Questions on fluid mechanics, numerics, and turbulence

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Note: In this note we compile a list of questions that might be suitable for a (mock) interview or oral (even written) exam on fluid mechanics, numerics, turbulence, or the combination of all, at a grad school level. They were from years of accumulation and from too many sources (lecture notes, textbooks, private studying/conversations, etc) to be cited and acknowledged individually, to which we are truly grateful. The length of answers may vary from one sentence to an extended/open-ended derivation and discussion, with some of them already given in the text to certain extent. We apologize in advance that the descriptions/wording of some questions might not be optimal/sufficient so please use with caution and let us know if you should have any feedback.

1 Fundemental fluid mechanics

1. What is potential flow? What is vorticity and in a potential flow and how to express velocity by a scalar potential? Derive the Laplace equation and explain why we can obtain potential solutions by superposing simple solutions. Note that when applying superposition, we also need to consider boundary conditions. What is the boundary conditions for velocity on a stationary solid wall and what about pressure? What boundary conditions for velocity are required if it is a viscous flow and why is the difference? Consider the order of derivatives.

2. Is the absolute value of pressure important in incompressible flows and compressible flows? Why?

3. What's the mass conservation equation and what's the incompressible condition? (Density does not change along fluid parcels) Are stratified flows compressible?

4. Write down the complex potential for a line vortex and obtain/write down u_r, u_θ. What's the circulation of this flow? (Explain what is circulation.) Write down another flow with $u_r = 0, u_\theta = \Omega r$. Which flow has non-zero vorticity? What's the relation between the flow is swirling and its vorticity is non-zero? (Unnecessary and insufficient. Consider a line vortex and a parallel shear flow.)

5. What's the relation between the existence of local pressure minima and the existence of a vortex? (Unnecessary and insufficient. Consider Kármán swirling flow which has zero radial pressure gradient, or an anticyclone which has a pressure maximum. Consider a stagnation point flow in section 2.1 of Jeong & Hussain 1995 JFM)

6. Write down the expression of vorticity in a 2D cylindrical coordinate and explain there is vorticity due to the shear and due to the curvature of streamlines.

7. Boundary layer theory. Consider a laminar Blasius boundary layer. What are the self-similar coordinates? Define and order the magnitude of the nominal, displacement, and momentum thicknesses in a Blasius BL. How does the momentum thickness θ scale with Reynolds number Re_x where x is the distance counting from the leading edge? ($\theta/x = 0.664/\sqrt{Re_x}$)

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8. Drag. Consider creeping (Stokes) flow around a spherical object. Which terms are neglected in the equation? How does the drag scale with velocity as $F_D \propto x^m$? (qualitative analysis, $F_D \propto \Pi D U \rho \nu$, with the constant obtained from solving the Stokes equation) Consider a high Reynolds wake past an object. How does drag change with velocity as $F_D \propto x^m$? ($F_D \propto c_D \rho U^2 D l$, l is the length in homogeneous direction taken to be unity, and assume c_D to be constant at high Re)

9. How does the skin friction $c_f = \tau_w/(1/2\rho U^2)$ scale with Re_x? ($c_f \sim 0.664/\sqrt{Re_x}$) Does this mean that we have zero c_f at high Re_x? (No, the BL will transition to turbulent BL.) Draw the c_f v.s. Re curve.

10. Sport. Two same-size gulf balls, one rough and one smooth, moving at the same speed. Which has less drag and why? What are the various sources of drag? How does the pressure (form) drag compare to the friction drag? Describe the drag crisis of a smooth sphere/cylinder.

11. Which term in N-S is the vortex stretching/tilting term coming from? Explain the terms corresponding to stretching/tilting, the corresponding physical picture, and smaller scales are generated in turbulence.

12. Describe what is nonlinearity and which term(s) in N-S are nonlinear. Why? Enumerate some flows/solutions to the N-S that are linear.

13. In incompressible flows, do we need the equations for energy conservation and angular momentum conservation? If not, what equations/assumptions they are contained in/consistent with? How about compressible flows?

14. Write down the nondimensionalized N-S equations and describe Reynolds similarity theorem. What is the controlling parameter and what's its physical meaning? Is the behavior of N-S at very-high Reynolds number (implying vanishing viscosity) the same as/similar to Euler equations, and why? Is viscosity not functioning at high Reynolds number?

2 Numerics and CFD

1. Write down the incompressible 2D Navier-Stokes equations. How many variables that we need to solve and how many equations we have? ($[u, v, p]$) Which equation we will use to solve for pressure? Derive the pressure Poisson. What type of PDE it is and what are the methods that can be used to solve the pressure Poisson? Why do we need staggered grids if we want to use central differences in space?

2. What if we don't want to solve for pressure? (Derive) the vorticity streamfunction formulation for 2D incompressible N-S. What type of equation are they? (Vorticity equation is elliptic-parabolic and the streamfunction Poisson is elliptic.) If we have solution of (ω, ψ) but want to solve for p, what's the equation that we need to solve and how do we give pressure boundary condition in a channel? (1) inflow and outflow? (Dirichlet and Neumann) (2) On the flate walls $y = 0, 2h$? (The x-momentum equation reduces to $\partial p/\partial x|_w = \mu \partial^2 u/\partial y^2|_w = -\mu \partial \omega/\partial y|_w$ and then discretize it to find Dirichlet BC's.) How do we get pressure after solving for ω, ψ?

3. Explain why only the non-diagonal terms of the Reynolds stress can change the mean flow. (normal components can be absorbed into the modified pressure) And if the force is conservative (can be written as the gradient of a potential), what we can do? (absorb it into the modified pressure)

4. In a spectral solver for turbulent channel flow the equations for ω_y and $\nabla^2 v$ (y is the vertical direction). There will be a viscous term for $\nabla^2 v$ and we have two walls in y-direction. How many boundary conditions we need for v and what are them? (ref. KMM87, JFM)

5. Previous problem is similar to the vorticity-streamfunction method in 2D. How do we give boundary conditions there and how do we obtain the pressure? There is no pressure term in the vorticity equation as an advantage of the method, and why is pressure not there?

6. Perform a modified wavenumber analysis for central difference scheme for the first derivative.

7. If all the eigenvalues of a matrix are zero, are all singular values zero? Is the opposite true? If so, prove, and if not, give counter examples.

8. What are primitive and conserved variables? Write down N-S in strong conservation form. How many equations and how many unknowns. EOS.

9. Burgers equation (1D). Draw the evolution at different instants and explain why. What's the dependence of phase velocity on magnitude? What is conserved in time? (Area. $d/dt(\int_{-\infty}^{\infty} u\,dx) = 0$) To capture discontinuities/shock waves, what form of equation that we need to use when doing CFD? Give the names of some schemes that are suitable for capturing shock wakes.

10. Stability of schemes. Forward-time central-space (FTCS) was used by Richardson to solve an advection problem for weather prediction but unfortunately it is unconditionally unstable for any arbitrarily small Δt. Show why. Show also, for a heat (diffusion equation), FTCS is conditionally stable.

11. Consider a heat equation. If I have a simple wave (single wavenumber), how would it change in time? If I have an initial wave with multiple wavenumbers, how would the decay rate depend on wavenumber? What is the time scale for diffusion?

12. Consider a backward heat equation. Write down the equation and show how an initially Gaussian function will evolve (graphically and via analysing the total energy). Infer the time-irreversibility of the heat equation. What does it imply to a fluids system?

13. Derive the optimality of singular value decomposition in terms of the least square approximation.

3 Stability and perturbation analysis

1. Derive the equation for the perturbation velocity u_i' from the full N-S. Assume $\boldsymbol{u}' = \epsilon\tilde{\boldsymbol{u}} + \mathcal{O}(\epsilon^2)$. Keep only the $\mathcal{O}(\epsilon)$ terms and derive the linearized perturbation equation and give the boundary conditions when it is applied to a bluff body wake. Assume bi-global modes $\tilde{\boldsymbol{u}}(x, y, t) = e^{\lambda t}\hat{\boldsymbol{u}}(x, y)$ and substitute it into the linearized equation to obtain the eigenvalue problem.

4 Turbulence theory

1. Write down Reynolds decomposition $u_i = \bar{u}_i + u_i'$. Define the mean of the total kinetic energy as $K = 1/2\overline{u_i u_i}$, the kinetic energy of the mean $\mathcal{K} = 1/2\bar{u}_i\bar{u}_i$, and turbulent kinetic energy $k = 1/2\overline{u_i' u_i'}$. Write down the transport equation for turbulent kinetic energy k, and explain the meaning of each term. Why is the production term always positive in shear flow in statistical sense? Why $-\overline{u'v'} > 0$ if $\partial_y U > 0$? How does the production term look like in the equation of $\mathcal{K}$? Explain how the production term extracts energy from the mean flow and adds it to the turbulent flow.

2. Describe homogeneous turbulence (mean turbulence quantities don't depend on time) and isotropic turbulence (all mean quantities invariant to rotation). Simplify the TKE equation in (unforced) isotropic turbulence and explain why the TKE is always decaying.

3. What is the dimension of $E(k)$? In the inertial range of turbulence what does the energy spectrum scale with? (ε, k) Use dimensional analysis to derive the K41 scaling law.

4. Define the two point correlation function in isotropic turbulence and how does it relate to the spectral density function?

5. What's the ratio between turbulent (Reynolds) stress and viscous stress and how does it change with Reynolds number? Scaling analysis.

6. What's the physical meaning of Reynolds number? How does dissipation and the gradient of perturbation velocity change as Re increases (or ν decreases)? Meaning of Reynolds number is inertia divided by viscosity. Turbulence has high Reynolds number, hence viscosity not important. Is this true?

7. What's the smallest scales (Kolmogorov scales) in turbulence? What quantities does it depend on (ν, ϵ)? Derive it $(\eta = (\nu^3/\varepsilon)^{1/4}, u_\eta = (\nu\varepsilon)^{1/4})$ from dimensional analysis. What's the Reynolds number defined by Kolmogorov scale? $(Re_\eta = 1)$ What's ratio between the Kolmogorov scale and the large scale? $(L/\eta = Re_L^{3/4}, Re_L = \frac{u'L}{\nu}$, assuming $\varepsilon = u^3/L$, the inertial scaling, and L is the forcing/large scale) How to interpret the meaning of Reynolds number in terms of separation between largest and smallest scales?

8. Consider (3DT) DNS. What scales do we need to resolve? If we want to keep $k\eta$ (k is the highest wavenumber) roughly a constant, and the box size/forcing scale the same, how would computational cost increase as a function of Re_L? $(N_x N_y N_z N_t \sim Re_L^3, N_t \sim N_x \sim Re_L^{3/4}$, for a constant CFL) Assume a decent DNS at $Re_\lambda = 100$ $(Re_\lambda = Re_L^{1/2})$, approximate the number of grid points we need (1024^3) and number of FLOPs (10^{12}). What's the memory requirement for storing one field? $(10^9$, i.e. GBytes).

9. What's the machinery/picture of energy transfer in turbulence and the relevant scales? (Injection, transport/cascade, dissipation) Can energy back-scatter from small scales to the large scales (in 3DT)?

10. Taylor microscale. Define Taylor scale λ_f as a function of ν, ε, u'. $(\lambda_f^2 = 30 u^2 \nu/\varepsilon)$ Is it larger or smaller than the Kolmogorov scale? Also give the definition in terms of correlation function $R(r)$ and explain its meaning $(\lambda_f = \sqrt{-R(0)/R''(0)}$, decorrelation length scale of a Gaussian equivalent to $R(r)$ near $r = 0$).

11. What determines the largest scale (forcing wavenumber and boundaries), and the smallest scale (Reynolds number) in turbulence?

12. Dissipation rate has an inertial scaling that doesn't depend on Reynolds number. What is this scaling? $(\varepsilon \sim u'^3/L)$ Write down ε and what happens if $\nu \to 0$ as $Re \to \infty$? If you extract an isosurface of velocity in turbulence, will it be rough or smooth?

13. Consider passive scalar in turbulence. We don't have a stretching term in turbulence. How are the small scales in temperature field generated?

14. What is statistical stationary and homogeneity? How do we determine in a simulation/experiment whether this state is reached? (Derive the scalar transport equation too.)

15. K41 has a $-5/3$ law for the energy spectrum in stationary isotropic turbulence. Do we see $-5/3$ in free shear turbulence or other types of turbulence and why? (small scale isotropy) K41 law is for $E(k)$ or $E_{11}(k_1)$ which is spatial spectrum. If we measure time series in a wake and perform an FFT to get temporal spectrum. Will we see a $-5/3$ range and why? (Taylor's frozen flow hypothesis)

16. Draw the spectrum $E(k)$ of 3D isotropic turbulence. What does it depend on? Where do we want to put the DNS filter? LES filter?

17. What's the difference between noise and turbulence? (Turbulence has cascade/interscale transfer.)

5 Wall-bounded and free shear flows

1. Give examples of wall bounded and free shear turbulence.

2. Consider Poiseuille channels (flow driven by constant pressure gradient). Draw the mean profile $U(y)$ for a laminar and a turbulent case and explain the difference between two profiles. Which is fuller and why? Derive the relationship between streamwise pressure gradient dp/dx and the wall shear stress τ_w using a control volume analysis. Note that you have two walls. ($\tau_w/h = -dp/dx$)

3. What's the main difference between a Poiseuille and a Coette channel? (The shear is non-zero in Coette) In which case it is more likely to see super-long structures?

4. Wall-bounded turbulence. Define the friction velocity. ($u_\tau = \sqrt{\tau_w/\rho}$) Give the viscous length scale near the wall. ($\delta_\nu = \nu/u_\tau$) (1) If you have a channel (half height $h = 1$ m) at $Re_\tau = 10000$, what is the height of the viscous sublayer? ($5^+ = 0.5$ mm) (2) The thickness of atmosphere surface (boundary) layer is about 100 m and $Re_\tau \approx 10^6$ (Wang & Zheng, JFM 2016, the QLOA facility). How high is the top of the buffer layer? ($100^+ \sim 0.01$ m) (3) Consider the Princeton super pipe (Hultmark et al. 2012 PRL), where $Re_\tau = 10^5$ and the pipe radius is about 0.13 m. What is the physical size of δ_ν there? ($\sim 1\mu$m) While the dimension of the smallest hot wire is at least ~ 0.2 mm. What's its size in plus units?

5. Consider wall-bounded turbulence. What are the different layers and what are the ranges? What are the velocity and length scales for the inner and the outer layer? (Velocity scale is always u_τ and length scales δ_ν, δ)

6. Write down the basic channel equation (momentum equation). How to measure u_τ base on it and statistics of what quantities are needed? Also, u_τ can be obtained from the linear wall layer from mean velocity profile. Which of the two is better in numerical and experimental channels and why?

7. Derive the laws of the wall based on scaling analysis. Use the possible velocity and length scales. Consider dimensional analysis: $\partial_y U = u_\tau/y\Phi(y/\delta, y/\delta_\nu)$. (1) near-wall turbulence: $\partial_{y+}U^+ = \Phi(y^+)/y^+$ hence $U^+ = f(y^+)$. Wall BC gives $f(0) = 0$ and we have $\partial_{y+}U^+|_w = \partial(U/u_\tau)/\partial(y/\delta_\nu) = \nu\partial_y U|_w/u_\tau^2 = 1$ by definition ($u_\tau^2 = \tau_w/\rho$). Hence $U^+ = y^+ + HOT$. (2) log layer. Φ is neither function of y/δ nor y/δ_ν. So $\partial_{y+}U^+ = \Phi/y^+ = \Pi/y^+$ with Π being a constant determined to be the Karman constant later. Hence $U^+ = \kappa^{-1}\ln y^+ + B$. Why do we have a integration constant? (Since we don't know where to integrate from).

8. At high Reynolds number, how would the thickness of the near-wall region (say $y^+ < 50$) change? An important characteristic of high Re wallturb is the overlap between the inner ($y < 0.1\ \delta$) and outer layer ($y^+ > 50$). Give the ranges of this two layers as well as the log layer. How high the Re_τ we need to have to see a log layer that spans a decade? ($Re_\tau = 5000$)

9. What are the non-zero components of the mean shear? Estimate the scale below which the turbulence is approximately isotropic. (Corrsin scale) Corrsin scale determines whether a turbulence structure will be destroyed by mean shear first or be dissipated by turbulence first, below which the flow cannot feel the presence of the mean shear. Define the Corrsin scale by using the local mean shear $S = \partial_y U$ and the turbulent dissipation rate ε. Use dimensional analysis. ($L_C = \sqrt{\varepsilon/S^3}$)

10. Similar question and argument for the Ozmidov scale ($L_O = \sqrt{\varepsilon/N^3}$), where N is the buoyancy frequency.

11. Log layer. Final goal: how does the Kolmogorov scale grow as a function of wall distance y? (1) How does production $\mathcal{P}$ compare to dissipation ε in the log layer? ($\mathcal{P} \sim \varepsilon$) (2) Using log law, derive $\partial_y U$. (3) Obtain $\eta^+ = (\kappa y^+)^{1/4}$ ($\partial_y U = u_\tau/(\kappa y)$, and $\varepsilon \sim \mathcal{P} \sim u_\tau^3/(\kappa y)$ then we have $\eta^4 = \kappa y\delta_\nu^3$).

12. Similar question in free shear flow. These answers are important when we need to design a simulation grid for a DNS. In a round jet, the characteristic scales for centerline velocity and half width are $U_0 \sim x, \delta \sim x^{-1}$. How does η change as a function of x? Show the dependence of η/δ on the local Reynolds number. ($\eta = (\nu^3/\varepsilon)^{1/4}$, inertial scaling $\varepsilon = U_0^3/\delta$ so we have $\eta/\delta = (\nu/(U_0\delta))^{3/4} = (1/Re_L)^{3/4} = \text{cst}$ since $U_0\delta$ is a constant. Hence $\eta \sim \delta \sim x$.) The local Reynolds number Re_L is also a constant which η/δ depends on.

13. Similarly, in an axisymmetric wake. The decay of velocity deficit is $U_s \sim x^{-2/3}$ and wake width grows as $\delta \sim x^{1/3}$. The implication is that the turbulent Reynolds number ($Re_L = U_s\delta/\nu \sim x^{-1/3}$) is decaying. Similarly, $\eta/\delta \sim Re_L^{-3/4} \sim x^{1/4}$ and hence $\eta \sim x^{7/12}$ is also growing but slower than in a jet.

14. Self-similar planer (2D) free shear flows. Derive (using momentum and continuity equations) the conservation of momentum flux for a jet and the conservation of momentum deficit for a wake. Starting point could be N-S or boundary-layer-like equation. Assuming the jet width δ is linearly increasing with x, and the wake width increases as $x^{1/2}$, derive the power laws for U_s and δ. What quntataties become self-similar in a high Re sufficiently far wake/jet? And derive the connection between drag coefficient and momentum thickness in a wake.

15. Qualitative and semi-quantitative analysis. Assume a turbulent building wake. Estimate the Reynolds number, the size of the largest scale, and the smallest scale. We may assume isotropic turbulence in a coordinate travelling with the wake. Consider the DNS requirement in the previous question. What is the cost to simulate such a wake at the estimated Reynolds with DNS?

6 GFD and EFD

1. Derive the expression of buoyancy frequency by examining the motion of an oscillating parcel due to a linear density variation.

2. Derive the expression of Coriolis frequency from the following idealized equations of motion:

$$\frac{du}{dt} = -fv \tag{1}$$

$$\frac{dv}{dt} = fu \tag{2}$$

What motion it will do if Coriolis force is the dominant force as we assumed? What assumptions do we need to reach the simplification to the above equations? Note that the convection terms and pressure terms scale as ρU^2 (inertial scaling) and we need the Rossby number and Ekman number to be small. Give the definitions of these two numbers and obtain them by comparing two relevant terms in the N-S equation.

3. Define the Rossby radius of deformation and explain the physical meaning of it.

4. Bousinessq approximation. What are the assumptions and in what condition will it fall?