Safe-by-design control with robust MPC

Certified uncertainty propagation II

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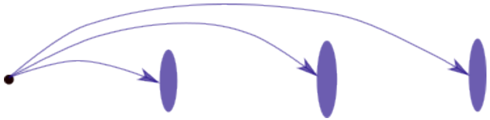
Sequential uncertainty propagation

Goal: *Retain the computational structure of nominal MPC while accounting for uncertainty.*

Direct reachable-set computation

Propagate the effect of *all* previous uncertainty realizations to future states:



Benefit: Less conservative uncertainty propagation

Limitation: No stage-wise structure in MPC optimization problems

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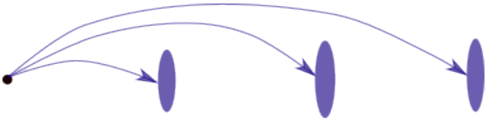

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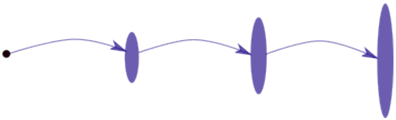
Sequential uncertainty propagation

Propagate an uncertainty description from one stage to the next:

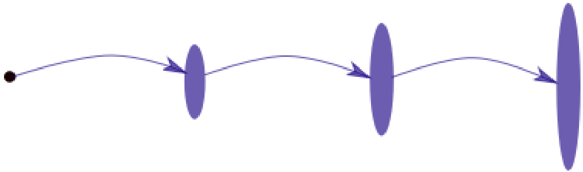


Challenge: Additional conservatism due to sequential over-approximation

Benefits: Preserves stage-wise structure in optimization problem

Sequential uncertainty propagation

Recursive set propagation



Recursive set propagation

$$\mathcal{X}_{k+1} = \mathcal{R}(\mathcal{X}_k, \pi_k) \supseteq \{f(x, \pi_k(x), w) \; | x \in \mathcal{X}_k, \; w \in \mathbb{W}\}.$$

Sequential uncertainty propagation

Tube MPC formulation

Sequential tube MPC

$\min_{\pi,\mathcal{X}} \ \mathcal{J}(\mathcal{X},\pi)$	Worst-case cost
$\text{s.t.} \ x(k) \in \mathcal{X}_0$	Initial tube constraint
$\mathcal{X}_{k+1} = \mathcal{R}(\mathcal{X}_k, \pi_k)$	Tube propagation
$\mathcal{X}_k \subseteq \mathbb{X}$	Robust state constraints
$\mathcal{R}(\mathcal{X}_N, \pi_{\mathrm{f}}) \subseteq \mathcal{X}_N$	Terminal invariance constraint

- Optimize over policy π and sets $\mathcal{X}$
- Recursive feasibility follows from shifting policies and sets.

- Preserves stage-wise structure: $\mathcal{X}_{k+1}$ only depends on $\mathcal{X}_k$ and π_k $\Rightarrow$ can be solved efficiently

Sequential uncertainty propagation

Parametrization

Required design choices

- Set parametrization: $\mathcal{X}(\xi)$, e.g., ellipsoid
- Policy parametrization: $u = \pi(x; \vartheta)$, e.g., linear feedback
- Set propagation law: $\xi_{k+1} = \mathcal{F}(\xi_k, \vartheta_k)$

$\Rightarrow$ Optimal control problem with state ξ and to be optimized control input ϑ_k

Sequential uncertainty propagation

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In this talk: We focus on two representative families.

Linearization-based propagation

[Althoff et al., 2008, Houska, 2011,
Cannon et al., 2011, Zanelli et al., 2021,
Kim et al., 2024, Frey et al., 2024,
Belvedere et al., 2025]

Contraction metrics

[Bayer et al., 2013, Singh et al., 2017,
Zhao et al., 2022, Köhler et al., 2021,
Sasfi et al., 2023, Dubied et al., 2025,
Chou et al., 2022, Köhler et al., 2021]

Linearization-based propagation

Linear error dynamics

$$\dot{e} = Ae + Ew, \qquad w(t) \in \mathbb{W} = \{w|\ w^\top Q^{-1} w \leq 1\}.$$

Parametrization: Ellipsoidal tube

$$\mathcal{X}(t) = \left\{e:\ e^\top \Sigma(t)^{-1} e \leq 1\right\}.$$

Sequential propagation law [Bertsekas and Rhodes, 1971]:

Matrix ODE: $$\dot{\Sigma} = \underbrace{{}_{A}\Sigma + \Sigma A^\top + \rho\Sigma + \frac{1}{\rho} EQE^\top}_{f_\Sigma(A,B,\Sigma,Q)}, \quad \rho > 0.$$

$\Rightarrow$ Uncertainty propagation easy-to-implement for linear dynamics

Linearization-based propagation

1. **Nominal prediction:** $\dot{z} = f(z, v)$
2. **Linearized error dynamics**: $\dot{e} = Ae + \underbrace{Ew + w_{\mathrm{nl}}}_{\tilde{w}},$

   Jacobian $A = \frac{\partial f}{\partial x}|_z$, Taylor remainder w_{nl}.

Linearization-based propagation

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3. **Bound lumped disturbance** $\tilde{w}$

   $\tilde{w}^\top Q^{-1}\tilde{w} \leq 1$, with $Q(\Sigma)$.[a]

Linearization-based propagation

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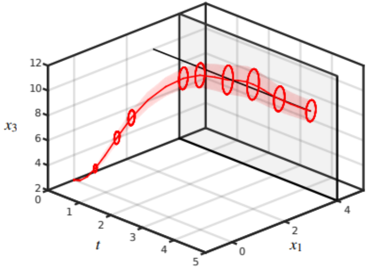
   $\tilde{w}^\top Q^{-1} \tilde{w} \leq 1$, with $Q(\Sigma)$.[a]
4. **Linear ellipsoidal propagation**

   $\Rightarrow x(t) \in \mathcal{X}(t) = \{x | \, \|x - z\|_{\Sigma^{-1}} \leq 1\}$



Ellipsoidal tube for quadrotor [Haimin et al., 2018]

[a] Bounds computed using Hessian bounds determined offline [Leeman et al., 2025] or with interval arithmetic [Althoff et al., 2008]; or using Lipschitz constants [Kim et al., 2024].

Linearization-based propagation

Tube MPC formulation

Robust MPC using ellipsoid propagation

$\min_u \mathcal{J}(z,u,\Sigma)$	Worst-case cost
s.t. $\dot{z} = f(z,u)$	Nominal prediction
$\dot{\Sigma} = f_\Sigma(A,B,\Sigma,Q)$	Ellipsoid propagation
$A(z,u),\ B(z,u),\ Q(\Sigma)$	Linearization & error bound
$\mathcal{X} = \{x \mid \|x-z\|^2_{\Sigma^{-1}} \leq 1\} \subseteq \mathbb{X}$	Robust constraint satisfaction

- Joint optimization of prediction, linearization, and uncertainty propagation[1]
- Provides rigorous robust over-approximation

[1] Joint optimization of feedback possible [Kim et al., 2024, Messerer and Diehl, 2021]

Linearization-based propagation

Discussion

Flexibility

- Everything optimized online
- No/Little offline design needed

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Computational aspects

- Preserves sequential MPC structure
- Propagates matrices $\Sigma \in \mathbb{R}^{n_x \times n_x}$
- Tailored solvers [Cannon et al., 2011], [Frey et al., 2024], [Kim et al., 2024]

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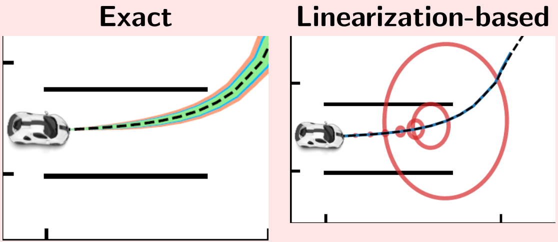
Limitation: linearization error



Reachable-set example [Prajapat et al., 2026]

Unstable tube dynamics

Linearization-error bounds can cause rapid tube growth and severe conservatism.

Sequential uncertainty propagation

Comparison

Linearization-based propagation

Parametrization

- Tube: ellipsoid (z, Σ), $\mathcal{O}(n_x^2)$
- Linear feedback (v, K), $\mathcal{O}(n_x \cdot n_u)$
- Propagation: matrix ODE for Σ

Benefits/limitations

+ Flexible and broadly applicable
– Optimizing over matrices
– Linearization error can be problematic

Sequential uncertainty propagation

Comparison

Can we avoid linearization and matrix ODEs?

Linearization-based propagation

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Contraction metrics

[Bayer et al., 2013, Singh et al., 2017, Zhao et al., 2022, Köhler et al., 2021, Sasfi et al., 2023, Dubied et al., 2025, Chou et al., 2022, Köhler et al., 2021]

Efficient tube parametrizations

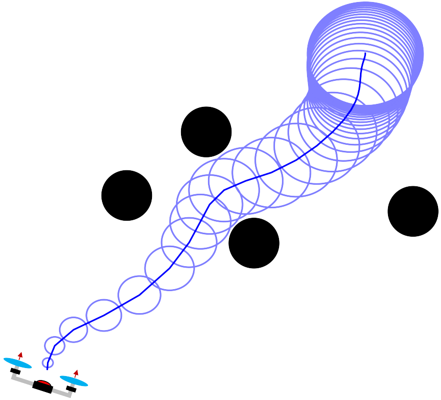
[Köhler et al., 2021, Sasfi et al., 2023, Dubied et al., 2025]

- Nominal prediction: $\dot{z} = f(z, v)$
- Feedback: $u = \kappa(x, z, v)$
- Parametrised tube:
  $\mathcal{X}(t) = \{x | \; V_\delta(x, z(t)) \leq \delta(t)\}$

Design steps:

- Choose "distance measure" V_δ
- Design feedback κ
- Find tube dynamics $\dot{\delta} = f_\delta(z, v, \delta)$

$\Rightarrow$ Contraction metrics can address this.

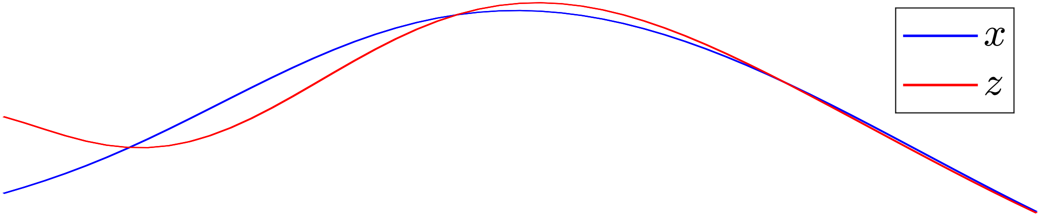


Plots from [Zhao et al., 2022, Sasfi et al., 2023]

Contraction metrics

Systematic way to construct incremental Lyapunov functions for nonlinear systems

[Lohmiller and Slotine, 1998, Manchester and Slotine, 2017, Tsukamoto et al., 2021]



Contraction metrics capture the stability between trajectories

Contraction metrics

Systematic way to construct incremental Lyapunov functions for nonlinear systems
[Lohmiller and Slotine, 1998, Manchester and Slotine, 2017, Tsukamoto et al., 2021]

Offline design

Find a contraction metric $M(x)$ and differential feedback $K(x)$ certifying LMIs:

$$\dot{M} + MA_{\mathrm{cl}} + A_{\mathrm{cl}}^\top M \preceq -2\rho M \quad \forall x$$

with Jacobians $A_{\mathrm{cl}} = A + BK$,

Design conditions mirror design of robust LPV/gain-scheduling synthesis
[Fromion and Scorletti, 2003, Koelewijn et al., 2019]

Contraction metric induces an incremental Lyapunov function $V_\delta(x,z)$ (Riemannian distance)

$$\|x - z\|_{\underline{M}} \leq V_\delta(x,z) \leq \|x - z\|_{\overline{M}}$$

Contraction metrics

Scalar uncertainty propagation

Stability properties of incremental Lyapunov function V_δ yield stable tube dynamics.

Uncertainty propagation

Predict tube scaling with a scalar ODE [Sasfi et al., 2023]:

$$\dot{\delta} = -\tilde{\rho}\,\delta + w(z,v)$$

ensures $x(t) \in \mathcal{X}(t)$.

- Contraction rate $\tilde{\rho} > 0$
- State/input-dependent model error bound $w(z,v) = \max_{d\in\mathbb{D}} \|E(z,v)d\|_{M(z)}$

Contraction metrics

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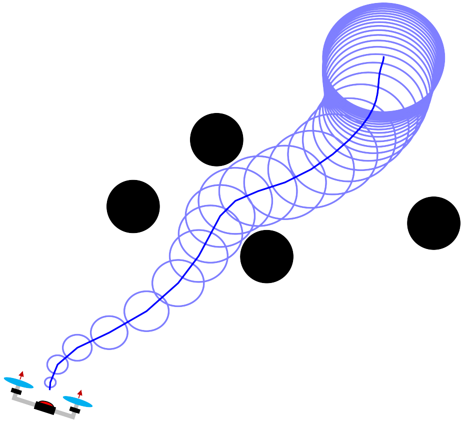
- Contraction rate $\tilde{\rho} > 0$
- State/input-dependent model error bound $w(z,v) = \max_{d\in\mathbb{D}} \|E(z,v)d\|_{M(z)}$
- Rigid tubes recovered as the special case $\delta(t) \equiv \bar{\delta}$ [Bayer et al., 2013, Singh et al., 2017, Zhao et al., 2022]

Numerical example - nonlinear quadrotor

Robust MPC design using contraction metrics [Sasfi et al., 2023]

- No quadratic Lyapunov function V_δ exists
  $\rightarrow$ ellipsoidal tubes get unstable
- Model mismatch depends on thrust/angle
- Offline design takes 20 minutes
- Online optimization: $4\times$ slower than nominal MPC; sufficient for real-time
- **Optimized trajectory:**
  Slows down near obstacles;
  reduces uncertainty growth;
  squeezes through gap between obstacles



Sequential uncertainty propagation

Comparison

Linearization-based propagation

Parametrization

- Ellipsoid tube (z, Σ), $\mathcal{O}(n_x^2)$
- Linear feedback (v, K), $\mathcal{O}(n_x \times n_u)$
- Propagation: matrix ODE for Σ

Benefits/limitations

+ Flexible and broadly applicable
– Optimizing over matrices
– Linearization error can be problematic

Contraction metrics

Parametrization

- Contraction metric tube (z, δ), $\mathcal{O}(n_x)$
- Nonlinear feedback κ offline designed
- Propagation: scalar ODE for δ

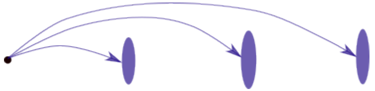
Benefits/limitations

– Offline synthesis required
+ Computationally efficient online
+ Stable tube propagation

Certified uncertainty propagation

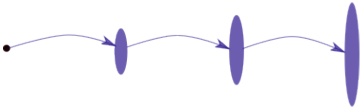
Comparison

Direct reachability



System-level synthesis & linearization

Sequential propagation



Linearization & ellipsoidal propagation

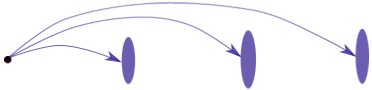
Contraction metrics

Different paradigms provide different complexity–conservatism trade-offs.

Certified uncertainty propagation

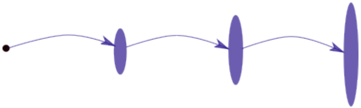
Comparison

Direct reachability



System-level synthesis & linearization

Sequential propagation



Linearization & ellipsoidal propagation

Contraction metrics

Different paradigms provide different complexity–conservatism trade-offs.

Other approaches not covered

- Stochastic [Mesbah, 2016]; Scenario [Campi et al., 2019]
- Sampling-based [Prajapat et al., 2026]; multi-stage [Lucia et al., 2013]
- Interval arithm. [Houska and Villanueva, 2019]

Outline of tutorial

1 Safe-by-design control with robust MPC
- Role of main components in robust MPC

2-3 Certified uncertainty propagation I-II
- Different methods for computationally tractable closed-loop robust predictions

4 Uncertainty quantification from data
- Different methods to extract model and uncertainty description

5 Safety under Interaction-Driven Distribution Shifts
- Statistical tools for safety assessment

6 Model predictive control for robot motion planning near humans
- Handle human-induced uncertainties

Tutorial material available online:
`https://kohlerjohannes.github.io/ECC_Tutorial_SafeByDesignMPC`

Appendix

Literature - contraction metrics

Contraction metrics

- Foundational work [Lohmiller and Slotine, 1998]
- Extension to design feedback [Manchester and Slotine, 2017]
- More recent overviews [Tsukamoto et al., 2021, Bullo et al., 2025]

Robust MPC using incremental Lyapunov functions/contraction metrics

- RPI set, given increm Lyap [Bayer et al., 2013]
- RPI design with contraction metrics [Singh et al., 2017, Singh et al., 2023, Zhao et al., 2022]
- Homothetic tube, given increm Lyap [Köhler et al., 2018, Köhler et al., 2021]
- Homothetic tube with contraction metric design and online model learning [Sasfi et al., 2023, Dubied et al., 2025]
- Output-feedback/perception-based [Chou et al., 2022, Köhler et al., 2021]
- Probabilistic safety with contraction metrics [Köhler and Zeilinger, 2025]

Appendix

Literature - linearization-based

Robust propagation for linear

- Foundational work [Bertsekas and Rhodes, 1971]

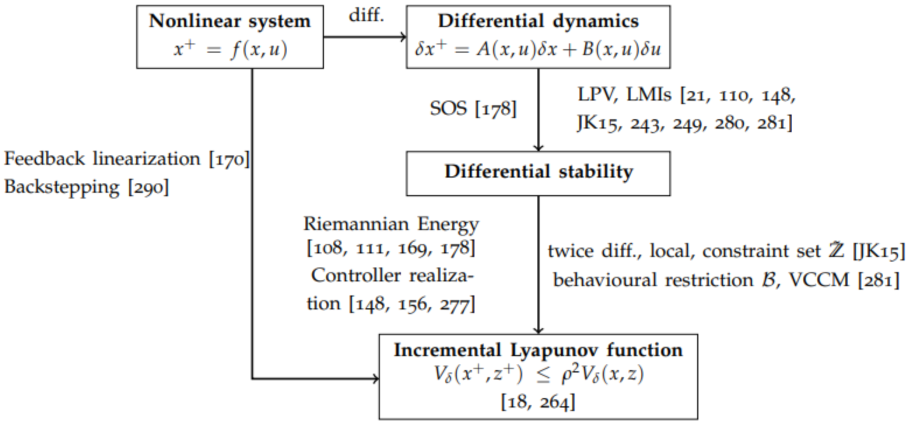
Linearization-based

- Handling of remainder [Althoff et al., 2008], [Houska, 2011, Chap. 3]
- convex algorithms [Cannon et al., 2011]
- zero-order algorithms [Zanelli et al., 2021, Frey et al., 2024]
- Joint feedback optimization [Messerer and Diehl, 2021],[Kim et al., 2024]
- Parametric uncertainty [Belvedere et al., 2025]

Appendix

Construction of incremm Lyap

Overview on different methods from [Köhler, 2021, App. C]



Appendix

Contraction metrics

Riemannian distance: $V_\delta(x,z) := \int_0^1 \|\gamma_s^\star(s)\|_{M(\gamma(s))}\mathrm{d}s.$

Feedback: $\kappa(x,z,v) := v + \int_0^1 K(\gamma^\star(s))\gamma_s^\star(s)\mathrm{d}s.$

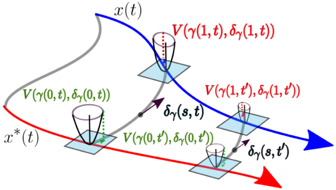


Figure from [Singh et al., 2023]. Geodesic connecting nominal and true state

Appendix

Example - linearization error bound [Leeman et al., 2025, Prop. III.1]
Consider coordinate i, with Hessian H_i of dynamics f w.r.t. x and uniform upper bound $\bar{H}_i \succeq H_i$.

Remainder satisfies

$$|w_{\mathrm{nl},i}| = |e^\top H_i(\xi)e| \leq \|e\|_{\bar{H}_i}^2, \tag{1}$$

Given a known bound $\|e\|_{\Sigma^{-1}} \leq 1$, we can compute a bound on $|w_{\mathrm{nl}}|$,

Quadratic dependence of linearization error based on prediction error implies unstable tube dynamics once error bound Σ is too large

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