_1^2 1\, dx + \int_2^3 2\, dx + \int_3^4 3\, dx$

$= 2 - 1 + 2(3-2) + 3(4-3) = 6.$

(ii) $\int_{-3}^4 |x|\, dx = \int_{-3}^0 |x|\, dx + \int_0^4 |x|\, dx$

$= \int_{-3}^0 (-x)\, dx + \int_0^4 x\,dx = \int_0^{-3} x\, dx + \int_0^4 x\,dx$

$= \left|\frac{x^2}{2}\right|_0^{-3} + \left|\frac{x^2}{2}\right|_0^4 = \frac{9}{2} - 0 + \frac{16}{2} - 0 = \frac{25}{2}.$

EXERCISE 9 (j)

Evaluate the following integrals.

1. (i) $\int_{-2}^3 x^4\, dx$ (ii) $\int_0^1 x^{\frac{1}{3}}\, dx$

(iii) $\int_1^4 \frac{dx}{\sqrt{x}}\, dx$ (iv) $\int_1^3 \frac{dx}{x^3}\, dx$

(v) $\int_1^2 \left(4x + \sqrt{x} + \frac{1}{x^{1/3}} + \frac{1}{x^2}\right) dx$ (vi) $\int_{-1}^2 (2x+1)(x-2)\, dx$

(vii) $\int_0^1 (2x+1)^4\, dx$ (viii) $\int_0^1 x^7(1+x^8)^{\frac{1}{3}}\, dx$

(ix) $\int_1^4 \frac{x^2 - 3x + 5}{\sqrt{x}}\, dx$ (x) $\int_1^4 \frac{(x+2)(x^2+3)}{\sqrt{x}}\, dx.$

2. (i) $\int_0^{\frac{\pi}{2}} (\cos x - \sin x)\, dx$ (ii) $\int_0^{\frac{\pi}{4}} \cos 2x\, dx$

(iii) $\int_0^{\frac{\pi}{4}} \tan^2 x\, dx$ (iv) $\int_0^{\pi} 3 \sin\frac{x}{2} \cos\frac{x}{2}\, dx$

(v) $\int_0^{\frac{\pi}{4}} \sin 2x \cos x\, dx$ (vi) $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1 + \cos 2x}{1 - \cos 2x}\, dx$

(vii) $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot x \, dx$ (viii) $\int_0^{\frac{\pi}{4}} \cos^2 2x \, \sin^3 4x \, dx.$

3. (i) $\int_1^2 e^{4x+1} \, dx$ (ii) $\int_0^2 3^{x+2} \, dx$

(iii) $\int_0^1 \cosh 2x \, dx$ (iv) $\int_0^1 \frac{e^x - e^{-x}}{e^x + e^{-x}} \, dx$

(v) $\int_0^2 x^2 e^{x^3} \, dx.$

4. (i) $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$ (ii) $\int_{\frac{2}{\sqrt{3}}}^{2} \frac{dx}{x\sqrt{x^2-1}}$

(iii) $\int_0^{\sqrt{3}} \frac{dx}{1+x^2} \, dx$ (iv) $\int_0^{\ln 2} \frac{dx}{\sqrt{e^{2x}-1}}.$

5. (i) $\int_0^1 \frac{dx}{3x+2}$ (ii) $\int_0^4 \frac{dx}{\sqrt{x^2+9}}$

(iii) $\int_2^3 \frac{dx}{\sqrt{x^2-2}} \, dx$ (iv) $\int_0^4 \sqrt{x^2+9} \, dx.$

6. (i) $\int_{-2}^{1} (|x| + x) \, dx$ (ii) $\int_2^5 [\, x \,] \, dx$

(iii) $\int_0^{\frac{3}{2}} [\, 2x \,] \, dx$ (iv) $\int_0^2 [\, x^2 \,] \, dx.$

7. (i) $\int_1^2 e^x \, (x+1) \, dx$ (ii) $\int_0^{\frac{\pi}{4}} x \, \sin x \, dx$

(iii) $\int_1^2 x \, \log x \, dx$ (iv) $\int_0^1 x \, \tan^{-1} x \, dx$

(v) $\int_0^1 \frac{x \, dx}{(2x+1)(x+1)}$ (vi) $\int_0^{\frac{\pi}{2}} \frac{\cos x \, dx}{(\sin x + 1)(\sin x + 2)}$

(vii) $\int_0^{\frac{\pi}{2}} e^x \, \cos x \, dx.$

9.14 SOME MORE PROPERTIES OF DEFINITE INTEGRALS.

The following properties are useful in enlarging the scope of definite integrals.

(i) $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

In particular $\int_0^a f(x)dx = \int_0^a f(a-x)dx.$

(ii) $\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx & \text{if } f \text{ is even function} \\ 0 & \text{if } f \text{ is odd function} \end{cases}$

(iii) $\int_0^{2a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx & \text{if } f(2a-x) = f(x) \\ 0 & \text{if } f(2a-x) = -f(x) \end{cases}$

(iv) If f is a periodic function with period T, then $\int_0^{nT} f(x)dx = n\int_0^T f(x)dx,\ n \in N.$

Proof :

(i) let a+b-$x = z$, then $x = a \Rightarrow z = \text{b}$ and $x = \text{b} \Rightarrow z = a$. Further $-dx = dz$.

$\therefore \quad \int_a^b f(a+b-x)\,dx = \int_b^a f(z)\,(-dz) = \int_a^b f(z)\,dz = \int_a^b f(x)\,dx$

(Replacing a by 0 and b by a in the above result we get $\int_0^a f(x)dx = \int_0^a f(a-x)dx.$)

(ii) $\int_{-a}^{0} f(x)\,dx = \int_a^0 f(-v)\,(-dv)$ (setting $x = -v$)

$= \int_0^a f(-v)\,dv = \int_0^a f(-x)\,dx$

For an even function $f(-x) = f(x)$

and for an odd function $f(-x) = -f(x)$

$\therefore \quad \int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_0^a f(x)\,dx = \int_0^a f(-x)\,dx + \int_0^a f(x)\,dx$

$= \int_0^a [f(-x) + f(x)]\,dx$

$= \begin{cases} \int_0^a [f(x) + f(x)]\,dx = 2\int_0^a f(x)\,dx, & \text{if } f \text{ is even function} \\ \int_0^a [-f(x) + f(x)]\,dx = 0 \text{ if } f \text{ is odd function} \end{cases}$

(iii) $\int_a^{2a} f(x)\,dx = \int_a^0 f(2a-t)(-dt)$ setting $2a - x = t$

$$= \int_0^a f(2a-t)\ dt = \int_0^a f(2a-x)\ dx.$$

$$\therefore \quad \int_0^{2a} f(x)\ dx = \int_0^a f(x)\ dx + \int_a^{2a} f(x)\ dx$$

$$= \int_0^a f(x)\ dx + \int_0^a f(2a-x)\ dx = \int_0^a [f(x) + f(2a-x)]\ dx$$

$$= \begin{cases} 2\int_0^a f(x)\,dx & \text{if } f(2a-x) = f(x) \\ 0 & \text{if } f(2a-x) = -f(x) \end{cases}$$

(For example $\int_0^{\pi} \sin x\ dx = 2\int_0^{\frac{\pi}{2}} \sin x\ dx$ since $\sin(\pi - x) = \sin x$; but $\int_0^{\pi} \cos x\ dx = 0$ as $\cos(\pi - x)$ $= -\cos x$.)

(iv) Since f is periodic with period T, $f(x+T)=f(x)$

We have, for $n \in N$,

$$\int_0^{nT} f(x)dx = \int_0^{T} f(x)dx + \int_T^{2T} f(x)dx + ... + \int_{(n-1)T}^{nT} f(x)dx$$

$$= \sum_{r=1}^{n} \int_{(r-1)T}^{rT} f(x)dx \qquad .... (1)$$

Now, $= \int_{(r-1)T}^{rT} f(x)dx = \int_0^{T} f(y-(r-1)T)dy$, putting $x=y+(r-1)T$

$$= \int_0^{T} f(y)dy \quad (\because f \text{ is periodic with period T})$$

$$= \int_0^{T} f(x)dx.$$

Hence from (i) $= \int_0^{nT} f(x)dx = \sum_{r=1}^{n} \int_0^{T} f(x)dx = n\int_0^{T} f(x)dx.$

Example 20 : Evaluate $\int_1^2 \frac{\sqrt{x}dx}{\sqrt{3-x}+\sqrt{x}}$

Solution : Let $I = \int_1^2 \frac{\sqrt{x}dx}{\sqrt{3-x}+\sqrt{x}} = \int_1^2 \frac{\sqrt{3-x}dx}{\sqrt{x}+\sqrt{3-x}}$. using (i)

$$\therefore \quad I + I = \int_1^2 \frac{\sqrt{x}dx}{\sqrt{3-x}+\sqrt{x}} + \int_1^2 \frac{\sqrt{3-x}}{\sqrt{x}+\sqrt{3-x}}dx$$

$$= \int_1^2 \frac{\sqrt{x}+\sqrt{3-x}}{\sqrt{3-x}+\sqrt{x}} = \int_1^2 dx = x]_1^2 = 2-1 = 1.$$

Hence $2\,I = 1 \Rightarrow I = \frac{1}{2}$.

Example 21

Evaluate $\int_0^{100} (x-[x])dx$

Solution

Let $f(x) = x-[x]$

$$\begin{aligned} f(x+1) &= (x+1)-[x+1] \\ &= (x+1) - ([x]+1) \\ &= x - [x] \\ &= f(x) \end{aligned}$$

$\Rightarrow f$ is periodic with period 1

$$\therefore \int_0^{100} f(x)dx = \int_0^{100} (x-[x])dx$$

$$= 100\int_0^1 (x-[x])dx, \text{ using (iv)}$$

$$= 100\int_0^1 x\,dx \quad (\because \int_0^1 [x]dx = 0)$$

$$= 100\left[\frac{x^2}{2}\right]_0^1 = 100\left(\frac{1}{2}-0\right) = 50.$$

Example 22

(i) $\int_0^{\frac{\pi}{4}} \log(1+\tan\theta)\,d\theta$

$$\text{Let} \quad I = \int_0^{\frac{\pi}{4}} ln\ (1+\tan\theta)\,d\theta = \int_0^{\frac{\pi}{4}} ln\left[1+\tan\left(\frac{\pi}{4}-\theta\right)\right]d\theta$$

$$= \int_0^{\frac{\pi}{4}} ln\left[1+\frac{1-\tan\theta}{1+\tan\theta}\right]d\theta = \int_0^{\frac{\pi}{4}} ln\ \frac{2}{1+\tan\theta}\,d\theta$$

$$= \int_0^{\frac{\pi}{4}} \log 2\,d\theta - \int_0^{\frac{\pi}{4}} ln\ (1+\tan\theta)\,d\theta$$

$$= \frac{\pi}{4}\ ln\,2 - I$$

$$\therefore 2I = \frac{\pi}{4}\ ln\,2$$

$$\text{or } I = \frac{\pi}{4}\ ln\,2.$$

(ii) $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos^3 x \, dx = 2\int_0^{\frac{\pi}{4}} \cos^3 x \, dx$ (as $\cos^3 x$ is an even function)

$$= 2\int_0^{\frac{\pi}{4}} (1-\sin^2 x)\cos x \, dx$$

$$= 2\int_0^{\frac{1}{\sqrt{2}}} (1-z^2)\, dz \qquad \text{(putting } \sin x = z\text{)}$$

$$= 2\left|z - \frac{z^3}{3}\right|_0^{\frac{1}{\sqrt{2}}} = 2\left(\frac{1}{\sqrt{2}} - \frac{1}{3.\,2\sqrt{2}}\right) = \frac{2}{\sqrt{2}}\left(1-\frac{1}{6}\right)$$

$$= \frac{5}{6}\sqrt{2}.$$

(iii) $\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \sin^3 x \, dx = 0$ (as $\sin^3 x$ is an odd function)

(iv) $\int_0^{\pi} \sin^3 x \, dx = 2\int_0^{\frac{\pi}{2}} \sin^3 x \, dx$ (as $\sin^3(\pi - x) = \sin^3 x$)

$$= 2\int_0^{\frac{\pi}{2}} (1-\cos^2 x)\sin x \, dx = 2\int_1^0 (1-z^2)(-dz) \text{ setting } \cos x = z$$

$$= 2\int_0^1 (1-z^2)\, dz = 2\left|z - \frac{z^3}{3}\right|_0^1 = 2\left(1-\frac{1}{3}\right) = \frac{4}{3}.$$

(v) Integrate $\int_0^{\frac{\pi}{2}} ln \sin x dx$.

$$I = \int_0^{\frac{\pi}{2}} ln \sin x dx = \int_0^{\frac{\pi}{2}} ln \sin\left(\frac{\pi}{2} - x\right) dx = \int_0^{\frac{\pi}{2}} ln \cos x dx$$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} ln \cos x dx + \int_0^{\frac{\pi}{2}} ln \sin x dx$$

$$= \int_0^{\frac{\pi}{2}} (ln \cos x + ln \sin x)\, dx = \int_0^{\frac{\pi}{2}} ln(\sin x \cos x)\, dx$$

$$= \int_0^{\frac{\pi}{2}} ln\left(\frac{\sin 2x}{2}\right) dx = \int_0^{\frac{\pi}{2}} ln \sin 2x \, dx - \int_0^{\frac{\pi}{2}} ln\, 2dx \qquad \text{........(1)}$$

Now $\int_0^{\frac{\pi}{2}} ln \sin 2x \, dx = \frac{1}{2}\int_0^{\pi} ln \sin z dz$ (Putting $z = 2x$)

$$= \frac{1}{2}\cdot\frac{1}{2}\int_0^{\frac{\pi}{2}} ln\,\sin z dz \quad \text{(Using 9.14 (iii))}$$

$$= \int_0^{\frac{\pi}{2}} ln\,\sin x dx = \text{I}$$

Hence from (1) we have

$$2\,\text{I} = \text{I} - \int_0^{\frac{\pi}{2}} ln\,2dx$$

$$= \text{I} - \frac{\pi}{2} ln2 \quad \Rightarrow \text{I} = \frac{\pi}{2} ln\,2 = \frac{\pi}{2} ln\left(\frac{1}{2}\right)$$

EXERCISE 9(k)

Evaluate the following integrals.

1. (i) $\int_0^{\frac{\pi}{2}} \frac{dx}{1+\tan x}\,dx$ (ii) $\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx$

   (iii) $\int_0^1 \frac{ln\,(1+x)}{1+x^2}\,dx\ (x = \tan\theta)$ (iv) $\int_0^{\pi} \frac{x\,dx}{1+\sin x}$.

2. (i) $\int_{-a}^{a} x^4\,dx$ (ii) $\int_{-a}^{a} (x^5+2x^2+x)\,dx$

   (iii) $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos^2 x\,dx$ (iv) $\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \sin^5 x\,dx$.

3. (i) $\int_0^{\pi} \cos^3 x\,dx$ (ii) $\int_0^{\pi} \cos^2 x\,dx$

   (iii) $\int_0^{\pi} \sin^3 x\,\cos x\,dx$ (iv) $\int_0^{\pi} \sin x\,\cos^2 x\,dx$.

4. Show that

   (i) $\int_0^1 \frac{ln\,x}{\sqrt{1-x^2}}dx = \frac{\pi}{2} ln\frac{1}{2}$ (ii) $\int_0^{\frac{\pi}{2}} \frac{\cos x-\sin x}{1+\sin x\cos x}dx = 0$

   (iii) $\int_0^{\pi} x ln\,\sin x\,dx = \frac{\pi^2}{2} ln\frac{1}{2}$

   (Use 9.14)

5. (i) $\int_0^{\pi/2} \ln\,(\tan x+\cot x)dx$ (ii) $\int_0^{\pi} \frac{x\,\tan x\,dx}{\sec x+\tan x}$

   (iii) $\int_1^3 \frac{\sqrt{x}\,dx}{\sqrt{4\text{-}x}+\sqrt{x}}$ (iv) $\int_0^{\pi} \frac{x\,\sin x\,dx}{1+\cos^2 x}$

   (v) $\int_0^1 x(1-x)^{100}dx$ (vi) $\int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\cot x}}$ (vii) $\int_0^{50} e^{x-[x]}dx$.

*9.15 REDUCTION FORMULAE

(For interested students, not for examination)

In the process of integration we come across integral of a function which involves certain parameter. In this case it is sometimes possible to relate the intregal with another which involves a smaller value of the parameter. A formula which relates the two integrals is called a **Reduction formula.**

The successive application of this formula enables us to evaluate an integral of the above type.

Reducation formulae are generally obtained by the application of the rule of integration by parts. In this section we shall mainly be concerned with the reduction formulae for integrals of some trigonometric functions.

Reduction Formulae for $\int \sin^n x dx$ where $n \geq 1$ is an integer :

Let $n \geq 2$

We write $\int \sin^n x dx = \int \sin^{n-1} x \sin x dx$

Integrating by parts, we obtian

$$\int \sin^n x dx = -\cos x \sin^{n-1} x - \int (-\cos x)(n-1)\sin^{n-2} x \cos x dx$$

$$= -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x(1-\sin^2 x)dx$$

$$= -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x dx - (n-1)\int \sin^n x dx$$

or $$n\int \sin^n x dx = -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x dx$$

or $$\int \sin^n x dx = -\frac{\cos x \sin^{n-1} x}{n} + \left(\frac{n-1}{n}\right)\int \sin^{n-2} x dx$$

which is the required reduction formula. Successive application of this formula enables us to integrate any positive integral power of sinx. We observe that this formula is also valid for n =1.

Reduction formula for $\int \cos^n x dx$ where $n \geq 1$ is an integer.

Let $n \geq 2$

We write $\int \cos^n x dx = \int \cos^{n-1} x \cos x dx$

Integrating by parts, we get

$$\int \cos^n x dx = \sin x \cos^{n-1} x + (n-1)\int \cos^{n-2} x \sin^2 x dx$$

$$= \sin x \cos^{n-1} x + (n-1)\int \cos^{n-2} x(1-\cos^2 x)dx$$

$$= \sin x \cos^{n-1} x + (n-1)\int \cos^{n-2} x dx - (n-1)\int \cos^n x dx$$

or $$n\int \cos^n x dx = \sin x \cos^{n-1} x + (n-1)\int \cos^{n-2} x dx$$

This implies that

$$\int \cos^n x dx = \frac{\sin x \cos^{n-1} x}{n} + \left(\frac{n-1}{n}\right)\int \cos^{n-2} x dx$$

which is the desired reduction formula. We observe that this formula is true for n = 1.

Walli's Formulae

(a) $\int_0^{\pi/2} \sin^n x dx = \begin{cases} \frac{n-1}{n}.\frac{n-3}{n-2}.\frac{n-5}{n-4}........\frac{2}{3} \text{ if n is odd} \\ \frac{n-1}{n}.\frac{n-3}{n-2}.\frac{n-5}{n-4}........\frac{1}{2}.\frac{\pi}{2} \text{ if n is even} \end{cases}$

(b) $\int_0^{\pi/2} \cos^n x dx = \begin{cases} \frac{n-1}{n}.\frac{n-3}{n-2}.\frac{n-5}{n-4}........\frac{2}{3} \text{ if n is odd} \\ \frac{n-1}{n}.\frac{n-3}{n-2}.\frac{n-5}{n-4}........\frac{1}{2}.\frac{\pi}{2} \text{ if n is even} \end{cases}$

Here $n \geq 1$ is an integer.

Proof: Let $I_n = \int_0^{\pi/2} \sin^n x dx$

Using the reduction formula for $\int \sin^n x dx$

we get $I_n = \left|\frac{-\cos x \sin^{n-1} x}{n}\right|_0^{\pi/2} + \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x dx$

$= \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x dx$

$= \frac{n-1}{n} I_{n-2}$

$= \frac{n-1}{n}.\frac{n-3}{n-2} I_{n-4}$

$= \frac{n-1}{n}.\frac{n-3}{n-2}.\frac{n-5}{n-4} I_{n-6}$

$= \begin{cases} \frac{n-1}{n}.\frac{n-3}{n-2}.\frac{n-5}{n-4}.....\frac{2}{3} I_1 \text{ if n is odd} \\ \frac{n-1}{n}.\frac{n-3}{n-2}.\frac{n-5}{n-4}.....\frac{1}{2} I_0 \text{ if n is even} \end{cases}$

we have $I_1 = \int_0^{\pi/2} \sin x dx = \left|-\cos x\right|_0^{\pi/2} = 1$

and $I_0 = \int_0^{\pi/2} dx = \frac{\pi}{2}$

So, $I_n = \int_0^{\pi/2} \sin^n x dx = \begin{cases} \frac{n-1}{n}.\frac{n-3}{n-2}.\frac{n-5}{n-4}........\frac{2}{3} \text{ if n is odd} \\ \frac{n-1}{n}.\frac{n-3}{n-2}.\frac{n-5}{n-4}........\frac{1}{2}.\frac{\pi}{2} \text{ if n is even} \end{cases}$

The formula for $\int_0^{\pi/2} \cos^n x dx$ follows from $\int_0^a f(x)dx = \int_0^a f(a-x)dx$.

So $\int_0^{\pi/2} \cos^n x dx = \int_0^{\pi/2} \cos^n\left(\frac{\pi}{2} - x\right)dx = \int_0^{\pi/2} \sin^n x dx$

Example 23 Evaluate $\int_0^{\pi/2} \sin^7 \theta d\theta$

Solution: 7 is an odd positive integer.

So using Walli's formula, we get

$$\int_0^{\pi/2} \sin^7 \theta d\theta = \frac{6}{7}.\frac{4}{5}.\frac{2}{3} = \frac{16}{35}$$

Example 24 Evaluate $\int_0^{\pi/2} \cos^8 \theta d\theta$

Solution: 8 is an even positive integer.

So using Walli's formula, we get

$$\int_0^{\pi/2} \cos^8 \theta d\theta = \frac{7}{8}.\frac{5}{6}.\frac{3}{4}.\frac{1}{2}.\frac{\pi}{2} = \frac{35\pi}{256}$$

Example 25 Evaluate $\int_0^1 \frac{x^5}{\sqrt{1-x^2}} dx$

Solution: Let $x = \sin\theta$

Then $dx = \cos\theta d\theta$

when x = 0, $\theta = 0$

when x = 1, $\theta = \frac{\pi}{2}$

So, $\int_0^1 \frac{x^5}{\sqrt{1-x^2}} dx = \int_0^{\pi/2} \frac{\sin^5\theta}{\cos\theta} \cos\theta d\theta$

$= \int_0^{\pi/2} \sin^5 \theta d\theta$

$= \frac{4}{5}.\frac{2}{3}$ by Walli's formula

$= \frac{8}{15}$

Example 26 Evaluate $\int_0^{\infty} \frac{dx}{\left(1+x^2\right)^n}$

Solution: We put $x = \tan\theta$

Then $\theta = 0$ when x = 0 and $\theta \to \frac{\pi}{2}$ from the left, when $x \to \infty$

Hence $\int_0^{\infty} \frac{dx}{(1+x^2)^n} = \int_0^{\pi/2} \frac{\sec^2\theta d\theta}{(1+\tan^2\theta)^n}$

$= \int_0^{\pi/2} \frac{\sec^2\theta}{\sec^{2n}\theta} d\theta$

$= \int_0^{\pi/2} \cos^{2n-2}\theta \, d\theta$

$$= \begin{cases} \frac{2n-3}{2n-2}.\frac{2n-5}{2n-4}.\frac{2n-7}{2n-6}.....\frac{1}{2}.\frac{\pi}{2} \text{ if } n>1 \\ \frac{\pi}{2} \text{ if } n=1 \end{cases}$$

(2n −2 is always even)

EXERCISES-9 (*l*)

Evaluate the following integrals:

1. $\int_0^{\pi/2} \sin^{10}\theta d\theta$
2. $\int_0^{\pi/2} \cos^{12}\theta d\theta$
3. $\int_0^{\pi/2} \sin^{11}\theta d\theta$
4. $\int_0^{\pi/2} \cos^{9}\theta d\theta$
5. $\int_0^{1} \frac{x^7}{\sqrt{1-x^2}} dx$
6. $\int_0^{1} \frac{x^5\left(4-x^2\right)}{\sqrt{1-x^2}} dx$
7. $\int_0^{a} x^3(a^2-x^2)^{\frac{5}{2}} dx$
8. $\int_0^{1} x^5\sqrt{\frac{1+x^2}{1-x^2}} dx$
9. $\int_0^{\infty} \frac{x^2}{(1+x^6)^n} dx$
10. $\int_0^{\pi} \sin^8\theta d\theta$

Additional Exercises

(1) $\sqrt{1-\sin 2x}\, dx$

(2) $\int \frac{dx}{1+\sin x}$

(3) $\int \frac{\sin x}{1+\sin x} dx$

(4) $\int \frac{\sec x}{\sec x+\tan x} dx$

(5) $\int \frac{1+\sin x}{1-\sin x} dx$

(6) $\int \tan^{-1}(\sec x+\tan x)dx$

(7) $\int \frac{\cos 2x-\cos 2\alpha}{\cos x-\cos\alpha} dx$

(8) $\int \tan^{-1}\sqrt{\frac{1-\cos 2x}{1+\cos 2x}} dx$

(9) $\int \frac{dx}{\sqrt{x+1}+\sqrt{x+2}}$

(10) $\int \frac{2+3x}{3-2x} dx$

(11) $\int \frac{dx}{\sqrt{x}+x}$

(12) $\int \frac{dx}{1+\tan x}$

(13) $\int \frac{x+\sqrt{x+1}}{x+2} dx$ (Hints put : $\sqrt{x+1}=t$)

(14) $\int \sin^{-1}\sqrt{\frac{x}{a+x}} dx$ (Hints put : $x=a\tan^2 t$)

(15) $\int e^x \left(\frac{2+\sin 2x}{1+\cos 2x}\right) dx$

(16) $\int \frac{(x^2+1)e^x}{(x+1)^2} dx$

(17) $\int \frac{x^2-1}{x^4+x^2+1} dx$

(18) $\int \frac{x^2\, dx}{x^4+x^2+1}$

(19) $\int \sqrt{\cot x}\, dx$

(20) $\int (\sqrt{\tan x}+\sqrt{\cot x})\, dx$

(21) $\int \frac{dx}{x(x^4+1)}$

(22) $\int \frac{dx}{e^x-1}$

(23) $\int \frac{(x-1)(x-2)(x-3)}{(x+4)(x-5)(x-6)} dx$

(24) $\int \frac{dx}{(e^x-1)^2}$

(25) $\int \frac{dx}{\sin x \cos^2 x}$

(26) $\int_2^4 \frac{(x^2+x)dx}{\sqrt{2x+1}}$

(27) $\int_{-a}^{a} \sqrt{\frac{a-x}{a+x}} dx$

(28) $\int_0^{\pi/2} \left(\sqrt{\tan x}+\sqrt{\cot x}\right) dx$

(29) $\int_0^{\pi/2} \frac{\cos x\, dx}{1+\cos x+\sin x}$

(30) $\int_0^1 x(1-x)^n dx$

(31) $\int_0^{\pi/2} \sin 2x \log(\tan x) dx$

(32) $\int_0^{\pi/2} \frac{\sin^2 x\, dx}{\sin x+\cos x}$

(33) $\int_0^{\pi/2} \frac{\sin^2 x\, dx}{1+\sin x \cos x}$

(34) $\int_0^{\pi/2} \frac{x\, dx}{\sin x+\cos x}$

(35) Prove that $\int_0^{\pi} x \sin^3 x\, dx = \frac{2\pi}{3}$

(36) $\int_{\pi/5}^{3\pi/10} \frac{\sin x\, dx}{\sin x+\cos x}$

(37) $\int_0^{\pi} |\cos x|\, dx$

(38) $\int_1^4 (|x-1|+|x-2|+|x-3|)\, dx$

(39) $\int_{-\pi/2}^{\pi/2} (\sin |x| + \cos |x|) dx$

(40) $\int_0^{\pi} \log(1+\cos x) dx$

●

Area Under Plane Curves

CHAPTER - 10

(Application of Definite Integrals)

All birds find shelter during a rain. But Eagle avoids rain by flying above the clouds. Problems are common, but attitude makes the difference.

- A.P.J. Abdul Kalam

10.1 Area under a plane curve between to ordinates

The definite integral was defined in 9.11 as the limit of a sum. If the limit exists, then

$$\int_a^b f(x)dx = \lim_{n\to\infty} h\sum_{r=0}^{n-1} f(a+b)$$

$$= \lim_{n\to\infty} h\sum_{r=1}^{n} f(a+bh)$$

where $h = \dfrac{b-a}{n}$

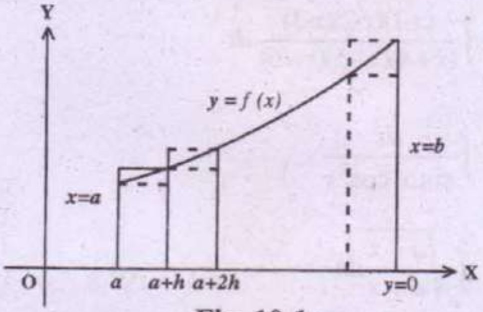


Fig. 10.1

Let us see what this fact means geometrically. Suppose $f(x)$ is positive and increasing in $[a, b]$ (fig 13.2). Let the ordinates corresponding to $x = a + rh$ ($r =$ 0, 1, 2, ... n) be drawn and the inner rectangles (rectangles drawn in the figure just below the curve) and outer rectangles (rectangles drawn just covering the curve) each of width h be completed.

Then $S_1 = h\sum_{r=0}^{n-1} f(a+rh)$ is the sum of the areas $hf(a), hf(a+r)$...............$hf(a+\overline{n-1}h)$ of the inner rectangles. Also $S_2 = h\sum_{r=1}^{n} f(a+rh)$ is the sum of the areas $hf(a+h)$......$hf(a+nh)$ of the outer rectangles. The actual area A under the curve $y = f(x)$, above the x-axis ($y = 0$) and between the ordincates at $x = a$ and $x = b$ lies between S_1 and S_2 i.e., $S_1 < A < S_2$. An $n\to\infty, h\to 0$, the difference between S_1 (or S_2) and A is reduced and ultimately S_1 (or S_2) approaches A.

Then $A = \lim_{n\to\infty} S_1 = \lim_{n\to\infty} h\sum_{r=0}^{n-1} f(a+rh)$

$= \lim_{n\to\infty} S_2 = \lim_{n\to\infty} h\sum_{r=1}^{n} f(a+rh)$

i.e. $A = \int_a^b f(x)dx.$

Thus the definite integral $A = \int_a^b f(x)dx.$ represnets the area under the curve $y = f(x)$, above the x-axis and between the ordinates $x = a$ and $x = b$.

If $f(x)$ is decreasing in $[a, b]$, then S_1 is the sum of the areas of the outer rectangles and S_2 that of the inner rectangles and $S_1 > A > S_2$. But $\lim_{n\to\infty} S_1 = \lim_{n\to\infty} S_2 = A$ and the same result follows.

If $f(x)$ is constant in $[a, b]$, the whole area is under one rectangle and then $S_1 = A = S_2$ for any mode of subdivision of the interval $[a, b]$.

Area between the curve $x = f(y)$, $x = 0$ and the abscissae $y = c$, $y = d$ can be similarly shown to be $\int_c^d f(y)\, dy$.

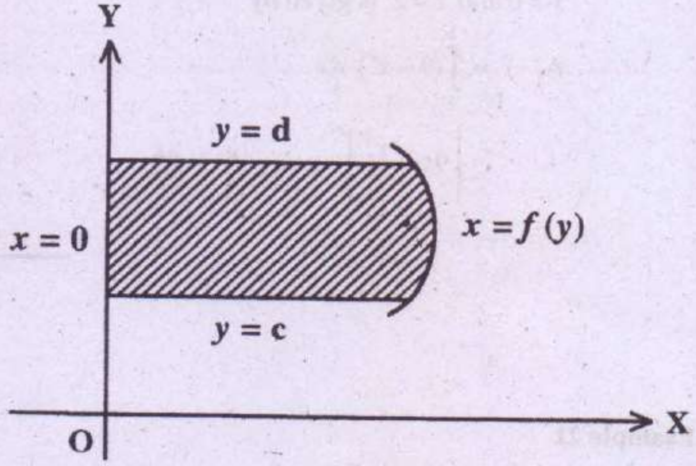


Fig 10.2

10.2 AREA BETWEEN TWO CURVES :

If there are two curves $y = f(x)$, $y = g(x)$ with $g(x) < f(x)$ in $[a, b]$, then the area between them and between the ordinates $x = a$ and $x = b$ is given by

$$A = \int_a^b f(x)\, dx - \int_a^b g(x)\, dx$$

$$= \int_a^b [f(x) - g(x)]\, dx$$

as can seen from fig 13.4. This formula holds good even if $g(x) < 0$ in $[a, b]$. In fact, this is the general formula of which $\int_a^b f(x)\, dx$ is a particular case as the latter is the area between the two curves $y = f(x)$ and $y = 0$.

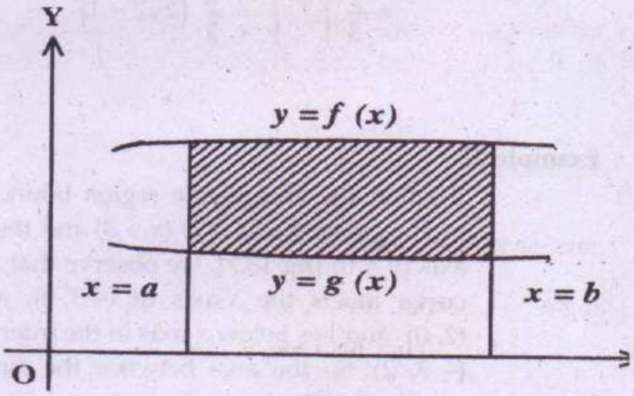


Fig. 10.3

Example 20:

Area of the region enclosed by $y = 9 - x^2$, $y = 0$ and the ordinates $x = 0$ and $x = 2$ is given by

$$A = \int_0^2 (9 - x^2)\, dx$$

$$= \left| 9x - \frac{x^3}{3} \right|_0^2 = 18 - \frac{8}{3} = \frac{46}{3}.$$

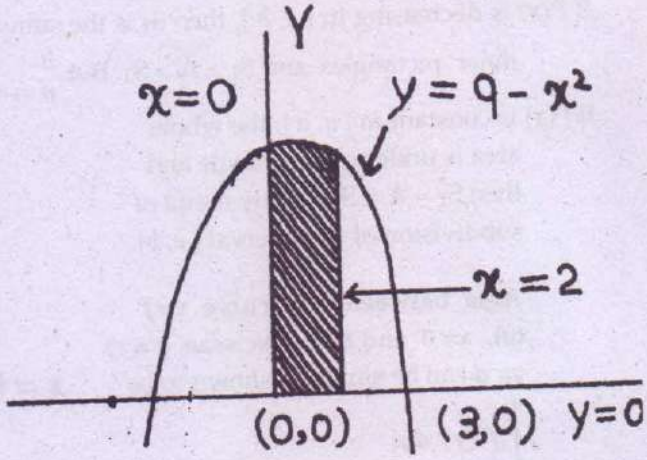


Fig. 10.4

Example 21

The area bounded by $x^2 = y$, the y-axis ($x = 0$) and the lines $y = 1$ and $y = 2$ is given by

$$A = \int_1^2 x\, dy = \int_1^2 \sqrt{y}\, dy$$

$$= \frac{2}{3} \left| y^{\frac{3}{2}} \right|_1^2 = \frac{2}{3}\left(2\sqrt{2} - 1\right).$$

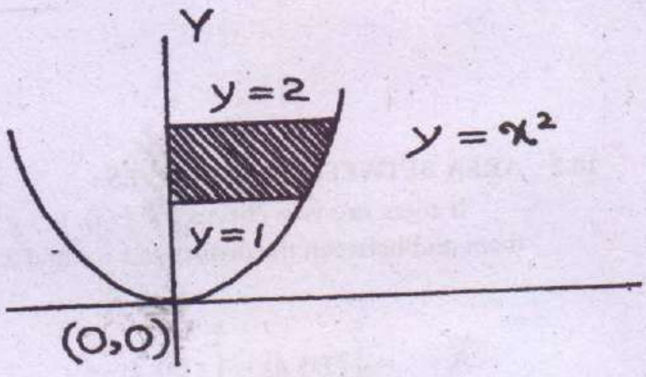
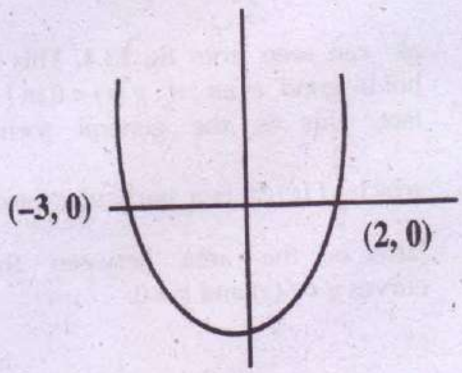


Fig. 10.5

Example 22

To find the area of the region bounded by the curve $y = (x - 2)(x + 3)$ and the x-axis ($y = 0$) [fig 13.7], we observe that the curve meets the x-axis at $(-3, 0)$, and $(2, 0)$ and lies below x-axis in the interval $(-3, 2)$. So the area between the curve and x-axis is given by $\int_{-3}^{2} (x - 2)(x + 3)\, dx$



$$= \int_{-3}^{2} (x^2 + x - 6)\, dx = \left| \frac{x^3}{3} + \frac{x^2}{2} - 6x \right|_{-3}^{2} = \frac{-125}{6}.$$

Fig. 10.6

The integral is negative as the area lies below the x-axis. However, the magnitude of the integral, $\frac{125}{6}$ is the required area.

$$A = \left| \int_{-3}^{2} (x^2 + x - 6)\, dx \right| = \frac{125}{6}.$$

Example 23

To find the area of a quadrant of the circle $x^2 + y^2 = a^2$, we observe that $y = \sqrt{a^2 - x^2}$ in the first quadrant (Fig 12.8).

$$\therefore\ A = \int_0^a \sqrt{a^2 - x^2}\, dx$$

$$= \left| \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} \right|_0^a$$

$$= \frac{a^2}{2}\ \sin^{-1}| = \frac{a^2}{2}.\frac{\pi}{2} = \frac{1}{4}\ \pi$$

a^2.

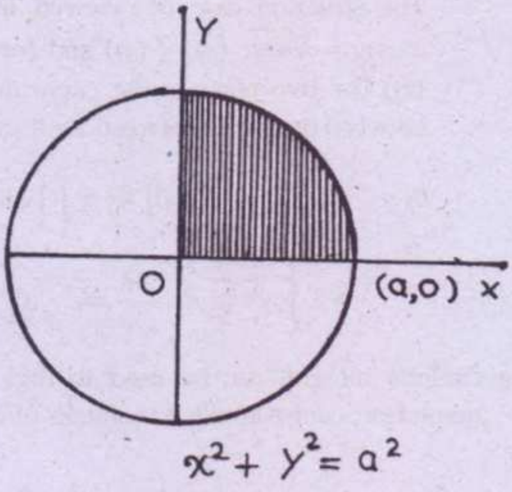


Fig. 10.7

As the curve (circle) is symmetrically situated about y-axis the area of the semicircle above the x-axis is twice the area of the quadrant i.e. $\frac{\pi a^2}{2}$. Also the circle is symmetrical about the x-axis. So the area of the circle is twice that of the part above x-axis i.e. πa^2.

Example 24

The area of region bounded by the parabola $y^2 = x$ and the straight line $y = x$ (they intersect at (0, 0), (1, 1)). (See Fig. 13.9) is given by

$$A = \int_0^1 \left(\sqrt{x} - x\right) dx$$

$$= \left| \frac{2}{3}x^{\frac{3}{2}} - \frac{x^2}{2} \right|_0^1 = \frac{2}{3} - \frac{1}{2} = \frac{1}{6}.$$

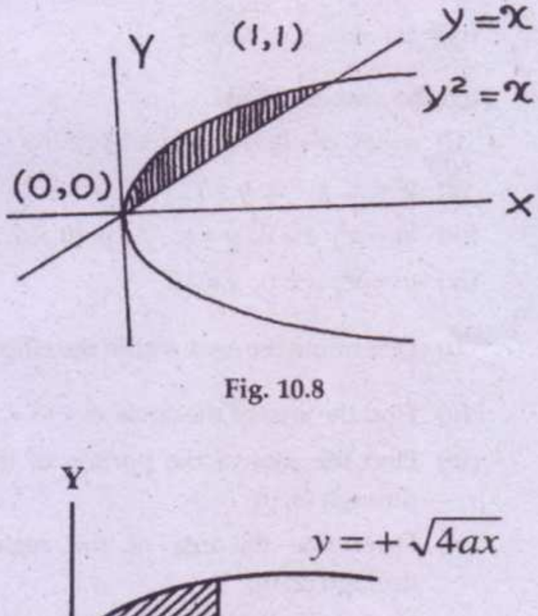


Fig. 10.8

Example 25

To find the area of the parabola $y^2 = 4ax$ bounded by its latus rectum $x = a$, we observe that the curve is symmetrical about x-axis and the region lies between $x = 0$ and $x = a$ (Fig 13.10)

$$\because\ A = 2\int_0^a y dx = 2\int_0^a \sqrt{4ax}\ dx$$

$$= 4\ \sqrt{a}\ \frac{2}{3} \left| x^{\frac{3}{2}} \right|_0^a = \frac{8}{3}\ a^2.$$

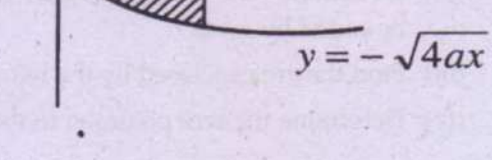


Fig. 10.9

The situation can be viewed in another way. For the part of the curve above x-axis $y = \sqrt{4ax}$ (say $f(x)$) and for the part of the curve below x-axis $y = -\sqrt{4ax}$ (say $g(x)$). The two parts of the curve may be thought of as two different curves and the area between them and between $x = 0$ and $x = a$ is given by

$$A = \int_0^a [f(x) - g(x)]\, dx = \int_0^a \left[\sqrt{4ax} - \left(-\sqrt{4ax}\right)\right] dx$$

$$= 2\int_0^a \sqrt{4ax}\, dx = \frac{8}{3}a^2.$$

Note: Definite integral can be used to find the length of a curve, volumes and surface areas of revolution, centre of mass, moments of inertia etc..

EXERCISES-10

1. Find the area bounded by
   (i) $y = e^x,\ y = 0,\ x = 4, x = 2$
   (ii) $y = x^2,\ y = 0, x = 1$
   (iii) $xy = a^2,\ y = 0,\ x = \alpha, x = \beta\ (\beta > \alpha > 0)$
   (iv) $y = \sin x, y = 0, x = \frac{\pi}{2}$.

2. Find the area enclosed by
   (i) $y = e^x,\ x = 0,\ y = 2,\ y = 3$
   (ii) $y^2 = x,\ x = 0,\ y = 1$
   (iii) $xy = a^2,\ x = 0,\ y = \alpha,\ y = \beta\ (\beta > \alpha > 0)$
   (iv) $y^2 = x^3,\ x = 0,\ y = 1$.

3. (i) Determine the area within the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.
   (ii) Find the area of the circle $x^2 + y^2 = 2ax$.
   (iii) Find the area of the portion of the parabola $y^2 = 4x$, bounded by the double ordinate through (3, 0).
   (iv) Determine the area of the region bounded by $y^2 = x^3$ and the double ordinate through (2, 0).

4. (i) Find the area of the regions into which the circle $x^2 + y^2 = 4$ is divided by the line $x + \sqrt{3}\,y = 2$.
   (ii) Determine the area of the region between the curves $y = \cos x$ and $y = \sin x$, bounded by $x = 0$.
   (iii) Find the area enclosed by the two parabolas $y^2 = 4ax$ and $x^2 = 4ay$.
   (iv) Determine the area common to the parabola $y^2 = x$ and the circle $x^2 + y^2 = 2x$.

CHAPTER - 11

Differential Equations

The science of Pure Mathematics, in its modern developments, may claim to be the most original creation of the human spirit.

- A.N. Whitehead

11.0 INTRODUCTION

We have already studied about derivatives and integrals of functions. We know that an integral of a function f is another function F whose derivative is f. In fact while integrating $f(x)$, we search for a solution of y of the equation $\frac{dy}{dx} = f(x)$ (which is also expressed as $dy = f(x)\,dx$).

Such an equation, involving derivatives (or differentials) of dependent variable(s) with respect to independent variable(s), is known as a differential equation.

Differential equations often arise in studies relating to engineering, physics, chemistry biology and other natural sciences. It finds application in economics, psychology and many other social sciences. In this chapter we shall study the nature of differential equations and some methods to solve them.

11.1 DIFFERENTIAL EQUATIONS AND THEIR CLASSIFICATION

First let us take some examples of differential equations :

$$x\,dx + y\,dy = 0 \quad \ldots\ldots\ldots\ldots \quad (1)$$

$$\frac{dy}{dx} + 3y = x \quad \ldots\ldots\ldots\ldots \quad (2)$$

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} = e^x \quad \ldots\ldots\ldots\ldots \quad (3)$$

$$\frac{d^ny}{dx^n} + \frac{dy}{dx} = e^{3x} \quad \ldots\ldots\ldots\ldots \quad (4)$$

$$\left(\frac{dy}{dx}\right)^2 = x^2 + \sqrt{x} \quad \ldots\ldots\ldots\ldots \quad (5)$$

$$\frac{d^3y}{dx^3} + \left(\frac{dy}{dx}\right)^5 = x \quad \ldots\ldots\ldots\ldots \quad (6)$$

$$\frac{d^2y}{dx^2} = \sqrt{1 + \frac{dy}{dx}} \quad \ldots\ldots\ldots\ldots \quad (7)$$

$$\frac{dy}{dx} = x + \frac{y}{\frac{dy}{dx}} \quad \ldots\ldots\ldots\ldots \quad (8)$$

When an equation contains derivatives with respect to a single independent variable, it is called an **ordinary differential equation**. In case of more than one independent variables involved in the problem, the derivatives occuring are partial and the equation is called **a partial differential equation**.

We shall have no occasion to discuss partial derivatives or partial differential equations within the scope of this book. We shall, in this chapter, only discuss ordinary differential equations.

The order of the highest order derivative occuring in it is known as the **order of the differential equation. The degree of a differential equation is** the highest positive integral power of the derivative that determines the order of the equation. The degree is, of course, determined after the equation is cleared of fractional indices with regard to all the derivatives involved and after the denominators are cleared of derivatives.

Equations (1) and (2) above are of first order and first degree; (5), (8) are of first order and second degree; (3) is of second order and first degree while (7) is an equation of second order, and second degree. Equation (6) is of third order and first degree and equation (4) is of nth order and first degree.

Note that in (6), the order of the equation is three and the third derivative has power one; so the degree of the equation is one. It is immaterial if any lower order derivative has a higher degree. In (7), a derivative is within radical sign. After making the equation free from fractional index, it takes the form $\left(\frac{d^2y}{dx^2}\right)^2 = 1 + \frac{dy}{dx}$ which is of second order, second degree. However, such consideration need not be carried to (5) where x is within radical sign but no derivative has fractional index. In (8), $\frac{dy}{dx}$ is in denominator. After clearing the fraction it becomes clear that the degree of the equation is two.

11.2 SOLUTION OF A DIFFERENTIAL EQUATION :

A relation like $y = f(x)$ or $f(x, y) = 0$ between the variables is called a **solution** of a differential equation if it reduces the equation to an identity when substituted into it.

For example, $y = e^x$ is a solution of the equation

$$\frac{dy}{dx} = y \qquad \ldots \qquad (a)$$

since taking $y = e^x$ and so $\frac{dy}{dx} = e^x$, (a) is reduced to $e^x = e^x$ which is identically true. But, in this case $y = 2e^x$, $y = 7e^x$, $y = e^x$ are each a solution of (a). In fact, $y = ce^x$ is a solution for any constant value of c. Here, $y = ce^x$ is known as the **general solution** of the equation (a), c being an arbitrary constant. $y = 2e^x$, $y = 7e^x$, $y = \sqrt{3}e^x$ are known as **particular solutions**. Particular solutions are obtained from the general solution by taking particular value (s) of the arbitrary constant(s). There are some solutions which cannot be obtained from the general solution by taking particular values of the arbitrary constant. Such solutions are known as **singular solutions** whose discussion is outside the scope of the book.

A differential equation of order n involves derivatives upto n^{th} order. A solution does not contain any derivative. Generally speaking, the process of solution of a differential equation involves as many integrations as the order of the equation. Since each integration introduces one arbitrary constant it is natural that the general solution would contain as many arbitrary constants as the order of the equation.

11.3 GEOMETRICAL MEANING OF SOLUTION OF DIFFERENTIAL EQUATION.

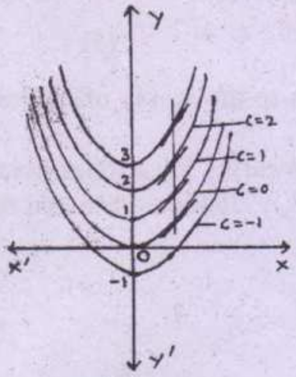
Consider the differential equation

$$\frac{dx}{dx} = 2x \qquad \ldots\ldots (a)$$

Its solution is a function $y = F(x)$ such that $\frac{dy}{dx} = F'(x) = 2x$. By integrating $2x$ w.r.t. x we get

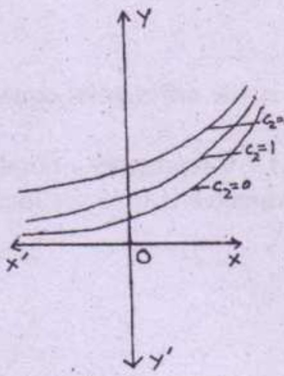
$y = x^2 + c$ ...... (b)

as the general solution of (a) where c is an arbitrary constant. Taking c=0, $y=x^2$ is a particular solution which represent a parabola with vertex at the origin. Taking values of c, we get a family of parabolas some of which are shown below.



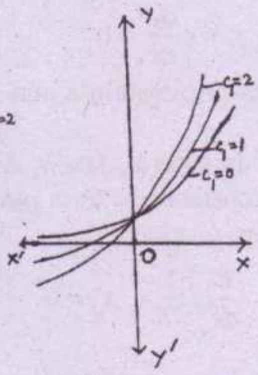
$y = x^2 + c$

Fig- A



$y = e^x + c_1 x + c_2$, $c_1 = 0$

Fig - B



$y = e^x + c_1 x + c_2$, $c_2 = 0$

Fig- C

It is pertinent to note that this family of parabolas are shifted parallel to one another. (see figure). Note that equation (*a*) is a first order differential equation and its solution (b) is a one parameter family of curves, c being the parameter.

Next consider the equation

$$\frac{d^2x}{dx^2} = e^x \quad \text{...... (c)}$$

Integrating once we get $\frac{dy}{dx} = e^x + c_1$ and integrating again we get

$$y = e^x + c_1 x + c_2 \quad \text{....... (d)}$$

Equation (d) which is the general solution of (c) is a two parameter family of curves. We get different systems of curves by varying the two parameters separately (see figures B,C) or simultaneously.

Thus we see that the general solution of an ordinary differential equation represents a family of curves whose equations involve one or more parameters.

On the other hand, given the equation of a one or more parameter family of curves we can obtain the differential equation free from the parameters whose solutions are the given family of curves. We describe it below.

11.4 FORMATION OF DIFFERENTIAL EQUATION :

For every real value of m, y = mx represents a straight line through (0, 0); taking all possible real values of m, one gets all possible straight lines through the origin.

$$y = mx \quad \text{... (1)}$$

is said to represent the family of straight lines through the origin where m is an arbitrary constant. Differentiating (1) with respect to x,

$$\frac{dy}{dx} = m \quad \text{... (2)}$$

Eliminating m between (1) and (2) one gets,

$$\frac{dy}{dx} = \frac{y}{x} \quad \text{... (3)}$$

which is free from the arbitrary constant m and is the differential equation for the family.
Similarly,

$$x^2 + y^2 = a^2 \quad \ldots \quad (4)$$

represents the family of circles with centre at origin where a is an arbitrary constant. Differentiating (4), we get

$$x + y\frac{dy}{dx} = 0 \quad \ldots \quad (5)$$

which does not contain a and is the differential equation to the family of circles with centre at origin.

In general, $f(x, y, A) = 0$ represents a family of curves in cartesian plane where A is an arbitrary constant (the term **parameter** is used for it). If A is eliminated between this equation and its derivative relation

$$\frac{d}{dx}f(x, y, A) = 0$$

one gets an equation free from the arbitrary constant A, which, containing $\frac{dy}{dx}$, is a differential equation of first order.

As a rule, one requires n + 1 relations to eliminate n quantities. If a relation

$$f(x, y, C_1, C_2, \ldots\ldots, C_n) = 0 \quad \ldots \quad (6)$$

containing n arbitrary constants is given, it is differentiated successively n times with respect to x to get n new relations. These together with the given relation (6) constitute a set of n+1 relations among which the n constants $C_1, C_2, \ldots, C_n$ can be eliminated. The resulting equation will evidently contain derivatives upto n^{th} order and, therefore, be a differential equation of n^{th} order.

Example 1 :

Let $y = A \sin x \quad \ldots \quad (7)$

be a relation where A is an arbitrary constant. To eliminate A another relation is required. Differentiating (7) with respect to x,

$$\frac{dy}{dx} = A \cos x \quad \ldots \quad (8)$$

Elimination of A between (7) and (8) yields

$$\frac{dy}{dx} = y \cot x \quad \ldots \quad (9)$$

which is a differential equation of order one.

Example 2 :

$$y = Ae^x + Be^{-x} \quad \ldots \quad (10)$$

is an equation with two arbitrary constants A and B. Two more relations are required to eliminate them. Differentiating (10) successively twice with respect to x,

$$\frac{dy}{dx} = Ae^x - Be^{-x} \quad \ldots \quad (11)$$

and $$\frac{d^2y}{dx^2} = Ae^x + Be^{-x} \quad \ldots \quad (12)$$

Comparing (10) and (12), one gets,

$$\frac{d^2y}{dx^2} = y \quad \ldots \quad (13)$$

which is free from the arbitrary constants and is a differential equation of order two.

Example 3 :-

Find the differential equation of the system of straight lines (in cartesian plane) with slope 3.

Solution :

With the given slope, there are infinite number of parallel lines. They constitute a system or family, all of them having the same slope but different y-intercepts (just like the members of a family having a common title but different names). If c be the y-intercept of any one of them, its equation is

$$y = 3x + c \qquad \ldots \qquad (14)$$

To include all members of the family, c is taken as arbitrary. Differentiating (14), one gets,

$$\frac{dy}{dx} = 3 \qquad \ldots \qquad (15)$$

which does not contain the arbitrary constant and is a differential equation of first order representing the given family of lines.

Example 4 :

Find the differential equation of the system of circles in xy-plane with centre at (1,2).

Solution :

With the given centre, there can be infinite number of concentric circles with different radii which form a family. If $|a|$ be the radius of any such circle, its equation is

$$(x-1)^2 + (y-2)^2 = a^2 \qquad \ldots \qquad (16)$$

Taking a arbitrary, all members of the family are represented by (16). To eliminate a (16) is differentiated with respect to x, yielding

$$(x-1) + (y-2)\frac{dy}{dx} = 0 \qquad \ldots \qquad (17)$$

which does not contain the arbitrary constant. (17) is a differential equation of first order and represents the family of circles (16).

11.5 METHODS OF SOLVING DIFFERENTIAL EQUATIONS

It is not always possible to solve a given differential equation. Some particular types of equations are amenable to solution. Only two of these types, one of first order and the other of second order, will be discussed in this section. They are

(i) $\frac{dy}{dx} = f(x)\, g(y)$ (ii) $\frac{d^2y}{dx^2} = h(x)$

Type I :

The equation

$$\frac{dy}{dx} = f(x) \qquad \ldots \qquad (1)$$

can be written in the form $dy = f(x)dx$. Integrating both sides one gets

$$y = \int f(x)\, dx + C \qquad \ldots \qquad (2)$$

as the general solution of (1). Similarly, the equation

$$\frac{dy}{dx} = g(y) \qquad \ldots \qquad (3)$$

can be written in the form $\frac{dy}{g(y)} = dx$. Integrating,

$$\int \frac{dy}{g(y)} = x + C \qquad \ldots \qquad (4)$$

is obtained as the general solution of (3). But both of these forms are special cases of the equation

$$\frac{dy}{dx} = f(x)\, g\,(y) \qquad \dots \quad (5)$$

which can be written in the form

$$\frac{dy}{g(y)} = f(x)\, dx \qquad \dots \quad (6)$$

This process of collecting all functions of x with dx and all functions of y with dy (so that integration may be possible) is known as the process of **separation of variables**. Integrating both sides of (6) one gets

$$\int \frac{dy}{g(y)} = \int f(x)\, dx + C \qquad \dots \quad (7)$$

as the general solution.

Example 5 :

Solve $\frac{dy}{dx} = x^2 + 2x + 5$

Solution :

The given equation can be written in the form

$$dy = (x^2 + 2x + 5)\, dx.$$

Integrating both sides,

$$\int dy = \int (x^2 + 2x + 5)\, dx.$$

or $y = \frac{x^3}{3} + x^2 + 5x + C$

This is the general solution, C being an arbitrary constant.

Example 6 :

Solve $\frac{dy}{dx} = \tan y$

Solution :

Writing the equation in the form of $\cot y\, dy = dx$ and integrating,

$$\ln \sin y = x + c$$

Or $\sin y = e^x . e^c = Ae^x$, taking $e^c = A$.

This is the general solution.

Example 7 :

Solve $\frac{dy}{dx} = \frac{2y}{x^2 + 1}$

Solution :

By separating the variables, the equation can be written as

$$\frac{dy}{y} = \frac{2\, dx}{x^2 + 1}$$

Integrating, $\int \frac{dy}{y} = 2 \int \frac{dx}{x^2 + 1}$

or $\ln y = 2 \tan^{-1} x + C$

This is the general solution, C being an arbitrary constant.

Differential equations of the form $\frac{dy}{dx} = f(ax + by)$ or $f(ax+by+c)$ can be reduced to the variable separable form by substitution $ax+by=V$ or $ax + by + c=V$.

Example 8

Solve $\frac{dy}{dx} = (x+y)^2$

Solution:

Let x+y = v, So $\frac{dy}{dx} = V^2$ ...... (i)

Differentiating $x+y=V$,

$$1+\frac{dy}{dx}=\frac{dV}{dx}$$

$$\Rightarrow \frac{dy}{dx}=\frac{dV}{dx}-1 \qquad \text{...... (ii)}$$

From (i) & (ii)

$$\frac{dV}{dx}-1=v^2$$

$$\Rightarrow \frac{dV}{dx}=1+V^2$$

$$\Rightarrow \int \frac{dv}{1+V^2}=\int dx$$

$$\Rightarrow \tan^{-1} V = x+c$$

$$\Rightarrow \tan^{-1}(x+y) = x+c.$$

Type II :

The second order equation

$$\frac{d^2y}{dx^2} = h(x) \qquad \text{... (8)}$$

can be solved in two steps each time a first order equation being solved. In the first step, taking $\frac{dy}{dx} = p$ and so

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{dp}{dx},$$

the equation (8) becomes $\frac{dp}{dx} = h(x)$. On integration,

$$p = \int h(x)\, dx + A$$

$$\text{or } \frac{dy}{dx} = \int h(x)\, dx + A \qquad \text{... (9)}$$

$$= \theta(x) + A \text{ where } \theta(x) = \int h(x)\, dx.$$

This is called a first integral or an intermediate integral. (9) can be integrated again to yield the final solution.

Thus $\int dy = \int[\theta(x) + A]\, dx$

or $y = \int \theta(x)\, dx + Ax + B$ ... (10)

which is the general solution containing two arbitrary constants A and B.

Example 9 :

Solve $\frac{d^2y}{dx^2} = 6x + 2$

Solution :

Taking $\frac{dy}{dx} = p$, the equation becomes

$$\frac{dp}{dx} = 6x + 2$$

On integration, $\int dp = \int (6x + 2)\,dx$

or $p = \frac{dy}{dx} = 3x^2 + 2x + A$

Integrating again, $\int dy = \int (3x^2 + 2x + A)dx$

or $y = x^3 + x^2 + Ax + B$

which is the general solution.

Example 10 :

Solve $\frac{d^2y}{dx^2} = \sin x - \cos x$

Solution :

The substitution $\frac{dy}{dx} = p$ may be avoided by thinking that $\frac{dy}{dx}$ is some quantity and its derivative is $\frac{d^2y}{dx^2}$, so that $\int \frac{d^2y}{dx^2}\,dx = \frac{dy}{dx}$

$$\text{So} \int \frac{d^2y}{dx^2}\,dx = \int (\sin x - \cos x)\,dx$$

$$\text{yields } \frac{dy}{dx} = -\cos x - \sin x + A$$

Integrating again, $\int dy = \int (-\cos x - \sin x + A)\,dx$

or $y = \cos x - \sin x + Ax + B$

which is the general solution.

PARTICULAR SOLUTION :

The general solution of a differential equation contains as many arbitrary constants as the order of the equation. Particular solutions are obtained by putting particular values for these constants. Sometimes, some conditions are required to be fulfilled by the solutions, often because of the need of physical situations. Usually the number of such conditions is equal to the order of the equation (and so to the number of arbitrary constants). The particular values for the arbitrary constants are determined so as to satisfy the given conditions.

Example 11 :

Solve the equation $\frac{dy}{dx} = \cos x$ subject to the condition, $y = 2$ when $x = 0$

Solution :

The equation is $dy = \cos x dx$

Integrating both sides,

$$y = \sin x + C$$

This is the general solution. Putting the given condition.

$$2 = \sin 0 + C$$

or $C = 2$

Hence the particular solution that will satisfy the given condition is $y = \sin x + 2$

Example 12 :

Find the particular solution of the equation

$$\frac{d^2y}{dx^2} = 2x, \text{ given that when } x = 0,\ y = 2 \text{ and } \frac{dy}{dx} = 3$$

Solution :

The equation is

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = 2x$$

On integration, $\frac{dy}{dx} = x^2 + A$ ... (1)

On further integration, $y = \frac{x^3}{3} + Ax + B$ ... (2)

This is the general solution containing two arbitrary constants A and B.

Using the condition, $\frac{dy}{dx} = 3$ when $x = 0$, from (1) we get

$$3 = 0 + A \Rightarrow A = 3$$

Using the condition, $y = 2$ when $x = 0$, we get from (2)

$$2 = 0 + 0 + B \Rightarrow B = 2$$

So the required particular solution is $y = \frac{1}{3}x^3 + 3x + 2$.

EXERCISE - 11 (a)

1. Determine the order and degree of each of the following differential equations.

(i) $y \sec^2 x\, dx + \tan x\, dy = 0$

(ii) $\left(\frac{dy}{dx}\right)^4 + y^5 = \frac{d^3y}{dx^3}$

(iii) $a\frac{d^2y}{dx^2} = \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}$

(iv) $\tan^{-1}\sqrt{\frac{dy}{dx}} = x$

(v) $ln\left(\frac{d^2y}{dx^2}\right) = y$

(vi) $\dfrac{\frac{dy}{dt}}{y + \frac{dy}{dt}} = \dfrac{yt}{\frac{dy}{dt}}$

(vii) $\frac{d^2y}{du^2} = \dfrac{3y + \frac{dy}{du}}{\sqrt{\frac{d^2y}{du^2}}}$

(viii) $e^{\frac{dy}{dx}} = x^2$

2. Form the differential equation by eliminating the arbitrary constants in each of the following cases.

(i) $y = A \sec x$

(ii) $y = C \tan^{-1} x$

(iii) $y = Ae^t + Be^{2t}$

(iv) $y = Ax^2 + Bx$

(v) $y = a \cos x + b \sin x$

(vi) $y = a \sin^{-1}x + b \cos^{-1}x$

(vii) $y = at + be^t$

(viii) $y = a \sin t + be^t$

(ix) $ax^2 + by = 1$

3. Find the general solution of the following differential equations.

(i) $\frac{dy}{dx} = \frac{e^{2x}+1}{e^x}$ (ii) $\frac{dy}{dx} = x\cos x$

(iii) $\frac{dy}{dt} = t^5 \ln t$ (iv) $\frac{dy}{dt} = 3t^2 + 4t + \sec^2 t$

(v) $\frac{dy}{dx} = \frac{1}{x^2 - 7x + 12}$ (vi) $\frac{dy}{du} = \frac{u+1}{\sqrt{3u^2+6u+5}}$

(vii) $(x^2 + 3x + 2)\, dy - dx = 0$ (viii) $\frac{dy}{dt} = \frac{\sin^{-1} t e^{\sin^{-1} t}}{\sqrt{1-t^2}}$

4. Solve the following differential equations.

(i) $\frac{dy}{dx} = y + 2$ (ii) $\frac{dy}{dt} = \sqrt{1-y^2}$

(iii) $\frac{dy}{dz} = \sec y$ (iv) $\frac{dy}{dx} = e^y$

(v) $\frac{dy}{dx} = y^2 + 2y$ (vi) $dy + (y^2 + 1)\, dx = 0$

(vii) $\frac{dy}{dx} + \frac{e^y}{y} = 0$ (viii) $dx + \cot x\, dt = 0$.

5. Obtain the general solution of the following differential equations.

(i) $\frac{dy}{dx} = (x^2 + 1)(y^2 + 1)$ (ii) $\frac{dy}{dt} = e^{2t+3y}$

(iii) $\frac{dy}{dz} = \frac{\sqrt{1-y^2}}{\sqrt{1-z^2}}$ (iv) $\frac{dy}{dx} = \frac{x \ln x}{3y^2 + 4y}$

(v) $x^2\sqrt{y^2+3}\, dx + y\sqrt{x^3+1}\, dy = 0$ (vi) $\tan y\, dx + \cot x\, dy = 0$

(vii) $(x^2 + 7x + 12)\, dy + (y^2 - 6x + 5)\, dx = 0$ (viii) $y\, dy + e^{-y} x \sin x\, dx = 0$.

6. Solve the following second order equations.

(i) $\frac{d^2y}{dx^2} = 12x^2 + 2x$

(ii) $\frac{d^2y}{dt^2} = e^{2t} + e^{-t}$

(iii) $\frac{d^2y}{d\upsilon^2} = -\sin\upsilon + \cos\upsilon + \sec^2\upsilon$

(iv) $\operatorname{cosec} x \frac{d^2y}{dx^2} = x$

(v) $x^2 \frac{d^2y}{dx^2} + 2 = 0$

(vi) $\sec x \frac{d^2y}{dx^2} = \sin 3x$

(vii) $\dfrac{d^2y}{dx^2} = \sec^2 x + \cos^2 x$

(viii) $e^{-x}\dfrac{d^2y}{dx^2} = x$.

7. Find the particular solutions of the following equations subject to the given conditions.

(i) $\dfrac{dy}{dx} = \cos x$, given that $y = 2$ when $x = 0$

(ii) $\dfrac{dy}{dt} = \cos^2 y$ subject to $y = \dfrac{\pi}{4}$ when $t = 0$.

(iii) $\dfrac{dy}{dx} = \dfrac{1+y^2}{1+x^2}$, given that $y = \sqrt{3}$ when $x = 1$

(iv) $\dfrac{d^2y}{dx^2} = 6x$, given that $y = 1$ and $\dfrac{dy}{dx} = 2$ when $x = 0$

8. (i) Solve : $\dfrac{dy}{dx} = \sec\,(x+y)$

(ii) Solve : $\dfrac{dy}{dx} = \sin(x+y) + \cos(x+y)$

(iii) Solve : $\dfrac{dy}{dx} = \cos(x+y)$

(iv) Solve : $\dfrac{dy}{dx} + 1 = e^{x+y}$

11.6 Linear Differential Equations

A differential equation is said to be **linear** if the dependent variable and its differential co-efficients occuring in the eqation are of first degree only and are not multiplied together.

The general form of a linear differential equation of the first order is

$$\mathbf{\frac{dy}{dx} + Py = Q} \qquad .....(1)$$

where P, Q are functions of x.

In this section we shall be concerned with linear differential equations of first order only.

The standard technique to slolve linear equations of the form (1) is to multiply both sides with $e^{\int Pdx}$

After multiplication, we get

$$e^{\int Pdx}\frac{dy}{dx} + \left(Pe^{\int Pdx}\right)y = Qe^{\int Pdx} \qquad .... (2)$$

If may be easily seen that the left hand side of (2) is the derivative of the product $ye^{\int pdx}$ with respect to x and the right hand side is a function of x alone.

So we can write (2) as

$$\frac{d}{dx}\left(ye^{\int Pdx}\right) = Qe^{\int Pdx} \qquad ....(3)$$

Integrating both sides of (3) with respect to x, we get

$$ye^{\int Pdx} = \int Qe^{\int Pdx}dx + C,$$

where C is an arbitary constant.

So $y = e^{-\int Pdx}\left(\int Qe^{\int Pdx}dx + C\right)$ .... (4)

is the general solution of the differential equation (1).

Note : The factor $e^{\int Pdx}$ on multiplication with the left hand side of (1) reduces it to an exact differential and is called the **integrating factor** of the differential equation (1).

We summarize below the steps involved is solving a first order linear differential equation.

(a) Write the equation in the standard form :

$$\frac{dy}{dx} + Py = Q$$

(b) Determine the integrating factor $e^{\int Pdx}$.

(c) Multiply both sides of the equation with $e^{\int Pdx}$ when the equation is in standard form.

(d) Integrate the resulting equation to obtain the required solution.

Note : (1) There are differential equations which are not linear in y but linear in x. Such equations in standard form look like

$$\frac{dx}{dy} + Px = Q$$

Where P, Q are functions of y alone.

In this case the integrating factor is $e^{\int Pdy}$

(2) A differential equation of the form

$$\frac{dy}{dx} + Py = Qy^n, n \neq 1 \quad .... (5)$$

where P, Q are functions of x is called **Bernoulli's equation**.

This equation can be reduced to the

form $\frac{dz}{dx} + (1 - n)\,Pz = (1 - n)\,Q$ which is linear with z as dependent variable, by putting $z = y^{-n+1}$.

Example 13 : Solve : $\left(1+x^2\right)\frac{dy}{dx} + 2xy - x^3 = 0$

Solution : $\left(1+x^2\right)\frac{dy}{dx} + 2xy - x^3 = 0$

$$\Rightarrow \frac{dy}{dx} + \frac{2x}{1+x^2}y = \frac{x^3}{1+x^2}$$

which is a linear differential equation of first order.

Here $P = \frac{2x}{1+x^2}$ and $Q = \frac{x^3}{1+x^2}$

So the integrating factor = $e^{\int P dx}$

$= e^{\int \frac{2x}{1+x^2} dx}$

$= e^{\log(1+x^2)} = 1 + x^2.$

Multiplying both sides of the equation with $1 + x^2$ we get

$$\left(1+x^2\right)\frac{dy}{dx} + 2xy = x^3$$

i.e., $$\frac{d}{dx}\left\{\left(1+x^2\right)y\right\} = x^3 \qquad \text{... (2)}$$

So $(1 + x^2)\, y = \int x^3\, dx + c$

$$= \frac{x^4}{4} + c,$$

where c is an arbitorary constant. Hence the general solution of the given differential equation is given by

$$y = \frac{x^4}{4\left(1+x^2\right)} + \frac{c}{1+x^2}$$

Example 14 : Solve : $(1 + y^2)\, dx + x\, dy = \tan^{-1}y\, dy$

Solution : $(1 + y^2)dx + xdy = \tan^{-1}y\, dy$

$$\Rightarrow \frac{dx}{dy} + \frac{1}{1+y^2}x = \frac{\tan^{-1}y}{1+y^2} \qquad \text{... (1)}$$

which is a linear differential equation of first order.

Here $P = \frac{1}{1+y^2}$, $Q = \frac{\tan^{-1}y}{1+y^2}$

So the integrating factor = $e^{\int P dy}$

$= e^{\int \frac{dy}{1+y^2}}$

$= e^{\tan^{-1}y}$

Multiplying both sides of the equation (1) with $e^{\tan^{-1}y}$, we get

$$e^{\tan^{-1}y}\frac{dx}{dy} + \frac{e^{\tan^{-1}y}}{1+y^2}x = e^{\tan^{-1}y}\frac{\tan^{-1}y}{1+y^2}$$

i.e. $$\frac{d}{dy}\left(x\, e^{\tan^{-1}y}\right) = e^{\tan^{-1}y}\frac{\tan^{-1}y}{1+y^2} \qquad \text{... (2)}$$

Integrating both sides of (2) with respect to y, we get

$$x\,e^{\tan^{-1}y} = \int e^{\tan^{-1}y}\frac{\tan^{-1}y}{1+y^2}\,dy + c$$

$$= \int te^t dt + c, \text{ where } t = \tan^{-1}y$$

$$= e^t(t-1) + c$$

$$= e^{\tan^{-1}y}(\tan^{-1}y - 1) + c.$$

So $x = \tan^{-1}y - 1 + c\,e^{-\tan^{-1}y}$ ... (3)

where c is an arbitrary constant.

(3) is the general solution of the given differential equation.

Example 15 : Solve : $\frac{dy}{dx}+\frac{y}{x} = xy^2$

Solution : The given differential equation is Benoulli's equation.

Dividing both sides of the equation by y^2,

we get

$$\frac{1}{y^2}\frac{dy}{dx}+\frac{1}{yx} = x \quad \text{... (1)}$$

We put $z = \frac{1}{y}$

Then $\frac{dz}{dx} = -\frac{1}{y^2}\frac{dy}{dx}$

So (1) becomes

$$-\frac{dz}{dx}+\frac{z}{x} = x,$$

or $\frac{dz}{dx}-\frac{z}{x} = -x$

which is a linear differential equation of first order.

The integrating factor $= e^{\int\left(-\frac{1}{x}\right)dx}$

$$= e^{-\log x}$$

$$= e^{\log\frac{1}{x}}$$

$$= \frac{1}{x}$$

Multiplying both sides of the equation (2) with $\frac{1}{x}$, we get

$$\frac{1}{x}\frac{dz}{dx}-\frac{z}{x^2} = -1$$

i.e. $\frac{d}{dx}\left(\frac{z}{x}\right) = -1$

So $\frac{z}{x} = -\int dx + c = -x + c$

Hence $\frac{1}{xy} = c - x$

Or $y = \frac{1}{x(c-x)}$

Thus $y = \frac{1}{x(c-x)}$ where c is an arbitrary constant, is the solution of the given differential equation.

EXERCISE 11 (b)

Solve the following differential equations :

1. $\frac{dy}{dx} + y = e^{-x}$
2. $(x^2 - 1)\frac{dy}{dx} + 2xy = 1$
3. $(1 - x^2)\frac{dy}{dx} + 2xy = x\sqrt{1-x^2}$
4. $x \log x \frac{dy}{dx} + y = 2 \log x$
5. $\left(1+x^2\right)\frac{dy}{dx} + 2xy = \cos x$
6. $\frac{dy}{dx} + y\sec x = \tan x$
7. $(x+\tan y)\, dy = \sin 2y\, dx$
8. $\left(x+2y^3\right)\frac{dy}{dx} = y$
9. $\sin x\frac{dy}{dx} + 3y = \cos x$
10. $(x+y+1)\frac{dy}{dx} = 1$
11. $(1+y^2)\, dx + (x-e^{-\tan^{-1}y})\, dy = 0$
12. $x\frac{dy}{dx} + y = xy^2$
13. $x\frac{dy}{dx} + y = y^2 \log x$
14. $\left(1+x^2\right)\frac{dy}{dx} = xy - y^2$
15. $\frac{dy}{dx} + \frac{y}{x-1} = xy^{\frac{1}{2}}$
16. $\frac{dy}{dx} + \frac{y}{x} = x^2, y(1) = 1$
17. $\frac{dy}{dx} + 2y\tan x = \sin x, y\left(\frac{\pi}{3}\right) = 0$

11.7 Homogeneous Equations.

A differential equation of the form

$$\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)} \quad \text{... (1)}$$

where f (x, y) and g (x,y) are homogeneous functions of x, y and of same degree, is said to be a homogeneous differential equation.

Such type of equations can be solved by putting

$y = vx,$

where v is a function of x.

If $y = vx$, then $\frac{dy}{dx} = v + x\frac{dv}{dx}$.

Then the given differential equation becomes

$$v + x\frac{dv}{dx} = \frac{f(x,y)}{g(x,y)}$$

$$= \frac{x^n f_1\left(\frac{y}{x}\right)}{x^n g_1\left(\frac{y}{x}\right)} \text{ if } f(x, y), g(x,y) \text{ are of degree n in x, y.}$$

$$= \frac{f_1(v)}{g_1(v)}$$

This implies that

$$\frac{g_1(v)dv}{f_1(v) - vg_1(v)} = \frac{dx}{x} \quad \text{... (2)}$$

In equation (2) the variables ϑ and x have been separated.

So integrating both sides of (2) and replacing v by $\frac{y}{x}$, we can obtain the required solution.

Example 16 : Solve $(x^2 + y^2)\,dx - 2xy\,dy = 0$.

Solution : The given differential equation can be written as

$$\frac{dy}{dx} = \frac{x^2 + y^2}{2xy} \quad \text{... (1)}$$

which is a homogeneous equation.

We put y = vx, where v is a function of x

Then $\frac{dy}{dx} = v + x\frac{dv}{dx}$

So from (1), we get

$$v + x\frac{dv}{dx} = \frac{1 + v^2}{2v}$$

or $x\frac{dv}{dx} = \frac{1 - v^2}{2v}$

So $\frac{2vdv}{1 - v^2} = \frac{dx}{x}$ ... (2)

Integrating both sides of (2), we get $-\log(1-v^2) = \log x - \log c$

where c is an arbitrary positive constant.

This gives

$$1 - v^2 = \frac{c}{x},$$

or $1 - \frac{y^2}{x^2} = \frac{c}{x}$,

or $x^2 - y^2 = cx$

which is the solution of the given differential equation.

11.8 Equations reducible to homogeneous form

The differential equations of the form

$$\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2} \quad \text{..... (1)}$$

where a_1, b_1, c_1, a_2, b_2, c_2 are constants

and $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$ can be reduced to homogeneous form by changing variables

x, y to the variables X, Y respectively by the substitutions

$$x = X + h,\ y = Y + k$$

where h and k are constants to be chosen later so as to make this reduction possible. We have

$$\frac{dy}{dx} = \frac{d}{dx}(Y + k) = \frac{dY}{dx} + \frac{dk}{dx}$$

$$= \frac{dY}{dX}\frac{dX}{dx} + 0$$

$$= \frac{dY}{dX}\left(\because \frac{dX}{dx} = \frac{d(x-h)}{dx} = 1\right)$$

Now the equation (1) becomes

$$\frac{dY}{dX} = \frac{(a_1X + b_1Y) + (a_1h + b_1k + c_1)}{(a_2X + b_2Y) + (a_2h + b_2k + c_2)} \quad \text{... (2)}$$

We choose h, k such that

$$a_1h + b_1k + c_1 = 0$$

and $a_2h + b_2k + c_2 = 0$

Then $h = \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}$, $k = \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}$

Thus if h, k have these values, then (2) becomes.

$$\frac{dY}{dX} = \frac{a_1X + b_1Y}{a_2X + b_2Y} \quad \text{... (3)}$$

which is a homogenous differential equation and can be solved by putting Y = vX where v is function of X.

In the solution of the equation (3), replacing X, Y, v by x–h, y – k and $\frac{y-k}{x-h}$ respectively we can get the solution of the original equation (1).

If $\frac{a_1}{a_2} = \frac{b_1}{b_2} = r$, then $a_1 = a_2 r$ and $b_1 = b_2 r$.

The equation (1) now becomes

$$\frac{dy}{dx} = \frac{r(a_2x + b_2y) + c_1}{a_2x + b_2y + c_2} \quad \text{... (4)}$$

Putting $a_2x + b_2y = z$ so that $a_2 + b_2\frac{dy}{dx} = \frac{dz}{dx}$, we can get from (4)

$$\frac{1}{b_2}\left(\frac{dz}{dx} - a_2\right) = \frac{rz + c_1}{z + c_2},$$

or, $$\frac{dz}{dx} = b_2\left(\frac{rz + c_1}{z + c_2}\right) + a_2 \quad \text{... (5)}$$

so that the variables are separable. In the solution of the equation (5) we replace z by $a_2x + b_2y$ in order to get the solution of the equation (1).

Example 17 : Solve (x–y + 1) dx – (x+y+5) dy = 0

Solution : The given differential equation can be written as

$$\frac{dy}{dx}=\frac{x-y+1}{x+y+5} \quad \ldots \quad (1)$$

Put x = X + h and y = Y + k in (1), where h, k are constants to be determined later.

Then $$\frac{dY}{dX}=\frac{(X-Y)+(h-k+1)}{(X+Y)+(h+k+5)} \quad \ldots \quad (2)$$

We choose the values of h, k such that

and $$h-k+1=0$$
$$h+k+5=0$$

Then h = – 3 and k = – 2

Substituting these values in (2), we get

$$\frac{dY}{dX}=\frac{X-Y}{X+Y} \quad \ldots \quad (3)$$

Put $Y = vX, \frac{dy}{dx}=v+X\frac{dv}{dX}$ in (3), where v is a function of X.

Then (3) becomes

$$v+X\frac{dv}{dx}=\frac{X(1-v)}{X(1+v)}=\frac{1-v}{1+v},$$

or $$X\frac{dv}{dx}=\frac{1-2v-v^2}{1+v},$$

or $$\frac{(1+v)}{1-2v-v^2}dv=\frac{dx}{x} \quad \ldots \quad (4)$$

Integrating both sides, we get

$$-\frac{1}{2}\log(1-2v-v^2)=\log X+\log c.$$

so $$\frac{1}{1-2v-v^2}=c^2X^2 \quad \ldots \quad (5)$$

We have $v=\frac{Y}{X}=\frac{y+2}{x+3}$

Replacing v by $\frac{y+2}{x+3}$ in (5), we get

$$\frac{1}{1-2\left(\frac{y+2}{x+3}\right)-\left(\frac{y+2}{x+3}\right)^2}=c^2(x+3)^2$$

or $$\frac{1}{(x+3)^2-2(x+3)(y+2)-(y+2)^2}=c^2$$

or $\quad x^2 + 2x - 2xy - 10y - y^2 - 7 = \frac{1}{c^2}$

or $\quad x^2 + 2x - 2xy - 10y - y^2 = 7 + \frac{1}{c^2} = \lambda$ (say)

where λ is an arbitrary positive constant.

Thus $x^2 + 2x - 2xy - 10y - y^2 = \lambda$ is the solution of the given differential equation.

EXERCISE 11(c)

Find the solutions of the following differential equations :

1. $(x+y)\,dy + (x-y)\,dx = 0$
2. $\frac{dy}{dx} = \frac{1}{2}\left(\frac{y}{x} + \frac{y^2}{x^2}\right)$
3. $(x^2 - y^2)\,dx + 2xy\,dy = 0$
4. $x\frac{dy}{dx} + \sqrt{x^2 + y^2} = y$
5. $x(x+y)\,dy = (x^2 + y^2)\,dx.$
6. $y^2 + x^2\frac{dy}{dx} = xy\frac{dy}{dx}$
7. $x \sin\frac{y}{x}dy = \left(y\sin\frac{y}{x} - x\right)dx.$
8. $x\,dy - y\,dx = \sqrt{x^2 + y^2}\,dx$
9. $\frac{dy}{dx} = \frac{y - x + 1}{y + x + 5}$
10. $(x-y)\,dy = (x + y + 1)\,dx$
11. $(x-y-2)\,dx + (x-2y-3)\,dy = 0$
12. $\frac{dx}{dy} = \frac{3x - 7y + 7}{3y - 7x - 3}$
13. $(2x + y + 1)\,dx + (4x + 2y - 1)\,dy = 0$
14. $(2x+3y-5)\frac{dy}{dx} + 3x + 2y - 5 = 0$
15. $(4x+6y+5)\,dx - (2x + 3y + 4)\,dy = 0$

Vectors

My pain may be reason for somebody's laugh. But my laugh must never be the reason for somebody's pain.

- Charlie Chaplin

12.0 Introduction:

In nineteenth century, the Irish mathematician Sir Willam Rowam Hamilton (1805 - 1865) introduced the "Theory of Quaternions", a new method for better understanding of both algebra and physics, which in its simple form is the vector analysis of today. Now a days, vector analysis has become a very important tool for Physics, Mathematics, Engineering and other branches of science.

In our day-to-day life we come across two types of physical quantities namely, (i) Scalar quantity, otherwise called a **scalar,** and (ii) Vector quantity, or a **vector**.

Some physical quantities are completely determined when their magnitudes are known in terms of specific units. These quantities are called scalars, for example, height, mass etc.

However, in the application of mathematics to physics and engineering, we come across certain quantities that possess both magnitude, as well as direction. These quantities cannot be specified or characterised by a single number. For example, complete motion of a moving object, say a car, with a given speed, can not be described until we know the direction of its motion. Such quantities which have a magnitude as well as direction are called vectors. Examples are force, velocity, acceleration, etc.

12.1 Representation of a Vector (Its magnitude and direcdtion)

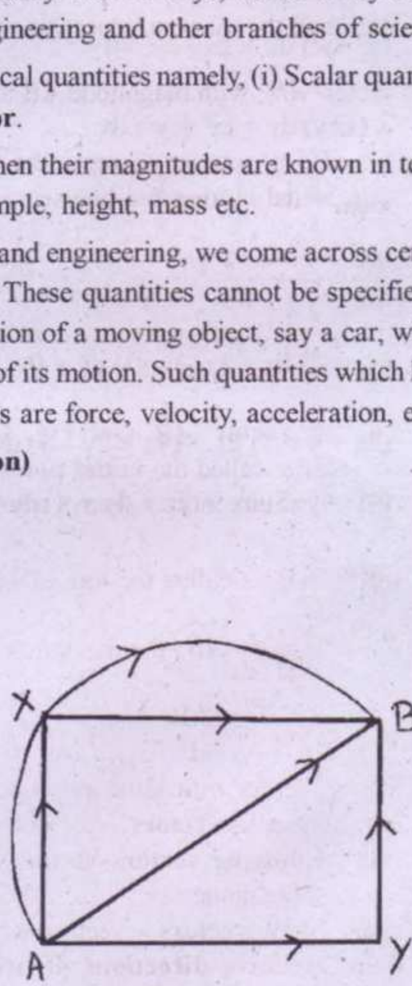
Suppose we want to move an object from position A to position B. It can be done in a number of ways as shown in Fig.12.1, i.e.

(i) from A to X and then from X to B

(ii) from A to B along the arc $\widehat{AXC}$

(iii) from A to Y and then from Y to B

(iv) directly from A to B as show by the line segment $\overline{AB}$, directed from A to B by an arrow mark.



(Fig. 12.1)

We have described four different paths along which we can move the object from A to B. There can be many others. The distance of movement of the object is different along different paths. However, along all these paths, the change of position of the object is from A to B and it is shown by the directed line segment as shown in the figure.

The change of position which is called **displacement** is thus associated with two entities, the magnitude which is the distance AB and the direction, which is from A to B. The displacement is represented as $\overrightarrow{AB}$.

$\overrightarrow{AB}$ is called a directed line segment. (which is different from a ray $\overrightarrow{AB}$ in geometry). The length AB is the magnitude of the displacement and the arrow mark above it signifies that the direction of the change of position of the object is from A to B. (So obviously $\overrightarrow{AB}$ and $\overrightarrow{BA}$ are displacements of same magnitude, but opposite directions).

Quantities as above such as displacement are called vectors which are characterized by both magnitude and direction.

It is customary to represent a vector by a directed line segment $\overrightarrow{AB}$.

Coming back to the above example, the distance covered in the given cases (i) to (iv) are

(i) AX + XB

(ii) length of the arc $\widehat{AXB}$

(iii) AY + YB

(iv) AB.

But in all the cases, irrespective of the distance of movement, change of position of the object is given by the vector $\overrightarrow{AB}$, with magnitude AB and direction as explained earlier.

N.B. It is also a practice to represent vectors as $\vec{u}$, $\vec{v}$, $\vec{w}$ etc., but in all cases they are elaborately represented by directed line segments. Thus we can write

$\vec{u} = \overrightarrow{AB}$, $\vec{v} = \overrightarrow{PQ}$, $\vec{w} = \overrightarrow{RS}$ etc. The arrow marks above the letters u, v, w simply indicate that $\vec{u}$, $\vec{v}$, $\vec{w}$ are vector quantities.

12.2 Further terminologies and notations

For the vector $\overrightarrow{AB}$ (see Fig. 12.1)

(i) A is called the **initial point** and B, the **terminal point**.
The direction is always from initial to terminal point.

(ii) $\overleftrightarrow{AB}$ is called the **line of support**

(iii) If $\vec{u} = \overrightarrow{AB}$, the magnitude of $\vec{u}$ is represented as $|\vec{u}|$.

So $|\vec{u}| = AB$.

In general

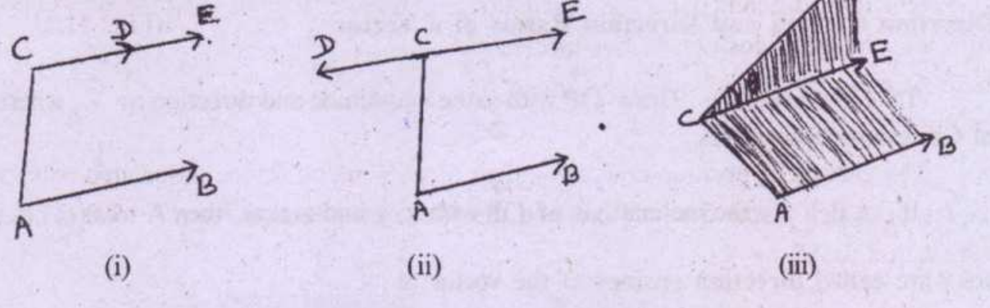
(iv) Vectors with same initial point are called **coinitial vectors.**

(v) **Parallel vectors** - Vectors whose lines of support are parallel.

(vi) **Coplanar vectors**-Vectors which lie on the same plane or are parallel to vectors lying on the same plane

(vii) **Skew vectors** - Vectors which are not coplanar

(viii) **Relative directions of two vectors**



(Fig 12.2)

Let $\vec{u} = \overleftrightarrow{AB}$ and $\vec{v} = \overrightarrow{CD}$ be any two vectors. To decide whether their directions are (i) the same, (ii) opposite or (iii) different, proceed as follows :

In the plane of A, B, C draw $\overrightarrow{CE}$ parallel to $\overleftrightarrow{AB}$ such that B and E lie on the same side of $\overleftrightarrow{AC}$ and also AB = CE, then

(i) If $\overrightarrow{CD}$ and $\overrightarrow{CE}$, considered as geometrical rays unite into one ray, then vector $\vec{u} = \overleftrightarrow{AB}$ and $\vec{v} = \overrightarrow{CD}$ are unidirectional, i.e. they have the same direction. [Fig. 12.2 (i)]. They are called **like vectors**.

(ii) If $\overrightarrow{CD}$ and $\overrightarrow{CE}$, considered as rays of geometry form a line as in Fig.12.2 (ii), then $\vec{u} = \overleftrightarrow{AB}$ and $\vec{v} = \overrightarrow{CD}$ have opposite directions. They are called **unlike vectors**.

(iii) It may so happen that $\overrightarrow{CD}$ and $\overrightarrow{CE}$ neither unite into a ray nor form a line [see Fig.12.2(iii)]. Here the plane of C,D,E and the plane of A,B,C (on which $\overrightarrow{CE}$ lies) may or may not be the same plane. If they are one and the same plane, then $\overleftrightarrow{AB}$ and $\overrightarrow{CD}$ are coplanar vectors. But if they are different planes, $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are non-coplanar, hence skew vectors. However $\overrightarrow{CE}$ has the same magnitude and direction as $\overrightarrow{AB}$. (You can confirm it by the procedure laid out in 12.2(i))

This shows that vectors which are not coinitial can be made coinitial.

In case of 12.2(iii) **vector $\vec{u} = \overrightarrow{AB}$ and $\vec{v} = \overrightarrow{CD}$ have different directions**.

(ix) Equal Vectors : Vectors with same magnitude and directions are called equal vectors.

(x) Zero (Null) Vector : It is a conceptual vector whose magnitude is supposed to be zero. It has indefinite direction.

(xi) Unit Vector : A vector whose magnitude is 1.

12.3 Definition (Inclination between two vectors)

(i) If two vectors $\vec{u}$ and $\vec{v}$ have the same direction, the inclination between them is 0 (degree or radian or grade).

(ii) If $\vec{u}$ and $\vec{v}$ have opposite directions then inclination between them is 180^0 or π.

(iii) If $\vec{u} = \overrightarrow{AB}$ and $\vec{v} = \overrightarrow{CD}$ have different directions [see Fig 12.2(iii)], then inclination between then is given by $\theta = m\angle DCE$.

Note that an inclination $\theta \in [0, \pi]$ whereas an angle-measure $\theta \in (0, \pi)$.

12.4 Direction Cosines and Direction Ratios of a Vector

Take any vector $\vec{v}$. Draw $\overrightarrow{OP}$ with same magnitude and direction as $\vec{v}$, where O is the origin of Cartesian co-ordinates.

If α, β, γ are the inclinations of $\overrightarrow{OP}$ with x, y and z-axes, then $l = \cos\alpha$, $m = \cos\beta$ and $n = \cos\gamma$ are called **direction cosines** of the vector $\vec{v}$.

If P has Cartesian co-ordinates (x,y,z) then obviously

$OP\cos\alpha = x$, $OP\cos\beta = y$ and $OP\cos\gamma = z$

$\Rightarrow OP^2(\cos^2\alpha + \cos^2\beta + \cos^2\gamma)$

$= OP^2(l^2+m^2+n^2) = x^2 + y^2 + z^2 = OP^2$

$\Rightarrow l^2+m^2+n^2 = 1$

Note : The symbols $\overrightarrow{AB}$, $\overrightarrow{PQ}$ etc. have already been used in geometry to mean rays, but here, in case of vectors, they are not rays. When $\overrightarrow{AB}$ is taken to be a vector, it has a finite length AB. The arrow-mark simply indicates that direction of the vector is from A towards B. Note that a ray $\overrightarrow{AB}$ does not have a finite length AB.

Types of Vectors :

Let $\vec{v}$ be a vector. If $|\vec{v}| = 0$, then $\vec{v}$ is called **zero vector ($\vec{0}$)** or a **null vector** or a point vector. (It has infinitely many directions.)

If $|\vec{v}| = 1$ then $\vec{v}$ is called a **unit vector**, and is usually denoted by $\hat{v}$.

12.5 Multiplication of a Vector by scalar :

If $\vec{v}$ is a vector and k ($\neq 0$) be a scalar, then multiplication of the vector $\vec{v}$ by the scalar k is a vector denoted by $k\vec{v}$ whose maginitude is $|k|$ times that of $\vec{v}$ and the direction is same as that of $\vec{v}$ if k is positive and opposite to $\vec{v}$ if k negative. If $k = 0$ then $k\vec{v}$ is zero vector $\vec{0}$.

Two vectors said to be **collinear** if they are parallel.

If $\vec{v}$ is a non-zero vector, i.e. $|\vec{v}| \neq 0$,

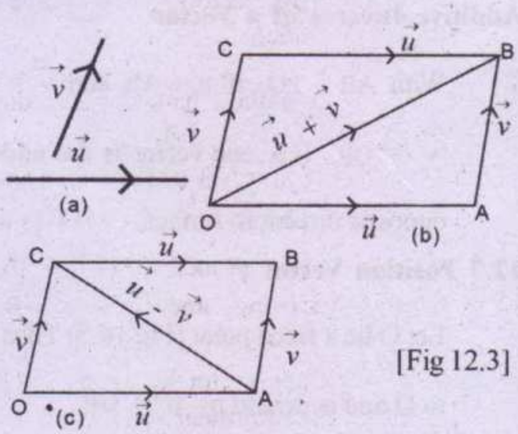
then $\hat{v} = \dfrac{\vec{v}}{|\vec{v}|}$ is **a unit vector in the direction of** $\vec{v}$, $\left|\dfrac{\vec{v}}{|\vec{v}|}\right| = 1$ Thus $\hat{v} = \dfrac{\vec{v}}{|\vec{v}|} \Rightarrow \vec{v} = |\vec{v}|\,\hat{v}$.

Thus a vector can be thought of as product of its magnitude with a unit vector in its direction.

12.6 Addition of Vectors :

If $\vec{u}$ and $\vec{v}$ are two vectors represented by $\vec{u} = \overrightarrow{OA}$ and $\vec{v} = \overrightarrow{AB}$, then the vector $\vec{u} + \vec{v}$ is represented by $\vec{u} + \vec{v} = \overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB}$

i.e. the sum of two vectors is a vector represented by the diagonal of the parallelogram, whose sides are the vectors $\vec{u}$ and $\vec{v}$. The initial point of $\vec{u} + \vec{v}$ being the same as that the $\vec{u}$.



[Fig 12.3]

The vector $\vec{u} - \vec{v}$ is represented by the other diagonal whose initial point is the terminal point of $\vec{v}$.

Since a vector is represented by a directed line segment, a vector $\vec{u} = \overrightarrow{OA}$ can be thought of as a force displacing point O to A.

If $\vec{u}$ and $\vec{v}$ are two forces, displacing the point O to A and A to B (respectively) then the final displacement is from the point O to B, which is represented by $\vec{u} + \vec{v}$.

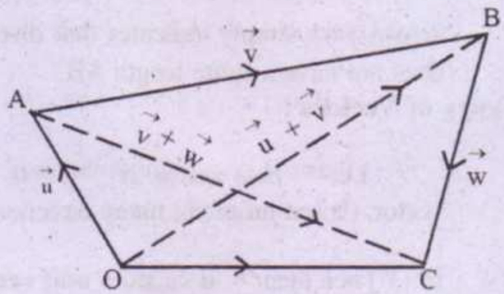
This is also known as **law of parallelogram of vectors.** From the above definition of addition of vectors we see that [fig - 12.3 (b)]

$\vec{v} + \vec{u} = \overrightarrow{OC} + \overrightarrow{CB} = \overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB} = \vec{u} + \vec{v}$

i.e. vector addition is **commutative**.

Also, if $\vec{u} = \overrightarrow{OA}$, $\vec{v} = \overrightarrow{AB}$ and $\vec{w} = \overrightarrow{BC}$, then

$$(\vec{u} + \vec{v}) + \vec{w} = (\overrightarrow{OA} + \overrightarrow{AB}) + \overrightarrow{BC}$$
$$= \overrightarrow{OB} + \overrightarrow{BC}$$
$$= \overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AC}$$
$$= \overrightarrow{OA} + (\overrightarrow{AB} + \overrightarrow{BC})$$
$$= \vec{u} + (\vec{v} + \vec{w})$$



[Fig 12.4]

i.e., addition of vectors is **associative**. So, vector addition satisfies **commutative** and **associative** laws.

From the above definitions of addition and multiplication of vectors by scalars it can be easily seen that, for scalars α, β and vectors $\vec{u}$ and $\vec{v}$, the following results hold good:

(i) $\alpha(\vec{u} + \vec{v}) = \alpha\vec{u} + \alpha\vec{v}$ (iv) $1\,\vec{u} = \vec{u}$

(ii) $(\alpha + \beta)\,\vec{u} = \alpha\vec{u} + \beta\vec{u}$ (v) $0\,\vec{u} = \vec{0}$

(iii) $(\alpha\beta)\,\vec{u} = \alpha(\beta\vec{u})$

Additive Inverse of a Vector

With $\overline{AB} \parallel \overline{PQ}$, if $\vec{u} = \overrightarrow{AB}$ and $\vec{v} = \overrightarrow{PQ}$ and AB = PQ then $\vec{u} = \vec{v}$, and $\vec{u} = -\vec{v}$ if $\vec{u} = \overrightarrow{AB}$ and $\vec{v} = \overrightarrow{QP}$. (i.e. one vector is the **additive inverse of other**, if they have the same magnitude and opposite direction) In fact, $-\vec{v} = (-1)\,\vec{v}$.

12.7 Position Vector :

Let O be a fixed point (Fig.16.5) Then the vector $\overrightarrow{OP}$ is called the **position vector** of point P, relative to O and is denotd by $\vec{p} = \overrightarrow{OP}$

Similarly position vectors of points Q, R ..... are $\vec{q}$, $\vec{r}$, ... respectively where $\overrightarrow{OQ} = \vec{q}$, $\overrightarrow{OR} = \vec{r}$ and so on.

[**Note :** The position vector of any point is dependent on the choice of the fixed point O and hence differs with choice of different fixed points.]

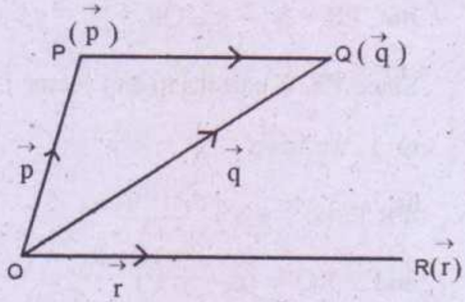
From Fig. 16.5, we have,

$$\overrightarrow{OP} + \overrightarrow{PQ} = \overrightarrow{OQ}$$

Hence,

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \vec{q} - \vec{p}$$

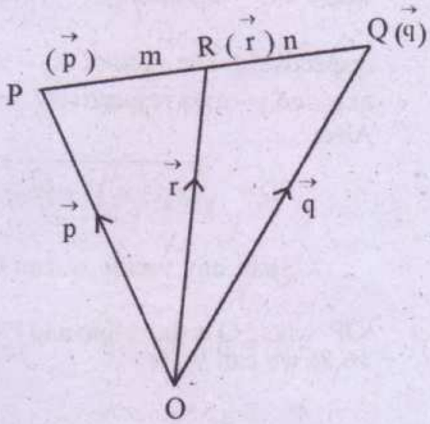
= position vector of Q – position vector of P.



[Fig 12.5]

So, any vector is given by the differnce of position vectors of its intial point from that of its terminal point.

Let P and Q be two points with position vectors $\vec{p}$ and $\vec{q}$ and R be a point on line segment $\overline{PQ}$, which divides it in ration m : n (fig 16.6).



[Fig 12.6]

Then, $\overrightarrow{PQ} = \vec{q} - \vec{p}$.

If $\vec{r}$ is the position vector of point R, then

$$\vec{r} = \overrightarrow{OR} = \overrightarrow{OP} + \overrightarrow{PR} = \vec{p} + \frac{m}{m+n}\overrightarrow{PQ}$$

$$= \vec{p} + \frac{m}{m+n}(\vec{q} - \vec{p}) = \frac{m\vec{q} + n\vec{p}}{m+n}.$$

If R ($\vec{r}$) divides $\overline{PQ}$ externally in ratio m : n, then it can be shown that,

$$\vec{r} = \frac{m\overrightarrow{q} - n\overrightarrow{p}}{m-n}.$$

If R ($\vec{r}$) is the midpoint of $\overline{PQ}$, then m = n

and, $\vec{r} = \dfrac{\vec{p} + \vec{q}}{2}$.

[**Note :** Three distinct points with position vectors $\vec{u}$, $\vec{v}$, $\vec{w}$ will be collinear iff we can find three nonzero scalars *l, m, n* such th at $l\vec{u} + m\vec{v} + n\overrightarrow{w} = \vec{0}$ and $l + m + n = 0$.]

12.8 Resolution of a Vector into components :

Consider the XY-plane and let $\hat{i}$ and $\hat{j}$ be unit vectors along $\overrightarrow{OX}$ and $\overrightarrow{OY}$ respectively. (Fig 16.7)

Let $\overrightarrow{PQ}$ be a vector in XY - plane whose initial point is P (x_1, y_1) and terminal point is Q (x_2, y_2). Then,

$$\overrightarrow{PQ} = \overrightarrow{PR} + \overrightarrow{RQ}$$

But, $PR = |x_2 - x_1|$, $QR = |y_2 - y_1|$

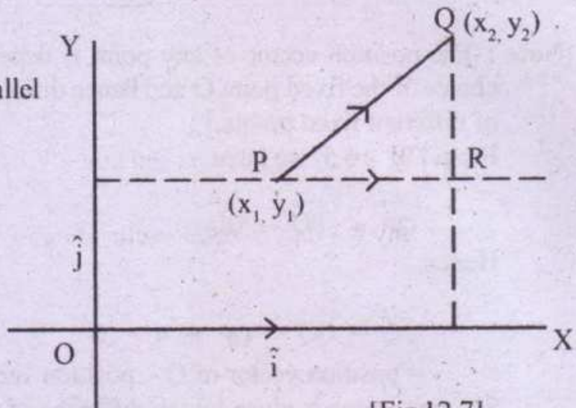
Since $\overrightarrow{PR}$ is parallel to unit vector $\hat{i}$ and $\overrightarrow{RQ}$ is parallel to $\hat{j}$, we have

$$\overrightarrow{PR} = (x_2 - x_1)\,\hat{i}$$

and $\overrightarrow{RQ} = (y_2 - y_1)\,\hat{j}$

Hence,

$$\overrightarrow{PQ} = (x_2 - x_1)\,\hat{i} + (y_2 - y_1)\,\hat{j};$$



[Fig 12.7]

where $(x_2 - x_1)$ and $(y_2 - y_1)$ are (scalar) components of vector $\overrightarrow{PQ}$ along x - axis and y - axis respectively. The vectors $(x_2 - x_1)\,\hat{i}$ and $(y_2 - y_1)\,\hat{j}$ are called component vectors of $\overrightarrow{PQ}$ along x - axis and y - axis respectively.

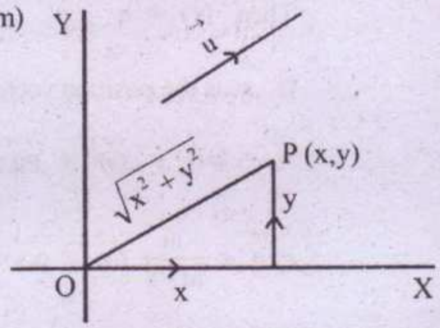
Also,

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \text{ (By Pythagoras Theorem)}$$

Since any vector $\vec{u}$ can be equal to some vector $\overrightarrow{OP}$ where O is the origin and P is some point (a, b) (fig. 16.8) we can write

$$\vec{u} = a\,\hat{i} + b\,\hat{j}$$

and $|\vec{u}| = \sqrt{a^2 + b^2}$



[Fig. 12.8]

Thus we see that every twodimensional vector $\vec{u}$ can be written as

$$\vec{u} = x\,\hat{i} + y\,\hat{j}$$

where $x\,\hat{i}$ and $y\,\hat{j}$ are vectors along x - axis and y - axis respectively and magnitude of $\vec{u}$ is

$$|\vec{u}| = \sqrt{x^2 + y^2}.$$

Further, if $\vec{a}$ and $\vec{b}$ are two equal vectors and $\vec{a} = a_1\,\hat{i} + a_2\,\hat{j}$, $\vec{b} = b_1\,\hat{i} + b_2\,\hat{j}$, then it can be shown that $a_1 = b_1$ and $a_2 = b_2$

Similarly, if $\vec{v}$ is a three-dimensional vector joining the points P (x_1, y_1, z_1) and Q (x_2, y_2, z_2) and $\hat{i}$, $\hat{j}$, $\hat{k}$ are unit vectors along x - axis, y - axis and z - axis repectively, then it can be proved that

$$\vec{v} = (x_2 - x_1)\,\hat{i} + (y_2 - y_1)\,\hat{j} + (z_2 - z_1)\,\hat{k}$$

and $|\vec{v}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

Note : Any vector $\vec{u}$ can be taken as

$\vec{u} = \overrightarrow{OP}$ where O is the origin and P = (x, y, z)

Hence any 3 - dimensional vector $\vec{u}$ can be written as

$\vec{u} = x\,\hat{i} + y\,\hat{j} + z\hat{k}$

and $|\vec{u}| = \sqrt{x^2 + y^2 + z^2} = r$ (say)

If $\vec{u}$ has inclinations α, β, γ with positive x - axis, y - axis and z - axis respectively (This means that the ray $\overrightarrow{OP}$ has inclinations α,β and γ with $\overrightarrow{OX}$, $\overrightarrow{OY}$ and $\overrightarrow{OZ}$ respectively) then the direction ratios of $\vec{u}$ are <x,y,z> that is, the scalar components along the co-ordinate axes. Therefore, every3-dimensional vector $\vec{u}$ can be represented by a triad< x,y, z >,where x, y, z are componets along co-ordinate axes and are direction ratios of $\vec{u}$ and modulus $\vec{u} = |\vec{u}| = \sqrt{x^2 + y^2 + z^2}$.

In case of a two-dimensional vector $\vec{v}$, (say in XY-plane) the third component is absent and hence $\vec{v}$ can be represented by an ordered pair (x, y) such that x and y are components along x-axis and y-axis and $|\vec{v}| = \sqrt{x^2 + y^2}$.

Example 1

Four vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{d}$ are given by $\vec{a} = (1, 1, 1)$, $\vec{b} = (2, 3, 0)$, $\vec{c} = (3, 5, -2)$ and $\vec{d} = (0, -1, 1)$. Prove that the vectors $(\vec{b} - \vec{a})$ and $(\vec{d} - \vec{c})$ are parallel and find the ratio of their moduli.

Solution :

$\vec{b} - \vec{a} = (2, 3, 0) - (1, 1, 1) = (1, 2, -1)$

and, $\vec{d} - \vec{c} = (0, -1, 1) - (3, 5, -2) = (-3, -6, 3) = (-3)(1, 2, -1)$

Hence

$(\vec{b} - \vec{a}) \parallel (\vec{d} - \vec{c})$

Also $|\vec{b} - \vec{a}| = \sqrt{1^2 + 2^2 + (-1)^2} = \sqrt{6}$

and $|\vec{d} - \vec{c}| = \sqrt{(-3)^2 + (-6)^2 + 3^2} = 3\sqrt{6}$

$\therefore |\vec{b} - \vec{a}| : |\vec{d} - \vec{c}| = 1 : 3$.

Example 2

If the position vectors of two given points A and B are $7\hat{i} + 3\hat{j} + \hat{k}$ = and $2\hat{i} - 5\hat{j} + 4\hat{k}$ respectively, find the magnitude and direction $\overrightarrow{AB}$.

Solution :

We have

$\overrightarrow{AB} = (2\hat{i} - 5\hat{j} + 4\hat{k}) - (7\hat{i} + 3\hat{j} - \hat{k}) = -5\hat{i} - 8\hat{j} + 5\hat{k}$

$|\overrightarrow{AB}| = \sqrt{(-5)^2 + (-8)^2 + (5)^2} = \sqrt{114}$

If $\overrightarrow{AB}$ has inclinations α, β, γ with positive x, y and z axes respectively, then

$\cos\alpha = \frac{-5}{\sqrt{114}}$, $\cos\beta = \frac{-8}{\sqrt{114}}$ and $\cos\gamma = \frac{5}{\sqrt{114}}$.

So the direction of $\overrightarrow{AB}$ is given by its direction cosines which are usually mentioned as $\left\langle \frac{-5}{\sqrt{114}}, \frac{-8}{\sqrt{114}}, \frac{5}{\sqrt{114}} \right\rangle$.

Example 3

Calculate the modulus and the unit vector in the direction of the sum of the vectors $\hat{i} + 4\hat{j} + 2\hat{k}$, $3\hat{i} - 3\hat{j} - 2\hat{k}$ and $-2\hat{i} + 2\hat{j} + 6\hat{k}$

Solution :

Then sum of the given vectors is

$\vec{r} = (\hat{i} + 4\hat{j} + 2\hat{k}) + (3\hat{i} - 3\hat{j} - 2\hat{k}) + (-2\hat{i} + 2\hat{j} + 6\hat{k}) = 2\hat{i} + 3\hat{j} + 6\hat{k}$

$\therefore |\vec{r}| = \sqrt{2^2 + 3^2 + 6^2} = 7$

Hence a unit vector parallel to $\vec{r}$ is $\hat{r} = \frac{\vec{r}}{|\vec{r}|} = \frac{2\hat{i} + 3\hat{j} + 6\hat{k}}{7} = \frac{2}{3}\hat{i} + \frac{3}{7}\hat{j} + \frac{6}{7}\hat{k}$

Example 4 :

Find the vector joining the points (2, –3) and (–1, 1). Find its magnitude and the unit vector along the same direction. Also determine the scalar components and component vectors along the co-ordinate axes.

Solution :

Let P be the point (2, –3) and Q (–1, 1). Then the position vector of P is $2\hat{i} - 3\hat{j}$ and that of Q is $-\hat{i} + \hat{j}$

Then $\overrightarrow{PQ} = (-\hat{i} + \hat{j}) - (2\hat{i} - 3\hat{j}) = -3\hat{i} + 4\hat{j}$

$|\overrightarrow{PQ}| = \sqrt{(-3)^2 + 4^2} = 5$

Thus unit vector $\widehat{PQ} = \frac{-3}{5}\hat{i} + \frac{4}{5}\hat{j}$

Therefore the scalar components are $\frac{-3}{5}$ and $\frac{4}{5}$

Whereas the component vectors along the x - and y - axes are respectively $\frac{-3}{5}\hat{i}$ and $\frac{4}{5}\hat{j}$.

Example 5 :

The position vectors of the points A, B and C are $2\hat{i} + \hat{j} - \hat{k}$, $3\hat{i} - 2\hat{j} + \hat{k}$, and $\hat{i} + 4\hat{j} - 3\hat{k}$ respectively. Show that A, B, and C are collinear.

Solution :

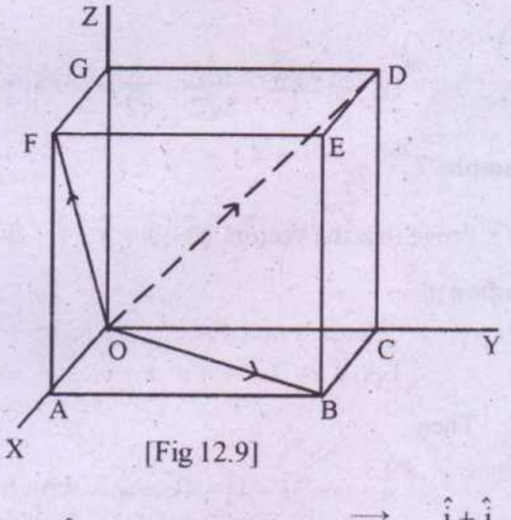
$$\overrightarrow{AB} = (3\hat{i} - 2\hat{j} + \hat{k}) - (2\hat{i} + \hat{j} - \hat{k}) = \hat{i} - 3\hat{j} + 2\hat{k}$$

$$\overrightarrow{AC} = (\hat{i} + 4\hat{j} - 3\hat{k}) - (2\hat{i} + \hat{j} - \hat{k}) = -\hat{i} + 3\hat{j} - 2\hat{k} = -\overrightarrow{AB}$$

$\Rightarrow \overrightarrow{AB}$ and $\overrightarrow{AC}$ are parallel with A as common point and hence, A, B, C are collinear.

Example 6 :

Three vectors of magnitude a, $2a$ and $3a$ act along the diagonals of three adjacent faces OABC, OCDG, OAFG of a cube. Find their sum and its direction cosines.



[Fig 12.9]

Solution :

Let the three vectors of magnitudes a, $2a$, $3a$ act along diagonals $\overline{OB}$, $\overline{OD}$ and $\overline{OF}$ and let us consider x-axis, y-axis, z-axis along sides $\overline{OA}$, $\overline{OC}$, $\overline{OG}$ of the cube.

The unit vectors along $\overrightarrow{OA}$ and $\overrightarrow{OC}$ being $\hat{i}$ and $\hat{j}$ the unit vector along $\overrightarrow{OB}$ is $\frac{\hat{i}+\hat{j}}{\sqrt{2}}$ and since OB = a, we have,

$$\overrightarrow{OB} = a\left(\frac{\hat{i}+\hat{j}}{\sqrt{2}}\right)$$

Similarly, unit vectors along $\overrightarrow{OD}$ and $\overrightarrow{OF}$ are $\frac{\hat{j}+\hat{k}}{\sqrt{2}}$ and $\frac{\hat{i}+\hat{k}}{\sqrt{2}}$ respectively. So

$$\overrightarrow{OD} = 2a\left(\frac{\hat{j}+\hat{k}}{\sqrt{2}}\right)$$

and $\overrightarrow{OF} = 3a\left(\frac{\hat{k}+\hat{i}}{\sqrt{2}}\right)$

Hence

$$\vec{r} = \overrightarrow{OB} + \overrightarrow{OD} + \overrightarrow{OF} = a\left(\frac{\hat{i}+\hat{j}}{\sqrt{2}}\right)+2a\left(\frac{\hat{j}+\hat{k}}{\sqrt{2}}\right)+3a\left(\frac{\hat{k}+\hat{i}}{\sqrt{2}}\right)$$

$$= \frac{4a\hat{i}+3a\hat{j}+5a\hat{k}}{\sqrt{2}} = \left(2\sqrt{2}a\right)\hat{i}+\left(\frac{3}{\sqrt{2}}a\right)\hat{j}+\left(\frac{5}{\sqrt{2}}a\right)\hat{k}$$

So,

$$|\vec{r}| = \sqrt{\left(2\sqrt{2}a\right)^2+\left(\frac{3a}{\sqrt{2}}\right)^2+\left(\frac{5}{\sqrt{2}}a\right)^2} = 5a$$

Its direction cosines are,

$$\left\langle \frac{2\sqrt{2}a}{5a}, \frac{3a}{5\sqrt{2}a}, \frac{5a}{5\sqrt{2}a} \right\rangle$$

$$\text{i.e.,} < \frac{2\sqrt{2}}{5}, \frac{3}{5\sqrt{2}}, \frac{1}{\sqrt{2}} >$$

Example 7

Prove that the vectors $2\hat{i}-\hat{j}+\hat{k}$, $\hat{i}-3\hat{j}-5\hat{k}$ and $3\hat{i}-4\hat{j}-4\hat{k}$ form a right angled triangle.

Solution :

Let $\vec{u} = 2\hat{i}-\hat{j}+\hat{k}$, $\vec{v} = \hat{i}-3\hat{j}-5\hat{k}$, and $\vec{w} = 3\hat{i}-4\hat{j}-4\hat{k}$.

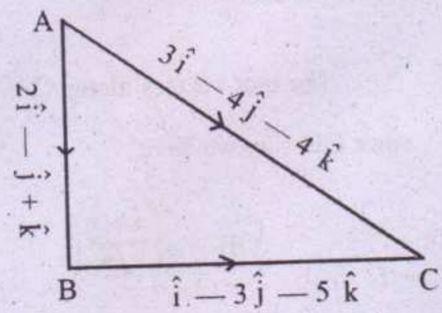
Then

$\vec{u}+\vec{\upsilon} = 3\hat{i}-4\hat{j}-4\hat{k} = \vec{w}$, which shows that $\vec{u}$, $\vec{\upsilon}$, $\vec{w}$ satisfy the triangle law of addition and hence form the sides $\overline{AB}, \overline{BC}, \overline{AC}$ of a triangle respectively.

$AB^2 = 6$, $BC^2 = 35$

And, $AC^2 = 41 = AB^2 + BC^2$

$\Rightarrow m\angle ABC = 90^o$.



[Fig 12.10]

EXERCISE – 12 (a)

1. Each question given below has four possible answers out of which only one is correct. Choose the correct one.

(i) If $\vec{a} = \hat{i} + 2\hat{j} + \hat{k}$, $\vec{b} = 2\hat{i} - 2\hat{j} + 2\hat{k}$ and $\vec{c} = -\hat{i} + 2\hat{j} + \hat{k}$ then

(a) $\vec{a}$ and $\vec{b}$ have the same direction

(b) $\vec{a}$ and $\vec{c}$ have opposite directions

(c) $\vec{b}$ and $\vec{c}$ have opposite directions

(d) no pair of vectors have same direction

(ii) If the vectors $\vec{a} = 2\hat{i} + 3\hat{j} - 6\hat{k}$ and $\vec{b} = \alpha\hat{i} - \hat{j} + 2\hat{k}$ are aarallel, then $\alpha =$ ——

(a) 2 (b) $\frac{2}{3}$ (c) $-\frac{2}{3}$ (d) $\frac{1}{3}$

(iii) If the position vectors of two points A and B are $3\hat{i} + \hat{k}$ and $2\hat{i} + \hat{j} - \hat{k}$, then the vector $\overrightarrow{BA}$ is

(a) $-\hat{i} + \hat{j} - 2\hat{k}$ (b) $\hat{i} + \hat{j}$ (c) $\hat{i} - \hat{j} + 2\hat{k}$ (d) $\hat{i} - \hat{j} - 2\hat{k}$

(iv) If $|k\vec{a}| = 1$, then

(a) $\vec{a} = \frac{1}{k}$ (b) $\vec{a} = \frac{1}{|k|}$ (c) $k = \frac{1}{|\vec{a}|}$ (d) $k = \frac{\pm 1}{|\vec{a}|}$

(v) The directoion cosines of the vector $\overrightarrow{PQ}$ where $\overrightarrow{OP} = (1, 0, -2)$ and $\overrightarrow{OQ} = (3, -2, 0)$ are

(a) 2, –2, 2 (b) 4, –2, –2 (c) $\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$ (d) $\frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}$

2. Rectify the mistakes, if any

(i) $\vec{a} - \vec{a} = 0$.

(ii) The vector $\vec{0}$ has unique direction.

(iii) All unit vectors are equal.

(iv) $|\vec{a}| = |\vec{b}| \Rightarrow \vec{a} = \vec{b}$

(v) Subtraction of vectors is not commutative.

3. (i) If $\vec{a} = (2, 1)$, $\vec{b} = (-1, 0)$, find $3\vec{a} + 2\vec{b}$.

(ii) If $\vec{a} = (1, 1, 1)$, $\vec{b} = (-1, 3, 0)$ and $\vec{c} = (2, 0, 2)$, find $\vec{a} + 2\vec{b} - \frac{1}{2}\vec{c}$

4. If A, B, C and D are the vertices of a square, find $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DA}$.

5. The given points A, B, C are the vertices of a triangle. Determine the vectors $\overrightarrow{AB} + \overrightarrow{BC}$ and $\overrightarrow{CA}$ and the lengths of these vectors in the following cases.
   (i) A (4, 5, 5), B (3, 3, 3), C (1, 2, 5)
   (ii) A(8, 6, 1), B(2, 0, 1) C(–4, 0, –5)

6. Find the vector from the origin to the mid-point of the vector $\overrightarrow{P_1P_2}$ joining the points $P_1(4, 3)$ and $P_2(8, -5)$.

7. Find the vectors from the origin to the points of trisection the vector $\overrightarrow{P_1P_2}$ joining $P_1(-4, 3)$ and $P_2(5, -12)$.

8. Find the vector from the origin to the point of intersection of the medians of the triangle whose vertices are :
   A(5, 2, 1) B (–4, 7, 0) and C (5, –3, 5)

9. Prove that the sum of all the vectors drawn from the centre of a regular octagon to its vertices is the null vector.

10. Prove that the sum of the vectors represented by the sides of a closed polygon taken in order is a zero vector.

11. (a) Prove that :
   (i) $|\vec{a} + \vec{b}| \le |\vec{a}| + |\vec{b}|$
   State when equality will hold;
   (ii) $|\vec{a} - \vec{b}| \ge |\vec{a}| - |\vec{b}|$.
   (b) What is the geometrical significance of the relation
   $|\vec{a} + \vec{b}| = |\vec{a}| - |\vec{b}|$?

12. Find the magnitude of the vector $\overrightarrow{PQ}$, its scalar components and the component vectors along the co-ordinate axes, if P and Q have the co-ordinates
   (i) P (–1, 3), Q (1, 2)
   (ii) P (–1, –2), Q(–5, –6)
   (iii) P(1, 4, –3), Q(2, –2, –1).

13. In each of the following find the vector $\overrightarrow{PQ}$, its magnitude and direction cosines, if P and Q have co-ordinates :
   (i) P(2, –1, –1), Q(–1, –3, 2);
   (ii) P(3, –1, 7), Q (4, –3, –1).

14. If $\vec{a} = (2, -2, 1)$, $\vec{b} = (2, 3, 6)$ and $\vec{c} = (-1, 0, 2)$ find the magnitude and direction of $\vec{a} - \vec{b} + \overrightarrow{2c}$.

15. Determine the unit vector having the direction of the given vector in each of the following problems:
   (i) $5\hat{i} - 12\hat{j}$ (ii) $2\hat{i} + \hat{j}$
   (iii) $3\hat{i} + 6\hat{j} - \hat{k}$ (iv) $3\hat{i} + \hat{j} - 2\hat{k}$

16. Find the unit vector in the direction of the vector $\vec{r_1} - \vec{r_2}$, where $\vec{r_1} = \hat{i} + 2\hat{j} + \hat{k}$ and $\vec{r_2} = 3\hat{i} + \hat{j} - 5\hat{k}$.

17. Find the unit vector parallel to the sum of the vectors $\vec{a} = 2\hat{i} + 4\hat{j} - 5\hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} + 3\hat{k}$. Also find its direction cosines.

18. If the sum of two unit vectors is a unit vector, show that the magnitude of their difference is $\sqrt{3}$.

19. The position vectors of the points A, B, C and D are $4\hat{i} + 3\hat{j} - \hat{k}$, $5\hat{i} + 2\hat{j} + 2\hat{k}$, $2\hat{i} - 2\hat{j} - 3\hat{k}$ and $4\hat{i} - 4\hat{j} + 3\hat{k}$ respectively. Show that $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are parallel.

20. In each of the following problems, show by vector method that the given points are collinear.
    (i) A(2, 6, 3), B(1, 2, 7) and C(3, 10, –1)
    (ii) P(2, –1, 3), Q(3, –5, 1) and R(–1, 11, 9).

21. Prove that the vectors $2\hat{i} - \hat{j} + \hat{k}$,
    $\hat{i} - 3\hat{j} - 5\hat{k}$, $3\hat{i} - 4\hat{j} - 4\hat{k}$ are the sides of a right angled triangle.

21. Prove by vector method that :
    (a) the medians of a triangle are concurrent;
    (b) the diagonals of a parallelogram bisect each other;
    (c) the line segment joining the mid points of two sides of a triangle is parallel to the third and half of it;
    (d) the lines joining the mid points of consecutive sides of a quadrilateral is a parallelogram;
    (e) in any triangle ABC, the point P being on the side $\overline{BC}$;
    if $\overrightarrow{PQ}$ is the resultant of the vectors $\overrightarrow{AP}$, $\overrightarrow{PB}$ and $\overrightarrow{PC}$, then ABQC is a parallelogram;
    (f) in a parallelogram, the line joining a vertex to the midpoint of an opposite side trisects the other diagonal.

12.9 Product of Vectors :

We have already defined the addition of two vectors and multiplication of a vector by a scalar. There are two types of physical situations in which two vectors occur in multiplicative combination. (even though ordinary product of two vectors is not possible). In one, the combination is a scalar, while it is a vector in the other case. The former is called **scalar product** (or **dot product**) and the latter is called **vector product** (or **cross product**).

12.10 Scalar Product (Dot Product) :

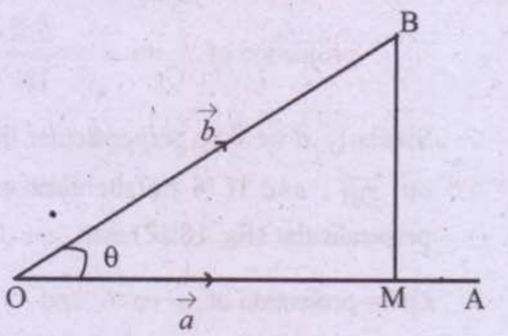
Let, $\vec{a} = \overrightarrow{OA}$ and $\vec{b} = \overrightarrow{OB}$ be two vectors and θ be the inclination between them.

Note that the indination between the vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$ is the same as the inclination between the rays $\overrightarrow{OA}$ and $\overrightarrow{OB}$.

Define

$\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta$ ... (1)

Then the right hand side being a scalar, the product is the scalar product (or dot product) of $\vec{a}$ and $\vec{b}$.



[Fig 12.11]

From definition,

$\vec{b} \cdot \vec{a} = |\vec{b}||\vec{a}| \cos\theta = |\vec{a}||\vec{b}| \cos\theta = \vec{a} \cdot \vec{b}$

This shows that the dot product is commutative.

Further, from definition, we get

$$\cos\theta = \frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}$$

and hence,

$$\theta = \cos^{-1}\left(\frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}\right) \qquad \text{.... (2)}$$

If $\vec{a}$ and $\vec{b}$ are perpendicular, then

$$\theta = 90^o \Leftrightarrow \vec{a} \cdot \vec{b} = 0$$

Hence, two vectors are perpendicular iff their dot product is zero.

This gives $\hat{i} \cdot \hat{j} = \hat{j} \cdot \hat{k} = \hat{k} \cdot \hat{i} = 0$

Further, $\vec{a} \cdot \vec{a} = |\vec{a}|\ |\vec{a}| \cos 0 = |\vec{a}|^2 = a^2$ (The magnitude of a vector $\vec{a}$ i.e. $|\vec{a}|$ is also written as a.)

$$\Rightarrow \hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1 \text{ (as } \hat{i}, \hat{j}, \hat{k} \text{ are unit vectors.)}$$

12.11 Geometrical Meaning of dot product :

In, fig. 16.11 we see that

$OM = OB \cos\theta = |\vec{b}| \cos\theta$

$= \text{projection of } \vec{b} \text{ on } \vec{a}.$

Hence,

$\vec{a} \cdot \vec{b} = |\vec{a}| (|\vec{b}| \cos\theta)$

$= \text{modulus of } \vec{a} \times \text{projection of } \vec{b} \text{ on } \vec{a}.$

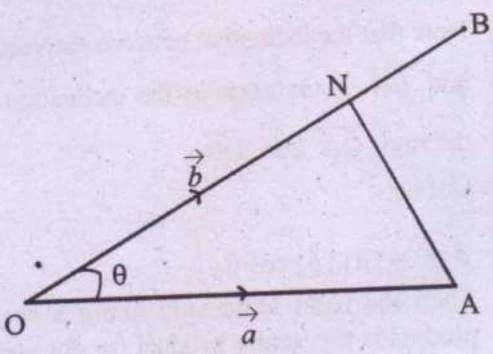
which gives,

$$\text{projection of } \vec{b} \text{ on } \vec{a} = \frac{\vec{a}.\vec{b}}{|\vec{a}|}.$$

Similarly, if we drop perpendicular from A on $\overrightarrow{OB}$, and if N be the foot of the perpendicular (fig. 16.12) then,

$ON = \text{projection of } \vec{a} \text{ on } \vec{b}$ and

$ON = OA \cos\theta = |\vec{a}| \cos\theta.$



[Fig 12.12]

Now,

$\vec{a}\,.\,\vec{b} = |\vec{b}|\,(|\vec{a}|\cos\theta)$ = magnitude of $\vec{b}$ × projection of $\vec{a}$ on $\vec{b}$

which gives, projection of $\vec{a}$ on $\vec{b} = \dfrac{\vec{a}.\vec{b}}{|\vec{b}|}$.

Hence, we have the following :

(1) The dot product of two vectors is equal to magnitude of one vector multiplied by the projection of the other vector on it.

(2) The (scalar) projection of one vector on another

$$= \frac{\text{their dot product}}{\text{magnitude of the vector on which projection is taken}}.$$

Also, from fig. 16.11 vector projection of $\vec{b}$ on $\vec{a}$ is

$$\overrightarrow{OM} = (|\overrightarrow{OM}|)\,\hat{a} = \left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}\right)\frac{\vec{a}}{|\vec{a}|} = \left(\frac{\vec{b}.\vec{a}}{|\vec{a}|^2}\right)\vec{a}$$

and from fig. 16.12 vector projection of $\vec{a}$ on $\vec{b}$ is

$$\overrightarrow{ON} = (|\overrightarrow{ON}|)\,\hat{b} = \left(\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}\right)\frac{\vec{b}}{|\vec{b}|} = \left(\frac{\vec{a}.\vec{b}}{|\vec{b}|^2}\right)\vec{b}$$

12.12 Distributive Law for scalar Product :

Theorem : Let $\vec{a}, \vec{b}, \vec{c}$ be three non-collinear vectors.
Then

$$\vec{a}\,.\,(\vec{b}+\vec{c}) = \vec{a}\,.\,\vec{b} + \vec{a}\,.\,\vec{c}$$

(This is the distributive law for scalar product.)

Proof :

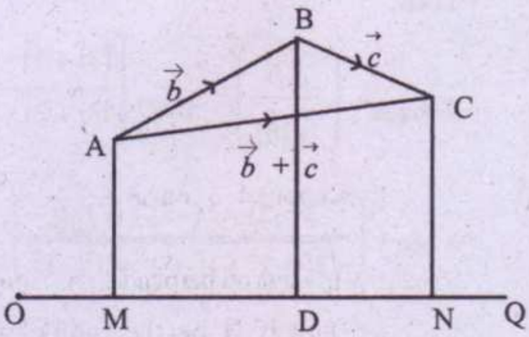
Let $\vec{a} = \overrightarrow{OQ}$

$\vec{b} = \overrightarrow{AB}$

$\vec{c} = \overrightarrow{BC}$

Then $\overrightarrow{AC} = \vec{b} + \vec{c}$

Now



[Fig 12.13]

$\vec{a}\,.\,(\vec{b}+\vec{c})$

$= |\vec{a}|$ (projection of $\vec{b} + \vec{c} = \overrightarrow{AC}$ on $\vec{a}$)

$= |\vec{a}|\,.\,MN = |\vec{a}|\,(MD + DN)$ (Fig. 16.13)

$= |\vec{a}|\,.\,MD + |\vec{a}|\,.\,DN$

$= |\vec{a}|$ (projection $\vec{b}$ on $\vec{a}$) $+ |\vec{a}|$ (projection of $\vec{c}$ on $\vec{a}$)

$= \vec{a}\,.\,\vec{b} + \vec{a}\,.\,\vec{c}$.

Corollary 1 It can be proved similarly that $\vec{a}\,.\,(\vec{b}-\vec{c}) = \vec{a}\,.\,\vec{b} - \vec{a}\,.\,\vec{c}$.

Corollary 2 If $\vec{a} = a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$

and

$$\vec{b} = b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$$

then applying distributive law,

$$\vec{a}\,.\,\vec{b} = (a_1\hat{i}+a_2\hat{j}+a_3\hat{k})\,.\,(b_1\hat{i}+b_2\hat{j}+b_3\hat{k})$$

$$= a_1b_1(\hat{i}.\hat{i})+a_2b_2(\hat{j}.\hat{j})+a_3b_3(\hat{k}.\hat{k})$$

(since $\hat{i}\,.\,\hat{j} = \hat{j}\,.\,\hat{k} = \hat{k}\,.\,\hat{i} = 0$)

$$= a_1b_1 + a_2b_2 + a_3b_3 \,.\, (\text{as } \hat{i}\,.\,\hat{i} = \hat{j}\,.\,\hat{j} = \hat{k}\,.\,\hat{k} = 1)$$

Corollary 3 Let θ be measure of the angle between $\vec{a}$ and $\vec{b}$, then

$$\cos\theta = \frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|} = \frac{a_1b_1+a_2b_2+a_3b_3}{\sqrt{a_1^2+a_2^2+a_3^2}\sqrt{b_1^2+b_2^2+b_3^2}}$$

So, $$\theta = \cos^{-1}\left(\frac{a_1b_1+a_2b_2+a_3b_3}{\sqrt{a_1^2+a_2^2+a_3^2}\sqrt{b_1^2+b_2^2+b_3^2}}\right)$$

Corollary 4 It can also be proved that dot product is associative with respect to scalar multiplication i.e. if α is a scalar and $\vec{a}$, $\vec{b}$ are two vectors, then $\alpha\,(\vec{a}\,.\,\vec{b}) = (\alpha\,\vec{a}\,.\,\vec{b}) = \vec{a}\,.(\alpha\,\vec{b})$

Example 8

Find the angle between the vectors

$$\vec{a} = 3\hat{i} + 2\hat{j} - \hat{k} \text{ and } \vec{b} = -2\hat{i} - 3\hat{j} + \hat{k}.$$

Solution :

Let θ measure the angle between $\vec{a}$ and $\vec{b}$

Then

$$\theta = \cos^{-1}\left(\frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}\right) = \cos^{-1}\left[\frac{(3\hat{i}+2\hat{j}-\hat{k}).(-2\hat{i}-3\hat{j}+\hat{k})}{|3\hat{i}+2\hat{j}-\hat{k}|\;|-2\hat{i}-3\hat{j}+\hat{k}|}\right]$$

$$= \cos^{-1}\left(\frac{3(-2)+2(-3)+(-1)1}{\sqrt{3^2+2^2+1^2}\sqrt{2^2+3^2+1^2}}\right)$$

$$= \cos^{-1}\left(\frac{-13}{\sqrt{14}\sqrt{14}}\right) = \cos^{-1}\left\{\frac{-13}{14}\right\}.$$

Example 9

Find the value of x, so that the vectors

$$\vec{a} = x\hat{i} - 3\hat{j} + 5\hat{k} \text{ and } \vec{b} = -x\hat{i} + x\hat{j} + 2\hat{k} \text{ are perpendicular.}$$

Solution :

Vectors $\vec{a}$ and $\vec{b}$ will be perpendicular if

$\vec{a} \cdot \vec{b} = 0$

i.e. if $(x\hat{i} - 3\hat{j} + 5\hat{k}) \cdot (-x\hat{i} + x\hat{j} + 2\hat{k}) = 0$
i.e. if $(x)(-x) + (-3)(x) + (5)(2) = 0$
i.e. if $-x^2 - 3x + 10 = 0$
i.e. if $x^2 + 3x - 10 = 0$
i.e. if $(x - 2)(x + 5) = 0$
i.e. if $x = 2$ or $x = -5$.

Example 10

Find the scalar and vector projections of the vector $2\hat{i} - 3\hat{j} - 6\hat{k}$ on the line joining the points (3, 4, –2) and (5, 6, –3).

Solution :

Let $\vec{u} = 2\hat{i} - 3\hat{j} - 6\hat{k}$ and the vector joining (3, 4, –2) to (5, 6, –3)

be $\vec{v}$. Then $\vec{v} = 2\hat{i} + 2\hat{j} - \hat{k}$

Now, scalar projection of $\vec{u}$ on $\vec{v}$

$$= \left(\frac{\vec{u} \cdot \vec{v}}{|\vec{v}|}\right) = \frac{(2\hat{i} - 3\hat{j} - 6\hat{k}) \cdot (2\hat{i} + 2\hat{j} - \hat{k})}{|2\hat{i} + 2\hat{j} - \hat{k}|}$$

$$= \frac{2 . 2 + (-3) . 2 + (-6) . (-1)}{\sqrt{2^2 + 2^2 + 1}} = \frac{4}{3}$$

and vector projection of $\vec{u}$ on $\vec{v}$.

$$= \left(\frac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2}\right)\vec{v} = \left(\frac{4}{3^2}\right)(2\hat{i} + 2\hat{j} - \hat{k}) = \frac{8}{9}\hat{i} + \frac{8}{9}\hat{j} - \frac{4}{9}\hat{k}.$$

Example 11

Prove by vector method, that in a traingle ABC,

$$\text{Cos A} = \frac{b^2 + c^2 - a^2}{2bc}$$

or $\quad a^2 = b^2 + c^2 - 2bc \text{ Cos A}$

Solution :

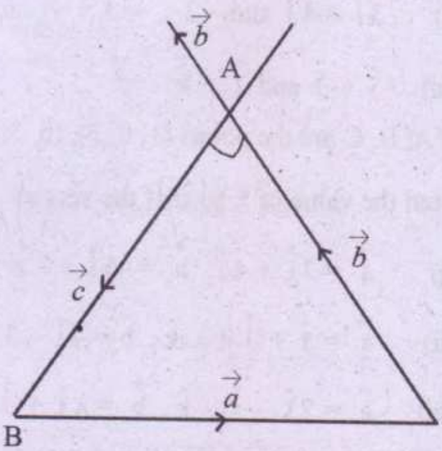
Let ABC be a triangle in which

$\overrightarrow{BC} = \vec{a}$, $\overrightarrow{CA} = \vec{b}$

and $\overrightarrow{AB} = \vec{c}$

then $\overrightarrow{BC} + \overrightarrow{CA} + \overrightarrow{AB} = \vec{0}$

or $\quad \vec{a} + \vec{b} + \vec{c} = \vec{0}$



[Fig 12.14]

or $\quad -\vec{a} = \vec{b} + \vec{c}$

or $\quad (-\vec{a}).(-\vec{a}) = (\vec{b} + \vec{c}).(\vec{b} + \vec{c})$

or $\quad a^2 = b^2 + c^2 + 2\,\vec{b}.\vec{c}$ (using distributive law)

$= b^2 + c^2 + 2\,|\vec{b}||\vec{c}|\cos(\pi - A)$ ... (Fig. 16. 14)

$= b^2 + c^2 - 2bc\cos A.$

EXERCISE 12 (b)

1. Each question given below has four possible answers, out of which only one is correct. Choose thee correct one.

(i) $(2\hat{i} - 4\hat{j}).(\hat{i} + \hat{j} + \hat{k}) =$ —

(a) –3 (b) +2

(c) –1 (d) –2

(ii) If $\vec{a} = \hat{i} + 2\hat{j} - \hat{k}$, $\vec{b} = \hat{i} + \hat{j} + 2\hat{k}$, $\vec{c} = 2\hat{i} - \hat{j}$, then

(a) $\vec{a} \perp \vec{b}$ (b) $\vec{b} \perp \vec{c}$

(c) $\vec{a} \perp \vec{c}$ (d) no pair of vectors are perpendicular

(iii) $(-3, \lambda, 1) \perp (1, 0, -3) \Rightarrow \lambda =$ —

(a) 0 (b) 1

(c) impossible to find (d) any real number

(iv) If $\vec{a}.\vec{b} = \vec{c}.\vec{a}$ for all vectors $\vec{a}$, then

(a) $\vec{a} \perp (\vec{b} - \vec{c})$ (b) $\vec{b} - \vec{c} = \vec{0}$

(c) $\vec{b} \neq \vec{c}$ (d) $\vec{b} + \vec{c} = \vec{0}$

2. Find the scalar product of the following pairs of vectors and the angle between them.

(i) $3\hat{i} - 4\hat{j}$ and $-2\hat{i} + \hat{j}$ (ii) $2\hat{i} - 3\hat{j} + 6\hat{k}$ and $2\hat{i} - 3\hat{j} - 5\hat{k}$

(iii) $\hat{i} - \hat{j}$ and $\hat{j} + \hat{k}$ (iv) $\vec{a} = (2, -2, 1)$ and $\vec{b} = (0, 2, 4)$

3. If A, B, C are the points (1, 0, 2), (0, 3, 1) and (5, 2, 0) arespectively, find m∠ABC.

4. Find the value of λ so that the vectors $\vec{a}$ and $\vec{b}$ are perpendicular to each other.

(i) $\vec{a} = 3\hat{i} + 4\hat{j}$, $\vec{b} = -5\hat{i} + \lambda\hat{j}$

(ii) $\vec{a} = \hat{i} + \hat{j} + \lambda\hat{k}$, $\vec{b} = 4\hat{i} - 3\hat{k}$

(iii) $\vec{a} = 2\hat{i} - \hat{j} - \hat{k}$, $\vec{b} = \lambda\hat{i} + \hat{j} + 5\hat{k}$

(iv) $\vec{a} = (6, 2, -3)$, $\vec{b} = (1, -4, \lambda)$.

5. Find the scalar and vector projections of $\vec{a}$ on $\vec{b}$

(i) $\vec{a} = \hat{i}$ $\vec{b} = \hat{j}$

(ii) $\vec{a} = \hat{i} + \hat{j}$ $\vec{b} = \hat{j} + \hat{k}$

(iii) $\vec{a} = \hat{i} - \hat{j} - \hat{k}$ $\vec{b} = 3\hat{i} + \hat{j} + 3\hat{k}$.

6. In each of the problems given below, find the work done by a force $\vec{F}$ acting on a particle, such that the particle is displaced from a point A to a point B.

[Hints : work done = $\vec{F} \cdot \vec{S}$ where $\vec{S}$ is the displacement.]

(i) $\vec{F} = 4\hat{i} + 2\hat{j} + 3\hat{k}$ A(1, 2, 0), B (2, –1, 3)

(ii) $\vec{F} = 2\hat{i} + \hat{j} - \hat{k}$ A(0, 1, 2), B (–2, 3, 0)

(iii) $\vec{F} = 4\hat{i} - 3\hat{k}$ A(1, 2, 0), B (0, 2, 3)

(iv) $\vec{F} = 3\hat{i} - \hat{j} - 2\hat{k}$ A(–3, –4, 1), B (–1, –1, –2).

7. If $(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = 0$ show that $|\vec{a}| = |\vec{b}|$.

8. (i) If $\vec{a}$ and $\vec{b}$ are perpendicular vectors show that

$(\vec{a} + \vec{b})^2 = (\vec{a} - \vec{b})^2$ [$(\vec{a} + \vec{b})^2$ means $(\vec{a} + \vec{b}) \cdot (\vec{a} + \vec{b})$, so does $(\vec{a} - \vec{b})^2$.]

(ii) Prove that two vectors are perpendicular iff

$|\vec{a} + \vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2$.

9. If $\vec{a}$, $\vec{b}$, $\vec{c}$ are mutually perpendicular vectors of equal magnitude, show that $\vec{a} + \vec{b} + \vec{c}$ is equally inclined to $\vec{a}$, $\vec{b}$, $\vec{c}$.

10. Prove the following by vector method

(i) Altitudes of a triangle are concurrent;

(ii) Median to the base of an isosceles triangle is perpendicular to the base;

(iii) The parallelogram whose diagonals are equal is a rectangle;

(iv) The diagonals of a rhombus are at right angles;

(v) An angle inscribed in a semi-circle is right angle;

(vi) In any triangle ABC, a = b cos C + c cos B;

(vii) In a traingle AOB, $m\angle AOB = 90^\circ$. If P and Q are the points of trisection of $\overline{AB}$, prove that

$$OP^2 + OQ^2 = \frac{5}{9} AB^2;$$

(viii) Measure of the angle between two diagonals of a cube is $\cos^{-1}\frac{1}{3}$.

12.13 Vector Product (Cross Porduct)

Let θ be the inclination between the vectors $\vec{a}$ an $\vec{b}$.
Define

$$\vec{a} \times \vec{b} = \left(|\vec{a}|\,|\vec{b}| \sin\theta\right)\hat{n}$$

where $\hat{n}$ is a unit vector perpendicular to the plane of $\vec{a}$ and $\vec{b}$ and the vectors $\vec{a}$, $\vec{b}$, $\hat{n}$ form a right handed system. (Fig. 16.15). Thus $\vec{a} \times \vec{b}$ is a vector perpendicular to the plane of $\vec{a}$ and $\vec{b}$.

So, a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ is $\dfrac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$.

Now, $\vec{b} \times \vec{a} = |\vec{a}|\ |\vec{b}| \sin\theta\,(-\hat{n})$ ($\vec{a}$, $\vec{b}$, $\hat{n}$ form a right handed system $\Rightarrow$ $\vec{b}$, $\vec{a}$, $-\hat{n}$ form a right handed system.)

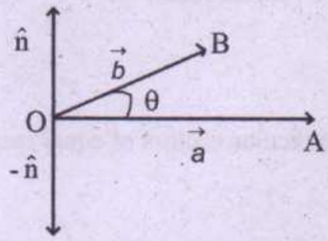
$$= -\vec{a} \times \vec{b}$$

Thus $\vec{a} \times \vec{b}$ and $\vec{b} \times \vec{a}$ are vectors of same magnitude but opposite in direction.

If $\vec{a}$ and $\vec{b}$ are perpendicular to each other, $\theta = 90^\circ$ and hence,

$$\vec{a} \times \vec{b} = \left(|\vec{a}|\ |\vec{b}|\right)\hat{n}$$

If $\vec{a}$ and $\vec{b}$ are unit vectors perpendicular to each other, then, $\vec{a} \times \vec{b} = \hat{n}$.



[Fig 12.15]

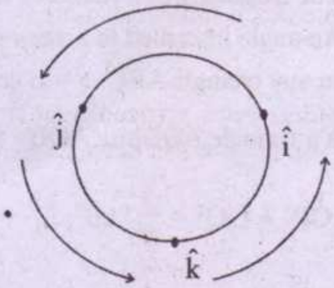
From the above discussion we conclude that cross product of two vectors is not commutative and cross product of two vectors is a vector perpendicular to both the vectors.

Also = $\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = \vec{0}$

and, $\hat{i} \times \hat{j} = \hat{k} = -\hat{j} \times \hat{i}$

$\hat{j} \times \hat{k} = \hat{i} = -\hat{k} \times \hat{j}$

$\hat{k} \times \hat{i} = \hat{j} = -\hat{i} \times \hat{k}$.



[Fig 12.16]

Properties of Vector Product :

(i) It is non-commutative. (already proved)

(ii) It is associative with respect to scalar,

i.e. $\alpha\left(\vec{a} \times \vec{b}\right) = \left(\alpha\vec{a}\right) \times \vec{b} = \vec{a} \times \left(\alpha\vec{b}\right)$.

If $\alpha > 0$, then

$$\alpha\left(\vec{a}\times\vec{b}\right)=\alpha\,|\vec{a}|\;|\vec{b}|\;\sin\theta\,\hat{n}=|\alpha\,\vec{a}|\;|\vec{b}|\;\sin\theta\,\hat{n}=|\vec{a}||\alpha\,\vec{b}|\sin\theta\,\hat{n}\text{; so}$$

$$\alpha\left(\vec{a}\times\vec{b}\right)=\left(\alpha\,\vec{a}\times\vec{b}\right)=\left(\alpha\,\vec{a}\right)\times\vec{b}=\vec{a}\times\left(\alpha\,\vec{b}\right)$$

Similarly it can be proved for $\alpha < 0$ by taking $\alpha = -\beta$, $\beta > 0$.

(iii) Vector product is distributive.

i.e. $\vec{a}\times\left(\vec{b}+\vec{c}\right)=\vec{a}\times\vec{b}+\vec{a}\times\vec{c}$

(Proof omitted).

Let $\vec{a} = a_1\,\hat{i} + a_2\,\hat{j} + a_3\,\hat{k}$

and $\vec{b} = b_1\,\hat{i} + b_2\,\hat{j} + b_3\,\hat{k}$

Then using the distributive law for vector product,

$$\vec{a}\times\vec{b} = (a_1\,\hat{i} + a_2\,\hat{j} + a_3\,\hat{k})\times(b_1\,\hat{i} + b_2\,\hat{j} + b_3\,\hat{k})$$

$$= a_1b_2(\hat{i}\times\hat{j}) + a_1b_3\,(\hat{i}\times\hat{k}) + a_2b_1\,(\hat{j}\times\hat{i}) + a_2b_3\,(\hat{j}\times\hat{k}) + a_3b_1\,(\hat{k}\times\hat{i}) + a_3b_2\,(\hat{k}\times\hat{j})$$

$$= (a_1b_2)\,\hat{k} - (a_1b_3)\,\hat{j} - (a_2b_1)\,\hat{k} + (a_2b_3)\,\hat{i} + (a_3b_1)\,\hat{j} - (a_3b_2)\,\hat{i}$$

$$= (a_2b_3 - a_3b_2)\,\hat{i} + (a_3b_1 - a_1b_3)\,\hat{j} + (a_1b_2 - b_1a_2)\,\hat{k}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

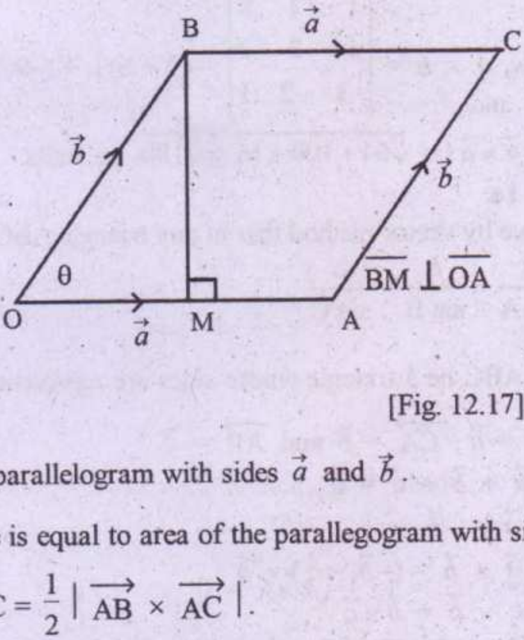
Geometrical meaning of Cross product :

Let $\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{OB} = \vec{b}$

Then $\vec{a}\times\vec{b}=\left(|\vec{a}|\;|\vec{b}|\sin\theta\right)\hat{n}$

$=|\vec{a}|\left(|\vec{b}|\sin\theta\right)\hat{n}$

$=|\vec{a}|\,\text{BM}\;\hat{n}$ (Fig. 16.17)



[Fig. 12.17]

$\Rightarrow |\vec{a}\times\vec{b}| = |\vec{a}|\,\text{BM}$ = area of the parallelogram with sides $\vec{a}$ and $\vec{b}$.

Hence, $\vec{a}\times\vec{b}$ is a vector whose magnitude is equal to area of the parallegogram with sides $\vec{a}$ and $\vec{b}$. From this it follows that area of a Δ ABC $= \frac{1}{2}\,|\,\overrightarrow{AB}\times\overrightarrow{AC}\,|$.

Example 12 :

Find a unit vector perpendicular to each of the vectors $2\,\hat{i} - \hat{j} + \hat{k}$ and $3\,\hat{i} + 4\,\hat{j} - \hat{k}$. Find the sine of angle between the two vectors.

Solution :

Let $\vec{a} = 2\,\hat{i} - \hat{j} + \hat{k}$,

$\vec{b} = 3\,\hat{i} + 4\,\hat{j} - \hat{k}$

Then $\frac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$ is the unit vector perpendicular to each of these vectors.

Now, $\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & -1 & 1\\ 3 & 4 & -1\end{vmatrix}=(1-4)\,\hat{i}+(3+2)\,\hat{j}+(8+3)\hat{k}$

$=-3\hat{i}+5\hat{j}+11\hat{k}$

$\therefore |\vec{a}\times\vec{b}|=\sqrt{9+25+121}=\sqrt{155}$

So, unit vector perpendicular to $\vec{a}$ and $\vec{b}$, is

$$\frac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}=\frac{-3\hat{i}+5\hat{j}+11\hat{k}}{\sqrt{155}}=\frac{-3}{\sqrt{155}}\hat{i}+\frac{5}{\sqrt{155}}\hat{j}+\frac{11}{\sqrt{155}}\hat{k}.$$

From definition of vector product,

$$\vec{a}\times\vec{b}=|\vec{a}||\vec{b}|\sin\theta\ \hat{n}$$

$$\Rightarrow \sin\theta=\frac{|\vec{a}\times\vec{b}|}{|\vec{a}|\times|\vec{b}|}=\frac{\sqrt{155}}{\sqrt{6}\sqrt{26}}=\sqrt{\frac{155}{156}}.$$

Example 13

Obtain the area of the parallelogram whose sides are vectors $\hat{i}+2\hat{j}+3\hat{k}$ and $-3\hat{i}-2\hat{j}+\hat{k}$.

Solution :

Let $\vec{a}=\hat{i}+2\hat{j}+3\hat{k}$

$\vec{b}=-3\hat{i}-2\hat{j}+\hat{k}$

$|\vec{a}\times\vec{b}|$ = area of the parallelogram whose sides are $\vec{a}$ and $\vec{b}$.

Now, $\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 2 & 3\\ -3 & -2 & 1\end{vmatrix}=(2+6)\hat{i}+(-9-1)\hat{j}+(-2+6)\hat{k}=8\hat{i}-10\hat{j}+4\hat{k}$

$\Rightarrow |\vec{a}\times\vec{b}|=\sqrt{64+100+16}=\sqrt{180}$ sq. units.

Example 14

Prove by vector method that in any triangle ABC,

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$$

Solution :

Let ABC be a triangle whose sides are represented by vectors (Fig 16.18).

$\overrightarrow{BC}=\vec{a}$, $\overrightarrow{CA}=\vec{b}$ and $\overrightarrow{AB}=\vec{c}$

$\Rightarrow \vec{a}+\vec{b}+\vec{c}=\vec{0}$

Or, $\vec{a}=-\vec{b}-\vec{c}$

Or, $\vec{a}\times\vec{b}=(-\vec{b}-\vec{c})\times\vec{b}$

$=-\vec{c}\times\vec{b}=\vec{b}\times\vec{c}$

Similialy, $\vec{b}\times\vec{c}=\vec{c}\times\vec{a}$.

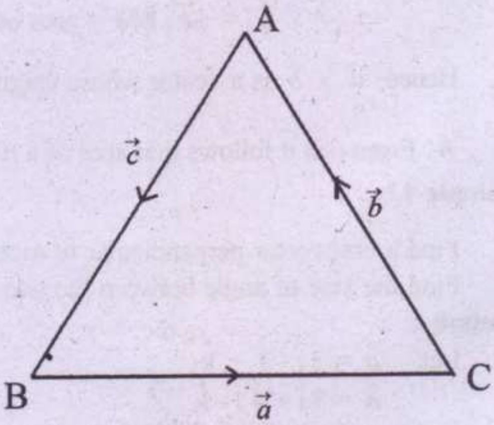
So, area of Δ ABC $=\frac{1}{2}|\overrightarrow{AB}\times\overrightarrow{BC}|=\frac{1}{2}|\vec{c}\times\vec{a}|$

$=\frac{1}{2}|\vec{a}\times\vec{b}|=\frac{1}{2}|\vec{b}\times\vec{c}|$

$\Rightarrow |\vec{a}\times\vec{b}|=|\vec{b}\times\vec{c}|=|\vec{c}\times\vec{a}|$

i.e. $ab\sin C=bc\sin A=ca\sin B$

$$\Rightarrow \frac{\sin C}{c}=\frac{\sin A}{a}=\frac{\sin B}{b}\Rightarrow \frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}.$$



[Fig 12.18]

Example 15

Find the area of the Δ ABC whose vertices are A (1, 2, 3), B (–1, –1, 0) and C (1, –1, 0).

Solution :

We have

$\overrightarrow{AB} = (-1-1)\,\hat{i} + (-1-2)\,\hat{j} + (0-3)\,\hat{k}$

$= -2\hat{i} - 3\hat{j} - 3\hat{k}$

and $\overrightarrow{AC} = (1-1)\hat{i} + (-1-2)\,\hat{j} + (0-3)\,\hat{k}$

$= 0\hat{i} - 3\hat{j} - 3\hat{k}$

$\therefore$ Area of $\Delta = \frac{1}{2}\,|\overrightarrow{AB} \times \overrightarrow{AC}|$

$$= \left|\frac{1}{2}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & -3 & -3 \\ 0 & -3 & -3 \end{vmatrix}\right| = \frac{1}{2}|-6\hat{j} + 6\hat{k}|$$

$$= \frac{1}{2}\sqrt{72} = 3\sqrt{2} \text{ sq. units.}$$

Example 16

Show that the vector area of the triangle whose vertices have position vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ is $\frac{1}{2}\,(\vec{a}\times\vec{b} + \vec{b}\times\vec{c} + \vec{c}\times\vec{a})$

Solution :

Let A B C be the triangle, whose vertices A, B, C have position vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ respectively.

Then $\overrightarrow{AB} = (\vec{b} - \vec{a})$ and $\overrightarrow{AC} = (\vec{c} - \vec{a})$

and vector area of Δ ABC $= \frac{1}{2}[\overrightarrow{AB} \times \overrightarrow{AC}]$

$$= \frac{1}{2}\left[\left(\vec{b}-\vec{a}\right)\times\left(\vec{c}-\vec{a}\right)\right]$$

$$= \frac{1}{2}\left[\vec{b}\times\vec{c} - \vec{b}\times\vec{a} - \vec{a}\times\vec{c} + \vec{a}\times\vec{a}\right]$$

$$= \frac{1}{2}\left[\vec{b}\times\vec{c} + \vec{a}\times\vec{b} + \vec{c}\times\vec{a}\right].$$

EXERCISE 12 (c)

1. Each question given below has four possible answers out of which only one is correct. Choose the correct one.

(i) $(\hat{i} + \hat{k}) \times (\hat{i} + \hat{j} + \hat{k}) =$ ——

(a) $\hat{i} - \hat{k}$ (b) $\hat{k} - \hat{i}$ (c) $\hat{k} - 2\hat{i} - \hat{j}$ (d) 2

(ii) A vector perpendicular to the vectors $\hat{i} + \hat{j}$ and $\hat{i} + \hat{k}$ is ——

(a) $\hat{i} - \hat{j} - \hat{k}$ (b) $\hat{j} - \hat{k} + \hat{i}$ (c) $\hat{k} - \hat{j} - \hat{i}$ (d) $\hat{j} + \hat{k} + \hat{i}$

(iii) The area of the triangle with vertices (1, 0, 0), (0, 1, 0) and (0, 0, 1) is ——

(a) $\frac{1}{2}$ (b) 1 (c) $\frac{\sqrt{3}}{2}$ (d) 2

(iv) If $\hat{a}$ and $\hat{b}$ are unit vectors such that $\hat{a} \times \hat{b}$ is a unit vector, then the angle between $\hat{a}$ and $\hat{b}$ is —

(a) of any measure (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{2}$ (d) π

(v) If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are non-zero vectors, then

$\vec{a} \times \vec{b} = \vec{a} \times \vec{c} \Leftrightarrow$ ——

(a) $\vec{b} = \vec{c}$ (b) $\vec{a} \parallel (\vec{b} - \vec{c})$ (c) $\vec{b} \parallel \vec{c}$ (d) $\vec{b} \perp \vec{c}$.

2. Let $\vec{a} = 2\hat{i} + \hat{j}$, $\vec{b} = -\hat{i} + 3\hat{j} + \hat{k}$ and $\vec{c} = \hat{i} + 2\hat{j} + 5\hat{k}$ be three vectors. Find

(i) $\vec{c} \times \vec{a}$ (ii) $\vec{a} \times (-\vec{b})$ (iii) $(\vec{a} - 2\vec{b}) \times \vec{c}$

(iv) $(\vec{a} - \vec{c}) \times \vec{c}$ (v) $(\vec{a} - \vec{b}) \times (\vec{c} - \vec{a})$.

3. Find the unit vector perpendicular to the vectors

(i) $\hat{i}, \hat{k}$ (ii) $\hat{i} + \hat{j}, \hat{i} - \hat{k}$

(iii) $2\hat{i} + 3\hat{k}, \hat{i} - 2\hat{j}$ (iv) $2\hat{i} - 3\hat{j} + \hat{k}, -\hat{i} + 2\hat{j} - \hat{k}$.

4. Determine the area of the parallelogram whose adjacent sides are the vectors

(i) $2\hat{i}, \hat{j}$ (ii) $\hat{i} + \hat{j}, -\hat{i} + 2\hat{j}$

(iii) $2\hat{i} + \hat{j} + 3\hat{k}, \hat{i} - \hat{j}$ (iv) (1, –3, 1), (1, 1, 1).

5. Calculate the area of the triangle ABC (by vector method) where

(i) A (1, 2, 4), B (3, 1, –2), C (4, 3, 1)

(ii) A (1, 1, 2), B (2, 2, 3), C (3, –1, –1)

6. Determine the sine of the angle between the vectors

(i) $5\hat{i} - 3\hat{j}, 3\hat{i} - 2\hat{k}$ (ii) $\hat{i} - 3\hat{j} + \hat{k}, \hat{i} + \hat{j} + \hat{k}$

7. Show that $(\vec{a} \times \vec{b})^2 = a^2b^2 - (\vec{a} . \vec{b})^2$

8. If $\vec{a} \times \vec{b} = \vec{b} \times \vec{c} \neq 0$, prove that

$\vec{a} + \vec{c} = m\vec{b}$, where m is scalar.

9. If $\vec{a} = 2\hat{i} + \hat{j} - \hat{k}$, $\vec{b} = -\hat{i} + 2\hat{j} - 4\hat{k}$,

$\vec{c} = \hat{i} + \hat{j} + \hat{k}$, find $(\vec{a} \times \vec{b}) \cdot (\vec{a} \times \vec{c})$

10. If $\vec{a} = 3\hat{i} + \hat{j} + 2\hat{k}$, $\vec{b} = 2\hat{i} - 3\hat{j} + 4\hat{k}$, then verify that $\vec{a} \times \vec{b}$ is perpendicular to both $\vec{a}$ and $\vec{b}$.

11. Fin the area of the parallelogram whose diagonals are vectors $3\hat{i} + \hat{j} - 2\hat{k}$ and $\hat{i} - 3\hat{j} + 4\hat{k}$.

12. Show that $(\vec{a} - \vec{b}) \times (\vec{a} + \vec{b}) = 2(\vec{a} \times \vec{b})$.

Interpret this result geometrically.

12.14 Scalar and Vector Triple Products.

The product of three vectors with the help of '.' and 'x' is called triple product of vectors. If the product is a scalar it a called a **scalar triple product** and if it is a vector, it is called a **vector triple product.**

The products $\vec{a}\,.\,(\vec{b}\times\vec{c})$ and $(\vec{a}\times\vec{b})\,.\,\vec{c}$ are scalar triple products whereas $(\vec{a}\times\vec{b})\times\vec{c}$ and $\vec{a}\times(\vec{b}\times\vec{c})$ are vector triple products.

The products $(\vec{a}\,.\,\vec{b})\,.\,\vec{c}$ and $\vec{a}\times(\vec{b}\,.\,\vec{c})$ are not defined. (Why ?)

The scalar triple product $\vec{a}\,.\,(\vec{b}\times\vec{c})$ is denoted by $[\vec{a}\ \vec{b}\ \vec{c}\,]$.

Scalar Triple Product :

Let $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$, $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$ and $\vec{c}=c_1\hat{i}+c_2\hat{j}+c_3\hat{k}$.

$$\therefore\ \vec{b}\times\vec{c}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}=(b_2c_3-c_2b_3)\ \hat{i}+(b_3c_1-c_3b_1)\ \hat{j}+(b_1c_2-c_1b_2)\ \hat{k}$$

Now $\vec{a}\,.\,(\vec{b}\times\vec{c})=(a_1\hat{i}+a_2\hat{j}+a_3\hat{k})\,.\,[(b_2c_3-c_2b_3)\ \hat{i}+(b_3c_1-c_3b_1)\ \hat{j}+(b_1c_2-c_1b_2)\ \hat{k}]$

$$=\begin{vmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}$$

Again, $(\vec{a}\times\vec{b}).\vec{c}=\vec{c}\,.\,(\vec{a}\times\vec{b})$ (by commutativity of dot product)

$$=\begin{vmatrix}c_1 & c_2 & c_3\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}=-\begin{vmatrix}a_1 & a_2 & a_3\\ c_1 & c_2 & c_3\\ b_1 & b_2 & b_3\end{vmatrix}=\begin{vmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}=\vec{a}\,.\,(\vec{b}\times\vec{c})$$

Hence $[\vec{a}\ \vec{b}\ \vec{c}\,]=[\vec{c}\ \vec{a}\ \vec{b}\,]$.

Similarly it can be shown that $[\vec{a}\ \vec{b}\ \vec{c}\,]=[\vec{b}\ \vec{c}\ \vec{a}\,]$.

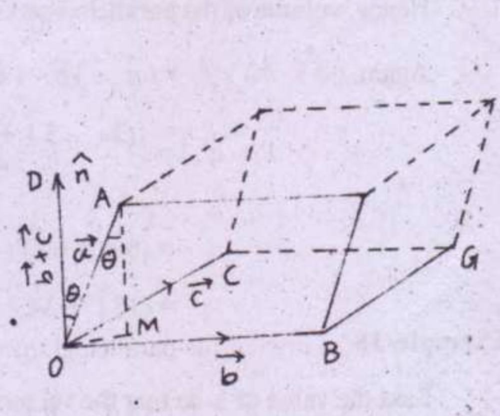
$\therefore\ [\vec{a}\ \vec{b}\ \vec{c}\,]=[\vec{b}\ \vec{c}\ \vec{a}\,]=[\vec{c}\ \vec{a}\ \vec{b}\,]$.

[**Note :** In a scalar triple product the position of '.' and 'x' can be interchanged.]

Geometrical Meaning of Scalar Triple product :

Let $\vec{a}$, $\vec{b}$, $\vec{c}$ be three no-coplanar vectors.

Construct a parallelopiped with sides $\vec{a}$, $\vec{b}$ and $\vec{c}$ (Fig 16.19) such that $\overrightarrow{OA}=\vec{a}$, $\overrightarrow{OB}=\vec{b}$ and $\overrightarrow{OC}=\vec{c}$. Then the vector $\vec{b}\times\vec{c}$, which is perpendicular to the plane of $\vec{b}$ and $\vec{c}$, is along $\overrightarrow{OD}$. Let θ be measure of the angle between $\vec{a}$ and $(\vec{b}\times\vec{c})$. Now, by definition of scalar product,



[Fig. 12.19]

$\vec{a}.(\vec{b}\times\vec{c}) = |\vec{a}|\,(|\vec{b}\times\vec{c}|\cos\theta)$

$= |\vec{b}\times\vec{c}|\,(|\vec{a}|\cos\theta)$

$= |\vec{b}\times\vec{c}|$ AM

= area of the parallelogram OCGB × AM

= area of the base × hight of the parallelopiped

= volume of the parallelopiped.

Hence, $\vec{a}.(\vec{b}\times\vec{c})$ = volume of the parallelopiped with sides $\vec{a}$, $\vec{b}$ and $\vec{c}$.

Note : 1. (i) $[\vec{a}\ \vec{b}\ \vec{c}] = 0$, iff the vectors are coplanar.

(ii) $[\vec{a}\ \vec{b}\ \vec{c}] = 0$, if any two vectors are eiter parallel or equal.

Vector Triple Product :

If $\vec{a}, \vec{b}, \vec{c}$ be three vectors, their vector triple products are given by

$$\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c}$$

and $(\vec{a}\times\vec{b})\times\vec{c} = (\vec{a}.\vec{c})\vec{b} - (\vec{b}.\vec{c})\vec{a}$

Example 17

Obtain the volume of the parallelopiped whose sides are vectors $\vec{a} = 2\hat{i} - 3\hat{j} + 4\hat{k}$,

$\vec{b} = \hat{i} + 2\hat{j} - \hat{k}$, $\vec{c} = 3\hat{i} - \hat{j} + 2\hat{k}$. Also find the vector $(\vec{a}\times\vec{b})\times\vec{c}$.

Solution :

The volume of thee parallelopiped with sides $\vec{a}$, $\vec{b}$, $\vec{c}$ is $\vec{a}.(\vec{b}\times\vec{c})$.

$$\text{Now, } \vec{b}\times\vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -1 \\ 3 & -1 & 2 \end{vmatrix} = (4-1)\hat{i} + (-3-2)\hat{j} + (-1-6)\hat{k} = 3\hat{i} - 5\hat{j} - 7\hat{k}$$

So, $\vec{a}.(\vec{b}\times\vec{c}) = (2\hat{i} - 3\hat{j} + 4\hat{k}).(3\hat{i} - 5\hat{j} - 7\hat{k})$

$= 6 + 15 - 28 = -7$

Hence, volume of the parallelopiped is 7 cubic units.

Again, $(\vec{a}\times\vec{b})\times\vec{c} = (\vec{a}.\vec{c})\vec{b} - (\vec{b}.\vec{c})\vec{a}$

$= \{(2\hat{i} - 3\hat{j} + 4\hat{k}).(3\hat{i} - \hat{j} + 2\hat{k})\}(\hat{i} + 2\hat{j} - \hat{k})$

$- \{(\hat{i} + 2\hat{j} - \hat{k}).(3\hat{i} - \hat{j} + 2\hat{k})\}(2\hat{i} - 3\hat{j} + 4\hat{k})$

$= (6+3+8)(\hat{i} + 2\hat{j} - \hat{k}) - (3-2-2)(2\hat{i} - 3\hat{j} + 4\hat{k})$

$= (17\hat{i} + 34\hat{j} - 17\hat{k}) - (-1)(2\hat{i} - 3\hat{j} + 4\hat{k}) = 19\hat{i} + 31\hat{j} - 13\hat{k}$

Example 18

Find the value of λ so that the vectors $\hat{i} - \hat{j} + \hat{k}$, $2\hat{i} + \hat{j} - \hat{k}$ and $\lambda\hat{i} - \hat{j} + \lambda\hat{k}$ are coplanar.

Solution :

We know that the the above three vectors will be coplanar iff their scalar triple product is zero i.e. if $\begin{vmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \\ \lambda & -1 & \lambda \end{vmatrix} = 0$

i.e. $(\lambda - 1)(1) + (-1)(-\lambda - 2\lambda) + (1)(-2 - \lambda) = 0$

i.e. $\lambda - 1 + 3\lambda - 2 - \lambda = 0$

i.e. $3\lambda = 3$ or $\lambda = 1$.

Example 19

Prove that the four points with position vectors $4\hat{i} + 5\hat{j} + \hat{k}, -\hat{j} - \hat{k}, 3\hat{i} + 9\hat{j} + 4\hat{k}$ and $-4\hat{i} + 4\hat{j} + 4\hat{k}$ are coplanar.

Solution :

Let the given points be A, B, C, D respectively. The four points will be coplanar, if vectors $\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{AD}$ are coplanar i.e. $\vec{0}\ .(\overrightarrow{AC} \times \overrightarrow{AD}) = 0$

Now, $\overrightarrow{AB} = (-\hat{j} - \hat{k}) - (4\hat{i} + 5\hat{j} + \hat{k}) = -4\hat{i} - 6\hat{j} - 2\hat{k}$

$\overrightarrow{AC} = (3\hat{i} + 9\hat{j} + 4\hat{k}) - (4\hat{i} + 5\hat{j} + \hat{k}) = -\hat{i} + 4\hat{j} + 3\hat{k}$

$\overrightarrow{AD} = (-4\hat{i} + 4\hat{j} + 4\hat{k}) - (4\hat{i} + 5\hat{j} + \hat{k}) = -8\hat{i} - \hat{j} + 3\hat{k}$

Now, $\overrightarrow{AB}.(\overrightarrow{AC} \times \overrightarrow{AD}) = \begin{vmatrix} -4 & -6 & -2 \\ -1 & 4 & 3 \\ -8 & -1 & 3 \end{vmatrix} = -4(12 + 3) + (-6)(-24 + 3) + (-2)(1 + 32)$

$= -60 + 126 - 66 = 0$

$\therefore$ Points A, B, C are coplanar.

EXERCISE 12 (d)

1. Each question given below has four possible answers out of which only one is correct. Choose the correct one.

(i) $\vec{a}.\vec{b} \times \vec{a} =$ ——

(a) $\vec{0}$ (b) 0 (c) 1 (d) $\vec{a}^2\ \vec{b}$

(ii) $(-\vec{a}).\ \vec{b} \times (-\vec{c}) =$ ——

(a) $\vec{a} \times \vec{b}.\vec{c}$ (b) $-\vec{a}.(\vec{b} \times \vec{c})$ (c) $\vec{a} \times \vec{c}.\vec{b}$ (d) $\vec{a}.(\vec{c} \times \vec{b})$

(iii) For the non-zero vectors $\vec{a}, \vec{b}$ and $\vec{c}$, $\vec{a}.(\vec{b} \times \vec{c}) = 0$ if

(a) $\vec{b} \perp \vec{c}$ (b) $\vec{a} \perp \vec{b}$ (c) $\vec{a} \parallel \vec{c}$ (d) $\vec{a} \perp \vec{c}$.

2. Find the scalar triple product $\vec{b}.(\vec{c} \times \vec{a})$ where $\vec{a}, \vec{b}$ and $\vec{c}$ are respectively

(i) $\hat{i} + \hat{j}, \hat{i} - \hat{j}, 5\hat{i} + 2\hat{j} + 3\hat{k}$

(ii) $5\hat{i} - \hat{j} + 4\hat{k}, 2\hat{i} + 3\hat{j} + 5\hat{k}, 5\hat{i} - 2\hat{j} + 6\hat{k}$

3. Find the volume of the parallelopiped whose sides given by the vectors

(i) $\hat{i} + \hat{j} + \hat{k}, \hat{k}, 3\hat{i} - \hat{j} + 2\hat{k}$

(ii) (1, 0, 0), (0, 1, 0), (0, 0, 1).

4. Show that the following vectors ar co-planar.

(i) $\hat{i} - 2\hat{j} + 2\hat{k}, 3\hat{i} + 4\hat{j} + 5\hat{k}, -2\hat{i} + 4\hat{j} - 4\hat{k}$

(ii) $\hat{i} + 2\hat{j} + 3\hat{k}, -2\hat{i} - 4\hat{j} + 5\hat{k}, 3\hat{i} + 6\hat{j} + \hat{k}$

5. Find the value of λ so that the three vectors are co-planar.

(i) $\hat{i} + 2\hat{j} + 3\hat{k}, 4\hat{i} + \hat{j} + \lambda\hat{k}$ and $\lambda\hat{i} - 4\hat{j} + \hat{k}$

(ii) (2, –1, 1), (1, 2, –3) and (3, λ, 5)

6. If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are mutually perpendicular, show that $\left[\vec{a}.(\vec{b} \times \vec{c})\right]^2 = a^2b^2c^2$

7. Show that $[\vec{a}+\vec{b} \;\; \vec{b}+\vec{c} \;\; \vec{c}+\vec{a}] = 2[\vec{a} \; \vec{b} \; \vec{c}]$

8. Prove that $[\vec{a}\times\vec{b} \;\; \vec{b}\times\vec{c} \;\; \vec{c}\times\vec{a}] = [\vec{a} \; \vec{b} \; \vec{c}]^2$.

9. For $\vec{a} = \hat{i} + \hat{j}, \vec{b} = -\hat{i} + 2\hat{k}, \vec{c} = \hat{j} + \hat{k}$,

obtain $\vec{a} \times (\vec{b} \times \vec{c})$ and also verify the formula $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c}$.

10. Prove that $\vec{a} \times (\vec{b} \times \vec{c}) + \vec{b} \times (\vec{c} \times \vec{a}) + \vec{c} \times (\vec{a} \times \vec{b}) = \vec{0}$ and hence prove that $\vec{a} \times (\vec{b} \times \vec{c})$, $\vec{b} \times (\vec{c} \times \vec{a})$, $\vec{c} \times (\vec{a} \times \vec{b})$ are coplanar.

11. If $\hat{a}$, $\hat{b}$, $\hat{c}$ be unit vectors and $\hat{a} \times (\hat{b} \times \hat{c}) = \frac{1}{2}\hat{b}$, find the angles that $\hat{a}$ makes with $\hat{b}$ and $\hat{c}$, where $\hat{b}$, $\hat{c}$ are not parallel.

Additional Exercises

1. Prove that the sum of the vectors directed from the vertices to the mid points of opposite sides of a triangle is zero.

2. Prove by vector method that the diagonals of a quadrilateral bisect each other iff it is a parallelogram.

3. If G is the centroid of a triangle ABC, prove that $\overrightarrow{GA} + \overrightarrow{GB} + \overrightarrow{GC} = \vec{0}$.

4. If M is the midpoint of the side $\overline{BC}$ of a triangle ABC, prove that $\overrightarrow{AB} + \overrightarrow{AC} = 2\overrightarrow{AM}$

5. If $\vec{a}$ and $\vec{b}$ are the vectors represented by the adjacent sides of a regular hexagon, taken in order, what are the vectors represented by the other sides taken in order ?

6. If the points with position vectors $10\hat{i}+3\hat{j}$, $12\hat{i}-5\hat{j}$ and $a\hat{i}+11\hat{j}$ are collinear, find the value of *a*.

7. Prove that the four points with position vectors $2\vec{a}+3\vec{b}-\vec{c}$, $\vec{a}-2\vec{b}+3\vec{c}$, $3\vec{a}+4\vec{b}-2\vec{c}$ and $\vec{a}-6\vec{b}+6\vec{c}$ are coplanar.

8. For any vector $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$, prove that $\vec{r}=(\vec{r}\cdot\hat{i})\hat{i}+(\vec{r}\cdot\hat{j})\hat{j}+(\vec{r}\cdot\hat{k})\hat{k}$

9. If two vectors $\vec{a}$ and $\vec{b}$ are such that $|\vec{a}|=3$, $|\vec{b}|=2$ and $\vec{a}\cdot\vec{b}=6$, find $|\vec{a}+\vec{b}|$ and $|\vec{a}-\vec{b}|$.

10. If $\vec{a}$ makes equal angles with $\hat{i}$, $\hat{j}$ and $\hat{k}$ and has magnitude 3, prove that the angle between $\vec{a}$ and each of $\hat{i}$, $\hat{j}$ and $\hat{k}$ is $\cos^{-1}(\frac{1}{\sqrt{3}})$

11. If $\vec{a}$, $\vec{b}$, $\vec{c}$ are such that $\vec{a}\cdot\vec{b}=\vec{a}\cdot\vec{c}$ then show that $\vec{a}=\vec{0}$ or $\vec{b}=\vec{c}$ or $\vec{a}$ is perpendicular to $\vec{b}-\vec{c}$.

12. If $\vec{a}+\vec{b}+\vec{c}=\vec{0}$, $|\vec{a}|=3$, $|\vec{b}|=5$ and $|\vec{c}|=7$, find the angle between $\vec{a}$ and $\vec{b}$.

13. If $\vec{a}$, $\vec{b}$, $\vec{c}$ are unit vectors such that $\vec{a}+\vec{b}+\vec{c}=\vec{0}$, find the value of $\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}$.

14. Find the angles which the vector $\vec{a}=\hat{i}-\hat{j}+\sqrt{2}\hat{k}$ makes with the coordinate axes.

15. Find the angle between $\vec{a}$ and $\vec{b}$ if $|\vec{a}\times\vec{b}|=\vec{a}\cdot\vec{b}$

CHAPTER - 13

Three Dimensional Geometry

Analytic geometry, far more than any of his metaphysical speculations, immortalized the name of Descartes and constitutes the greatest single step ever made in the progress of exact sciences.

- John Stuart Mill

Preliminary Ideas

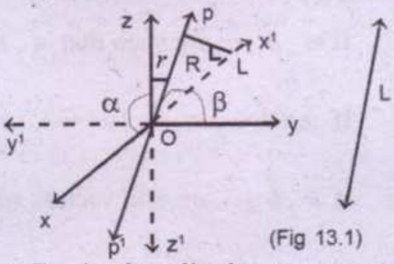
13.0 Inclination between two rays with a common vertex

Let $\overrightarrow{OA}$ and $\overrightarrow{OB}$ be two rays with common vertex O. We define :

1. If $\overrightarrow{OA}$ and $\overrightarrow{OB}$ are coincident rays then the inclination between them in zero.
2. If $\overrightarrow{OA}$ and $\overrightarrow{OB}$ are opposite rays i.e. $\overleftrightarrow{AB}$ is a line then inclination between them is π i.e. 180^o.
3. If $\overrightarrow{OA}$ and $\overrightarrow{OB}$ are noncollinear rays (i.e. neither coincident nor opposite) then the inclination between them is $\theta = m\angle AOB$.

Direction cosines and Direction ratios of a line in space.

Let L be a line in space. Consider a ray R paralled to L with vortex at orign. (R can be taken as either $\overrightarrow{OP}$ or $\overrightarrow{OP'}$) Let α, β and γ be the inclinations between the ray R and $\overleftrightarrow{OX}$, $\overleftrightarrow{OY}$ and $\overleftrightarrow{OZ}$ respectively. Then we define the direction cosines of L as $\cos\alpha$, $\cos\beta$ and $\cos\gamma$.



(Fig 13.1)

Notes :

1. Usually direction cosines (written as d.cs) of a line are denoted as $< l, m, n >$. For the above line $l = \cos\alpha$, $m = \cos\beta$, $n = \cos\gamma$.
2. In the definition of the d.cs of L the ray R can be either $\overrightarrow{OP}$ or $\overrightarrow{OP'}$. Therefore if $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are the d.cs of L, then $\cos(\pi-\alpha)$, $\cos(\pi-\beta)$, $\cos(\pi-\gamma)$ can also be considered as d.cs. of L. The two sets of d.cs correspond to the two opposite directions of a line L.
3. The d.cs of the ray $\overrightarrow{OP}$ are $\cos\alpha$, $\cos\beta$, $\cos\gamma$ and of the ray $\overrightarrow{OP'}$ are $\cos(\pi-\alpha)$, $\cos(\pi-\beta)$, $\cos(\pi-\gamma)$. The d.c's of a segment are same as those of the line containing it.
4. If θ is an angle measure, then $0 < \theta < \pi$
   But for an inclination θ , $0 \le \theta \le \pi$.

A property of Direction cosines.

Let O be the origin and direction cosines of $\overrightarrow{OP}$ be l, m, n. If OP = r and P has coordinates (x, y, z), then **x = lr, y = mr, z = nr.**

Prof : Let L be the foot of the perpendicular from P on x-axis. (see figure 13.1)

$$l = \cos\alpha = -\frac{OL}{OP} = \frac{x}{OP} = \frac{x}{r}$$

(By definition of space coordinates of P, x = –OL in this case, as L falls on $\overrightarrow{OX'}$)

$\therefore x = lr$

Similarly it can be shown that y = mr, z = nr.

Corollary : If l, m, n are direction cosines of a line then

$$\boxed{l^2+m^2+n^2=1}$$

Proof : Let $\overrightarrow{OP}$ be paralled to L where O is origin (Fig. 15.10). If P has coordinates (x, y, z) and the d.c.s. of $\overrightarrow{OP}$ are l, m, n then they are also d.c.s of L.

We have x = lr, y = mr, z = nr, where r = OP.

By the distance formula.

$r^2 = OP^2 = x^2 + y^2 + z^2 = (l^2 + m^2 + n^2) r^2$ ....(i)

Since $\overrightarrow{OP}$ is a ray, P is different from O so that $OP = r \neq 0$. Therefore it follows from (i) that $l^2 + m^2 + n^2 = 1$.

Direction Ratios :

Let l, m, n be the direction cosines of a line L such that none of the d.cs is zero.

If a, b, c are non zero real numbers such that $\frac{a}{l} = \frac{b}{m} = \frac{c}{n}$ then *a*, b, c are called the direction ratios of the line L.

Usually direction ratios are abbreviated as d.rs.

Exceptional cases :

1. If one of the d.cs of a line L, say $l = 0$ and $m \neq 0$, $n \neq 0$ then d.rs of L are given by (o, b, c) where $\frac{b}{m} = \frac{c}{n}$; b and c are nonzero real numbers.

   Similarly d.rs corresponding to d.cs (l, o, n) $(l \neq o, n \neq o)$ are given by (*a*, o, c) such that $\frac{a}{l} = \frac{c}{n}$ $(a \neq o, c \neq o)$ and d.rs corresponding to d.cs (l, m, o) $(l \neq o, m \neq o)$ are given by (*a*, b, o) such that $\frac{a}{l} = \frac{b}{m}$; where *a*, b are nonzero real numbers.

2. If two of the d.cs are zero e.g. $l = m = o$ and $n \neq o$ then obviously $n = \pm 1$ and the d.rs are given by (0, 0, c); $c \in R$, $c \neq 0$.

   Direction ratios in the other cases can be similarly decided.

N.B.: (i) **The case of all the d.c.s or d.rs being zero does not arise (Why ?)**

(ii) **As in case of d.cs, d.rs *a*, b, c of a line are also written as <a, b, c>.**

(iii) **A set of direction cosines of a line can be regarded as its direction ratios, but not the converse. A triple of real numbers *a*, b, c can be direction cosines of a line if and only if $a^2 + b^2 + c^2 = 1$.**

(iv) **Direction ratios of parallel lines are proportional.**

Finding direction cosiness from direction ratios :

If *a*, b, c are direction ratios of a line then its direction cosines are given by

$$l = \frac{a}{\pm\sqrt{a^2+b^2+c^2}},\ m = \frac{b}{\pm\sqrt{a^2+b^2+c^2}},\ n = \frac{c}{\pm\sqrt{a^2+b^2+c^2}}$$

Proof : If any of the direction ratios is zero, then the corresponding direction cosine is also zero. Therefore we consider the case when none of *a*, b, c is zero.

Let l, m, n be the d.c.s of the line. Obviously each of l, m, n is nonzero.

We have $\frac{a}{l} = \frac{b}{m} = \frac{c}{n} = \pm\frac{\sqrt{a^2+b^2+c^2}}{\sqrt{l^2+m^2+n^2}} = \pm\sqrt{a^2+b^2+c^2}$

$$\Rightarrow l = \frac{a}{\pm\sqrt{a^2+b^2+c^2}}, m = \frac{b}{\pm\sqrt{a^2+b^2+c^2}}, n = \frac{c}{\pm\sqrt{a^2+b^2+c^2}}$$

The signs of the d.cs are determined according to the position of the line with respect to the coordinate axes.

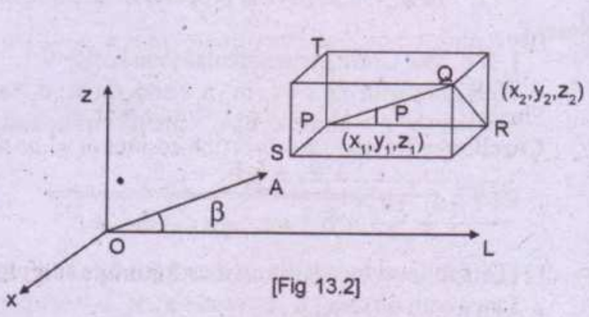
N.B: The formula for d.cs from d.rs holds good in the exceptional cases considered in the definition of direction ratios.

For example d.cs corresponding to $< a, b, 0 >$, $a \neq 0$ Or $b \neq 0$ are given by

$$< \frac{a}{\pm\sqrt{a^2+b^2}}, \frac{b}{\pm\sqrt{a^2+b^2}}, 0 >.$$

13.1 Direction ratios of the line joining two points :

By passing planes through P and Q parallel to the coordinate planes a cuboid can be formed as shown in figure 13.2. If α β, γ be the inclinations of $\overrightarrow{PQ}$ with $\overrightarrow{PS}$, $\overrightarrow{PR}$ and $\overrightarrow{PT}$, then the inclinations of a ray $\overrightarrow{OA}$ parallel to $\overrightarrow{PQ}$ (as shown in figure) with $\overrightarrow{OX}$, $\overrightarrow{OY}$, and $\overrightarrow{OZ}$ are also α, β and γ (Ref. Fact-8, Art.13.3 on properties of paralles lines in space discussed in elements of mathematics (EOM) for previous class)



[Fig 13.2]

3. Since parallel lines have same direction cosines, if follows from definition of direction ratios that lines with direction ratios

$<a_1, b_1, c_1>$ and $<a_2, b_2, c_2>$ are parallel if and only if

$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$ ....(A)

Here, of course, we assume that none of the d.rs is zero. However, in case of zero direction ratio (A) holds between the corresponding non zero direction ratios. For example, lines with d.rs $<a, 0, b>$ and $<a_1, 0, b_1>$ are parallel if

$\frac{a}{a_1} = \frac{b}{b_1}$

13.3 Inclination between two lines :

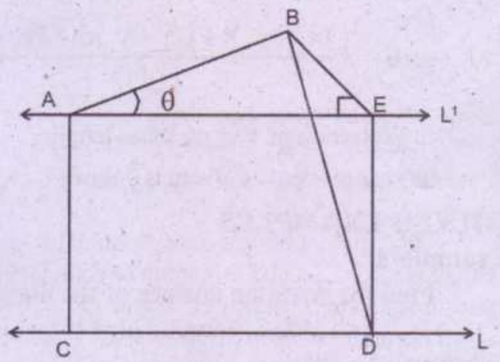
Definition : Let L_1 and L_2 be two lines in space.

1. If L_1 and L_2 are parallel or coincident then the inclination between them is zero.

2. If L_1 and L_2 are intersecting or skew then their inclination is the measure of angle between them.

Length of projection of a segment on a line

Let $\overline{AB}$ be a segment and L, a line such that θ is the inclination between L and $\overleftrightarrow{AB}$. When L and $\overleftrightarrow{AB}$ are neither parallel, nor perpendicular we take θ to be the measure of the acute angle between them. The length of the projection of $\overline{AB}$ on L is ABcosθ.



[Fig. 13.4]

Proof : If $\overleftrightarrow{AB}$ and L are intersecting the length of the projection of $\overline{AB}$ on L can be easily seen to be

ABcosθ. So suppose that $\overleftrightarrow{AB}$ and L are skew and the inclination between them is θ such that $0 < \theta < \frac{\pi}{2}$. Let L′ be a line passing through A such that L′ || L. Suppose C, E and D are respectively the feet of the perpendiculars from A on L, B on L′ and E on L.

Since L′ is parallel to L, L and L′ are coplanar. ($\overleftrightarrow{AB}$ and L being skew, $\overline{AB}$ does not lie in the plane of L and L′).

It now follows from plane-geometry that $\overleftrightarrow{AE} \perp \overleftrightarrow{ED}$,

$\overleftrightarrow{AE} \perp \overleftrightarrow{BE}$ (Construction).

∴ $\overleftrightarrow{AE}$ is perpendicular to the plane of $\overleftrightarrow{BE}$ and $\overleftrightarrow{ED}$ (Fact-1)

But $\overleftrightarrow{CD}$ (i.e. L) is parallel to $\overleftrightarrow{AE}$ (i.e. L′). Therefore $\overleftrightarrow{CD}$ is perpendicular to the plane of $\overleftrightarrow{BE}$ and $\overleftrightarrow{ED}$. (Fact-6, Art-13.3, EOM-prev. class)

Hence $\overleftrightarrow{CD} \perp \overleftrightarrow{BD}$ (Definition of perpendicular to a plane)
Thus D is the foot of the perpendicular from B on L.

So, by definition of projection of a segment on a line, $\overline{CD}$ is the projection of $\overline{AB}$ on L.

Now consider the triangle ABE. Since ∠AEB is a right angle (∵ $\overline{BE} \perp$ L′), AE = AB cosθ.

But in the rectangle ACDE, CD = AE.

∴ CD = length of projection of $\overline{AB}$ on L = AE = ABcosθ. □

Notes :

1. If $\overleftrightarrow{AB}$ || L, then θ = 0 and hence it can be easily seen that projection of $\overline{AB}$ on L has length AB = ABcosθ.

2. If $\overline{AB} \perp L$, it can be shown that length of the projection of $\overline{AB}$ on L is zero i.e. ABcosθ ($\because \theta = \frac{\pi}{2}$ in this case).

Corollary : The length of the projection of $\overline{PQ}$ joining $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ on a line L with direction cosines $<l, m, n>$ is given by $|l(x_2-x_1)+m(y_2-y_1)+n(z_2-z_1)|$

Proof : $\overleftrightarrow{PQ}$ has d.rs $<x_2-x_1, y_2-y_1, z_2-z_1>$

So d.cs of $\overleftrightarrow{PQ}$ are $\pm\frac{x_2-x_1}{PQ}, \pm\frac{y_2-y_1}{PQ}, \pm\frac{z_2-z_1}{PQ}$

$\therefore$ Measure θ of the angle between $\overleftrightarrow{PQ}$ and L is given by

$$\cos\theta = \pm\frac{(x_2-x_1)l+(y_2-y_1)m+(z_2-z_1)n}{PQ}$$

$\therefore$ Projection of $\overline{PQ}$ on L has length

$|PQ\cos\theta| = |(x_2-x_1)l+(y_2-y_1)m+(z_2-z_1)n|$.

SOLVED EXAMPLES :

Example 1

Find the direction cosines of the line which is perpendicular to the lines whose direction ratios are $<1, -2, 3>$ and $<2, 2, 1>$.

Solution :

Let l, m, n, be the direction cosines of the line perpendicular to the given lines. Then we have,

$l.1 + m.(-2) + n.3 = 0$ and $l.2 + m.2 + n.1 = 0$

By cross-multiplication, we get,

$$\frac{l}{-2-6} = \frac{m}{6-1} = \frac{n}{2+4}$$

or, $$\frac{l}{-8} = \frac{m}{5} = \frac{n}{6}$$

$$\therefore l = \frac{-8}{5\sqrt{5}}, m = \frac{1}{\sqrt{5}}, n = \frac{6}{5\sqrt{5}}$$

Example 2

Prove that the two lines whose direction cosines are connected by the equations $l + 2m + 3n = 0$, $3lm - 4ln + mn = 0$, are perpendicular to each other.

Solution :

The two given equations are

$l + 2m + 3n = 0$ ...(1)

$3lm - 4ln + mn = 0$ ...(2)

From equation (1), we get $l = -(2m + 3n)$ and hence from (2)

we get, by eliminating l,

$-3(2m + 3n)m + 4(2m + 3n)n + nm = 0$

or, $-6m^2 + 12n^2 = 0$ or, $m = \pm\sqrt{2}\,n$ and hence,

$$l = -(2m + 3n) = -\left(3 \pm 2\sqrt{2}\right)n$$

so, the direction ratios of the two lines are

$$<-\left(3\pm2\sqrt{2}\right), \pm\sqrt{2}, 1>$$

i.e. the two given lines have the direction ratios,

$<-(3+2\sqrt{2}), \sqrt{2}, 1>$ and $<(-3+2\sqrt{2}), -\sqrt{2}, 1>$.

If these be taken as a_1, b_1, c_1 and a_2, b_2, c_2 respectively, then we have $a_1a_2 + b_1 b_2 + c_1 c_2$

$= -(3+2\sqrt{2})(2\sqrt{2}-3) + (-\sqrt{2})\sqrt{2} + 1 = -1(8-9) - 2 + 1 = 0$

Hence the two lines are perpendicular to each other.

Example 3

Find the co-ordinates of the foot of the perpendicular drawn from the point A (1, 3, 4) to the line joining the points B (3, 0, –1) and C (0, 1, –2).

Solution :

Let the foot of the perpendicular D, drawn from the point A on the line segment, $\overline{BC}$ divide $\overline{BC}$ in a ratio k : 1.

Then the co-ordinates of D are $\left(\frac{3}{k+1}, \frac{k}{k+1}, \frac{-2k-1}{k+1}\right)$

Hence the d.rs of $\overleftrightarrow{AD}$ are $\left\langle\frac{2-k}{k+1}, \frac{-2k-3}{k+1}, -\frac{6k+5}{k+1}\right\rangle$. Now the d.rs of $\overleftrightarrow{BC}$ are <3, –1, 1>. Since $\overleftrightarrow{BC}$ is perpendicular to $\overleftrightarrow{AD}$, we have,

$\left(\frac{2-k}{k+1}\right)(3)+\left(\frac{-2k-3}{k+1}\right)(-1)-\left(\frac{6k+5}{k+1}\right)(1)=0$ or $k = \frac{4}{7}$

Hence, co-ordinates of D are $\left(\frac{21}{11}, \frac{4}{11}, \frac{-15}{11}\right)$

Example 4

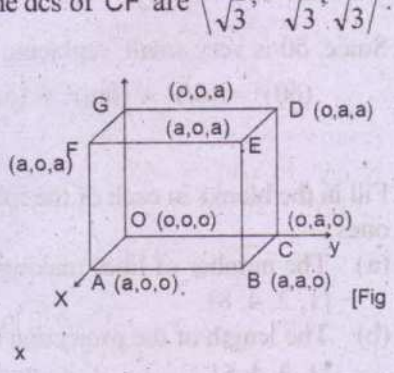
A line makes angles α, β γ, δ with the four main diagonals of a cube.

Prove that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = 4/3$.

Solution :

Let the cube be of side 'a'. Then the coordinates of the vertices of the cube are as shown in Fig. (15.14) Now the main diagonals of the cube are $\overline{OE}, \overline{AD}, \overline{BG}$ and $\overline{CF}$. The d.rs of $\overline{OE}$ are <a, a, a> and hence dc's of $\overline{OE}$ are $\left\langle\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right\rangle$. Similarly, the dcs of $\overline{AD}$ are

$\left\langle-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right\rangle$. The dcs of $\overline{BG}$ are $\left\langle-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right\rangle$ and the dcs of $\overline{CF}$ are $\left\langle\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right\rangle$. If the line L makes angles α, β, γ, δ with $\overline{OE}, \overline{AD}, \overline{BG}$, and $\overline{CF}$ respectively and <*l, m, n*> be the *dcs* of L, then,



[Fig 13.5]

$\cos\alpha = \frac{l+m+n}{\sqrt{3}}$, $\cos\beta = \frac{-l+m+n}{\sqrt{3}}$

$\cos\gamma = \frac{-l-m+n}{\sqrt{3}}$, and $\cos\delta = \frac{l-m+n}{\sqrt{3}}$

Hence,

$\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta$

$= \frac{(l+m+n)^2}{3} + \frac{(-l+m+n)^2}{3} + \frac{(-l-m+n)^2}{3} + \frac{(l-m+n)^2}{3} = \frac{4}{3}(l^2+m^2+n^2) = \frac{4}{3}$

Example 5

If $l_1, m_1, n_1; l_2, m_2, n_2; l_3, m_3, n_3$ be the direction cosines of the three mutually perpendicular lines, then prove that the straight line L with d.cs $\frac{l_1+l_2+l_3}{\sqrt{3}}, \frac{m_1+m_2+m_3}{\sqrt{3}}, \frac{n_1+n_2+n_3}{\sqrt{3}}$ makes equal angles with each of them.

Solution :

Since the three lines are mutually perpendicular,

$l_1l_2 + m_1m_2 + n_1 n_2 = 0$

$l_1l_3 + m_1m_3 + n_1 n_3 = 0$

$l_3l_2 + m_3m_2 + n_3 n_2 = 0$

Also $l_1^2 + m_1^2 + n_1^2 = l_2^2 + m_2^2 + n_2^2 = l_3^2 + m_3^2 + n_3^2 = 1$

If α, be the angle between the line L and the straight line having dcs $<l_1, m_1, n_1>$ we have

$$\cos\alpha = l_1\frac{l_1+l_2+l_3}{\sqrt{3}} + m_1\frac{m_1+m_2+m_3}{\sqrt{3}} + n_1\frac{n_1+n_2+n_3}{\sqrt{3}}$$

$$= \frac{l_1^2+m_1^2+n_1^2}{\sqrt{3}} = \frac{1}{\sqrt{3}}$$

Similarly, if β and γ are the angles which L makes with lines having d.c's $< l_2, m_2, n_2>$ and $< l_3, m_3, n_3>$ respectively then $\cos\beta = \frac{1}{\sqrt{3}}$ and $\cos\gamma = \frac{1}{\sqrt{3}}$. Hence, $\alpha = \beta = \gamma$.

Example 6

The direction cosines of a straight line in two neighbouring positions are $< l, m, n >$ and $<l + \delta l, m$ $\delta m, n + \delta n>$. If $\delta\theta$ is a small angle between them, then prove that $(\delta\theta)^2 = (\delta l)^2 + (\delta m)^2 + (\delta n)^2$.

Solution :

Since $\delta\theta$ is the angle between them, we have,

$$\cos\delta\theta = l(l + \delta l) + m(m + \delta m) + n(n + \delta n)$$
$$= (l^2 + m^2 + n^2) + (l\delta l + m\delta m + n\delta n)$$
$$= 1 + (l\delta l + m\delta m + n\delta n),$$

which gives, $l\delta l + m\delta m + n\delta n = \cos\delta\theta - 1$

Again, $1 = (l + \delta l)^2 + (m + \delta m)^2 + (n + \delta n)^2$

$$= (l^2 + m^2 + n^2) + (\delta l^2 + \delta m^2 + \delta n^2) + 2(l\delta l + m\delta m + n\delta n)$$
$$= 1 + (\delta l)^2 + (\delta m)^2 + (\delta n)^2 + 2(\cos\delta\theta - 1)$$
$$= 1 + (\delta l)^2 + (\delta m)^2 + (dn)^2 - 4\sin^2\left(\frac{\delta\theta}{2}\right)$$
$$\therefore 4\sin^2\left(\frac{\delta\theta}{2}\right) = (\delta l)^2 + (\delta m)^2 + (\delta n)^2$$

Since, $\delta\theta$ is very small, replacing $\sin\frac{\delta\theta}{2}$ by $\frac{\delta\theta}{2}$, we have,

$$(\delta\theta)^2 = (\delta l)^2 + (\delta m)^2 + (\delta n)^2.$$

EXERCISE – 13(a)

1. Fill in the blanks in each of the following questions by choosing the appropriate answer from the given ones.
   - (a) The number of lines making equal angles with coordinate axes is ____.
     [1, 2, 4, 8]
   - (b) The length of the projection of the line segment joining (1, 3, –1) and (3, 2, 4) on z axis is —.
     [1, 3, 4, 5]
   - (c) If a line is perpendicular to z-axis and makes an angle measuring 60^0 with x-axis, then the angle it makes with y-axis measures ____.
     [$30^0, 60^0, 90^0, 120^0$]

(d) If the distance between the points $(-1, -1, z)$ and $(1, -1, 1)$ is 2 then z = ——.

[1, √2, 2, 0]

2. Which of the following statements are true (T) or false (F) :

(a) The line through (1 – 1, 2) and (–2, –1, 2) is always perpendicular to z - axis.

(b) The line passing through (0, 0, 0) and (1, 2, 3) has direction cosines (–1, –2, –3).

(c) If l, m, n be three real mumbers proportional to the direction cosines of a line L, then $l^2 + m^2 + n^2 = 1$.

(d) If α, β, γ be any three arbitrary angles then $\cos\alpha, \cos\beta, \cos\gamma$ can always be considered as the direction cosines of a line.

(e) If two lines are perpendicular to a third line, then the direction ratios of the two lines are proportional.

3. (a) Show that the points (3, –2, 4), (1, 1, 1) and (–1, 4, –1) are collinear.

(b) Show that points (0, 1, 2), (2, 5, 8), (5, 6, 6) and (3, 2, 0) form a parallelogram.

4. (a) Find the co-ordinates of the foot of the perpendicular from the point (1, 1, 1) on the line joining (1, 4, 6) and (5, 4, 4).

(b) Find the co-ordinates of the point where the perpendicular from the origin meets the line joining the points (–9, 4, 5) and (11, 0, –1).

(c) Prove that the points P (3, 2, –4), Q(5, 4, –6) and R(9, 8, –10) are collinear.

(d) If P (1, y, z) lies on the line through (3, 2, –1) and (–4, 6, 3), find y & z.

5. (a) If A, B, C, D are the points (6, 3, 2), (3, 5, 7), (2, 3, –1) and (3, 5, –3) respectively, then find the projection of $\overline{AB}$ on $\overleftrightarrow{CD}$.

(b) The projections of a line segment $\overline{OP}$, through origin O, on the co-ordinate axes are 6, 2, 3. Find the length of the line segment $\overline{OP}$ and its direction cosines.

(c) The projections of a line segment on x, y and z-axis respectively are 12, 4, 3. Find the length and the direction cosines of the line segment.

6. (a) If A, B C are the points (1, 4, 2), (–2, 1, 2) and (2, –3, 4) respectively then find the angles of the triangle ABC.

(b) Find the acute angle between the lines passing through (–3, –1, 0), (2, –3, 1) and (1, 2, 3), (–1, 4, –2) respectively.

(c) Prove that measure of the angle between two main diagonals of a cube is $\cos^{-1}\frac{1}{3}$.

(d) Prove that measure of the angle betwen the diagonal of a face and the diagonal of a cube, drawn from a vertex is $\cos^{-1}\sqrt{\left(\frac{2}{3}\right)}$.

(e) Find the angle which a diagonal of a cube makes with one of its edges.

(f) Find the angle between the lines whose d.cs l, m, n are connected by the relation, $3l + m + 5n = 0$ and $6mn - 2nl + 5lm = 0$.

7. Show that the measures of the angles between the four diagonals of a rectangular parallelopiped whose edges are a, b, c are

$$\cos^{-1}\left(\frac{a^2 \pm b^2 \pm c^2}{a^2 + b^2 + c^2}\right)$$

8. If l_1, m_1, n_1, and l_2, m_2, n_2 are the direction cosines of two mutually perpendicular lines show that the d.cs of the line perpendicular to both of them are $m_1 n_2 - n_1 m_2,\ n_1 l_2 - l_1 n_2,\ l_1 m_2 - m_1 l_2$.

PLANE

13.4 Vector Equation of a Plane :

Let O be the origin (fixed point) and p be the length of perpendicular drawn from O to the plane. If $\hat{n}$ be the unit vector along $\overrightarrow{ON}$,

then $\overrightarrow{ON} = p\hat{n}$. If P be any point on the plane with position vector $\vec{r}$,

then $\overrightarrow{OP} = \overrightarrow{ON} + \overrightarrow{NP}$

Or, $\vec{r} = p\hat{n} + \overrightarrow{NP}$

Or, $\overrightarrow{NP} = \vec{r} - p\hat{n}$

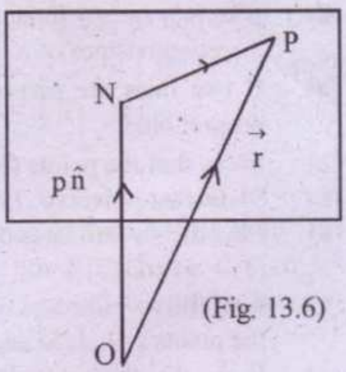
Since $\overrightarrow{NP} \perp \overrightarrow{ON}$, we have

$\overrightarrow{NP} \cdot \overrightarrow{ON} = 0$

Or, $(\vec{r} - p\hat{n}) \cdot p\hat{n} = 0$

Or, $\vec{r} \cdot p\hat{n} = p\hat{n} \cdot p\hat{n} = p^2$

Or, $\vec{r} \cdot \hat{n} = p$



(Fig. 13.6)

which is the equation of the plane whose distance from the fixed point is p and $\hat{n}$ is the unit vector along its normal.

Note : 1. If the plane passes through a given point Q with position vector $\vec{a}$ and is perpendicular to a given vector $\vec{n}$, then for any point P on the plane with position vector $\vec{r}$,

$\overrightarrow{PQ} \perp \vec{n} \Rightarrow (\vec{r} - \vec{a}) \cdot \vec{n} = 0$ Or, $\vec{r} \cdot \vec{n} = \vec{a} \cdot \vec{n}$

In particular, if it passes through the origin, $\vec{a} = \vec{0}$ and so the equation of the plane becomes $\vec{r} \cdot \hat{n} = 0$.

2. If the plane passes through a given point with position vector $\vec{a}$ and is parallel to vectors $\vec{b}$ and $\vec{c}$ then for any point P ($\vec{r}$) on it, ($\vec{r} - \vec{a}$) is perpendicular to $\vec{b} \times \vec{c}$ and hence,

$(\vec{r} - \vec{a}) \cdot (\vec{b} \times \vec{c}) = 0$

Or, $\vec{r} \cdot (\vec{b} \times \vec{c}) = \vec{a} \cdot (\vec{b} \times \vec{c})$

which is an equation of the form $\vec{r} \cdot \vec{n} = q = [\vec{a}\ \vec{b}\ \vec{c}]$, where $\vec{n} = \vec{b} \times \vec{c}$

Hence, length of perpendicular from the origin is $p = \vec{r} \cdot \hat{n} = \dfrac{\vec{r} \cdot \vec{n}}{|\vec{n}|} = \dfrac{[\vec{a}\,\vec{b}\,\vec{c}]}{|\vec{b} \times \vec{c}|}$.

3. Transformation to Cartesian form from vector form

We can transform the vector form of the equation to a plane to cartesian form by taking $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $\hat{n} = l\hat{i} + m\hat{j} + n\hat{k}$

where *l*, m, n are direction cosines of the normal $\hat{n}$, in the vector equation $\vec{r} \cdot \hat{n} = p$ as follows :

$\vec{r} \cdot \hat{n} = p$

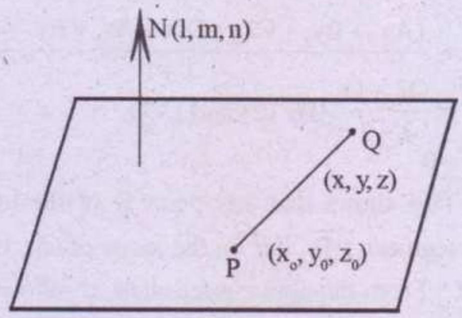
$\Rightarrow (x\hat{i} + y\hat{j} + z\hat{k}) \cdot (l\hat{i} + m\hat{j} + n\hat{k}) = p$

$\Rightarrow lx + my + nz = p$ which becomes the Cartesian form of the equation to the plane.

We shall discuss the Cartesian equation of planes in details in the subsequent articles.

13.5 Cartesian Equation of a Plane :

Let the plane pass through a given point P (x_0, y_0, z_0) and the normal to the plane have direction cosines l, m, n. If Q (x, y, z) be any variable point on the plane (Fig. 13.7) then the line segment $\overline{PQ}$ lies on the plane (Fig 13.7) and hence is perpendicular to the normal to the plane. [Since normal to a plane is perpendicular to every line in that plane.] Now the direction ratios of $\overline{PQ}$ are $<x - x_0, y - z_0, z - z_0>$ and the direction ratios of the normal are l, m, n, (as direction cosines of any line can be considered as direction ratios of itself). Hence by condition of perpendicularity,



[Fig 13.7]

$$\boxed{l(x - x_0) + m(y - y_0) + n(z - z_0) = 0}$$

is the equation of the plane passing through the point (x_0, y_0, z_0) and whose normal has direction cosines l, m, n.

Note : If instead of the direction cosines, a, b, c are the direction ratios of the normal to the plane, then the equation of plane is

$$\boxed{a(x - x_0) + b(y - y_0) + c(z - z_0) = 0}$$

Since it is possible to determine the direction cosines when the direction ratios are given, we usually consider the cases where the direction ratios of the normal are given.

13.6 General Equation of a plane

We have already seen that the equation of the plane passing through a given point (x_0, y_0, z_0) and whose normal has direction cosines l, m, n, is

$$l(x - x_0) + m(y - y_0) + n(z - z_0) = 0$$

Or $lx + my + nz - (lx_0 + my_0 + nz_0) = 0$

which is a first degree equation in x, y and z.

We now proceed to show that the general equation of first degree in x, y, and z always represents a plane.

Consider the most general equation of first degree in x, y, z.

$$Ax + By + Cz + D = 0 \qquad .... (1)$$

Let $P_1 (x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ be two points on the locus of (1) and R be any point on the line sement $\overline{P_1P_2}$. Then R divides $\overline{P_1P_2}$ in a ratio $\lambda : 1$ for some real value of λ Hence the co - ordinates of R are $\left(\frac{\lambda x_2 + x_1}{\lambda + 1}, \frac{\lambda y_2 + y_1}{\lambda + 1}, \frac{\lambda z_2 + z_1}{\lambda + 1}\right)$

Since P_1 and P_2 are on the locus of (1), we have

$$Ax_1 + By_1 + Cz_1 + D = 0 \qquad .... (2)$$

$$Ax_2 + By_2 + Cz_2 + D = 0 \qquad .... (3)$$

Now $A\left(\frac{\lambda x_2 + x_1}{\lambda+1}\right) + B\left(\frac{\lambda y_2 + y_1}{\lambda+1}\right) + C\left(\frac{\lambda z_2 + z_1}{\lambda+1}\right) + D$

$= \frac{A(\lambda x_2 + x_1) + B(\lambda y_2 + y_1) + C(\lambda z_2 + z_1) + D(\lambda+1)}{\lambda+1}$

$= \frac{(Ax_2 + By_2 + CZ_2 + D)\lambda + Ax_1 + By_1 + Cz_1 + D}{\lambda+1}$

$= \frac{O\lambda + O}{\lambda+1}$ [By (2) and (3)]

$= 0$

This shows that any point R of the line segment $\overline{P_1P_2}$ lies on the locus of (1) and hence the line segment $\overline{P_1P_2}$ lies on the locus of (1). Hence the equation Ax + By + Cz + D = 0 represents a plane.

Note 1 : From the above discussion, it follows that we can represent a plane by an equation of first degree in x, y, and z.

Again, if the plane is Ax + By + Cz + D = 0 and $D \neq 0$, then this can further be written as

$$\frac{A}{D}x + \frac{B}{D}y + \frac{C}{D}z + 1 = 0$$

Or $A_1x + B_1y + C_1z + 1 = 0$

This shows that it really contains three independent constants which can be determined by three given conditions.

2: If the plane passes through the origin, the equation becomes

Ax + By + Cz = 0

3: Let a plane be given by

Ax + By + Cz + D = 0

If P (x_1, y_1, z_1) and Q (x_2, y_2, z_2) are two points on the plane, then

$Ax_1 + By_1 + Cz_1 + D = 0$ .... (1)

$Ax_2 + By_2 + Cz_2 + D = 0$ ... (2)

Subtracting (2) from (1) we get

$A(x_1 - x_2) + B(y_1 - y_2) + C(z_1 - z_2) = 0$ ... (3)

Now equation (3) shows that a line with d.rs <A, B, C> is perpendicular to a line with d.rs $<x_2 - x_1, y_2 - y_1, z_2 - z_1>$, i.e. to $\overleftrightarrow{PQ}$. But P and Q being any two points one the plane, we see that, the line with d.rs <A, B, C> is perpendicular to every line lying in the plane and hence, is normal to the plane.

Thus the **direction ratios of the normal to the plane, Ax + By + Cz + D = 0 can be taken as <A, B, C>. (i.e. coefficients of x, y, z respectively).**

13.7 Exquation of plane through three given points.

Let (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) be three given points and the required plane be

Ax + By + Cz + D = 0 .... (1)

Since it passes through (x_1, y_1, z_1) we have

$Ax_1 + By_1 + Cz_1 + D = 0$ .... (2)

Subtracting (2) from (1), we get

$A(x - x_1) + B(y - y_1) + C(z - z_1) = 0$ .... (3)

Since this plane also passes through (x_2, y_2, z_2) and (x_3, y_3, z_3) we have,

$A(x_2 - x_1) + B(y_2 - y_1) + C(z_2 - z_1) = 0$ .... (4)

and

$A(x_3 - x_1) + B(y_3 - y_1) + C(z_3 - z_1) = 0$ .... (5)

Eliminating A, B, C from equations (3), (4) and (5) we get.

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0 \quad \text{.... (6)}$$

which is the equation of the plane.

Corollary 1 : If the plane makes intercepts, a, b, c on the co-ordinate axes $\overleftrightarrow{OX}$, $\overleftrightarrow{OY}$ and $\overleftrightarrow{OZ}$ repectively, then the plane passes through the points, (a, 0, 0), (0, b, 0) and (0, 0, c). Hence the equation (6) gives.

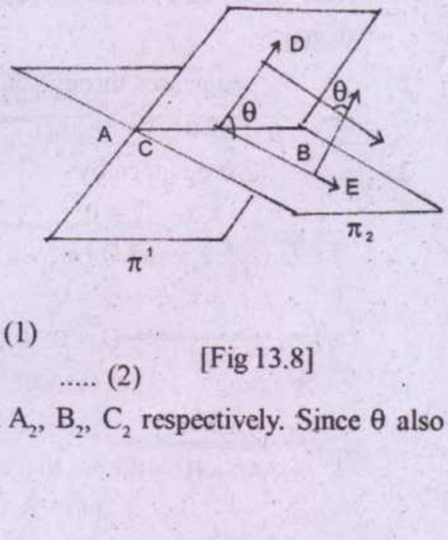
$$\begin{vmatrix} x-a & y-0 & z-0 \\ 0-a & b-0 & 0-0 \\ 0-a & 0-0 & c-0 \end{vmatrix} = 0$$

This on simplification gives, $\boxed{\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1}$ .... (7)

which is the equation of the plane in ***intercept from***.

13.8 Angle between two planes.

Let two planes π_1 and π_2 intersect along the line $\overleftrightarrow{AB}$ (Fig 13.8). Let C be any point on $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ and $\overleftrightarrow{CE}$ be lines lying respectively on π_1 and π_2 such that they are both perpendicular to $\overleftrightarrow{AB}$. If $\theta = m\angle DCE$, then the angle between planes π_1 and π_2 is defined to have measure θ. The angle between the planes is independent of the choice of C on $\overleftrightarrow{AB}$. (Follows from fact–8, Art-13.3, EOM for prev. class).



[Fig 13.8]

It is obvious that θ also measures an angle between the normals to the planes. The two angles between the intersecting planes are of measures θ and $\pi - \theta$.

Let planes π_1 and π_2 have equations respectively

$A_1x + B_1y + C_1z + D_1 = 0$ .... (1)

and $A_2x + B_2y + C_2Z + D_2 = 0$ ..... (2)

Then the direction ratios of their normals are A_1, B_1, C_1, and A_2, B_2, C_2 respectively. Since θ also measures an angle between their normals, we have

$$\cos\theta = \pm\frac{A_1A_2 + B_1B_2 + C_1C_2}{\sqrt{A_1^2+B_1^2+C_1^2}\ \sqrt{A_2^2+B_2^2+C_2^2}}$$

Notes :

1. If the planes $A_1x + B_1y + C_1Z + D_1 = 0$ and $A_2x + B_2y + C_2z + D_2 = 0$ are parallel, then their normals are also parallel. So $\boxed{\frac{A_1}{A_2}=\frac{B_1}{B_2}=\frac{C_1}{C_2}}$ is the condition that the planes be **parallel**.

   The planes are **perpendicular** if $\cos\theta = 0$

   i.e. $\boxed{A_1A_2 + B_1B_2 + C_1C_2 = 0}$

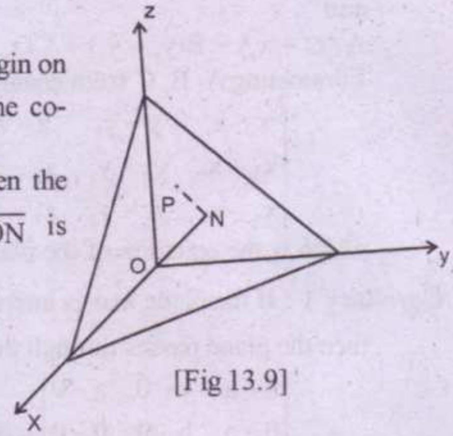
2. The equation of any plane **parallel** to the plane $Ax + By + Cz + D = 0$ is of the form $Ax + By + Cz + D_2 = 0$.

3. Two planes (1) and (2) will be **identical** if

$$\boxed{\frac{A_1}{A_2}=\frac{B_1}{B_2}=\frac{C_1}{C_2}=\frac{D_1}{D_2}}$$

13.9 Equation of plane in Normal form

Let p be the length of the perpendicular $\overline{ON}$ from the origin on the plane and let $<l, m, n>$ be its direction cosines. Then the co-ordinates of the foot of the perpendicular N, are *(lp, mp, np)*.



[Fig 13.9]

If P (x, y, z) be any point on the plane (fig. 13.9) then the direction ratios of $\overline{NP}$ are (x-*l*p, y-*m*p, z-*n*p). Since $\overline{ON}$ is perpendicular to the plane, it is also perpendicular to $\overline{NP}$

Hence,

$$l(x-lp) + m(y-mp) + n(z-np) = 0$$

$$\text{or, } lx + my + nz = (l^2 + m^2 + n^2)\, p$$

$$\text{or, } \boxed{lx + my + nz = p} \quad \text{.... (1)}$$

is the equation of the plane in ***normal form***.

13.10 Transformation of general form to normal form

Let the plane

$$Ax + By + Cz + D = 0 \quad \text{.... (1)}$$

be represented in its normal form by

$$lx + my + nz - p = 0 \quad \text{.... (2)}$$

Since equations (1) and (2) represent the same plane, we have

$$\frac{D}{-p} = \frac{A}{l} = \frac{B}{m} = \frac{C}{n} = \pm\frac{\sqrt{A^2+B^2+C^2}}{\sqrt{l^2+m^2+n^2}} = \pm\sqrt{A^2+B^2+C^2}$$

$$\text{This gives, } p = \frac{-D}{\pm\sqrt{A^2+B^2+C^2}}$$

$$l = \frac{A}{\pm\sqrt{A^2+B^2+C^2}}$$

$$m = \frac{B}{\pm\sqrt{A^2+B^2+C^2}}$$

$$n = \frac{C}{\pm\sqrt{A^2+B^2+C^2}}$$

$\therefore$ The normal form of equation of plane (1) becomes,

$$\frac{A}{\pm\sqrt{A^2+B^2+C^2}}x + \frac{B}{\pm\sqrt{A^2+B^2+C^2}}y + \frac{C}{\pm\sqrt{A^2+B^2+C^2}}z + \frac{D}{\pm\sqrt{A^2+B^2+C^2}} = 0$$

Since p is positive, from the relation $p = \frac{-D}{\pm\sqrt{A^2+B^2+C^2}}$, the sign of the denominator is positive or negative according as D is negative or positive.

13.11 System of Planes

Consider the equation of a plane represented by

$$Ax + By + Cz + D = 0$$

If A, B, C are given, then for different values of D, we get different parallel planes, and hence a system of parallel planes.

Thus, $2x + 3y + z + k = 0$ represents a system of parallel planes for different real values of k.

Let a pair of planes be given by,

$A_1x + B_1y + C_1z + D_1 = 0$... (1)

$A_2x + B_2y + C_2z + D_2 = 0$... (2)

Consider the equation,

$(A_1x + B_1y + C_1z + D_1) + k(A_2x + B_2y + C_2z + D_2) = 0$ ... (3)

for any real value of k. This being an equation of first degree in x, y, and z, represents a plane. If (α, β, γ) is a point lying in the intersection of planes (1) and (2), then

$$A_1\alpha + B_1\beta + C_1\gamma + D_1 = 0$$

and,

$$A_2\alpha + B_2\beta + C_2\gamma + D_2 = 0.$$

Hence,

$(A_1\alpha + B_1\beta + C_1\gamma + D_1) + k(A_2\alpha + B_2\beta + C_2\gamma + D_2) = 0$ for all real values of k. This proves that the point with co-ordinates (α, β, γ) satisfies the equation (3). Hence (3) represents a plane passing through the line of intersection of (1) and (2) for all real values of k.

Position of two given points with respect to a plane.

Let,

$$Ax + By + Cz + D = 0 \quad ... (1)$$

be a given plane and $P(x_1, y_1, z_1)$, $Q(x_2, y_2, z_2)$ be two given points. Let $\overleftrightarrow{PQ}$ meet the plane (1) at a point R. Then the co-ordinates of the point R are,

$$\left(\frac{\lambda x_1 + x_2}{1+\lambda}, \frac{\lambda y_1 + y_2}{1+\lambda}, \frac{\lambda z_1 + z_2}{1+\lambda}\right) \text{ for some real number } \lambda \neq -1.$$

Since, this point R lies on the plane (1) we have

$$A\left(\frac{\lambda x_1 + x_2}{1+\lambda}\right) + B\left(\frac{\lambda y_1 + y_2}{1+\lambda}\right) + C\left(\frac{\lambda z_1 + z_2}{1+\lambda}\right) + D = 0$$

$$\text{or, } \lambda(Ax_1 + By_1 + Cz_1 + D) + (Ax_2 + By_2 + Cz_2 + D) = 0$$

$$\text{or, } \lambda = -\frac{Ax_2 + By_2 + Cz_2 + D}{Ax_1 + By_1 + Cz_1 + D}$$

This proves that, λ is positive or negative according as the quantities $(Ax_1 + By_1 + Cz_1 + D)$ and $(Ax_2 + By_2 + Cz_2 + D)$ are of opposite signs or same sign.

If λ is positive, then R divides line segment $\overline{PQ}$ internally and hence P and Q lie on different sides of the plane. If λ is negative, then R divides line segment $\overline{PQ}$ externally, and hence P and Q lie on the same side of the plane.

Thus, two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ will lie on the same side or opposite sides of the plane, $Ax + By + Cz + D = 0$, according as the quantities $(Ax_1 + By_1 + Cz_1 + D)$ and $(Ax_2 + By_2 + Cz_2 + D)$ have same sign or opposite signs.

Note :

Consider two planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$, where both d_1 and d_2 are positive (i.e origin lies on same side of both planes). Then the direction cosines of their normals are

$$\left\langle \frac{a_1}{-\lambda_1}, \frac{b_1}{-\lambda_1}, \frac{c_1}{-\lambda_1} \right\rangle \text{ and } \left\langle \frac{a_2}{-\lambda_2}, \frac{b_2}{-\lambda_2}, \frac{c_2}{-\lambda_2} \right\rangle$$

where $\lambda_1 = \sqrt{a_1^2 + b_1^2 + c_1^2}$ and $\lambda_2 = \sqrt{a_2^2 + b_2^2 + c_2^2}$

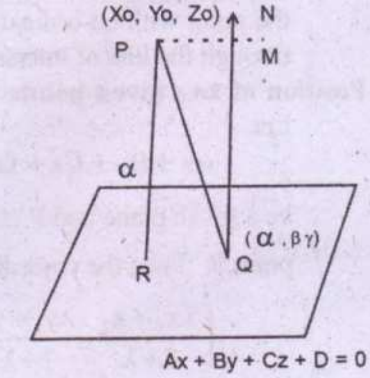
If θ measures the angle between the two planes, then,

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\lambda_1\lambda_2}$$

which is positive or negative according as $(a_1a_2 + b_1b_2 + c_1c_2)$ is positive or negative. (i.e angle is acute or obtuse.) Hence the or origin lies in the interior of the acute or obtuse angle between the planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$, where both d_1, d_2 are positive, according as $a_1 a_2 + b_1 b_2 + c_1 c_2$ is negative or positive.

13.12 Distance of a point from a plane.

Let $P(x_0, y_0, z_0)$ be a given point and $Ax + By + Cz + D = 0$ be the equation of a given plane. Consider a point $Q(\alpha, \beta, \gamma)$ on the given plane. Draw $\overrightarrow{QN}$ normal to the plane at Q and $\overline{PM}$ perpendicular to $\overrightarrow{QN}$. Join $\overline{PQ}$. If R be the foot of the perpendicular drawn from the point P to the given plane, then



[Fig 13.10]

$d = PR = QM =$ projection of $\overline{PQ}$ on $\overrightarrow{QN}$. $\overrightarrow{QN}$ being normal to the given plane $Ax + By + Cz + D = 0$, the direction ratios of $\overrightarrow{QN}$ are <A, B, C> and hence the direction cosines are

$$\left\langle \frac{A}{\pm\sqrt{A^2+B^2+C^2}}, \frac{B}{\pm\sqrt{A^2+B^2+C^2}}, \frac{C}{\pm\sqrt{A^2+B^2+C^2}} \right\rangle$$

∴ d = projection of line segment $\overline{PQ}$ on $\overrightarrow{QN}$

$$= \frac{A}{\pm\sqrt{A^2+B^2+C^2}}(x_0-\alpha) + \frac{B}{\pm\sqrt{A^2+B^2+C^2}}(y_0-\beta) + \frac{C}{\pm\sqrt{A^2+B^2+C^2}}(z_0-\gamma)$$

$$= \frac{A(x_0-\alpha) + B(y_0-\beta) + C(z_0-\gamma)}{\pm\sqrt{A^2+B^2+C^2}}$$

$$= \frac{Ax_0 + By_0 + Cz_0 - (A\alpha + B\beta + C\gamma)}{\pm\sqrt{A^2+B^2+C^2}}$$

Now, $Q(\alpha, \beta, \gamma)$ lies on the given plane. Hence

$A\alpha + B\beta + C\gamma + D = 0$

or, $A\alpha + B\beta + C\gamma = -D$

Thus,

$$\boxed{d = \frac{Ax_0 + By_0 + Cz_0 + D}{\pm\sqrt{A^2+B^2+C^2}}}$$

the sign of the denominator chosen accordingly so as to make the whole quantity positive.

In particular the distance of the plane from the origin is given by,

$$\boxed{d = \frac{D}{\pm\sqrt{A^2+B^2+C^2}}}$$

13.13 Equations of planes bisecting the angle between two given planes.

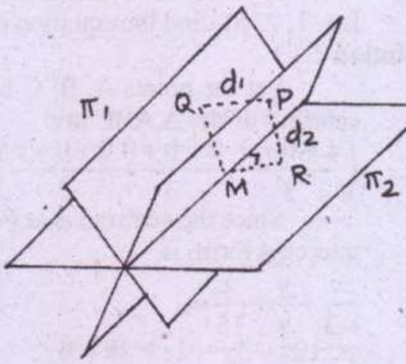
Consider two planes given by the equations

$A_1x + B_1y + C_1z + D_1 = 0$

and,

$A_2x + B_2y + C_2z + D_2 = 0$

If P (x, y, z) is any point on the bisector plane, then it is equidistant from both the given planes (which can be proved from the similarity of Δs PMQ and PRM), i.e $d_1 = d_2$. Hence the equation of the bisecting planes are,



$$\frac{A_1x + B_1y + C_1z + D}{\sqrt{A_1^2 + B_1^2 + C_1^2}} = \pm\frac{A_2x + B_2y + C_2z + D_2}{\sqrt{A_2^2 + B_2^2 + C_2^2}}$$

[Fig 13.11]

One of the planes given by these equations is the internal bisector and the other is the external bisector of the angle between the given planes. To distinguish between these two planes, we find out the angle between one of the bisector planes and a given plane. If measure of this angle is greater than 45^0, then the concerned angle is obtuse and the bisector plane is the external bisector. Otherwise, it is the internal bisector. We illustrate this in example 14.

SOLVED EXAMPLES :

Example 7

Find the equation of the plane through the points (1, 3, 4), (2, 1, –1) and (1, –4. 3).

Solution :

Any plane passing through (1, 3, 4) is given by

$A (x-1) + B (y-3) + C (z-4) = 0$ ... (1)

where A, B, C, are drs. of the normal to the plane.

Since the plane passes through points (2, 1, –1) and (1, –4, 3), we have

$A (2-1) + B (1-3) + C (-1-4) = 0,$

i.e. $A1 + B (-2) + C (-5) = 0$

and $A (1-1) + B (-4-3) + C (3-4) = 0$, i.e. $A.0 + B (-7) + C (-1) = 0$

By cross multiplication, we get,

$$\frac{A}{(-2)(-1)-(-5)(-7)} = \frac{B}{(-5)0-(1)(-1)} = \frac{C}{1.(-7)-0.(-2)}$$

or, $\frac{A}{-33} = \frac{B}{1} = \frac{C}{-7}$

Hence the drs. of the normal to the plane are 33, –1, 7 and putting these values in (1), the equation of the required plane is

$33 (x-1) -1 (y-3) + 7 (z-4) = 0$

or, $33x - y + 7z - 58 = 0$..

Example 8

A plane meets the co-ordinate axes at A, B, C respectively. If the centroid of the triangle ABC is (–1, 2, 5), find the equation of the plane.

Solution :

Let the points A, B, C be A (a, 0, 0), B (o, b, o) and C (0, 0, c). Then the co-ordinates of the centroid of the Δ ABC are

$$\left(\frac{a+0+0}{3}, \frac{0+b+0}{3} \frac{0+0+c}{3}\right) = \left(\frac{a}{3}, \frac{b}{3}, \frac{c}{3}\right)$$

Since the centroid is at (–1, 2, 5), we have a = – 3, b = 6, c = 15, and so the equation of plane by intercept form, is

$$\frac{x}{-3} + \frac{y}{6} + \frac{z}{15} = 1$$

or, $10x - 5y - 2z + 30 = 0$

Example 9

Obtain the normal form of equation of the plane $3x + 2y + 6z + 1 = 0$ and find the d.cs and length of the perpendicular from the origin to this plane.

Solution :

The drs. of the normal to the plane are <3, 2, 6> and hence the direction cosines are,

$$\left\langle \frac{3}{\pm\sqrt{9+4+36}}, \frac{2}{\pm\sqrt{9+4+36}}, \frac{6}{\pm\sqrt{9+4+36}} \right\rangle$$

Length of the perpendicular from origin is

$p = \dfrac{-D}{\pm\sqrt{A^2+B^2+C^2}} = \dfrac{-1}{\pm\sqrt{9+4+36}} = \dfrac{1}{7}$ [∵ D is positive, we choose – sign before the rdical sign to make p > 0]

The equation of plane in normal form is

$$\frac{A}{-\sqrt{A^2+B^2+C^2}}x + \frac{B}{-\sqrt{A^2+B^2+C^2}}y + \frac{C}{-\sqrt{A^2+B^2+C^2}}z + \frac{D}{-\sqrt{A^2+B^2+C^2}} = 0$$

or, $\dfrac{3}{-7}x + \dfrac{2}{-7}y + \dfrac{6}{-7}z + \dfrac{1}{-7} = 0$

or, $-\dfrac{3}{7}x - \dfrac{2}{7}y - \dfrac{6}{7}z = \dfrac{1}{7}$

Example 10

Find the equation of the plane through the points (2, 2,1) and (9,3,6) and perpendicular to the plane $2x + 6y + 6z + 9 = 0$.

Solution : Any plane through (2, 2, 1) is given by

$A(x-2) + B(y-2) + C(z-1) = 0$ ... (1)

where A, B, C are drs. of the normal to the plane. This plane passes through (9, 3, 6) and hence putting the values in (1) we get,

$A(9-2) + B(3-2) + C(6-1) = 0$

i.e. $7A + 1.B + 5.C = 0$ ... (2)

Again, the plane (1) is perpendicular to $2x + 6y + 6z + 9 = 0$. So by condition of perpendicularity of the normals of both the planes, we get,

$2A + 6B + 6C = 0$ ...(3)

From (2) and (3), by cross-multiplication, we have

$$\frac{A}{1.6-5.6} = \frac{B}{5.2-7.6} = \frac{C}{7.6-1.2}$$

or, $\dfrac{A}{-24} = \dfrac{B}{-32} = \dfrac{C}{40}$ or, $\dfrac{A}{3} = \dfrac{B}{4} = \dfrac{C}{-5}$

Hence, from (1) the equation of plane is

$3(x-2) + 4(y-2) - 5(z-1) = 0$

or, $3x + 4y - 5z - 9 = 0$

Example 11

Find the equation of the plane passing through the line of intersection of the planes $2x + y + 3z - 4 = 0$ and $x+y = 0$, which is perpendicular to $3x - y + z = 3$.

Solution :

Any plane passing through the line of intersection of $2x + y + 3z - 4 = 0$ and $x+y = 0$ is

$(2x + y + 3z - 4) + \lambda\,(x+y) = 0$

i.e $(2+\lambda).x+(1+\lambda)\,y + 3z - 4 = 0$ ... (1)

This plane is perpendicular to the plane

$3x - y + z - 3 = 0$

Hence, by condition of perpendicularity,

$(2+\lambda).\,3 + (1+\lambda)\,(-1) + 3.1 = 0$

or, $2\lambda = -8$. i.e. $\lambda = -4$

So, the equation of the required plane from (1) is $-2x - 3y + 3z - 4 = 0$

or, $2x + 3y - 3z + 4 = 0$.

Example 12

Find the equation of the plane through the line of intersection of the planes $x + 3y + 6 = 0$, $3x - y - 4z = 0$ which is at a unit distance from the origin.

Solution :

Any plane through the intersection of $3x-y-4z = 0$, $x+3y + 6 = 0$ is given by

$(3x-y-4z) + \lambda\,(x+3y+6) = 0$

i.e. $(3+\lambda)\,x - (1-3\lambda)\,y - 4z + 6\lambda = 0$

The perpendicular distance of this plane from the origin is

$$p = \frac{6\lambda}{\pm\sqrt{(3+\lambda)^2 + (1-3\lambda)^2 + 4^2}} = \frac{6\lambda}{\pm\sqrt{10\lambda^2 + 26}}$$

Since it is given that the plane is at a unit distance from the origin, we have $p = 1$. So,

$$1 = \frac{6\lambda}{\pm\sqrt{10\lambda^2 + 26}} \text{ i.e. } 10\lambda^2 + 26 = 36\lambda^2$$

or $\lambda^2 = 1$

or $\lambda = \pm 1$.

Hence, the required planes are,

$(3x - y - 4z) \pm (x+3y+6) = 0$

or, $4x+2y-4z + 6 = 0$ i.e. $2x+y-2z+3 = 0$,

and

$2x - 4y - 4z - 6 = 0$, i.e. $x-2y - 2z - 3 = 0$

Example 13

A variable plane meets the co-ordinate axes at A, B, C and is at a constant distance d from origin. Prove that the locus of the centroid of the triangles ABC is

$$\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = \frac{9}{d^2}$$

Solution :

Let the plane meet the x-axis, y-axis and z-axis at A $(\alpha, 0,0)$, B $(o, \beta, 0)$ and C $(0, 0, \gamma)$.

Then the equation of the plane is :

$$\frac{x}{\alpha} + \frac{y}{\beta} + \frac{z}{\gamma} - 1 = 0$$

whose distance d from the origin is given by

$$d = \frac{1}{\pm\sqrt{\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}}} \text{ or } \frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}=\frac{1}{d^2} \quad ...(1)$$

If (x_1, y_1, z_1) are the co-ordinates of the centroid of the Δ ABC, then

$$x_1 = \frac{\alpha+0+0}{3}=\frac{\alpha}{3},\ y_1 = \frac{0+\beta+0}{3}=\frac{\beta}{3}, z_1 = \frac{0+0+\gamma}{3}=\frac{\gamma}{3}$$

i.e $\alpha = 3x_1, \beta = 3y_1, \gamma = 3z_1$.

So, equation (1) gives

$$\frac{1}{9x_1^2}+\frac{1}{9y_1^2}+\frac{1}{9z_1^2}=\frac{1}{d^2}$$

or, $$\frac{1}{x_1^2}+\frac{1}{y_1^2}+\frac{1}{z_1^2}=\frac{9}{d^2}$$

Replacing (x_1, y_1, z_1) by (x, y, z) the locus of the centroid is

$$\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=\frac{9}{d^2}$$

Example 14

Find the bisector of the acute angle between the planes $4x - 3y + 5z + 1 = 0$ and $12x - 5y + 13z + 2 = 0$.

Solution :

The equations of planes bisecting the angle between the given planes are,

$$\frac{4x-3y+5z+1}{\sqrt{4^2+3^2+5^2}} = \pm\frac{12x-5y+13z+2}{\sqrt{12^2+5^2+13^2}}$$

or, $$\frac{4x-3y+5z+1}{5\sqrt{2}} = \pm\frac{12x-5y+13z+2}{13\sqrt{2}}$$

or, $13(4x - 3y + 5z + 1) = \pm 5(12x - 5y + 13z + 2)$

Taking +sign, the equation of one bisector is,

$8x + 14y - 3 = 0$ ... (1)

and taking –sign, the equation of the other bisector is,

$112x - 64y + 130z + 23 = 0$ ... (2)

Now consider one of the given planes, say

$4x - 3y + 5z + 1 = 0$

and one of the bisectors, say

$8x + 14y - 3 = 0$

If θ measures the angle between these two planes, then,

$$\theta = \cos^{-1}\left(\frac{32-42}{\pm\sqrt{50}\,\sqrt{260}}\right)$$

$$\text{i.e. } \cos\theta = \frac{-10}{\pm\sqrt{50}\,\sqrt{260}} = \frac{-1}{\pm\sqrt{130}}$$

Hence, $\sin\theta = \pm\frac{\sqrt{129}}{\sqrt{130}}$ which gives $\tan\theta = \pm\sqrt{129}$

This being numerically greater than $1, \theta > 45^0$ and hence $2\theta > 90^0$. Thus the plane $8x + 14y - 4 = 0$, bisects the obtuse angle. So the equation of the plane bisecting the acute angle is $112x - 64y + 130z + 23 = 0$.

EXERCISE – 13 (b)

1. State, which of the following statements are true (T) or false (F) :
   (a) Through any four points one and only one plane can pass.
   (b) The equation of xy – plane is $x + y = 0$.
   (c) The plane $ax + by + c = 0$ is perpendicular to z - axis.
   (d) The equation of the plane parallel to xz-plane and passing through (2, -4, 0) is $y + 4 = 0$
   (e) The planes $2x - y + z - 1 = 0$
   and $6x - 3y + 3z = 1$ are coincident.
   (f) The planes $2x + 4y - z + 1 = 0$
   and $x - 2y - 6z + 3 = 0$ are perpendicular to each other.
   (g) The distance of a point from a plane is same as the distance of the point from any line lying in that plane.
2. Fill in the blanks by choosing the appropriate answer from the given ones :
   (a) The equation of a plane passing through (1, 1, 2) and parallel to $x + y + z - 1 = 0$ is ——
   [$x + y + z = 0$, $x + y + 2z - 1 = 0$
   $x + y + z = 2$, $x + y + z = 4$]
   (b) The equation of plane perpendicular to z - axis and passing through (1, -2, 4) is ——
   [$x = 1$, $y + 2 = 0$, $z - 4 = 0$, $x + y + z - 3 = 0$]
   (c) The distance between the parallel planes
   $2x - 3y + 6z + 1 = 0$ and
   $4x - 6y + 12z - 5 = 0$ is ——
   $\left[\frac{1}{2}, \frac{1}{7}, \frac{4}{7}, \frac{6}{7}\right]$
   (d) The plane $y - z + 1 = 0$ is ——
   [paralles to x -axis, perpendicular to x -axis, parallel to xy -plane, perpendicular to yz -plane.]
   (e) A plane whose normal has direction ratios <3, –2, k> is parallel to the line joining (–1, 1, –4) and (5, 6, –2). Then the value of k = —— [6, –4, –1, 0]
3. Find the equation of planes passing through the points :
   (a) (6, –1, 1), (5, 1, 2) and (1, –5, –4);
   (b) (2, 1, 3), (3, 2, 1) and (1, 0, –1);
   (c) (–1, 0, 1), (–1, 4, 2) and (2, 4, 1);
   (d) (–1, 5, 4), (2, 3, 4) and (2, 3, –1);
   (e) (1, 2, 3), (1, –4, 3) and (–1, 3, 2);
4. Find the equation of plane in each of the following cases :
   (a) Passing through the point (2, 3, –1) and parallel to the plane $3x - 4y + 7z = 0$.
   (b) Passing through the points (2, –3, 1) and (–1, 1, –7) and perpendicular to the plane $x - 2y + 5z + 1 = 0$.
   (c) Passing through the foot of the perpendiculars drawn from P (a, b, c) on the coordinate planes.
   (d) Passing through the point (–1,3,2) perpendicular tothe planes $x + 2y + 2z = 5$ and $3x + 3y + 2z = 8$.
   (e) Bisecting the line segment joining (–1, 4, 3) and (5, –2, –1) at right angles.
   (f) Parallel to the plane $2x - y + 3z + 1 = 0$ and at a distance 3 units away from it.
5. (a) Write the equation of the plane $3x - 4y + 6z - 12 = 0$ in intercept form and hence obtain the co-ordinates

of the points where it meets the co-ordinate axes.

(b) Write the equation of the plane $2x - 3y + 5z + 1 = 0$ in normal form and find its distance from the origin. Find also the distacne from the point (3, 1, 2).

(c) Find the distance between the parallel planes $2x - 2y + z + 1 = 0$ and $4x - 4y + 2z + 3 = 0$.

6. In each the following cases, verify whether the four given points are coplanar or not.

(a) (1, 2, 3), (–1, 1, 0), (2, 1, 3), (1, 1, 2)

(b) (1, 1, 1), (3, 1, 2), (1, 4, 0), (–1, 1, 0)

(c) (0, –1, –1), (4, 5, 1), (3, 9, 4), (–4, 4, 4)

(d) (–6, 3, 2), (3, –2, 4), (5, 7, 3), (–13, 17, –1)

7. Find the equation of plane in each of the following cases :

(a) Passing through the intersection of planes $2x + 3y - 4z + 1 = 0$ and $3x - y + z + 2 = 0$ and passing through the point (3, 2, 1).

(b) Which contains the line of intersection of the planes $x + 2y + 3z - 4 = 0$, $2x + y - z + 5 = 0$ and perpendicular to the plane $5x + 3y + 6z + 8 = 0$.

(c) Passing through the intersection of $ax + by + cz + d = 0$ and $a_1x + b_1y + c_1z + d_1 = 0$ and perpendicular to xy - plane.

(d) Passing through the intersection of the planes $x + 3y - z + 1 = 0$ and $3x - y + 5z + 3 = 0$ and is at a distance $\frac{2}{3}$ units from origin.

8. Find the angle between the following pairs of planes.

(a) $x + 3y - 5z + 1 = 0$ and $2x + y - z + 3 = 0$

(b) $x + 2y + 2z - 3 = 0$ and $3x + 4y + 5z + 1 = 0$

(c) $x + 2y + 2z - 7 = 0$ and $2x - y + z = 6$

9. (a) Find the equation of the bisector of the angles between the following pairs of planes and specify the ones which bisects the acute angles ;

(i) $3x - 6y + 2z + 5 = 0$ and $4x - 12y + 3z - 3 = 0$

(ii) $2x + y - 2z - 1 = 0$ and $4x - 12y + 3z + 3 = 0$

(b) Show that the origin lies in the interior of the acute angle between planes $x + 2y + 2z = 9$ and $4x - 3y + 12z + 13 = 0$. Find the eqation of bisector of the acute angle.

10. (a) Prove that the line joining (1, 2, 3), (2, 1, –1) intersects the line joining (–1, 3, 1) and (3, 1, 5).

(b) Show that the point $\left(-\frac{1}{2}, 2, 0\right)$ is the circumecentre of the triangle formed by the points (1, 1, 0), (1, 2,1) and (–2, 2, –1)

11. Show that the plane $ax + by + cz + d = 0$ divides the line segment joining (x_1, y_1, z_1) and (x_2, y_2, z_2) in a ratio $-\frac{ax_1 + by_1 + cz_1 + d}{ax_2 + by_2 + cz_2 + d}$

12. A variable plane is at a constant distance p from the origin and meets the axes at A, B, C. Through A, B, C planes are drawn parallel to the co-ordinate planes. Show that the locus of their points of intersection is $\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = \frac{1}{p^2}$

13. A variable plane passes through a fixed point (a, b, c) and meets the co-ordinate axes at A, B, C. Show that the locus of the point common to the planes drawn through A, B and C parallel to the co-ordinate planes is $\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 1$.

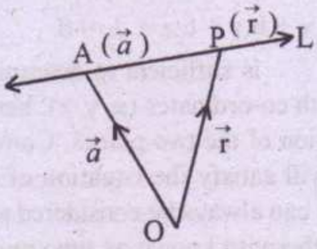
14. The plane $4x + 7y + 4z + 81 = 0$ is rotated through a right angle about its line of intersection with the plane $5x + 3y + 10z - 25 = 0$. Find the equation of the plane in new position.

15. The plane $lx + my = 0$ is rotated about its line of intersection with the plane $z = 0$ through angle measure α. Prove that the equation of the plane in new position is $lx + my \pm z\sqrt{l^2 + m^2}\tan\alpha = 0$.

LINE

13.14 Vector equation of a line :

Let L be a given line passing through a given point A with position vector $\vec{a}$ and parallal to a vector $\vec{b}$. If O be refered to as the origin of vectors (a fixed pt). and $\vec{a}$ be the position vector of the point A on L, then for anypoint P on L (Fig. 16.20)



(Fig 13.12)

$\vec{r} = \overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP} = \vec{a} + \overrightarrow{AP}$

But $\overrightarrow{AP} = t\vec{b}$ where t is a scalar ($\because \overrightarrow{AP} \parallel \vec{b}$)

$\Rightarrow \vec{r} = \vec{a} + t\vec{b}$ is the vector equation of the line L.

Note 1 : In particular, if the line passes through the origin, i.e. fixed point O and is parallel to $\vec{b}$, then its equation is $\vec{r} = t\vec{b}$.

2. If the line passes through two points A and B with position vectors $\vec{a}$, $\vec{b}$ respectively, then $\overrightarrow{AB} = \vec{b} - \vec{a}$.

If P is any point on L having position vector $\vec{r}$, then $\overrightarrow{AP}$ is parallel to $\overrightarrow{AB}$ and hence, for some scalar t,

$\overrightarrow{AP} = t(\overrightarrow{AB})$

Or, $(\vec{r} - \vec{a}) = t(\vec{b} - \vec{a})$

Or, $\vec{r} = \vec{a} + t(\vec{b} - \vec{a}) = (1-t)\vec{a} + t\vec{b}$ is the equation of the line.

3. Cartesian equation from vector equation

Putting $\vec{r} = x\hat{i} + y\hat{i} + z\hat{k}$, $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

[Where (a_1, a_2, a_3) and (b_1, b_2, b_3) are the Cartesian co-ordinates of the points with positive vectors $\vec{a}$ and $\vec{b}$ respectively],

we get

$x\hat{i} + y\hat{j} + z\hat{k} = \{(1\text{-}t)a_1 + tb_1\}\hat{i} + \{(1\text{-}t)a_2 + tb_2\}\hat{j} + \{(1\text{-}t)a_3 + tb_3\}\hat{k}$

Equating components

$x = a_1 + t(b_1 - a_1)$

$y = a_2 + t(b_2 - a_2)$

$z = a_3 + t(b_3 - a_3)$

$$\Rightarrow \quad \frac{x - a_1}{b_1 - a_1} = \frac{y - a_2}{b_2 - a_2} = \frac{z - a_3}{b_3 - a_3}$$

which become the Catesian form of the equations of the line passing through the points (a_1, a_2, a_3) and (b_1, b_2, b_3).

We shall know more about Cartesian form of equations to a line in the subsequent articles.

13.15 Cartesian Equation of a Line

Since a straight line is the intersection of two planes, a pair of equations of first degree in, x, y and z, such as

$$a_1x + b_1y + c_1z + d_1 = 0$$
$$a_2x + b_2y + c_2z + d_2 = 0$$

is sufficient to determine a line. If a set a values of x, y, z satisfies the two equations then the point with co-ordinates (x, y, z), lies on these planes and hence give the co-ordinates of a point on the line of intersection of the two planes. Conversely, if we take a point on the line of intersection of these two planes, then it will satisfy the equation of both the planes. Hence the combined equation of a pair of planes (not parallel) can always be considered as the general equation of a straight line. This form of equation of straight line is otherwise known as **unsymmetric form** of equation of line.

13.16 Symmetric form of Equations of a line

Consider a line L passing through a point Q (x_0, y_0, z_0) and having direction ratios $<a, b, c>$.

Then for any variable point P (x, y, z) on the line L, the direction ratios of $\overleftrightarrow{PQ}$ are $<x - x_0, y - y_0, z - z_0>$. Hence we get

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

which are the equations of the line in **symmetric form.**

Now, equating each of the above fraction to a parameter r, we get

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c} = r.$$

Hence the co-ordinates of any variable point, on the line L are given by $(x_0 + ar, y_0 + br, z_0 + cr)$ for some real value of the parameter r.

13.17 Two-point Form.

If a line passes through two given points P_1 (x_1, y_1, z_1) and P_2 (x_2, y_2, z_2) then its direction ratios are $<x_2 - x_1, y_2 - y_1, z_2 - z_1>$. Hence its equation is given by

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$ (compare this with the equation that we have derived from the vector form)

which are known as the equations of line in **two - point form**.

13.18 Transformation of unsymmetrical form to symmetrical form.

Consider the unsymmetrical form of equation of line, represented by the pair of equations,

$$a_1x + b_1y + c_1z + d_1 = 0 \quad \text{... (1)}$$
$$a_2x + b_2y + c_2z + d_2 = 0 \quad \text{... (2)}$$

To reduce it to symmetric form, we require the direction ratios of the line and the co-ordinates of a point on it. Now if $<a, b, c>$ are the direction ratios of the line, then the line being common to both the planes, the normals to both the planes are perpendicular to the line. Hence by condition of perpendicularity,

$aa_1 + bb_1 + cc_1 = 0$ and $aa_2 + bb_2 + cc_2 = 0$.

By cross-multiplication, we get

$$\frac{a}{b_1c_2 - c_1b_2} = \frac{b}{c_1a_2 - a_1c_2} = \frac{c}{a_1b_2 - b_1a_2}$$

Thus the direction ratios of the line are $<b_1c_2 - c_1b_2, c_1a_2 - a_1c_2, a_1b_2 - b_1a_2>$. To obtain the co-ordinates of a point on the line, for the sake of convenience, we consider one of the co-ordinates to be zero, say $z = 0$.

Then putting $z = 0$, in equations (1) and (2) we have

$$a_1x + b_1y + d_1 = 0$$
$$a_2x + b_2y + d_2 = 0$$

By cross multiplication, we get

$$\frac{x}{b_1d_2 - d_1b_2} = \frac{y}{d_1a_2 - a_1d_2} = \frac{1}{a_1b_2 - b_1a_2}$$

and hence, $x = \dfrac{b_1d_2 - b_2d_1}{a_1b_2 - b_1a_2}$

$$y = \frac{d_1a_2 - a_1d_2}{a_1b_2 - b_1a_2}$$

and $z = 0$, are co-ordinates of a point on the line. So the symmetric form of equations of the line are,

$$\frac{x - \dfrac{b_1d_2 - d_1b_2}{a_1b_2 - a_2b_1}}{b_1c_2 - c_1b_2} = \frac{y - \dfrac{d_1a_2 - a_1d_2}{a_1b_2 - b_1a_2}}{c_1a_2 - a_1c_2} = \frac{z - 0}{a_1b_2 - a_2b_1}$$

13.19 Condition for a line to lie on a plane.

Consider the line

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

and the plane

$$Ax + By + Cz + D = 0$$

It the line lies on the plane then the normal to the plane is normal to the line and hence by condition of perpendicularity,

$$Aa + Bb + Cc = 0.$$

But this condition also implies that the line may be parallel to the plane (as in that case also the normal to the plane is perpendicular to the line.)

So, if along with this we take the condition that at least one point of the line lies on the plane, then the line will lie on the plane. The point (x_0, y_0, z_0) being a given point on the line, lies on the plane, if

$$Ax_0 + By_0 + Cz_0 + D = 0$$

Hence the required conditions are

(i) $Aa + Bb + Cc = 0$

(ii) $Ax_0 + By_0 + Cz_0 + D = 0$

13.20 Condition for two lines to be Coplanar.

Consider the lines

$$\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}$$

and

$$\frac{x - x_2}{a_2} = \frac{y - y_2}{b_2} = \frac{z - z_2}{c_2}$$

Let the lines lie on the plane

$$Ax + By + Cz + D = 0$$

By the conditions of sec 13.19, we have,

$$Aa_1 + Bb_1 + Cc_1 = 0, \qquad \text{... (1)}$$
$$Ax_1 + By_1 + Cz_1 + D = 0 \qquad \text{... (2)}$$

and

$$Aa_2 + Bb_2 + Cc_2 = 0, \qquad \text{... (3)}$$

$$Ax_2 + By_2 + Cz_2 + D = 0 \quad \ldots (4)$$

From (2) and (4), we get

$$A(x_2 - x_1) + B(y_2 - y_1) + C(z_2 - z_1) = 0 \quad \ldots (5)$$

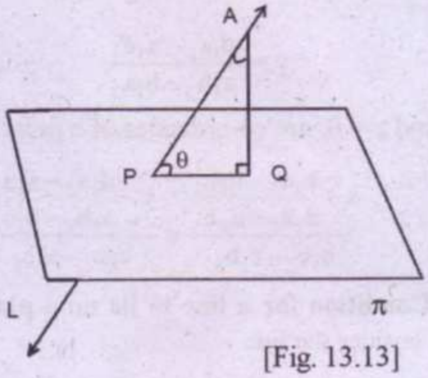
Eliminating A, B, C from (5), (1) and (3) we get,

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$$

which is the required condition.

13.21 Angle between a line and a plane.

If the line L intersects plane π at P and Q is the foot of the perpendicular from a point A on L onto the plane π, the angle between L and π is $\angle APQ$. [Fig 13.13]



[Fig. 13.13]

Let the angle between the line

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

and the plane,

$$Ax + By + Cz + D = 0$$

have measure θ.

Then the angle between the line and the normal to the plane is $\left(\frac{\pi}{2} - \theta\right)$. Hence,

$$\cos(\pi/2 - \theta) = \frac{Aa + Bb + Cc}{\pm\sqrt{A^2 + B^2 + C^2}\sqrt{a^2 + b^2 + c^2}}$$

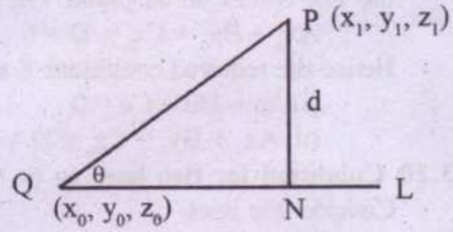
$$\text{or, } \sin\theta = \frac{Aa + Bb + Cc}{\pm\sqrt{A^2 + B^2 + C^2}\sqrt{a^2 + b^2 + c^2}}$$

$$\text{or, } \theta = \sin^{-1}\left\{\frac{Aa + Bb + Cc}{\pm\sqrt{A^2 + B^2 + C^2}\sqrt{a^2 + b^2 + c^2}}\right\}$$

13.22 Distance of a point from a line.

Let the line L be given by,

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$



[Fig. 13.14]

and the given point be P (x_1, y_1, z_1). Join PQ,

where Q (x_0, y_0, z_0) is the point on the line and drop perpendicular $\overline{PN}$ from P on L (Fig. 13.14).

Then, $d = PN = PQ \sin\theta$

where θ measures the angle between $\overrightarrow{QP}$ and L.

Now, the *dr's* of the line are $<a, b, c>$ and those of $\overleftrightarrow{PQ}$ are $<x_1 - x_0, y_1 - y_0, z_1 - z_0>$. Hence,

$$\cos\theta = \frac{a(x_1 - x_0) + b(y_1 - y_0) + c(z_1 - z_0)}{\sqrt{a^2 + b^2 + c^2}\sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2 + (z_1 - z_0)^2}} \quad \ldots (1)$$

Again,

$$PQ = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2 + (z_1 - z_0)^2} \quad \ldots (2)$$

Thus, $d = PQ \sin\theta = PQ\ \sqrt{1 - \cos^2\theta}$

can be determined from (1) and (2).

Alternative method :

Let the line be,

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

and the point P be (x_1, y_1, z_1). If N be the foot of the perpendicular drawn from P on L then N being a point on L, its co-ordinates are $(x_0 + ar, y_0 + br, z_0 + cr)$, for some value of the parameter r. Now, the direction ratios of $\overline{PN}$ are, $< x_0 + ar - x_1, y_0 + br - y_1, z_0 + cr - z_1 >$ and those of L are $< a, b, c >$. So by condition of perpendicularity,

$a(x_0 + ar - x_1) + b(y_0 + br - y_1) + c(z_0 + cr - z_1) = 0$, which gives the value of r and hence the co-ordinates of N.

Thus,

$$d = \sqrt{\sum (x_0 + ar - x_1)^2}$$

is found out.

13.23 Shortest distance between two lines.

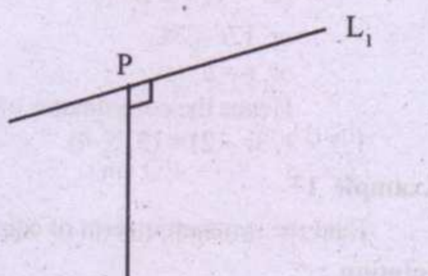
The shortst distance between two lines is defined to be the length of the line segment wich is perpendicular to both of them.

If the two lines are coplanar, then they will be either intersecting or parallel. If they are intersecting, then the shortest distance between them is zero. If they are parallel, then the shortest distance between them is the distance of any point on one of the lines from the other.

Let us consider two skew lines (i.e. non-coplanar lines) L_1 and L_2, given by,

$$\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}$$

and, $$\frac{x - x_2}{a_2} = \frac{y - y_2}{b_2} = \frac{z - z_2}{c_2}$$



[Fig. 13.15]

If PQ is the shortest distance between L_1 and L_2 (Fig. 13.15), then the line segment $\overline{PQ}$ is perpendicular to both L_1 and L_2. Since P and Q are points on the lines L_1 and L_2 respectively, their co-ordinates are $(x_1 + a_1r_1, y_1 + b_1r_1, z_1 + c_1r_1)$ and $(x_2 + a_2r_2, y_2 + b_2r_2, z_2 + c_2r_2)$, for some values of the parameters r_1 and r_2. Hence the direction ratios of $\overline{PQ}$ are

$$< x_1 - x_2 + a_1r_1 - a_2r_2,\ y_1 - y_2 + b_1r_1 - b_2r_2,\ z_1 - z_2 + c_1r_1 - c_2r_2 >.$$

Now, by condition of perpendicularity of $\overline{PQ}$ with L_1 and L_2 we get,

$$a_1(x_1 - x_2 + a_1r_1 - a_2r_2) + b_1(y_1 - y_2 + b_1r_1 - b_2r_2) + c_1(z_1 - z_2 + c_1r_1 - c_2r_2) = 0$$

and $$a_2(x_1 - x_2 + a_1r_1 - a_2r_2) + b_2(y_1 - y_2 + b_1r_1 - b_2r_2) + c_2(z_1 - z_2 + c_1r_1 - c_2r_2) = 0$$

Solving these two equations, we get the values of r_1 and r_2 and the co-ordinates of P and Q. Then by distance formula, the shortest distance PQ is found out.

SOLVED EXAMPLES :

Example 15

Find the equation of the line passing through the point (4, –6, 1) and parallel to the line

$$\frac{x-1}{1} = \frac{y+2}{3} = \frac{z-1}{-1}$$

Solution :

Since the line is parallel to the line

$$\frac{x-1}{1} = \frac{y+2}{3} = \frac{z-1}{-1}$$

its dr's are $< 1, 3, -1 >$. It also passes through the point (4, –6, 1). Hence the equation of the required line is,

$$\frac{x-4}{1} = \frac{y+6}{3} = \frac{z-1}{-1}.$$

Example 16

Find the point of intersection of the line passing through the points (1, 3, –2) and (3, 4, 1) with the plane $x - 2y + 4z = 11$.

Since the line passes through (1, 3, –2) and (3, 4, 1), its equation by two-point form is

$$\frac{x-1}{3-1} = \frac{y-3}{4-3} = \frac{z+2}{1-(-2)}$$

Or, $\frac{x-1}{2} = \frac{y-3}{1} = \frac{z+2}{3} = r$ (say).

Then any point on the line is of the form $(2r + 1, r + 3, 3r - 2)$, for some value of r.

If this point be the point of intersection of this line with the given plane, then these co-ordinates must satisfy the equation of the given plane. Hence,

$(2r + 1) - 2(r + 3) + 4(3r - 2) = 11,$

or, $12r = 24,$

or, $r = 2$

Hence the co-ordinates of the point of intersection of the given line and the given plane are $(2r + 1, r + 3, 3r - 2) = (5, 5, 4)$.

Example 17

Find the symmetric form of equation of the line $x + 2y + z - 3 = 0 = 6x + 8y + 3z - 10$.

Solution :

The given line is the intersection of the planes,

$x + 2y + z - 3 = 0$ ... (1)

and $6x + 8y + 3z - 10 = 0$ ... (2)

Mutltiplying equation (1) by 3 and subtracting from (2), we get,

$3x + 2y - 1 = 0$

or, $3x = 1 - 2y = -2\left(y - \frac{1}{2}\right)$

or, $\frac{x}{2} = \frac{y - \frac{1}{2}}{-3}$ ... (3)

Again, multiplying equation (1) by 4 and subtracting from (2), we get,

$2x - z + 2 = 0$

or, $2x = z - 2$

or, $\frac{x}{2} = \frac{z-2}{4}$ ... (4)

From (3) and (4) the symmetric form of equation of the line is

$$\frac{x}{2} = \frac{y - \frac{1}{2}}{-3} = \frac{z-2}{4}.$$

Example 18

Prove that the lines $\frac{x+5}{3} = \frac{y+4}{3} = \frac{z-7}{-2}$ and, $3x + 2y + z - 2 = 0 = x + 5y + 2z + 3$ are coplanar. Find their point of intersection and the equation of the plane in which they lie.

Solution :

The equation of the second line in unsymmetric form is

$3x + 2y + z - 2 = 0$ ... (1)

and $x + 5y + 2z + 3 = 0$ ... (2)

Multiplying equation (1) by – 2 and adding to (2) we obtain,

$-5x + y + 7 = 0$

or $\frac{x}{1} = \frac{y+7}{5}$

Multiplying equation (1) by – 5 and equation (2) by 2 and adding, we obtain,

$-13x - z + 16 + 0$

or, $\frac{x}{1} = \frac{z-16}{-13}$

So the symmetric equation of the second line is

$\frac{x}{1} = \frac{y+7}{5} = \frac{z-16}{-13}$ ... (3)

and the first line is

$\frac{x+5}{3} = \frac{y+4}{1} = \frac{z-7}{-2}$ ... (4)

To verify co-planarity of (3) and (4), we have (by condition of sec. 13.20),

$$\begin{vmatrix} -5-0 & -4-(-7) & 7-16 \\ 3 & 1 & -2 \\ 1 & 5 & -13 \end{vmatrix}$$

$$= \begin{vmatrix} -5 & 3 & -9 \\ 3 & 1 & -2 \\ 1 & 5 & -13 \end{vmatrix}$$

$= -5\,(-13 + 10) - 3\,(-39 + 2) - 9\,(15 - 1)$
$= 15 + 111 - 126 = 0$

Hence the two lines are coplanar. To find their point of intersection, we see that any point on the line (3) is of the form,

$(r_1, 5r_1 - 7, 16 - 13r_1)$

and any point on the line (4) is of the form,

$(3r_2 - 5, r_2 - 4, 7 - 2r_2)$

for some values of the parameters r_1 and r_2. At the point of intersection,

$3r_2 - 5 = r_1$
$r_2 - 4 = 5r_1 - 7$
$7 - 2r_2 = 16 - 13r_1$.

Solving the first two we obtain $r_1 = 1$ and $r_2 = 2$ which also satisfy the third equation.

Hence the point of intersection is $(1, -2, 3)$.

The plane containing both of them contains the point $(1, -2, 3)$. So its equation is,

$$A\,(x - 1) + B\,(y + 2) + C\,(z - 3) = 0 \qquad \dots (5)$$

where A, B, C are d.rs of its normal.

Since both the lines (3) and (4) lie on the plane, the normal to the plane is perpendicular to both of them.

Hence,

$$A \cdot 1 + B \cdot 5 + C \cdot (-13) = 0 \qquad \dots (6)$$

and $$A \cdot 3 + B \cdot 1 + C\,(-2) = 0 \qquad \dots (7)$$

Eliminating A, B, C from (5), (6), (7) we get the equation of the plane as

$$\begin{vmatrix} x-1 & y+2 & z-3 \\ 1 & 5 & -13 \\ 3 & 1 & -2 \end{vmatrix} = 0$$

or, $3x - 37y - 14z - 35 = 0$

Example 19

Find the image of the point (3, 5, 7) with respect to the plane $2x + y + z = 6$.

Solution :

Let P be the point (3, 5, 7) and A, the foot of the perpendicur drawn form P to the plane. Extend $\overline{PA}$ to B Such that PA = AB. Then B is called the image of the point P, with respect to the given plane.

Since $\overline{PA}$ is perpendicular to the plane $2x + y + z = 6$, its drs. are $< 2, 1, 1 >$. Here the equation of the line $\overleftrightarrow{PA}$ is,

$$\frac{x-3}{2} = \frac{y-5}{1} = \frac{z-7}{1}$$

Any point on this line is given by $(2r + 3, r + 5, r + 7)$ for some value of r. This will be the foot of the perpendicular on the plane from P, if the point lies on the plane, i.e.

$$2(2r + 3) + (r + 5) + (r + 7) - 6 = 0$$

Or, $6r + 12 = 0$

Or, $r = -2$

Hence, co-ordinates of A are $(-1, 3, 5)$.

If the co-ordinates of the image B are (α, β, γ), then

$$\frac{\alpha+3}{2}=-1\Rightarrow\alpha=-5,$$
$$\frac{\beta+5}{2}=3\Rightarrow\beta=1,$$
$$\frac{\gamma+7}{2}=5\Rightarrow\gamma=3.$$

$\therefore$ The image of the point (3,5,7) with respect to the plane $2x+y+z-6=0$ is the point $(-5, 1, 3)$.

Example 20

Find the foot of the perpendicular drawn from the point (5, 7, 3) to the line $\frac{x-15}{3}=\frac{y-29}{8}=\frac{z-5}{-5}$.

Find the length of the perpendicular and its equation.

Solution :

Let the foot of the perpendicular from the given point P (5, 7, 3) on the given line be N. Then co-ordinates of N are $(3r+15, 8r+29, 5-5r)$ for some value of r. Hence the d.rs of the line $\overleftrightarrow{PN}$ are $< 3r+10, 8r+22, 2-5r >$. Since $\overline{PN}$ is perpendicular to the given line, we have,

$$(3r+10).\,3+(8r+22).\,8+(2-5r)(-5)=0$$

Or, $98r+196=0$

Or, $r=-2$

$\therefore$ N is the point (9, 13, 15) so that the length of the perpendicular is

$$PN=\sqrt{(9-5)^2+(13-7)^2+(15-3)^2}$$
$$=\sqrt{16+36=144}=14.$$

Also, by two point form, the equation of $\overline{PN}$ is,

$$\frac{x-5}{2}=\frac{y-7}{3}=\frac{z-3}{6}.$$

Example 21

Find the distance of the point (3, –4, 5) from the plane $2x+5y-6z-19=0$ measured parallel to the line

$$\frac{x-1}{2}=\frac{y}{1}=\frac{z+3}{-2}.$$

Solution :

A line through (3, –4, 5) parallel to the line,

$$\frac{x-1}{2}=\frac{y}{1}=\frac{z+3}{-2} \quad ...\ (1)$$

is given by,

$$\frac{x-3}{2}=\frac{y+4}{1}=\frac{z-5}{-2} \quad ....\ (2)$$

Any point on this line has co-ordinates $(2r+3, r-4, 5-2r)$, for some value of r. This point lies on the plane

$2x+5y-6z-19=0$

if $2(2r+3)+5(r-4)-6(5-2r)-19=0$

i.e. if, $(4r+6)+(5r-20)+(12r-30)-19=0$

i.e. $21r=63$

Or, $r=3$.

Hence, the point of intersection of the line (2) with the plane $2x+5y-6z-19=0$ is $(9,-1,-1)$. So the distance of the point (3, –4, 5) from the line (1) measured parallel to the plane $2x + 5y - 6z - 19 = 0$ is,

$$\sqrt{(9-3)^2 + (-1+4)^2 + (-1-5)^2} = \sqrt{36+9+36} = 9.$$

Example 22

Find the shortest distance between the lines and the equation of the line measuring shortest distance:

$$\frac{x-3}{2} = \frac{y+15}{-7} = \frac{z-9}{5} \quad \text{... (1)}$$

and $$\frac{x+1}{2} = \frac{y-1}{1} = \frac{z-9}{-3} \quad \text{... (2)}$$

Solution :

The shortest distance between two lines is the line segment which is perpendicular to both the lines, having both its ends on the given lines. Let $\overleftrightarrow{PQ}$ be the line of shortest distance between the lines (1) and (2) where P is on the line (1) and Q is on the line (2). The co-ordinates of P are

$(2r_1 + 3, -7r_1 - 15, 5r_1 + 9)$,

for some value of r_1 and co-ordinates of Q are

$(2r_2 - 1, r_2 + 1, 9 - 3r_2)$

for some value of r_2. Then the drs of the line segment $\overline{PQ}$ are,

$< 2r_2 - 2r_1 - 4, r_2 + 7r_1 + 16, -3r_2 - 5r_1 >$

Since $\overline{PQ}$ is perpendicular to lines (1) and (2) we have,

$(2r_2 - 2r_1 - 4) .2 + (r_2 + 7r_1 + 16) (-7) + (-3r_2 - 5r_1) . 5 = 0.$

and $(2r_2 - 2r_1 - 4).2 + (r_2 + 7r_1 + 16) .1 + (-3r_2 - 5r_1) (-3) = 0$

On simplification we get,

$3r_2 + 13r_1 + 20 = 0,$

$7r_2 + 9r_1 + 4 = 0,$

which gives, $r_1 = -2, r_2 = 2.$

Hence the co-ordinates of P are (–1, –1, –1) and co-ordinates of Q are (3, 3, 3). So the shortest distance between the lines is,

$$PQ = \sqrt{(4^2 + 4^2 + 4^2)} = 4\sqrt{3}$$

The equation of the line of S.D., by two-point form, is

$$\frac{x+1}{4} = \frac{y+1}{4} = \frac{z+1}{4}$$

or, $x = y = z$.

EXERCISES – 13(C)

1. State which of the following statements are true (T) or false (F) :

(a) The line $\frac{x-1}{2} = \frac{y-1}{2} = \frac{z-1}{2}$ passes through the origin.

(b) The lines $\frac{x+2}{-k} = \frac{y-3}{2} = \frac{z+4}{k}$ and $\frac{x-4}{4} = \frac{y-3}{k} = \frac{z+1}{2}$ are perpendicular for every value of k.

(c) The line $\frac{x+5}{-2}=\frac{y-3}{1}=\frac{z-2}{3}$ lies on the plane $x-y+z+1=0$.

(d) The line $\frac{x-2}{3}=\frac{1-y}{4}=\frac{5-z}{1}$ is parallel to the plane $2x-y-2z=0$.

(e) The line $\frac{x+3}{-1}=\frac{y-2}{3}=\frac{z-1}{4}$ is perpendicular to the plane $3x-3y+3z-1=0$.

2. Fill in the blanks by choosing the correct alternative from the given ones :

(a) The lines $\frac{x+2}{-4}=\frac{y+3}{5}=\frac{z-1}{3}$
and, $\frac{1-x}{-4}=\frac{y-1}{5}=\frac{2-z}{3}$
are — (parallel, perpendicular, coincident)

(b) The line passing through $(-1, 0, 1)$ and perpendicular to the plane $x+2y+1=0$ is —

$$\left[\frac{x+1}{1}=\frac{y-0}{2}=\frac{z-1}{1},\ \frac{x+1}{1}=\frac{y}{2}=\frac{z-1}{0},\ \frac{x-1}{1}=\frac{y-0}{2}=\frac{z-1}{.0}\right]$$

(c) The line $\frac{x+1}{2}=\frac{y-6}{1}=\frac{z-4}{0}$ is ....
(parallel to x-axis, perpendicular to y-axis, perpendicular to z - axis)

(d) If the line $\frac{x-3}{2}=\frac{y+k}{-1}=\frac{z+1}{-5}$ lies on the plane $2x-y+z-7=0$, then $k=$ –(2, –1, –2)

(e) If l, m, n, be dcs of a line, then the line is perpendicular to the plane $x-3y+2z+1=0$ if –

[(i) $l=1, m=-3, n=2$ (ii) $\frac{l}{1}=\frac{m}{-3}=\frac{n}{2}$ (iii) $l-3m+2n=0$].

3. Find the equation of lines joining the points
(i) $(4, -6, 1)$ and $(0, 3, -1)$,
(ii) (a, a, a) and $(a, 0, a)$,
(iii) $(2, 1, 3)$ and $(4, -2, 5)$.

4. Write the symmetric form of equation of the following lines :
(i) x – axis;
(ii) $y=b, z=c$;
(iii) $ax+by+d=0, 5z=0$;
(iv) $x-2y=3, 2x+y-5z=0$;
(v) $4x+4y-5z-12=0=8x+12y-13z-32$;
(vi) $3x-2y+z=1, 5x+4y-6z=2$.

5. (a) Obtain the equation of the line through the point $(1, 2, 3)$ and parallel to the line $x-y+2z-5=0=3x+y+z-6$.

(b) Find the equation of the line through the point $(3, -1, 2)$ and parallel to the planes $x+y+2z-4=0$ and $2x-3y+z+3=0$.

6. Obtain the equation of the line through the point $(0, 2, -3)$ and perpendicular to each of the lines $x+4y-3z=0=2x-5y+7$ and $y+3z-2=0=x+2z+5$.

7. (a) Show that the line passing through the points (a_1, b_1, c_1) and (a_2, b_2, c_2) passes through the origin, if $a_1a_2+b_1b_2+c_1c_2=p_1p_2$, where p_1 and p_2 are distances of the points from origin.

(b) Prove that the lines $x=az+b, y=cz+d$ and $x=a_1z+b_1, y=c_1z+d_1$ are perpendicular if $aa_1+cc_1+1=0$

8. Find the point of intersection of the line $\frac{x-1}{1}=\frac{y+2}{3}=\frac{z-1}{-1}$ and the plane $2x+y+z=9$.

9. Find the co-ordinates of the point where the line joining (3, 4, –5) and (2, –3, 1) meets the plane $2x + y + z - 7 = 0$.

10. (a) Find the distance of the point (–1, –5, –10) from the point of intersection of the line $\frac{x-2}{2} = \frac{y+1}{4} = \frac{z-2}{12}$ and the plane $x - y + z = 5$.

(b) Find the image of the point (2, –1, 3) in the plane $3x - 2y + z - 9 = 0$.

11. (a) Prove that the lines.

$\frac{x+3}{2} = \frac{y+5}{3} = \frac{z-7}{-3}$ and $\frac{x+1}{4} = \frac{y+1}{5} = \frac{z+1}{-1}$ are co-planar. Find the equation of the plane containing them.

12. Prove that the lines

$\frac{x+4}{3} = \frac{y+6}{5} = \frac{z-1}{-2}$

and $3x - 2y + z + 5 = 0 = 2x + 3y + 4z - 4$ are co-planar.

13. Show that the lines $7x - 4y + 7z + 16 = 0 = 4x + 3y - 2z + 3$ and $x - 3y + 4z + 6 = 0 = x - y + z + 1$ intersect. Find the co-ordinates of their point of intersection and equation of the plane containing them.

14. Show that the line joining the points (0, 2, –4) and (–1, 1, –2) and the lines joining the points (–2, 3, 3) and (–3, –2, 1) are co-planar. Find their point of intersection.

15. Show that the lines $x - mz - a = 0 = y - nz - b$ and $x - m'z - a' = 0 = y - n'z - b'$ intersect, if $(a - a')(n - n') = (b - b')(m - m')$.

16. Prove that the line $\frac{x-1}{2} = \frac{y+2}{-3} = \frac{z-3}{1}$ lies on the plane $7x + 5y + z = 0$.

17. (a) Find the angle between the plane $x + y + 4 = 0$ and the line $\frac{x+3}{2} = \frac{y-1}{1} = \frac{z+4}{-2}$.

(b) Find the angle between the plane $4x + 3y + 5z - 1 = 0$ and the line $\frac{x+3}{2} = \frac{y-1}{3} = \frac{z+4}{6}$.

18. (a) Find the equation of the line passing through the point (1, 0, –1) and intersecting the lines $x = 2y = 2z$; $3x + 4y - 1 = 0 = 4x + 5z - 2$.

(b) A line with direction ratios < 2, 1, 2 > meets each of the line $x = y + a = z$ and $x + a = 2y = 2z$. Find the co-ordinates of the points of intersection.

19. Obtain the co-ordinates of the foot of the perpendicular drawn from the point (3, –1, 11) to the line $\frac{x}{2} = \frac{y-2}{3} = \frac{z-3}{4}$. Obtain the equation of the perpendicular also.

20. Find the perpendicular distance of the point (–1, 3, 9) from the line $\frac{x-13}{5} = \frac{y+8}{-8} = \frac{z-31}{1}$.

21. Find the distance of the point (1, –2, 3) from the plane $x - y + z = 5$, measured parallel to the line $\frac{x}{2} = \frac{y}{3} = \frac{z}{-6}$.

22. Find the distance of the point (1, –1, –10) from the line $\frac{x-4}{1} = \frac{y+3}{-4} = \frac{z+1}{7}$ measured parallel to the line $\frac{x+2}{2} = \frac{y-3}{-3} = \frac{z-4}{8}$.

23. Find the equation of plane through the point (2, 0, –3) and containing the line $3x + y + z - 5 = 0 = x - 2y + 4z + 4$.

24. Find the equation of the plane containing the line $x + 2 = 2y - 1 = 3z$ and parallel to the line $x = 1 - 5y$

$= 2z - 7$. Also find the shortest distance between the two lines.

25. Find the equation of the two planes through the origin, parallel to the line $\frac{x-1}{2} = \frac{y+3}{-1} = \frac{z+1}{-2}$ and at a distance $\frac{5}{3}$ from it.

26. Find the equation of the straight line perpendicular to the line $\frac{x-2}{3} = \frac{y+1}{4} = \frac{z-6}{7}$ and lying in the plane $x - 2y + 4z - 51 = 0$.

27. Find the shortest distance between the lines $\frac{x-3}{3} = \frac{y-8}{-1} = \frac{z-3}{1}$ and $\frac{x+3}{-3} = \frac{y+7}{2} = \frac{z-6}{4}$.

    Find also the equation of the line of shortest distance.

28. Show that the shortest distance between the lines $x + a = 2y = -12z$ and $x = y + 2a = 6z - 6a$, is $2a$.

29. Find the length and equation of the line of shortest distance between the lines $3x - 9y + 5z = 0 = x + y - z$ and $6x + 8y + 3z - 13 = 0 = x + 2y + z - 3$.

ADDITIONAL EXERCISES

1. Find the equation in vector and Cartesian form of the plane passing through the point (3,-3,1) and normal to the line joining the points (3,4,-1) and (2,-1,5)

2. Find the vector equation of the plane whose Cartesian form of equation is $3x - 4y + 2z = 5$

3. Show that the normals to the planes $\vec{r}\cdot(\hat{i} - \hat{j} + \hat{k}) = 3$ and $\vec{r}\cdot(3\hat{i} + 2\hat{j} - \hat{k}) = 0$ are perpendicular to each other.

4. Find the angle between the planes $\vec{r}\cdot(2\hat{i} - \hat{j} + 2\hat{k}) = 6$ and $\vec{r}\cdot(3\hat{i} + 6\hat{j} - 2\hat{k}) = 9$

5. Find the angle between the line $\vec{r} = (\hat{i} + 2\hat{j} - \hat{k}) + \lambda(\hat{i} - \hat{j} + \hat{k})$ and the plane $\vec{r}.(2\hat{i} - \hat{j} + \hat{k}) = 4$.

6. Prove that the acute angle between the lines whose direction cosines are given by the relations $l + m + n = 0$ and $l^2 + m^2 - n^2 = 0$ is $\frac{\pi}{3}$.

7. Prove that the three lines drawn from origin with direction cosines l_1, m_1, n_1 ; l_2, m_2, n_2 ; l_3, m_3, n_3 are coplanar if $\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} = 0.$

8. Prove that three lines drawn from origin with direction cosines proportional to (1, -1, 1), (2,-3,0), (1,0,3) lie on one plane.

9. Determine k so that the lines joining the points $P_1(k,1,-1)$ and $P_2(2k,0,2)$ shall be perpendicular to the line from P_2 to $P_3(2+2k,k,1)$.

10. Find the angle between the lines whose direction ratios are proportional to a,b,c and b-c, c-a, a-b.

11. O is the origin and A is the point (a,b,c). Find the equation of the plane through A at right angies to $\overline{OA}$.

12. Find the equation of the plane through (6,3,1) and (8,-5,3) parallel to x-axis.

OBJECTIVE AND SHORT TYPE QUESTIONS

Functions and Derivatives

Objective Type Questions:

1. Is the function $[x]$ differentiable at $x=2$?
2. Is the function $[x]$ differentiable at $x=2.5$?
3. Is the function $|x|$ differentiable at $x=0$?
4. Is the function $|x|$ differentiable at $x=2$?
5. If $f(x)=|x|$, what is the value of $f'(0-)$?
6. If the first derivative of a function vanishes at all points and if $f(0)=1$, then what is $f(x)$?
7. Give example of a function which is continuous but not differentiable at $x=1$.
8. Give example of a function whose first derivative is not differentiable at $x=2$?
9. If $f(x+y)=f(x)f(y)$ for all x,y and if $f(5)=2$ and $f'(0)=3$, then what is the value of $f'(5)$?
10. A differentiable function f defined for all $x>0$ and satisfies $f(x^2)=x^3$ for all $x>0$. What is the value of $f'(16)$?
11. If $f(x)=[\tan^2 x]$, what is $f'(0)$?
12. If $f(x)=(3x+2)^{100}$ and $f'(x)=n(3x+2)^{99}$, then what is the value of n?
13. If $x\in\left(\frac{3\pi}{4},\pi\right)$ what is $\frac{dy}{dx}$ for $y=|\cos x|+|\sin x|$?
14. What is the derivative of $f(\ell nx)$ with respect to x where $f(x)=\ell nx$?
15. If $f'(x)=\sqrt{2x^2-1}$ and $y=f(x^2)$ then what is $\frac{dy}{dx}$ at $x=1$?
16. What is the derivative of $\tan^{-1}\frac{2x}{1-x^2}$ with respect to $\sin^{-1}\frac{2x}{1+x^2}$?
17. What is the differential co-efficient of $\tan^{-1}\frac{\sqrt{1+x^2}-1}{x}$ with respect to $\tan^{-1}x$?
18. If $y=at^2$, $x=2at$ where a is a constant what is the value of $\frac{d^2y}{dx^2}$ at $x=\frac{1}{2}$?
19. Is the function $f(x)=\sin^{-1}\frac{2x}{1+x^2}$ differentiable at $x=\pm 1$?
20. Write the value of $\frac{d}{dx}\left(\sin^{-1}x+\cos^{-1}x\right)$ for $x\in(-1,1)$.
21. Write the value of $\frac{d}{dx}\sec^{-1}\left(\frac{1}{2x^2-1}\right)$, for $x\in\left(0,\frac{1}{\sqrt{2}}\right)$.
22. Write the interval in which the function $f(x)=\sin^{-1}(1-x)$ is differentiable.
23. Write the values of x for which $\frac{d}{dx}\sin\left(\sin^{-1}x\right)=1$.

24. Write the values of x for which $\frac{d}{dx}\sin^{-1}(\sin x)=1$.
25. Write the derivative of $\sec^{-1} x$ with respect to x at $x=-\frac{1}{3}$.
26. Find the derivative of $\ell n_x a$.
27. Write a logarithmic function which is differentiable at every point in R.
28. Give example of two functions which are not derivable at $x=0$, but their sum is derivable at $x=0$.
29. Write the values of x for which $\frac{d}{dx}\tan^{-1}\left(\frac{2x}{1-x^2}\right)=\frac{2}{1+x^2}$.
30. If $y=5^t$ and $t=e^{3x}$. Write the value of $\frac{dy}{dx}$ at $x=0$.
31. Write the derivative of $\sin^{-1} x$ with respect to $\cos^{-1} x$.
32. Write the value of $\frac{dy}{dx}$ if $y=\sin x+\sqrt{\sin x+\sqrt{\sin x+\sqrt{\sin x+}}}$ ....∞.
33. Write the value of $\frac{dy}{dx}$ if $y=x^{x^x}$.
34. Write the value of $\frac{dy}{dx}$ at (1, 0) where $x^2y+y^2+x=0$.
35. Write the minimum value of y_2 where $y=\sin^2 x\cos^2 x$.
36. If $x=\log t, y=t^2-1$, then what is y_2 at $t=1$?
37. What is the scope of the tangent to the curve $y=3x^2+2x-1$ at $x=2$?
38. If $y=\frac{1}{x(x+1)}$, then what is y_3?
39. If $f(x)=e^{ax}\sin ax$ and $f'''(0)=2$, then what is a?
40. A balloon is pumped at the rate of 10 cubic cm/min. What is the rate of increase of its radius when its radius is 15 cm.
41. For what values of a the function e^{ax} is increasing?
42. What is the interval in which $\log_{.5} x$ is decreasing?
43. What is the interval in which $f(x)=x^3-3x^2+3x-10$ is strictly increasing?
44. Give example of a function which is increasing in $(-\infty,2)$ and $(3,\infty)$ and decreases in (2, 3).
45. Write the least value of a for which the function f defined by $f(x)=x^2+ax+1$ increases.
46. For what value of K, Kf is increasing if f is increasing.
47. At what point of $x^2=2y$ the point (0, 3) is nearest to the curve?
48. If $\theta+\phi=\frac{\pi}{3}$, then for what value of a $\sin\theta.\sin\phi$ is maximum?

49. Write the absolute maximum and absolute minimum of the function $f(x)=\dfrac{x}{|x|}$ in [–2, 2].

50. Which condition of Rolle's theorem is violated

(i) by the function $f(x)=\sin x$ in $\left[0,\dfrac{3\pi}{4}\right]$

(ii) by the function $f(x)=|x|$ in [–1, 1]

51. Is there any tangent to the curve $y=|2x-1|$ at $\left(\dfrac{1}{2},0\right)$?

52. Write the subinterval of $(0,\pi)$ in which $\sin\left(x+\dfrac{\pi}{4}\right)$ is increasing.

Short Answer type Questions:

1. Given $f'(2)=6, f'(1)=4$ find $\lim\limits_{h\to 0}\dfrac{f(2+2h+h^2)-f(2)}{f(1+h-h^2)-f(1)}$.

2. If $(f(x))^n = f(nx)$, find $\dfrac{f'(nx)}{f'(x)}$

3. Evaluate $\dfrac{d}{dx}\left(a^{\sin x}+e^{-x}\right)$ by chain rule.

4. If $f(x)=\sin|x|-|x|$, find $f'(0+)$

5. Find the derivative of $\ell n\sqrt{\dfrac{1-\cos x}{1+\cos x}}$

6. If $y=\cos^{-1}\left(\dfrac{a+b\cos x}{b+a\cos x}\right)$ find $\dfrac{dy}{dx}$.

7. If $y=\tan^{-1}(\cot x)+\cot^{-1}(\tan x)$, find $\dfrac{dy}{dx}$.

8. If $y=\tan^{-1}\left(\dfrac{3x-x^3}{1-x^2}\right)$, find $\dfrac{dy}{dx}$.

9. If $x=a\left(\dfrac{1-t^2}{1+t^2}\right), y=\dfrac{2at}{1+t^2}$, find $\dfrac{dy}{dx}$ at $t=\dfrac{1}{\sqrt{3}}$.

10. If $x=a\cos^3\theta, y=b\sin^3\theta$, find $\dfrac{dy}{dx}$.

11. If $y=e^{e^{e^{x^{\cdot^{\cdot}}}}}$, find $\dfrac{dy}{dx}$.

12. If $y=x+\cfrac{1}{x+\cfrac{1}{x+\ldots\infty}}$, find $\dfrac{dy}{dx}$.

13. If $x^y+y^x=1$, find $\dfrac{dy}{dx}$.

14. If $e^x+e^y=e^{x+y}$, find $\dfrac{dy}{dx}$.

15. If $x^y y^x=1$, find $\dfrac{dy}{dx}$.

16. Find the derivative of $\tan^{-1}\dfrac{\sqrt{1-x^2}}{x}$ with respect to $\cos^{-1}x$.

17. Find $\dfrac{d^2y}{dx^2}$ if $x=a\cos\theta, y=b\sin\theta$.

18. If $y=e^{ax}\sin bx$ show that $y_2-2ay_1+\left(a^2+b^2\right)y=0$

19. If $x^7y^3=(x+y)^{10}$, then find $\dfrac{d^2y}{dx^2}$.

20. If $u = f\left(\frac{x^2+y^2}{xy}\right)$ prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$

21. If $f(x+y) = f(x)f(y)\ \forall\ x, y, f(5) = 2$ and $f'(0) = 3$, then show that $f'(5) = 6$.

22. If $(fog)'(1) = 3, g(1) = 2, g'(1) = 1$, then show that $f'(2) = 3$.

23. If $f(a) = 2, f'(a) = 1, g(a) = -1$ and $g'(a) = 2$, then find $\lim_{x\to a}\frac{g(x)f(a) - g(a)f(x)}{x-a}$.

24. Find the rate of change of the area of circle w.r.t. r when r = 8 cm.

25. The side of a square is increasing at the rate of 0.1 cm/sec and at the same time the area is increasing at the rate of 30 sq. cm/sec. Find the length of side of the square.

26. A particle moves along a straight line according to the law $s = t^3 - 3t^2 + 5t$. Find its velocity and acceleration at the end of 1 sec.

27. Show that the function $f(x) = \frac{1}{x}$ is decreasing in $(0, \infty)$.

28. Find the intervals where the function $f(x) = x^3 - 12x + 10$ is increasing.

29. Find the slope of the normal to the curve $y = xe^{-x}$ at $x = 2$.

30. Find the angle between the tangents to the curve $y = x^2 - 5x + 6$ at the points (2,0) and (3, 0).

31. Find the points on the curve $y - x^2 + 2x - 1 = 0$, where the tangent is parallel to the x-axis.

32. Find the points on the curve $9y^2 = x^3$, where normal to the curve makes equal intercepts with axes.

33. If $y = x^4 - 12$ and if x changes from 2 to 1.99, find the approximate error in y.

34. Using differential find the value of $\sqrt{16.2}$.

35. Show that $f(x) = x^3 - 6x^2 + 24x + 4$ has neither a maximum nor a minimum value.

36. Find the points where $f(x) = 8x^2 - x^4 - 4$ has local maximum or minimum.

37. Find the absolute maximum and absolute minimum value of the function $f(x) = 2x^3$ in [–2, 2].

38. Find the absolute maximum and absolute minimum value of $f(x) = x - x^3$ in [0, 1].

39. Using mean value theorem, prove that $\sin x < x, x \in (0, \pi/2)$.

Integration

Objective type of Question:

1. Write antiderivative of $\tan^2 x$.

2. Write the primitive of $\frac{x-1}{x\sqrt{x^2-1}}$.

3. Write the value of $\int xa^{x^2+1}dx$

4. Write the value of $\int \sqrt{1-\cos 2x}\,dx$

5. If $f'(x) = e^x + \frac{1}{1+x^2}$, what is $f(x)$?

6. Write the value of $\int e^{\log\cot^2 x}dx$

7. Write the value of $\int \frac{e^x - 1}{1-e^{-x}}dx$

8. Write the value of $\int x^{20}\sec^2 dx - \int x^{20}\tan^2 dx$

9. Write the value of $\int \frac{\sec^2 x}{\cos ec^2 x}dx$

10. Write the value of $\int \frac{1}{\sqrt{x}e^{\sqrt{x}}}dx$

11. Write the primitive of $\sin^3 x$.

12. Write the value of $\int e^x e^{e^x} e^{e^{e^x}} dx$

13. Write the value of $\int \frac{d\left(x^2+1\right)}{x^2+5}$

14. Write the value of $\int \frac{d\left(x^2+1\right)}{1+x^4}$

15. Write the value of $\int x\sec^2 x dx$

16. Write the value of $\int e^{-x}(1-\tan x)\sec x\, dx$

17. Write the value of $\int \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)dx$

18. Write the value of $\int e^x\left(\ell n\sin x+\cot x\right)dx$

19. $\int \sin^{-1}\sqrt{\frac{x}{a+x}}dx =$ ________.

20. $\int \tan^{-1}\frac{x-5}{1+5x}dx =$ ________.

21. $\int \tan^{-1}\sqrt{x}dx =$ ________.

22. $\int \cot x\log\sin x dx =$ ________.

23. $\int e^{\cos^2 x}\sin 2x dx =$ ________.

24. $\int \frac{dx}{e^x+e^{-x}} =$ ________.

25. Write the value of $\int \frac{1}{x\ell nx}dx$

26. Write the value of $\int \frac{x^2(5x^3+4x^2+3x)}{x^5+x^4+x^3+1}dx$

27. Write the value of $\int \frac{dx}{x^{1/5}\left(1+x^{4/5}\right)^{1/2}}$

28. Write the value of $\int \frac{\{f(x)\phi'(x)+f'(x)\phi(x)\}\{\log\phi(x)+\log f(x)\}}{f(x)\phi(x)}dx$

29. Write the value of A if $\int \frac{dx}{(x+1)(x+2)} = A\log(x+1)-\log(x+2)+C$

30. Write the value of $\int \frac{\cos x-\sin x}{\sin x+\cos x}dx$

31. Write the value of $\int \frac{1}{1+\sin^2 x}dx$

32. $\int_{-\pi/2}^{\pi/2}(\sin|x|-\cos|x|)dx =$ ________.

33. $\int_{\pi/6}^{\pi/3}\frac{dx}{1+\sqrt{\tan x}}dx =$ ________.

34. $\int_0^{\pi/2}\frac{\sin x-\cos x}{1+\sin x.\cos x} =$ ________.

35. $\int_{-1}^{1} e^{|x|}dx =$ ________.

36. $\int_0^2 |2x-1|\,dx =$ ________.

37. $\int_{-1}^{1}\log\frac{4-x}{4+x}dx =$ ________.

38. If $f(0)=1,\ f(2)=3,\ f'(2)=5,$

then the value of $\int_0^1 xf''(2x)dx =$ ________.

39. $\int_0^{\pi/2}\log\tan x\, dx =$ ________.

40. If $F(x)=\int_0^x e^{2t}\cos 5t\,dt$, what is $F'(x)$?

41. Write the value of $\frac{d}{dx}\int_0^{x^2}\sin t\,dt$.

42. Write the value of $\lim_{x\to 0}\left[\frac{d}{dx}\left(\int_a^x\sqrt{1+t^2}\,dt\right)\right]$.

43. Write the vale of $\int_{-1}^{1}\left(\sin^5 x+x\right)dx$.

44. Write the value of $\int_0^{\pi/2}\frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx$.

45. If $f(3-x)=f(x)$, then $\int_1^2 xf(x)dx=$ ——————.

46. $\lim_{x\to 0}\frac{\int_0^x \sec^2 t\,dt}{x\sin x}=$ ——————.

47. $\int_0^1 x(1-x)^n\,dx=$ ——————.

48. $\int_0^{3/2}[2x]dx=$ ——————.

49. $\int_0^{\pi}\cos^3 x\,dx=$ ——————.

50. Area under the curve $x+y=1$ in the first quadrant is ——————.

51. The value of $\int_1^2\left[f\{g(x)\}\right]^{-1}f'\{g(x)\}g'(x)dx$ for $g(1)=g(2)$ is ——————.

52. If $f(x)$ is a quadratic polynomial such that $f(0)=2$, $f'(0)=-3$ and $f''(0)=4$, then $\int_{-1}^{1}f(x)dx=$ ——————.

53. The value of $\int_0^{\pi/2}[\cos x]dx=$ ——————.

54. If $\int_0^a f(x)dx=\lambda$ and $\int_0^a f(2a-x)dx=\mu$

then $\int_0^{2a}f(x)dx=$ ——————.

Short Answer Type Questions:

1. Evaluate $\int(2x+1)(x^2+x+1)^{10}dx$

2. Evaluate $\int\left\{\frac{e^{2\log x}+e^{3\log x}}{x+x^2}\right\}dx$

3. Evaluate $\int\left[\cos^{-1}\left(\frac{1-\tan^2 x}{1+\tan^2 x}\right)+\sin^{-1}\left(\frac{2\tan x}{1+\tan^2 x}\right)\right]dx$

4. Evaluate $\int\frac{1-\cos 2x}{1+\cos 2x}dx$

5. Evaluate $\int\frac{1}{\sin^2 x.\cos^2 x}dx$.

6. If $f'(x)=e^x+\frac{1}{1+x^2}$ and $f(0)=1$, then find $f(x)$.

7. Evaluate $\int 2^{\cos^2 x}\sin 2x\,dx$.

8. Evaluate $\int\frac{\sin x}{\cos(x-\alpha)}dx$

9. Evaluate $\int\frac{1}{1+\tan x}dx$

10. Evaluate $\int\frac{\cos 4x+\cos 2x}{\sin 4x+\sin 2x}dx$

11. Evaluate $\int\tan x\sec^4 x\,dx$

12. Evaluate $\int\frac{x^9}{x^{20}+4}dx$

13. Evaluate $\int \frac{3x+4}{x^2+4}dx$

14. Evaluate $\int \frac{\cos 3x.\cos x}{1+\cos 2x}dx$

15. Integrate $\int \frac{x^5}{x^2+1}dx$

16. Integrate $\int \frac{2x+5}{(x+2)^{7/2}}dx$

17. Integrate $\int \tan^{-1} x dx$

18. Integrate $\int \sec\theta \tan\theta \sqrt{\tan^2\theta - 3d\theta}$

19. Evaluate $\int \frac{1}{x+x^{1/3}}dx$

20. Evaluate $\int (6x+1)\sqrt{3x+4dx}$

21. If $\int \frac{x}{(x-1)(2x-1)}dx = \int \left[\frac{A}{x-1}+\frac{B}{2x-1}\right]dx$ find A and B.

22. Evaluate $\int \frac{1}{2\sin x + 3\cos x}dx$

23. Evaluate $\int \cos^5 x\, dx$

24. Evaluate $\int \frac{\cos 2x}{\sin 7x.\cos 5x}dx$

25. Evaluate $\int \frac{d\theta}{\cos^2\theta + 2\sin^2\theta}$

26. Evaluate $\int e^x\left(\frac{1+\sin x}{1+\cos x}\right)dx$

27. Evaluate $\int e^{\tan^{-1}x}\left(\frac{1+x+x^2}{1+x^2}\right)dx$

28. Evaluate $\int \frac{\sin^{-1}x}{\left(1-x^2\right)^{3/2}}dx$

29. Evaluate $\int \sin^{-1}\sqrt{\frac{x}{a+x}}\,dx$

30. Evaluate $\int \frac{\sqrt{\tan x}}{\sin x.\cos x}$

31. Evaluate $\int_0^1 \frac{x^2}{x^2+1}dx$

32. Evaluate $\lim_{x\to 0}\frac{\int_0^{x^2}\sin\sqrt{t}\,dt}{x^3}$

33. Evaluate $\int_0^{\pi/2} x\sin x dx$

34. Evaluate $\int_{-1}^{3}\{|x|+[x]\}\,dx$

35. Evaluate $\int_{-3}^{3}|x+1|\,dx$

36. $\int_0^{10}\sin(x-[x])\,\pi dx$

37. If $f(x)=\int_1^{x^2}\tan^{-1}\sqrt{t}\,dt,\ t>0$, find $f'(1)$.

38. If $f(x)=\frac{1}{x^2}\int_2^x\left[t^2+f'(t)\right]dt$, find $f'(2)$.

39. If $f(x)=\cos x-\int_0^x (x-1)f(t)dt$, then find $f''(x)+f(x)$.

40. Evaluate $\int_0^4\left[\sqrt{x}\right]dx$

41. Evaluate $\int_0^{\pi/2}\frac{\cos x dx}{(2-\sin x)(3+\sin x)}$

42. Evaluate $\int_0^5\sqrt{25-x^2}\,dx$

43. Evaluate $\int_3^6\frac{\sqrt{x}}{\sqrt{9-x}+\sqrt{x}}dx$.

44. Evaluate $\int_0^{\pi/2} \ell n \cot x\, dx$.

45. Find the area bounded by the curve $x = y^2$ and the straight lines $x = 0, y = 1$.

46. Find the area bounded by the curve $y = \sin x$ between $x = 0$ and $x = 2\pi$.

47. Find the area of the parabola $y^2 = 36x$ bounded by its latus rectum.

48. Find the area of the region bounded by $y = 6x - x^2$, x–axis and between ordinates $x = 0$ and $x = 6$.

49. Find the area of the trapezium bounded by the sides $y = x, x = 0, y = 3, y = 4$.

Differential Equation

Objective Type Questions:

1. Write the order and degree of the differential equations given by

(i) $\frac{d^2y}{dx^2} + 3\left(\frac{dy}{dx}\right)^4 + y = 0$ (ii) $\left\{y + \left(\frac{dy}{dx}\right)^3\right\}^{\frac{1}{2}} = 1 + x$

(iii) $\frac{d^2y}{dx^2} = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{4}{3}}$ (iv) $\left(\frac{d^2y}{dx^2}\right)^{\frac{3}{2}} = 1 + \left(\frac{dy}{dx}\right)^5$

2. The degree of the differential equation satisfying $\sqrt{1-x^2} + \sqrt{1+y^2} = a(x-y)$ is ———.

3. Write the differential equation corresponding to $v = \frac{a}{r} + b$ is ———.

4. The differential equation of $y = a\cos 2x + b\sin 2x$ is ———.

5. The differential equation of the family of straight lines parallel to x-axis is ———.

6. The differential equation of the family of straight lines passing through origin is ———.

7. The differential equation of the family of parabolas with axis along x-axis is ———.

8. Write the differential equation whose **general solution is** $y = ce^{2x}$.

9. Write the differential equation of parabolas $y^2 = 8x + c$.

10. Write the differential equation whose general solution is $y = a\cos 3x + b\sin 3x$.

11. Write the general solution of the differential equations:

(i) $\frac{dy}{dx} = \cos x - x$ (ii) $\frac{dy}{dx} = \frac{2}{x^2}$ (iii) $\frac{dy}{dx} = \cot^2 y$

(iv) $\frac{dy}{dx} = 4x^3 + 2x + \sec^2 x$ (v) $\frac{dy}{dx} = \frac{1+y^2}{1+x^2}$

(vi) $\frac{d^2y}{dx^2} = 0$ (vii) $\frac{d^2y}{dx^2} = x$

12. Write the solution of $\frac{d^2y}{dx^2} = 0, y(0)$ and $y'(0) = 1$

13. Write the solution of $\frac{dy}{dx} = 2x, y = 2$ when $x = 1$.

14. Write the general solution of $\frac{ydx - xdy}{y} = 0$.

15. Write solution of $\sqrt[3]{4 + \frac{dy}{dx}} = 2$.

16. Write solution of $\frac{dy}{dx} = \frac{1}{1+x^2}, x = 0, y = 1$.

17. Write solution of $\frac{dy}{dx} = 2y, y(0) = 2$.

18. Write solution of $\frac{dy}{dx} = \frac{2}{y}, y(0) = 0$.

19. Write the solution of $ydx - xdy = x^2 ydx$.

20. Write integrating factor of $(1 + y^2)dx + xdy = \tan^{-1} ydy$.

Short Answer Type Questions:

1. Find the solution of $\frac{dy}{dx} = e^x \sin x$.

2. Find the solution of $\frac{dy}{dx} = \frac{y}{x}$.

3. Find integrating factor of $(x + \tan y)dy = \tan ydx$.

4. Find integrating factor of $(x - \ell ny)\frac{dy}{dx} = -y\ell ny$.

5. Solve $e^{-2x}\frac{dy}{dx} = x$.

6. Solve $\frac{dy}{dx} = x + y$.

7. Solve $\frac{dy}{dx} = \frac{x+y+1}{x+y+2}$.

8. Find the equation of the curve whose slope is given by $\frac{dy}{dx} = \frac{2y}{x}$ and which passes through (1, 1).

9. If $y + \frac{dy}{dx} = 0$ and $y(0) = 2$, find y.

10. Find differential equation of the curve $y = ae^{3x} + be^{5x}$.

3-D Geometry

Objective Type Questions:

1. Write the projection of the point (1, 2, 3) on *xy*-plane.
2. Write the projection of the point (2, 3, 1) on *y*- axis.
3. Write the image of the point (2, 1, 3) with respect to *yz*-plane.
4. Write the distance of the point (3, 1, 5) from *y*-axis.
5. A line is perpendicular to *xy*-plane. Write the angle made by the line with *z*-axis.
6. If the distance between the points (1, 2, z) and (–1, 2, 1) is 3, then find *z*.
7. If the direction angles of a line are $\alpha = 30°$, and $\beta = 60°$, find the other direction angle γ.
8. Write the direction cosine of the line whose direction ratios are $\langle 1, 2, 3 \rangle$.
9. Write the direction cosines of the line joining (1, 2, 3) and (1, 1, 2).
10. Write the distance of the point (1, 1, 2) from *x*- axis.
11. Write the distance of the point (2, 3, 6) from *zx*-plane.
12. Write the locus of a point p which moves in space such that its distance from origin is 4 units.
13. Write the centroid of the triangle with vertices (1, 2, 3), (2, 1, 2), (3, 0, 1).
14. Write the ratio in which the line joining the points (2, 3, 4) and (–3, 5, –4) is divided by *yz*-plane.
15. Write the value of *y* so that the points (1, y, 2), (3, 2, –1) and (–4, 6, 3) are collinear.

16. Write the projection of the line segment joining the points (2, 1, 3) and (3, 2, 4) on z-axis.
17. Write the projection of the line segment joining (2, 4, 3) and (3, 2, 4) on yz-plane.
18. If d.c.s of a line be $\left(\frac{1}{2}, \frac{3}{4}, \frac{k}{4}\right)$ what is the value of k?
19. If α, β, γ be direction angles of a line, what is the value of $\sin^2\alpha + \sin^2\beta + \sin^2\gamma$.
20. Write the direction cosines of the normal to the plane $x - y + 1 = 0$.
21. Write the equation of the plane passing through the point (1, 2, 3), the direction ratios of the normal to the plane being $\langle 3, 5, 7 \rangle$.
22. Write the value of x-intercept of the plane $x + y + 2z = 1$.
23. The equation $ax + by + c = 0$ represents a plane parallel to _______ axis.
24. Write the equation of the plane parallel to x-axis having intercepts 5 and 6 on y and z- axis respectively.
25. The plane having equation $2x + 5z + 1 = 0$ is parallel to ____ plane.
26. What is the distance from origin of the plane $3x + 4y = 1$.
27. Write the equation of the plane passing through (1, 2, 3) and parallel to the plane $x + 2y + 5z = 0$.
28. Write the distance between the planes $x - 2y + z = 6$ and $2x - 4y + 2z = 8$.
29. Write the equation of the plane through origin and passing through the intersection of the planes $3x - 2y + z - 1 = 0$ and $x - 2y + 3z - 1 = 0$.
30. Write the position of the points A (2, 4, –3) and B(2, –6, 2) with respect to the plane $4x + 7y + 6 = 0$.
31. Write the equation of the plane $3x - 4y + z + 5 = 0$ in normal form.
32. Write the equation of the plane $x + 3y - 7z + 2 = 0$ in the intercept form.
33. Write the equation of the plane passing through x-axis and y-axis.
34. Write the equation of the plane perpendicular to z axis and passing through (1, –2, 4).
35. If the planes $2x + 4y + z + 2 = 0$ and $x - 2y + kz + 5 = 0$ are perpendicular to each other what is the value of k?
36. Write down the equation of x-axis.
37. Write the vector equation of a line through the point (1, 2, 3) and parallel to the vector $3\hat{i} + 2\hat{j} - 2\hat{k}$.
38. Write parametric equation of a line through (1, –1, 2) parallel to the vector $3\hat{i} + \hat{j} - \hat{k}$.
39. Write the equation of the line passing through the points (3, –2, –5) and (3, –2, 6).
40. Write the equation of the line $\frac{x-5}{3} = \frac{y+4}{7} = \frac{z-6}{2}$ in vector form.
41. Write the equation of the line in symmetric form through the point (1, –2, 3) having direction ratios $\langle 3, -4, 5 \rangle$.
42. Write the equation of the line $x = ay + b, z = cy + d$ in symmetric form.
43. Write the equation of the plane passing through the point (2, 3, 1) and perpendicular to the line $\frac{x-1}{1} = \frac{y-2}{-1} = \frac{z+1}{3}$.
44. What is the number of independent constants that occur in the general equation of a plane.
45. The angle between the planes $x + y + z + 1 = 0$ and $2x + y + z + 2 = 0$ is ———.

46. The angle between the plane $3x+3z-5=0$ and the line $\frac{x-1}{1}=\frac{y-2}{-1}=\frac{z-3}{0}$ is ______.

47. If the lines $\frac{x-1}{k}=\frac{y-3}{2}=\frac{2-z}{-5}$ and $\frac{x+1}{3}=\frac{2-y}{-4}=\frac{z}{k}$ are perpendicular to each other, then what is the value of K?

48. Find the vector equation of the line passing through the point with position vector $2\hat{i}-3\hat{j}-5\hat{k}$ and perpendicular to the plane $\vec{r}\cdot(6\hat{i}-3\hat{j}-5\hat{k})+2=0$.

49. Find the equation of the plane passing through the intersection of the planes $\vec{r}\cdot(\hat{i}+\hat{j}+\hat{k})=1$ and $\vec{r}\cdot(2\hat{i}+3\hat{j}-\hat{k})+4=0$ and parallel to *x*-axis.

Short Type Questions:

1. Find the co-ordinates of the foot of the perpendicular drawn from the point (1, 3, 4) to the line joining the points (3, 0, –1) and (0, 1, –2).
2. Prove that the points (3, –2, 4), (1, 1, 1) and (–1, 4, –2) are collinear.
3. Find *a, b* such that the points (–1, 1, 3), (2, *a*, 4) and (1, 2, *b*) are collinear.
4. Two vertices of a triangle are (1, 2, 3) and (3, –2, 7), and its centroid is (2, 1, –2). Find the remaining vertex of the triangle.
5. Show that the points (1, 2, 3), (2, 3, 1) and (3, 1, 2) form an equilateral triangle.
6. Find the direction cosines of the line segment joining (1, –1, 2) and (2, 1, 1).
7. Find the projection of the line segment joining (1, 2, 3) and (2, 3, 4) on a straight line having d.r's $\langle 2,1,3\rangle$.
8. If P, Q, R, S are points (1, 2, 5), (–2, 1, 3), (4, 4, 2) and (2, 1, –4) respectively, find the projection of PQ on RS.
9. If A, B, C are points (0, 4, 1), (2, 3, –1), (4, 5, 0), find the angle that AB makes with BC.
10. Find the acute angle between the lines whose d.rs are (1, 1, 2) and $(\sqrt{3}-1,-\sqrt{3}-1,4)$ respectively.
11. Find the equation of the plane passing through the points (2, 1, 3), (3, 2, 1) and (1, 0, –1).
12. Find the equation of the plane parallel to z-axis and with intercepts 3 and 4 on x and y axes respectively.
13. Find the equation of the plane through the point (–1, 3, 0), which is perpendicular to both the planes $x+2y+2z-5=0$ and $3x+3y+2z-8=0$.
14. Find the equation of the plane if the point (5, –3, 4) is the foot of the perpendicular drawn from origin to the plane.
15. Find the equation of a plane which is at a distance 3 units from the origin and which is normal to the vector $2\hat{i}+3\hat{j}-6\hat{k}$.
16. Find the components of the unit vector perpendicular to the plane $\vec{r}.(2\hat{i}+3\hat{j}-6\hat{k})-6=0$.
17. Writing the equation of the plane $3x-2y+z+2=0$ in normal form find its distance from origin.
18. Find the angle between the planes $x+2y+3z+1=0$ and $3x+2y+z+2=0$.
19. Find the equation of the plane through the point (2, 1, 0) and passing through the intersection of the planes $3x-2y+z-1=0$ and $x-2y+3z-1=0$
20. Find the position of the points (1, 2, –1) and (2, –1, 3) with respect to the plane $x+3y+z+1=0$.
21. Find the intercepts of a plane $3x+4y-7z=84$ on co-ordinate axes.
22. Find the equation of the plane through the points (1, 1, 0), (–2, 2, 1) and (1, 2, 1).
23. Find the equation of the plane passing through the points (1, –1, 1) and (1, 1, –1) and perpendicular to *xy*- plane.

24. A plane meets the co-ordinate axes at A, B, C such that the centroid of ΔABC is (2, –2, 3). Find the equation of the plane.
25. Find the equation of the plane through the feet of the perpendicular drawn from P(2, 3, 5) on co-ordinate planes.
26. Find equation of a plane through (2, –3, 1) and perpendicular to the line joining the points (3, 4, –1) and (2, –1, 5).
27. Find the equation of a plane bisecting the line segment joining (–1, 4, 3) and (5, –2, –1) at right angle.
28. Find the equations of planes parallel to the plane $6x-3y-2z+5=0$ and at a distance 2 units from origin.
29. Find the point where the line $\frac{x-2}{1}=\frac{y}{-1}=\frac{z-1}{2}$ meets the plane $2x+y+z=2$.
30. If the points (–1, 3, 2), (–4, 2, –2) and (5, 5, λ) are collinear, find λ.
31. Find the angle between the plane $x+y+z-2=0$ and the line $\frac{x+3}{2}=\frac{y-1}{1}=\frac{z+2}{4}$.
32. Find the equation of the plane passing through the point (2, 3, 1) and perpendicular to the line $\frac{x-1}{1}=\frac{y-2}{-1}=\frac{z+1}{2}$.
33. Find the image of the point (3, –2, 1) in the plane $x-y+3z=2$.
34. Find the equation of the plane passing through the line $\frac{x-8}{3}=\frac{y+19}{-16}=\frac{z-10}{7}$ and the point (1, 2, –4).
35. Find the length of the perpendicular from (2, 0, 1) on the line $x=y=z$.
36. Find the equation of the line through (–1, 0, 1) and perpendicular to the plane $x+2y+1=0$.
37. Show that the lines $\frac{x+3}{2}=\frac{y+5}{3}=\frac{z-7}{-3}$ and $\frac{x+1}{4}=\frac{y+1}{5}=\frac{z+1}{-1}$ are co-planer.
38. Show that the plane $2x-y-2z-4=0$ touches the sphere $x^2+y^2+z^2+2x-6y+1=0$.
39. Find the equation of the sphere concentric with the sphere $x^2+y^2+z^2-4x-6y+8z-5=0$ and passing through origin.
40. Find equation of the sphere whose centre is (2, –3, 4) and which passes through the point (1, 2, –1).
41. Find equation of the sphere with centre (3, 6, –4) and touching the plane $2x-2y-z-10=0$.
42. Find equation of the sphere passing through the points (0, 0, 0), (2, 0, 0), (0, 3, 0) and (0, 0, 4).
43. If one end of a diameter of a sphere $x^2+y^2+z^2-4x-2y+2z-30=0$ is (4, 5, –5) find the other end.
44. Find the equation of the sphere on the join of (2, 3, 5) and (4, 9, –3) as ends of diameter.

VECTORS

Objective Type Questions:

1. If the vectors $\vec{a}=\hat{i}+2\hat{j}-3\hat{k}$ and $\vec{b}=2\hat{i}+\alpha\hat{j}+6\hat{k}$ are parallel, write the value of α.
2. If $|\alpha\vec{a}|=2$, what is the value of α?
3. If the position vectors of two points A and B are $3\hat{i}+2\hat{j}+\hat{k}$ and $2\hat{i}-5\hat{j}+4\hat{k}$ respectively, what is the magnitude of $\overrightarrow{AB}$?
4. Given position vectors of P and Q, as (1,0,–2) and (3,–2,0) respectively, what is the unit vector parallel to $\overrightarrow{PQ}$?
5. Given co-ordinates of the points A and B as (–1, –2) and (–5, –6) respectively, what is the magnitude of $\overrightarrow{AB}$?

6. Write the unit vectors parallel to the vector $3\hat{i}+\hat{j}-2\hat{k}$.
7. Find the direction cosines of the vector. $\vec{r}_1-\vec{r}_2$ where $\vec{r}_1=\hat{i}+\hat{j}+\hat{k}$ and $\vec{r}_2=2\hat{i}+\hat{j}+2\hat{k}$.
8. Are the points A(2, 6, 3), B(1, 2, 7) and (3, 10, –1) collinear?
9. What is the angle between the vectors $\vec{a}=\hat{i}+\hat{j}-\hat{k}$ and $\vec{b}=2\hat{i}-\hat{j}+\hat{k}$.
10. What is the scalar product of the vectors $\vec{a}=2\hat{i}+\hat{j}+2\hat{k}$ and $\vec{b}=\hat{i}-2\hat{j}+\hat{k}$?
11. What is the scalar projection of the vector $\vec{a}=\hat{i}+\hat{j}-\hat{k}$ on $\vec{b}=2\hat{i}+2\hat{j}-\hat{k}$?
12. What is the vector projection (component) of the vector $\vec{a}=2\hat{i}+\hat{j}-\hat{k}$ on $\vec{b}=\hat{i}+2\hat{j}+\hat{k}$?
13. What is the unit vector perpendicular to each of the vectors $\hat{i}+\hat{j},\ \hat{j}+\hat{k}$?
14. What is the area of the parallelogram whose sides are vectors $2\hat{i}+\hat{j}$, and $2\hat{j}+\hat{k}$?
15. The area of the triangle with vectors (2, 0, 0), (0, 1, 0) and (0, 0, 3) is ________.
16. If the vectors $\hat{i}+2\hat{j}+\hat{k}$ and $2\hat{i}+3\hat{j}+\alpha\hat{k}$ are perpendicular, then what is α?
17. If the vectors $\alpha\hat{i}+3\hat{j}-\hat{k}$ and $2\hat{i}+\hat{j}+\hat{k}$ are parallel, what is α?
18. If $\vec{a}=\hat{i}-2\hat{j},\ \vec{b}=\hat{j}+\hat{k}$, what is the component of $\vec{a}$ perpendicular to $\vec{b}$?
19. If $|\vec{a}|=10, |\vec{b}|=1$ and $\vec{a}.\vec{b}=0$, then $|\vec{a}\times\vec{b}|=$ ________.
20. If $\vec{a}.\vec{b}=0$ and $\vec{a}\times(\vec{b}\times\vec{c})=\vec{0}$, then $\vec{a}.\vec{c}=$ ________.
21. If two vectors $\vec{a}$ and $\vec{b}$ are such that $|\vec{a}+\vec{b}|=|\vec{a}-\vec{b}|$, then what is the angle between $\vec{a}$ and $\vec{b}$?
22. If $[\vec{a}\ \vec{b}\ \vec{c}]=10$, what is the value of $[\vec{a}+\vec{b}\ \ \vec{b}+\vec{c}\ \ \vec{c}+\vec{a}]$?
23. If $\vec{a}=4\hat{i}+n\hat{j}-3\hat{k}$ and $|\vec{a}|=13$, what is the value of n?
24. If $[\vec{a}\ \vec{b}\ \vec{c}]=5$, then what is $[\vec{a}\times\vec{b}\ \ \vec{b}\times\vec{c}\ \ \vec{c}\times\vec{a}]$?
25. If $\vec{a}$ and $\vec{b}$ are non-collinear and $\vec{p}=x\vec{a}+(y+1)\vec{b},\ \vec{q}=y\vec{a}+x\vec{b}$, then what is the value of x and y if $\vec{p}=2\vec{q}$?
26. What is the value of x if the vectors $3\hat{i}-7\hat{j}-4\hat{k},\ 3\hat{i}-2\hat{j}+\hat{k}$ and $\hat{i}+\hat{j}+x\hat{k}$ are coplanar?
27. If $(\vec{a}+\vec{b})(\vec{a}-\vec{b})=12$ and $|\vec{a}|=2|\vec{b}|$ then what is $|\vec{a}|$?
28. If $\vec{a},\ \vec{b},\ \vec{c}$ are unit vectors such that $\vec{a}+\vec{b}+\vec{c}=\vec{0}$, then what is the value of $\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a}$?
29. If θ is angle between $\vec{a}$ and $\vec{b}$ and $|\vec{a}\times\vec{b}|=|\vec{a}.\vec{b}|$, then what is the value of θ?
30. If $\vec{a}=2\vec{b}$ and $\vec{c}=-3\vec{b}$ what is angle between $\vec{a}$ and $\vec{c}$?

Short Answer Type Questions:

1. Show that the points A, B, C with position vectors $\vec{a}+2\vec{b}+3\vec{c}, 2\vec{a}+3\vec{b}-4\vec{c}$, $-7\vec{b}+10\vec{c}$ are collinear.
2. Find a vector in the direction of the vector $\vec{a}=5\hat{i}-\hat{j}+2\hat{k}$ which has magnitude 8 units.
3. Write the values of m and n for which the vectors $(m-1)\hat{i}+(n+2)\hat{j}+4\hat{k}$ and $(m+1)\hat{i}+(n-2)\hat{j}+8\hat{k}$ are parallel.
4. Find the magnitude of $\vec{a}+\vec{b}-2\vec{c}$ where $\vec{a}=(2,3,4),\ \vec{b}=(1,-1,2)$ and $\vec{c}=(1,0,3)$.

5. Show that the vectors $\hat{i}+\hat{j}-2\hat{k}, \hat{i}-2\hat{j}+\hat{k}$ and $2\hat{i}-\hat{j}-\hat{k}$ are the sides of an equilateral triangle.
6. If $\vec{a}=(2,3,6), \vec{b}=(2,-2,1), \vec{c}=(-1,0,2)$ find the direction cosines of $\vec{b}-\vec{a}+2\vec{c}$.
7. If the sum of two unit vectors is a unit vector, find the magnitude of their difference.
8. Show that the vectors $4\hat{i}+4\hat{j}+4\hat{k}, 7\hat{i}+6\hat{j}-\hat{k}$ and $3\hat{i}+2\hat{j}-5\hat{k}$ form a right angled triangle.
9. Find the value of k for which A(1, 0, 3), B(–1, 2, 4), C(1, 2, 1) and D (k, 2, 5) are coplaner.
10. Find the scalar components of the unit vector which is perpendicular to the vectors $\hat{i}+2\hat{j}-\hat{k}$ and $3\hat{i}-\hat{j}+2\hat{k}$.
11. Find a unit vector perpendicular to each of the vectors $(\vec{a}+\vec{b})$ and $(\vec{a}-\vec{b})$ where $\vec{a}=\hat{i}+\hat{j}+\hat{k}$ and $\vec{b}=\hat{i}+2\hat{j}+3\hat{k}$.
12. Find the vector of magnitude 5 units and parallel to the resultant of the vectors $\vec{a}=2\hat{i}+3\hat{j}-\hat{k}$ and $\vec{b}=\hat{i}-2\hat{j}+\hat{k}$.
13. If $\vec{a}=\hat{i}+2\hat{j}-3\hat{k}$ and $\vec{b}=3\hat{i}-\hat{j}+2\hat{k}$ then find the angle between $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$.

LINEAR PROGRAMMING

Objective Type Questions:

Write True or False

1. The region given by $2x+5y\geq 1$ is a bounded region.
2. The region $x+2y\leq 8, 2x+y\leq 8$ and $x\geq 0, y\geq 0$ is unbounded.
3. The feasible region $x+y\geq 0, 2x+y\leq 0$ is a bounded set.
4. (1, 5) is a point in the region $2x-y\geq 4$, $2x+5y\leq 2$.
5. The minimum value of $2x+5$ subject to $3x-1\geq 1$ is ———.

Short Type Questions:

1. Shade the feasible region given by the inequation $2x+3y\leq 6,\ x\geq 0, y\geq 0$.
2. Find the feasible region satisfying the inequation $2x+y\leq 4,\ x\geq 0, y\geq 0$.
3. For the LPP:

   Maximize $z=5x_1+7x_2$

   subject to $x_1+x_2\leq 4$

   $3x_1+8x_2\leq 24$

   $10x_1+7x_2\leq 35$

   $x\geq 0, y\geq 0,$

   Examine whether

   $\left(\frac{10}{3},\frac{11}{5}\right)$ is a feasible solution or not.
4. Find the feasible solution for the system $x+y\geq 1, 2x+y\leq 4, x\geq 0, y\geq 0$.
5. Find the solution of the LPP:

   Maximize $z=3x+4y$

   subject to $x+y\leq 1$

   $x,y\geq 0$

Answers

EXERCISE 1(a)

1. (i) Symmetric, transitive (ii) Reflexive, Symmetric, transitive
   (iii) Symmetric (iv) transitive (v) reflexive (R) symmetric (S) transitive (T)
   (ii) and (v) are equivalince relations.

2. (i) {(1,2), (2,4), (3,6), (4,8), (5,10), (6,12)}, $\not R, \not S, \not T$
   (ii) {(1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (2,2), (2,4), (2,6), (3,3), (3,6), (4,4), (5,5), (6,6)}, $\not R, \not S, \not T$
   (iii) {(1,1), (1,2), (1,3), (1,4), (1,5), (2,2), (2,4), (3,2), (3,5), (4,2), (5,2)}, $\not R, \not S, \not T$
   (iv) {(1,1), (2,1), (2,2), (3,1), (3,2), (3,3), (4,1), (4,2), (4,3), (4,4)}, $\not R, \not S, \not T$

3. (i) $\not R, \not S, \not T$ (ii) R, S, T (iii) $\not R$, S, $\not T$, (iv) R, S, T, (v) $\not R$, S, $\not T$, (vi) R, S, T

4. (i) R = {(1,1), (2,2), (3,3), (3,4), (4,3), (4,4)} [1]={1}, [2]={2}, [3]={3,4}=[4]
   (ii) R = {(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3), (4,4)}
   [1]=[2]=[3]={1,2,3}, [4]={4}
   (iii) R={{1,2,3,4}×{1,2,3,4}}, [1]=[2]=[3]=[4]={1,2,3,4}

7. xRx may not be true for all x. Hence R may not be reflexive.;   8. 4 ;   9. 5

10. [1]=Z, $[\frac{1}{2}] = \{n+\frac{1}{2} : n \in Z\}$,   11. (i) 3, (ii) 10, (iii) 1   12. (i) 7 (ii) 4 (iii) 2 (iv) 0 ; 14. 3, 10, 7

EXERCISE 1(b)

1. {(x,u), (y,u)}, {(x,u),(y,v)}, {(x,v),(y,u)}, {(x,v),(y,v)}
   2 one-one, 2 onto, 2 one-one onto

2. (i) n^n (ii) n(n-1)(n-2)...(n-m+1), 0, n! ; 3. (a) iv, (b) ii, (c) i, (d) i, ii, iii (e) (i), (ii), (iii) (f) (iv) (g) (iv) (h) (iv) (i) (i), (ii), iii

8. gof = {(1,x),(2,x), (3,4), (4,x), fog not defined.

9. gof = {(1,2), (2,3), (3,1)}, fog = {(2,4), (4,7), (7,3)}

10. (i) $domf$=Ro$\{x \in R : x \geq 0\}$, $domg$=R, (ii) $(fog)(x) = \sqrt{1-x^2}$
    $(gof)(x) = 1-x$. $dom\ fog = \{x \in R : |x| \leq 1\}$, $dom\ gof = R_0$
    (iii) dom h = R.

11. (i) $(fog)(x) = (x^2-2)^3+1$, $(gof)(x) = (x^3+1)^2-2$, $gof \neq fog$.
    (ii) $(fog)(x) = \sin x^5$, $(gof)(x) = \sin^5 x$, $fog \neq gof$
    (iii) $(fog)(x) = \cos(\sin x^2)$, $(gof)(x) = \sin(\cos^2 x)$, $fog \neq gof$
    (iv) $(fog)(x) = x = (gof)(x)$, $fog = gof$

12. (i) $(1+x)^2+x$ (ii) x^2+0 (iii) $\frac{1}{2}(e^x+e^{-x})+\frac{1}{2}(e^x+e^{-x})$

(iv) $\left\{\frac{1}{2}(e^x+e^{-x})\right\}+\left\{\frac{1}{2}(e^x-e^{-x})+\sin x\right\}$

13. (i) $f^{-1}=\{(1,4),(3,2),(2,3),(1,4)\}$ (ii) No inverse
(iii) $f^{-1}=\{(1,4),(2,1),(3,2),(4,3)\}$ (iv) $f^{-1}=f$ (v) no inverse

EXERCISE 1(c)

2. (i) yes (ii) yes (iii) yes (iv) yes (v) yes (vi) yes (vii) No (viii) No (ix) yes (x) yes (xi) yes

3. (i) Not associative, Not commutative, (ii) Not associative Not commutative (iii) associative, commutative, 0 is identity element, inverse elements ex is to. $e^1 = 4$ etc. (iv) associative, commutative (v) associative, commutative (vi) associatative, commutative, identity = 1, inverse elements do not exist except for 1 (ix) associative, commutative, identity = 1, inverse elements exist. (x) Not associative, commutative (xi) associative, commutative, identity = 0, inverse of $a=\frac{a}{a-1}$.

4.

X_5	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

$2^{-1}=3,\ 4^{-1}=4$

EXERCISE - 2

1. (i) $\frac{2x}{1-x^2}$ (ii) $\frac{3\pi}{10}$ (iii) $\frac{\pi}{4}$ (iv) $\frac{\pi}{2}$ (v) $\frac{\pi}{4}$ (vi) $\frac{\pi}{3}$ (vii) 3 (viii) l (ix) $\pi-\frac{x}{2}$ (x) π (xi) $(0, \frac{\pi}{2}]$ (xii) 15

2. (i) F (ii) T (iii) T (iv) F (v) T (vi) F (vii) F (viii) T (ix) F (x) F (xi) F (xii) T

3. (i) 0.96 (ii) 7 (iii) $1-2x^2$ (iv) $\frac{\sqrt{1-x^2}}{x}$ (v) $\frac{\pi}{4}$ (vi) 1 (vii) $\frac{\pi}{4}$ (viii) 0 (ix) $\sqrt{1-x^2}$ (x) $\frac{x+y}{1-xy}$

(xi) $\sqrt{\frac{x^2+y}{x^2+2}}$ (xii) $\frac{x}{2}-\cot^{-1}x$

9. (i) $\pm\frac{2}{3}$ (ii) 0 or 1 (iii) 0 (iv) 1 (v) $\pm\frac{1}{\sqrt{2}}$ (vi) 3 (vii) $\frac{1}{\sqrt{3}}$ (viii) $\pm\frac{1}{2}$ (ix) $\frac{a-b}{1+ab}$ (x) $\frac{a+b}{1-ab}$ (xi) $\frac{\sqrt{3}}{2}$

(xii) $\frac{1}{2}\sqrt{\frac{3}{7}}$; 10. since $\left(\frac{4}{5}\right)^2+\left(\frac{12}{13}\right)^2>1$ sum of the first two lines is wrong

13. (i) $\frac{4}{3}$ (ii) $\frac{1}{8}$ (iii) 0 (iv) 2

EXERCISE- 3(a)

1. Max $Z = 15000x + 2000y$ s.t. $x+y \leq 20$, $x+2y \leq 24$, $x, y \geq 0$;
2. Max $Z = 15x + 10y$ s.t. $zx+3y \leq 600$, $2x+y \leq 12$, $4x \leq 16$, $x, y \geq 0$;
3. Max $Z = 20x + 30y$ s.t. $2x+4y \leq 20$, $2x+2y \leq 12$, $4x \leq 16$, $x,y \geq 0$;
4. Max $Z = 15x + 17y$ s.t. $x+y \leq 30$, $15x+17y \geq 300$, $80x+140y \geq 3000$;
5. Max $Z = 2x + 4y$ s.t. $3x+2y \leq 10$, $10x+25y \leq 75$, $10x+12y \leq 42$; $x, y \geq 0$;
6. Max $Z = 1000x+800y$ s.t. $x+y \leq 5$, $4x+2y \leq 18$, $200x+500y \leq 2000$, $x, y \geq 0$;
7. Min $Z= 210x_1+160x_2+250x_3+170(8-x_1)+180(8-x_2)+140(6-x_2)$ s.t. $x_1+x_2+x_3=12$, $x_1+x_2 \leq 12$, $x_1+x_2 \geq 6$, $x_1 \leq 8$, $x_2 \leq 8$, $x_2 \leq 8$, $x_1, x_2 \geq 0$;
8. Min $Z = 16x + 20y$, s.t. $x+2y>10$, $2x+2y>12$, $3x+y>8$ $x,y>0$;
9. Min $Z = 500x_1 + 800x_2 + 50x_3$ s.t. $x_1+x_2+x_3=10$, $x_1 \leq 7$, $x_2 \leq 3$, $x_3=\frac{x_1}{4}$, $x_1x_2x_3 \geq 0$;
10. Max Z = 30 x1+50x2+40x3 s.t. 2x1+3x2<80, 2x2+5x3<100 3x1+2x2+4x3<150, x1, x2, x3>0;
11. Min $Z = 45x_1+40x_2+85x_3$ s.t. $3x_1+4x_2+8x_3>1000$, $2x_1+2x_2+7x_3 \geq 200$, $6x_1+3x_2+7x_3 \geq 800$, x_1, x_2, $x_3 \geq 0$.

EXERCISE- 3(b)

1. (3,0), max = 15; 2. (0,2), min=14; 3. (0,1) max = 40
4. $(\frac{11}{13}, \frac{5}{13})$, min = $\frac{555}{13}$; 5. (2,1), max = 8; 6. (0.5) max = 300; 7. $(\frac{2}{3}, \frac{10}{3})$, max = $\frac{80}{3}$;
8. 8. $(\frac{19}{5}, \frac{4}{5})$, (5,5), $(\frac{22}{5}, \frac{29}{10})$, max=50; 9. (0, 15,5), max = 220; 10. (6,12), min=240;
11. (50,0), max = 200; 12. $(\frac{5}{6}, \frac{7}{6})$ min = $\frac{100}{3}$, no max. value; 13. (0,2), min = 4, no max. value;
14. $(\frac{7}{2}, \frac{5}{2})$ min = 30, (0,5) max = 10
15. (1) solution = (16,4) max = 32000; (2) (210,60) max = 3750; (3) (2,4) max = 160; (4) (20,10) max = 470; (5) $(\frac{15}{13}, \frac{33}{13})$ max = $\frac{162}{13}$; (6) (4,1) max = 4800; (7) (4,8) min = 3640; (8) (2,4) min = 112; (9) $(\frac{28}{5}, 3, \frac{7}{5})$ min = 5270.

Exercises 4 (a)

1. (i) 1×3 (ii) 2×1 (iii) 3×2 (iv) 3×4; 2. (i) 9 (ii) 12 (iii) pq (iv) p^2

3. (i) $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ (ii) $\begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}$ (iii) $\begin{bmatrix} a_1 b_1 \\ a_2 b_2 \\ a_3 b_3 \\ a_4 b_4 \end{bmatrix}$ (iv) $[a\,b\,c]$

4. (i) 3×5 (ii) 3, 2, 6 (iii) $\begin{bmatrix} 1 & 4 & 3 \\ 2 & 5 & 9 \\ 3 & 6 & 1 \\ 4 & 1 & 1 \\ 1 & 2 & 6 \end{bmatrix}$ (iv) 5×3

6. (i) $X = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ (ii) $x = -9, y = 4, z = 1$ (iii) $x_1 = 5, x_2 = 8, y_1 = 1, y_2 = 3$ (iv) $\begin{bmatrix} 2 & 4 \\ 7 & -5 \end{bmatrix}$

7. (i) $\begin{bmatrix} 8 \\ 18 \end{bmatrix}$ (ii) not possible (iii) $\begin{bmatrix} 5 & 3 \\ 7 & 3 \end{bmatrix}$ (iv) $\begin{bmatrix} -3 & -4 & -5 \\ 4 & 5 & 6 \end{bmatrix}$

8. (i) $AB = \begin{bmatrix} 5 & 10 \\ 13 & 22 \end{bmatrix}$ (ii) $BA = \begin{bmatrix} 9 & 14 \\ 13 & 18 \end{bmatrix}$ (iii) $\begin{bmatrix} 8 & 12 \\ 6 & 14 \end{bmatrix}$ (iv) $\begin{bmatrix} 8 & 12 \\ 6 & 14 \end{bmatrix}$ (v) $\begin{bmatrix} 4 & 8 \\ 10 & 18 \end{bmatrix}$

(vi) $\begin{bmatrix} 8 & 12 \\ 10 & 14 \end{bmatrix}$ 9. (i) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ (iii) $\begin{bmatrix} 3 & 11 \\ 7 & 25 \end{bmatrix}$ (iv) $\begin{bmatrix} 10 & 14 \\ 13 & 18 \end{bmatrix}$ (v) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ (vi) $\begin{bmatrix} c & d \\ a & b \end{bmatrix}$

(vii) $\begin{bmatrix} ck & dk \\ a & b \end{bmatrix}$ (viii) $\begin{bmatrix} b & a \\ d & c \end{bmatrix}$ (ix) $\begin{bmatrix} a & b \\ ck & dk \end{bmatrix}$ (x) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

10. (i) false (ii) false (iii) false (iv) true (v) true (vi) false (vii) false (viii) true

11. $\begin{bmatrix} 2-\alpha & 4 \\ 3 & 13-\alpha \end{bmatrix}$ 12. (i) $x = 1, y = 4$ (ii) $x = -2, y = 5$ (iii) $x = -2, y = -7$ (iv) $x = -1, y = -5$ (v) $x = 1, y = 3$

13. $\begin{bmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \\ 5 & 6 & 7 & 8 \\ 6 & 7 & 8 & 9 \end{bmatrix}$ 14. $\begin{bmatrix} 5 & 8 & 11 \\ 7 & 10 & 13 \end{bmatrix}$ 15. (i) $\begin{bmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$ (ii) $\begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & -1 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix}$ (iv) $\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ 2 & 1 & \frac{2}{3} \end{bmatrix}$

16. $x = 2, y = 1$; 17. $\begin{bmatrix} -1 & -1 & -5 \\ 1 & -1 & 2 \\ -2 & 2 & 3 \end{bmatrix}$; 18. $x = 1, y = 1, z = -1$; 19. 3×5; 20. $A = [-1\ 2\ 1]$

22. $B = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$ or $\begin{bmatrix} 1 & 4 \\ 4 & 1 \end{bmatrix}$ 22. $x = y = 1$; 23. (i) $AB = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$, $BA = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$

(ii) $AB = \begin{bmatrix} 7 & 5 \\ 6 & 4 \end{bmatrix} = BA$; (iii) $AB = \begin{bmatrix} 5 & 4 \\ 5 & -2 \end{bmatrix}$, $BA = \begin{bmatrix} 8 & 2 \\ -5 & -5 \end{bmatrix}$ (iv) $AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

24. (i) [1 8 6] (ii) $\begin{bmatrix} 4 \\ 20 \end{bmatrix}$ 26. $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}, B = \begin{bmatrix} 0 & -2 & -1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ 27. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

31. $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$

36. $\begin{bmatrix} 21 & 22 & 15 \\ 7 & 14 & 23 \end{bmatrix}, \begin{bmatrix} -21 & -14 & -3 \\ 1 & -10 & -7 \end{bmatrix}$ 37. (i) $\begin{bmatrix} -7 & -3 & -11 \\ 4 & 5 & 6 \end{bmatrix}$ (ii) $\begin{bmatrix} 3 & 1 \\ 2 & 1 \\ 1 & -5 \end{bmatrix}$ (iii) $\begin{bmatrix} 2 & -1 \\ 2 & -2 \\ 3 & -4 \end{bmatrix}$

39. $x = y = -1$ 41. $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, B = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} \end{bmatrix} C = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$ 42. 0, 1, 2, 3

43. $A = \begin{array}{c} \\ A \\ B \end{array} \begin{array}{cc} \text{Cal} & \text{Prot.} \\ \left[\begin{array}{c} 24,600 \\ 15,800 \end{array} \right. & \left. \begin{array}{c} 576 \\ 332 \end{array} \right] \end{array}$ 44. ₹22,000, ₹28,000.

EXERCISE- 4(b)

1. (i) Symmetric, (ii) Not either, (iii) Symmetric, (iv) Skew symmetric, (v) Both, (vi) Not either, (vii) Skew symmetric
2. (i) True, (ii) True, (iii) False, (iv) True, (v) False, (vi) True, (vii) False
3. (i) Skew Symmetric, 6. No

9. (i) $\begin{bmatrix} 1 & 3 & 1 \\ 3 & 0 & 3 \\ 1 & 3 & -2 \end{bmatrix} + \begin{bmatrix} 0 & -1 & 2 \\ 1 & 0 & -2 \\ -2 & 2 & 0 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & 2 & 2 \\ 2 & 7 & 1 \\ 2 & 1 & 6 \end{bmatrix} + \begin{bmatrix} 0 & -3 & 1 \\ 3 & 0 & -3 \\ -1 & 3 & 0 \end{bmatrix}$

(iii) $\begin{bmatrix} x & a & b \\ a & y & c \\ b & c & z \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ (iv) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & x \\ -x & 0 \end{bmatrix}$

(v) $\begin{bmatrix} 1 & 6 \\ 6 & -3 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ (vi) $\begin{bmatrix} 4 & -1 \\ -1 & 2 \end{bmatrix} + \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$ (vii) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

10. (i) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

11. (i) $\begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}$ (ii) $\begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$ (iii) $\begin{bmatrix} \frac{1}{10} & \frac{1}{5} \\ \frac{-3}{10} & \frac{2}{5} \end{bmatrix}$ (iv) $\begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$ (v) $\begin{bmatrix} 1 & 0 \\ \frac{2}{3} & -\frac{1}{3} \end{bmatrix}$ (vi) $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

12. (i) $\begin{bmatrix} 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 \end{bmatrix}$ (ii) $\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -4 & 3 & -1 \\ \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \end{bmatrix}$

(iii) $\begin{bmatrix} \frac{1}{17} & \frac{5}{17} & \frac{1}{17} \\ \frac{8}{17} & \frac{6}{17} & \frac{-9}{17} \\ \frac{10}{17} & -\frac{1}{17} & -\frac{7}{17} \end{bmatrix}$ (iv) $\begin{bmatrix} 1 & -1 & 0 \\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \end{bmatrix}$ (v) $\begin{bmatrix} -2 & 4 & -5 \\ 0 & 1 & -2 \\ 1 & -2 & 3 \end{bmatrix}$

EXERCISE- 5(a)

1. (i) 1, (ii) -5, (iii) 1, (iv) $-2x$, (v) -1, (vi) 14, (vii) $\cos 2\theta$, (viii) 0, (ix) 1, (x) 0 (xi) $\sin(x-y)$ (xii) 0, (xiii) 0, (xiv) 0 (xv) 0 (xvi) 666 (xvii) 4 (xviii) 183
2. (i) True (ii) False (iii) False (iv) False (v) False (vi) True (vii) True (viii) True.
3. (i) 200 (ii) 0, (iii) 0 (iv) a (v) 4 (vi) 0, (vii) $\begin{vmatrix} 2 & 1 & 1 \\ 2 & 2 & 3 \\ 2 & 3 & 6 \end{vmatrix}$ (viii) 6
4. (i) 8, (ii) $x = a, b$ (iii) 2, 7, -9 (iv) 0, (v) $-(a+b+c)$ or 0 (vi) 0, -3 (vii) $-1, 2$ (viii) 0, (ix) $-1, 2$ (x) 2, 3
5. (i) 0 (ii) $-4x + 2y - 5z$ (iii) $xyz + x + 3y - 2z + 5$
   (iv) $abc + 2fgh - af^2 - bg^2 - ch^2$ (v) -148
   (vi) 0 (vii) -6 (viii) -109
   (ix) -204 (x) 1
10. (i) $(x+a+b+c)(x^2 - a^2 - b^2 - c^2 + ab + bc + ca)$
    (ii) $(a+b+c)(b-c)(a-b)(a-c)$ (iii) $(x+5)(x-1)(x-3)$
15. $bc + ca + ab + 1 = 0$ 17. $\frac{1}{ab-h^2}\left\| \begin{matrix} a & h & g \\ h & b & f \\ g & f & c \end{matrix} \right\|$

Exercises 5 (b)

1. (i) infinite (ii) no solution (iii) no solution (iv) one (v) one
   (vi) no solution (vii) one (viii) one (ix) infinite.
3. (i) a, (ii) -12, (iii) a, (iv) c, (v) b, (vi) a
5. (i) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (ii) $\frac{1}{7}\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$ (iii) $\frac{1}{10}\begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$ (iv) $\begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$

(v) $-\frac{1}{3}\begin{bmatrix}-3 & 0\\-2 & 1\end{bmatrix}$ (vi) $-\begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix}$ (vii) $-\frac{1}{2}\begin{bmatrix}i & i\\-i & i\end{bmatrix}$ (viii) $\frac{1}{x^3+x^2}\begin{bmatrix}x^2 & x\\-x & x\end{bmatrix}$

6. (i) $\begin{bmatrix}-4 & -1 & 1\\6 & -1 & -4\\7 & -2 & -3\end{bmatrix}$ (ii) $\begin{bmatrix}10 & -11 & -8\\-5 & 4 & 7\\3 & -10 & -10\end{bmatrix}$ (iii) $\begin{bmatrix}2 & 2 & -3\\-3 & 2 & 2\\2 & -3 & 2\end{bmatrix}$ (iv) $\begin{bmatrix}-13 & -3 & 18\\28 & 1 & -6\\9 & 13 & -7\end{bmatrix}$

7. (i) invertible (ii) not invertible (iii) invertible (iv) invertible

8. (i) 2, – 1, 1 (ii) 2, 7, 4 (iii) Infinite number of solutions

(iv) Incosistent (v) –1, 3, 2 (vi) $6, \frac{11}{2}, \frac{11}{2}n$

(vii) $\frac{12}{10}, -\frac{17}{20}, \frac{5}{2}$ (viii) 1, 2, 3 (ix) 3, 1, 2

9. $x + 2y + 3z = 1, 3x - 2y + z = 2, 4x + 2y + z = 3$; Solution is $\frac{7}{10}, \frac{3}{40}, \frac{1}{20}$ 10. $X = \begin{bmatrix}1\\2\\3\end{bmatrix}$

11. (i) 125 (ii) 6 (iii) $x = 5, y = 3$ (iv) 0 (v) 0 (vi) 0 (vii) – 2 (viii) [2, 4] (ix) 0

12. (i) 0 (ii) 0 (iii) 0 (iv) 1 (v) 0 (vi) –8
(vii) 20000 (viii) 0 (ix) $a^5 + b^5$ (x) 0 (xi) 0 (xii) 0

13. $x = 0$ or $y = 0$ 14. $-\frac{3}{8}$ 15. $x = -a, -b, -c$ 16. $x = 0, x = \frac{a}{3}$

17. $x = 0$ or $x = -(a + b + c)$ 18. $x = 2, x = -3$ 19. 0 20. 0 21. 3

22. $\begin{bmatrix}2 & 0 & 0\\0 & 2 & 0\\0 & 0 & 2\end{bmatrix}$ 30. 0 31. –2 32. 0 33. 4

EXERCISE- 6 (a)

1. (i) $\frac{25}{144}$; (ii) $\frac{5}{12} \times \frac{4}{11}$ 2. (i) $\frac{52}{52} \times \frac{39}{51}$ (ii) $\frac{52}{52} \times \frac{48}{51}$ 3. (i) $\frac{52}{52} \times \frac{39}{51} \times \frac{26}{50}$ (ii) $\frac{52}{52} \times \frac{48}{51} \times \frac{44}{50}$

4. (i) $\frac{52}{52} \times \frac{39}{51} \times \frac{26}{50} \times \frac{13}{49}$ (ii) $\frac{52}{52} \times \frac{48}{51} \times \frac{44}{50} \times \frac{40}{49}$ 5. (i) $\frac{5}{15} \times \frac{4}{14} \times \frac{3}{13}$ (ii) $\frac{10}{15} \times \frac{9}{14} \times \frac{8}{13}$

6. $\frac{5}{11}$ 7. (i) $\frac{4}{30}$ (ii) $\frac{8}{30}$ (iii) $\frac{10}{30}$ 8. (i) $\frac{1}{2}$ (ii) $\frac{1}{3}$ (iii) $\frac{1}{10}$ 9. (i) $\frac{1}{4}$ (ii) $\frac{1}{3}$ (iii) $\frac{1}{3}$ (iv) $\frac{3}{7}$

10. (i) $\frac{1}{2}$ (ii) $\frac{1}{3}$ (iii) $\frac{2}{3}$ (iv) $\frac{1}{2}$; 12. (i) $\frac{1}{4}$ (ii) $\frac{1}{5}$ (iii) $\frac{1}{2}$ (iv) $\frac{1}{2}$ 13. (i) $\frac{1}{5}$ (ii) $\frac{4}{5}$ (iii) $\frac{6}{10}$ (iv) $\frac{6}{10}$

(v) $\frac{88}{100}$ 14. $\frac{8}{10}$ 15. $\frac{24}{52} \times \frac{23}{51}$;

16. (i) $P(BBB) = \frac{5}{12} \times \frac{4}{11} \times \frac{3}{10} = \frac{1}{22}$; $P(WWW) = \frac{7}{12} \times \frac{6}{11} \times \frac{5}{10} = \frac{7}{44}$

hence P (BBB or WWW) $= \frac{9}{44}$

(ii) When each colour is represented we must have at least one white and one black.
Thus we have two cases :
Case 1 : 2 white, 1 black and three subcases, that is, WWB, WBW, BWW
Case 2 : 1 white, 2 black and three sub-cases, namely, BBW, BWB, WBB

Use the multiplication rule to find the probability of each of these six mutually exclusive cases. A smarter way of looking at this problem : Let A be the event that each colour is represented; then A^c represents the event that all balls are of the same colour

17. (i) $\frac{5}{36}$ (ii) $\frac{25}{216}$ 18. $\frac{4}{5}$

19. For (i) consider the five mutually exclusive cases :
SSSS, SSSF, SSFS, SFSS, FSSS

answer : $p^3 + \frac{3}{2}p^3(1-p)$ (ii) $\frac{5}{2}p^3(1-p)$ 21. (i) $\frac{1}{4}$ (ii) $\frac{3}{7}$ 22. (i) $\frac{11}{24}$ (ii) $\frac{6}{11}$

EXERCISE- 6 (b)

2. (i) $\frac{115}{216}$ (ii) $\frac{59}{126}$

4. P (A wins) $= \frac{1296}{3373}$, P (B wins) $= \frac{1116}{3373}$, P (C wins) $= \frac{961}{3373}$

5. $\frac{1}{7}$ 6. $\frac{5}{42}$ 7. $1-\left(\frac{5}{6}\right)^n$

8. Let A be the event that your alarm goes off and let B be the event that you make your 8 a.m. class. Since $S = A \cup A^c$,

$$B = (B \cap A) \cup (B \cap A^c),$$

a union of two disjoint sets. Therefore

$$P(B) = P(A \cap B) + P(A^c \cap B) = P(A).\ P(B/A) + P(A^c).\ P(B/A^c)$$
$$= .9 \times .8 + .1 \times .5 = .77$$

9. $\frac{6}{2^6}$ 10. $\frac{{}^6C_2}{2^6}$

EXERCISE- 6 (c)

1. $\frac{773}{1512}$ 2. $\frac{8}{13}$ 3. 21% 4. $\frac{5}{21}$ 5. (i) $\frac{8}{41}$ (ii) $\frac{14}{41}$ (iii) $\frac{15}{41}$ (iv) $\frac{4}{41}$

EXERCISE- 7 (a)

1. (i) discontinuous (ii) continuous (iii) – (v) continuous (vi) – (viii) discontinuous (ix) continuous (x) discontinuous (xi) for x = 0, discontinuous, for x = 1, continuous (xii) continuous (xiii) – (xiv) discontinuous.

2. (i) f(a) (ii) 0 3. ±1 4. $a = \frac{2}{3}$, $b = \frac{1}{3}$

EXERCISE 7 (b)

1. (i) $3e^{3x}$ (ii) $2^{x^2+1} x \ln 2$ (iii) $3/(3x+1)$ (iv) $-\ln 5/x(\ln x)^2$ (v) cot x (vi) $2xa^{2x} + 2x^2 a^{2x} \ln a$

EXERCISE 7 (c)

1. $16x(x^2+5)^7$ 2. $-2(x^3+\sin x)^{-3}(3x^2+\cos x)$ 3. $\frac{1}{2\left(x+\sqrt{x}\right)}$ 4. $5\cos 5x - 7\sin 7x$

5. $e^{\sin t}\cos t$ 6. $\frac{(2ax+b)}{2\sqrt{ax^2+bx+c}}$ 7. $\frac{3(x^2+3)^2(x+3)(x-1)}{(x+1)^4}$. 8. $\sec(\tan\theta)\tan(\tan\theta)\sec^2\theta$

9. $\frac{-4x\cos\left\{\frac{(1-x^2)}{(1+x^2)}\right\}}{(1+x^2)^2}$ 10. $\frac{3\sec^2 3z}{2\sqrt{\tan 3z}}$ 11. $3\tan^2 x\sec^2 x$ 12. $4\sin^3 x\cos x$ 13. $\sin 2x\cos 2x$

14. $5\cos 5x\cos 7x - 7\sin 7x\cos 5x$ 15. $-\tan x\sec^2 x$ 16. $\frac{\cos\sqrt{x}}{4\sqrt{x}\sqrt{\sin\sqrt{x}}}$

17. $\sqrt{\sec}(2x+1)\tan(2x+1)$ 18. $-2a(ax+b)\operatorname{cosec}(ax+b)^2\cot(ax+b)^2$ 19. $\frac{1}{x}a^{\ln x}.\ln a$

20. $xa^{x^2}b^{x^3}(2\ln a+3x\ln b)$ 21. $\cot x+\tan x$ 22. $2x\cos x^2 5^{\sin x^2}\ln 5$ 23. $\sec x$

24. $\frac{\ln a}{4}\left(\frac{a\sqrt{x}}{x}\right)^{\frac{1}{2}}$ 25. $\frac{n\cdot(e^{nx}-e^{-nx})}{(e^{nx}+e^{-nx})}$ 26. $\frac{\sqrt{a}\,e^{\sqrt{ax}}}{2\sqrt{x}}$ 27. $\frac{1}{2x\sqrt{\log x}}$

28. $e^{\sin x}\cos x + a^{\cos x}\sin x\ln a$ 29. $\frac{e^{3x^2}(6x\ln\sin x-\cot x)}{(\ln\sin x)^2}$

EXERCISE – 7 (d)

2. $\frac{2}{\sqrt{1-4x^2}}$ 3. $\frac{-1}{\left[2\sqrt{x}(1+x)\right]}$ 4. $\frac{1}{\left[12x+11\sqrt{(x^2+x)}\right]}$ 5. $\frac{-1}{2\sqrt{1-x^2}}$ 6. $\frac{-2}{(x^2+1)}$

7. $\frac{-\sin\sqrt{x}}{\left[2\sqrt{x}\left(1+\cos^2\sqrt{x}\right)\right]}$ 8. $\frac{2x\operatorname{cosec}^{-1}\left(\frac{1}{\ln x}\right)+x}{\sqrt{\left[1-(\ln x)^2\right]}}$ 9. $\frac{1}{\sqrt{1-x^2}}$

10. $15(x\sin^{-1}x)^{14}\left[\sin^{-1}x + x/\sqrt{1-x^2}\right]$ 11. $\frac{1}{\sqrt{2x(1-x)}(1+x)}$

EXERCISE – 7 (e)

1. $\frac{2}{\sqrt{(1-x^2)}}$ 2. $\frac{2}{(1+x^2)}$ 3. $\frac{-1}{2\sqrt{1-t^2}}$ 4. $\frac{2(1+t^2)}{(1-t^2)^2}$ 5. $\frac{1}{2\sqrt{x}(1+x)}$

6. $\frac{2}{(1+x^2)}$ 7. $\frac{a}{(a^2+x^2)}$ 8. $\frac{2}{t\sqrt{|t^2-1|}}$ 9. $\frac{2}{(1+t^2)}$ 10. $\frac{-2}{\sqrt{(1-t^2)}}$

EXERCISE – 7 (f)

1. $x^x \ln ex$ 2. $\left(1+\frac{1}{x}\right)^x\left[\ln\left(1+\frac{1}{x}\right)-\frac{1}{1+x}\right]$ 3. $x^{\sin x}$ [cos x ln x + sin x/x]

4. $(\log x)^{\tan x}$ [$\sec^2 x$ ln ln x + tan x/x ln x]; 5. $2^{2^x}\, 2^x (\ln 2)^2$; 6. $(1+\sqrt{x})^{x^2}$ [2x ln(1+√x) + x√x/2(1+√x)]

7. $(\sin^{-1}x)^{\sqrt{1-x^2}-1}\left[\sqrt{1-x^2}-x\sin^{-1}x\ln\sin^{-1}x\right]/\sqrt{(1-x^2)}$

8. 3 $(\tan x)^{\log x^3}$ [In tan x + (tan x + cotx) x In x]/x

9. $x^{\frac{1}{x}-2}\ \ln(e/x)+(\sin x)^x(x\cot x+\ln\sin x)$

10. $(\cos x)^x$ (in cosx – x tan x) + $x^{\cos x-1}$ (cos x – x sin x In x)

11. $(x^2+1)^{\frac{2}{3}}(3x+1)^{\frac{1}{4}}\sqrt{x}\left\{\frac{4x}{3(x^2+1)}+\frac{3}{4(3x+1)}+\frac{1}{2x}\right\}$

12. $-\left(\frac{2}{x^2-1}+\frac{8}{x^2-4}+\frac{18}{x^2-9}\right)\frac{(x+1)(x+2)^2(x+3)^3}{(x-1)(x-2)^2(x-3)^3}$

13. $\left\{(1+x^2)\sin x\right\}^{x+\frac{1}{2}}\ \{\ln(1+x^2)+\ln\sin x+\frac{x+2x^2}{1+x^2}+\left(x+\frac{1}{2}\right)\cot x\}$

14. $(\sec x+\tan x)^{\cot x}$ cosec x [1–cosecx. In (sec x + tan x)] 15. $2^{\sqrt{x}+x}\left(1+\frac{1}{2\sqrt{x}}\right)\ln 2$

EXERCISE – 7 (g)

1. $-(y^2+2xy)/(x^2+2xy)$; 2. $\frac{y}{x}$; 3. – x/3y ; 4. $(2x\cot y+y^2\operatorname{cosec}^2 x)/(2y\cot x+x^2\operatorname{cosec}^2 y)$

5. $y\sec^2 xy/(1-x\sec^2 xy)$ 6. (x–y) /x *l*n (exy) 7. $-y(\cos x+e^{xy})/(\sin x+x e^{xy})$

8. (x+y) /(x–y) 9. y (siny – x in y) / x(x-y cos y *l*n x)

EXERCISE – 7 (h)

1. –cot θ 2. 2 3. –1 4. 1 5. cot t

EXERCISE – 7 (i)

1. $\frac{1}{4}x^{-3/2}$ 2. $-\sin^2 x\cos x$ 3. $-\tan x(1+\sin x)^2/(1+\cos x)^2$. 4. $(2+x^2)/x(1+x^2)$ 5.1

EXERCISE – 7 (j)

1. Not differentiable, continuous ; 2. Differentiable, continuous ; 3. Not differentiable, not continuous
4. Differentiable, continuous ; 5. Not differentiable, continuous ; 6. Differentiable, continuous
7. Not differentiable, continuous ; 8. Differentiable, continuous

EXERCISE – 7 (k)

1. (i) F (ii) F (iii) F (iv) T (v) F (vi) T (vii) T (viii) F

2. (i) u^v Inu (ii) $t\,x^{t-1}$ (iii) $\cos t^2$ (iv) – 8 (v) $\frac{dg}{df}\frac{df}{dx}$ (vi) 0 (vii) does not exist (viii) 0

3. (i) $2e^{2x}$ (ii) $\sin 2x$ (iii) $-2x \sin x^2$ (iv) $2x e^{x^2}$ (v) $\sec^2 x / 2\sqrt{\tan x}$ (vi) $x^2 \cos x + 2x \sin x$ (vii) $\cot x$ (viii) $\cos\sqrt{x} / 2\sqrt{x}$ (ix) $-\sin \ln x / x$

4. (i) Differentiable (ii) Not differentiable (iii) Not differentiable

5. (i) $-(3x+2)/2x(x+1)\left[\ln\left(x\sqrt{x+1}\right)\right]^2$ (ii) $[\sin x - ln\, x(\sin x + \cos x)] / e^x \sin^2 x$

(iii) $e^x[\sec^2 x + \operatorname{cosec}^2 x + \tan x - \cot x]$ (iv) $\frac{\tan x}{2\sqrt{x}}(3x+1) + \sqrt{x}(x+1)\sec^2 x$

(v) $\sec 2x \tan 2x$ (vi) $x e^x \operatorname{cosec} x(2 + x - x\cot x)$

(vii) $\left[(x+3)\ln x + 2\left(x + \frac{2}{x} + 3\right)\right] / (x+2)^{3/2}$

(viii) $(x^3-1)^8 \sec^2 x\,[27x^2 + 2(x^3-1)\tan x]$ (ix) $-2x$ (x) $\left(x+\frac{1}{x}\right)^9 a^x\left[\left(x+\frac{1}{x}\right)ln\, a - \frac{10}{x^2} + 10\right]$

(xi) $2/x\sqrt{x+4}$ (xii) $\frac{2}{x} + \frac{6}{2x-7} - \frac{30x}{3x^2+7}$ (xiii) $5^{\ln \sin x}\cot x \ln 5$ (xiv) $\cos\sqrt{x} / 4\sqrt{x}\sqrt{\sin\sqrt{x}}$

(xv) $x^{\sin x}[\cos x \ln x + \sin x / x] + (\tan x)^x[\ln \tan x + x(\tan x + \cot x)]$ (xvi) $e^x e^{e^x}$

(xvii) $x^{\sqrt{x}}\ln\left(e\sqrt{x}\right)/\sqrt{x}$ (xviii) $(e^x+1)/(e^x+x)\sqrt{\left(e^x+x\right)^2-1}$ (xix) $e^x \tan e^x$

(xx) $a^{\sin^{-1}x^2}\, 2x ln\, a / \sqrt{1-x^4}$ (xxi) $-4x/(1+x^4)$ (xxii) $(x^e)e^x e^{x+1}\left(\ln x + \frac{1}{x}\right) + (e+1)x^e (e^x)^{x^e}$

(xxiii) $x^{(x^x)}\left[x^x \ln ex \ln x + x^{x-1}\right]$ (xxiv) $\frac{(x+1)^2\sqrt{x-1}}{\left(x^2+3\right)^3 3^x}\left[\frac{2}{x+1} + \frac{1}{2(x-1)} - \frac{6}{x+3} - \ln 3\right]$

(xxv) $\frac{2}{3}\left\{5\ln\left(x^3+1\right) - x^4\right\}^{-\frac{1}{3}} \frac{x^2\left(15 - 4x - 4x^4\right)}{x^3+1}$ (xxvi) $\cot x \log_{10} e - \frac{\log_e 10}{x(\log_e x)^2}$

6. (i) $-2/(x^2+1)$ (ii) $2x e^{\tan^{-1}x^2} / (1+x^4)$

(iii) $(\sin^{-1}x + x\sqrt{1+x^2}) / (1+x^2)^{3/2}$ (iv) $2e^{2x}/(1+e^{4x})$

(v) $-\frac{1}{2}$ (vi) -1 (vii) $\frac{4a}{a^2+16x^2} + \frac{3a}{a^2+9x^2}$ (viii) $-x/\sqrt{1-x^4}$ (ix) πx (x) $\frac{1}{2\sqrt{1-x^2}}$

(xi) $\sin\alpha/(1-2x\cos\alpha + x^2)$

7. (i) $\frac{x^2-4y}{4x-y^2}$ (ii) $-(ay/bx)^{1/3}$ (iii) $-y/x \ln x$

(iv) $-y\, ln y / x$ (v) $(y \operatorname{cosec} x \cot x - \cot y) / (\operatorname{cosec} x - x \operatorname{cosec}^2 y)$ (vi) $y(1-2x^2)/x(2y^2-1)$ (vii) $y \tan x / (\ln \cos x - \cot y)$ (viii) $a^{\sqrt{x}} \ln a / 4y\sqrt{x}$ (ix) $n x^{n-1}/(m+2n)y^{m+2n-1}$

(x) $\frac{y}{x}$, (xi) $2y \cos 2x \ln \sin y / (1 - y \cot y \sin 2x)$ (xii) $x^{y-1}/(2 - y\, ln x)$ (xiii) $\frac{(x+y)\sin x \ln(x+y)}{\cos x - x - y} - 1$

(xiv) $-(y\sec^2 x + \tan y) / (x \sec^2 y + \tan x)$ (xv) $\left(ky + x\sqrt{x^2+y^2}\right) / \left(kx - y\sqrt{x^2+y^2}\right)$

8. (i) 1 (ii) 2/x (iii) $-\frac{1}{4}\sin x$ 9. (i) tan t (ii) 1. (iii) 2 (iv) $-(\tan 2u)^{3/2}$ (v) – tan 3 t.

10. (i) cos x / (2y–1) (ii) – y/ (x+2y) (iii) 1 / (x+y–1) 12. No. The function is undefined for $x \in R$

18. (i) x = 0 (ii) x = ± 2 (iii) x = 1, x = 2 (iv) x = 0

EXERCISE– 8(a)

1. (i) 11units /sec, 4 units / sec^2 (ii) $\frac{1}{2\sqrt{2}}$ units /sec– $\frac{1}{8\sqrt{2}}$ unit /sec^2

(iii) $\frac{-6}{25}$ units / sec, $\frac{24}{125}$ units /sec^2 (iv) 3 units / sec, 0

2. 6 sq. cm /sec 3. 15.08 cu m./min. 4. $\frac{1}{4}$ cm /sec

EXERCISE– 8(b)

	Tangent	Normal
1.	(i) 4x +y –1 = 0;	x – 4y +21 = 0
	(ii) 11x –y –16 = 0;	x+11y –68 = 0
	(iii) 9x –4y +28 = 0;	4x + 9y –160 = 0
	(iv) y = 2;	$x = \pi / 3$
	(v) 2 ex + y =3 ;	x - 2 ey +2e $-\frac{1}{e} = 0$
	(vi) x+2 $(\log 2)^2$y - 2(log 2+1) =0	$2(\log 2)^2 x - y = 4(\log 2)^2 - \frac{1}{\log 2}$

(vii) y = x ; x+y =0

(viii) $x - y = \frac{\pi a}{2} - 2a$ $x + y = \frac{\pi a}{2}$

(ix) $\frac{x}{a\cos\theta} + \frac{y}{b\sin\theta} = 1$ $\frac{x}{b\sin\theta} - \frac{y}{a\cos\theta} = \frac{a^2\cos^4\theta - b^2\sin^4\theta}{ab\sin\theta}$

2. (1,0), 3. $\left(\frac{9a}{10}, \frac{3a}{10}\right)$ and $\left(-\frac{9a}{10}, \frac{27a}{10}\right)$ 4. $\pm\left(\sqrt{\frac{2}{3}}, 2\sqrt{\frac{2}{3}}\right)$

10. $y - \left(2 \pm \sqrt{3}\right) = \pm 2\sqrt{3}(x-2)$ 13. $2x - 4y - \pi = 0$ 15. y = x

EXERCISE – 8 (c)

1. (i) (a) $[0, \frac{\pi}{2}] \cup \left(\frac{3\pi}{2}, 2\pi\right)$ (b) $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ (ii) (a) $(0, \infty)$ (b) nowhere

(iii) (a) $(-\infty, \infty)$ if a > 1 (b) $(-\infty, \infty)$ if a <1 (iv) (a) $[0, \frac{\pi}{4}) \cup (5\frac{\pi}{4}, 2\pi]$ (b) $\left(\frac{\pi}{4}, 5\frac{\pi}{4}\right)$

(v) (a) $(-\infty, -3) \cup (2, \infty)$ (b) $(-3, 2)$ (vi) (a) nowhere (b) R - {1}

(vii) (a) $\left(-3, -\sqrt{\frac{8}{3}}\right) \cup \left(\sqrt{\frac{8}{3}}, \infty\right)$ (b) $(-\infty, -3) \cup \left(-\sqrt{\frac{8}{3}}, \sqrt{\frac{8}{3}}\right)$ (viii) (a) $\left(\frac{1}{2}, \infty\right)$ (b) $\left(-\infty, \frac{1}{2}\right)$

(ix) (a) $(-\infty,-1)\cup(1,\infty)$ (b)$(-1,1)$ (x) (a) $(0, e)$ (b) (e, ∞)

(xi) (a) $\left(\frac{-\pi}{2},\frac{-\pi}{3}\right)\cup\left(\frac{\pi}{3},\frac{\pi}{2}\right)$ (b) $\left(-\frac{\pi}{3}, \frac{\pi}{3}\right)$

(xii) (a) $\left[\left(0,\frac{3\pi}{8}\right)\cup\left(\frac{7\pi}{8},\frac{11}{8}\pi\right)\cup\left(\frac{15}{8}\pi,2\pi\right)\right]$ (b) $\left(\frac{3\pi}{8}, \frac{7\pi}{8}\right)\cup\left(\frac{11\pi}{8}, \frac{15\pi}{8}\right)$

EXERCISE– 8 (d)

1. (i) Minimum at $x = -1$, minimum value $=2$

(ii) min. at $x = 0$, min value $= 0$

max. at $x = 1$, max value $=3$

(iii) min. at $x = -1$, min . value $= -3/2$

max. at $x= 1$, max value $= 3/2$

(iv) min. at $x = 0$, min value $= 0$

max .at $x = \pm\sqrt{\frac{2}{3}}$, max value $= \frac{2\sqrt{3}}{9}$

(v) min. at $x = 6$, min. value $= -306$

max. at $x = -1$, max, value $= 37$

(vi) min. at $x = 0$, min value $= 12/5$

max. at $x = \pm 2$, max, value $= 20/3$

(vii) No extreme point, $x = 1$ is a point of inflexion

(viii) min at $x = -\frac{1}{7}$, min value $= -655.703$

max. at $x = -3$, max value $= 0$

(ix) min. at $x = 1$, min value $=2$

max . at $x = -1$, max value $= -2$

(x) min at $x = \frac{\pi}{2}-\frac{1}{2}\tan^{-1}\left(\frac{3}{4}\right)$ min . value $= -5$

max. at $x = -\frac{1}{2}\tan^{-1}\left(\frac{3}{4}\right)$ max . value $= 5$

(xi) max. at $x = \frac{\pi}{4}$, max. value $= \frac{1}{2}$

No min. value in the domain.

(xii) min. at $x = \frac{5\pi}{6}$ min. value $= -\frac{3\sqrt{3}}{4}$

max. at $x = \frac{\pi}{6}$, max. value $= \frac{3\sqrt{3}}{4}$, $y(\frac{3\pi}{2})$ is a point of inflexion

(xiii) max. for $\tan x = \left(\frac{p}{q}\right)^{\frac{1}{2}}$ max value $= \left(\frac{p}{p+q}\right)^{p/2}\left(\frac{q}{P+q}\right)^{q/2}$.

(xiv) max. at x =1, max, value = $\frac{1}{e}$

5. $a = 2, b = -\frac{1}{2}$

7. Absolute min. at $x = \pm 1, f(1) = f(-1) = 0$
Absolute max. at $x = 0$, $f(0) = 1$ although f is not differentiable at $x = 0$

8. No max. point . Min. at $x = \frac{1}{\sqrt{3}}, f\left(\frac{1}{\sqrt{3}}\right) = -\frac{2\sqrt{3}}{9}$

9. 5 and 10

11. A square of side 5 units. 12. $2r/\sqrt{3}$

15. Radius of base = Height of the tank = $\sqrt{\frac{10}{\pi}}$ units

16. Radius = Height = $\left(\frac{500}{\pi}\right)^{1/3}$ 17. (0,0) 18. $\left(\pm\frac{1}{\sqrt{2}}, \frac{3}{2}\right)$

EXERCISE– 8(e)

1. (i) $3x^2$ dx (ii) 2 sin x cos x dx (iii) $x\, x^{-\frac{1}{2}}\, (1-x^{1/2})^{-2}$ dx
(iv) -2 (sin 2 t – cosec^2 t)dt (v) $-4\cos\theta\, d\theta/(1+\sin\theta)^2$

(vi) $-4x^{-3}$ dx (vii) $-\frac{y^2+2xy}{x^2+2xy}dx$

2. (i) $\delta f = .0808$ $df = .08$ (ii) $\delta f = .0373$ $df = .0375$
(iii) $\delta f = 9.7632$ $df = 9.72$ (iv) $\delta f = .0198$ $df = .02$

3. (i) 3.0370, (ii) 1.9947 (iii) 6. 9971 (iv) 123.52 (v) 8.1109 (vi) .85729

4. 2/7 5. 1.08 cu.cm. 6. 0.049

EXERCISE– 8(f)

1. a 2. $\frac{a}{b}$ 3. $\frac{1}{2}$ 4. 0 5. $\frac{1}{6}$ 6. $\frac{3}{5}$ 7. $\frac{1}{2}$ 8. 0 9. $\frac{1}{12}$ 10. $\frac{1}{2}$ 11. $16(\ln 2)^2$ 12. 1. 13. 0 14. 0
15. 0 16. 1 17. 2 18. 0 19. 1 20. 1 21. e^{-1} 22. 1 23. –1/2 24.0 25. -1 26. 1/3 27. ∞ 28. $\sqrt{(\ln a - \ln b)}$
29. e^3 30. e 31.1 32.1 32. 1 33. 0 34. $e^{1/6}$ 35. e^{-2} 36. –e /2 37. e. 38. 0 39. 2 ln2 40. $2/\pi$ 41. –1/3 42.0

EXERCISE– 9(a)

1. (i) $2x + c$ (ii) $x^3 + c$ (iii) $x^4 + c$ (iv) $\frac{1}{6}x^6 + c$ (v) $\frac{1}{32}x^{32} + c$ (vi) $\frac{4}{3}x\sqrt{x} + 6\sqrt{x} + c$ (vii) $\frac{-2}{\sqrt{x}} + c$

(viii) $\frac{7}{11}x^{\frac{11}{7}} + \frac{3}{2}x^{\frac{2}{3}} + c$ (ix) $4\ln x + c$ (x) $\frac{-3}{x} - \frac{1}{11}\frac{1}{x^{11}} + c$ (xi) $\frac{x^5}{5} + \frac{4}{7}x^{\frac{7}{2}} + \frac{x^2}{2} + c$

(xii) $6x - \frac{x^2}{2} - \frac{x^3}{3} + c$ (xiii) $-\frac{1}{2x^2} - \frac{8}{5}x^{\frac{-5}{x^2}} - \frac{4}{3}x^3 + c$ (xiv) $2x + \ln|x| + 4\sqrt{x} - \frac{2}{\sqrt{x}} + c$

2. (i) $\sin x + c$ (ii) $\tan x + c$ (iii) $-\cot x + c$ (iv) $\sec x + c$ (v) $-2 \operatorname{cosec} x + c$
(vi) $-\cot x + \cos x + c$ (vii) $x - \sin x + c$ (viii) $\sin x + \cos x + c$ (ix) $\sin x - \cos x + c$
(x) $-\cot x - \tan x + c$ (xi) $\frac{1}{4}(a^2 \tan x - b^2 \cot x) + c$ (xii) $\tan x - \cot x + c$ (xiii) $\tan x - x + c$
(xiv) $\tan x - \cot x + c$ (xv) $a \sec x - b \operatorname{cosec} x + c$ (xvi) $\sin x - \cos x + c$ (xvii) $-\sqrt{2} \cos x + c$
(xviii) $\sqrt{2} \sin x + c$ (xix) $-\operatorname{cosec} x + c$ (xx) $b^2 \tan x - a^2 \cot x + c$

3. (i) $e^x + 2x + c$ (ii) $\frac{3^x}{\ln 3} + c$ (iii) $\frac{a^{x+2}}{\ln a} + c$ (iv) $\frac{a^{3x}}{3 \ln a} + c$ (v) $e^x - e^{-x} + c$

4. (i) $5 \sin^{-1} x + 7 \tan^{-1} x + c$ (ii) $3(x - \tan^{-1} x) + c$ (iii) $\frac{x^5}{5} - \frac{x^3}{3} + x - \tan^{-1} x + c$ (iv) $\frac{x^3}{3} + \frac{x^2}{2} + 2 \tan^{-1} x + c$
(v) $\sin^{-1} x + c$ (vi) $\sec^{-1} x - \frac{1}{2x^2} + c$ 5. $\frac{2}{3}x^3 + x - 2$

EXERCISE – 9 (b)

1. (i) $-\frac{1}{3} \cos 3x + c$ (ii) $\frac{1}{a} \sin ax + c$ (iii) $-\frac{1}{7} \sin (2-7x) + c$ (iv) $-2 \cos \frac{x}{2} + c$ (v) $\frac{1}{4} \tan 4x + c$
(vi) $-3 \cot \frac{x}{3} + c$ (vii) $\sec (x+2) + c$ (viii) $-\operatorname{cosec} \left(x + \frac{\pi}{4}\right)$ (ix) $\frac{1}{3} \sin x^3 + c$ (x) $\sec e^x + c$
(xi) $2 \tan \sqrt{x} + c$

2. (i) $\frac{1}{2} \sin^2 x + c$ (ii) $\frac{1}{4} \tan^4 x + c$ (iii) $-\ln |1 + \cot x| + c$ (iv) $\frac{1}{2} \sec^2 x + c$ (v) $-\frac{1}{4} \operatorname{cosec}^4 x + c$
(vi) $-\cot (\ln x) + c$ (vii) $-\frac{2}{3}(1 - \sin x)^{\frac{3}{2}} + c$

3. (i) $\frac{1}{3}(x^2 + 3)^{3/2} + c$ (ii) $-\frac{7}{3} \ln |2 - 3x| + c$ (iii) $\sqrt{x^2 - a^2} + c$ (iv) $-\frac{1}{6}(x^3 + 3x + 7)^{-2} + c$
(v) $\frac{1}{10}\left(x^4 - 3x^2 + 1\right)^5 + c$

4. (i) $\frac{1}{3}e^{3x} + c$ (ii) $\frac{1}{2}e^{2x+7} + c$ (iii) $3e^{x/3} + c$ (iv) $\frac{1}{3}e^{x^3} + c$ (v) $\frac{a^{2x}}{2 \ln a} + c$ (vi) $\frac{a^{x^2}}{\ln a} + c$
(vii) $\frac{1}{2}e^{2 \tan x} + c$ (viii) $-\frac{1}{e^x - 2} + c$ (ix) $-e^{\cos^2 x} + c$

5. (i) $\frac{1}{2}\left(\sin^{-1} x\right)^2 + c$ (ii) $\sin^{-1}(x-1) + c$ (iii) $\frac{1}{3}(\sec^{-1} x)^3 + c$ (iv) $\tan^{-1}(\ln x) + c$ (v) $\tan^{-1}(x+1) + c$

6. (i) $\frac{1}{3} \ln |\sec 3x| + c$ (ii) $3 \ln |\sin x/3| + c$ (iii) $\frac{1}{2} \ln |\sec (2x + 1) + \tan (2x+1)| + c$
(iv) $\frac{1}{7} \ln |\operatorname{cosec} 7x - \cot 7x| + c$ (v) $\ln |\sin (x^2 + 3)| + c$ (vi) $\ln |\sec e^x| + c$

(vii) $\frac{1}{2}\tan 2x - 3\ ln\ |\sec 2x + \tan 2x| + 9x + c$

7. (i) $ln\ |e^x - e^{-x}| + c$ (ii) $\frac{3^x \cdot e^{2x}}{2+\ln 3} + c$ (iii) $\frac{1}{4}\left[ln\left(x^2+2x+2\right)\right]^2 + c$

8. (i) $x\cos\alpha - \sin\alpha\ ln\ |\sin(x+\alpha)| + c$ (ii) $\cos\alpha\ ln\ |\sec(x-\alpha)| - x\sin\alpha + c$

(iii) $x\cos 2\alpha + \sin 2\alpha\ ln\ |\sin(x-\alpha)| + c$

EXERCISE – 9 (c)

1. (i) $-\frac{1}{14}\cos 7x - \frac{1}{2}\cos x + c$ (ii) $\frac{1}{14}\sin 7x + \frac{1}{6}\sin 3x + c$ (iii) $\frac{1}{6}\cos 3x - \frac{1}{10}\cos 5x + c$

(iv) $\frac{1}{6}\sin 3x - \frac{1}{18}\sin 9x + c$ (v) $\frac{1}{28}\cos 7x - \frac{1}{44}\cos 11x - \frac{1}{12}\cos 3x - \frac{1}{4}\cos x + c$

(vi) $-\frac{2}{5}\cos\frac{5}{4}x - 2\cos\frac{x}{4} + c$ (vii) $\frac{1}{5}\sin\frac{5x}{2} + \frac{1}{3}\sin\frac{3x}{2} + c$

(vii) $\frac{3}{5}\sin\frac{5}{13}x - \frac{3}{13}\sin\frac{13}{12}x + 3\sin\frac{x}{12} - \frac{3}{7}\sin\frac{7x}{12} + c$

2. (i) $\frac{1}{2}(x + \sin x\cos x) + c$ (ii) $\frac{1}{12}\cos 3x - \frac{3}{4}\cos x + c$ (iii) $\frac{3}{8}x + \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + c$

(iv) $-\cos x + \frac{2}{3}\cos^3 x - \frac{1}{5}\cos^5 x + c$ (v) $\sin x - \sin^3 x + \frac{3}{5}\sin^5 x - \frac{1}{7}\sin^7 x + c$

(vi) $\frac{1}{192}(60x - 45\sin 2x + 9\sin 4x - \sin 6x) + c$ (vii) $\frac{1}{8}\cos^8 x - \frac{1}{6}\cos^6 x + c$

(viii) $\frac{1}{21}\sin^{21} x - \frac{1}{23}\sin^{23} x + c$ (ix) $\frac{1}{5}\sec^5 x - \frac{1}{3}\sec^3 x + c$ (x) $\frac{1}{16}\mathrm{cosec}^{16}x - \frac{1}{18}\mathrm{cosec}^{18}x + c$

(xi) $\frac{1}{30}\sec^{30} x + c$ (xii) $\frac{1}{13}\sec^{13} x - \frac{1}{11}\sec^{11} x + c$

3. (i) $\frac{1}{128}\left(3x - \sin 4x + \frac{1}{8}\sin 8x\right) + c$ (ii) $-\frac{1}{6}\cos 3x - \frac{1}{20}\cos 5x - \frac{1}{4}\cos x + c$

(iii) $\frac{1}{40}(\cos 5x + 20\cos x - 5\cos 3x) + c$ (iv) $\frac{1}{192}(12x - 3\sin 2x - 3\sin 4x + \sin 6x) + c$

4. (i) $\frac{1}{6}\tan^6\theta + \frac{1}{8}\tan^8\theta + c$ (ii) $-\frac{1}{5}\cot^5\theta - \frac{1}{7}\cot^7\theta + c$ (iii) $\frac{1}{11}\sec^{11}\theta + c$ (iv) $-\frac{1}{7}\mathrm{cosec}^7\theta + c$

(v) $\frac{1}{2}\tan^2\theta + ln\ |\cos\theta| + c$ (vi) $-\frac{1}{3}\cot^3\theta + \cot\theta + \theta + c$

(vii) $\frac{1}{4}\tan^4\theta - \frac{1}{2}\tan^2\theta + \ln|\sec\theta| + c$ (viii) $-\frac{1}{5}\cot^5\theta + \frac{1}{3}\cot^3\theta - \cot\theta - \theta + c$

5. (i) $-\frac{2}{a+b}\ln\left|\sin\frac{a+b}{2}x\right| + c$ (ii) $\frac{2}{p+q}\ln\left|\sin\frac{p+q}{2}x\right| + c$ (iii) $4\cos x - \left(\frac{2}{3}\right)\cos 3x + c$

(iv) $\frac{1}{b-a} \ln |a \cos^2 x + b \sin^2 x + c| + k$

EXERCISE – 9 (d)

1. (i) $\frac{1}{2}\sin^{-1}\frac{2x}{\sqrt{11}} + c$ (ii) $\frac{1}{3}\sin^{-1}\left(\frac{e^{3x}}{2}\right) + c$ (iii) $\sin^{-1}\left(\frac{\ln x}{5}\right) + c$ (iv) $\sin^{-1}\left(\frac{\sin\theta}{2}\right) + c$ (v) $\frac{1}{3}\sin^{-1}\left(\frac{x^3}{6}\right) + c$

   (vi) $3\sin^{-1}(x/3) - \sqrt{9-x^2} + c$ (vii) $\sin^{-1}\frac{x+2}{3} + c$ (viii) $\sin^{-1}\frac{x+2}{3} - \sqrt{5-x^2-4x} + c$

2. (i) $\frac{1}{\sqrt{21}}\tan^{-1}\frac{\sqrt{3}x}{\sqrt{7}} + c$ (ii) $\frac{1}{8}\tan^{-1}\left(\frac{e^{4x}}{2}\right) + c$ (iii) $\frac{1}{5}\tan^{-1}\left(\frac{\ln x}{5}\right) + c$ (iv) $\frac{1}{2}\tan^{-1}\left(\frac{\sec\theta}{2}\right) + c$

   (v) $\frac{1}{20}\tan^{-1}\left(\frac{x^{10}}{2}\right) + c$ (vi) $\frac{3}{2}\ln(x^2+4) + 2\tan^{-1}\left(\frac{x}{2}\right) + c$ (vii) $\frac{1}{2}\tan^{-1}\frac{x+3}{2} + c$

   (vii) $\frac{1}{2}\ln(x^2+6x+13) + \tan^{-1}\frac{x+3}{2} + c$

3. (i) $\frac{1}{3}\sec^{-1}\frac{2x}{3} + c$ (ii) $\frac{1}{2\sqrt{5}}\sec^{-1}\left(\frac{e^{2x}}{\sqrt{5}}\right) + c$ (iii) $\frac{1}{2}\sec^{-1}\left(\frac{\ln x}{2}\right) + c$ (iv) $\sec^{-1}\left(\sqrt{3}\tan\theta\right) + c$

   (v) $\frac{1}{7b}\sec^{-1}\left(\frac{x^7}{b}\right) + c$ (vi) $\sqrt{x^2-4} + \frac{3}{2}\sec^{-1}\left(\frac{x}{2}\right) + c$ (vii) $\frac{1}{2}\sec^{-1}\frac{x+1}{2} + c$

   (viii) $\sqrt{x^2+2x-3} + \frac{3}{2}\sec^{-1}\frac{x+1}{2} + c$

4. (i) $\frac{1}{\sqrt{3}}\ln\left(\sqrt{3}x + \sqrt{3x^2+4}\right) + c$ (ii) $\frac{4}{\sqrt{3}}\ln\left(\sqrt{3}e^x + \sqrt{3e^{2x}+4}\right) + c$

   (iii) $\ln\left(\ln x + \sqrt{(\ln x)^2 + 8}\right) + c$ (iv) $-\ln\left(\cot\theta + \sqrt{\cot^2\theta + 2}\right) + c$ (v) $\frac{1}{3}\ln\left(x^3 + \sqrt{x^6 + a^6}\right) + c$

   (vi) $\frac{3}{5}\sqrt{5x^2+8} + \frac{4}{\sqrt{5}}\ln\left(\sqrt{5}x + \sqrt{5x^2+8}\right) + c$ (vii) $\ln\left(\sin e^x + \sqrt{\sin^2 e^x + 9}\right) + c$ (viii) $2\sqrt{x^2+10x+29}$

   $+ \ln\left(x+5+\sqrt{x^2+10x+29}\right) + c$

5. (i) $\frac{1}{2}\ln\left|2x + \sqrt{4x^2-6}\right| + c$ (ii) $\frac{1}{5}\ln\left(e^{5x} + \sqrt{e^{10x-4}}\right) + c$ (iii) $\ln\left|\ln x + \sqrt{(\ln x)^2 - 4}\right| + c$

   (iv) $-\ln\left|\operatorname{cosec}\theta + \sqrt{\operatorname{cosec}^2\theta - 4}\right| + c$ (v) $2\ln\left|\sqrt{x} + \sqrt{x-a^2}\right| + c$

   (vi) $\frac{1}{3}\sqrt{3x^2-8} - \frac{2}{\sqrt{3}}\ln\left|\sqrt{3}x + \sqrt{3x^2-8}\right| + c$ (vii) $\frac{3}{\sqrt{2}}\ln\left|\sqrt{2}x + \sqrt{2x^2-5}\right| + \frac{4}{\sqrt{5}}\sec^{-1}\frac{\sqrt{2}x}{\sqrt{5}} + c$

   (viii) $\sqrt{x^2-4} + 2\ln|x + \sqrt{x^2-4}| + \sec^{-1}(x/2) + c$

(ix) $\ln\left|x+4+\sqrt{x^2+8x}\right|+c(x)\sqrt{x^2+8x}+3\ln\left|x+4+\sqrt{x^2+8x}\right|+c$

EXERCISE – 9 (e)

1. (i) $xe^x + c$ (ii) $e^x(x^3-3x^2+6x-6)+c$ (iii) $\frac{e^{ax}}{a^3}\left(a^2x^2-2ax+2\right)+c$ (iv) $\frac{e^{2x}}{4}\left(18x^2+6x+5\right)+c$

2. (i) $\sin x - x\cos x + c$ (ii) $x^2\sin x + 2x\cos x - 2\sin x + c$ (iii) $\frac{1}{a^3}\,[(2-a^2x^2)\cos ax + 2ax\sin ax]+c$

(iv) $\frac{1}{8}\left(2x^2+2x\sin 2x+\cos 2x\right)+c$ (v) $\frac{1}{4}\left(3\sin x-3x\cos x+\frac{x}{3}\cos 3x-\frac{1}{9}\sin 3x\right)+c$

(vi) $\sin x+\frac{1}{9}\sin 3x - x(\cos x+\frac{1}{3}\cos 3x)+c$

(vii) $\cos x+\frac{1}{25}\cos 5x+x\left(\sin x+\frac{1}{5}\sin 5x\right)+c$

(viii) $x^2\sin x^2+\cos x^2+c$

(ix) $\ln|\sin x| - x\cot x + c$ (x) $x\tan x+\ln|\cos x| - x^2/2 + c$

3. (i) $\frac{x^2-1}{2}\ln|1+x|-\frac{1}{4}\left(x^2-2x\right)+c$ (ii) $\frac{x^8}{64}(8\ln x-1)+c$

(iii) $x[(\ln x)^3-3(\ln x)^2+6\ln x-6]+c$ (iv) $x\ln(x^2+1)-2x+2\tan^{-1}+x+c$

(v) $-\frac{1}{16x^4}(1+4\ln x)+c$

(vi) $\left(x+\frac{1}{2}\right)\ln(x+x+2)-2x+\sqrt{7}\tan^{-1}\frac{2x+1}{\sqrt{7}}+c$

(vii) $x\ln\left(x+\sqrt{x^2+a^2}\right)-\sqrt{x^2+a^2}+c$ (viii) $x\ln|x+\sqrt{x^2-a^2}|-\sqrt{x^2-a^2}+c$

4. (i) $x\sin^{-1}x+\sqrt{1-x^2}+c$ (ii) $\left(\frac{x^2}{2}-\frac{1}{4}\right)\sin^{-1}x+\frac{1}{4}x\sqrt{1-x^2}+c$

(iii) $x\cos^{-1}x-\sqrt{1-x^2}+c$ (iv) $\frac{x^2+1}{2}\tan^{-1}x-\frac{1}{2}x+c$

(v) $\frac{1}{6}\left[2x^3\tan^{-1}x-x^2+\ln\left(x^2+1\right)\right]+c$ (vi) $x\sec^{-1}x-\ln|x+\sqrt{x^2-1}|+c$

(vii) $x\,\mathrm{cosec}^{-1}x+\ln|x+\sqrt{x^2-1}|+c$

5. (i) $\frac{e^{2x}}{13}(3\cos 2x+2\sin 2x)+c$ (ii) $\frac{e^x}{5}(2\sin x-\cos x)+c$

(iii) $\frac{e^x}{10}(5+\cos 2x+2\sin 2x)+c$ (iv) $\frac{e^{x^2}}{4}(\sin x^2-\cos x^2)+c$

(v) $\frac{e^{ax}}{a^2+b^2}\,[a\sin(bx+c)-b\cos(bx+c)]+K$ (vi) $xe^{x^2}+c$

6. (i) $\frac{x}{2}\sqrt{9-x^2}+\frac{9}{2}\sin^{-1}\frac{x}{3}+c$ (ii) $\frac{1}{4}\left[2x\sqrt{5-4x^2}+5\sin^{-1}\frac{2x}{\sqrt{5}}\right]+c$

(iii) $\frac{x+1}{2}\sqrt{1-x^2-2x}+\sin^{-1}\frac{x+1}{\sqrt{2}}+c$ (iv) $\frac{1}{2}e^z\sqrt{4-e^{2z}}+2\sin^{-1}\frac{e^z}{2}+c$

(v) $\frac{1}{2}\sin\theta\sqrt{5-\sin^2\theta}+\frac{5}{2}\sin^{-1}\left(\frac{\sin\theta}{\sqrt{5}}\right)+c$

7. (i) $\frac{x}{2}\sqrt{x^2+4}+2\ln\left(x+\sqrt{x^2+4}\right)+c$ (ii) $\frac{x}{2}\sqrt{7x^2+2}+\frac{1}{\sqrt{7}}\ln\left(\sqrt{7}x+\sqrt{7x^2+2}\right)+c$

(iii) $\frac{2x+3}{4}\sqrt{4x^2+12x+13}+\ln\left(2x+3+\sqrt{4x^2+12x+13}\right)+c$

(iv) $\frac{1}{4}\left[e^{2z}\sqrt{e^{4z}+6}+6\ln\left(e^{2z}+\sqrt{e^{4z}+6}\right)+c\right]$

(v) $\frac{1}{2}\tan\theta\sqrt{\sec^2\theta+3}+2\ln\left(\tan\theta+\sqrt{\sec^2\theta+3}\right)+3$

8. (i) $\frac{x}{2}\sqrt{x^2-8}-4\ln|x+\sqrt{x^2-8}|+c$ (ii) $\frac{x}{2}\sqrt{3x^2-2}-\frac{1}{\sqrt{3}}\ln|\sqrt{3x}+\sqrt{3x^2-21}|+c$

(iii) $\frac{x-2}{2}\sqrt{x^2-4x+2}-\ln|x-2+\sqrt{x^2-4x+2}|+c$ (iv) $\frac{1}{2\ln a}\left[a^z\sqrt{a^{2z}-4}-4\ln|a^z+\sqrt{a^{2z}-4}|+c\right.$

(v) $\frac{1}{2}\sec\theta\sqrt{\tan^2\theta-3}-2\ln|\sec\theta+\sqrt{\tan^2\theta-3}+c$

9. (i) $e^x\ln(\sec x)+c$ (ii) $e^x\ln(\sin x)+c$ (iii) $e^x\ln(x)+c$ (iv) $\frac{e^x}{x+1}+c$

10. (i) $\frac{x}{\ln x}$ (ii) $\frac{x}{2}\left[\sin(\ln x)-\cos(\ln x)\right]+c$

(iii) $\ln|\sin x|-\cos x\,\ln\,|\operatorname{cosec} x-\cot x|+c$

EXERCISE – 9 (f)

1. (i) $\ln|(x-2)(x-3)^3|+c$ (ii) $\ln|(x+2)(x-4)^2|+c$

(iii) $\ln(x-6)^2\sqrt{2x-3}$ (iv) $\ln|2x+1|+\frac{7}{3}\ln|3x-2|+c$

(v) $\ln|x-1|-8\ln|x-2|+9\ln|x-3|+c$ (vi) $\frac{2}{3}x^2+\frac{1}{2}\ln|2x+1|-\frac{1}{3}\ln|3x-2|+c$

2. (i) $2\ln|x+3|-\frac{3}{x+3}+c$ (ii) $2\ln|x-2|+3\ln|x+2|+\frac{4}{x+2}+c$

(iii) $3\ln|x-1|-2\ln|x+1|-\frac{1}{x+1}+c$ (iv) $\frac{x^2}{2}+2x+\ln\left|\frac{x-1}{x+1}\right|-\frac{1}{x+1}+c$

3. (i) $\frac{1}{2}\ln(x^2+1)+3\ln|x-1|+c$

(ii) $3\tan^{-1}(x-1)+\frac{1}{2}\ln(x^2-2x+2)-\ln|x+1|+c$

(iii) $\ln|x-1|-\frac{1}{2}\ln(x^2+x+1)-\sqrt{3}\tan^{-1}\frac{2x-1}{\sqrt{3}}+c$

(iv) $\frac{x^3}{3}+\frac{x^2}{2}+x+\frac{1}{2}\ln\left(x^2-x+1\right)-\ln|x+1|+\sqrt{3}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)+c$

4. (i) $\frac{1}{2\sqrt{5}}\ln\left|\frac{x-\sqrt{5}}{x+\sqrt{5}}\right|+C$ (ii) $\frac{1}{2\sqrt{2}}\ln\left|\frac{\sqrt{2x}+2\sqrt{2}-1}{\sqrt{2x}+2\sqrt{2}+1}\right|+C$

(iii) $\frac{1}{4}\ln|2x^2+8x+7|\frac{1}{2\sqrt{2}}\ln\left|\frac{\sqrt{2x}+2\sqrt{2}-1}{\sqrt{2x}+2\sqrt{2}+1}\right|+c$

(iv) $2x+\ln|2x^2+8x+7|+\frac{3}{2\sqrt{2}}\ln\left|\frac{\sqrt{2x}+2\sqrt{2}-1}{\sqrt{2x}+2\sqrt{2}+1}\right|+c$

(v) $\frac{1}{\sqrt{5}}\ln\frac{2e^x+3-\sqrt{5}}{2e^x+3+\sqrt{5}}+c$ (vi) $\frac{1}{2}\ln\left|\frac{\tan\theta-1}{\tan\theta+1}\right|+c$

5. (i) $\frac{1}{2\sqrt{3}}\ln\left|\frac{\sqrt{3}+x}{\sqrt{3}-x}\right|+c$ (ii) $\frac{1}{8}\ln\left|\frac{x+1}{7-x}\right|+c$ (iii) $-\frac{1}{4}\ln|(7-x)(x+1)^3|+c$

(iv) $\frac{1}{2\sqrt{3}}\ln\frac{\sqrt{3}+\sin\theta}{\sqrt{3}-\sin\theta}+c$ (v) $\sqrt{3}\tan^{-1}\frac{x}{\sqrt{3}}-\sqrt{2}\tan^{-1}\frac{x}{\sqrt{2}}$ (vi) $\ln(x^2+2)-\frac{1}{2}\ln(x^2+1)$

(vii) $-\frac{1}{2}\ln(1+\cos x)+\frac{1}{10}\ln(1-\cos x)+\frac{2}{5}\ln(3+2\cos x)$

EXERCISE – 9 (g)

1. (i) $2\sqrt{2x+3}+\sqrt{3}\ln\left|\frac{\sqrt{2x+3}-\sqrt{3}}{\sqrt{2x+3}+\sqrt{3}}\right|+c$ (ii) $\sqrt{x^2-7}-\sqrt{7}\tan^{-1}\frac{\sqrt{x^2-7}}{\sqrt{7}}+c$

(iii) $2\sqrt{x}-2\sqrt{2}\tan^{-1}\sqrt{\frac{x}{2}}+c$ (iv) $\frac{1}{14}(3x+2)^{4/3}(2x-1)+c$

(v) $\frac{3}{40}(4x+23)(2x-1)^{2/3}+c$ (vi) $\frac{4}{15}(5x+14)(x+1)^{5/4}+c$

(vii) $\frac{4}{5}(x-13)(x+2)^{\frac{1}{4}}+c$ (viii) $2x^{\frac{1}{2}}+3x^{\frac{1}{3}}+6x^{\frac{1}{6}}+6\ln|x^{\frac{1}{6}}-1|+c$

2. (i) $(x+7)\sqrt{2x-3}+c$ (ii) $\frac{2}{135}(63x+32)(3x+2)^{3/2}+c$

(iii) $\frac{2}{143}(33x+25)(x-2)^{\frac{11}{2}}+c$ (iv) $-\frac{2}{15}(10x+23)(x+2)^{-\frac{5}{2}}+c$

(v) $\frac{2}{5}(x^2-2x+21)(x+4)^{\frac{1}{2}}+c$ (vi) $\frac{2}{7}(x^2+2x+15)(x+1)^{3/2}+c$

3. (i) $\frac{1}{2}\ln(2x-1+\sqrt{4x^2-4x+5})+c$ (ii) $\frac{1}{4}(2x-1)\sqrt{4x^2-4x+5}+\ln(2x-1+\sqrt{4x^2-4x+5})+c$

(iii) $\ln\left|x-3+\sqrt{x^2-6x+51}+c\right|$ (iv) $\frac{1}{2}(x-3)\sqrt{x^2-6x+5}-2\ln\left|x-3+\sqrt{x^2-6x+5}\right|+c$

(v) $\sin^{-1}\frac{x-1}{\sqrt{2}}+c$ (vi) $\frac{1}{2}(x-1)\sqrt{1+2x-x^2}+\sin^{-1}\frac{x-1}{\sqrt{2}}+c$

4. (i) $-\sqrt{\frac{1-x}{1+x}}+c$ (ii) $\ln\left|\frac{1+\sqrt{5-4x+x^2}}{2-x}\right|+c$

5. (i) $\sqrt{2}\tan^{-1}\sqrt{2x+4}+c$ (ii) $\ln\left|\frac{x^2-1+\sqrt{x^4+1}}{x}\right|+c$

(iii) $\frac{1}{2\sqrt{2}}\left[\sqrt{(2x-2)(2x+1)}-3\ln\left(\sqrt{2x-2}+\sqrt{2x+1}\right)\right]+c$ (iv) $\sin^{-1}\sqrt{\frac{x^2-b^2}{a^2-b^2}}+c$

EXERCISE – 9 (h)

1. (i) $\frac{1}{3}\ln\left|\frac{3+\tan\frac{x}{2}}{3-\tan\frac{x}{2}}\right|+c$ (ii) $\frac{1}{\sqrt{2}}\tan^{-1}\frac{\tan\frac{x}{2}}{\sqrt{2}}+c$ (iii) $\frac{1}{\sqrt{2}}\tan^{-1}\frac{3\tan\frac{x}{2}+1}{2\sqrt{2}}+c$

(iv) $\frac{1}{\sqrt{3}}\ln\left|\frac{\tan\frac{x}{2}+2-\sqrt{3}}{\tan\frac{x}{2}+2+\sqrt{3}}\right|+c$ (v) $-\frac{1}{\sqrt{13}}\ln\left|\frac{\sqrt{13}-3\tan\frac{x}{2}+2}{\sqrt{13}+3\tan\frac{x}{2}-2}\right|+c$ (vi) $\ln\left|1+\tan\frac{x}{2}\right|+c$

2. (i) $3x+2\ln\left|5\sin x+6\cos x\right|+c$ (ii) $5x-7\ln\left|\sin x+\cos x\right|+\frac{3}{\sqrt{2}}\ln\left|\frac{\sqrt{2}-1+\tan\frac{x}{2}}{\sqrt{2}+1-\tan\frac{x}{2}}\right|$

(iii) $3\sqrt{5}\tan^{-1}\left(\frac{3\tan\frac{x}{2}-c}{\sqrt{5}}\right)-\frac{5x}{2}+c$ (iv) $\frac{14}{\sqrt{15}}\tan^{-1}\frac{4\tan\frac{x}{2}-1}{\sqrt{15}}-2\ln(4-\sin x)+c$

3. (i) $\frac{1}{3}\ln\left|\sec x+\tan x\right|-\frac{4}{3\sqrt{5}}\tan^{-1}\frac{\tan\frac{x}{2}}{\sqrt{5}}+c$ (ii) $\frac{4}{\sqrt{15}}\ln\left|\frac{\sqrt{15}-4+\tan\frac{x}{2}}{\sqrt{15}+4-\tan\frac{x}{2}}\right|-\ln\left|\operatorname{cosec}x-\cot x\right|+c$

(iii) $\frac{1}{2}\ln(\sin^2x-2\sin x+3)+\frac{1}{\sqrt{2}}\tan^{-1}\frac{\sin x-1}{\sqrt{2}}+c$ (iv) $\frac{1}{8}[\ln\frac{1+\sin x}{1-\sin x}-2\operatorname{cosec}x]+c$

4. (i) $\frac{1}{2\sqrt{7}}\tan^{-1}\left(\frac{\sqrt{7}\tan\theta}{2}\right)+c$ (ii) $\frac{1}{2\sqrt{2}}\ln\left|\frac{\sqrt{2}\tan\theta-1}{\sqrt{2}\tan\theta+1}\right|+c$

(iii) $\frac{1}{6}\tan^{-1}\left(\frac{3\tan\theta}{2}\right)+c$ (iv) $\frac{1}{2\sqrt{10}}\ln\left|\frac{\sqrt{5}+\sqrt{2}\tan\theta}{\sqrt{5}-\sqrt{2}\tan\theta}\right|+c$

5. (i) $\frac{1}{7}\ln|\sec 7x|-\frac{1}{4}\ln|\sec 4x|+c$ (ii) $\frac{1}{7}\ln|\sin 7x|+\frac{1}{5}\ln|\sec 5x|+c$

6. (i) $\frac{1}{5}\ln\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)-\frac{3}{10}\tan^{-1}(\frac{1}{2}\tan\frac{x}{2})$ (ii) $\frac{2}{3}\ln(1+2\sin)-\frac{1}{2}\ln(1+\sin x)-\frac{1}{6}\ln(1-\sin x)$

EXERCISE – 9 (i)

1. $\frac{130}{3}$ 2. $e-\frac{1}{e}$ 3. 18 4. $\frac{160}{3}$

EXERCISE – 9 (j)

1. (i) 55 (ii) $\frac{3}{4}$ (iii) 2 (iv) $\frac{4}{9}$ (v) $\frac{13+4\sqrt{2}}{3}+\frac{3}{2}2^{\frac{2}{3}}$ (vi) $-\frac{9}{2}$

(vii) 24.2 (viii) $\frac{3}{32}(2^{\frac{4}{3}}-1)$ (ix) 8.4 (x) $87\frac{3}{35}$

2. (i) 0 (ii) $\frac{1}{2}$ (iii) $1-\pi/4$ (iv) 3 (v) $\frac{1}{3}\left(2-\frac{1}{\sqrt{2}}\right)$ (vi) $\frac{2}{\sqrt{3}}-\pi/6$ (vii) $\frac{1}{2}\ln 2$ (viii) $\frac{1}{6}$

3. (i) $\frac{1}{4}\left(e^9-e^5\right)$ (ii) $\frac{72}{\ln 3}$ (iii) $\frac{1}{4}\left(e^2-\frac{1}{e^2}\right)$ (iv) $\ln\frac{e^2+1}{2e}$ (v) $\frac{1}{3}\left(e^8-1\right)$

4. (i) $\frac{\pi}{2}$ (ii) $\frac{\pi}{6}$ (iii) $\pi/3$ (iv) $\pi/3$

5. (i) $\frac{1}{3}\ln(5/2)$ (ii) ln3 (iii) $\ln\left(\frac{3+\sqrt{7}}{2+\sqrt{2}}\right)$ (iv) $10+\frac{9}{2}\ln 3$

6. (i) 1 (ii) 9 (iii) 3/2 (iv) $5-\sqrt{3}-\sqrt{2}$

7. (i) $2e^2-e$ (ii) $\frac{1}{\sqrt{2}}(1-\pi/4)$ (iii) $2\ln 2-\frac{3}{4}$ (iv) $\frac{\pi}{4}-1/2$

(v) $\ln\left(2/\sqrt{3}\right)$ (vi) ln (4/3) (vii) $\frac{1}{2}\left(e^{\frac{\pi}{2}}-1\right)$

EXERCISE – 9 (k)

1. (i) $\frac{\pi}{4}$ (ii) $\pi/4$ (iii) $\frac{\pi}{8}\ln 2$ (iv) π 2. (i) $\frac{2}{5}a^5$ (ii) $\frac{4}{3}a^3$ (iii) $\frac{1}{4}(\pi+2)$ (iv) 0

3. (i) 0 (ii) $\pi/2$ (iii) 0 (iv) 2/3

5. (i) $\pi\ln 2$ (ii) $\frac{\pi(\pi-2)}{2}$ (iii) 1 (iv) $\frac{\pi^2}{4}$ (v) $\frac{1}{10302}$ (vi) $\frac{\pi}{12}$ (vii) $50(e-1)$

ADDITIONAL EXERCISE

1. $\sin x + \cos x + c$ 2. $\tan x - \sec x + c$ 3. $x - \tan x + \sec x + c$

4. $\tan x - \sec x + c$ 5. $2\tan x - x + 2\sec x + c$ 6. $\frac{\pi x}{4} + \frac{1}{4}x^2 + c$

7. $2\sin x + 2x\cos\alpha + c$ 8. $\frac{x^2}{2} + c$ 9. $\frac{2}{3}(x+2)^{3/2} - \frac{2}{3}(x+1)^{3/2} + c$

10. $-\frac{13}{4}\log|3-2x| + \frac{3}{4}(3-2x) + c$ 11. $2\log|1+\sqrt{x}| + c$ 12. $\frac{1}{2}x + \frac{1}{2}\log|\cos x + \sin x| + c$

13. $(x+1) + 2\sqrt{x+1} - 2\log|x+2| - 2\tan^{-1}\sqrt{x+1} + c$ 14. $x\tan^{-1}\sqrt{\frac{x}{a}} - \sqrt{9x} + a\tan^{-1}\sqrt{\frac{x}{a}} + c$

15. $e^x\tan x + c$ 16. $e^x\left(\frac{x-1}{x+1}\right) + c$ 17. $\frac{1}{2}\log\left|\frac{x^2-x+1}{x^2+x+1}\right| + c$

18. $\frac{1}{2}\log\left|\frac{x^2-x+1}{x^2+x+1}\right| + \frac{1}{2\sqrt{3}}\tan^{-1}\left(\frac{x^2+1}{\sqrt{3}x}\right) + c$

19. $-\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\cot x - 1}{\sqrt{2\cot x}}\right) - \frac{1}{2\sqrt{2}}\log\left|\frac{\cot x - \sqrt{x\cot x + 1}}{\cot x + \sqrt{2\cos x + 1}}\right| + c$ 20. $\sqrt{2}\tan^{-1}\left(\frac{\tan x - 1}{\sqrt{2\tan x}}\right) + c$

21. $\log|x| - \frac{1}{4}\log(x^4+1) + c$ 22. $\log\left|\frac{e^x-1}{e^x}\right| + c$

23. $x + 3\log|x-4| - 24\log|x-5| + 30\log|x-6| + c$ 24. $x - \log|e^x - 1| - \frac{1}{e^x-1} + c$

25. $\sec x - \frac{1}{2}\log\left|\frac{\cos x + 1}{\cos x - 1}\right| + c$ 26. $\frac{57}{5} - \sqrt{5}$ 27. $a\pi$

28. $\sqrt{2}\,\pi$ 29. $\frac{\pi}{4} - \frac{1}{2}\log 2$ 30. $\frac{1}{(n+1)(n+2)}$

31. 0 32. $\frac{1}{\sqrt{2}}\log(\sqrt{2}+1)$ 33. $\frac{\pi}{3\sqrt{3}}$

34. $\frac{\pi}{4\sqrt{2}}\log\left|\frac{\sqrt{2}+1}{\sqrt{2}-1}\right|$ 36. $\frac{\pi}{20}$ 37. 2 38. $\frac{19}{2}$ 39. 4 40. $-\pi\ln 2$

EXERCISE – 10

1. (i) $e^4 - e^2$ (ii) $\frac{1}{3}$ (iii) $a^2\ln\beta/\alpha$ (iv) 1 2. (i) $\ln\left(\frac{27}{4e}\right)$ (ii) $\frac{1}{3}$ (iii) $a^2\ln\beta/\alpha$ (iv) 3/5

3. (i) πab (ii) πa^2 (iii) $8\sqrt{3}$ (iv) $\frac{16\sqrt{2}}{5}$ 4. (i) $\frac{4}{3}\pi - \sqrt{3}; \frac{8}{3}\pi + \sqrt{3}$ (ii) $\sqrt{2} - 1$ (iii) $\frac{16}{3}a^2$ (iv) $4/3 + \frac{\pi}{2}$

EXERCISE – 11 (a)

1. (i) 1,1 (ii) 3, 1 (iii) 2, 2 (iv) 1, 1 (v) 2, 1 (vi) 1, 2 (vii) 2, 3 (viii) 1, 1

2. (i) $y_1 = y \tan x$ (ii) $(1+x^2)\tan^{-1} x y_1 = y$ (iii) $y_2 - 3y_1 + 2y = 0$
(iv) $x^2 y_2 - 2xy_1 + 2y = 0$ (v) $y_2 + y = 0$ (vi) $y_2(1-x^2) = xy_1$
(vii) $y_2(1-t) + ty_1 - y = 0$ (viii) $y_2(\cot t - 1) + 2y_1 - y(\cot t + 1) = 0$ (ix) $xy_2 = y_1$

3. (i) $y = e^x - e^{-x} + c$ (ii) $y = x\sin x + \cos x + c$ (iii) $y = \frac{t^6}{36}(6\ln t - 1) + c$
(iv) $y = t^3 + 2t^2 + \tan t + c$ (v) $y = \ln\left|\frac{x-4}{x-3}\right| + c$ (vi) $3y = \sqrt{3u^2+6u+5} + c$
(vii) $y = \ln\left|\frac{x+1}{x+2}\right| + c$ (viii) $y = e^{\sin^{-1}t}(\sin^{-1} t - 1) + c$

4. (i) $y + 2 = ce^x$ (ii) $\sin^{-1} y = t + c$ (iii) $\sin y = z + c$
(iv) $e^{-y} = c - x$ (v) $y = c(y+2)e^{2x}$ (vi) $y = \tan(c-x)$
(vii) $x = (y+1)e^{-y} + c$ (viii) $\cos x = ce^t$

5. (i) $\tan^{-1} y = \frac{x^3}{3} + x + c$ (ii) $2e^{-3y} + 3e^{2t} = c$ (iii) $\sin^{-1} y = \sin^{-1} z + c$
(iv) $y^3 + 2y^2 = \frac{x^2}{4}(2\ln x - 1) + c$ (v) $2\sqrt{x^3+1} + 3\sqrt{y^2+3} = c$ (vi) $\sin y = c\cos x$
(vii) $(y-5)(x+3)^4 = c(y-1)(x+4)^4$ (viii) $e^y(y-1) - x\cos x + \sin x = c$

6. (i) $y = x^4 + \frac{x^3}{3} + Ax + B$ (ii) $y = \frac{e^{2t}}{4} + e^{-t} + At + B$
(iii) $y = \sin v - \cos v + \ln|\sec v| + Av + B$ (iv) $y = -x\sin x - 2\cos x + Ax + B$
(v) $y = 2\ln|x| + Ax + B$ (vi) $y = Ax + B - \frac{1}{32}(\sin 4x + 4\sin 2x)$
(vii) $y = \ln|\sec x| + \frac{1}{8}(2x^2 - \cos 2x) + Ax + B$ (viii) $y = e^x(x-2) + Ax + B$

7. (i) $y = \sin x + 2$ (ii) $\tan y = t + 1$, (iii) $\tan^{-1} y = \tan^{-1} x + \pi/12$
(iv) $y = x^3 + 2x + 1$

8. (i) $y - \tan\frac{1}{2}(x+y) = c$ (ii) $\ln\left[1 + \tan(\frac{x+y}{2})\right] = x + c$
(iii) $\tan\frac{x+y}{2} = x + c$ (iv) $(x+c)e^{x+y} + 1 = 0$

EXERCISE – 11 (b)

1. $y = (x+c)e^{-x}$
2. $y(1-x^2) = -x + c$
3. $y = c(1-x^2) + \sqrt{1-x^2}$
4. $y\log x = c + (\log x)^2$
5. $y(1+x^2) = \sin x + c$
6. $(y-1)(\sec x + \tan x) + x = c$
7. $x\sqrt{\cot y} = c + \sqrt{\tan y}$
8. $x = y(c+y^2)$
9. $\left(\frac{1}{3} + y\right)\tan^3\frac{x}{2} = c + 2\tan\frac{x}{2} - x$
10. $x = ce^y - y - 2$
11. $xe^{\tan^{-1}y} = c + \tan^{-1} y$
12. $xy\log\frac{c}{x} = 1$
13. $y(1 + cx + \log x) = 1$
14. $\sqrt{1+x^2} = y(c + \sinh^{-1} x)$.

15. $y^{2/3} = c(x-1)^{-2/3} + \frac{1}{4}(x-1)^2 + \frac{2}{5}(x-1)$; 16. $4xy = x^4 + 3$; 17. $y = \cos x - 2\cos^2 x + \frac{1}{8}$

EXERCISE–11(c)

1. $\frac{1}{2}\log(x^2+y^2) + \tan^{-1}\frac{y}{x} = c$; 2. $y = x + c\sqrt{xy}$; 3. $x^2 + y^2 = cx$; 4. $c - y = \sqrt{x^2+y^2}$;

5. $(x-y)^2 = cxe^{-\frac{y}{x}}$; 6. $\frac{y}{x} = c + \log y$; 7. $\log x = \cos\left(\frac{y}{x}\right) + c$; 8. $y + \sqrt{x^2+y^2} = cx^2$

9. $\tan^{-1}\frac{y+3}{x+2} + \log c\sqrt{(x+2)^2+(y+3)^2} = 0$ 10. $\tan^{-1}\frac{2y+1}{2x+1} = \log c\sqrt{x^2+y^2+x+y+\frac{1}{2}}$

11. $\log(x-1) + c = -\frac{1}{2}\log\left\{2\left(\frac{y+1}{x-1}\right)^2 - 1\right\} + \frac{1}{2\sqrt{2}}\log\left(\frac{\sqrt{2}y - x + \sqrt{2} + 1}{\sqrt{2}y + x + \sqrt{2} - 1}\right)$

12. $(c-y-x-1)^2(y+x-1)^5 = 1$ 13. $\log(2x+y-1) + x + 2y = c$

14. $\frac{3}{2}(x^2+y^2) + 2xy - 5(x+y) = c$ 15. $\frac{1}{8}(2x+y^3) + \frac{9}{64}\log(16x+24y+23) = x + c$

EXERCISE–12(a)

1. (i) d, (ii) c, (iii) c, (iv) d, (v) c

2. (i) $\vec{0}$, (ii) indefinite direction, (iii) equal in magnitude, (iv) $\vec{a} = \vec{b} \Rightarrow |\vec{a}| = |\vec{b}|$ (v) true

3. (i) (4, 3), (ii) (–2,7, 0) ; 4. $\vec{0}$

5. (i) $\overrightarrow{AB} = -\hat{i}-2\hat{j}-2\hat{k}$, $\overrightarrow{BC} = -2\hat{i}-\hat{j}+2\hat{k}$, $\overrightarrow{CA} = 3\hat{i}+3\hat{j}$,

$|\overrightarrow{AB}| = 3$, $|\overrightarrow{BC}| = 3$, $|\overrightarrow{CA}| = 3\sqrt{2}$

(ii) $\overrightarrow{AB} = -6\hat{i}-6\hat{j}$, $\overrightarrow{BC} = -6\hat{i}-6\hat{k}$, $\overrightarrow{CA} = 12\hat{i}+6\hat{j}+6\hat{k}$

$|\overrightarrow{AB}| = 6\sqrt{2}$, $|\overrightarrow{BC}| = 6\sqrt{2}$, $|\overrightarrow{CA}| = 6\sqrt{6}$

6. $6\hat{i}-\hat{j}$; 7. $-\hat{i}-2\hat{j}$; 8. $2\hat{i}-7\hat{j}$, $2\hat{i}+2\hat{j}+2\hat{k}$

11. (a) (i) When $\vec{a}$ and $\vec{b}$ are collinear vectors or when both are zero vectors.

(b) The parallelogram formed by the vectors $\vec{a}$ and $\vec{b}$ as adjacent sides is a rectangel (i.e the two vectors are at right angles).

12. (i) $\sqrt{5}$;2, –1; $2\hat{i}-\hat{j}$ (ii) $4\sqrt{2}$; –4, –4; $-4\hat{i}-4\hat{j}$ (iii) $\sqrt{41}$; 1, –6, 2; $\hat{i}-6\hat{j}+2\hat{k}$

13. (i) $-3\hat{i}-2\hat{j}+3\hat{k}; \sqrt{22}; \frac{-3}{\sqrt{22}}, \frac{-2}{\sqrt{22}}, \frac{3}{\sqrt{22}}$ (ii) $\hat{i}-2\hat{j}-8\hat{k}; \sqrt{69}; \frac{1}{\sqrt{69}}, \frac{-2}{\sqrt{69}}, \frac{-8}{\sqrt{69}}$; 14. $\sqrt{30}; \frac{-2}{\sqrt{30}}, \frac{-5}{\sqrt{30}}, \frac{-1}{\sqrt{30}}$

15. (i) $\frac{5}{13}\hat{i} - \frac{12}{13}\hat{j}$, (ii) $\frac{2}{\sqrt{5}}\hat{i} + \frac{1}{\sqrt{5}}\hat{j}$, (iii) $\frac{3}{\sqrt{46}}\hat{i} + \frac{6}{\sqrt{46}}\hat{j} - \frac{1}{\sqrt{46}}\hat{k}$ (iv) $\frac{3}{\sqrt{14}}\hat{i} + \frac{1}{\sqrt{14}}\hat{j} - \frac{2}{\sqrt{14}}\hat{k}$; 16. $\frac{-2}{\sqrt{41}}\hat{i} + \frac{1}{\sqrt{41}}\hat{j} + \frac{6}{\sqrt{41}}\hat{k}$

17. $\frac{3}{7}\hat{i} + \frac{6}{7}\hat{j} - \frac{2}{7}\hat{k}$; $\frac{3}{7}, \frac{6}{7}, \frac{-2}{7}$

EXERCISE–12(b)

1. (i) d, (ii) c, (iii) c (iv) b ; 2. (i) –10, $\cos^{-1}\frac{-2}{\sqrt{5}}$ (ii) –17, $\cos^{-1}\frac{-17}{7\sqrt{38}}$ (iii) –1, $\frac{2\pi}{3}$ (iv) 0, $\frac{\pi}{2}$

3. (i) $\cos^{-1}\frac{7}{3\sqrt{33}}$; 4. (i) $\frac{15}{4}$, (ii) $\frac{4}{3}$ (iii) 3 (iv) −2/3 ; 5. (i) 0, $\vec{0}$ (ii) $\frac{1}{\sqrt{2}}, \frac{1}{2}(\hat{j}+\hat{k})$ (iii) $\frac{-1}{\sqrt{19}}, \frac{1}{19}(3\hat{k}+\hat{j}+3\hat{i})$

6. (i) 7 units, (ii) 0 unit, (iii) −13 units, (iv) 9 units.

EXERCISE 12 (c)

1. (i) b, (ii) a, (iii) c, (iv) c, (v) b
2. (i) $-5\hat{i}+10\hat{j}-3\hat{k}$, (ii) $-\hat{i}+2\hat{j}-7\hat{k}$, (iii) $-21\hat{i}-22\hat{j}+13\hat{k}$, (iv) $5\hat{i}-10\hat{j}+3\hat{k}$, (v) $-9\hat{i}-14\hat{j}+\hat{k}$
3. (i) $\pm\hat{j}$, (ii) $\pm\frac{1}{\sqrt{3}}(\hat{i}-\hat{j}+\hat{k})$, (iii) $\pm\frac{1}{\sqrt{61}}(6\hat{i}+3\hat{j}-4\hat{k})$, (iv) $\pm\frac{1}{\sqrt{3}}(\hat{i}+\hat{j}+\hat{k})$
4. (i) 2 sq. units (ii) 3sq.units (iii) $3\sqrt{3}$ sq. units (iv) $4\sqrt{2}$ sq. units
5. (i) $\frac{5}{2}\sqrt{10}$ sq.units, (ii) $\frac{1}{2}\sqrt{42}$ sq.units ; 6. (i) $\sqrt{\frac{217}{442}}$ (ii) $\sqrt{\frac{32}{33}}$; 9. −26 ; 11. $5\sqrt{3}$ sq.units

EXERCISE-12 (d)

1. (i) b (ii) a (iii) c 2. (i) −6 (ii) 51 ; 3. (i) 4 cubic units (ii) 1 cubic unit
5. (i) 5 or $\frac{-11}{2}$ (ii) −4 ; 9. $-\hat{i}+\hat{j}+3\hat{k}$; 11. $\frac{\pi}{2}, \frac{\pi}{3}$

ADDITIONAL EXERCISES

5. $\vec{a}-\vec{b}$, $-\vec{a}$, $-\vec{b}$, $\vec{b}-\vec{a}$; 6. $a=8$; 9. $|\vec{a}+\vec{b}|=5$, $|\vec{a}-\vec{b}|=1$; 12. $\frac{\pi}{3}$; 13. $-\frac{3}{2}$; 14. $\frac{\pi}{3}, \frac{2\pi}{3}, \frac{\pi}{4}$; 15. $\frac{\pi}{4}$

EXERCISE–13(a)

1. (a) 8 (b) 5 (c) 30^0; 2. (a) T (b) F (c) F (d) F (e) F. ; 4. (a) (3,4,5) (b) (1,2,2) (d) $y=\frac{22}{7}$ $z=\frac{1}{7}$
5. (a) 3 (b) 7, $\frac{6}{7},\frac{2}{7},\frac{3}{7}$ (c) 13, $\frac{12}{13},\frac{4}{13},\frac{3}{13}$
6. (a) $\cos^{-1}\frac{1}{\sqrt{3}}, 90^0, \cos^{-1}\sqrt{2/3}$ (b) $\cos^{-1}\frac{19}{3\sqrt{110}}$ (e) $\cos^{-1}\left(\pm\frac{1}{\sqrt{3}}\right)$ (f) $\cos^{-1}\left(\frac{1}{6}\right)$

EXERCISE–13(b)

1. (a) F (b) F (c) F, (d) T (e) F (f) T (g) F
2. (a) $x+y+z=4$ (b) $z-4=0$. (c) $\frac{1}{2}$ (d) parallel to x-axis (e) $k=-4$.
3. (a) $3x+5y-7z=6$ (b) $x-y-1=0$ (c) $4x-3y+12z=8$ (d) $2x+3y=13$ (e) $x-2z+5=0$
4. (a) $3x-4y+7z+13=0$ (b) $4x+7y+2z+11=0$ (c) $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=2$ (d) $2x-4y+3z+8=0$

   (e) $3x-3y-2z-1=0$ (f) $2x-y+3z=-1\pm 3\sqrt{14}$
5. (a) $\frac{x}{4}+\frac{y}{-3}+\frac{z}{2}=1$, (4, 0, 0), (0, −3, 0), (0, 0, 2). (b) $\frac{2}{\sqrt{38}}x-\frac{3}{\sqrt{38}}y+\frac{5}{\sqrt{38}z}=0, \frac{13}{\sqrt{38}}$ (c) $\frac{1}{6}$:
6. (a) Coplanar, (b) Non-coplanar, (c) co-planar, (d) coplanar
7. (a) $7x-39y+49z+8=0$ (b) $51x+15y-50z+173=0$ (c) $(ac_1-a_1c)x+(bc_1-b_1c)y$
   (d) $(dc_1-cd_1)=0$ (e) $2x+y+2z+2=0$; $82x+71y+58z+82=0$

8. (a) $\pi/2$ (b) $\cos^{-1}\left(\frac{7}{5\sqrt{2}}\right)$ (c) $\cos^{-1}\left(\frac{2}{3\sqrt{6}}\right)$

9. (a) (i) $11x + 6y + 5z + 86 = 0; 67x - 162y + 47z + 44 = 0;$

(ii) $38x - 23y - 17z - 4 = 0, 14x + 49y - 35z - 22 = 0.$

(b) $25x + 17y + 62z - 78 = 0.$

14. $x - 4y + 6z = 106$

EXERCISE 13 (c)

1. (a) T (b) T (c) F (d) F (e) F

2. (a) perpendicular, (b) $\frac{x+1}{1} = \frac{y}{2} = \frac{z-1}{0}$ (c) perpendicular to z-axis, (d) 2, (e) $\frac{l}{1} = \frac{m}{-3} = \frac{n}{2}$

3. (a) $\frac{x-4}{-4} = \frac{y+6}{9} = \frac{z-1}{-2}$, (b) $\frac{x-a}{0} = \frac{y-a}{-a} = \frac{z-a}{0}$, (c) $\frac{x-2}{2} = \frac{y-1}{-3} = \frac{z-3}{2}$

4. (i) $\frac{x}{1} = \frac{y}{0} = \frac{z}{0}$ (ii) $\frac{y-b}{0} = \frac{z-c}{0} = \frac{x}{k}$ (iii) $\frac{x}{b} = \frac{y+d/b}{-a} = \frac{z-c}{0}$ (iv) $\frac{x-3/5}{2} = \frac{y+6/5}{1} = \frac{z}{1}$

(v) $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z}{4}$ (vi) $\frac{x}{8} = \frac{y+1}{23} = \frac{z+1}{22}$

5. (a) $\frac{x-1}{-3} = \frac{y-2}{5} = \frac{z-3}{4}$ (b) $\frac{x-3}{7} = \frac{y+1}{3} = \frac{z-2}{-5}$; 6. (a) $\frac{x}{-45} = \frac{y-2}{41} = \frac{z+3}{33}$

8. $(3, 4, -1)$; 9. $\left(\frac{11}{3}, \frac{26}{3}, -9\right)$; 10. (a) 13, (b) $\left(\frac{8}{7}, \frac{-3}{7}, \frac{19}{7}\right)$ 11. $6x - 5y - z = 0$

13. $\left(-\frac{1}{3}, -3, -\frac{11}{3}\right)$, $3x - 7y + 9z + 13 = 0$; 14. $\left(\frac{-11}{4}, \frac{-3}{4}, \frac{3}{2}\right)$; 17. (a) 45^0, (b) $\sin^{-1}\left(\frac{47}{35\sqrt{2}}\right)$

18. (a) $\frac{x-1}{-6} = \frac{y}{1} = \frac{z+1}{9}$, (b) (a, a, a), $(3a, 2a, 3a)$, (c) $(2, 8, -3)$, $(0, 1, 2)$, $\sqrt{78}$

19. $(2, 5, 7)$, $\frac{x-3}{1} = \frac{y+1}{-6} = \frac{z-11}{4}$; 20. 21 ; 21. 1 ; 22. $\sqrt{308}$; 23. $8x + 5y - z - 19 = 0$

24. $19x - 10y - 42z + 43 = 0$; 25. $x - 2y + 2z = 0, 2x + 2y + z = 0$; 26. $\frac{x-5}{-6} = \frac{y-3}{1} = \frac{z-13}{2}$

27. $3\sqrt{30}, \frac{x-3}{2} = \frac{y-8}{5} = \frac{z-3}{1}$; 29. $\frac{11}{\sqrt{342}}$, $10x - 29y + 16z = 0 = 13x + 82y + 55z - 109.$

ADDITIONAL EXERCISES

1. $\vec{r}.(-\hat{i} - 5\hat{j} + 6\hat{k}) = 18$, $x + 5y - 6z + 18 = 0$; 2. $\vec{r}.(3\hat{i} - 4\hat{j} + 2\hat{k}) = 5$

4. $\cos^{-1}\left(\frac{-4}{21}\right)$ 5. $\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)$ 9. 3 ; 10. $\frac{\pi}{2}$;

11. $ax + by + cz = a^2 + b^2 + c^2$; 12. $y + 4z - 7 = 0$