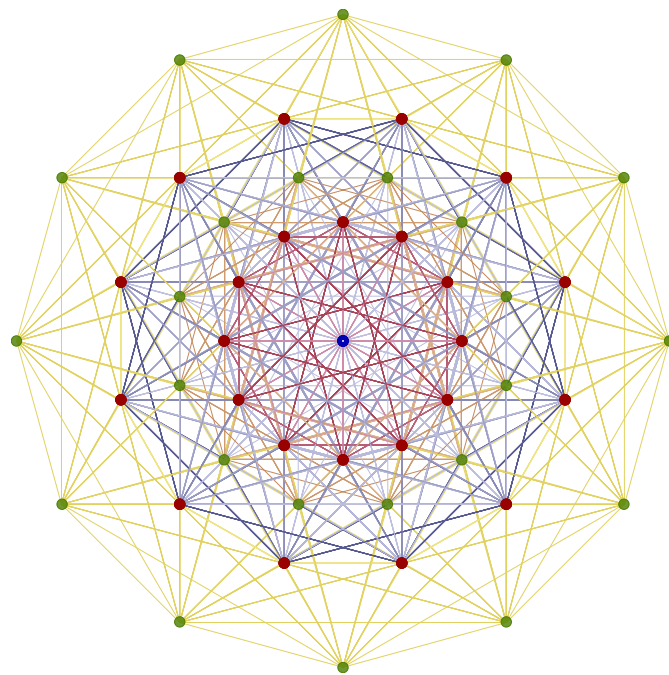


Group Theory Methods in Physics

PERSONAL NOTES



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1 Grouping things together...

Symmetry is one of the many things which both mathematicians and physicists crave to understand. And *Group Theory* is a mathematically formal way of learning about these symmetries. The basic idea is to ‘cluster’ the symmetries into some ‘groups’ and then do all kind of nasty things to them. For that, first let us define the main hero of this act: a group.

Definition 1 (Group):

A group g is a ordered pair $(G, *)$ where G is a set and $*$: $G \times G \rightarrow G$ is a binary operation satisfying:

- **Associativity:** $g_1 * (g_2 * g_3) = (g_1 * g_2) * g_3 \quad \forall g_1, g_2, g_3 \in G$
- **Identity:** There exists a unique $e \in G$ such that $\forall g \in G, e * g = g * e = g$
- **Existence of Inverse:** For every $g \in G$, there exists g^{-1} such that $g * g^{-1} = g^{-1} * g = e$
- **Closure:** $g * h \in G \quad \forall g, h \in G$

A few points to note:

- We sometimes (almost everytime) omit $*$ when the context is clear and just write gh for $g * h$ where $g, h \in G$.
- Technically $(G, *)$ is called a group but when the context is clear, we just simply refer to G as the group.
- A group G is called *Abelian* if the elements commute, that is, $\forall g_1, g_2 \in G \quad g_1 g_2 = g_2 g_1$

Let us see some examples of groups, all of which can be checked to satisfy the properties specified in the definition:

1. $(\mathbb{R} \setminus \{0\}, \times), (\mathbb{R}, +)$: the real numbers under multiplication¹ and addition form a group.
2. $GL(n, \mathbb{R}), GL(n, \mathbb{C})$: the set of all invertible $n \times n$ matrices over \mathbb{R} or \mathbb{C} field, form a general linear group under matrix multiplication.
3. $SL(n, \mathbb{R}), SL(n, \mathbb{C})$: the set of all invertible $n \times n$ matrices with determinant 1, over \mathbb{R} or \mathbb{C} field, form a special linear group under matrix multiplication.
4. $O(n), SO(n)$: set of all orthogonal $n \times n$ matrices and orthogonal matrices with determinant 1 form orthogonal and special orthogonal group.
5. S_n : the set of all permutations of n objects form a group. Permutation means arranging the same objects in some way, so basically it can be thought of as a bijection of a set onto itself. We represent these maps by π such that $\pi : \{1, 2, \dots, n\} \rightarrow \{1, 2, \dots, n\}$ and

$$\pi \equiv \begin{pmatrix} 1 & 2 & \cdots & n \\ \pi(1) & \pi(2) & \cdots & \pi(n) \end{pmatrix}$$

The above representation means that the first row, after the arrangement (acting on by map π), changes to the second row. As an example, let us consider:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 2 & 5 & 6 & 1 & 4 & 3 \end{pmatrix}$$

¹Note that under multiplication, no inverse exist for zero, so we exclude 0

The red elements are how the objects are initially there, blue are how the objects change place after permutation. Note that, under the map π , $1 \rightarrow 7, 7 \rightarrow 3, 3 \rightarrow 5, 5 \rightarrow 1$ and also $4 \rightarrow 6, 6 \rightarrow 4$ while 2 is mapped to 2 itself. This hints writing the thing in a cyclic structure, like $(1\ 7\ 3\ 5)(2)(4\ 6)$. We say 1, 7, 3, 5 form a *4-cycle* while 4, 6 form a *2-cycle*.

1.1 Symmetries of the Equilateral Triangle

This is one of the typical examples which is always mentioned in any group theory introduction, so as a reverence to the old-age custom, we include it too. To describe this, we will consider three actions on the triangle:

- Leave it alone
- Rotate it by some angle
- Flip (reflect) it about some axis.

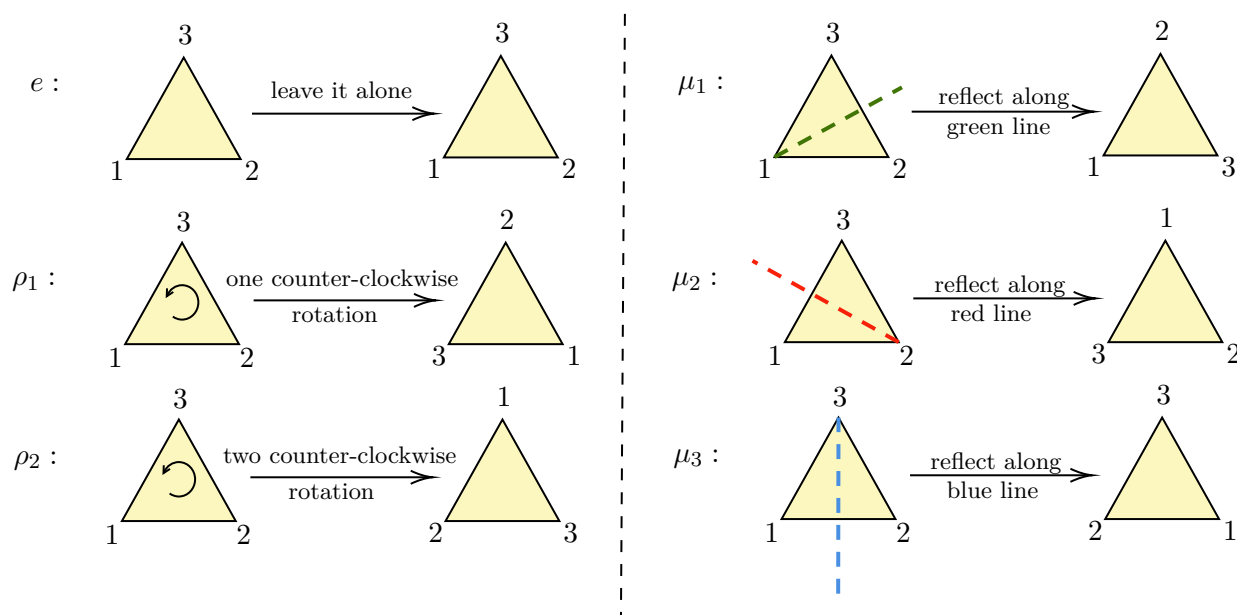


Figure 1: Diagram showing the different actions of reflection and rotations to the equilateral triangle.

Okay, so these six actions form a group for the symmetries of an equilateral triangle. To check that, we can try to form the multiplication table for these operations, which is a table demonstrating the result of the composition of the elements with each other. The table looks like something like this:

*	e	ρ_1	ρ_2	μ_1	μ_2	μ_3
e						
ρ_1						
ρ_2						
μ_1						
μ_2						
μ_3						

Now, note the following things: First, the first row and first column will be filled as it is, since acting with identity does not change anything. Second, $\rho_1^2 \equiv \rho_2$ (since it is how we defined it).

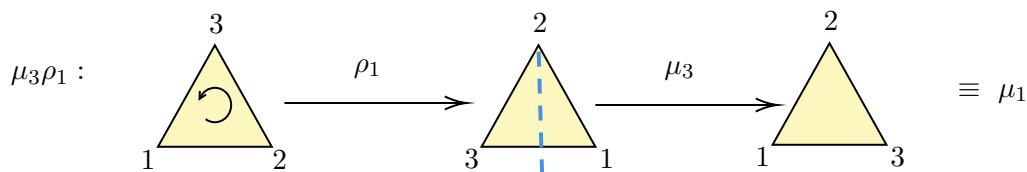
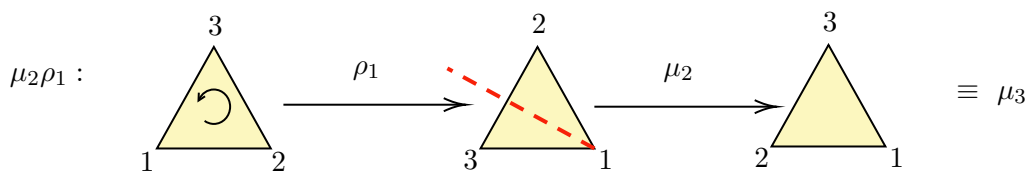
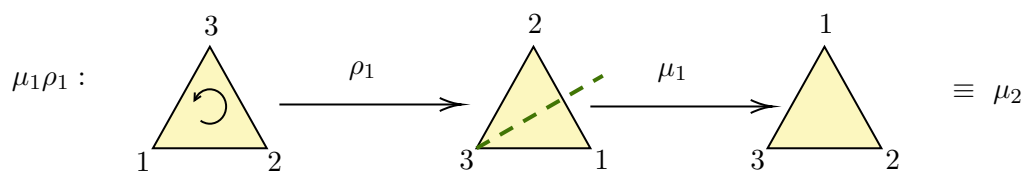
Also, note that rotations will commute with themselves and one anti-clockwise rotation is rotation by 120° , so accordingly $\rho_1 \equiv 120^\circ$, $\rho_2 \equiv 240^\circ$ and hence $\rho_2\rho_1 = \rho_1\rho_2 \equiv 360^\circ \equiv e$, $\rho_2^2 \equiv 480^\circ \equiv 120^\circ \equiv \rho_1$.

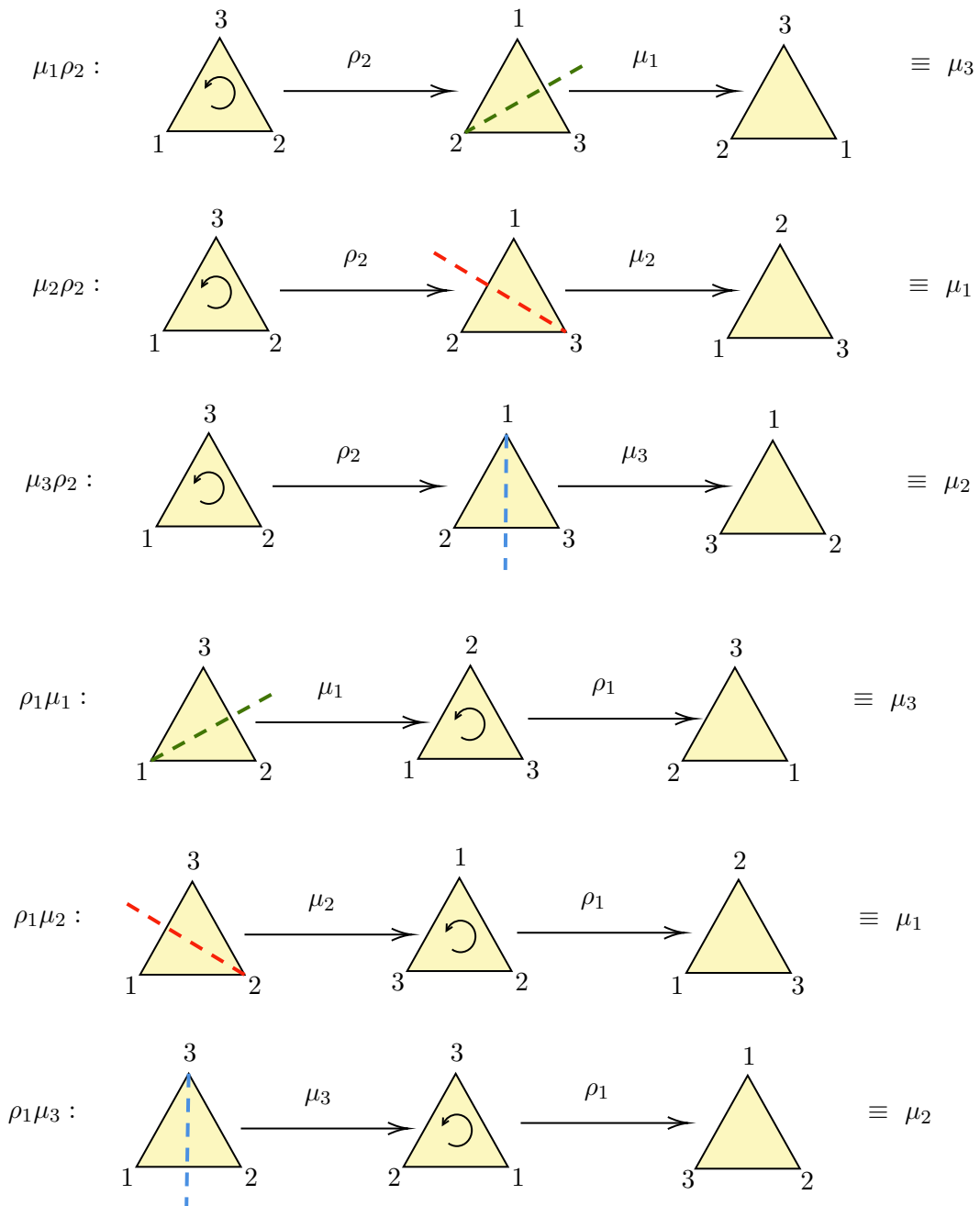
Also, reflecting along the same line twice will result in the initial configuration, so $\mu_i^2 \equiv e \forall i$. Let us now fill the table with these information:

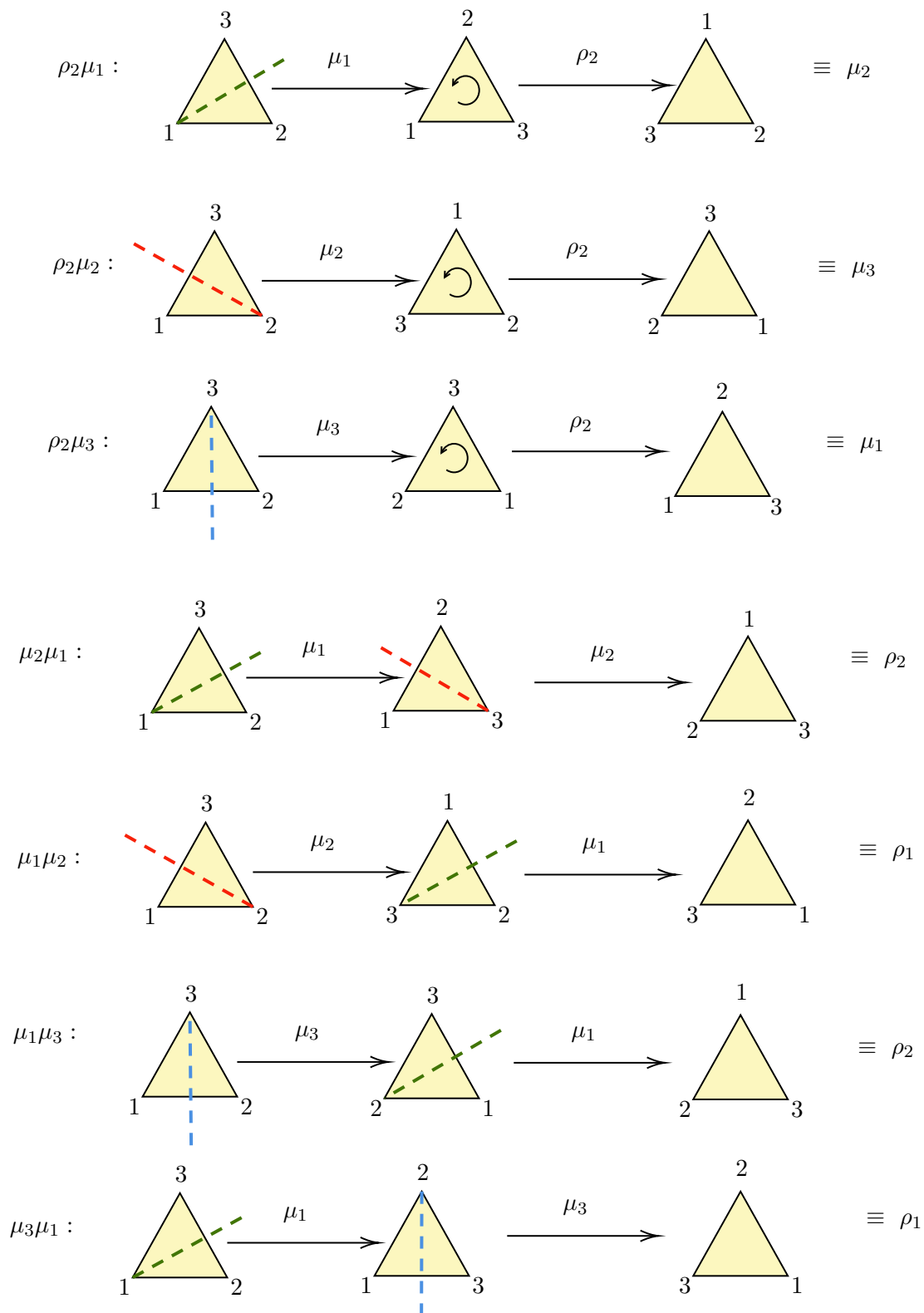
*	e	ρ_1	ρ_2	μ_1	μ_2	μ_3
e	e	ρ_1	ρ_2	μ_1	μ_2	μ_3
ρ_1	ρ_1	ρ_2	e			
ρ_2	ρ_2	e	ρ_1			
μ_1	μ_1			e		
μ_2	μ_2				e	
μ_3	μ_3					e

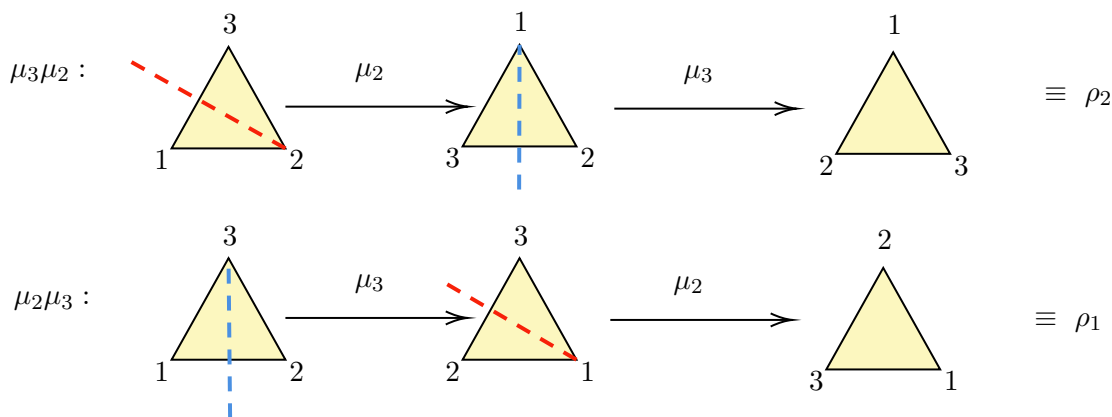
Now there are the non-intuitive actions left, combining rotation with reflections and reflections with reflections. Let us see these actions. We have to be careful with the following :

- ab means b is acted first and then a .
- When taking reflections, we should take them along the axes specified initially, not based on the current position of the index. So, μ_2 does not mean taking reflection along the axis passing through the vertex with '2' on it. μ_2 is taking reflection from the bottom-right vertex of the triangle, irrespective of what index is there on the vertex.









So, from the above diagrams, we can fill in the rest of the table. The final multiplication table thus becomes:

*	e	ρ_1	ρ_2	μ_1	μ_2	μ_3
e	e	ρ_1	ρ_2	μ_1	μ_2	μ_3
ρ_1	ρ_1	ρ_2	e	μ_3	μ_1	μ_2
ρ_2	ρ_2	e	ρ_1	μ_2	μ_3	μ_1
μ_1	μ_1	μ_2	μ_3	e	ρ_1	ρ_2
μ_2	μ_2	μ_3	μ_1	ρ_2	e	ρ_1
μ_3	μ_3	μ_1	μ_2	ρ_1	ρ_2	e

Table 1: Final multiplication table for the symmetries of an equilateral triangle

We can now check the group properties from this table. By the way, the group of symmetries of the equilateral triangle is denoted by D_3 ¹.

We will come back to this table again and again! Well, notice one thing, all elements of the group occur exactly once in each row or column of the multiplication table. This leads to a pretty nice observation that each row and column of the multiplication table is a permutation of the group elements.

Fun Fact:

Consider a finite group G with h elements. For any $g_k \in G$ the sequence $\{g_i g_k\}_{i=1}^h$ contains each group element exactly once.

Proof. Let $g_k \in G$ and choose any $g_i \in G$. Then, there exists an element $g_r = g_i g_k^{-1}$ since $g_k^{-1} \in G$ and elements of G satisfy closure property.

Then we have $g_i = g_r g_k$ and then g_i must appear in the sequence atleast once, since $r \leq h$ and g_i has the form of terms appearing in the sequence. Now, this happens for all $i = 1, \dots, h$ and there are only h terms in the sequence. Thus, each element can occur atmost once, since if any element has occurred twice, then atleast one of the elements will not have appeared in the sequence.

¹The symmetries of a regular polygon are known as dihedral groups. This group is usually denoted D_n for the symmetries of a regular n-gon. It so happens that the dihedral group of degree 3 (the group of symmetries of an equilateral triangle) is 'isomorphic' to the symmetric group, denoted by, $D_3 \cong S_3$

1.2 Subgroup

We saw what a group is. Now, let us see what a subgroup is ¹. A subset $H \subseteq G$ is called a *subgroup* if it is a group in its own right, where the binary operation is the same operation as in G but restricted to $H \times H$, that is, $*$ $\Big|_{H \times H}$

Examples

Let us see some example of subgroups.

- Consider the multiplication table of the dihedral group D_3 , again. Note that, any rotation ρ_i , when multiplied with any other rotation ρ_j will always produce another rotation. Thus, we can say that rotations are closed among themselves and the set of all rotations indeed form a subgroup of the dihedral group.
- Consider the group of integers under addition, $G = \mathbb{Z}, +$. Then all the even integers form a subgroup of G . That is, $(2\mathbb{Z}, +)$ is a subgroup of G .

Subgroups are pretty important. Let us see how to identify whether a subset of a group is indeed a subgroup or not.

Lemma 1 (One-step subgroup test):

Let G be a group and if $H \subseteq G, H \neq \emptyset$ and if $h_1 h_2^{-1} \in H \forall h_1, h_2 \in H$, then H is a subgroup of G .

Proof. Note that the restricted binary operation on H is still associative. Now, we show that H follows all the properties of a group. For that, consider $h_1, h_2 \in H \subseteq G$. As $h_1, h_2 \in G$, then h_1^{-1} and h_2^{-1} exist.

Take $h_1 = h_2 \implies h_1 h_1^{-1} = e$ and hence, from the given property, $e \in H$ (existence of identity). Then take $h_1 = e, h_2 = h \in H \implies e h^{-1} = h^{-1} \in H$ (existence of inverse). Now, take $h_1 = h', h_2 = h^{-1}$ for $h', h \in H$, then $h' (h^{-1})^{-1} = h' h \in H$ (closure property). Thus, H follows all properties of a group and hence is a subgroup of G .

Using the lemma above, we can say that the special linear group is a subgroup of the general linear group. Obviously, $SL(n, \mathbb{R}) \subseteq GL(n, \mathbb{R})$. For the other part, note that, $A, B \in SL(n, \mathbb{R}) \implies \det(A) = \det(B) = 1$. Now, $\det(AB^{-1}) = \det(A) \det(B^{-1}) = \det(A) \frac{1}{\det(B)} = 1 \implies AB^{-1} \in SL(n, \mathbb{R})$, hence proved.

1.2.1 Cyclic Subgroups

Consider a group G and let $g \in G$. Then the *cyclic subgroup* of G generated by g , is given by the set $\langle g \rangle = \{g^k | k \in \mathbb{Z}\} \subseteq G$ ². Note that the cyclic subgroup is element-specific. Let us see an example using the symmetry group of the equilateral triangle.

So, $G \equiv \{e, \rho_1, \rho_2, \mu_1, \mu_2, \mu_3\}$ is the group of symmetries of the equilateral triangle. Now,

- $\langle e \rangle = \{e\}$ (since powers of identity is always identity)
- $\langle \rho_1 \rangle = \{\rho_1, \rho_1^2 = \rho_2, \rho_1^3 = e, \dots \text{repetitions}\} \equiv \{\rho_1, \rho_2, e\}$
- Similarly, $\langle \mu_1 \rangle = \{\mu_1, e\}$
- and so on...

¹Think about it, it is natural that smaller groups are always formed within a bigger group

²We can check that this set is indeed a subgroup.

For a finite group, the cyclic subgroup must have some repetition and hence there exists $m, n \in \mathbb{N}$ such that $g^m \equiv g^{m+n}$ and hence $g^n = e$. If there always exist this kind of n , then the set $\{n \in \mathbb{N} | g^n = e\} \neq \emptyset$ has a minimum element and this is called the *order* of g (basically this is the number of elements in the subgroup generated by g). If no such finite n exist (which happens mostly when infinite groups are considered), then order is taken to be infinite.

For finite groups, we have $g^m g^n = g^{(m+n) \bmod r}$ where r is the order of g . This mimics the group of integers under addition modulo r , denoted by $\mathbb{Z}/r\mathbb{Z}$.

NOTE: The order of the group G is the number of elements in G and is denoted by $|G|$. If G is infinite, then $|G|$ is also taken to be infinite.

1.2.2 Centre of a Group

Definition 2 (Centre):

If G is a group, then the centre of the group denoted by $Z(G)$ is given by:

$$Z(G) = \{a \in G | ab = ba \forall b \in G\}$$

Basically, we are looking for the set of all elements which commute with every other element in the group. Let us see an example for \mathbb{Z}_4 , the group of integers under addition modulo 4. The multiplication table is given by:

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

Then we can check that $Z(\mathbb{Z}_4) = \{0, 1, 2, 3\} = \mathbb{Z}_4$. Note that G itself was abelian and hence the centre of the group is the entire group itself. We can also check that $Z(D_3) = \{e\}$. So we saw the centre of a group can be the entire set as well as just the identity. Can we have something in between?

It turns out that $Z(D_4) = \{e, \rho_2\}$ where D_4 is the group of symmetries of the square and ρ_2 being the action of rotation by 180° anti-clockwise.

Lemma 2:

The centre of a group $Z(G)$ is a subgroup of the group G .

Proof. We will use lemma 1 for this. Let us take $a, b \in Z(G)$ and let $k \in G$ be any arbitrary element. Then, from the definition of centre of group,

$$ak = ka \implies a = kak^{-1} \quad bk = kb \implies b = kbk^{-1} \implies b^{-1} = kb^{-1}k^{-1}$$

Then we have:

$$(ab^{-1})k = (kak^{-1})(kb^{-1}k^{-1})k = kak^{-1}kb^{-1}k^{-1}k = kaeb^{-1}e = k(ab^{-1}) \implies ab^{-1} \in Z(G)$$

1.3 Cyclic Group

Definition 3 (Cyclic Group):

A group G is called *cyclic* if there exists $g \in G$ such that $\langle g \rangle = G$. The element g is called the generator of the group.

For example, if we consider \mathbb{Z}_6 (integers under addition modulo 6), then:

$$\begin{aligned} 1^1 &\equiv 1 = 1 \\ 1^2 &\equiv 1 + 1 = 2 \\ 1^3 &\equiv 1 + 1 + 1 = 3 \\ 1^4 &\equiv 1 + 1 + 1 + 1 = 4 \\ 1^5 &\equiv 1 + 1 + 1 + 1 + 1 = 5 \\ 1^6 &\equiv 1 + 1 + 1 + 1 + 1 + 1 = 6 \equiv 0 \end{aligned}$$

After that repetitions start occurring, however, note that, we have obtained all the elements of the group. Hence we can say $\langle 1 \rangle = \mathbb{Z}_6$ and hence 1 is a generator of the cyclic group \mathbb{Z}_6 . Similarly, we can check that $\langle 5 \rangle = \mathbb{Z}_6$ ¹

Fun Fact:

For any integer $n > 1$, $\mathbb{Z}_n = \langle 1 \rangle = \langle n - 1 \rangle$

Proof. 1 and $n - 1$ both belong to \mathbb{Z}_n . Note that given $k \in \mathbb{Z}$, the operation, $1^k = k \pmod{n} \equiv -k \pmod{n} \equiv nk - k \pmod{n} \equiv (n - 1)k \pmod{n} = (n - 1)^k$. Hence, by definition of cyclic subgroup, $\langle 1 \rangle = \langle n - 1 \rangle$. We will now see a nice relation between cyclic and abelian groups. It turns out, every cyclic group is abelian.

NOTE: Sometimes, a group might not be cyclic, so it cannot be generated by a single element. However, it can have two or more generators, that is, any element of the group can be written as a finite combination of these generators. The set of all such generators is called the *generating set of a group*.

For example, let G be a group with two generators, a and b , each of order 2. Thus $a^2 = e$ and $b^2 = e$. Using the property of the group and the generators, the possible elements in the group are $G = \{e, a, b, ab, ba\}$. Thus the generating set of the group G is $\{a, b\}$.

Suppose we further impose the condition, $ab = ba$, then $G = \{e, a, b, ab\}$ is called the *Klein four-group* and is denoted by V_4 . Note that V_4 is Abelian. This is the unique non-cyclic group of order 4. Surprisingly, this is also the dihedral group of order 4, D_4 , that is, set of symmetries of a square.

Lemma 3:

Every cyclic group is an abelian group but converse is not true.

Proof. Let G be a cyclic group. Then $G = \langle a \rangle = \{a^k | k \in \mathbb{Z}\}$ for some $a \in G$. Let us take $g_1, g_2 \in G \implies \exists m, n \in \mathbb{Z}$ s.t. $g_1 = a^m, g_2 = a^n$. Then we have:

$$g_1 g_2 = a^m a^n = a^{m+n} = a^{n+m} = a^n a^m = g_2 g_1$$

For the converse, let us take the group of integers under multiplication modulo 12, $U(12) = \{1, 5, 7, 11\}$. We can check that the multiplication table is:

\times	1	5	7	11
1	1	5	7	11
5	5	1	11	7
7	7	11	1	5
11	11	7	5	1

¹A group can have multiple generators.

Since the table is symmetric, the group is abelian. However, note that:

$$\begin{aligned}\langle 1 \rangle &= \{1\} \\ \langle 5 \rangle &= \{5, 1\} \\ \langle 7 \rangle &= \{7, 1\} \\ \langle 11 \rangle &= \{11, 1\}\end{aligned}$$

Thus, this is not a cyclic group, since none of the elements generate the entire group and hence, converse of the lemma does not hold true.

1.3.1 Infinite Cyclic Groups

Let us consider the infinite group $(\mathbb{Z}, +)$ ¹. Then, note that $\mathbb{Z} = \langle 1 \rangle$, that is, \mathbb{Z} is cyclic group. And, order of every element is infinite, apart from 0.

Lemma 4:

If G is an infinite cyclic group generated by $a \in G$, then order of g is infinite for all $g \in G, g \neq e$.

Proof. We have $G = \{a^k | k \in \mathbb{Z}\}$. Consider $e \neq g \in G$, then $g = a^m$ for some $m \in \mathbb{Z} \setminus \{0\}$. Now, for any $l \in \mathbb{Z}$, $g^l = (a^m)^l = a^{ml}$. For order to be finite, $g^l = e \implies a^{ml} = e \implies ml = 0$ but $m \neq 0$ forces l to be zero. Thus, no positive powers exist such that $g^l = e$ and hence, the order of the group is infinite.

Lemma 5:

Let G be an infinite cyclic group. If $a \neq e \in G$ and a has infinite order, then $\forall u, v \in \mathbb{Z}, a^u = a^v \iff u = v$. If G is finite and order of a is n , then $a^u = a^v \iff n | (u - v)$

Proof. (\implies) For infinite case: Let $a^u = a^v \implies a^u(a^v)^{-1} = e \implies a^u a^{-v} = e \implies a^{u-v} = e$. Since order of a is infinite, $a^{u-v} = e$ iff $u - v = 0 \implies u = v$.

For finite case: $a^{u-v} = e \implies (u - v) = 0 \pmod n$

(\impliedby) $u = v \implies a^u = a^v$ lol 😊

1.3.2 Group Presentation

1.4 Cosets

For this, let us focus on something related to lemma 1. Given a group G and its subgroup H , consider a relation $\sim_L \subseteq G \times G$ such that for $a, b \in G, a \sim_L b \iff a^{-1}b \in H$. We show the following:

Proposition 1:

Let G be a group and let H be a subgroup of G . Define the relations \sim_L and \sim_R such that $a \sim_L b \iff a^{-1}b \in H$ and $a \sim_R b \iff ab^{-1} \in H$. Then, \sim_L and \sim_R are equivalence relations on G .

Proof. For \sim_L :

- $a \in G \implies a^{-1}a = e \in H \implies a \sim_L a$ (reflexive)

¹This can be thought of as group of integers under addition modulo 1

- $a \sim_L b \implies a, b \in G, a^{-1}b \in H \implies (a^{-1}b)^{-1} \in H \implies b^{-1}a \in H \implies b \sim_L a$
(symmetric)
- $a \sim_L b, b \sim_L c \implies a^{-1}b \in H, b^{-1}c \in H \implies (a^{-1}b)(b^{-1}c) = a^{-1}c \in H$ (transitive)

Similarly we can show for \sim_R . Whenever we define any equivalence relation, we at once go into the equivalence classes (since these form a partition of the set). We will now see how the equivalence classes look for these relations.

Lemma 6:

The equivalence class of an element $a \in G$ under the relation \sim_L is given by $[a] = \{ah \mid h \in H\}$

Proof. Let $x \in [a]$, then $x \sim_L a$. Then, by symmetricity, $a \sim_L x \implies a^{-1}x \in H$. Hence $a^{-1}x = h$ for some $h \in H$.

From this, we have $x = ah$ which implies that $x \in \{ah \mid h \in H\} \implies [a] \subseteq \{ah \mid h \in H\}$. Now, let us prove the opposite way:

Let $x \in \{ah \mid h \in H\} \implies x = ah$ for some $h \in H$. This implies that $a^{-1}x = h \in H \implies x \sim_L a \implies x \in [a]$. Thus, $\{ah \mid h \in H\} \subseteq [a]$. From these two things, we can say:

$$[a] = \{ah \mid h \in H\}$$

Similarly, for \sim_R , we will have $[b] = \{hb \mid h \in H\}$. This motivates us to the definition of cosets.

Definition 4 (Cosets):

Let G be a group and H be a subgroup of G . Let $a \in G$, then the *left coset* of H in G is given by $aH = \{ah \mid h \in H\}$ and the *right coset* of H is given by $Ha = \{ha \mid h \in H\}$

We will mostly consider left cosets only. Things about right cosets follow similarly. We find that cosets are nothing but the equivalence classes for the above defined relations. Since equivalence classes are disjoint unless two elements are related, then, two cosets will either coincide or are disjoint. Since we know that equivalence classes partition the set, the set of all distinct cosets partition the group G . Let us denote the set of distinct cosets by $\{g_1H, g_2H \dots g_kH\}$, considering that H is finite. Thus we can write,

$$G = \bigcup_{i=1}^k g_iH$$

since distinct cosets partition the group. Since each set g_iH is distinct, the order of their union is equal to the sum of order of each set, which is equal to the order of the subgroup $|H|$. Then we obtain $|G| = k|H|$, where k is the number of distinct cosets. This is *Lagrange's Theorem* which states that $|H|$ divides $|G|$. The integer k is called the *index* of the group and is denoted by $[G : H]$.

Lemma 7:

Order of an element divides the order of the group

Proof. Consider a group G of order k and $g \in G$. Then, $\langle g \rangle$ is the cyclic subgroup generated by g which has an order, say n . From Lagrange's theorem, $|\langle g \rangle|$ divides k . Observe that $|\langle g \rangle| = n$ and hence $n|k$.

Having seen a lot of derivations, let us see an example of coset. For this, consider the dihedral group again, however, we will write it in a different form. $G = \{e, \rho_1, \rho_2, \mu_1, \mu_2, \mu_3\}$. Consider ρ_2 and μ_1 . Then $\rho_1 = \rho_2^2 \equiv$ rotation by 60° . Also, from the multiplication table, we can see $\mu_3 = \mu_1\rho_2$ and $\mu_2 = \mu_1\rho_2^2$. Hence we can write each element in the group using μ_1 and ρ_2 . Renaming them as a and b , we get the group as,

$$G = \{e, a, b, b^2, ab, ab^2\}$$

What is the use for this alternate renaming? Well, now we can say the G is generated by two generators a and b with the following relations, $a^2 = b^3 = e$ and $ab = b^2a$. From the second relation, we can obtain $b = ab^2a$ and $ba = ab^2a^2 = ab^2$, which we will use below.

Now, consider the subgroup $H = \{e, a\} \subseteq G$. Then $aH = \{e, a\}$, $bH = \{b, ba\}$, $b^2H = \{b^2, b^2a = ab\}$, $(ab)H = \{ab, aba = b^2a^2 = b^2\} = b^2H$, $(ab^2)H = \{ab^2, ab^2a = b\}$. The red coloured ones are the distinct cosets and we can write,

$$G = aH \cup b^2H \cup (ab^2)H$$

1.5 Conjugate Subgroups

In physics we often do similarity transformations. Given an operator \hat{O} , we perform $S\hat{O}S^{-1}$ which keeps the eigenvalues invariant. There's a similar notion for groups too. Suppose we have a group G and let $H \subseteq G$ be a subgroup. Then, for $a \in G$, $aHa^{-1} \subseteq G$ is also a subgroup. This is called the subgroup *conjugate* to H . Why is this a subgroup?

Let us use the subgroup test for this. Since H is already given to be a subgroup, $\forall h_1, h_2 \in H \implies h_1h_2^{-1} \in H$. Let $x, y \in aHa^{-1}$ then,

$$x = aha^{-1} \quad y = aka^{-1} \quad \text{for some } h, k \in H$$

Then, $xy^{-1} = (aha^{-1})(ak^{-1}a^{-1}) = a(hk^{-1}a^{-1})$ and since H is a subgroup, hk^{-1} belongs to H and hence $xy^{-1} \in aHa^{-1}$.

If in any case, it happens that $aHa^{-1} = H$, then H is called *self-conjugate*. Trivially,

$$gHg^{-1} = H \iff gH = Hg$$

and hence self-conjugate subgroups have their left and right cosets equal. These are also called *normal subgroups* or *invariant subgroups*.

1.6 Homomorphism

Definition 5:

Let $(G, *)$ and (H, \cdot) be two groups. Then, a map $f : G \rightarrow H$ is called a *group homomorphism* if $f(a * b) = f(a) \cdot f(b) \forall a, b \in G$

The set of all homomorphisms between two algebraic structures (like a group) A and B is denoted by $Hom(A, B)$ while the set of all bijective homomorphisms is called an *isomorphism* and denoted by $Iso(A, B)$. If the homomorphism is on the same space, we call it an *endomorphism* and the set is denoted by $End(A, B)$. Bijective endomorphisms are called *automorphisms* and the set is denoted by $Aut(A)$.

1.7 Lie Groups

Definition 6 (Lie Group):

A *Lie Group* is a group $(G, *)$ where G is a differentiable manifold and the maps $\mu : G \times G \rightarrow G, (g, h) \mapsto g * h$ and $\iota : G \rightarrow G, g \mapsto g^{-1}$ are both smooth.

Hence, for a Lie group, multiplication and inverse maps are smooth. Examples of Lie groups include $GL(n, \mathbb{R})$ or $SL(n, \mathbb{R})$ and many more. In general, study of Lie groups becomes important (atleast in the physical sense) since these provide a way to express the concept of a continuous family of symmetries for geometric objects.

Definition 7 (Lie Algebra):

Let G be a Lie group. Since G is also a manifold, then the tangent space at identity $e \in G$ is defined to be the *Lie Algebra* of G and is denoted by:

$$\mathfrak{g} := T_e G$$

2 When Groups said, "represent us!"

Well, most often we do not see the group itself, but rather, its manifestation. Like, say an element of the rotation group causes rotation of some object, due to which we are able to see its effects. To this extent, we define a group action.

Definition 8 (Group Action):

A group $(G, *)$ is said to act on some set X if there exists a map $\phi : G \times X \rightarrow X$ such that:

1. $\phi(e, x) = x$, that is, action of identity element of group is to map the element of X to itself.
2. $\phi(g, \phi(h, x)) = \phi(g * h, x)$, that is, action of a group element on another action is combination of the two actions.

Group actions just cause permutations to the elements of X . Consider the example of \mathbb{R}^3 , where the inner-product preserving group actions are just *translations* and *rotations*.

Let f be a function on X . Then the group action on the function is defined by:

$$(g \cdot f)(x) = f(g^{-1}x)$$

It is trivial that $e \cdot f(x) = f(x)$. Also,

$$\begin{aligned} g_1 \cdot (g_2 \cdot f)(x) &= g_2 \cdot f(g_1^{-1}x) \\ &= f(g_2^{-1} * g_1^{-1}x) \\ &= f((g_1 * g_2)^{-1}x) \\ &= (g_1 * g_2)f(x) \end{aligned}$$

Hence, using the definition as above, it is indeed a group action on the functions on X .

Definition 9 (Representation):

A *representation* (π, V) of a group G is a group homomorphism defined as,

$$\pi : G \rightarrow GL(V)$$

where $GL(V)$ is the set of invertible linear maps $V \rightarrow V$ and V is a vector space.

Note that, if V is finite-dimensional and a basis of V is chosen, then the set of invertible linear maps can be seen in terms of matrices, which provide an isomorphism:

$$GL(V) \cong GL(n, \mathbb{R})$$

where $GL(n, \mathbb{R})$ is the group of $n \times n$ invertible, real matrices. In most cases, G will be a Lie group and the representation π will be a smooth map.

There can be different representations of the group. The *trivial* representation is such that $g \xrightarrow{\pi} \text{id}$, where id is the identity map. Given two representations π_1 and π_2 of dimensions n_1 and n_2 respectively, we can define the *direct-sum* representation which is of dimension $n_1 + n_2$ by the homomorphism:

$$(\pi_1 \oplus \pi_2) : G \rightarrow GL(V_{\pi_1} \oplus V_{\pi_2}) \quad \text{s.t.} \quad g \mapsto \begin{pmatrix} \pi_1(g) & 0 \\ 0 & \pi_2(g) \end{pmatrix}$$

This can be thought of as the representation matrix being a block diagonal matrix, with each sub-representation matrix occupying the diagonal blocks.

Definition 10 (Irreducible Representation):

An *irreducible representation* (or irr rep.) π is one which has no sub-representations, that is, $\nexists W \subset V$ such that the restricted function $\pi|_W$ is a representation.

It turns out that if π is an irreducible representation, it cannot be written as a direct-sum of two other representations. It makes sense, since direct-sum representation does not convey any new message and is not that interesting. Hence, looking at irreducible representations makes more sense.

Lemma 8 (Schur's First Lemma):

Let $\pi : G \rightarrow GL(V)$ and $\phi : G \rightarrow GL(U)$ be two irreducible representations of a finite group G . Then, if a matrix A is such that:

$$A\pi(g) = \phi(g)A \quad \forall g \in G$$

then, either $A = 0$ or A is invertible (and hence two representations are equivalent).

Lemma 9 (Schur's Second Lemma):

Let π be an irreducible, complex, representation of a group G and M is a linear map over the complex field, that commutes with $\pi(g)$ for all $g \in G$. Then, M can atmost be a scalar multiplication with the identity, that is,

$$M = \lambda \mathbf{1}, \quad \lambda \in \mathbb{C}$$

Proof. Since M is over the field of complex numbers, it must have atleast one eigenvalue which we call α . Then $\det(M - \lambda \mathbf{1}) = 0$ and hence $M - \lambda \mathbf{1}$ is not invertible.

Since $M\pi(g) = \pi(g)A$ for all g , we also have $(M - \lambda \mathbf{1})\pi(g) = \pi(g)(M - \lambda \mathbf{1})$ for all g . Then from Schur's first lemma, we have $M - \lambda \mathbf{1} = 0$ since it is not invertible. Thus, $M = \lambda \mathbf{1}$ for some $\lambda \in \mathbb{C}$

A corollary of the above lemma says that if G is a commutative group, then all of its irreducible representations will be one-dimensional.

Proof. Let G be a commutative group and π be its irreducible representation. Then, since group representation is a homomorphism, we have:

$$\pi(gh) = \pi(g)\pi(h) = \pi(hg) = \pi(h)\pi(g) \quad \forall g, h \in G$$

The third step comes from the commutativity of the group elements. Now, fix h and hence, for all g , we have the commutation relation holding true. This implies that $\pi(h) = \lambda_h \mathbb{1}$ from second lemma. Since h was arbitrary, all irreducible representations are one-dimensional $\pi(h) = \lambda_h \in \mathbb{C}$.

NOTE: The proof depends on existence of eigenvalues which will only be true for complex fields. For real vector spaces, this will in general not be true!

2.1 Representations of $U(1)$

Consider rotations in 2D plane by some angle θ . The most common entity we are reminded of is $e^{i\theta}$. It turns out that the group of rotations is a Lie group, denoted by $U(1)$, since those can be thought of as elements of a 1×1 matrices.

The elements of the group $U(1)$ are points on the unit-circle, labelled as $e^{i\theta}$ where θ can be any real number but θ and $\theta + 2n\pi$ denote the same element, where $n \in \mathbb{Z}$.

Since $U(1)$ is a commutative group, all its representations will be one-dimensional as seen before and is given by:

$$\pi : U(1) \rightarrow GL(1, \mathbb{C})$$

where $GL(1, \mathbb{C})$ is the group of all invertible complex numbers, that is, non-zero complex numbers. All the irreducible representations of $U(1)$ is unitary, that is, $\pi_k(\theta) = e^{ik\theta}$, $k \in \mathbb{Z}$

2.1.1 Charge Operator and Symmetry

3 Lie Algebra and their representations

We had earlier defined Lie Groups and Lie algebras using a differential geometric viewpoint. Here, we will consider some matrix groups G , where the elements considered will be invertible matrices defined over some field, that is, $G \subseteq GL(n, K)$, is a closed subgroup of $GL(n, K)$. Actually definition comes from the [Closed Subgroup Theorem](#) which states that a closed subgroup of a Lie group is also a Lie group. Since $GL(n, \mathbb{C}$ or $\mathbb{R})$ is a Lie group, the definition follows. We will consider the complex field only for the definitions. Then we have the definition of the Lie algebra as:

Definition 11 (Lie Algebra):

Let $G \subseteq GL(n, \mathbb{C})$ be a Lie group, then its corresponding Lie algebra is defined as:

$$\mathfrak{g} := \{X \in M_n(\mathbb{C}) \mid e^{tX} \in G \forall t \in \mathbb{R}\}$$

where the map e^A for any matrix A is defined in the following way:

$$e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!}$$

NOTE: The Lie Algebra of a group A is represented by the fraktur symbol of the same letter as that of the group, that is, \mathfrak{a} .

Now let us try to reconcile with the manifold definition a bit. For that, using the definition of the exponential map, we can obtain,

$$\left. \frac{d}{dt} e^{tX} \right|_{t=0} = X$$

Thus, if we define the curve $\gamma_X : \mathbb{R} \rightarrow G$ such that $t \mapsto e^{tX}$, then we have that $\frac{d}{dt} \gamma_X(t) = X$. Thus, if we think of G as a surface, then the tangent direction at identity $\mathbb{1} \in G$ (note that for $t = 0, t \mapsto e^0 = \mathbb{1}$) is given by its Lie algebra X .

Definition 12 (Adjoint Representation):**Definition 13 (Lie Bracket):**

The Lie bracket associated with a Lie algebra \mathfrak{g} is the bilinear anti-symmetric map,

$$[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$$

such that $(X, Y) \mapsto XY - YX$

We can indeed check that given $X, Y \in \mathfrak{g}$, $XY - YX$ belongs to the Lie algebra too.

The Lie algebra of a group forms a real vector space and the basis elements of this vector space are called the *generators* of the group since any group element can be generated by exponentiating these generators (although some subtleties are there, it is only applicable for group elements in the identity components for groups with multiple connected components)

Now, since the Lie bracket of two elements of the Lie algebra is also an element of the Lie algebra, the same follows for the generators also as these form the basis of the Lie algebra.

Suppose T^a and T^b are generators of a Lie algebra \mathfrak{g} , then $[T^a, T^b] \in \mathfrak{g}$ and hence this can be written as a linear combination of the basis elements. Thus, we have:

$$[T^a, T^b] = i f^{ab}_c T^c$$

where f^{ab}_c are called the *structure constants* of the Lie algebra and are independent of the representation used, although the generators are heavily dependent on the representation.

3.1 Examples of Lie algebras

- Let us consider $G = \text{GL}(n, \mathbb{C})$ and take any $X \in \text{M}_n(\mathbb{C})$. It can be shown that for any such X , $\exp(X)$ is invertible and hence belongs to $\text{GL}(n, \mathbb{C})$. Thus, the Lie algebra of $\text{GL}(n, \mathbb{C})$ is given by $\mathfrak{gl}(n, \mathbb{C}) := \text{M}_n(\mathbb{C})$
- Let us consider the orthogonal group $G = \text{O}(n)$ which consists of matrices Ω such that

$$\Omega^T \Omega = \mathbb{1}$$

From this we can have $\det \Omega = \pm 1$. Suppose $X \in \mathfrak{o}(n) \implies e^{tX} \in \text{O}(n)$ from the definition of Lie algebra. From this, we have the following:

$$(e^{tX})^T (e^{tX}) = \mathbb{1} \implies (e^{tX^T}) (e^{tX}) = \mathbb{1} \implies \frac{d}{dt} ((e^{tX^T}) (e^{tX})) = 0 \implies A^T e^{tA^T} e^{tA} + e^{tA^T} A e^{tA} = 0$$

Taking $t = 0$, we see that the Lie algebra then consists of real anti-symmetric matrices.

$$\mathfrak{o}(n) := \{X \in \text{M}_n(\mathbb{R}) | X^T = -X\}$$

- For $\text{SO}(n)$ also, the Lie algebra is the same as that of $\text{O}(n)$. Note that $\text{SO}(n)$ is the component of $\text{O}(n)$ containing the identity and since Lie algebras are concerned with what happens close to the identity, intuitively we can see that the Lie algebras for both the groups are the same.

4 Few things about Lorentz Group

We will discuss about Lorentz group here. The Lorentz group, denoted by $O(1, 3)$, is defined to be the group of all transformations (Lorentz transformation or LT) that preserves the Minkowski metric, that is,

$$\Lambda^T \eta \Lambda = \eta \iff \eta_{\mu\nu} \Lambda^\mu_\alpha \Lambda^\nu_\beta = \eta_{\alpha\beta} \quad (1)$$

Note that, as long as we are not using the indices, we are talking about the *abstract* group element. However, when we consider the expression using index notation, we are essentially considering the matrix representation of the group element using the *standard basis* of the Minkowski space¹.

From the above definition, we can instantly see that for any $\Lambda \in O(1, 3)$, $\det \Lambda = \pm 1$. For $\det \Lambda = +1$, we refer to them as *proper LT* (denoted by L_+) and for $\det \Lambda = -1$ (denote by L_-), we refer to them as *improper LT*.

In Eqn. 1, if we take $\alpha = \beta = 0$, we get:

$$1 = \eta_{\mu\nu} \Lambda^\mu_0 \Lambda^\nu_0 = (\Lambda^0_0)^2 - \sum_{i=1}^3 (\Lambda^i_0)^2 \implies (\Lambda^0_0)^2 \geq 1$$

Thus, we can have $\Lambda^0_0 \geq 1$ which we refer to as *orthochronous* (denoted by L^\uparrow) or $\Lambda^0_0 \leq -1$ which we call *antichronous* (denoted by L^\downarrow).

From the above two classifications using the signs of the determinant and Λ^0_0 , we can thus have four disconnected components of the Lorentz group:

- L^\uparrow_+ : These are the proper orthochronous LT. The identity $\mathbb{1}$ is contained in this subgroup. It is denoted by $SO^+(1, 3)$ and is continuously connected to the identity, that is, we can build up any finite Lorentz transformation by making many consecutive small Lorentz transformations. Sometimes, as an abuse of terminology, this subgroup is itself called the *Lorentz group*, since this contains the only physical transformations.
- L^\uparrow_- : These are written as a product of the *parity* and L^\uparrow_+ transformations.
- L^\downarrow_+ : These contain both time and space inversion and hence are written as a product of *PT* and L^\uparrow_+ transformation.
- L^\downarrow_- : These contain only time inversion and hence are written as a product of *time reversal* and L^\uparrow_+ transformation.

Thus, the entire Lorentz group can be written symbolically:

$$L = L^\uparrow_+ \cup PL^\uparrow_+ \cup PTL^\uparrow_+ \cup TL^\uparrow_+$$

Note that except the proper orthochronous LT, none of the other component can form subgroups since these do not contain the identity element.

4.1 Spacetime Representation

For the time being, we will consider only the proper, orthochronous Lorentz group, that is, $SO^+(1, 3)$ which is a Lie group. Since this is continuously connected to the identity, let us consider an infinitesimal transformation from the identity given by,

$$\Lambda^\mu_\nu = \delta^\mu_\nu + \omega^\mu_\nu$$

¹The Minkowski space is denoted by $\mathbb{R}^{1,3}$ which is equal to \mathbb{R}^4 along with the associated metric $\eta_{\mu\nu}$. (1, 3) refers to the metric signature, that is, 1 positive entry and 3 negative entries viz. $\text{diag}(+1, -1, -1, -1)$

Substituting this in Eq. 1 and retaining terms upto the first order in ω , we have the following:

$$\begin{aligned}
\eta_{\alpha\beta} &= \eta_{\mu\nu}(\delta^\mu_\alpha + \omega^\mu_\alpha)(\delta^\nu_\beta + \omega^\nu_\beta) \\
&= \eta_{\alpha\beta} + \delta^\mu_\alpha \eta_{\mu\nu} \omega^\nu_\beta + \eta_{\mu\nu} \delta^\nu_\beta \omega^\mu_\alpha \\
&= \eta_{\alpha\beta} + \delta^\mu_\alpha \omega_{\mu\beta} + \delta^\nu_\beta \omega_{\nu\alpha} \\
&= \eta_{\alpha\beta} + \omega_{\alpha\beta} + \omega_{\beta\alpha}
\end{aligned}$$

From this, we get that infinitesimal displacement ω must be anti-symmetric, that is,

$$\boxed{\omega_{\alpha\beta} = -\omega_{\beta\alpha}}$$

We know that antisymmetric matrices have six independent elements (since diagonals are zero and upper-diagonal has $(16 - 4)/2 = 6$ elements and lower diagonal are negative of the upper diagonal) and hence the basis of antisymmetric matrices consists six elements. Of these six, three corresponds to *rotations* while the other three corresponds to *boosts*

Let us consider this basis of the second rank anti-symmetric tensor $\{\mathcal{J}^{\rho\sigma}\}$ which forms the *generators* of the transformation. These are antisymmetric in the indices ρ, σ and hence there are six of them. These can be written as:

$$(\mathcal{J}^{\rho\sigma})^\mu_\nu := \eta^{\mu\rho} \delta^\sigma_\nu - \eta^{\mu\sigma} \delta^\rho_\nu$$

As ω is antisymmetric, we can write it as a linear combination of the basis elements, that is,

$$\omega^\mu_\nu = -i\Omega_{\rho\sigma} (\mathcal{J}^{\rho\sigma})^\mu_\nu$$

Then we can write the infinitesimal transformation in the following way:

$$\Lambda = \mathbb{1} - i\Omega_{\rho\sigma} \mathcal{J}^{\rho\sigma}$$

Any finite Lorentz transformation can be generated by exponentiating the generators, that is,

$$\Lambda_{\text{finite}} = \exp\left(-\frac{i}{2}\Omega_{\rho\sigma} \mathcal{J}^{\rho\sigma}\right)$$

Once we found the generators, any element of the Lie algebra of the Lorentz group will be a linear combination of them. It is a good time for us to calculate the Lie bracket since it is an essential characteristic of the Lie algebra. Since any element is a linear combination of the generators, it is sufficient to calculate the Lie bracket for the generators only, which we can do by the explicit form that we wrote earlier.

$$\begin{aligned}
[\mathcal{J}^{\mu\nu}, \mathcal{J}^{\rho\sigma}]^\alpha_\beta &= (\mathcal{J}^{\mu\nu} \mathcal{J}^{\rho\sigma})^\alpha_\beta - (\mathcal{J}^{\rho\sigma} \mathcal{J}^{\mu\nu})^\alpha_\beta \\
&= (\mathcal{J}^{\mu\nu})^\alpha_\theta (\mathcal{J}^{\rho\sigma})^\theta_\beta - (\mathcal{J}^{\rho\sigma})^\alpha_\theta (\mathcal{J}^{\mu\nu})^\theta_\beta \\
&= (\eta^{\mu\alpha} \delta^\nu_\theta - \eta^{\nu\alpha} \delta^\mu_\theta)(\eta^{\rho\theta} \delta^\sigma_\beta - \eta^{\sigma\theta} \delta^\rho_\beta) - (\eta^{\rho\alpha} \delta^\sigma_\theta - \eta^{\sigma\alpha} \delta^\rho_\theta)(\eta^{\mu\theta} \delta^\nu_\beta - \eta^{\nu\theta} \delta^\mu_\beta) \\
&= \left\{ \eta^{\mu\alpha} \delta^\nu_\theta \eta^{\rho\theta} \delta^\sigma_\beta - \eta^{\mu\alpha} \delta^\nu_\theta \eta^{\sigma\theta} \delta^\rho_\beta - \eta^{\nu\alpha} \delta^\mu_\theta \eta^{\rho\theta} \delta^\sigma_\beta + \eta^{\nu\alpha} \delta^\mu_\theta \eta^{\sigma\theta} \delta^\rho_\beta \right\} \\
&\quad - \left\{ \eta^{\rho\alpha} \delta^\sigma_\theta \eta^{\mu\theta} \delta^\nu_\beta - \eta^{\rho\alpha} \delta^\sigma_\theta \eta^{\nu\theta} \delta^\mu_\beta - \eta^{\sigma\alpha} \delta^\rho_\theta \eta^{\mu\theta} \delta^\nu_\beta + \eta^{\sigma\alpha} \delta^\rho_\theta \eta^{\nu\theta} \delta^\mu_\beta \right\} \\
&= \left\{ \eta^{\mu\alpha} \eta^{\rho\nu} \delta^\sigma_\beta - \eta^{\mu\alpha} \eta^{\sigma\nu} \delta^\rho_\beta - \eta^{\nu\alpha} \eta^{\rho\mu} \delta^\sigma_\beta + \eta^{\nu\alpha} \eta^{\sigma\mu} \delta^\rho_\beta \right\} \\
&\quad - \left\{ \eta^{\rho\alpha} \eta^{\mu\sigma} \delta^\nu_\beta - \eta^{\rho\alpha} \eta^{\nu\sigma} \delta^\mu_\beta - \eta^{\sigma\alpha} \eta^{\mu\rho} \delta^\nu_\beta + \eta^{\sigma\alpha} \eta^{\nu\rho} \delta^\mu_\beta \right\} \text{complete proof!}
\end{aligned}$$

From this, we find out the Lie bracket between the generators of the Lie Algebra of $\text{SO}(1, 3)$:

$$[\mathcal{J}^{\mu\nu}, \mathcal{J}^{\rho\sigma}] = \eta^{\nu\rho} \mathcal{J}^{\mu\sigma} - \eta^{\mu\rho} \mathcal{J}^{\nu\sigma} - \eta^{\nu\sigma} \mathcal{J}^{\mu\rho} + \eta^{\mu\sigma} \mathcal{J}^{\nu\rho}$$

We had obtained the six generators for the Lie algebra and now let us rearrange these into two separate parts:

$$J^i = \frac{i}{2} \varepsilon^{ijk} \mathcal{J}^{jk} \quad K^i = i \mathcal{J}^{i0}$$

In terms of these newly defined quantities, the Lie bracket becomes **check!**:

$$\begin{aligned} [J^i, J^j] &= i \varepsilon^{ijk} J^k \\ [K^i, K^j] &= -i \varepsilon^{ijk} J^k \\ [J^i, K^j] &= i \varepsilon^{ijk} K^k \end{aligned}$$