Certified Uncertainty Propagation I

linear, LTV, nonlinear via linearized propagation

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From Part 1 to Part 2

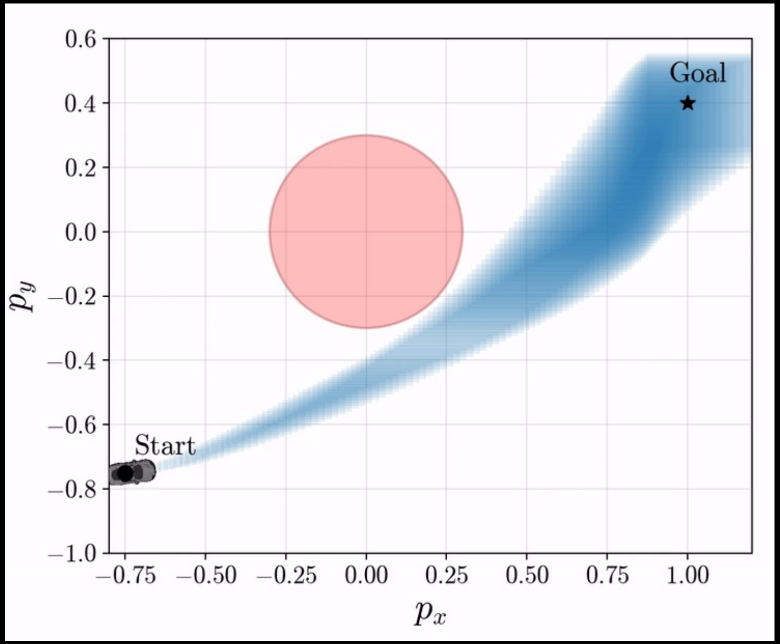
Part 1: Robust MPC ingredients

- Model + uncertainty bounds
- Closed-loop robust prediction sets
- Tightened constraints imply safety
- Terminal ingredients ensure recursive feasibility

Part 2: Computing reachable tubes

- Linear systems: min–max robust control
- LTV systems: exact closed-loop tubes via SLS
- Nonlinear systems: include linearization error
- Real-time computation: Riccati, GPU solvers

Goal: make robust predictions tractable while retaining guarantees.



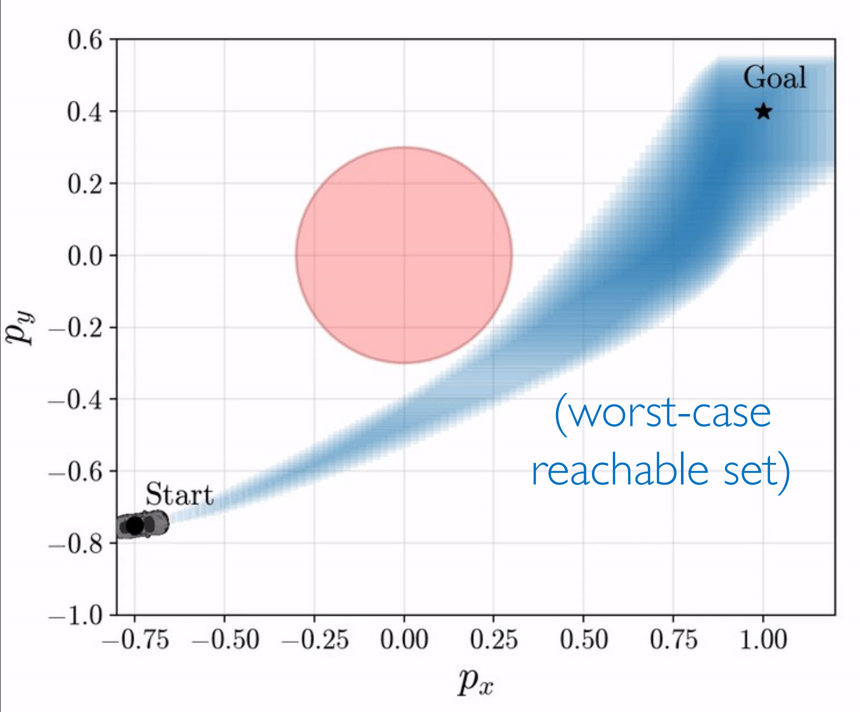
Linear Robust Control: Min-Max

Uncertain linear system

$$x_{k+1} = Ax_k + Bu_k + Ew_k, \qquad w_k \in \mathbb{W},$$
$$x_k \in \mathbb{X}, \quad u_k \in \mathbb{U}.$$

Finite-horizon robust control

$$\min_{\pi} \max_{w_0,\dots,w_{N-1}\in\mathbb{W}} \sum_{k=0}^{N-1} \ell(x_k, u_k) + \ell_f(x_N)$$
$$\text{s.t.} \quad x_{k+1} = Ax_k + B\pi_k(x_{0:k}) + Ew_k,$$
$$x_k \in \mathbb{X}, \quad \pi_k(x_{0:k}) \in \mathbb{U}, \quad \forall w_{0:k-1} \in \mathbb{W}.$$



- Worst-case disturbances ⇒ most damaging realization.
- Controller must keep the entire tube inside constraints.

Examples: references [1-7].

Linear Robust Control: Challenges

Two sources of intractability

- **Policy space:** arbitrary causal policies $\pi_k(x_{0:k})$ are infinite-dimensional.
- **Nested adversary:** the feasible set must hold for every disturbance sequence.

Key design question: which policy parameterization gives 1) tight reachable tubes and 2) efficient optimization?

Common tractable restriction (affine *causal* feedback)

$$u_k = v_k + \sum_{j=0}^{k-1} K_{k,j} w_j \quad \text{or} \quad u_k = v_k + K_k(x_k - z_k).$$

Common tractable choices for $\mathbb{W}$

- boxes / intervals
- zonotopes
- ellipsoids
- polytopes

Linear Robust Control: Tube MPC

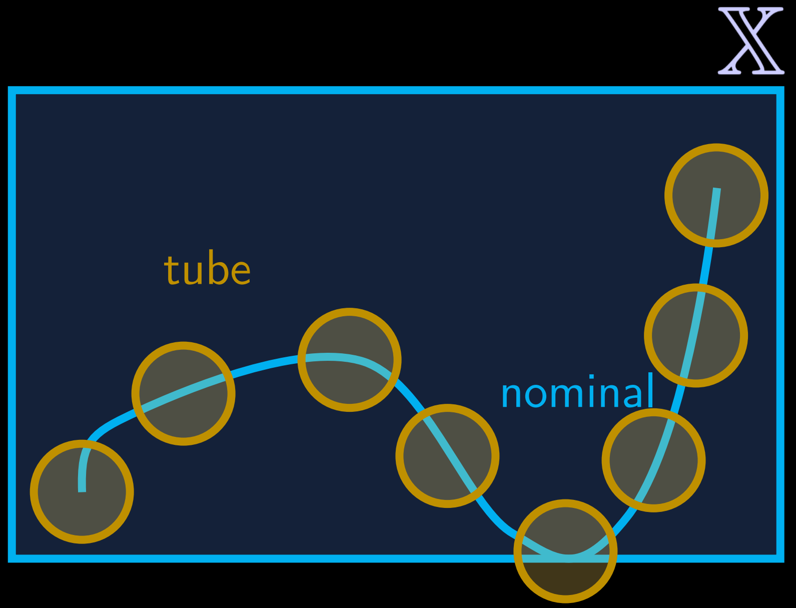
Split state and input

$$
\begin{aligned}
x_k &= z_k + e_k, \qquad u_k = v_k + Ke_k,\\
z_{k+1} &= Az_k + Bv_k,\\
e_{k+1} &= (A+BK)e_k + Ew_k.
\end{aligned}
$$

Pick a robust invariant error set

$$(A+BK)\mathcal{R} \oplus E\mathbb{W} \subseteq \mathcal{R}.$$

If $e_0 \in \mathcal{R}$, then $e_k \in \mathcal{R}$ for all admissible disturbances.



Reference [2].

Linear Robust Control: Tube MPC

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$$z_{k+1} = Az_k + Bv_k,$$
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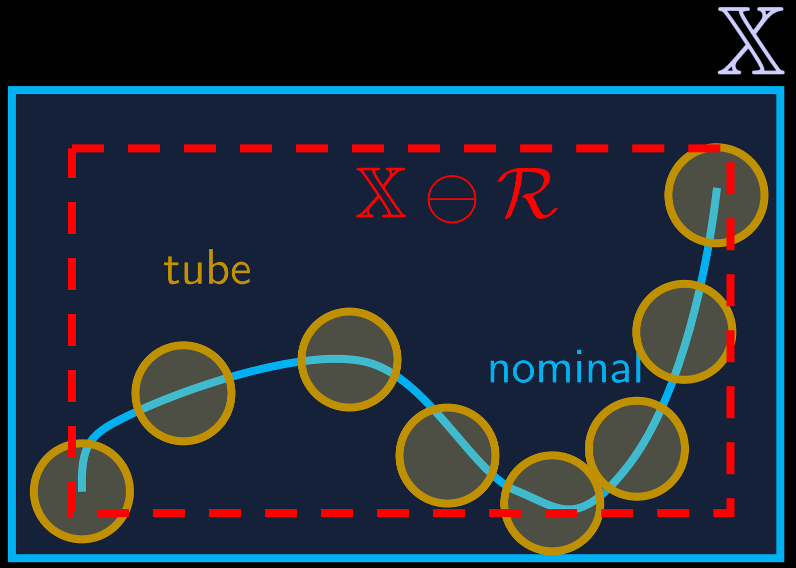
If $e_0 \in \mathcal{R}$, then $e_k \in \mathcal{R}$ for all admissible disturbances.

Fixed-tube MPC turns robust feasibility into nominal MPC with *tightened constraints*
$\Rightarrow$ fixed tube causes conservativeness



Tighten constraints

$$z_k \in \mathbb{X} \ominus \mathcal{R},$$
$$v_k \in \mathbb{U} \ominus K\mathcal{R}.$$

LTV Systems

LTV prediction model

$$x_{k+1} = A_k x_k + B_k u_k + E_k w_k, \qquad w_k \in \{w : \|w\|_\infty \leq 1\}.$$

$\mathbb{W}$

Where LTV appears

- linearized nonlinear dynamics along a nominal trajectory

What we need from the solver

- optimize nominal path and feedback together
- get exact closed-loop reachable tubes for the LTV model
- keep constraints robustly satisfied

System Level Synthesis (SLS) does this by optimizing closed-loop responses.

LTV Systems: System Level Synthesis

Causal disturbance feedback

$$u_k = v_k + \sum_{j=0}^{k-1} \Phi^u_{k,j} E_j w_j,$$

$$x_k = z_k + \sum_{j=0}^{k-1} \Phi^x_{k,j} E_j w_j.$$

System-level constraints

$$z_{k+1} = A_k z_k + B_k v_k,$$

$$\Phi^x_{j+1,j} = I,$$

$$\Phi^x_{k+1,j} = A_k \Phi^x_{k,j} + B_k \Phi^u_{k,j}.$$

nominal dynamics

the closed-loop effect of disturbance w_j on state x_k

causal ⇒ cannot influence past response

convex constraints in (Φ, z, v)

simple to extend to *output* feedback

Closed-loop reachable sets are explicit

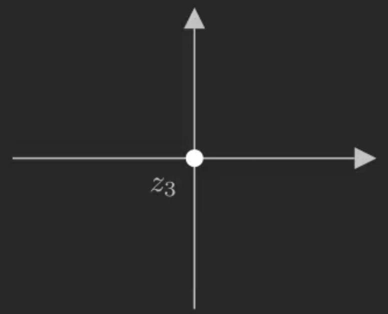
$$\mathcal{X}_k = z_k \oplus \bigoplus_{j=0}^{k-1} \Phi^x_{k,j} E_j \mathbb{W}, \qquad \mathcal{U}_k = v_k \oplus \bigoplus_{j=0}^{k-1} \Phi^u_{k,j} E_j \mathbb{W}.$$

Reference [6, 8].

Disturbance propagation in System Level Synthesis

$$x_k = z_k + \sum_{j=0}^{k-1} \Phi^x_{k,j} w_j \qquad u_k = v_k + \sum_{j=0}^{k-1} \Phi^u_{k,j} w_j$$

reachable zonotope at state x_3



$$\mathcal{R}_3 = z_3 \oplus \bigoplus_{i=0}^{2} \Phi^x_{3,i} \mathcal{W}_i$$

LTV Systems: Exact Reachability via SLS

Closed-loop reachable sets are explicit

$$\mathcal{X}_k = z_k \oplus \bigoplus_{j=0}^{k-1} \Phi^x_{k,j} E_j \mathbb{W}, \qquad \mathcal{U}_k = v_k \oplus \bigoplus_{j=0}^{k-1} \Phi^u_{k,j} E_j \mathbb{W}.$$

For ℓ_∞-ball disturbances

$$G_x x \leq g_x \ \forall x \in \mathcal{X}_k \iff G_x z_k + \sum_{j=0}^{k-1} \left\| G_x \Phi^x_{k,j} E_j \right\|_{1,r} \leq g_x.$$

One-shot, not step-by-step

- compute the whole map $w_{0:k-1} \mapsto (x_k, u_k)$ at once
- no sequential one-step set propagation
- avoids repeated overapproximation and wrapping effects

Reference [6, 8].

LTV SLS: overall optimization

Assume polytopic constraints $\mathbb{X} = \{x : G_x x \leq g_x\}$, $\mathbb{U} = \{u : G_u u \leq g_u\}$, and ℓ_∞-ball disturbances $\|w_k\|_\infty \leq 1$.

LTV SLS Robust Control

$$\min_{z,v,\Phi^x,\Phi^u} \quad \sum_{k=0}^{N-1} \ell(z_k, v_k) + \ell_f(z_N) + \lambda\, J_{\text{tube}}(\Phi^x, \Phi^u)$$

s.t. $z_{k+1} = A_k z_k + B_k v_k,$ — nominal dynamics

$\Phi^x_{j+1,j} = I, \qquad \Phi^x_{k+1,j} = A_k \Phi^x_{k,j} + B_k \Phi^u_{k,j}, \quad j < k,$ — SLS propagation constraints

$$G_x z_k + \sum_{j<k} \left\| G_x \Phi^x_{k,j} E_j \right\|_{1,r} \leq g_x,$$

$$G_u v_k + \sum_{j<k} \left\| G_u \Phi^u_{k,j} E_j \right\|_{1,r} \leq g_u,$$

robust stage constraints

$z_N \in \mathbb{X}_f \ominus \mathcal{R}_N.$ — robust terminal constraints

Reference [6, 8, 11].

Nonlinear Systems: SLS and *Linearization Error*

True dynamics and nominal trajectory

$$x_{k+1} = f(x_k, u_k) + E_k w_k, \qquad z_{k+1} = f(z_k, v_k).$$

First-order model around (z_k, v_k)

$$\begin{aligned} x_{k+1} &= f(z_k, v_k) + A_k(x_k - z_k) + B_k(u_k - v_k) + E_k w_k + r_k, \\ r_k &= \underbrace{f(x_k, u_k) - f(z_k, v_k) - A_k(x_k - z_k) - B_k(u_k - v_k)}_{\text{Taylor remainder}}. \end{aligned}$$

Bound Taylor remainder $\Longrightarrow$ inflates E_k, propagate as LTV system

Reference [9, 10, 13].

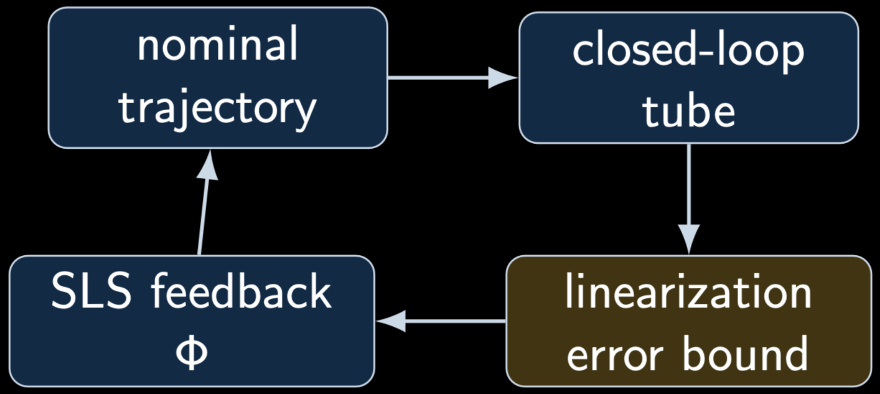
Nonlinear SLS: Linearization Error Tubes

Bound the residual on the current tube

$$r_k(x,u) \in \bar{R}_k \mathbb{W}, \qquad (x,u) \in \mathcal{X}_k \times \mathcal{U}_k.$$

Augment the LTV disturbance

$$\bar{E}_k = \begin{bmatrix} E_k & \bar{R}_k \end{bmatrix},$$
$$\tilde{w}_k \in \mathbb{W}^{2n_x}, \qquad \tilde{d}_k = \bar{E}_k \tilde{w}_k.$$



- Tight residual bounds shrink tubes.
- The tube determines where the residual must be certified.
- The optimizer can trade nominal performance and robustness.

Reference [9, 10, 13].

Nonlinear SLS: Computing Linearization Bounds

Classical Hessian bound

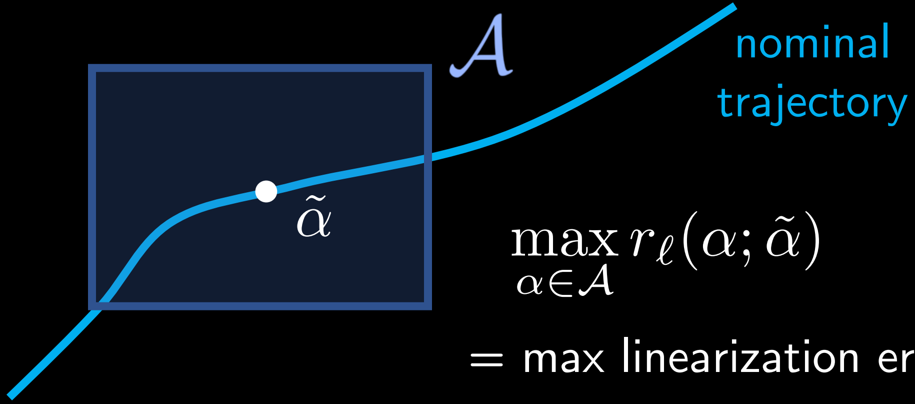
For $p(\alpha) = f(x, u)$, $\alpha = (x, u)$, and local set $\mathcal{A}$ around a nominal $\tilde{\alpha}$,

$$r_\ell(\alpha; \tilde{\alpha}) = \frac{1}{2}(\alpha - \tilde{\alpha})^\top H_\ell(\xi)(\alpha - \tilde{\alpha}), \quad \forall \ell \in [n_x].$$

- Interval Hessian bounds are sound.
- Conservativeness grows with tube size, dimension.

Recent improvements

- **Dynamic over-bounds:** optimize linearization-error bounds jointly with nominal path and feedback [9].
- **Differentiable bounds:** use gradients of the bound to move the nominal trajectory to low-error regions.



$$\max_{\alpha \in \mathcal{A}} r_\ell(\alpha; \tilde{\alpha})$$

$=$ max linearization error

Safety depends on <u>sound</u> bounds;
performance depends on <u>tight</u> bounds.

Reference [13].

Nonlinear SLS: Optimizing Error Bounds

Robust nonlinear SLS optimization

$$\begin{aligned}
\min_{z,v,\Phi} \quad & J(z,v) + \lambda \underbrace{J_{\text{tube}}(\Phi, \bar{E})}_{\text{tube size}} \\
\text{s.t.} \quad & z_{k+1} = f(z_k, v_k), \quad \Phi \text{ satisfies SLS constraints,} \\
& g_k(z_k, v_k) + h_k(\Phi, \bar{E}) \leq 0, \\
& \bar{R}_k \mathbb{W} \text{ bounds linearization error over the tube.}
\end{aligned}$$

What is optimized?

- nominal trajectory (z, v)
- feedback response Φ
- tube size
- (potentially) the residual bound itself

What is guaranteed?

If the residual bound is sound and the tightened constraints are feasible, then the true nonlinear closed-loop trajectory remains inside the computed tube.

Reference [9, 10, 13].

Nonlinear SLS: overall guarantee

Residual bound + feasible robust SLS $\Rightarrow$ robust nonlinear constraint safety

If for each time k,

$$r_k(x,u) \in \bar{R}_k \mathbb{W} \qquad \forall (x,u) \in \bar{\mathcal{X}}_k \times \bar{\mathcal{U}}_k,$$

then the augmented disturbance model produces tubes

$$\bar{\mathcal{X}}_k = z_k \oplus \bigoplus_{j<k} \Phi^x_{k,j} \bar{E}_j \mathbb{W}, \qquad \bar{\mathcal{U}}_k = v_k \oplus \bigoplus_{j<k} \Phi^u_{k,j} \bar{E}_j \mathbb{W},$$

that overapproximate the true nonlinear closed-loop reachable set.

Interpretation

SLS enables exact propagation for the *surrogate* LTV system, and linearization error bounds make the surrogate contain the *nonlinear* dynamics at each step.

Reference [9, 10, 13].

SLS: computational challenges

Main computations

1. nominal trajectory optimization
2. controller/tube update via SLS
3. residual-error bound evaluation
4. constraint tightening and dual updates

$$
\begin{aligned}
\min_{z,v,\Phi^x,\Phi^u} \quad & \sum_{k=0}^{N-1} \ell(z_k, v_k) + \ell_f(z_N) + \lambda\, J_{\text{tube}}(\Phi^x, \Phi^u) \\
\text{s.t.} \quad & z_{k+1} = A_k z_k + B_k v_k, \\
& \Phi^x_{j+1,j} = I, \qquad \Phi^x_{k+1,j} = A_k \Phi^x_{k,j} + B_k \Phi^u_{k,j}, \quad j < k, \\
& G_x z_k + \sum_{j<k} \left\| G_x \Phi^x_{k,j} E_j \right\|_{1,r} \le g_x, \\
& G_u v_k + \sum_{j<k} \left\| G_u \Phi^u_{k,j} E_j \right\|_{1,r} \le g_u, \\
& z_N \in \mathbb{X}_f \ominus \mathcal{R}_N.
\end{aligned}
$$

Algorithmic choices

- **Set representation:** boxes are cheap; zonotopes reduce conservativeness.
- **SLS solve:** exploit problem structure instead of generic conic/QP solvers.
- **Hardware:** CPU or GPU.

Convex, but can still be slow $\Rightarrow$ massive number of decision variables: $\mathcal{O}(N^2(n_x^2 + n_u^2))$

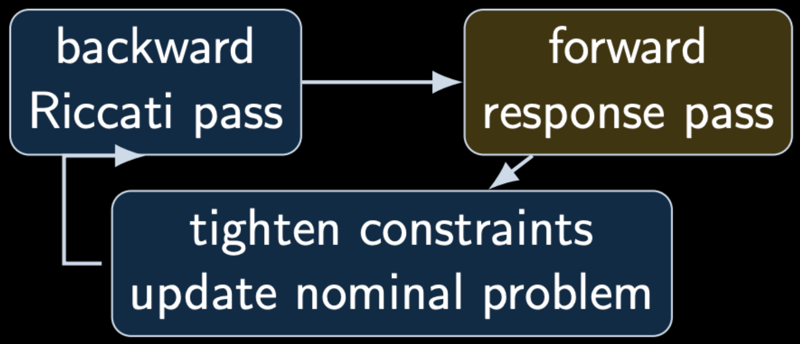
Reference [11, 12, 13].

Fast SLS via Riccati recursions

Controller optimization has dynamic-programming structure

Fast SLS uses Riccati recursions to solve the LTV disturbance-feedback optimization efficiently, rather than passing the full response problem to a generic solver [11].

$$
\begin{aligned}
z_{k+1} &= A_k z_k + B_k v_k, \\
\Phi^x_{j+1,j} &= I, \\
\Phi^x_{k+1,j} &= A_k \Phi^x_{k,j} + B_k \Phi^u_{k,j}.
\end{aligned}
$$



FastSLS loop

Scaling

- original complexity: $\mathcal{O}(N^4(n_x^3 + n_u^3)n_x^3)$
- new per-iterate complexity: $\mathcal{O}\big(N(n_x^3 + n_u^3)\big)$
- up to $10^3\times$ speedup over general-purpose commercial solvers

- Exploit sparse temporal structure.
- Avoid forming huge dense optimization problems.

Reference [11].

GPU-SLS: scaling robust MPC to robotics

Core idea

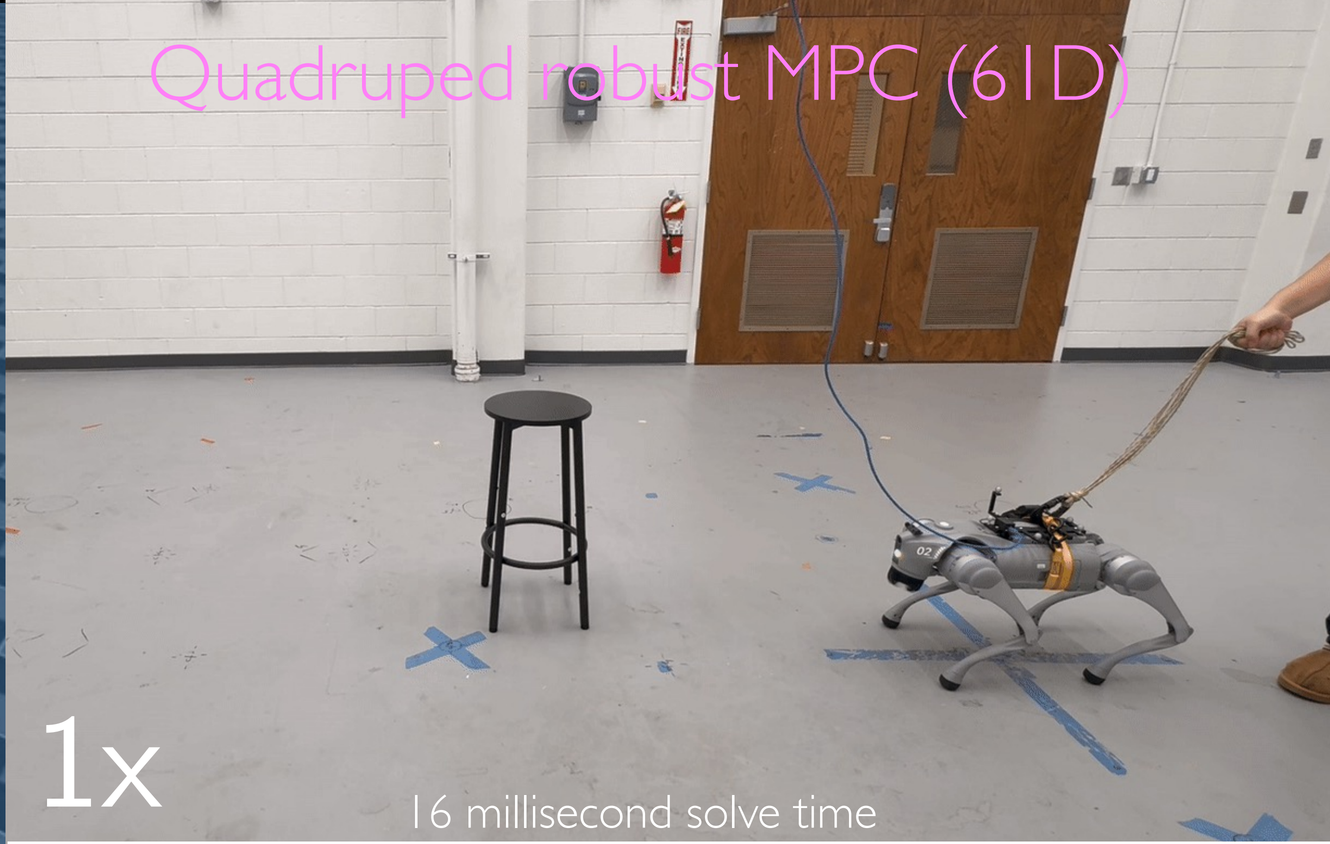
GPU-SLS parallelizes both constrained nominal trajectory optimization and SLS-based reachability/control updates on the GPU [12].

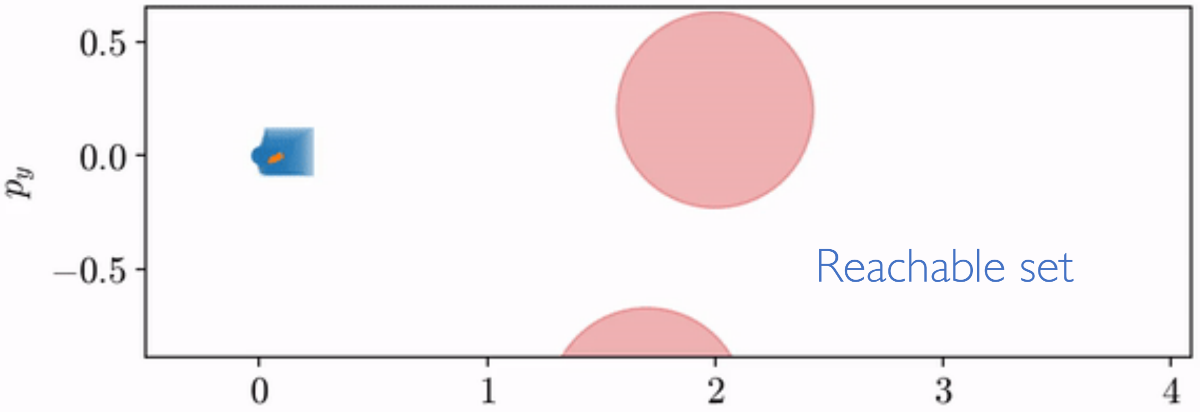
Computational ingredients

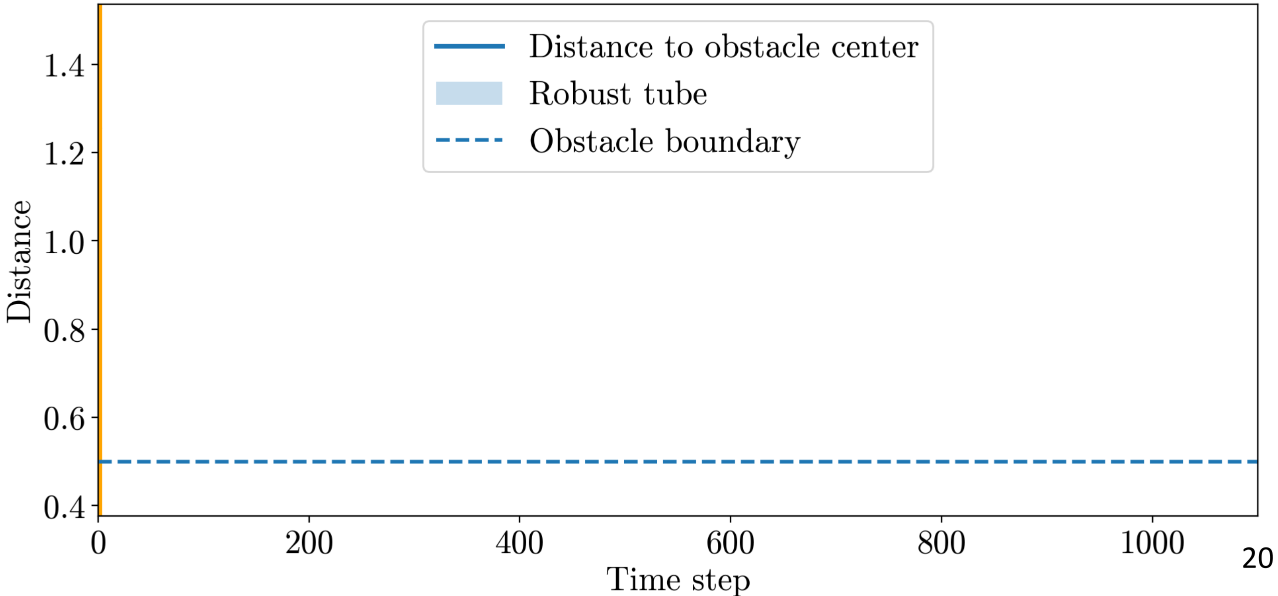
- SQP outer loop for nonlinear dynamics and constraints
- ADMM QP solver with parallel associative scans
- shared GPU backend for nominal trajectory and SLS updates
- differentiable linearization error bounds that can be optimized to reduce conservativeness
- Fast-SLS $\mathcal{O}(N(n_x^3 + n_u^3)) \rightarrow$ GPU-SLS $\mathcal{O}(\log N \log^2 n_x + \log^2 n_u)$

Horizon N	10	25	50	100	200	500	1000
FastSLS (ms)	10	65	180	630	2,538	35,057	189,052
GPU-SLS (ms)	9	11	15	27	50	198	796
Speedup	1.11	5.91	12	23.3	50.8	193.5	237.5

Reference [12, 13].



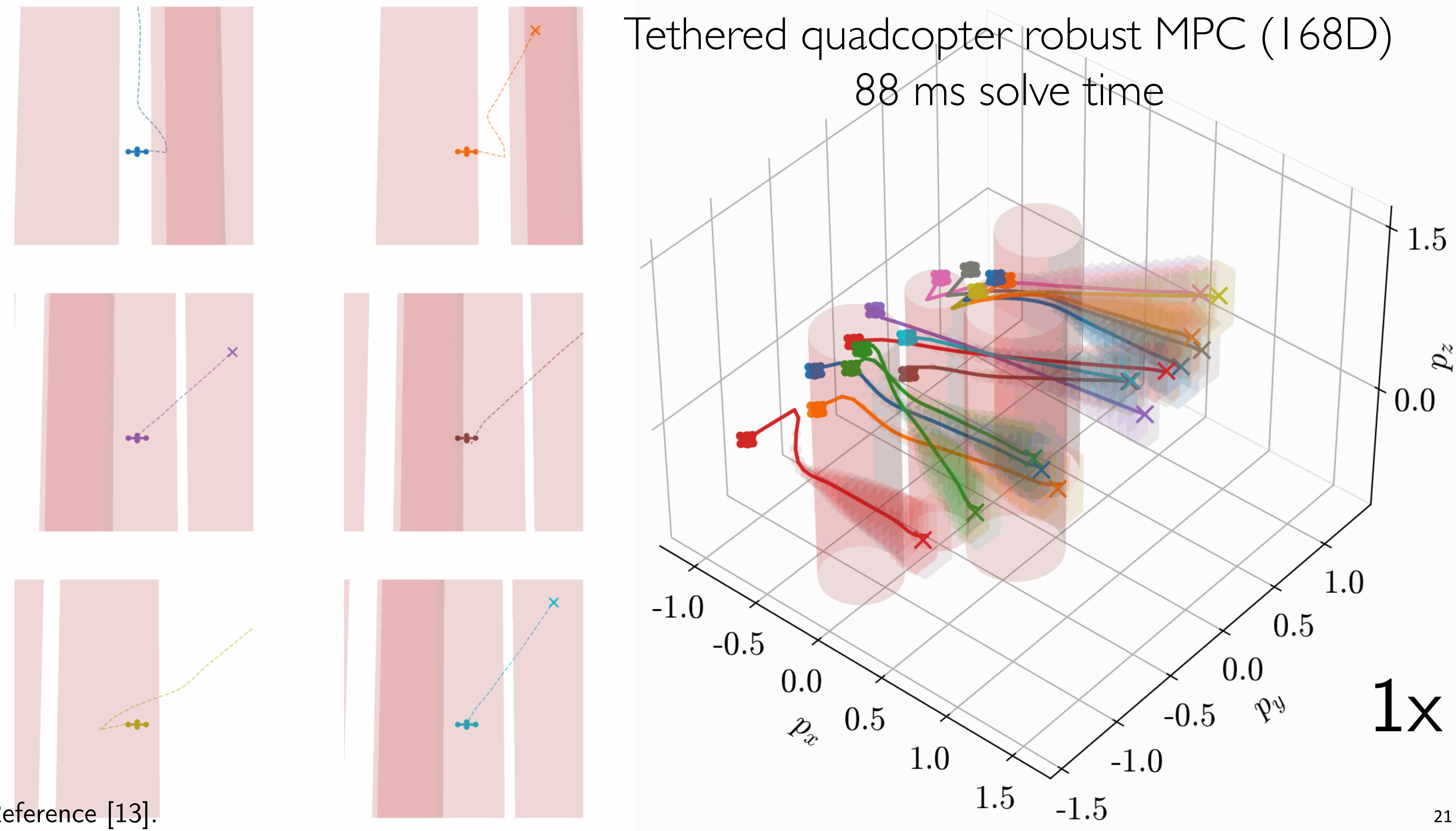




Reference [12, 13].

Tethered quadcopter robust MPC (168D)

88 ms solve time

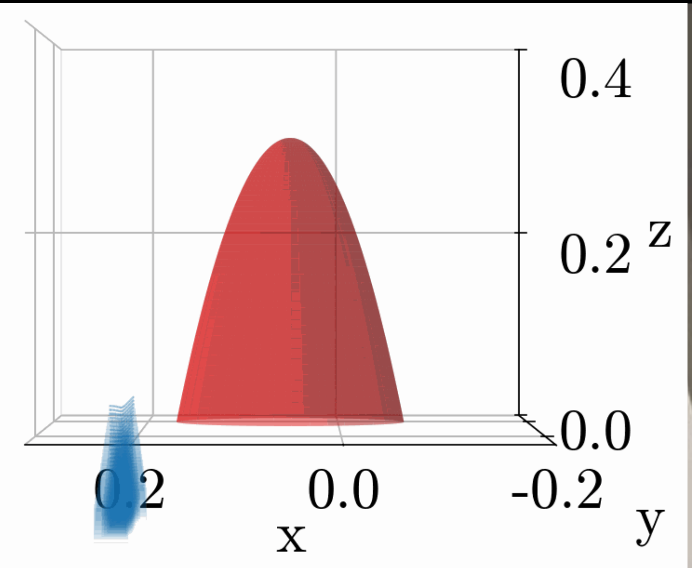


1x

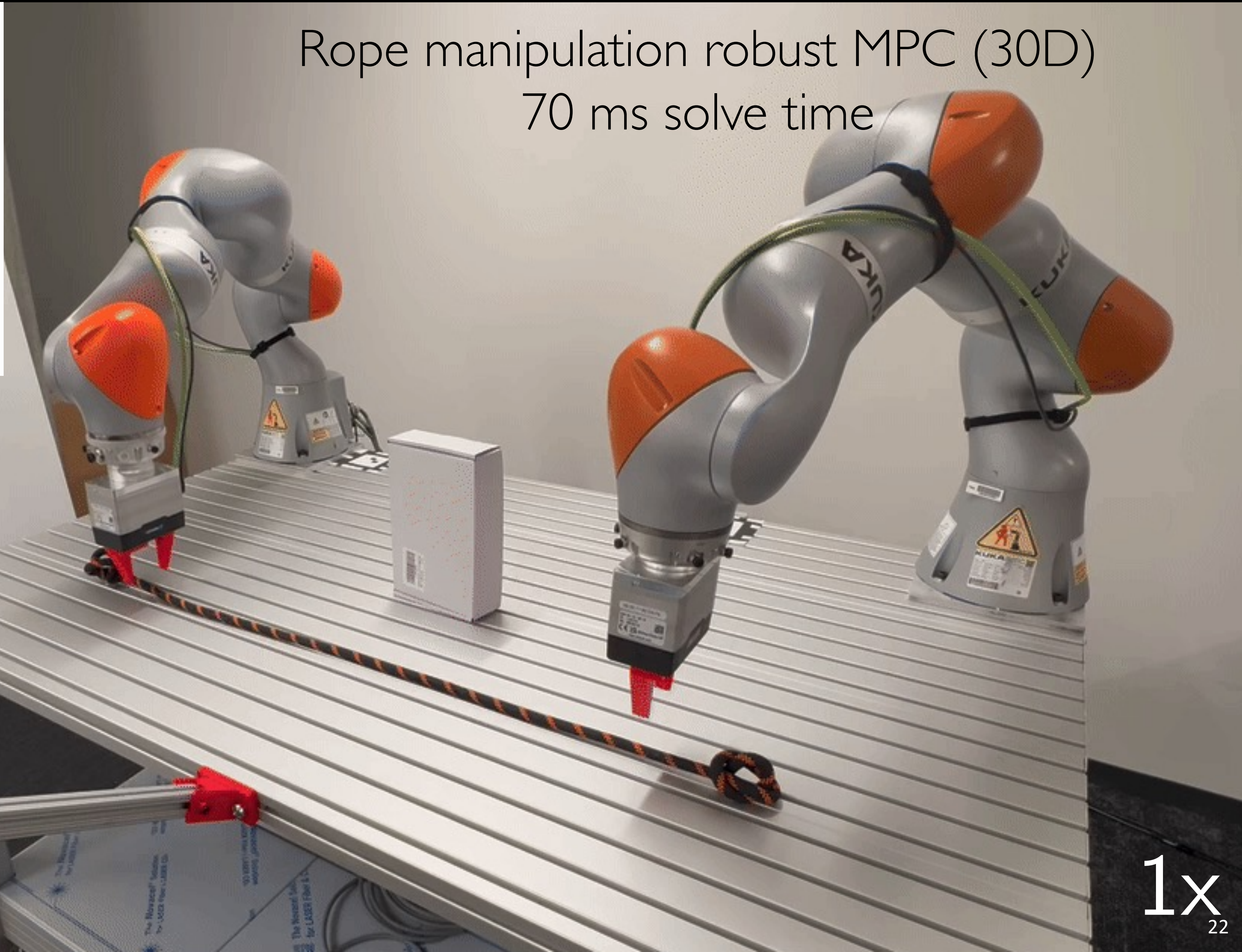
Reference [13].

Rope manipulation robust MPC (30D)

70 ms solve time



Reachable tubes



1x

Reference [14].

Cloth manipulation robust MPC (300D)
760 ms solve time

2x

Reference [14].

Model class	Robust prediction method	Guarantee
LTI or LTV	SLS closed-loop responses	exact reachable tubes for the LTV model and chosen disturbance set
Nonlinear	LTV SLS + linearization-error bounds	true nonlinear trajectories are contained if residual bounds are sound

Advantages:

- LTV structure enables massive parallelization on the GPU ⇒ scales to huge problems (2×10^5 variables)
- Real-time performance
- Sound robust constraint satisfaction guarantees

Limitations:

- Requires controller to be re-designed for each nominal trajectory planned
- Can be conservative for nonlinear systems (tradeoff for being fast w/ linearized propagation)

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