Frayed Demazure weaves for Poisson-compatible cluster structures on Bott–Samelson charts

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Dramatis Personae

- G : simple algebraic group over $\mathbb{C}$.
- B : Borel subgroup.
- I : index set of simple roots.
- $\varphi_i : SL_2 \hookrightarrow G$ root embedding.
- $\underline{w} = (i_1, i_2, \ldots, i_\ell) \in I^\ell$: word

Bott–Samelson varieties

For $i \in I$, $P_i = B \sqcup Bs_iB$ is a minimal parabolic subgroup of G.
Consider the product $P_{\underline{w}} := P_{i_1} \times P_{i_2} \times \cdots \times P_{i_\ell} \subset G^\ell$ with B^ℓ-action

$$(g_1, g_2, \ldots, g_\ell) * (b_1, \ldots b_\ell) = (g_1b_1, b_1^{-1}g_2b_2, \ldots, b_{l-1}^{-1}g_\ell b_\ell).$$

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The *Bott–Samelson variety* is the quotient $Z_{\underline{w}} := P_{\underline{w}}/B^\ell$.

Configurations of flags

A B^ℓ-equivariant map $P_{\underline{w}} \hookrightarrow G^\ell$, gives an embedding

$$Z_{\underline{w}} \hookrightarrow (G/B)^\ell, \quad [g_1, g_2, \ldots, g_\ell] \mapsto \big(g_1 B, g_1 g_2 B, \ldots, g_1 \cdots g_\ell B\big).$$

Each point is a sequence of flags where the j-th flag differs from the previous at most in position i_j.

Affine charts

$$B_i(z) := \varphi_i \begin{pmatrix} z & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad B_{-i}(z) := \varphi_i \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}.$$

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For $\gamma \in \{i_1, -i_1\} \times \{i_2, -i_2\} \times \cdots \times \{i_\ell, -i_\ell\}$ a subexpression of $\underline{w}$, the *Bott–Samelson chart* $\mathcal{O}^\gamma$ is the image of the embedding

$$\Phi_\gamma : \mathbb{A}^\ell \hookrightarrow Z_{\underline{w}}, \quad (z_1, \ldots, z_l) \mapsto [B_{\gamma_1}(z_1), B_{\gamma_2}(z_2), \ldots, B_{\gamma_\ell}(z_\ell)].$$

Standard Poisson structures

- The group G has a *standard Poisson structure*.

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- Product Poisson structure on $P_{\underline{w}} = P_{i_1} \times \cdots \times P_{i_\ell}$.
- Pushforward to quotient $Z_{\underline{w}} = P_{\underline{w}}/B^\ell$.

Poisson compatibility

Definition (Gekhtman–Shapiro–Vainshtein)

A cluster structure and a Poisson structure are *compatible* if every (extended) cluster is log-canonical.

The Elek–Lu formula

Theorem (Elek–Lu, '21)

For $j < k$, the Poisson structure on $\mathcal{O}^\gamma \subset Z_{\underline{w}}$ has the form

$$\{z_j, z_k\}_\gamma = \begin{cases} C^\gamma_{j,k} z_j z_k & \text{if } \gamma_j < 0, \\ & \end{cases}$$

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Here, $C^\gamma_{j,k}$ are integer constants determined by Cartan integer data, and the polynomial $\sigma^\gamma_{j,k} \in \mathbb{C}[z_{j+1}, \dots z_k]$ is determined by the action of a particular vector field.

$\underline{w} = (1,2,1,2)$ and $\gamma = (1,2,1,-2)$

Example

In the coordinates for the Bott–Samelson charts $\mathcal{O}^{\underline{w}}$ and $\mathcal{O}^{\gamma}$ respectively:

$$\begin{aligned}
\{z_1, z_2\}_{\underline{w}} &= -z_1 z_2, & \{z_1, z_3\}_{\underline{w}} &= z_1 z_3 - 2z_2, & \{z_1, z_4\}_{\underline{w}} &= 2z_1 z_4 - 2, \\
& & \{z_2, z_3\}_{\underline{w}} &= -z_2 z_3, & \{z_2, z_4\}_{\underline{w}} &= z_2 z_4 - 2z_3, \\
& & & & \{z_3, z_4\}_{\underline{w}} &= -z_3 z_4.
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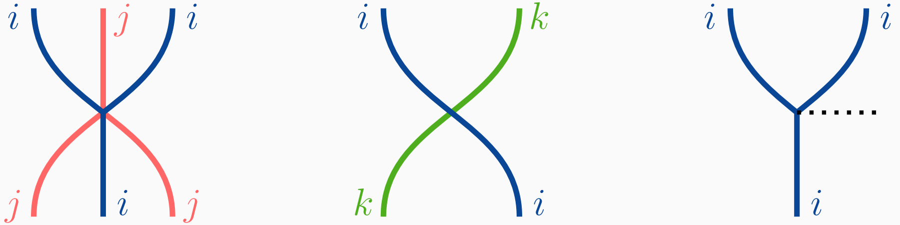
$$\{z_1', z_2'\}_{\gamma} = -z_1' z_2', \quad \{z_1', z_3'\}_{\gamma} = z_1' z_3' - 2z_2', \quad \{z_1', z_4'\}_{\gamma} = -2z_1' z_4' + (z_4')^2,$$
$$\{z_2', z_3'\}_{\gamma} = -z_2' z_3', \quad \{z_2', z_4'\}_{\gamma} = -z_2' z_4' + 2z_3'(z_4')^2,$$
$$\{z_3', z_4'\}_{\gamma} = z_3' z_4'.$$

Demazure weaves [CGGS24,CGGLSS25]



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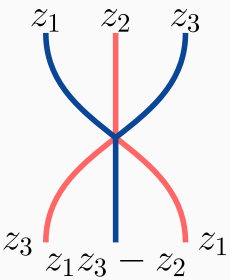

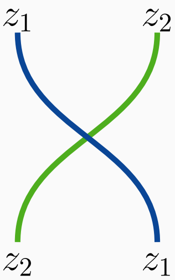

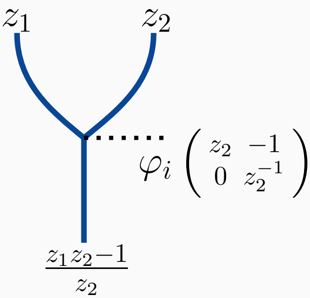

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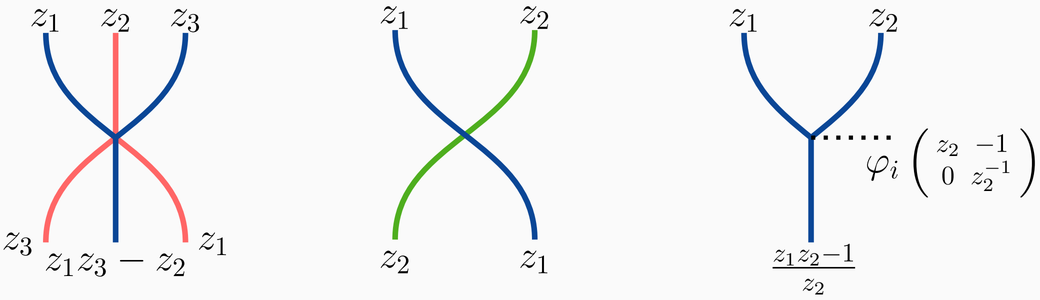


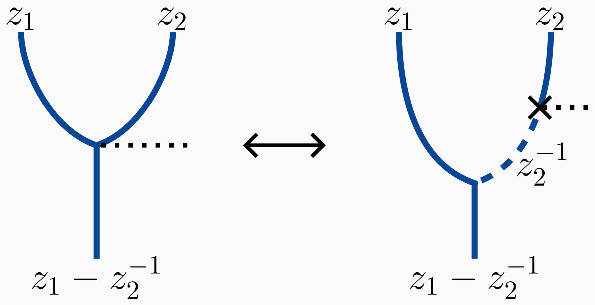
These give Poisson maps between Bott–Samelson cells.

$$f^*(\{z_j, z_k\}_{\underline{w}}) = \{f^*(z_j), f^*(z_k)\}_{\underline{w'}}$$

Easy to compute for the first two.

Poisson-ness of weaves

For the third map, it factors via the change of coordinates to an adjacent Bott–Samelson chart.



Weaves and the standard Poisson structure

Theorem (C. ’26)

- *Weave tori are log-canonical in the standard Poisson structure.*

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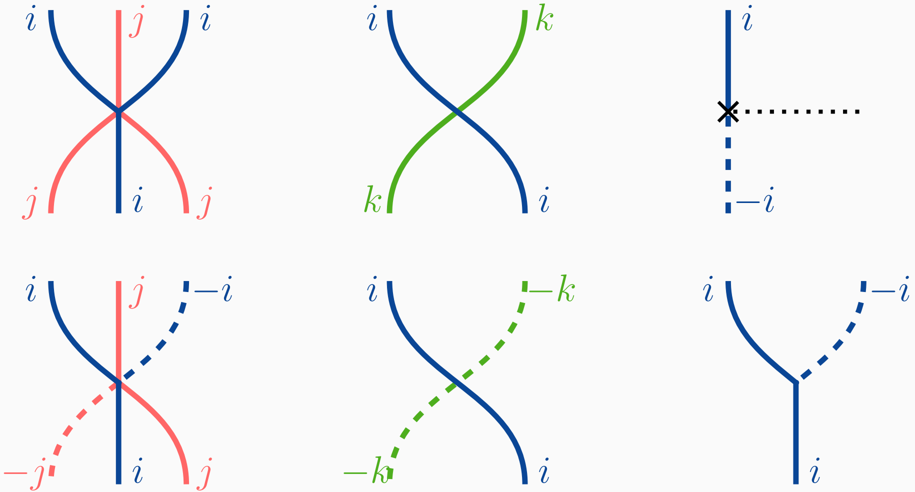
- *Weave tori are log-canonical in the standard Poisson structure.*
- *The cluster structures on Bott–Samelson cells and braid varieties are compatible with the standard Poisson structure.*

Frayed strands for weaves



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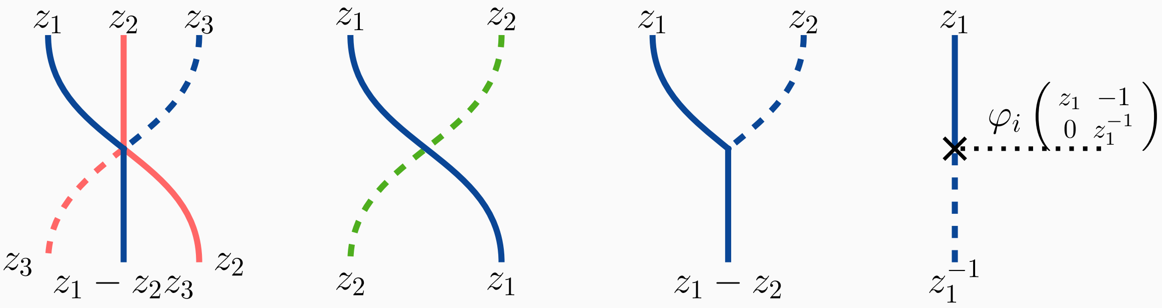

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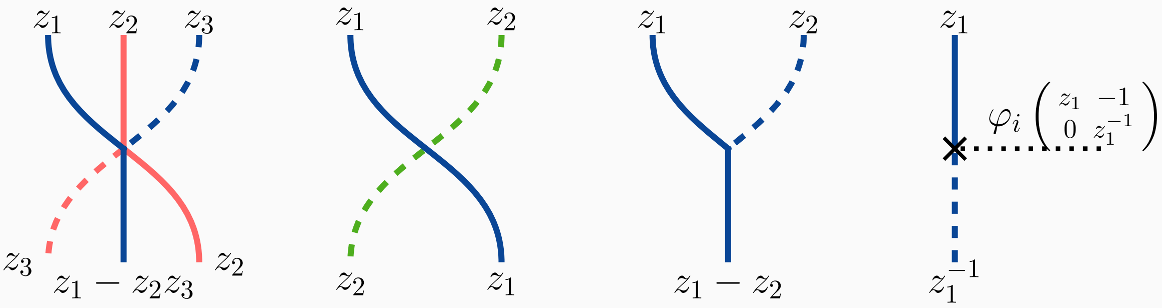

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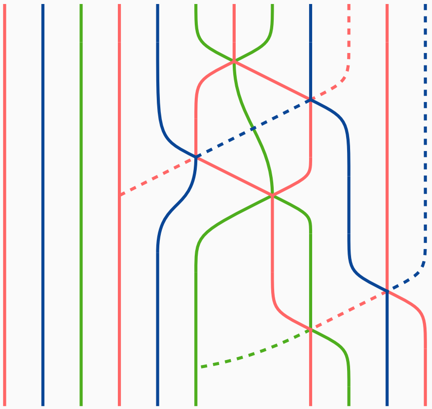
Proposition

For $(\underline{w_0}; \gamma)$, the all but last are enough to twine all frayed strands.

The map $\rho : \mathcal{O}^{\gamma} \to \mathcal{O}^{\gamma^+}$



$$(\underline{w}_0; 2, 3, 1, -2, 2, -1)$$

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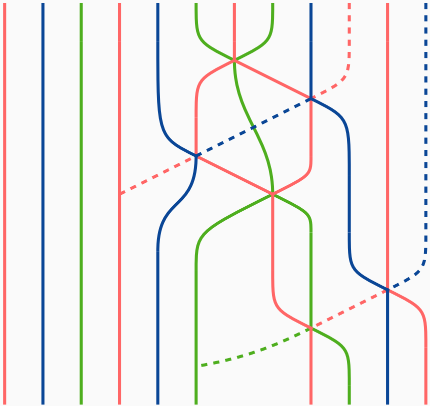

$$(z_1, z_2, z_3, z_4, z_5, z_6)$$

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The map $\rho : \mathcal{O}^{\gamma} \to \mathcal{O}^{\gamma^+}$

Theorem (C. '26, also Lu–Yakimov (unpublished))

If γ has m negative letters, we have a Poisson isomorphism

$$\mathcal{O}^{\gamma} \xrightarrow{\cong} \mathcal{O}^{\gamma^+} \times \mathbb{A}^m,$$

$$\Phi_\gamma(\underline{z}) \mapsto \left(\rho(\Phi_\gamma(\underline{z}))\ ,\ (z_j)_{\gamma_j<0}\right)$$

where the RHS has a log-canonical product structure.

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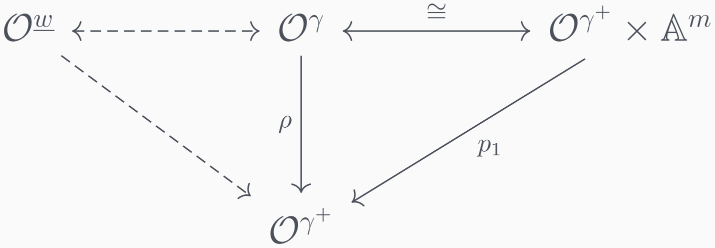

Cluster structures on $\mathcal{O}^\gamma$

Theorem (C. '26)

- *There is a Poisson-compatible cluster structure on* $\mathcal{O}^\gamma$ *with seed*

$$\mathbf{s}(\gamma) := \rho^*(\mathbf{s}(\gamma^+)) \sqcup \left\{ \boxed{z_j} \,\middle|\, \gamma_j < 0 \right\}.$$

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- *The transition function* $\mathcal{O}^{\underline{w}} \dashrightarrow \mathcal{O}^\gamma$ *gives a rational quasi-cluster morphism between seeds* $\mathrm{s}(\gamma)$ *and* $\overrightarrow{\mu_\gamma}(\mathrm{s}(\underline{w}))$.

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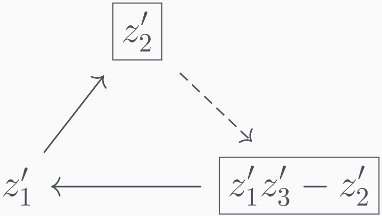
The sequence $\overrightarrow{\mu_\gamma}$ is defined by negating letters from left to right.

$\gamma = (1, 2, 1, -2)$

The seed $\mathrm{s}(\gamma^+)$ on $\mathcal{O}^{\gamma^+}$ in the BS coordinates z'_i for $\mathcal{O}^{\gamma^+}$:



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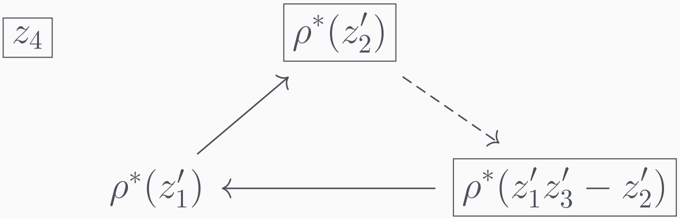

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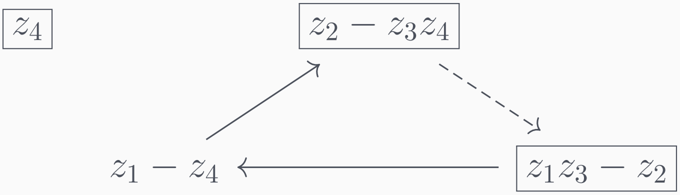
The seed $\mathrm{s}(\gamma)$ in the BS coordinates z_i for $\mathcal{O}^\gamma$:



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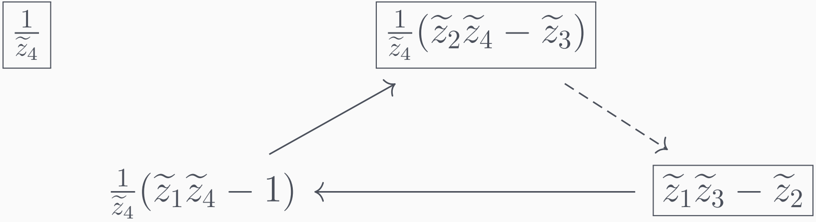
The seed $\mathrm{s}(\gamma)$ in the BS coordinates $\widetilde{z}_i$ for $\mathcal{O}^{\underline{w}}$:



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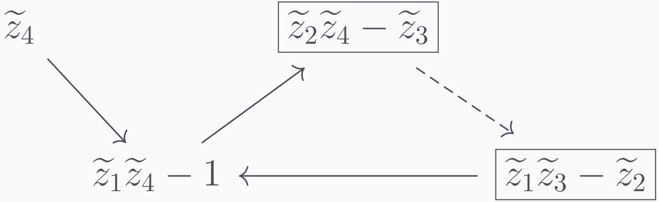

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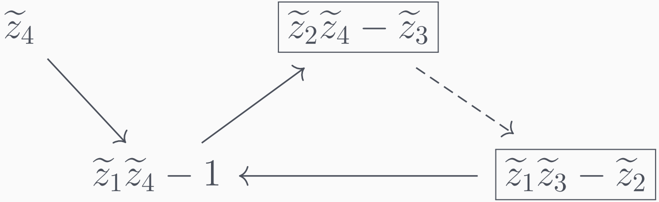

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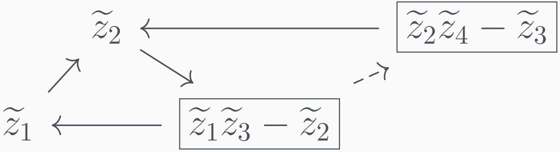

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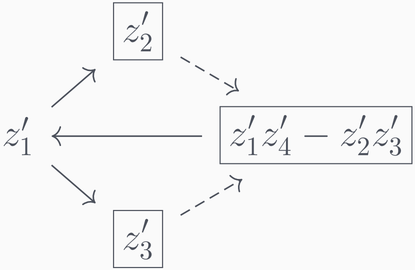
The seed $\mathrm{s}(\gamma^+)$ on $\mathcal{O}^{\gamma^+}$ in the BS coordinates z_i' for $\mathcal{O}^{\gamma^+}$:



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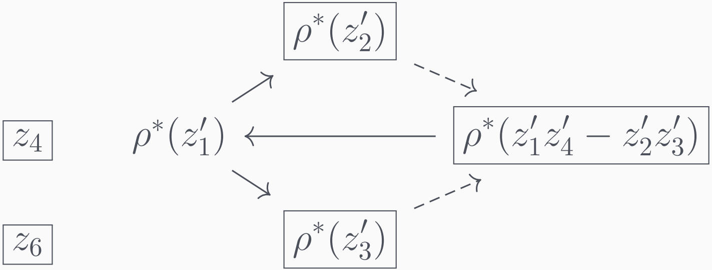
The seed $\mathrm{s}(\gamma)$ on $\mathcal{O}^\gamma$:



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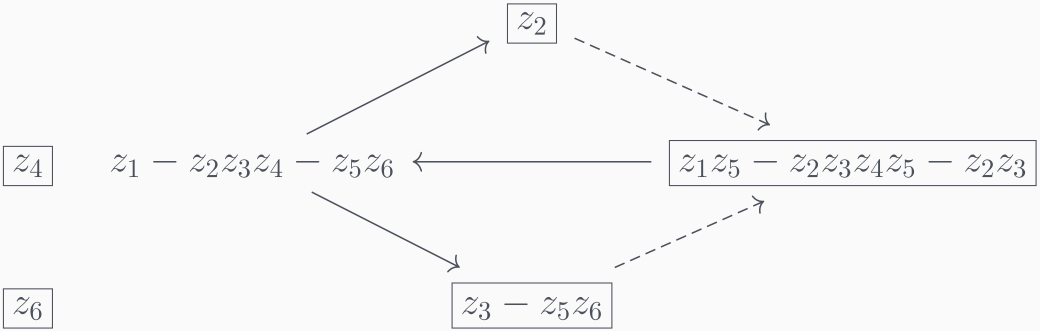
The seed $s(\gamma)$ in the BS coordinates z_i for $\mathcal{O}^\gamma$:



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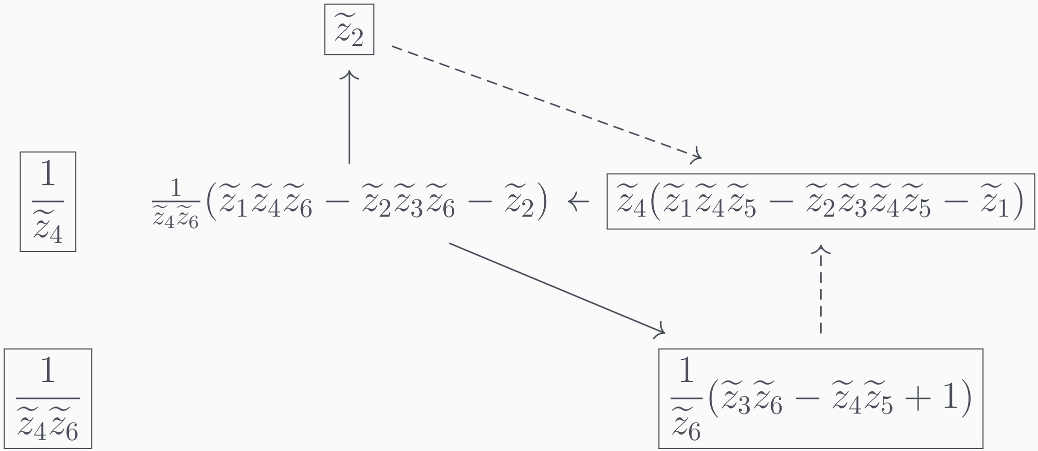

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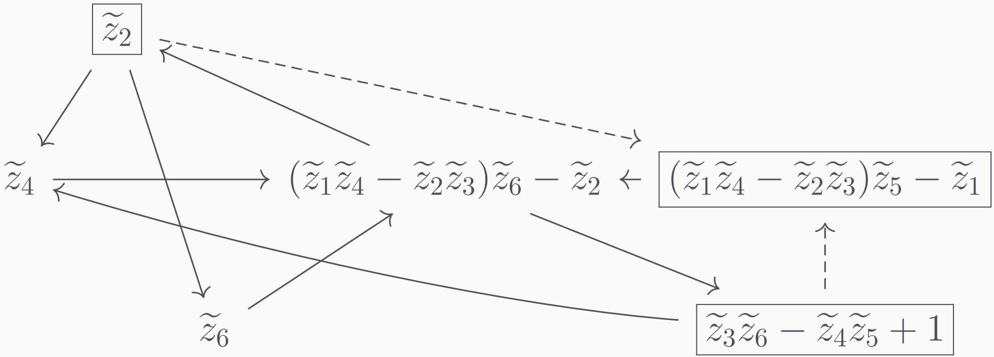
The seed $\overrightarrow{\mu_\gamma}(\mathrm{s}(\underline{w}))$ in the BS coordinates $\widetilde{z}_i$ for $\mathcal{O}^{\underline{w}}$:

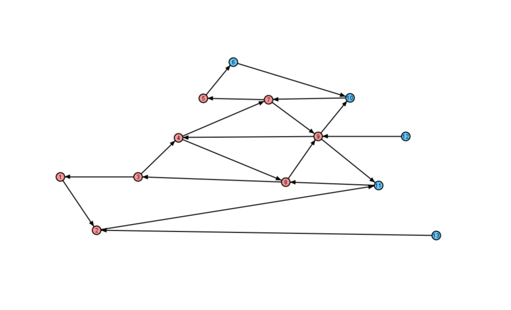


Mutation sequences

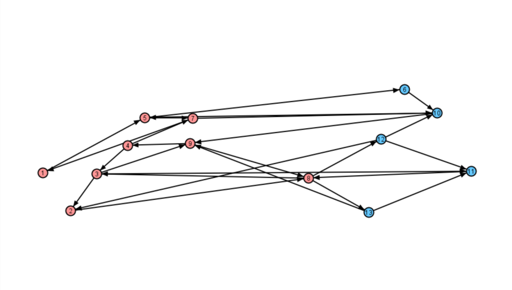
Remark

This seed $\overrightarrow{\mu_\gamma}(\mathrm{s}(\underline{w}))$ differs from Ménard's open Richardson seed $M(\gamma)(\mathrm{s}(\underline{w}))$ by an application of $\overrightarrow{\mu_{\gamma^+}}$.

$\gamma = (2, 1, -2, 3, -4, 5, 4, 2, -3, -4, -2, -3, -1)$



$\mathbf{s}(\underline{w})$



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