# Data related to the $\mathbb{ATLAS}$ of Finite Groups

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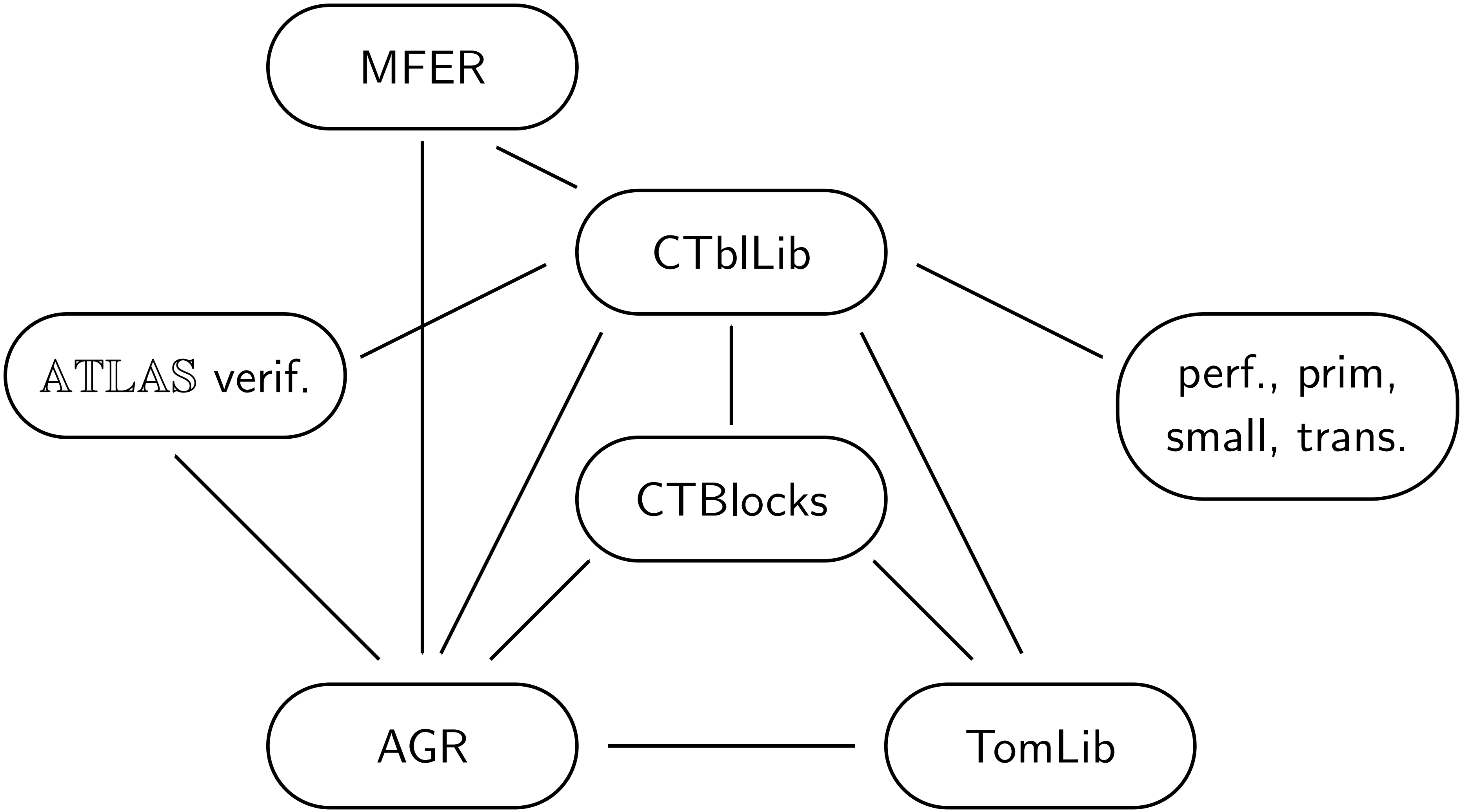
# ATLAS related data



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$\chi, \psi \in \mathrm{Irr}(G)$ with $\chi(1) = \psi(1)$ and $\mathrm{ind}(\chi) = \mathrm{ind}(\psi) \implies \psi = \overline{\chi}$ $(*)$

# Character table of $\mathrm{M}_{11}$

M11

| | 2 | 1a | 2a | 3a | 4a | 5a | 6a | 8a | 8b | 11a | 11b |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\chi_1$ | + | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $\chi_2$ | + | 10 | 2 | 1 | 2 | . | -1 | . | . | -1 | -1 |
| $\chi_3$ | o | 10 | -2 | 1 | . | . | 1 | $A$ | $\bar{A}$ | -1 | -1 |
| $\chi_4$ | o | 10 | -2 | 1 | . | . | 1 | $\bar{A}$ | $A$ | -1 | -1 |
| $\chi_5$ | + | 11 | 3 | 2 | -1 | 1 | . | -1 | -1 | . | . |
| $\chi_6$ | o | 16 | . | -2 | . | 1 | . | . | . | $B$ | $\bar{B}$ |
| $\chi_7$ | o | 16 | . | -2 | . | 1 | . | . | . | $\bar{B}$ | $B$ |
| $\chi_8$ | + | 44 | 4 | -1 | . | -1 | 1 | . | . | . | . |
| $\chi_9$ | + | 45 | -3 | . | 1 | . | . | -1 | -1 | 1 | 1 |
| $\chi_{10}$ | + | 55 | -1 | 1 | -1 | . | -1 | 1 | 1 | . | . |

$A = \zeta_8^3 + \zeta_8$

$\bar{A} = -\zeta_8^3 - \zeta_8$

$B = \zeta_{11}^9 + \zeta_{11}^5 + \zeta_{11}^4 + \zeta_{11}^3 + \zeta_{11}$

$\bar{B} = -\zeta_{11}^9 - \zeta_{11}^5 - \zeta_{11}^4 - \zeta_{11}^3 - \zeta_{11} - 1$

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**Claim**: Property $(*)$ does *not* hold for these $\hat{G} = M.G$.

# Regular orbits

**Sufficient condition:**

If $G$ has at least 5 regular orbits on $M^*$
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$$\begin{array}{lclcl} M^* & = & \{v \in M^*; |\mathrm{Stab}_G(v)| = 1\} & \cup & \bigcup_g \mathrm{Fix}_{M^*}(g) \\ |M| & \leq & |G| \cdot R(G, M) & + & \sum_g |\mathrm{Fix}_{M^*}(g)| \end{array}$$

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Get upper bounds for the dimension and the characteristic of $M$
such that $M.G$ can satisfy $(*)$.

# Regular orbits: The group Th

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We have $\max\{r(g); g \in G\} = 3$.

$R(G, M) \geq 5$ if $\dim(M) \geq 148$.

$\dim(M) \geq 248$ by [Jansen 2005].

In the other cases, it is not that easy . . .

# Use the table of marks of $G$

For the remaining candidates $M$,
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If available then use the **table of marks** of $G$:

Decompose $M^*$ into $G$-orbits,
compute the exact number of regular orbits in $M^*$.

# Table of marks of $A_5$

```
julia> tom = table_of_marks("A5")
A5

1: 60
2: 30 2
3: 20 . 2
4: 15 3 . 3
5: 12 . . . 2
6: 10 2 1 . . 1
7:  6 2 . . 1 . 1
8:  5 1 2 1 . . . 1
9:  1 1 1 1 1 1 1 1 1

julia> representative(tom, 7)
Permutation group of degree 5 and order 10
```

# Use the table of marks of $G$

```
julia> function fix_dim(matgroup::MatrixGroup)
       mats = map(matrix, gens(matgroup))
       length(mats) == 0 && return degree(matgroup)
       m = vcat([transpose(x)-one(x) for x in mats]...)
       return nullspace(m)[1]
       end;

julia> info = all_atlas_group_infos("A5", dim => 4,
                   characteristic => 0);

julia> mats = gens(atlas_group(info[1]))
2-element Vector{MatrixGroupElem{QQFieldElem, QQMatrix}}:
 [1 0 0 0; 0 0 1 0; 0 1 0 0; -1 -1 -1 -1]
 [0 1 0 0; 0 0 0 1; 0 0 1 0; 1 0 0 0]

julia> s = [representative(tom,i) for i in 1:length(tom)];
```

# Use the table of marks of $G$

```
julia> for p in [2, 3, 5, 7, 11, 13, 17, 19]
       f = GF(p)
       g = matrix_group([matrix(f, matrix(x)) for x in mats])
       iso = hom(group(tom), g, gens(g); check = false)
       v = [fix_dim(image(iso, x)[1]) for x in s]
       vv = marks_vector(tom, [p^x for x in v])
       println( "$p: $(coordinates(vv))")
       end
2: ZZRingElem[0, 0, 0, 0, 0, 1, 0, 1, 1]
3: ZZRingElem[0, 1, 1, 0, 0, 2, 0, 2, 1]
5: ZZRingElem[4, 6, 6, 0, 2, 4, 0, 4, 1]
7: ZZRingElem[26, 15, 15, 0, 0, 6, 0, 6, 1]
11: ZZRingElem[204, 45, 45, 0, 0, 10, 0, 10, 1]
13: ZZRingElem[418, 66, 66, 0, 0, 12, 0, 12, 1]
17: ZZRingElem[1288, 120, 120, 0, 0, 16, 0, 16, 1]
19: ZZRingElem[2040, 153, 153, 0, 0, 18, 0, 18, 1]
```

# Check our candidates

Apply the table of marks idea to the groups on our list,
this excludes the “big” modules.

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this excludes the “big” modules.

If $M^*$ does *not* have at least 5 regular orbits:

- Take the $G$-module $M$ (from the AGR or constructed ad hoc),
- compute the cohomology of $G$ and $M$,
- compute the possible extensions $M.G$,
- compute their character tables,
- find characters that violate property $(*)$.

# Check our candidates

| $G$ | $N(G)$ | $p$ | $M$ | $R_1(R_2)$ | $C$ | deg. |
|---|---|---|---|---|---|---|
| $A_8$ | 45 | 2 | 4a | | 1 | $+105^3$ |
| | | | 4b | | 1 | $+105^3$ |
| | | | 6a | | 0 | $35^5$ |
| | | | 14a | | 2 | $210^5$ |
| | | | 20a | 37 | | |
| | | | 20b | 37 | | |
| | | 3 | 7a | | 1 | $28^5$ |
| | | | 13a | 23 | | |
| | | 5 | 7a | | 0 | $28^9$ |
| | | 7 | 7a | (15) | | |
| | | 11 | 7a | 240 | | |
| | | 13 | 7a | 1122 | | |

# What have we seen in this example?

**Data used:**

- available character tables,
- values $r(g)$ ($\leq 5$ or better),
- minimal degree information for Th,
- available tables of marks,
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- available character tables,
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- available tables of marks,
- available matrix representations of the groups.

**Consistency issues:**

- Do the data for a given group fit together at all?
  (character table, table of marks, representations from the AGR)
- Are the group generators used in the AGR compatible with the ones in the table of marks?

# More consistency issues

- "Generality problems" between $p$-Brauer character tables for a group $G$:
  Can the tables be used simultaneously?
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- Does the group $\mathrm{U}_4(5)$ have a primitive permutation representation on 1575 points?
  (In GAP? In MAGMA?)

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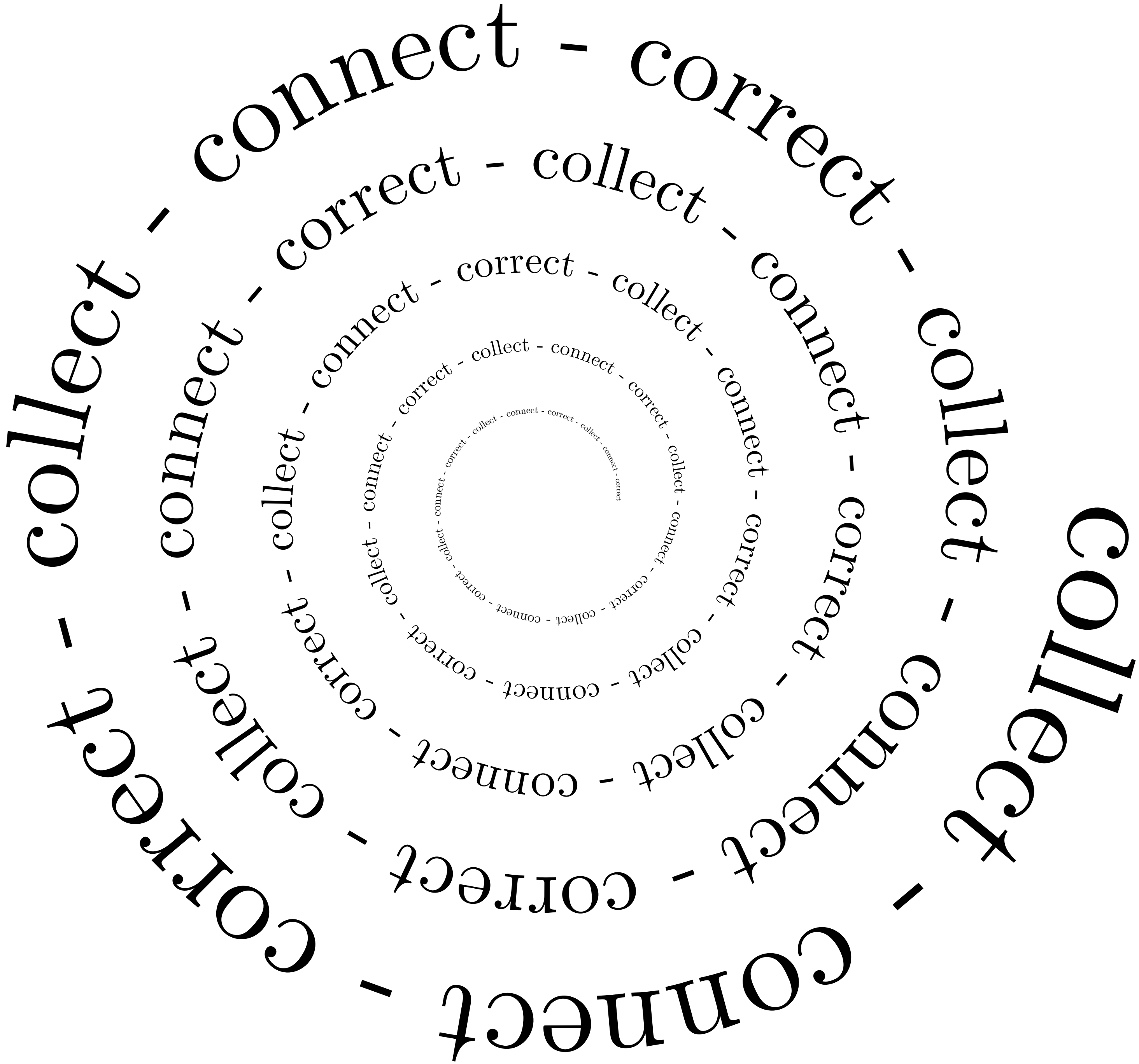
**correct** errors/inconsistencies which we find


collect - connect - correct

# Relevant Web Addresses

**AGR:**
`http://atlas.math.rwth-aachen.de/Atlas`

**AtlasRep.:**
`http://www.math.rwth-aachen.de/~Thomas.Breuer/atlasrep`

**ATLAS verif.:**
`http://www.math.rwth-aachen.de/~Thomas.Breuer/atlasverify`

**CTblLib:**
`http://www.math.rwth-aachen.de/~Thomas.Breuer/ctbllib`

**CTBlocks:**
`http://www.math.rwth-aachen.de/~Thomas.Breuer/ctblocks`

**MFER:**
`http://www.math.rwth-aachen.de/~mfer`

**TomLib:**
`https://gap-packages.github.io/tomlib`

# $\mathbb{ATLAS}$ related data