Question 1

$$\max_{x_1,x_2} \; 3x_1^{1/3}x_2^{1/3} - x_1 - 2x_2$$

Taking the partials and setting them to zero we get:

$$\begin{aligned} 3\frac{1}{3}x_1^{-2/3}x_2^{1/3} &= 1 \\ 3x_1^{1/3}(\frac{1}{3}x_2^{-2/3}) &= 2 \end{aligned}$$

As in the general case we have two equations in the two unknowns, x_1 and x_2. The simplest way to proceed is once again to divide the equations through:

$$\frac{x_2}{x_1} = \frac{1}{2}$$

i.e. $x_2 = x_1/2$. Substitute that into the first equation to get:

$$x_1^{-2/3}(x_1/2)^{1/3} = 1$$

or

$$(2)^{-1/3}x_1^{-1/3} = 1$$

raising both sides to power 3:

$$x_1^{-1} = 2$$

or, $x_1^* = 1/2$. Thus $x_2^* = x_1^*/2 = 1/4$. Output is: $y^* = (1/2)^{1/3}(1/4)^{1/3} = (1/8)^{1/3} = 1/2$, optimal profit is: $\pi^* = 3y^* - x_1^* - 2x_2^* = 1/2$.

Question 2

- For $f(K,L) = K^2L^2$:

$$f(\lambda K, \lambda L) = (\lambda K)^2(\lambda L)^2 = \lambda^4 K^2 L^2 = \lambda^4 f(K,L) > \lambda f(K,L) \implies IRS$$

- For $f(K,L) = K^{\frac{1}{3}}L^{\frac{2}{3}}$:

$$f(\lambda K, \lambda L) = (\lambda K)^{\frac{1}{3}}(\lambda L)^{\frac{2}{3}} = \lambda^{\frac{1}{3}+\frac{2}{3}} K^{\frac{1}{3}} L^{\frac{2}{3}} = \lambda f(K,L) \implies CRS$$

- For $f(K,L) = K^{\frac{1}{4}}L^{\frac{1}{4}}$:

$$f(\lambda K, \lambda L) = (\lambda K)^{\frac{1}{4}}(\lambda L)^{\frac{1}{4}} = \lambda^{\frac{1}{4}+\frac{1}{4}} K^{\frac{1}{4}} L^{\frac{1}{4}} = \lambda^{\frac{1}{2}} f(K,L) < \lambda f(K,L) \implies DRS$$

- For $f(K,L) = \sqrt{\min\{K,L\}}$:

$$f(\lambda K, \lambda L) = (\min\{\lambda K, \lambda L\})^{\frac{1}{2}} = \lambda^{\frac{1}{2}}(\min\{K,L\})^{\frac{1}{2}} = \lambda^{\frac{1}{2}} f(K,L) < \lambda f(K,L) \implies DRS$$