

An explicit bound for Erdős problem 417

Put $\mathcal{T} = \varphi(\mathbb{N})$ and

$$V(x) = \#\{v \in \mathcal{T} : v \leq x\}, \quad V'(x) = \#\{\varphi(n) : n \leq x\}, \quad \ell(v) = \min\{n : \varphi(n) = v\}.$$

Theorem 1. *With*

$$M_* = \frac{\log 2}{2} + \sum_{\substack{p \geq 3 \\ p \text{ prime}}} \frac{p}{(p-1)^2} \log \frac{p}{p-1},$$

we have

$$\limsup_{x \rightarrow \infty} \frac{V(x)}{V'(x)} \leq e^{M_*} < 2.195202.$$

Proof. Set $W(x) = \sum_{v \in \mathcal{T}, v \leq x} v/\ell(v)$. We first deduce from the common-prefix estimates of [1, Proposition 5.3] that

$$W(x) = V'(x) + o(V(x)). \tag{1}$$

For fixed H , these estimates, including their bounds for exceptional representations, leave an error $O(\varepsilon_H V(x)) + o_H(V(x))$, where $\varepsilon_H \rightarrow 0$. For each remaining prefix, write

$$v = D(p-1), \quad \ell(v) = Ap, \quad 1 \leq D \leq A \leq \exp((\log x)^{0.8}),$$

where p ranges over primes $x^{0.9} \leq p \leq 1+x/D$. Uniformly in the fixed prefix, the prime number theorem gives

$$\sum_{x^{0.9} \leq p \leq 1+x/D} \frac{D}{A} \left(1 - \frac{1}{p}\right) \sim \frac{x}{A \log x} \sim \#\{p : x^{0.9} \leq p \leq x/A\}.$$

Here $\frac{x}{\log x} \sum D^{-1} = O_H(V(x))$ over the remaining prefixes, so the errors may be summed. Removing the exceptional representations and then letting $H \rightarrow \infty$ proves (1).

For a prime p , define

$$C_p(x) = \#\{v \in \mathcal{T} : v \leq x, p \mid \ell(v)\}.$$

Deleting $p^a \parallel \ell(v)$ gives

$$C_p(x) \leq \sum_{a \geq 1} V\left(\frac{x}{p^{a-1}(p-1)}\right). \tag{2}$$

By [1, Theorem 2.1], $V(cx)/V(x) \rightarrow c$ for each fixed $c > 0$. Iteration at $c = 2$ also gives, uniformly in $k \geq 1$,

$$\frac{V(x/k)}{V(x)} \ll k^{-1/2}.$$

Consequently (2) and dominated convergence yield

$$\limsup_{x \rightarrow \infty} \frac{C_p(x)}{V(x)} \leq \frac{p}{(p-1)^2}. \tag{3}$$

If $2 \mid \ell(v)$, minimality forces $4 \mid \ell(v)$; hence $\varphi(\ell(v)/2) = v/2$, giving the sharper bound

$$C_2(x) \leq V(x/2), \quad \limsup_{x \rightarrow \infty} \frac{C_2(x)}{V(x)} \leq \frac{1}{2}. \quad (4)$$

The uniform estimate above implies

$$\frac{C_p(x)}{V(x)} \log \frac{p}{p-1} \ll p^{-3/2},$$

so the prime tails are uniformly summable. Since

$$\log \frac{\ell(v)}{v} = \sum_{p \mid \ell(v)} \log \frac{p}{p-1},$$

(3)–(4) give

$$\limsup_{x \rightarrow \infty} \frac{1}{V(x)} \sum_{\substack{v \in \mathcal{T} \\ v \leq x}} \log \frac{\ell(v)}{v} \leq M_*.$$

Jensen's inequality and (1) now imply

$$\liminf_{x \rightarrow \infty} \frac{V'(x)}{V(x)} \geq \liminf_{x \rightarrow \infty} \frac{W(x)}{V(x)} \geq e^{-M_*},$$

which proves the series bound.

For the numerical bound, let $N = 999990$ and

$$U(z) = 2 \sum_{j=0}^8 \frac{z^{2j+1}}{2j+1} + \frac{2z^{19}}{19(1-z^2)} \quad (0 < z < 1).$$

Then $\log \frac{1+z}{1-z} \leq U(z)$, and exact rational summation over primes $3 \leq p \leq N$ gives

$$\frac{1}{2} U(1/3) + \sum_{3 \leq p \leq N} \frac{p}{(p-1)^2} U\left(\frac{1}{2p-1}\right) < 0.786273785.$$

For $t > 1$, $t/(t-1)^3$ decreases, and every prime greater than 5 lies in one of eight reduced residue classes modulo 30. As $30 \mid N$,

$$\begin{aligned} \sum_{p > N} \frac{p}{(p-1)^2} \log \frac{p}{p-1} &\leq \sum_{\substack{n > N \\ (n, 30) = 1}} \frac{n}{(n-1)^3} \\ &\leq \frac{8}{30} \int_{N-29}^{\infty} \frac{t}{(t-1)^3} dt \\ &= \frac{4}{15} \left(\frac{1}{N-30} + \frac{1}{2(N-30)^2} \right). \end{aligned}$$

Thus $M_* < 0.786274052$ and $e^{M_*} < 2.195202$. □

References

- [1] OpenAI, *An asymptotic formula for the number of totients*, preprint, September 25, 2026. <https://github.com/openai/math/tree/main/preprints/An-asymptotic-formula-for-the-number-of-totients-September-25-2026>.