One-relator groups and the coherence property

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KIAS workshop on One RElator groups and other Aspects of GGT

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Theorem 2 (The Freiheitssatz)

If $G = F/\langle\langle w \rangle\rangle$ is a one-relator group and $A \leqslant F$ is a free factor not containing a conjugate of w, then the natural map $A \to G$ is injective.

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In the decades that followed, significant contributions to the theory came from work of G. Baumslag, Lyndon, Magnus, Moldavanskii, Newman, Schupp, Solitar and many others.

Some well known examples:

1. Free groups.
2. Surface groups:

$$\left\langle a_1, b_1, \ldots, a_g, b_g \mid \prod_{i=1}^{g} [a_i, b_i] \right\rangle, \quad \left\langle a_1, \ldots, a_n \mid \prod_{i=1}^{n} a_i^2 \right\rangle.$$

3. Many knot groups. Torus knot and torus link groups:

$$\langle a, b \mid a^m = b^n \rangle.$$

4. Many 2-knot groups (i.e. $\pi_1(S^4 - S^2)$: if $w \in [F(a,b), F(a,b)]$ is any element, then $\langle a, b \mid b = w \rangle$ is a 2-knot group.

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$$\langle a, b, c \mid a^5 b^3 c^2 \rangle, \quad \langle a, b, c \mid a = [a,b][c,b] \rangle.$$

7. Baumslag–Gersten groups:

$$\mathrm{BG}(m,n) = \langle a, t \mid (a^t)^{-1} a^m (a^t) = a^n \rangle.$$

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At the same time, they have a long history of study and a substantial enough theory so that many conjectures are approachable within the class of one-relator groups.

Often, resolutions of conjectures for one-relator groups have led to significant development of theory and techniques that are not specific to one-relator groups.

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Often, resolutions of conjectures for one-relator groups have led to significant development of theory and techniques that are not specific to one-relator groups.

The main aim of these talks will be to explain the main ingredients that go into the proof of the following:

Theorem 3 (Jaikin-Zapirain–L '23)

One-relator groups are coherent. That is, all their finitely generated subgroups are finitely presented.

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The Magnus hierarchy

The main tool used to study one-relator groups over the last century is the *Magnus hierarchy*, introduced by Magnus in 1930 to prove the Freiheitssatz:

Take a one-relator group G. Embed G into another one-relator group G_0' which splits as a HNN-extension over a one-relator group G_1 with shorter relator length. Repeat until we reach a one-relator group $G_N \cong \mathbb{Z}/n\mathbb{Z}$:

$$\begin{tikzcd}
G = G_0 \arrow[r, hook] & G_0' \\
G_1 \arrow[r, hook] \arrow[ur, hook] & G_1' \\
\cdots \arrow[r, hook] \arrow[ur, hook] & \cdots \\
G_N \arrow[ur, hook] &
\end{tikzcd}$$

Most classical results are proved for each individual step in the hierarchy, assuming they hold for all one-relator groups lower down.

A topological hierarchy

Using basic topology, we can simplify the hierarchy somewhat.

A 2-complex X is a 2-dimensional CW-complex. We assume the attaching maps of 2-cells are immersions. An *immersion* of 2-complexes $Y \looparrowright X$ is a locally injective combinatorial map.

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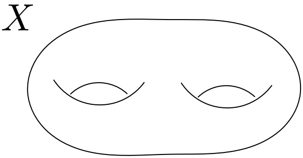
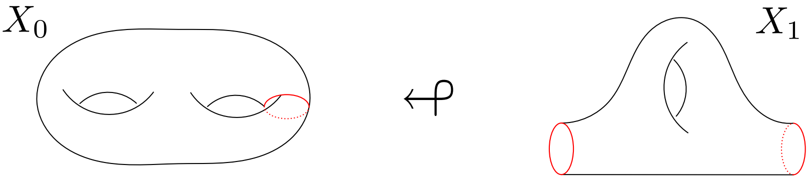
Let us take a surface of genus two as inspiration.



We want to 'split' X as a HNN-extension.

A topological hierarchy

We can cut X over a non-separating curve:



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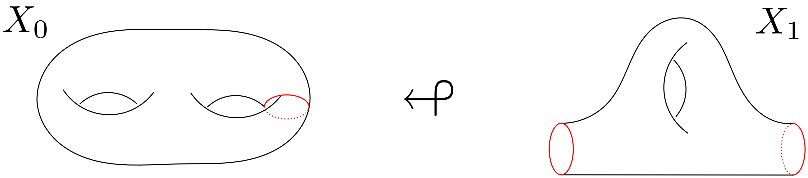
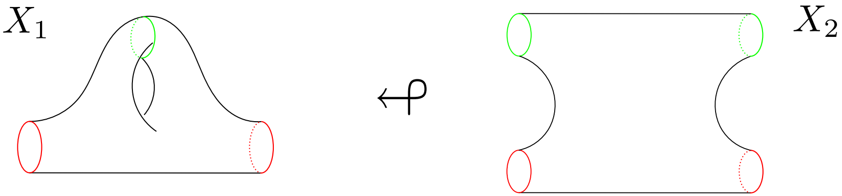
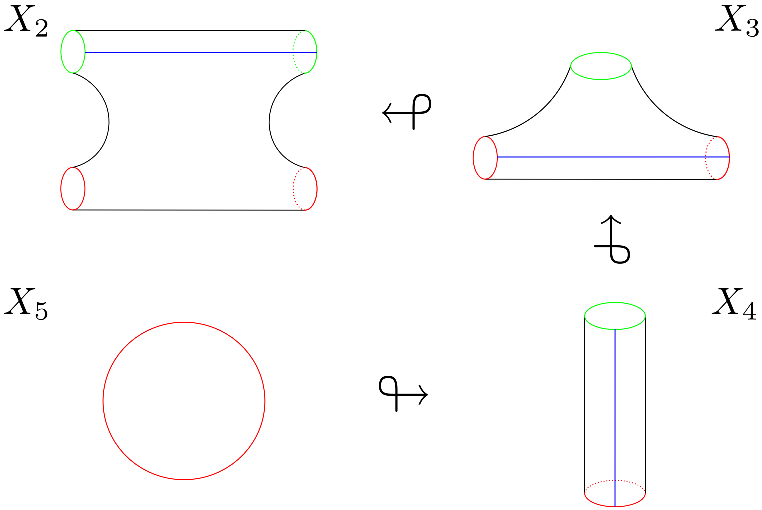


and repeat:



A topological hierarchy

Until eventually we reach a disc:



A topological hierarchy

In general, we want to consider one-relator complexes:

Definition 4

A *one-relator complex* $X = (\Gamma, \lambda)$ is a combinatorial 2-complex with 1-skeleton Γ and a single 2-cell with attaching map $\lambda : S^1 \looparrowright \Gamma$.

- A surface has a one-relator complex structure.
- The presentation complex of a one-relator group is a one-relator complex.
- Going down the topological hierarchy, we will 'unravel' a given one-relator complex, so we cannot just consider presentation complexes.
- Morally, the hierarchy for a one-relator complex is not so different from the surface case.

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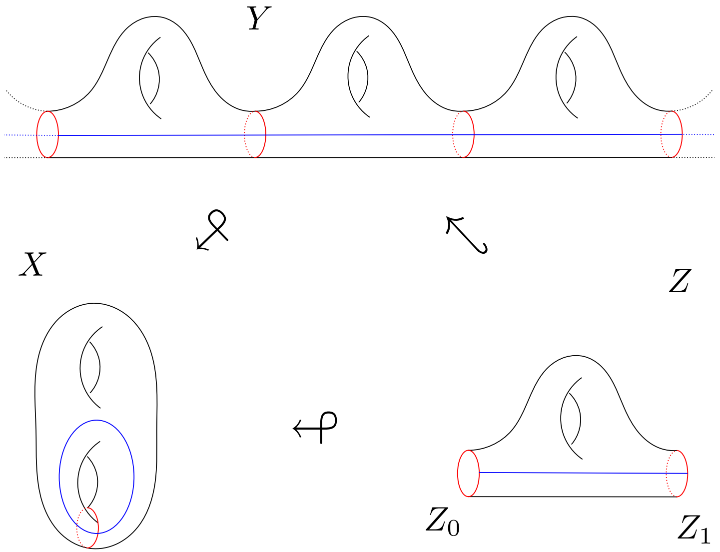
- Morally, the hierarchy for a one-relator complex is not so different from the surface case.

However, we cannot split a one-relator complex X along a subcomplex in general, so we unwrap X via a $\mathbb{Z}$-cover.

Recall that if X is a 2-complex, a $\mathbb{Z}$-cover is a covering space $\rho \colon Y \to X$ such that $\mathrm{Deck}(\rho) \cong \mathbb{Z}$.

A topological hierarchy: an example

In the surface case:



The topological Magnus hierarchy

Definition 5

A *Magnus subgraph* of a one-relator complex $M \subset X$ is one that does not support the attaching map of the 2-cell.

We may recast the Freiheitssatz in terms of one-relator complexes:

Theorem 6 (Freiheitssatz)

If X is a one-relator complex and $M \subset X$ is a Magnus subgraph, then the map $\pi_1(M) \to \pi_1(X)$ induced by inclusion is injective.

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Theorem 7 (The hierarchy)

If $X = (\Gamma, \lambda)$ is a finite one-relator complex, there exists a finite sequence of immersions

$$X_N \looparrowright \ldots \looparrowright X_1 \looparrowright X_0 = X$$

where:

- *X_i is a finite one-relator subcomplex of a $\mathbb{Z}$-cover $Y_i \to X_{i-1}$ for each i,*
- *$\pi_1(X_N) \cong F * \mathbb{Z}/n\mathbb{Z}$ where F is free and $n = \deg(\lambda)$,*
- *We have:*

  $$\pi_1(X_{i-1}) \cong \pi_1(X_i)*_{\psi_i}$$

  where ψ_i is induced by identification of two Magnus subgraphs.

Some consequences

Let $G = F/\langle\langle w^n \rangle\rangle$ be a one-relator group with w not a proper power in F and $n \geqslant 1$. Let $X = (\Gamma, \lambda)$ be the presentation complex of G so that $\pi_1(X) = G$.

Then classical consequences of the Magnus hierarchy are:

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- (Karrass–Magnus–Solitar '60) G has torsion if and only if $n \geqslant 2$ and every torsion subgroup is conjugate into $\langle w \rangle$.
- (Newman '68) If $n \geqslant 2$, then the presentation $F/\langle\langle w^n \rangle\rangle$ is a Dehn presentation, hence G is hyperbolic.

Magnus splittings

Theorem 8 (Magnus splitting)

Let X be a finite one-relator complex and $\rho\colon Y \to X$ a $\mathbb{Z}$-cover, then there is a finite one-relator subcomplex $Z \subset Y$ such that

$$\pi_1(X) \cong \pi_1(Z)*_\psi$$

where $\psi\colon \pi_1(Z_0) \to \pi_1(Z_1)$ identifies fundamental groups of two Magnus subgraphs $Z_0, Z_1 \subset Z$.

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Lemma 9

Let X be a finite one-relator complex, let $\rho\colon Y \to X$ be a $\mathbb{Z}$-cover and let $t \in \mathrm{Deck}(\rho)$ be a generator. There is a finite connected one-relator subcomplex $Z \subset Y$ such that the following properties hold:

1. *Z is a fundamental domain: $Y = \bigcup_{i\in\mathbb{Z}} t^i(Z)$.*
2. *$Z \cap t(Z)$ is a connected graph that does not support the attaching map of a 2-cell.*
3. *$Z \cap t^i(Z) \subset t^{i-1}(Z)$ for all $i \geqslant 1$.*

Proof of Theorem 8, assuming the Freiheitssatz

Proof of Theorem 8.

Take $Z \subset Y$ to be the one-relator complex from Lemma 9 and let t be a generator of $\mathrm{Deck}(\rho)$.

Denote by $Z_0 = t^{-1}(Z) \cap Z$ and $\iota\colon Z_0 \hookrightarrow Z$ the inclusion. Denote by

$$\mathcal{X} = Z \sqcup (Z_0 \times [-1, 1]) / \{\iota(z) \sim (z, -1),\ (t \circ \iota)(z) \sim (z, 1),\ z \in Z_0\},$$

and consider the map

$$h\colon \mathcal{X} \to X$$

given by ρ when restricted to Z or $Z_0 \times \{i\}$ for all $i \in (-1, 1)$.

Since Z is a fundamental domain for Y (Property (1)), h is surjective.

By Property (2) and the Freiheitssatz, the maps $\iota_*\colon \pi_1(Z_0) \to \pi_1(Z)$ and $(t \circ \iota)_*\colon \pi_1(Z_0) \to \pi_1(Z)$ are injective. Hence,

$$\pi_1(\mathcal{X}) \cong \pi_1(Z)*_\psi$$

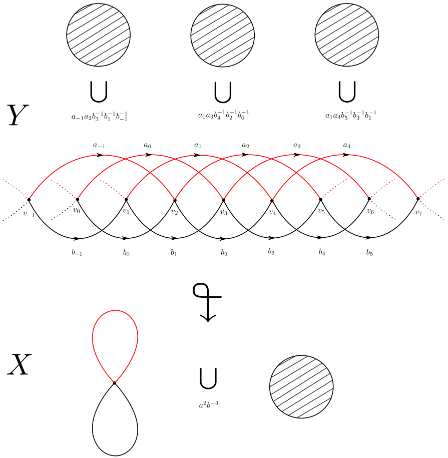
where $\psi\colon \pi_1(Z_0) \to \pi_1(t \cdot Z_0)$ is given by t_*.

By Property (3), $h^{-1}(x)$ is homeomorphic to a closed subset of $\mathbb{R}$ and so h is a homotopy equivalence, yielding the required splitting for $\pi_1(X)$. $\square$

A topological hierarchy: an example

Consider the presentation complex X of $G = \langle a, b \mid a^2 = b^3 \rangle$.

Let $\rho\colon Y \to X$ be the $\mathbb{Z}$-cover induced by the homomorphism $\phi\colon G \to \mathbb{Z}$ given by $\phi(a) = 3, \phi(b) = 2$.



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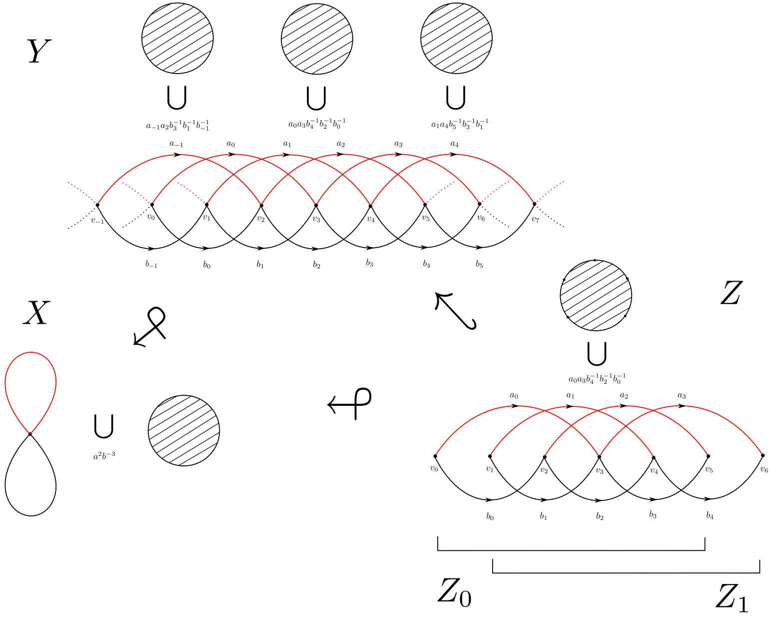

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Define the *degree* of a cycle $\lambda\colon S^1 \looparrowright \Gamma$ as the minimal degree of a covering map $S^1 \looparrowright S^1$ that λ factors through.

Define the *complexity* of a one-relator complex $X = (\Gamma, \lambda)$ as:

$$c(X) := \frac{|\lambda|}{\deg(\lambda)} - \left|\mathrm{Im}(\lambda)^{(0)}\right|.$$

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$$c(X) := \frac{|\lambda|}{\deg(\lambda)} - \left|\mathrm{Im}(\lambda)^{(0)}\right|.$$

In this way, $c(X) = 0$ if and only if either of the following equivalent conditions hold:

- $\mathrm{Im}(\lambda) \cong S^1$.
- $\pi_1(X) \cong F * \mathbb{Z}/n\mathbb{Z}$ where F is a free group and $n = \deg(\lambda)$.

A topological hierarchy

By proving that $c(Z) < c(X)$ in Theorem 8 and using induction, one obtains a hierarchy of one-relator complexes:

Theorem 10

If $X = (\Gamma, \lambda)$ is a finite one-relator complex, there exists a finite sequence of immersions

$$X_N \looparrowright \ldots \looparrowright X_1 \looparrowright X_0 = X$$

where:

- X_i *is a finite one-relator complex for each* i,
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A topological hierarchy

Proof.

If $\mathrm{Im}(\lambda) \cong S^1$, then $\pi_1(X_N) \cong F * \mathbb{Z}/n\mathbb{Z}$ where $n = \deg(\lambda)$ and we are done.

If this is not the case, then let $X_\lambda \subset X$ be the minimal subcomplex containing the 2-cell. Since $\mathrm{Im}(\lambda)$ is not a circle, it follows that $b_1(X_\lambda) \geqslant 1$. Hence, there is a $\mathbb{Z}$-cover $\rho\colon Y \to X$ such that $\rho^{-1}(X_\lambda) \to X_\lambda$ is a non-trivial $\mathbb{Z}$-cover.

Let $Z = (\Lambda, \widetilde{\lambda})$ be the one-relator subcomplex from Theorem 8. Since $\widetilde{\lambda}$ is a lift of λ to Y, we have

$$\frac{|\widetilde{\lambda}|}{\deg(\widetilde{\lambda})} = \frac{|\lambda|}{\deg(\lambda)}.$$

Since $\rho^{-1}(X_\lambda) \to X_\lambda$ is a non-trivial $\mathbb{Z}$-cover, the map $\mathrm{Im}(\widetilde{\lambda}) = Z_{\widetilde{\lambda}} \to X_\lambda = \mathrm{Im}(\lambda)$ is not injective. Hence

$$\left|\mathrm{Im}(\widetilde{\lambda})^{(0)}\right| < \left|\mathrm{Im}(\lambda)^{(0)}\right|.$$

and so $c(Z) < c(X)$. By induction on complexity, there is a hierarchy $X_N \looparrowright \ldots \looparrowright X_1 = Z \looparrowright X_0 = X$ as required.

$\square$

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Coherence

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A group G is *coherent* if all of the finitely generated subgroups of G are finitely presented.

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5. Free-by-cyclic groups (Feighn–Handel).
6. Limit groups (Sela).

Incoherence

Definition 12

A group G is *incoherent* if it contains a finitely generated subgroup which is not finitely presented.

Lemma 13

$F_2 \times F_2$ is incoherent.

Proof.

Consider the subgroup $H = \langle a, c, bd \rangle \leqslant F(a,b) \times F(c,d)$. Then

$$\begin{aligned} H &\cong (\langle\langle a \rangle\rangle_{F(a,b)} \times \langle\langle c \rangle\rangle_{F(c,d)}) \rtimes \langle bd \rangle \\ &\cong \left\langle x, y, z \mid \left[x^{z^i}, y^{z^j}\right], i, j \in \mathbb{Z} \right\rangle . \end{aligned}$$

If H were finitely presented, then there would be a finite subset the above relations from which all others would be derivable. This is not possible.

One can also compute that $H_2(H, \mathbb{Z}) \cong \mathbb{Z}^\infty$ to conclude that H is not finitely presented. $\square$

Road to coherence: understanding 2-complexes via immersions

A 2-complex X is a 2-dimensional CW-complex. An *immersion* of 2-complexes $Y \looparrowright X$ is a locally injective combinatorial map. We always assume that attaching maps of 2-cells are immersions.

We write $X = (\Gamma, \lambda_X)$ where $\lambda_X : \mathbb{S}_X = \bigsqcup S^1 \looparrowright X^{(1)}$ is the attaching map of the 2-cells.

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Definition 14

A 2-complex X has *non-positive immersions* if for every immersion $Z \looparrowright X$ with Z compact and connected, either $\chi(Z) \leqslant 0$ or Z is contractible.

Some examples of 2-complexes with non-positive immersions are:

- Graphs.
- Aspherical surfaces.
- Mapping tori of graphs.
- Spines of aspherical 3-manifolds with non-empty boundary.

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Wise conjectured in '03 that presentation complexes of torsion-free one-relator groups have non-positive immersions.

Dani Wise predicted that non-positive immersions would be a useful property to solve Baumslag's conjecture that all one-relator groups are coherent.

Theorem 15 (Helfer–Wise, Louder–Wilton '14)

If X is a one-relator complex with $\pi_1(X)$ torsion-free, then X has non-positive immersions.

If $Y = (Y^{(1)}, \lambda_Y)$, $X = (X^{(1)}, \lambda_X)$ are 2-complexes, an immersion $\gamma\colon Y \looparrowright X$ can be described by a commutative diagram:

$$\begin{tikzcd} \mathbb{S}_Y \arrow[r, "\sigma"] \arrow[d, "\lambda_Y"'] & \mathbb{S}_X \arrow[d, "\lambda_X"] \\ Y^{(1)} \arrow[r, "\gamma"] & X^{(1)} \end{tikzcd}$$

of immersions, where $\lambda_Y : \mathbb{S}_Y = \bigsqcup S^1 \looparrowright Y^{(1)}, \lambda_X : \mathbb{S}_X = \bigsqcup S^1 \looparrowright X^{(1)}$ are the attaching maps of the 2-cells in Y and X. The cover σ factors through the pullback:

$$\begin{tikzcd} \mathbb{S}_Y \arrow[r, hook] \arrow[rr, bend left, "\sigma"] & Y^{(1)} \times_{X^{(1)}} \mathbb{S}_X \arrow[r, "p_{\mathbb{S}_X}"'] & \mathbb{S}_X \end{tikzcd}$$

Understanding this pullback was key in confirming Wise's conjecture.

Subgroups via immersions

Lemma 16

Let X be a compact 2-complex and let $H \leqslant \pi_1(X)$ be a finitely generated subgroup. There is a sequence of π_1-surjective immersions of compact connected 2-complexes:

$$Z_0 \looparrowright Z_1 \looparrowright \ldots \looparrowright Z_k \looparrowright \ldots \looparrowright X$$

such that

$$\lim_{i \to \infty} \pi_1(Z_i) = H.$$

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such that

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Fact: H is finitely presented if and only if $\pi_1(Z_i) \to \pi_1(Z_{i+1})$ is an isomorphism for all $i \gg 0$.

When X has non-positive immersions, $\chi(Z_i) \leqslant 0$ for all i. One might hope that this is enough to guarantee that the sequence stabilises so that H is finitely presented.

Subgroups via immersions

A 2-complex X is *reducible* if there is a homotopy equivalence $Z \to X$ with the following properties:

1. $Z \to X$ restricts to a homotopy equivalence $Z^{(1)} \to X^{(1)}$.
2. $Z \to X$ is a homeomorphism on the interiors of 2-cells.

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3. One of the following holds:
   3.1 $Z = Z_1 \vee Z_2$ such that $Z_1^{(1)}$ and $Z_2^{(1)}$ are not contractible
   3.2 There is some 1-cell $e \subset Z$ such that e is traversed by the attaching map of precisely one 2-cell precisely once.

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Lemma 17 (Louder–Wilton/Scott)

Let X be a compact 2-complex and let $H \leqslant \pi_1(X)$ be a finitely generated non-cyclic and freely indecomposable subgroup. There is a sequence of π_1-surjective immersions of compact connected irreducible 2-complexes:

$$Z_0 \looparrowright Z_1 \looparrowright \ldots \looparrowright Z_k \looparrowright \ldots \looparrowright X$$

such that

$$\lim_{i \to \infty} \pi_1(Z_i) \cong H.$$

Uniform negative immersions

Definition 18

A 2-complex X has *uniform negative immersions* if there exists an $\epsilon > 0$ such that for every immersion $Z \looparrowright X$ with Z compact, connected and irreducible, we have

$$\frac{\chi(Z)}{\#\{\text{2-cells in } Z\}} \leqslant -\epsilon.$$

If X has uniform negative immersions, then the sequence $Z_0 \looparrowright Z_1 \looparrowright \ldots \looparrowright X$ always stabilises! More can be said:

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Theorem 19 (Louder–Wilton '21)

If X has uniform negative immersions, then $\pi_1(X)$ is coherent and, for every integer $k \geqslant 0$, there are finitely many conjugacy classes of one-ended subgroups $H \leqslant \pi_1(X)$ with $b_1(H) \leqslant k$.

Proof of Theorem 19.

Let $H \leqslant \pi_1(X)$ be a finitely generated non cyclic freely indecomposable subgroup.

Consider a sequence of immersions

$$Z_0 \looparrowright Z_1 \looparrowright \ldots \looparrowright X$$

for H as in Lemma 17. Each Z_i is irreducible and hence by definition we have

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$$\chi(Z_i) \leqslant -\epsilon \cdot \#\{\text{2-cells in } Z_i\}.$$

Since $\chi(Z_i) = 1 - b_1(Z_i) + b_2(Z_i)$, we have

$$1 + b_2(Z_i) + \epsilon \cdot \#\{\text{2-cells in } Z_i\} \leqslant b_1(Z_i).$$

Since $\pi_1(Z_i) \to \pi_1(Z_{i+1})$ is surjective for all i, we have that

$$b_1(H) \leqslant b_1(Z_{i+1}) \leqslant b_1(Z_i)$$

for all i.

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$$b_1(H) \leqslant b_1(Z_{i+1}) \leqslant b_1(Z_i)$$

for all i. Hence there is some k such that for all $i \geqslant k$ we have $b_1(H) = b_1(Z_i)$ and so

$$\#\{\text{2-cells in } Z_i\} \leqslant \frac{b_1(H) - 1}{\epsilon}.$$

Proof of Theorem 19 continued.

Thus, there is a bound on the number of 2-cells in the 2-complexes Z_i for all $i \geqslant k$. We need a simple lemma.

Lemma 20

If X is a finite 2-complex and k a positive integer, there are finitely many immersions $Z \looparrowright X$ with Z irreducible (and core) and with $\#\{$2-cells in $Z\} \leqslant k$.

Proof of Theorem 19 continued.

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By Lemma 20, for n large enough, each $Z_i \looparrowright Z_{i+1}$ is an isomorphism for all $i \geqslant n$. Hence, $\pi_1(Z_n) \to \pi_1(X)$ is injective.

We have shown that for any finitely generated non-cyclic freely indecomposable subgroup $H \leqslant \pi_1(X)$ with $b_1(H) \leqslant k$, there exists a π_1-injective immersion $Z \looparrowright X$ with Z compact, irreducible and with at most $\frac{b_1(H)-1}{\epsilon}$ many 2-cells and with π_1-image precisely (a conjugate of) H. This proves the second statement.

Proof of Theorem 19 continued.

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Now let H be an arbitrary finitely generated subgroup, By Grushko's theorem, $H \cong H_1 * \ldots * H_n * F$ where each H_i is non-cyclic, freely indecomposable and finitely generated and where F is free. By the above, each H_i is finitely presented and hence so is H.

$\square$

Primitivity rank

A deep theorem of Louder–Wilton characterises when a one-relator group is k-*free*; that is, when every k-generated subgroup is free.

Theorem 21 (Louder, Wilton '18)

A one-relator group $G = F/\langle\langle w \rangle\rangle$ *is* k*-free if and only if* $\pi(w) \geqslant k+1$.

The *primitivity rank* of an element $w \in F(\Sigma)$ is defined as:

$$\pi(w) = \min\{\mathrm{rk}(K) \mid w \in K < F(\Sigma),\ w \text{ imprimitive in } K\} \in \mathbb{N} \cup \{\infty\}$$

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- $\pi(w)$ is computable (Puder '11),
- Generically $\pi(w) = |\Sigma|$ (Puder '12).
- $\pi(w) = 1$ if and only if w is a proper power (hence, $F/\langle\langle w \rangle\rangle$ has torsion).

Example 22

$w = a^2b^2c^2b^2c^{-2}b^{-2}a^{-2}b^3$ has primitivity rank 2 as w is imprimitive in $\langle a^2b^2c^2, b\rangle$.

Uniform negative immersions and strong coherence

Louder–Wilton characterise uniform negative immersions for one-relator complexes.

Theorem 23 (Louder–Wilton '21)

If X is the presentation complex of the one-relator group $G = F/\langle\langle w \rangle\rangle$, then the following are equivalent:

- *X has (uniform) negative immersions.*
- *$\pi_1(X) = G$ is 2-free.*
- *$\pi(w) \geqslant 3$.*

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Lemma 24

If X is a one-relator complex such that $\pi_1(X)$ has torsion, then there is a finite sheeted cover $Y \looparrowright X$ and an inclusion $\iota\colon Z \hookrightarrow Y$ inducing an isomorphism on π_1 such that Z has uniform negative immersions.

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Theorem 25 (Louder–Wilton '21)

If $G = F/\langle\langle w \rangle\rangle$ is a one-relator group with $\pi(w) \neq 2$, then G is coherent and, for every integer $k \geqslant 0$, there are finitely many conjugacy classes of one-ended subgroups $H \leqslant \pi_1(X)$ with $b_1(H) \leqslant k$.

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The relation module

Let $G = F/N$ be a presentation. Recall that the group ring is the ring:

$$\mathbb{Z}G = \left\{ \sum_{g \in G} z_g g \,\middle|\, z_g \in \mathbb{Z} \text{ and } z_g = 0 \text{ for all but finitely many } g \in G \right\}.$$

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The action of the free group F on

$$N_{\mathrm{ab}} = N/[N, N]$$

by conjugation extends linearly to an action of the group ring $\mathbb{Z}F$ on N_{ab}.

Since N acts trivially on N_{ab}, the $\mathbb{Z}F$ action descends to an action

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With this action, N_{ab} is a $\mathbb{Z}G$-module called the *relation module*.

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If $G = F/N$ has cohomological dimension two, then N_{ab} is a projective $\mathbb{Z}G$-module and there is resolution P_*:

$$0 \longrightarrow N_{\mathrm{ab}} \longrightarrow \bigoplus_{s \in S} \mathbb{Z}G \longrightarrow \mathbb{Z}G \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0$$

where $S \subset F$ is a free generating set. Then
$H^i(G, -), H_i(G, -) = H^i(\hom_{\mathbb{Z}G}(P_*, -)), H_i(P_* \otimes_{\mathbb{Z}G} -)$.

Lyndon's identity theorem

Theorem 26 (Lyndon's Identity Theorem)

Let $G = F/\langle\langle w^n \rangle\rangle$ be a one-relator group with w not a proper power and with $n \geqslant 1$. Denoting by $N = \langle\langle w^n \rangle\rangle$, we have

$$N_{\mathrm{ab}} \cong \mathbb{Z}G/(w-1)\cdot\mathbb{Z}G.$$

In particular, if $n = 1$ then

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Lyndon's theorem allows us to easily compute a resolution of $\mathbb{Z}$:

$$0 \longrightarrow \mathbb{Z}G \xrightarrow{\left(\frac{\partial}{\partial s}\right)} \bigoplus_{s\in S}\mathbb{Z}G \xrightarrow{(s-1)} \mathbb{Z}G \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0 \qquad \text{if } n = 1$$

$$\ldots \xrightarrow{\sum_{i=0}^{n-1} w^i} \mathbb{Z}G \xrightarrow{w-1} \mathbb{Z}G \xrightarrow{\left(\frac{\partial}{\partial s}\right)} \bigoplus_{s\in S}\mathbb{Z}G \xrightarrow{(s-1)} \mathbb{Z}G \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0 \qquad \text{if } n \geqslant 2$$

and so to compute $H^*(G,-)$ and $H_*(G,-)$ when G is a one-relator group.

Homological coherence

A group G is of *type* $\mathrm{FP}_2(\mathbb{Z})$ if any of the following equivalent conditions hold:

- For any presentation $G = F/N$ with F finitely generated, the relation module N_{ab} is finitely generated.

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- There is a projective resolution $P_* \to \mathbb{Z}$ with P_0, P_1, P_2 finitely generated.
- The augmentation ideal $I_G \leqslant \mathbb{Z}G$ is finitely presented.
- There is a finitely presented group H and a surjection $H \twoheadrightarrow G$ with perfect kernel.

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Definition 27

A group is *homologically coherent* if every finitely generated subgroup is of type $\mathrm{FP}_2(\mathbb{Z})$.

Bestvina–Brady constructed the first groups of type $\mathrm{FP}_2(\mathbb{Z})$ that are not finitely presented (but they are all homologically incoherent).

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Question 28

Does there exist a homologically coherent group that is not coherent?

A criterion for homological coherence

A ring R is a *division ring* if every non-zero element is a unit.

Lemma 29

If G is a finitely generated group of cohomological dimension two and $\mathbb{Z}G \hookrightarrow \mathcal{D}$ an embedding into a division ring, then

$$G \text{ has type } \mathrm{FP}_2(\mathbb{Z}) \iff \dim_{\mathcal{D}} H_2(G, \mathcal{D}) < \infty.$$

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With some work, we will show how to use Lemma 29 to prove:

Theorem 30 (Jaikin-Zapirain–L '23)

If X has non-positive immersions, then $\pi_1(X)$ is homologically coherent.

Division ring embedding example: the Mal'cev–Neumann completion

Theorem 31 (Mal'cev '48, Neumann '49)

Let G a group, let $<$ be a bi-ordering on G. The space of Mal'cev–Neumann series

$$\mathbb{Q}(\!(G,<)\!) = \left\{ r = \sum_{g \in G} r_g \cdot g \mid r_g \in \mathbb{Q}, \operatorname{supp}(r) \textit{ is well-ordered} \right\}$$

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1. The set $S_1 \cup S_2$ is well-ordered.
2. The set $S_1 S_2 = \{s_1 s_2 \mid s_1 \in S_1, s_2 \in S_2\}$ is well-ordered.
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3. For any given element $g \in S_1 S_2$, there are only finitely many products $s_1 s_2$ equal to g.

Let $r \in \mathbb{Q}(\!(G,<)\!)$ and let $g \in \operatorname{supp}(r)$ be the minimal element. Then for some $a \in \mathbb{Q} - \{0\}$, we have $r = (ag)(1-s)$ where $s \in \mathbb{Q}(\!(G,<)\!)$ such that for all $h \in \operatorname{supp}(s)$, we have $h > 1$. Then

$$\left(\sum_{i \in \mathbb{N}} s^i \right) a^{-1} g^{-1}$$

is a left and right inverse for r.

Embeddings into division rings

Conjecture 32 (Kaplansky)

The group ring of a torsion-free group contains no (non-zero) zero-divisors.

A ring that embeds into a division ring clearly has no (non-zero) zero-divisors.

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Lewin–Lewin solved the Kaplansky conjecture for torsion-free one-relator groups:

Theorem 33 (Lewin–Lewin '78)

If G is a torsion-free one-relator group and k a division ring, then $k[G]$ embeds into a division ring $\mathcal{D}$.

The proof explicitly constructs the division ring $\mathcal{D}$ inductively using the hierarchy and using Cohn's universal matrix inverting ring construction. In the base case of the hierarchy one has a free group and can appeal to the theorem of Mal'cev and Neumann (free groups are bi-ordered).

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Jaikin-Zapirain–López-Álvarez proved something much more general:

Theorem 34 (Jaikin-Zapirain–López-Álvarez)

If G is locally indicable, then $\mathbb{Q}[G]$ embeds in a division ring $\mathcal{D}$.

Proof that X has non-positive immersions $\implies$ $\pi_1(X)$ homologically coherent

We will need some facts before starting the proof:

1. $G = F/N$ has cohomological dimension at most two if and only if N_{ab} is a projective $\mathbb{Z}G$-module.
2. If F is f.g. then $G = F/N$ is of type $\mathrm{FP}_2(\mathbb{Z})$ if and only if N_{ab} is finitely generated as a $\mathbb{Z}G$-module.
3. (Wise '20) If X has non-positive immersions, then:
   3.1 X is aspherical and so $\pi_1(X)$ has cohomological dimension at most 2.
   3.2 $\pi_1(X)$ is locally indicable.

Proof that X has non-positive immersions $\implies$ $\pi_1(X)$ homologically coherent

We will need some facts before starting the proof:

1. $G = F/N$ has cohomological dimension at most two if and only if N_{ab} is a projective $\mathbb{Z}G$-module.
2. If F is f.g. then $G = F/N$ is of type $\mathrm{FP}_2(\mathbb{Z})$ if and only if N_{ab} is finitely generated as a $\mathbb{Z}G$-module.
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   3.1 X is aspherical and so $\pi_1(X)$ has cohomological dimension at most 2.
   3.2 $\pi_1(X)$ is locally indicable.

Proof.

If X has non-positive immersions and $H \leqslant \pi_1(X)$ is a finitely generated subgroup, let

$$Z_0 \looparrowright Z_1 \looparrowright \ldots \looparrowright Z_k \looparrowright \ldots \looparrowright X$$

be a sequence of π_1-surjective immersions such that

$$\lim_{i \to \infty} \pi_1(Z_i) = H$$

as in Lemma 17.

Proof continued.

Let $F = \pi_1\left(Z_0^{(1)}\right)$, $N_i = \ker(F \to \pi_1(Z_i))$ for each i and $N = \ker(F \to H)$. Note that $\pi_1(Z_i) = F/N_i$ for each i and $H = F/N$. In order to complete the proof, we need to show that N_{ab} is finitely generated as a left $\mathbb{Z}[H]$-module.

Proof continued.

Let $F = \pi_1\left(Z_0^{(1)}\right)$, $N_i = \ker(F \to \pi_1(Z_i))$ for each i and $N = \ker(F \to H)$. Note that $\pi_1(Z_i) = F/N_i$ for each i and $H = F/N$. In order to complete the proof, we need to show that N_{ab} is finitely generated as a left $\mathbb{Z}[H]$-module.

We obtain a dual sequence

$$N_0 \hookrightarrow N_1 \hookrightarrow \ldots \hookrightarrow N_k \hookrightarrow \ldots \hookrightarrow F$$

with the following properties:

1. The homomorphisms $N_i \to N_{i+1} \to \ker(F \to H) \leqslant F$ are injective (using the fact that $\pi_1(Z_i) \to \pi_1(Z_{i+1})$ is surjective) for all i.

Proof continued.

Let $F = \pi_1\left(Z_0^{(1)}\right)$, $N_i = \ker(F \to \pi_1(Z_i))$ for each i and $N = \ker(F \to H)$. Note that $\pi_1(Z_i) = F/N_i$ for each i and $H = F/N$. In order to complete the proof, we need to show that N_{ab} is finitely generated as a left $\mathbb{Z}[H]$-module.

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2. We have:

$$\lim_{i\to\infty} N_i = \bigcup_{i\in\mathbb{N}} N_i = \ker(F \to H) = N$$

(using the fact that $\lim_{i\to\infty} \pi_1(Z_i) = H$).

Proof continued.

Let $F = \pi_1\left(Z_0^{(1)}\right)$, $N_i = \ker(F \to \pi_1(Z_i))$ for each i and $N = \ker(F \to H)$. Note that $\pi_1(Z_i) = F/N_i$ for each i and $H = F/N$. In order to complete the proof, we need to show that N_{ab} is finitely generated as a left $\mathbb{Z}[H]$-module.

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2. We have:
$$\lim_{i\to\infty} N_i = \bigcup_{i\in\mathbb{N}} N_i = \ker(F \to H) = N$$
(using the fact that $\lim_{i\to\infty} \pi_1(Z_i) = H$).
3. The relation module $(N_i)_{\mathrm{ab}}$ for $\pi_1(Z_i) = F/N_i$ is a finitely generated (left) $\mathbb{Z}[\pi_1(Z_i)]$-module (using the fact that Z_i is compact) for all i.

Proof continued.

4. If $\mathcal{D}$ is a division ring which is also a right $\mathbb{Q}[H]$-module, then

$$\begin{aligned}\dim_{\mathcal{D}}(\mathcal{D} \otimes_{\mathbb{Q}[F/N_i]} (N_i)_{\mathrm{ab}}) - \mathrm{rk}(F) + 1 &= \sum_{j=0}^{2} (-1)^j \dim_{\mathcal{D}} H_j(\pi_1(Z_i), \mathcal{D}) \\ &= \chi(Z_i) \\ &\leqslant 0\end{aligned}$$

(using the fact that Z_i is aspherical and $\chi(Z_i) \leqslant 0$) for all i.

Proof continued.

4. If $\mathcal{D}$ is a division ring which is also a right $\mathbb{Q}[H]$-module, then

$$\begin{aligned}\dim_{\mathcal{D}}(\mathcal{D} \otimes_{\mathbb{Q}[F/N_i]} (N_i)_{\mathrm{ab}}) - \mathrm{rk}(F) + 1 &= \sum_{j=0}^{2} (-1)^j \dim_{\mathcal{D}} H_j(\pi_1(Z_i), \mathcal{D}) \\ &= \chi(Z_i) \\ &\leqslant 0\end{aligned}$$

(using the fact that Z_i is aspherical and $\chi(Z_i) \leqslant 0$) for all i.

Since X has non-positive immersions, $\pi_1(X)$ is locally indicable and so by Jaikin-Zapirain–López-Álvarez's result, there is a division ring embedding $\mathbb{Q}[H] \hookrightarrow \mathcal{D}$.

Proof continued.

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Since X has non-positive immersions, $\pi_1(X)$ is locally indicable and so by Jaikin-Zapirain–López-Álvarez's result, there is a division ring embedding $\mathbb{Q}[H] \hookrightarrow \mathcal{D}$.

We have

$$\dim_{\mathcal{D}}(\mathcal{D} \otimes_{\mathbb{Q}H} N_{\mathrm{ab}}) \leqslant \sup_{i \in \mathbb{N}} \dim_{\mathcal{D}} \left(\mathcal{D} \otimes_{\mathbb{Q}[F/N_i]} (N_i)_{\mathrm{ab}}\right) \leqslant \mathrm{rk}(F) - 1.$$

Since H has cohomological dimension at most 2, $H_2(H, \mathcal{D}) \subseteq \mathcal{D} \otimes_{\mathbb{Q}H} N_{\mathrm{ab}}$.

Hence

$$\dim_{\mathcal{D}} H_2(H, \mathcal{D}) \leqslant \dim_{\mathcal{D}}(\mathcal{D} \otimes_{\mathbb{Q}H} N_{\mathrm{ab}}) < \infty.$$

By Lemma 29, H has type $\mathrm{FP}_2(\mathbb{Z})$. $\square$

From homological coherence to coherence

Theorem 35 (Jaikin-Zapirain–L '23)

If G has a hierarchy (e.g. splits as an iterated amalgamated free product of HNN-extension) terminating at coherent groups, then

$$G \text{ is homologically coherent} \iff G \text{ is coherent}$$

From homological coherence to coherence

Theorem 35 (Jaikin-Zapirain–L '23)

If G has a hierarchy (e.g. splits as an iterated amalgamated free product of HNN-extension) terminating at coherent groups, then

$$G \text{ is homologically coherent} \iff G \text{ is coherent}$$

Using the Magnus hierarchy, we obtain:

Theorem 36 (Jaikin-Zapirain–L '23)

One-relator groups are coherent.

Group ring coherence

A ring R is *coherent* if all of its finitely generated ideals are finitely presented.

This is a property of interest to ring theorists and algebraic geometers which predates the corresponding definition for groups.

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Using the properties of the division ring $\mathcal{D}(G)$, one can prove that the group ring of a torsion-free one-relator group G is coherent.

The torsion case is handled by appealing to the fact that they are virtually free-by-cyclic (Kielak–L '23).

Theorem 37 (Jaikin-Zapirain–L '23)

If G is a one-relator group, then $\mathbb{Q}[G]$ is coherent.

Concluding questions

Question 38

If X is an aspherical 2-complex, which of the following properties are equivalent:

1. *X has non-positive immersions.*
2. *$b_2^{(2)}(X) = 0$.*
3. *$\pi_1(X)$ is coherent.*
4. *$\pi_1(X)$ is homologically coherent.*
5. *$\mathbb{Q}[\pi_1(X)]$ is coherent.*

Question 39 (Effective coherence)

Is there an algorithm that, on input a one-relator group G and a finite set of elements $g_1, \ldots, g_n \in G$, computes a presentation for the subgroup $\langle g_1, \ldots, g_n \rangle \leqslant G$?

Thank you