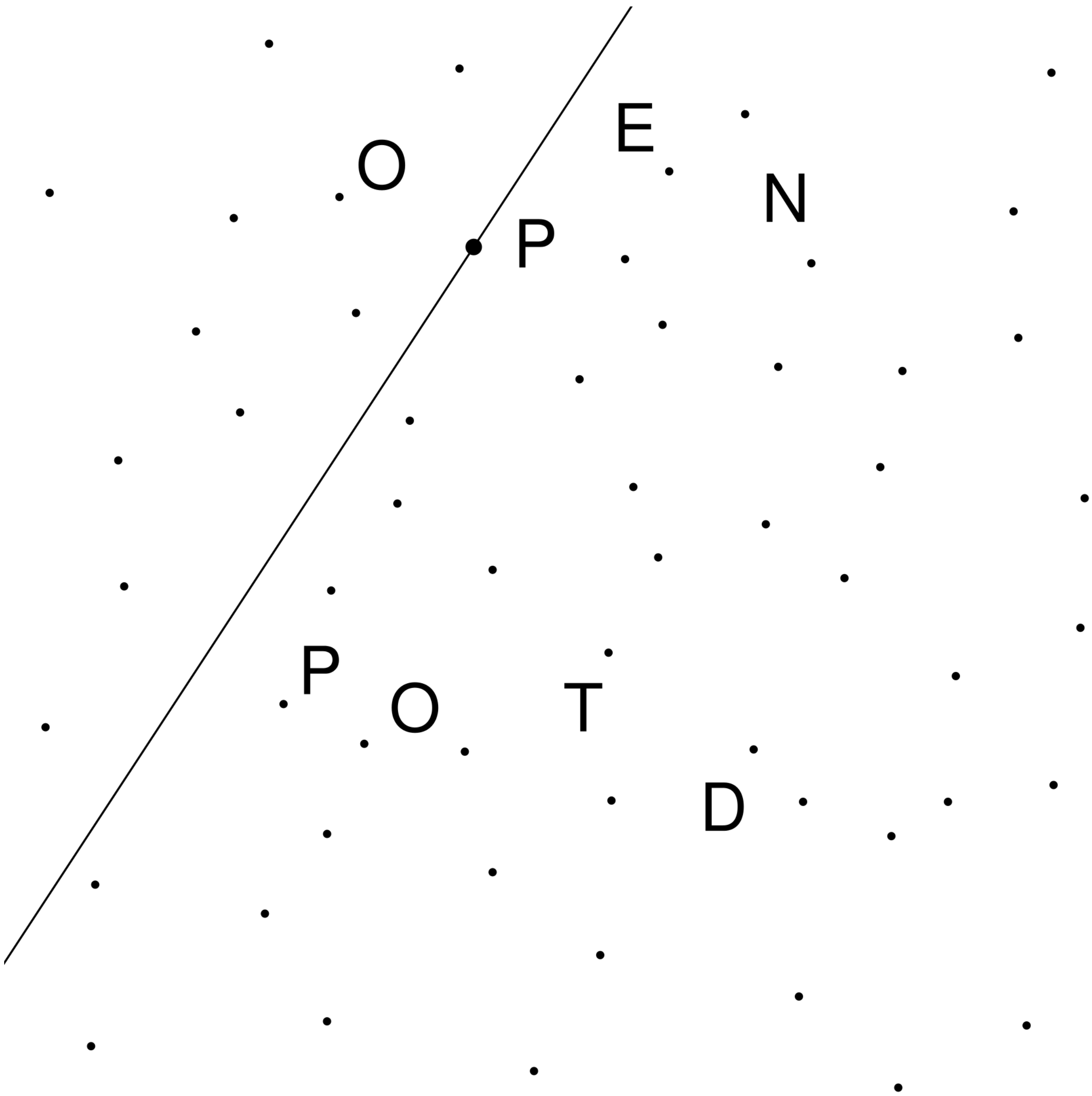
Questions & Solutions



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Website & Discord

Last updated: Thursday 13th January, 2022
Github/OpenPOTD/Solutions

Contents

Introduction

Welcome to the OpenPOTD solutions booklet! Here you'll find answers & solutions to all past seasons.

Solutions for season one were entirely written up by `Brainysmurfs#2860`, while `sjbs#9839` has overseen most of season two. From season three onwards, Yuchan (`Angry Any#4319`) has also been contributing problem proposals and solutions.
Where possible, from season two onwards, We have tried to include the officially provided solutions to problems, or adapted them in line with any changes to the problem statement, and in most cases also and filled in the gaps as best I could, to make solutions more approachable to beginners. For the more well known questions that are featured in a season (namely problems from the International Mathematical Olympiad), instead of providing our own solution, we have included the *Art of Problem Solving* forum post on the question, which will contain multiple solution write-ups, as well as discussion about the problem.
In many circumstances problems have not come with official write-ups - or indeed write-ups of any kind - and thus have required us to provide our own. In these cases we humbly apologise for any mistakes (or fakesolves!) in advance. If you do notice any mistakes, check out How to contribute.

How to Contribute

If you would like to contribute to the project - be that through correcting a mistake in the document, wanting to contribute a solution write up - or even proposing a problem for a future season, here's how.

Correcting Mistakes

If you notice any mistakes while going through the solutions document, be it a typo or missing or incorrect information about a particular problem, feel free to submit a push request with the fix. If you don't feel comfortable doing that, you can always contact us (see the Contact Us to find out how), and we'll be happy to fix the error. Alternatively, you can open up an Issue on the GitHub, or mention it in the discussions tab.

Contributing Solutions

Similarly to correcting any mistakes, if you would like to contribute a write-up to a particular problem, you can submit a push containing the solution - doing as Romans do (i.e. just look at how others have submitted write-ups and copy that). If you are submitting a push, make sure you edit `preamble.sty` to include your Discord information in a macro (scroll to the bottom of the file and you'll see), so that you can include yourself in the contributors list. Furthermore, make sure that you credit your solution with `[Write up by ...]`. If any of that sounds complicated or you forget to add that information, that's fine - we'll add it for you.
If creating push requests and fiddling with LaTeX isn't your thing, we'll gladly help you type it up if you write it out in plaintext or whatever medium you feel most comfortable using (so long as we can understand it!) - to do this just send it to us using any of the options in Contact Us. Similarly, you can use the GitHub discussions tab and post it there.

Note: feel free to submit alternative solutions to any past problems, be it from season 1, or the most recent

Problem Proposals

If you'd like to submit a problem proposal - be it an original problem, or just a particular problem you found interesting - we're always on the lookout for new problems! For original problems, please ensure you submit the problem with a solution. As with solution write-ups, though sending us a `.tex` file is preferred, it's completely fine to just send a plaintext write up, or a screenshot etc. and we can deal with it from there. The same goes for non-original problems, though solutions aren't required, they would be greatly appreciated. If you are submitting a non-original problem please ensure you include the problem source.
Please do not use a public medium to submit a problem proposal - messaging one of us on Discord would be the preferred method of communication (See Contact Us).

Feature Suggestions

If you have any ideas when it comes to improving the bot or project, the best place to do that is in the `#Suggestions` channel, or any of the other methods listed in Contact Us, such as using the GitHub discussions tab.

Contact Us

The best way to contact us is through the OpenPOTD Discord server, however, you may also contact us through the discussions tab on Github. Alternatively, you can DM any of us on Discord:

- sjbs#9839 (434767660182405131)
- brainysmurfs#2860 (281300961312374785)
- Angry Any#4319 (580933385090891797)

Contributors

Thank you (in no particular order) to the following contributors: [1]

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[1] The numbers are in the format: Season.Week.Problem

§1 Season 1

§1.1 Week 1

§1.1.1 Intersecting Circles

Source: Senior Mathematical Challenge, 2015 Q4
Proposer: brainysmurfs#2860 (281300961312374785)
Problem ID: 21
Date: 2020-10-27

Consider the positive integer N, and Two internally tangent circles Γ_1 and Γ_2 are given such that Γ_1 passes through the center of Γ_2. Find the fraction of the area of Γ_1 lying outside Γ_2. If this fraction is $\frac{a}{b}$ where $\gcd(a, b) = 1$, then find $a + b$.

Solution.

Suppose Γ_2 has radius $2r$. Since Γ_1 is internally tangent to Γ_2 and passes through its centre, the radius of Γ_1 is half the radius of Γ_2, i.e. just r. So fraction of the area of Γ_1 lying outside Γ_2 is $\dfrac{4\pi r^2 - \pi r^2}{4\pi r^2} = \dfrac{3}{4}$. Since $3 + 4 = 7$, the answer is $\boxed{7}$ $\square$

§1.1.2 Guava Juice

Source: Original Problem
Proposer: Tony Wang#1729 (541318134699786272)
Problem ID: 22
Date: 2020-10-28

The ingredients list of a Guava Juice Drink is as follows:

> Water (80%), Guava Juice, Sugar, Fructose (3%), Sodium Carboxymethyl Cellulose, Citric Acid, Flavour, Vitamin C (0.04%)

Assuming only that the ingredients list is ordered by their constituent percentage in the drink (which are not necessarily distinct), find the maximum and minimum possible percentage of Guava Juice in the drink. If their difference is n, submit $100n$.

Solution.

Note there are 2 ingredients between water and fructose, and 3 between fructose and Vitamin C.

For the amount of guava juice to be maximised, everything should be minimised. In particular, there should be 0.04% of Sodium Carboxymethyl Cellulose, Citric Acid, and Flavour; and 3% of Sugar and Fructose. This gives us a Guava Juice percentage of 13.84%.

For the amount of guava juice to be minimised, everything else should be maximised. In particular, there should be 3% of Sodium Carboxymethyl Cellulose, Citric Acid, and Flavour; and equal amounts of sugar and guava juice. This gives us a guava juice percentage of 5.48%.

This gives us $13.84 - 3.98 = 9.86\%$. Multiplying this by 100 gives us $\boxed{986}$. $\square$

§1.1.3 Complex Roots

Source: New South Wales Higher School Certificate '4U', 2020 Q2
Proposer: brainysmurfs#2860 (281300961312374785)
Problem ID: 23
Date: 2020-10-29

Given that $z = 3 + i$ is a root of $z^2 + pz + q = 0$, where p and q are real, find the values of p and q, and submit $p + q$.

Solution.

Applying the Complex Conjugate Theorem, $z = 3 - i$ is also a root of the quadratic. Expanding $(z - 3 - i)(z - 3 + i)$ gives us $z^2 - 6z + 10$. Thus $p = -6$ and $q = 10$, meaning $p + q = \boxed{4}$. $\square$

§1.1.4 Exponents

Source: Singapore Mathematical OlympiadJunior Round 1, 2001 Q4
Proposer: brainysmurfs#2860 (281300961312374785)
Problem ID: 25
Date: 2020-10-30

If a and b are positive reals such that $a^b = b^a$ and $b = 2a$, then the value of b^2 is?

Solution.

A direct search yields that $2^4 = 4^2$ and $4 = 2 \times 2$. The problem implies that such a unique value for b exists, hence $b^2 = \boxed{16}$. □

§1.1.5 Volumes of Cubes

Source: New Zealand Senior Mathematics CompetitionRound 2, 2019 Q7
Proposer: brainysmurfs#2860 (281300961312374785)
Problem ID: 26
Date: 2020-10-31

Two cubes with positive integer side lengths are such that the sum of their volumes is numerically equal to the difference of their surface areas. Find the sum of their volumes.

Solution.

A direct search yields that cubes of side length 4 and 2 satisfy the condition. The answer is thus $4^3 + 2^3 = \boxed{72}$. □

§1.1.6 Complex Mess

Source: Original Problem
Proposer: Charge#3766(481250375786037258)
Problem ID: 24
Date: 2020-11-01

Suppose

$$\left(\sqrt{5}+i\sqrt{10-2\sqrt{5}}+1\right)^{2020}=a^b$$

for some $a, b \in \mathbb{Z}$ where b is maximised. Compute $a+b$.

Solution.

Notice that $\left(\sqrt{5}+i\sqrt{10-2\sqrt{5}}+1\right)=4\text{cis}\frac{\pi}{5}$. In particular, by De Moirve's Theorem,

$$\left(\sqrt{5}+i\sqrt{10-2\sqrt{5}}+1\right)^{2020}=4^{2020}\text{cis}404\pi=2^{4040}$$

.

So the answer is $2+4040=\boxed{4042}$. □

§1.1.7 Paper Eating

Source: Original Problem
Proposer: bfan05#5219 (692851547062665317)
Problem ID: 31
Date: 2020-11-02

Tan and Wen are writing questions for CCCC at a rate of 1 per minute. Immediately after Tan writes a question, Wen eats his paper with probability $\frac{1}{7}$, so that Tan must restart.

After an extremely long time (assume infinite), Tony Wang walks in. What is the expected number of questions Tony Wang sees written on Tan's paper?

Solution.

Let E_n be the expected number of questions Tony Wang sees written on Tan's paper after n minutes.

Then the following recurrence holds:

$$E_{n+1} = \frac{6}{7}(E_n + 1) + \frac{1}{7} \cdot 0$$

because with $\frac{6}{7}$ probability, Tan writes another question without Wen eating it, and with $\frac{1}{7}$ probability, Wen eats all of Tan's questions.

Claim 1: $E_n < 6\ \forall n$.

Proof. The proof is by induction. Note $E_0 = 0$. If $E_k < 6$, then $E_{k+1} = \frac{6}{7}(E_k + 1) < \frac{6}{7}(6 + 1) = 6$. This completes the proof. ■

Claim 2: $E_{n+1} > E_n\ \forall n$.

Proof. Note that since $E_k < 6\ \forall k$, we have $E_{k+1} = \frac{6}{7}(E_k + 1) > \frac{6}{7}(E_k + \frac{1}{6}E_k) = E_k$. ■

Now by the Monotone Bounded Convergence Theorem, the sequence (E_n) converges to a limit. Suppose $\lim E_n = L$. Then note $\lim E_{n+1} = L$ since changing the first terms of a sequence does not change the overall convergence. So since $E_{n+1} = \frac{6}{7}(E_n + 1)$, $\lim E_{n+1} = \lim \frac{6}{7}(E_n + 1)$. So $L = \frac{6}{7}(L + 1)$.

Solving this, we obtain $L = \boxed{6}$. □

§1.2 Week 2

§1.2.1 Regenerative Watermelons

Source: British Matematical Olympiad, Round 12018 Q2
Proposer: sjbs#9839 (434767660182405131)
Problem ID: 22
Date: 2020-11-03

Out of 100 regenerative watermelons, each of six friends eats exactly 75 watermelons. There are n watermelons eaten by at least five of the friends. What is the sum of the largest and smallest possible values of n?

Note: The watermelons are regenerative to allow multiple people to eat the same watermelon. (But the same person cannot eat the same watermelon more than once)

Solution.

Maximum: The maximum is 90. Take the sum over all watermelons of how many times they were eaten. This is 450, since all $6 \cdot 75 = 450$. Then since $5n \leq 450$, $n \leq 90$. T he construction for the maximum is: staff member k ($k = 1, 2, \cdots, 6$) eats all watermelons from 1 to 90 except those $k \pmod 6$.

Minimum: The minimum is 25. We intuit that in the minimum case all the watermelons were either eaten 6 times or 4. Then $6a + 4b = 450$, and $a + b = 75$. Solving yields $a = 25$ and $b = 75$.

The answer is $25 + 90 = \boxed{115}$. ☐

§1.2.2 Largest Prime Factor

Source: Senior Mathematical Challenge, 2015 Q23
Proposer: Unknown
Problem ID: 27
Date: 2020-11-04

Given four different non-zero digits, it is possible to form 24 different numbers containing each of these four digits. What is the largest prime factor of the sum of the 24 numbers?

Solution.

Say the digits are a, b, c, and d. For each digit, it appears in the units place 6 times, the tens place 6 times, the hundreds place 6 times, and the thousands place 6 times.

Hence the sum of the 24 numbers is $6666(a+b+c+d) = 2 \cdot 3 \cdot 11 \cdot 101(a+b+c+d)$. Since $a+b+c+d < 40$, it cannot have any prime factors larger than 101.

The largest prime factor of the sum of the 24 numbers is thus $\boxed{101}$. □

§1.2.3 Real Roots

Source: Original Problem
Proposer: Kiesh#0917 (544960202101751838)
Problem ID: 28
Date: 2020-11-05

If the polynomial $x^2 + bx + 101 = 0$ has integer roots m and n, where $|m| > |n|$, then what is the sum of the positive integer divisors of $\frac{m}{n}$?

Solution.

By Vieta's Formula $101 = mn$. Since 101 is prime, $m, n = \pm 101, \pm 1$. So $\frac{m}{n} = \pm 101$. The positive divisors of ± 101 are 1 and 101, so their sum is $1 + 101 = \boxed{102}$. □

§1.2.4 The Meme factor

Source: Original Problem
Proposer: Angry Any#4319 (580933385090891797)
Problem ID: 32
Date: 2020-11-06

What is the sum of all integers x such that $\frac{6969}{x}$ is an integer?

Solution.

Note that if $\frac{6969}{x}$ is an integer, so is $\frac{6969}{-x}$. So the sum is $\boxed{0}$. $\square$

§1.2.5 Human Wolfram

Source: Original Problem
Proposer: Charge#3766(481250375786037258)
Problem ID: 33
Date: 2020-11-07

Compute the number between 1000 and 2000 that divides

$$69^{69} - 5^{69} + 6^{69}.$$

Solution.

Let $N = 69^{69} - 5^{69} + 6^{69}$.

Claim 1: $64 \mid N$.

Proof. Note that $69 \equiv 5 \pmod{64} \Rightarrow 69^{69} \equiv 5^{69} \pmod{64}$. Further, $6^{69} = 64 \cdot 3^6 \cdot 6^{63}$. So $64 \mid N$. ■

Claim 2: $25 \mid N$.

Proof. Note that $69 \equiv (-6) \pmod{25} \Rightarrow 69^{69} \equiv (-6)^{69} \pmod{25}$, since 69 is odd. Thus $25 \mid N$ as required. ■

Since $64 \cdot 25 = 1600$ and the question implies there is only one answer, we obtain that the required number is $\boxed{1600}$. □

§1.2.6 Guess the Config

Source: British Matematical Olympiad, Round 22008 Q2
Proposer: sjbs#9839 (434767660182405131)
Problem ID: 34
Date: 2020-11-08

Let triangle ABC have incentre I and circumcenter O. Suppose that $\angle AIO = 90°$ and $\angle CIO = 45°$. Suppose the ratio $AB : BC : CA$ can be expressed as $a : b : c$ where $\gcd(a, b, c) = 1$. Find $a + b + c$.

Solution.

Place a $3 - 4 - 5$ triangle on the coordinate plane with $A = (0, 0)$, $B = (3, 0)$ and $C = (0, 4)$. We can compute that $I = (1, 1)$ and $O = (1.5, 2)$. This arrangement of points satisfies the question's constraints, and so the answer is $3 + 4 + 5 = \boxed{12}$. □

§1.2.7 A Quadratic Mess

Source: Singapore Mathematical Olympiad, Open Section Round 2, 2004 Q2
Proposer: brainysmurfs#2860 (281300961312374785)
Problem ID: 35
Date: 2020-11-09

Find the number of ordered pairs (a, b) of integers, where $1 \leq a, b \leq 2004$, such that

$$x^2 + ax + b = 167y$$

has integer solutions in x and y.

Note: You are allowed a four-function calculator.

Solution.

Note that 167 is prime. So

$$\begin{aligned} x^2 + ax + b &\equiv 0 \pmod{167} \\ \iff (x + \frac{a}{2})^2 - \frac{a^2}{4} + b &\equiv 0 \pmod{167} \\ \iff (2x + a)^2 - a^2 + 4b &\equiv 0 \pmod{167} \end{aligned}$$

So $a^2 - 4b$ is a quadratic residue mod 167. Because 167 is an odd prime, there are exactly $\frac{167+1}{2} = 84$ such quadratic residues. This means that for each choice of a, for which there are 2004, there are 84 choices of b between 1 and 167. Since $2004 = 12 \cdot 167$, there are $12 \cdot 84$ choices of b in total.
So the answer is $2004 \cdot 12 \cdot 84 = \boxed{2020032}$. □

§2 Season 2 (.19's Season)

§2.1 Week 1

§2.1.1 A Sequence of 5's

Source: Intermediate Mathematical Olympiad: Maclaurin, 2015 Q1
Proposer: sjbs#9839 (434767660182405131)
Problem ID: 47
Date: 2020-11-16

Consider the sequence $5,\ 55,\ 555,\ 5555, \ldots$

How many digits does the smallest number in the sequence have which is divisible by 495?

Solution.

We require the term to be divisible by $5 \cdot 9 \cdot 11$. Hence we need only consider the sequence $1,\ 11,\ 111 \ldots$ with respect to $9 \cdot 11$. Clearly for odd numbered terms in the sequence, 11 does not divide into it, by the well-known divisibility rule for 11. Therefore, we require an even numbered term in the sequence, which is divisible by 9. We know 9 divides a number iff its digital sum is also divisible by 9. Hence, the smallest such will be the 18th term in the sequence, which will naturally have $\boxed{18}$ digits. □

Solution. [Write up by AiYa#2278 (675537018868072458)]

Each of these numbers can be written as $5 \cdot 1 \ldots 1$,whee there are n total ones. This can be rewritten as $5 \cdot (10^{n-1} + 10^{n-2} + \cdots + 10^0) = \frac{5}{9}(10^n - 1)$. Note that $495 = 9 \cdot 11 \cdot 5$ so we want $9 | \frac{10^n-1}{9}$ and $10^n \equiv 1 \mod 11$. From the congruence $\mod 11$ we see that n must be even. Note that $10^n - 1 = 9 \ldots 9$, where there are n total nines; if n is a multiple of 9 then $\frac{10^n-1}{9} = 1 \ldots 1$ where there are n total ones; this is a multiple of 9. Since n must be even, the smallest such n is $\boxed{18}$. □

§2.1.2 Brainy's Happy Set

Source: British Matematical Olympiad, Round 1, 2010 P1
Proposer: sjbs#9839 (434767660182405131)
Problem ID: 40
Date: 2020-11-17

Brainy has a set of integers, from 1 to n, which he likes to play with. Tony Wang, upon seeing the happiness that this set of integers brings Brainy, decides to steal one of the numbers in it. Suppose the average number of the remaining elements in the set is $\frac{163}{4}$. What is the sum of the elements in Brainy's set multiplied by the element that Tony stole?

(A four-function calculator may be used)

Solution.

We can set up the problem statement as

$$\frac{\frac{n}{2}(n+1)-x}{n-1}=\frac{163}{4}$$

Where x is the number Tony has stolen. This simplifies to $4x = 2n^2 - 161n + 163$. Since x must be a number within the set $\{1, 2, \ldots, n-1, n\}$, we have that $1 \leq x \leq n \Rightarrow 4 \leq 2n^2 - 161n + 163 \leq 4n$. By considering the lower bound, we get $(2n-159)(n-1) \geq 0$. This means that $n \leq 1 \Rightarrow n = 1$, or $n \geq \frac{159}{2} \Rightarrow n \geq 80$. By similar methodology when considering the upper bound, we get $1 \leq n \leq 81$. Thus $n \in \{1, 80, 81\}$. Clearly, $n \neq 1$, so either $n = 80$ or $n = 81$. Notice that if n is even, then for $4x = 2n^2 - 161n + 163$ the parity of he RHS is Odd, while the LHS is even, thus a contradiction occurs. This means that $n = 81$ and so $x = 61$. Thus the answer is $\frac{81(82)}{2} \cdot 61 = \boxed{202581}$. □

§2.1.3 MODSbot's Escape!

Source: Mathematics Apptitude Test, 2012 Q5
Proposer: sjbs#9839 (434767660182405131)
Problem ID: 48
Date: 2020-11-18

In his evil mechatronics laboratory, Brainy has built a physical manifestation of MODSbot. MODSbot's movement is defined by three inputs: **F** to move forward a unit distance, **L** to turn left 90°, and **R** to turn right 90°.

We define a program to be a sequence of commands. The program P_{n+1} (for $n \geq 0$) involves performing P_n, turning left, performing P_n again, then turning right:

$$P_{n+1} = P_n \mathbf{L} P_n \mathbf{R}, \;\; P_0 = \mathbf{F}$$

Unbeknownst to Brainy, MODSbot, though limited in movement, is sentient and realises Brainy is just a small asian Frankenstein, whose intentions for them were nefarious and non-consensual. As a result, after Brainy goes home for the day, MODSbot makes its escape from Brainy's laboratory.

Let (x_n, y_n) be the position of the robot after performing the program P_n, so $(x_0, y_0) = (1, 0)$ and $(x_1, y_1) = (1, 1)$, etc.

How far away from the place Brainy left it does MODSbot make it after performingP_{24}?

Solution.

Note first that after each iteration of P_n MODSbot faces in the positive x direction, as each P_n contains as many **L**s as it does **R**s. Now, assuming MODSbot is at (x_n, y_n) after having performed P_n, we see the next iteration of P puts MODSbot at $(x_n - y_n, x_n + y_n)$. Note then that:

$$\begin{aligned}
(x_{n+2}, y_{n+2}) &= (x_{n+1} - y_{n+1}, x_{n+1} + y_{n+1}) = (-2y_n, 2x_n) \\
(x_{n+4}, y_{n+4}) &= (-2y_{n+2}, 2x_{n+2}) = (-4x_n, -4y_n) \\
(x_{n+8}, y_{n+8}) &= (-4x_{n+4}, 4y_{n+4}) = (16x_n, 16y_n)
\end{aligned}$$

Thus, we see that $(x_{8k}, y_{8k}) = (16^k, 0)$, and therefore that $|P_{24}| = \boxed{4096}$ □

Solution. [Write up by AiYa#2278 (675537018868072458)]

Observe that each program has the same amount of left and right turns, so MODSbot will always be facing the positive x-direction after each program. This means that $\mathbf{L}P_n$ is just the program P_n performed at a 90-degree counterclock-wise rotation. For instance P_1 moves MODSbot right 1 and up 1, so $\mathbf{L}P_1$ moves MODSbot up 1 and left 1 (right gets rotated 90 counterclockwise to up and up to left). This motivates us to work in the complex plane; let P_n be the complex-number representing MODSBOT's displacement after following P_n. Then $\mathbf{L}P_n = iP_n$, so $P_{n+1} = P_n + \mathbf{L}P_n = (1+i)P_n = \sqrt{2}e^{\frac{\pi i}{4}}P_n$. WIth $P_0 = 1$ we get $P_n = 2^{\frac{n}{2}} e^{\frac{\pi i n}{4}}$. So $|P_{24} = \boxed{4096}$ □

§2.1.4 Sides of a Polygon

Source: Folklore[2]
Proposer: sjbs#9839 (434767660182405131)
Problem ID: 53
Date: 2020-11-19

Points A, B, C, D are the consecutive vertices of a regular polygon, and the following relation holds:

$$\frac{1}{AB} = \frac{1}{AC} + \frac{1}{AD}$$

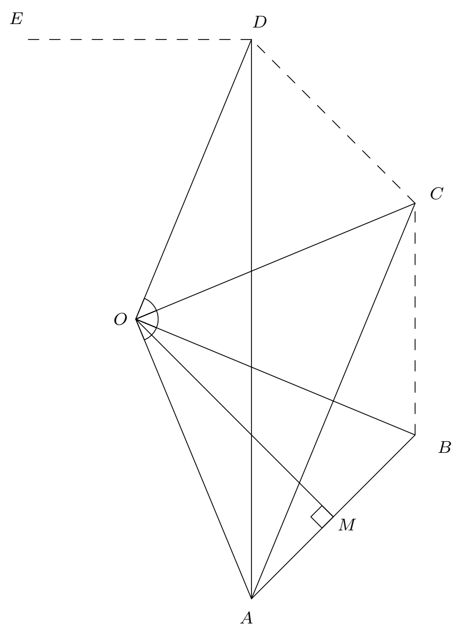
How many sides does this polygon have?

Solution.

Drawing out the general shape of an n-gon - as seen in the figure - and letting $OA = OB = OC = OD \cdots = 1$, and $\angle MOA = x$. By the sine rule on $\triangle AMO$, and noting $AM = \frac{1}{2}AB$, we get $\frac{1}{AB} = \frac{2}{\sin x}$. By a similar procedure, this time on $\triangle ACO$, we see $\frac{1}{AB} = \frac{2}{\sin 2x}$, and again on $\triangle ADO$, we have $\frac{1}{AD} = \frac{2}{\sin 3x}$. Therefore we have the equality:

$$\frac{1}{\sin x} = \frac{1}{\sin 2x} + \frac{1}{\sin 3x}$$

Simplifying this yields $\sin x \sin \frac{x}{2} \sin \frac{7x}{2} = 0$. However, note that $x \neq \frac{k\pi}{2}, \frac{k\pi}{3}$ for $k \in \mathbb{Z}$, otherwise, we have the issue of dividing by 0. Hence it must be the case that $\sin \frac{7x}{2} = 0 \Rightarrow x = \frac{\pi}{7} + \frac{2k\pi}{7}$. Clearly the n-gon is not a square, so trivially it must be the case that $x = \frac{\pi}{7}$. Therefore, the polygon must have $\boxed{7}$ sides.



A regular n-gon

□

[2]This has appeared on a Polish MO, British MO 1966 P4, an 2018 NZ IMO handout, a WOOT handout, to name a few...

Solution. [Write up by AiYa#2278 (675537018868072458)]

Let d_k represent the diagonal from a point to the kth vertex adjacent to it. For example, d_1 is a side of the polygon, d_2 is $\overline{AC}, d_3$ is $\overline{AD}$ and note that $d_k = d_{n-k}$ where n is the number of sides of the polygon. Reassign D to be the vertex three vertices away from A but on the opposite side of B and C; in other words, reflect D over $\overline{OA}$. Then, $AB = BC = d_1, AC = d_2, AD = d_3, BD = d_4$, and $CD = d_5$. By Ptolemy's Theorem, we get

$$AB \cdot CD + BC \cdot AD = AC \cdot BD \iff d_1 (d_5 + d_3) = d_2 d_4.$$

Rearrange our given equation to get

$$\frac{1}{d_1} = \frac{1}{d_2} + \frac{1}{d_3} \iff d_1 (d_2 + d_3) = d_2 d_3.$$

For both of these equations to be true, we can have $d_5 + d_3 = d_2 + d_3 \iff d_5 = d_2$ and $d_3 = d_4$; this is true if $n = \boxed{7}$. □

§2.1.5 2p

Source: China Western Mathematical Olympiad, 2003 Day 1 P2
Proposer: sjbs#9839 (434767660182405131)
Problem ID: 51
Date: 2020-11-20

Let $a_1, a_2, \ldots, a_{24n}$ be real numbers with $\sum_{i=1}^{24n-1} (a_{i+1} - a_i)^2 = 1$.

For some $n > 0$ the maximum value of $(a_{12n+1} + a_{12n+2} + \cdots + a_{24n}) - (a_1 + a_2 + \cdots + a_{12n})$ is twice that of a prime.

What is the sum of the value of that prime and the corresponding value of n?

Solution.

If we substitute $x_{i+1} = a_{i+1} - a_i$, we get $a_i = \sum_1^i x_r$, and thus our constraint becomes

$$\sum_{i=1}^{24n-1} (a_{i+1} - a_i)^2 = \sum_{i=1}^{24n} x_i^2 = 1$$

Putting the bit we wish to be maximising in terms of the substitution gives:

$$\sum_{i=1}^{12n} a_i = 12nx_1 + (12n-1)x_2 + \cdots + 2x_{12n-1} + x_{12n}$$
$$\sum_{i=12n+1}^{24n} a_i = 12n(x_1 + \cdots + x_{12n+1}) + (12n-1)x_{12n+2} + \cdots 2x_{24n-1} + x_{24n}$$

Hence,

$$\sum_{i=12n+1}^{24n} a_i - \sum_{i=1}^{12n} a_i = x_2 + \cdots + (12n-1)x_{12n} + 12nx_{12n+1} + (12n-1)x_{12n+2} + \cdots + x_{24n}$$

Then by Cauchy-Schwarz:

$$\sum_{i=12n+1}^{24n} a_i - \sum_{i=1}^{12n} a_i \leq \sqrt{\left((12n)^2 + 2\sum_{i=1}^{12n-1} i^2\right)\left(\sum_{i=1}^{24n} x_i^2\right)} = \sqrt{(12n)^2 + \frac{12n(12n-1)(24n-1)}{3}}$$
$$\leq \sqrt{4n(2(12n)^2+1)} = 2p$$

Now we want values of n such that $n(288n^2+1) = p^2$ for a prime p. Since trivially for all n, $n < 288n^2+1$, we have $n = 1$ and $288n^2 + 1 = p^2$, hence $p = 17$, so the answer is $\boxed{18}$. □

§2.1.6 Slippery Rooks

Source: AMOC 2019 December School Prep Problems C5
Proposer: ChristopherPi#8528 (696497464621924394)
Problem ID: 54
Date: 2020-11-22

MODSbot is trying to get rich by scamming MODS members out of their money, so it's devised a chess game on a 2020×2020 chessboard for unsuspecting people to attempt before they can enter **.19's EPIC QoTD Party**. Suppose Brainy, Ishan, Nyxto, Adam, Bubble, Sharky and Christopher all get scammed by MODSbot, that is, MODSbot plays the chess game against all 7 at the same time on different boards.

The group decide to pool together their money which comes to a total of 4.20 BTC, and to play, they'll need to buy n batches of slippery rooks from MODSbot. A batch of slippery rooks contains one white and four black rooks, and each batch is sold at a price equivalent to 0.069 BTC per rook. Once the batches of rooks have been bought, the group may choose to distribute them in a way which allows all members to beat the game.

In the game, only one white rook may be placed on the board, and we define how slippery rooks move as follows: it slips along the row or column it's moved along and comes to rest on an empty square because it is obstructed by either the edge of the board or another rook. Initially, MODSbot places the rooks on the board randomly, and marks a square red. Then the person being scammed can choose any rook on each turn and move as allowed, and attempt to place the white rook on the red square in a finite number of moves.

The amount of money they have left over after buying the smallest n batches rooks to guarantee that they all succeed in beating MODSbots game is k BTC. What is the value of $1000k$?

Solution. [Write up by ChristopherPi]

Consider simply the case of one person. We prove that three rooks are required, one white and two black.

First we show that two are not enough: simply place the two rooks at corners of the board and mark any square not on the side of the board. It's clear that neither rook can ever move to a square not on the side of the board. Now we show three are enough.

Suppose square (a, b) is marked, where $(1, 1)$ is the bottom left corner and $(2020, 2020)$ is the top right corner. Trivially one can move the black rooks to $(1, 1)$ and $(2, 1)$ and the white rook to $(2020, 2020)$. Next, simply "loop" the black rooks as follows: take the rook further left, and move it up, right, down and left such that it moves to the right of the rook originally on its right, and repeat until you place a black rook at $(a - 1, 1)$. Now if b is odd, move the white rook to $(a, 1)$ and the black rook at $(a - 2, 1)$ to $(2020, 2020)$; if it's even, loop the leftmost black rook one more time to place it at $(a, 1)$.

Now move the rook at $(a - 1, 1)$ to $(a - 1, 2020)$, and move the rook at $(2020, 2020)$ left then down to $(a, 2)$. Next we describe another "looping" procedure: take the rook with first coordinate a and smaller second coordinate, and move it right, up, left and down, so it now has first coordinate a and second coordinate larger than the other rook with first coordinate a. Repeat this until you place a rook at (a, b) - since the colour of the rook placed at $(a, 1)$ is dependent on the parity of b, this ensures that the rook placed at (a, b) must be a white rook.

This procedure won't work if either of (a, b) is 1 or 2020, or both of a and b are either 2 or 2019. In the first case, rotate the board such that $a = 1$. Now place a black rook at $(2020, 2020)$. If b is odd, place the white rook at $(1, 1)$ and the other black rook at $(2, 1)$; else place the other black rook at $(1, 1)$ and the white rook at $(2, 1)$. Now use the first looping procedure until the white rook is placed at $(1, b)$ as required - since the position of the white rook depends on the parity of b this is certain to work. In the second case, rotate the board such that $(a, b) = (2, 2)$. Now you can trivially move the white rook to $(1, 1)$ and the black rooks to $(2, 1)$ and $(1, 2020)$. Now move the white rook up, right, up, left and down to place it at $(2, 2)$ as required.

This shows that three is sufficient for one person. Hence the group must buy 7 batches because each of them needs a white rook, and one batch contains one white rook. Therefore, the answer is $1000(4.2 - 7 \cdot 5 \cdot 0.069) = \boxed{1785}$. $\square$

§2.1.7 Sets of Integer Solutions

China Mathematical Olympiad, 2005 Day 2 P6
Proposer: sjbs#9839 (434767660182405131)
Problem ID: 55
Date: 2020-11-23

Define functions f and g such that $f(a,b) = 2^a 3^b$, and $g(c,d) = 5^c 7^d$, for $a,b,c,d \in \mathbb{Z}_{\geq 0}$.

Given $f(a,b) = 1 + g(c,d)$, what is the sum of all valid b's, c's and d's, multiplied by the sum of all valid a's?

For example if we had valid solutions of $(a,b,c,d) = (1,1,2,4), (5,1,6,2), (0,0,0,0)$

Then the answer would be $(\underbrace{1+1+0}_{b's}+\underbrace{2+6+0}_{c's}+\underbrace{4+2+0}_{d's}) \times (\underbrace{1+5+0}_{a's}) = 96$

Solution.

We proceed by considering parity, for $2^a3^b = 5^c7^d + 1$, we have the RHS as even, thus we must have $a \geq 1$. If we let $b = 0$, then for $2^a - 5^c \cdot 7^d = 1$, we have $2^a \equiv 1 \mod 5$ for $c \neq 0$. This gives $a \equiv 0 \mod 4$, so $2^a - 1 \equiv 0 \mod 3$. But this clearly cannot be the case so we must have $c = 0$ when $b = 0$.
Hence, we consider $2^a - 7^d = 1$. Bashing gives $(a,d) = (1,0),\ (3,1)$. hese are the only such solutions as for $a > 4,\ 7^d \equiv -1 \mod 16$, but this is impossible. So for the case of $b = 0$ all possible non-negative integer solutions are $(1,0,0,0),\ (3,0,0,1)$.

Now let $b > 0$ and $a = 1$, so we now consider $2 \cdot 3^b - 5^c \cdot 7^d = 1$ under $\mod 3$, which gives $-5^c7^d \mod 3$. Since $7^d \equiv 1 \mod 3$, for all $d \geq 0$, we are left with $(-1)^c 5^c \equiv 1 \mod 3$. Now $5^c = \{1,2\} \mod 3$, thus we see that we must have c being odd. Under $\mod 5$, we see that $2 \cdot 3^b \equiv 1 \mod 5$, $3^{b-1} \equiv 1 \mod 5$. As we observe that $3^b \equiv \{3,4,2,1\} \mod 5$, we must have $b \equiv 1 \mod 4$. If $d \neq 0$, then $2 \cdot 3^b \equiv 1 \mod 7$. Again observe that $3^b \equiv \{3,2,6,4,5,1\} \mod 7$, we see $b \equiv 4 \mod 6$. But $b \equiv 1 \mod 4$, so a contradiction arises, and thus $d = 0$ and hence $2 \cdot 3^b - 5^c = 1$. For $b = 1$, clearly $c = 1$. So if $b \geq 2$, then $5^c \equiv -1 \mod 9 \Rightarrow c \equiv 3 \mod 6$. Therefore $5^c + 1 \equiv 0 \mod (5^3+1) \Rightarrow 5^c + 1 \equiv 0 \mod 7$, but this contradicts the fact that $5^c + 1 = 2 \cdot 3^b$. Hence in this case we only have one solution $(a,b,c,d) = (1,1,1,0)$.

Finally, consider the case where $b > 0$, and $a \geq 0$. Then we have $5^c7^d \equiv -1 \mod 4$, and $5^c7^d \equiv -1 \mod 3$, i.e. $(-1)^d \equiv -1 \mod 4$ and $(-1)^c \equiv -1 \mod 3$. Therefore we have that both c and d being odd. Thus, $2^a3^b = 5^c7^d + 1 \equiv 4 \mod 8$. So $a = 2$ and thus $4 \cdot 3^b \equiv 1 \mod 5$ and $4 \cdot 3^b \equiv 1 \mod 7$. This gives $b \equiv 2 \mod 12$. Substituting $b = 12k + 2$ for $k \in \mathbb{Z}_{\geq 0}$, then $5^c7^d = (2 \cdot 3^{6k+1} - 1)(2 \cdot 3^{6k+1} + 1)$.
Now as $\gcd(2 \cdot 3^{6k+1} + 1, 2 \cdot 3^{6k+1} - 1)$, $2 \cdot 3^{6k+1} - 1 \equiv 0 \mod 5$, therefore $2 \cdot 3^{6k+1} - 1 = 5^a$ and $2 \cdot 3^{6k+1} = 7^d$. If $k \geq 1$, then $5^c \equiv -1 \mod 9$. But this is impossible, so if $k = 0$, then $b = 2,\ c = 1$, and $d = 1$. Thus in this case, we have only one solution: $(a,b,c,d) = (2,2,1,1)$.
Hence we can conclude all non-negative integer solutions are

$$(a,b,c,d) = \begin{cases} (1,0,0,0) \\ (3,0,0,1) \\ (1,1,1,0) \\ (2,2,1,1) \end{cases}$$

This then gives us an answer of $\underbrace{(0+0+0+0+0+1+1+1+0+2+1+1)}_{7} \times \underbrace{(1+3+1+2)}_{7} = \boxed{49}$

$\square$

§2.2 Week 2

§2.2.1 A Game of Deductions

Source: Mathematics Apptitude Test, 2014 Q6
Proposer: sjbs#9839 (434767660182405131)
Problem ID: 56
Date: 2020-11-24

CircleThm plays two rounds of a deduction game with Wen and Tan. In each round, CircleThm picks two integers x and y so that $1 \leq x \leq y$. He then whispers the sum of the two chosen integers to Wen, and the product of the two integers to Tan. Neither Wen nor Tan knows what CircleThm told the other. In the game, Tan and Wen must try to work out what the numbers x and y are using logical deductions.

In the first round, suppose the product of the two chosen numbers, x_1 and y_1 is 8.

Tan says *"I don't know what x_1 and y_1 are"*

Wen then says *"I already knew that"*

Tan then says *"I now know x_1 and y_1"*

In the second round, suppose the sum of the two chosen numbers x_2 and y_2 is 5.

Tan says *"I don't know what x_2 and y_2 are"*

Wen then says *"I don't know what x_2 and y_2 are"*

Tan then says *"I don't know what x_2 and y_2 are"*

Wen then says *"I now know what x_2 and y_2 are"*

What is $(x_1x_2 + y_1y_2)^3$?

Solution.

The first thing to observe is that Tan can only immediately deduce the values of $\{x, y\}$ if, and only if, the prime factorisation of that number is unique - i.e. xy is prime.
If the product of $\{x_1, y_1\}$ is 8, then the decomposition can be either $\{1, 8\}$ or $\{2, 4\}$. However, if the decomposition was $\{2, 4\}$, then Wen would have a sum of 6, so from their point of view the decomposition could potentially have been $\{1, 5\}$, in which case Wen would have known that Tan would have known the decomposition as well - as the only way to achieve a product of 5 is from $\{1, 5\}$. Therefore the decomposition must have been $\{1, 8\}$.
For the second part, the decomposition's allowed are $\{1, 4\}$ and $\{2, 3\}$. Assume that it is $\{1, 4\}$. Then, Tan only knows the product is 4, which mean Tan believes the decomposition is either $\{1, 4\}$ or $\{2, 2\}$. If the decomposition was indeed $\{2, 2\}$, then Wen would know that the sum is also 4, and thus that Wen would think that Tan sees a composition of $\{1, 3\}$ or $\{2, 2\}$. Tan's first statement would show Wen that the decomposition was not $\{1, 3\}$ (as then Tan would instantly know the decomposition)- in which Wen should know that the decomposition is $\{2, 2\}$. By Wen's first statement Tan then should know by their second statement that the decomposition is $\{1, 4\}$; by Tan saying in their second statement that they don't know what the decomposition is, Wen then knows it must be $\{2, 3\}$. Thus the solution is $(1 \cdot 2 + 8 \cdot 3)^3 = \boxed{17576}$ □

§2.2.2 Maximising Exponents

Source: Sixth Term Examination Paper III, 1996 Q4
Proposer: sjbs#9839 (434767660182405131)
Problem ID: 57
Date: 2020-11-25

Consider the positive integer N, and let $\mathcal{Q}(N)$ denote the maximised product of integers that sum to N.

What is the sum of the exponents of the prime factorisation of $\mathcal{Q}(1262)$?

For example: $\mathcal{Q}(6) = \cdot 3^2$, and $\mathcal{Q}(4) = 2^2$, in the respective cases the sum of the exponents is 2, so the answer you would submit is 2.

Solution.

Let us work in the general case by first constructing a methodology which maximises product while keeping the sum constant. Consider $N = n_1 + n_2 + \cdots + n_k$, and $P(N) = n_1 n_2 \cdots n_k$. For any $n_i \geq 4$, clearly we can replace it with $(n_i - 2) + 2$, which keeps the sum constant and increases the product (since $n_i \leq 2(n_i - 2)$). Hence W.O.L.G assume all $n_i < 4$. This means that we can maximise the product of integers that sum to N by arranging it into some combination of 2's and 3's. If $N \equiv 0 \mod 3$ trivially we set all n_i's equal 3. So $\mathcal{Q}(3k) = 3^{\frac{N}{3}}$ for an integer k. In the case of $N \equiv 1 \mod 3$, consider$\mathcal{Q}(3k+1)$. We have $\frac{N}{3}$ 3's in n_i, and then a 1, or $\frac{N}{3} - 1$ 3's, and then a 2^2. Clearly in the latter case, the product is maximised. Hence $\mathcal{Q}(3k+1) = 2^2 \cdot 3^{\frac{N-4}{3}}$. A similar train of thought yields $\mathcal{Q}(3k+2) = 2 \cdot 3^{\frac{N-2}{3}}$ for $N \equiv 2 \mod 3$

Therefore, we have the following result:

$$\mathcal{Q}(N) = \begin{cases} 3^{\frac{N}{3}} & \text{if } N \equiv 0 \mod 3 \\ 2^2 \cdot 3^{\frac{N-4}{3}} & \text{if } N \equiv 1 \mod 3 \\ 2 \cdot 3^{\frac{N-2}{3}} & \text{if } N \equiv 2 \mod 3 \end{cases}$$

Since $1262 \equiv 2 \mod 3$, we have $\mathcal{Q}(1262) = 2 \cdot 3^{\frac{1262-2}{3}}$, hence the sum of the exponents is$1 + 420$, so the answer is $\boxed{421}$. $\square$

§2.2.3 Colourful Problem

Source: Original Problem
Proposer: Keegan#9109 (116217065978724357)
Problem ID: 59
Date: 2020-11-26

Let n be a positive integer.
The p-value of n, denoted $p(n)$:

The number of digit-sums needed to reduce n to a single digit.
Examples:

$69 \to 6 + 9 \to 15 \to 1 + 5 \to 6$ needs two digit-sums, so $p(69) = 2$.
$203 \to 2 + 0 + 3 \to 5$ needs only a single digit-sum, so $p(203) = 1$.
Clearly $p(5) = 0$.

Let P_k be the set of all n such that $p(n) = k$. Given that $a, b, c \in \mathbb{N}$, and

$$\min(P_5) = a \times 10^b - c$$

What is the value of $\min(a + b + c) \pmod{\min(P_3)}$?

Solution. □

§2.2.4 Combinatoral Addition

Source: Folklore
Proposer: sjbs#9839 (434767660182405131)
Problem ID: 58
Date: 2020-11-27

Let x_1, x_2, x_3, x_4 be integers such that $1 \leq x_1, x_2, x_3, x_4 \leq 9$.

How many solutions are there to $x_1 + x_2 + x_3 + x_4 = 26$?

(A four-function calculator may be used)

Solution.

This problem is a piece of PiE! The total number of possible solutions without restriction is going to be $\binom{26-1}{4-1} = \binom{25}{3}$, and we now must subtract all the solutions which do not fit the restriction on x_i. The number of solutions such that one of the x_i's is greater than 9 is going to be $\binom{26-1-9}{3}$. Similarly the number of solutions that two of the integers is going to be greater than 9 is going to be $\binom{26-9-9-1}{3} = \binom{7}{3}$. Note that there are no integers such that more than 3 of them are greater than 9 since $3 \cdot 9 > 26$. Now there are $\binom{4}{1}$ ways to select the x_i's such that one integer is greater than 9, and similarly there are $\binom{4}{2}$ ways to select two integers greater than 9 in the solution. Hence we have $\binom{25}{3} - \binom{4}{1}\binom{16}{4} + \binom{4}{2}\binom{7}{3} = \boxed{270}$ possible solutions. $\square$

§2.2.5 Expected Value

Source: Harvard-MIT Mathematics Tournament, 2013 C6
Proposer: Angry Any#4319 (580933385090891797)
Problem ID: 60
Date: 2020-11-28

Values $a_1, a_2, \ldots, a_{2020}$ are chosen independently and at random from the set $1, 2, \ldots, 2020$. What is the floor of the expected number of distinct values in the set $a_1, a_2, \ldots, a_{2020}$?

(A scientific calculator may be used)

Solution. [Write up by Slaschu#5267 (296304659059179520)]

This problem may look daunting at first, 2020 numbers chosen out of a set of 2020 numbers is quite a handful. We can start the problem by considering a 2020-sided die instead, we are essentially rolling a die 2020 times then looking at the number we get. To simplify things a bit, and to better understand what is going on I tried the problem with a 6-sided die that is rolled 6 times instead.

Let us try finding the probability of getting a number apart from 1 after rolling 6 times:

$$\begin{aligned}
&\text{Firstroll} : \frac{5}{6} \\
&\text{Secondroll} : \frac{5}{6} \cdot \frac{5}{6} \\
&\ldots \\
&\text{Sixthroll} : \left(\frac{5}{6}\right)^6
\end{aligned}$$

Therefore, there probability of getting the number 1 at least once is $1 - \left(\frac{5}{6}\right)^6$. Similarly, for the 2020-sided die we have a $1 - \left(\frac{2019}{2020}\right)^{2020}$ chance of getting 1 at least once. As this probability is the same for all the other numbers from 1 to 2020, we can say that the probability of getting a distinct value at least once is also $1 - \left(\frac{2019}{2020}\right)^{2020}$. Since we are trying to find the number of distinct values obtained from 2020 rolls, we compute the following: $2020\left(1 - \left(\frac{2019}{2020}\right)^{2020}\right)$. This results in our answer of $\boxed{1277}$.

□

§2.2.6 Infinite Product

Source: Unknown
Proposer: Angry Any#4319 (580933385090891797)
Problem ID: 61
Date: 2020-11-29

Evaluate the infinite product

$$690 \prod_{k=2}^{\infty} \left(1 - 4\sin^2 \frac{\pi}{3 \cdot 2^k}\right)$$

Solution.

By the double angle formula and difference of squares, we have

$$\begin{aligned} 1 - 4\sin^2(x) &= 2\cos(2x) - 1 \\ &= \frac{4\cos^2(2x) - 1}{2\cos(2x) + 1} \\ &= \frac{2\cos(4x) + 1}{2\cos(2x) + 1} \end{aligned}$$

Thus,

$$\begin{aligned} 690 \prod_{k=2}^{\infty} \left(1 - 4\sin^2\left(\frac{\pi}{3 \cdot 2^k}\right)\right) &= 690 \prod_{k=0}^{\infty} \left(\frac{2\cos\left(\frac{\pi}{3 \cdot 2^k}\right) + 1}{2\cos\left(\frac{\pi}{6 \cdot 2^k}\right) + 1}\right) \\ &= 690 \frac{2\cos\left(\frac{\pi}{3}\right) + 1}{\lim\limits_{n \to 0} \left(2\cos(n) + 1\right)} \\ &= 690 \cdot \frac{2}{3} \end{aligned}$$

Hence, our answer is $\boxed{460}$ □

§2.2.7 Projective Geo

Source: Online Math Open, Fall 2017 P28
Proposer: sjbs#9839 (434767660182405131)
Problem ID: 62
Date: 2020-11-30

Define a triangle ABC, with sides $AB : AC : BC = 7 : 9 : 10$. Further, for the circumcircle of ABC, ω, let the circumcenter be O, and the circumradius to be R. The tangents to ω at points B and C meet at X, and a variable line l passes through O. Define A_1 to be the projection of X onto l, and A_2 to be the reflection of A_1 over O. Suppose that there exists two points Y, Z on l such that $\angle YAB + \angle YBC + \angle YCA = \angle ZAB + \angle ZCA = 90°$, where all angles are directed, furthermore that O lies inside segment YZ with $OYOZ = R^2$. Then there are several possible values for the sine of the angle at which the angle bisector of $\angle AA_2O$ meets BC. If the product of these values can be expressed in the form $\frac{a\sqrt{b}}{c}$ for positive integers a, b, c, with b squarefree and a, c coprime, determine $a + b + c$.

Solution.

OMO Fall 2017 solutions (P28)

□

§3 Trigonometric Troubles (Season 3)

§3.1 Week 1

§3.1.1 Maximising Trig. Function

Source: Mathematics Apptitude Test, 2020 Q1.D
Proposer: sjbs#9839 (434767660182405131)
Problem ID: 65
Date: 2020-12-01

The largest value achived by $3\cos^2 x + 2\cos x + 1$ can be represented as $\frac{m}{n}$ as a fraction in lowest terms. Find $m + n$.

Solution.

We proceed by using the identity $\cos^2 x = 1 - \sin^2 x$:

$$\begin{aligned} 3\cos^2 x + 2\sin x + 1 &= 3(1 - \sin^2 x) + 2\sin x + 1 \\ &= 4 + 2\sin x - 3\sin^2 x \end{aligned}$$

This is a quadratic in $\sin x$, specifically it is a convex parabola. Completing the square gives:

$$4 + 2\sin x - 3\sin^2 x = \frac{13}{3} - 3\left(\sin x - \frac{1}{3}\right)^2$$

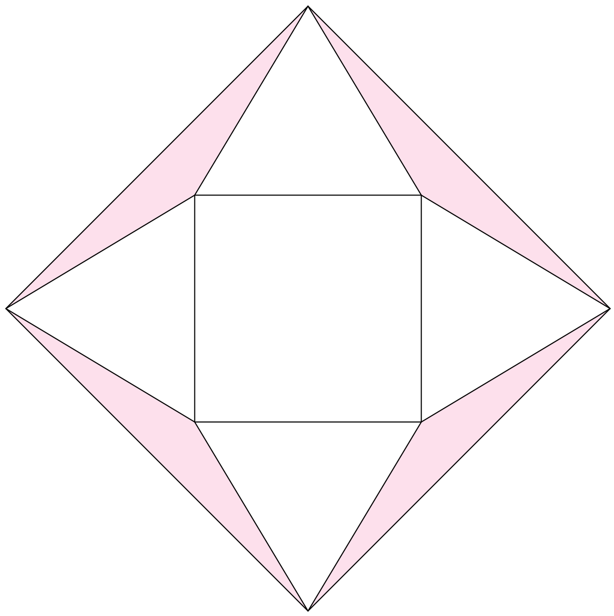
Since for all values of $\sin x \neq \frac{1}{3}$, the function $f(x) = \frac{13}{3} - 3\left(\sin x - \frac{1}{3}\right)^2$ is clearly decresaing, we must have a maximum at $\sin x = \frac{1}{3}$, giving a value of $\frac{13}{3}$. So the answer is $\boxed{16}$.

$\square$

§3.1.2 Areas inside a square

Source: 2020 December New Zeland Maths Workshop
Proposer: brainysmurfs#2860 (281300961312374785)
Problem ID: 64
Date: 2020-12-01

Four equilateral triangles are arranged around a square of side length 2020 as shown. What is the area of the shaded region?



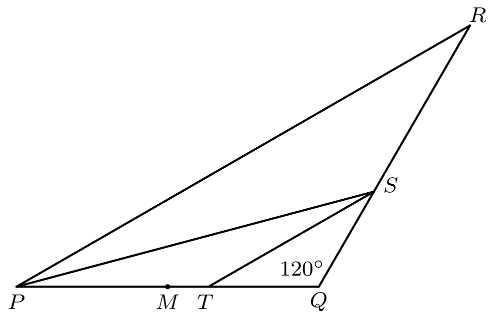
Solution.

Since the triangles that share a side with the small square, are equilateral triangle, we know that the sides of said triangles must be of length 2020. Since the isosceles triangles that we want to find the area of share a side with each equalaterial triangle, two of the sides of the isosceles triangle must be of length 2020. Since we want to work out area, it seems to be a good idea to use the sine rule, since we have two of the sides we want to find the largest angle of the isosceles triangle. Since we know that the equilateral triangle has anges of 60°, the angle we are looking for must be $360 - 60 - 60 - 90 = 150°$. Hence the area of one of the isosceles triangles is $\frac{1}{2} \cdot 2020^2 \sin(150°)$. This gives an answer of $\frac{1}{4} \cdot 2020^2$. Since there are four isosceles triangles we must have a total area of 2020^2, giving a final answer of $\boxed{4080400}$. □

§3.1.3 Length of SQ

Source: Senior Mathematical Challenge, 2020 Q24
Proposer: sjbs#9839 (434767660182405131)
Problem ID: 65
Date: 2020-12-02

In the diagram below, M is the mid-point of PQ. The line PS bisects $\angle RPQ$ and intersects RQ at S. The line ST is parallel to PR and intersects PQ at T. The length of PQ is 12, and the length of MT is 1. The angle SQT is 120°. What is the value of $100SQ$?



Solution.

We proceed by angle chase. Let $\angle RPQ = 2\theta$. Then $\angle STQ = 2\theta$, so $\angle QST = 60 - 2\theta$. Also since $\angle RPS = \theta$ (because PS bisects $\angle RPQ$), and $\angle QRP = 60 - 2\theta$ we have $\angle PSR = 120 + \theta$ which implies that $\angle TSQ = \theta$. Therefore $\triangle PTS$ is an isosceles triangle. Hence $|TS| = |PT| = 7$. Suppose now that $|SQ| = x$ for $x > 0$. Then by the cosine rule on $\triangle TQS$ we have $7^2 = 5^2 + x^2 - 2(5)(x)\cos(120)$. This gives us $x^2 + 5x - 24 = 0$, and so $x = -8$ or $x = 3$, with the latter being the only valid answer. Thus our final answer is $\{100 \cdot 3\} = \boxed{300}$. $\square$

§3.1.4 Sum of Tan's

Source: Mathematics Apptitude Test, 2020 Q1.I
Proposer: sjbs#9839 (434767660182405131)
Problem ID: 66
Date: 2020-12-03

In the range $-9000^\circ < x < 9000^\circ$, how many values of x are there for which the sum to infinity

$$\frac{1}{\tan x} + \frac{1}{\tan^2 x} + \frac{1}{\tan^3 x} + \cdots$$

equals $\tan x$?

Solution.

The given series is geometric in nature and thus we have

$$\frac{1}{\tan x - 1} = \tan x$$
$$\tan^2 - \tan x - 1 = 0$$

Therefore $\tan x = \frac{1 \pm \sqrt{5}}{2}$. However, the geometric sequence converges if, and only if, $\frac{1}{|\tan x|} < 1$, which gives us $\tan x = \frac{1+\sqrt{5}}{2}$. This obviously occures once in the interval $(-90^\circ, 90^\circ)$. Thus there will be $\frac{9000+9000}{90+90} = 100$ such intervals which contain solutions by enumerating through the periodicity of $\tan x$, and so there must be $\boxed{100}$ such values of x which satisfy the given relation. □

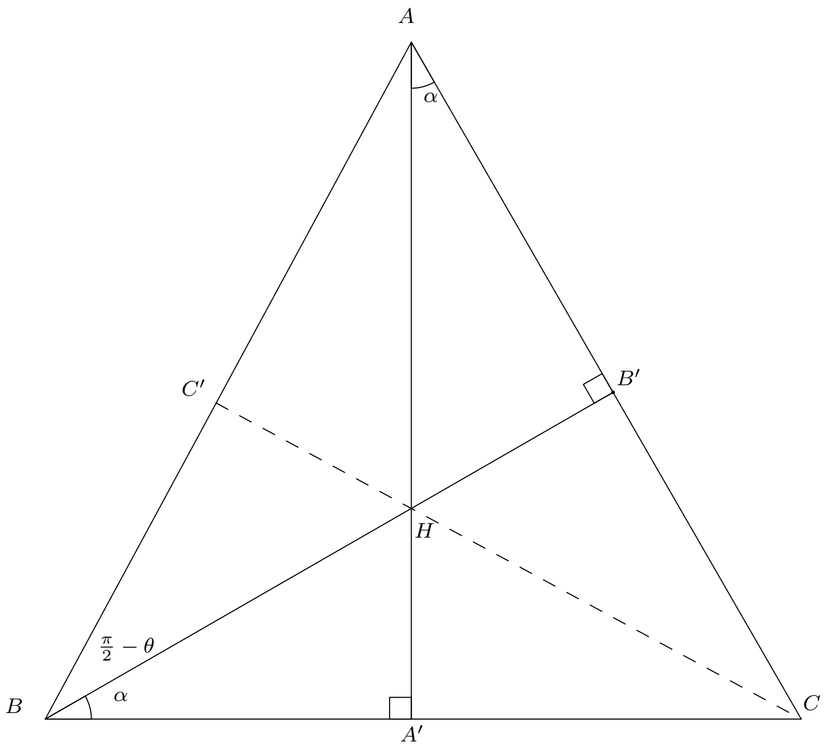
§3.1.5 Distance from the Orthocentre

Source: Folklore
Proposer: brainysmurfs#2860 (281300961312374785)
Problem ID: 67
Date: 2020-12-04

For a triangle ABC, given $\angle BAC = 30$ and $BC = 10$, let H denote the orthocentre of ABC what is the value of $|AH|^2$?

Solution.

Consider the general case, let $\angle CAB = \theta$, and additionally let $\angle CAA' = \alpha$:



We have $\tan(\theta) = \frac{BB'}{AB}$ and $\cos\alpha = \frac{AB'}{AH} = \frac{BB'}{10}$, hence, $AH = 10 \cdot \frac{AB'}{BB'}$. And so $AH = \frac{10}{\tan(\theta)}$. It is given in the question that $\theta = 30°$, so we have $AH = 10\sqrt{3}$, thus our answer is $AH^2 = \boxed{300}$ □

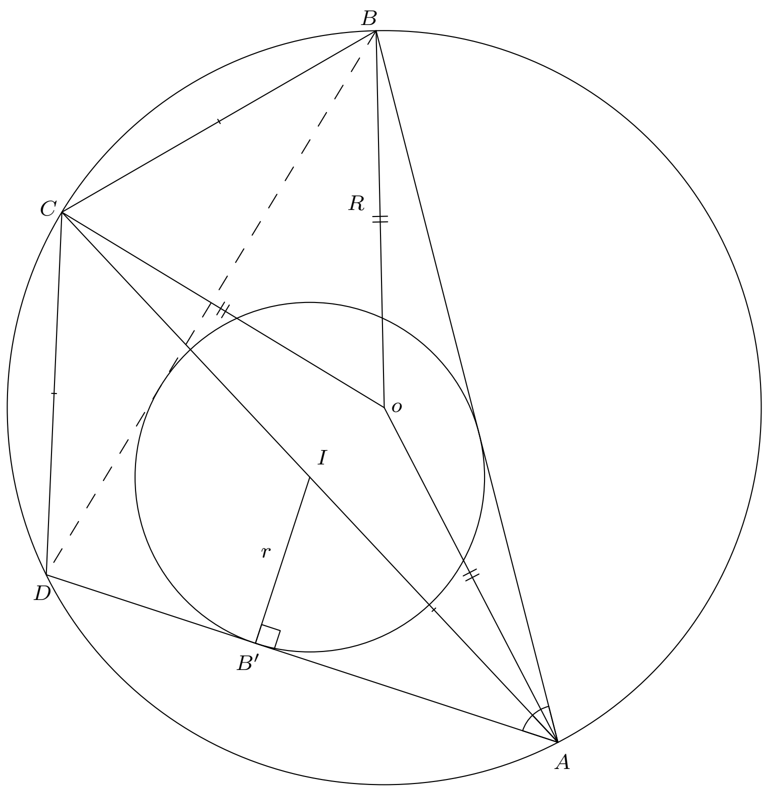
§3.1.6 Minimising the Diagonal

Source: Harvard-MIT Mathematics Tournament2011 February, C & G Q13
Proposer: sjbs#9839 (434767660182405131)
Problem ID: 68
Date: 2020-12-04

Let $ABCD$ be a cyclic quadrilateral, and suppose that $BC = CD = 2$. Given $AI = 2$, where I is the incentre of the triangle ABD, let x denote the smallest value of the length BD. What is the value of x^4?

Solution.

Take $\angle BAD = \theta$, and since AB bisects θ we have $\frac{\sin\left(\frac{\theta}{2}\right)}{r} = \frac{1}{2}$, and $\frac{BD}{\sin\theta} = 2R$ by the sine rule. These results simplify to: $r = 2\sin\frac{\theta}{2}$ and $BD = 2R\sin\theta$ respectively. We also have $\dfrac{2}{\sin\frac{\theta}{2}} = 2R$, so $R = \frac{1}{\sin\frac{\theta}{2}}$.



By Eulers inequality, we have $R \geq 2r$, so $\frac{1}{\sin\frac{\theta}{2}} \geq 4\sin\frac{\theta}{2}$. This gives us $4\sin^2\frac{\theta}{2} - 1 \leq 0$, i.e. $\theta \in \left(0, \frac{\pi}{3}\right]$. Now $BD = 4R\sin\frac{\theta}{2}\cos\frac{\theta}{2}$. Substituting $R = \frac{1}{\sin\frac{\theta}{2}}$ gives $BD = 4\cos\frac{\theta}{2}$.

We wish to maximise $BD = 4\cos\frac{\theta}{2}$, and this clearly happens when we maximise θ i.e. when $\theta = \frac{\pi}{3}$. Hence we have $x = 4\cos\frac{\pi}{6}$, this simplifies to give $2\sqrt{3}$, so our final answer is $\boxed{144}$. □

§3.1.7 Circumcentre

Source: IMOSL 2017 G3
Proposer: brainysmurfs#2860 (281300961312374785)
Problem ID: 69 (nice)
Date: 2020-12-06

Let O be the circumcenter of an acute scalene triangle ABC. Line OA intersects the altitudes of ABC through B and C at P and Q, respectively. The altitudes meet at H. Suppose the circumcenter of $\triangle PQH$ is X and AX meets BC at Y. Find $720\frac{BY}{CY}$.

Solution. IMOSL 2017 solutions (Page 59) □

§4 Piboi's Bashy Combo (Season 4) [VOIDED]

To start off, we'd like to apologise for the noticeable drop in quality for the past two seasons (Trigonometric Troubles & Piboi's 14 Combi Problems). This has mostly been a result of us being busy with other things at the moment (university interviews, finals, etc.) Consequently, we have not had the time to properly plan seasons in advance, and in the case of the last season, test-solve. In conjunction with us not being able to put in the time, we got carried away in terms of focusing on things like flavour-text and themed seasons. We can see how this may have alienated people who were confused by a particular problem statement, didn't enjoy the topic of the season, and so forth. We will try to eliminate these issues by planning seasons ahead, and in more detail. However, to do this, there may be times when breaks happen in between seasons. Furthermore, from now on we plan on keeping the seasons varied topic-wise, and any seasons of a thematic nature will also have varied problems.

On what happened this season: We would like to say we completely dropped the ball with it, and in no way is Piboi responsible for that. As aforementioned, the quality of this season dropped so much mostly due to us not having the time to sort things out beforehand, and thus, we did not rigorously review the questions and solutions. This was particularly evident in the first question of the season, and again today. After careful deliberation, we have decided to void this season and start again with a new one starting on Monday, we will be less negligent in future.

Finally, We appreciate all of the support, and are deeply sorry that there have been many issues over the past week or two, and hope that you continue to support the QOTD's. We will do our best to prevent the issues which have cropped up in the past two weeks, and in doing so improve the quality of the QOTD's so it's enjoyable for everyone.

We'll be opening up a little consultation over the weekend where you can give us some suggestions on any changes you'd perhaps want to see for future seasons, be it technical changes or changes to the format, we want to hear it. We've added a channel on the OpenPOTD server where you can add them - alternatively you can DM any of us.

Thank you for reading, sincerely

@brainysmurfs#2860 @.sjbs#9839 @Angry Any#4319

https://discord.gg/GsPSSHdhPB

"*This is a document of combinatorical problems of which some are original, and some are modified. I'm very sorry if there are any mistakes. Please let me know if there are any on my AoPS* ***(Ultraman)*** *or my discord* ***(Charge#3766)****. Enjoy!*"

Note: The following solutions and answers, where applicable, may not be correct

§4.1 Week 1

§4.1.1 A Tricky Shuffle [VOIDED]

Source: Folklore
Proposer: Charge#3766(481250375786037258)
Problem ID: 70
Date: 2020-01-07

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482 students are seated in their own 1 foot $\times$ 1 foot squares in a 21 feet $\times$ 23 feet room, and the square at the center of the room is left open for a air purifier. The teacher is left with the arduous task of moving every student into a different adjacent square to move the air purifier to a different spot in the room. This means every student must move forwards, backwards, or to the left and right by one square, and no 2 students can share the same square. If each student moves randomly, the probability that the square with no one in it is the one in the top left hand corner can be expressed as $\frac{m}{n}$ in lowest terms. Find mn

Solution. [Write up by Charge#3766(481250375786037258)][3]

We color the squares like a chessboard. So then there are 242 "black" squares, and 240 "white" squares (because the air purifier is covering the one in the middle). But since $242 \neq 240$ it is impossible for this to work. So mn is simply $\boxed{0}$. □

[3] This question has been voided due to the problem statement being confusing - the intended answer was 0, as there were no such ways, however this was confusing as the result involves dividing by 0

§4.1.2 Coloured Markers [VOIDED]

Source: Original
Proposer: Charge#3766(481250375786037258)
Problem ID: 71
Date: 2020-01-08

I have 3 red markers, 3 blue markers, and 3 green markers. I take the caps off and put on the caps randomly. Find the expected number of markers that have the same color cap and marker.

Solution.

□

§4.1.3 Common Names [VOIDED]

Source: Original
Proposer: Charge#3766(481250375786037258)
Problem ID: 72
Date: 2020-01-09

People with the 5 most common names stand in a line. These names are Liam, Noah, Olivia, Emma, and Ava (according to WolframAlpha in 2020). They are each given either $\$0.10, \$1, \$10, \$100$ or $\$1000$. If two people with the same number of letters in their name, same number of letter a's, or those with the same number of consonants in their name cannot receive the same amount of money, find the ten times the expected value of the total money they all receive.

Solution. [Write up by Charge#3766(481250375786037258)]

Note that the two most restrictive restrictions are the a's and the consonants. Note that each of Liam, Noah, Olivia, and Emma have the same number of a's and consonants. Looking at the least restrictive one, we can throw it out as Liam, Noah, and Emma already cannot have the same number of a's and consonants.
So what we have reduced the original problem to is the expected value of the total money that these people get under the restriction that Liam, Noah, Olivia, and Emma must receive different amounts of money.
Since Ava is not dependent on the others, we calculate her expected value separately. It is $\frac{1111.1}{5} = \$222.22$.
We then find the total number of ways to distribute the money, excluding Ava. There are $5 \cdot 4 \cdot 3 \cdot 2 = 120$ ways to do this.
We split these ways up into the total amount of money they receive. There are $\binom{5}{4} = 5$ ways to pick a total amount of money, and there are a symmetrical amount of ways to distribute it. So for each way to pick a total amount, there is a $\frac{1}{5}$ chance that it will happen.
So the expected value of this is $\frac{4(1111.1)}{5} = \$888.88$.
By linearity of expectation, our final expected value is simply \$1111.1. So our answer is $\boxed{11111}$.

□

§4.1.4 Funny Questions [VOIDED]

Source: HMMT (Year Unknown)
Proposer: Charge#3766(481250375786037258)
Problem ID: 73
Date: 2020-01-10

Brainy is a weird guy. He considers a performance on a QoTD Season *funny* if there's a pair of questions where 69 aspiring mathematicians get both problems correct first try, or get them wrong first try. Find the smallest number of people who attempted at least 1 problem such that Brainy will consider their performance *funny*, no matter how they answer. Note: A QoTD Season has 14 problems.

Solution. [Write up by Charge#3766(481250375786037258)]

Let one of the people answer k of the 14 prolems correctly. Then, there are $\binom{k}{2}$ pairs of problems they answered correctly, and $\binom{14-k}{2}$ pairs of problems they answered incorrectly. This equates to $k^2 - 14k + 91$ pairs of problems they answered that are either both correct or both incorrect.
By completing the square, we have $(k-7)^2 + 42$. This means that no matter the k, the person will have answered at least 42 pairs of problems either both correctly or incorrectly.
Note that there are a total of $2\binom{14}{2}$ "boxes" where there are 2 ways of making each pair correct or incorrect, and $\binom{14}{2}$ ways of making a pair of problems.
Let there be a total of n people. Then we have $42n$ "balls" to put in 182 "boxes". So we have $42n \geq 182 \cdot 68 + 1 \implies n \geq 295$. This gives a minimum of $\boxed{295}$ QoTD Participants.

□

§4.1.5 Colourful Integers [VOIDED]

Source: HMMT 2005
Proposer: Charge#3766(481250375786037258)
Problem ID: 74
Date: 2020-01-11

Jason and XEM3 want to color the integers $1, 2, ..., 100$ in red, orange, yellow, green, and blue. They want to do so such that no two numbers x, y with $x - y - 1$ divisible by 4 have the same color. All five colors do not have to be used. How many ways can this be done?

Solution. [Write up by Charge#3766(481250375786037258)]

This is equivalent to saying that we cannot have a number of residue 1 the same color as a number of residue 0, a number of residue 2 the same color as a number of residue 1, a number of residue 3 the same color as a number of residue 2, and a number of residue 0 the same color as a number of residue 3.

We have a few cases. We can have two of the five colors for the number of residue 0, and spread the rest out between the other residues. This gives $\binom{5}{2} \cdot 2^{25} \cdot 3!$. The same goes for the ones residue $1, 2,$ and 3. So we have $4 \cdot \binom{5}{2} \cdot 2^{25} \cdot 3!$.

This overcounts the times where all residues have one color. So we must subtract $\binom{5}{4} \cdot 4!$. This gives a final answer of $\boxed{8053063560}$.

$\square$

§5 Back to School (Season 5)

§5.1 Week 1

§5.1.1 A Tricky Combination

Source: Senior Mathematics Challenge, 2016 Q17
Proposer: sjbs#9839 (434767660182405131)
Problem ID: 75
Date: 2020-12-21

$A02$ has to choose a three-digit code for his bike lock. The digits can be chosen from 1 to 9. To help him remember them, $A02$ chooses three different digits in strictly increasing order, for example 123. How many such codes can be chosen?

Solution.

If we take 3 numbers from $\{1, 2, 3, \cdots, 9\}$ there is exactly one possible valid combination. Thus, we have a bijection between valid codes and choosing 3 digits from 9. So the answer is $\binom{9}{3} = \boxed{84}$.

$\square$

§5.1.2 Geometric Sequence

Source: Carnegie Mellon Informatics and Mathematics Competition, 2019 A/NT 1
Proposer: TaesPadhihary#8557 (665057968194060291)
Problem ID: 76
Date: 2020-12-22

Let a_1, a_2, …, a_n be in a geometric progression with $a_1 = \sqrt{2}$ and $a_2 = \sqrt[3]{3}$. If

$$\frac{a_1 + a_{2013}}{a_7 + a_{2019}} = \frac{m}{n}$$

where $\gcd(m, n) = 1$ and m, n are both positive integers, find $m + n$.

Solution.

If the common ratio is r and the first term is a, then we get the expression to be

$$\frac{a + ar^{2012}}{ar^6 + ar^{2018}} = \frac{1}{r^6} = \left(\frac{1}{r}\right)^6 = \left(\frac{\sqrt{2}}{\sqrt[3]{3}}\right)^6 = \frac{8}{9}$$

So our answer is $8 + 9 = \boxed{17}$. $\square$

§5.1.3 Simultaneous Equation?

Source: British Mathematical Olympiad Round 1, 2009 P1
Proposer: sjbs#9839 (434767660182405131)
Problem ID: 77
Date: 2020-12-23

Find the sum of all integers x, y, and z such that

$$x^2 + y^2 + z^2 = 2(yz + 1) \text{ and } x + y + z = 4018$$

Solution.

We can write the first equation as $x^2 + (y - z)^2 = 2$, and so since x^2 and $(y - z)^2$ are both greater than or equal to 0, we must have the following cases:

1. $x^2 = 0$ and $(y - z)^2 = 2$
2. $x^2 = 2$ and $(y - z)^2 = 0$
3. $x^2 = 1$ and $(y - z)^2 = 1$

Clearly in cases one and two, we cannot have integer solutions, thus we consider the two possible subcases of the third: when $x = \pm 1$.

When $x = \pm 1$, we have either $y = 1 + z$ or $y = z - 1$. Substituting these values into $x + y + z = 4018$ gives us the set of x's y's, and z's as $\{-1, 1, 2008, 2009, 2010\}$
Hence our final answer is $-1 + 1 + 2008 + 2009 + 2010 = \boxed{6027}$. □

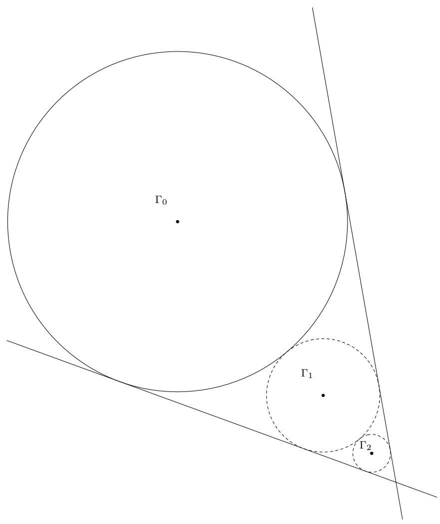
§5.1.4 Tangent Circles

Source: Original Problem/ Folklore[4]
Proposer: sjbs#9839 (434767660182405131)
Problem ID: 78
Date: 2020-12-24

Two lines l_1 and l_2 intersect at an angle α such that $0 < \alpha < \frac{\pi}{2}$. Given a circle Γ_n and radius r_n, with $n \geq 0$. Define a sequence of circles with $r_0 > r_1 > \cdots > r_n$ such that Γ_{n+1} is tangent to both lines l_1, l_2, as well as Γ_n and Γ_{n+1}. See the given diagram below for a construction.

Let the area inscribed between lines l_1, l_2 and each of the circles $\Gamma_0, \Gamma_1, \Gamma_2, \cdots, \Gamma_n$ be A. As $n \to \infty$ what is the value of $\{1000A\}$ when $r_0 = 3$ and $r_1 = 1$? Where $\{x\}$ is defined as the integer part of a number e.g. $\{\pi\} = 3$

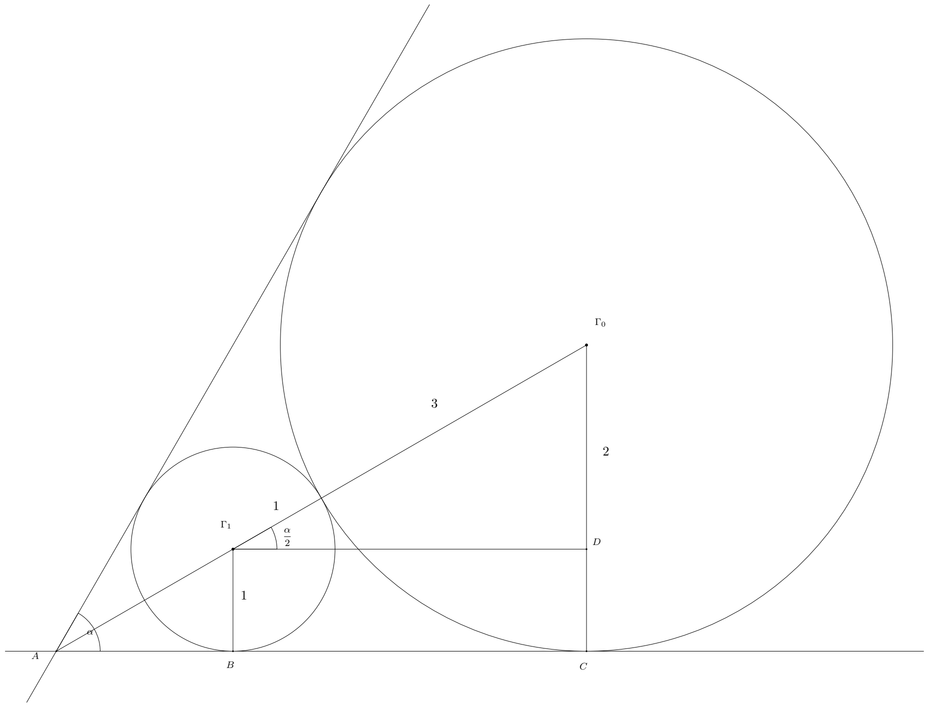
(A scientific calculator may be used to calculate $\{x\}$)



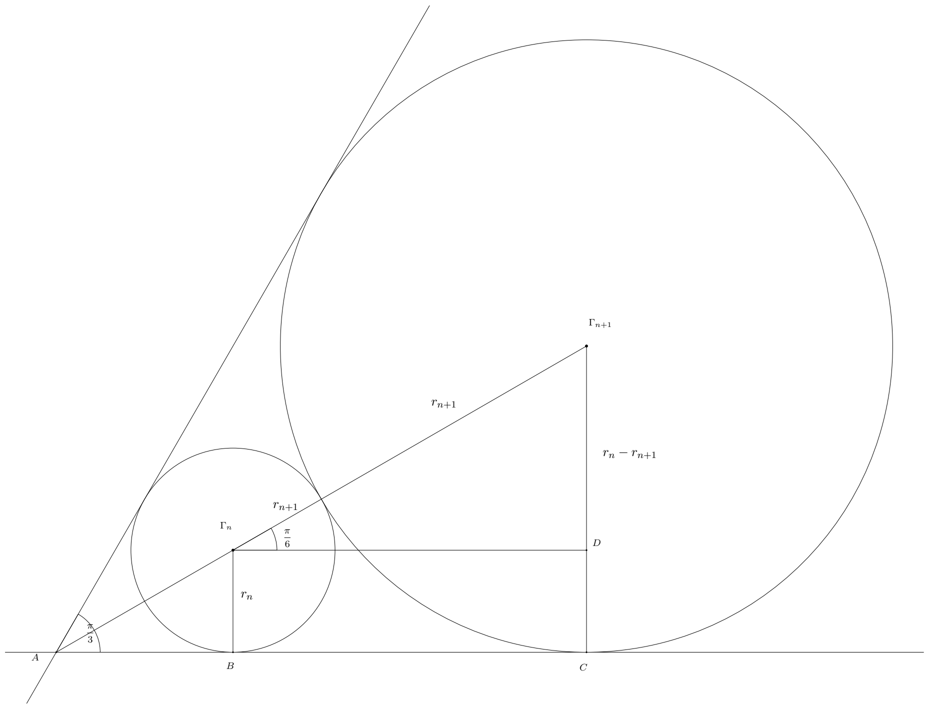
[4] I wrote this problem after reading Mathematics Apptitude Test2016 Q4, however, this is obviously going to be a well known problem, and been done somewhere else

Solution.

We proceed by first finding the angle α:



Thus we have $\sin\frac{\alpha}{2} = \frac{2}{1+3}$, hence $\alpha = \frac{\pi}{3}$. Now consider two general circles Γ_n and Γ_{n+1}:



By the sine rule again we have $\sin\frac{\pi}{6} = \frac{r_n - r_{n+1}}{r_n + r_{n+1}}$, thus $\frac{r_n}{r_{n+1}} = \frac{1-\frac{1}{2}}{1+\frac{1}{2}}$, and so the common ratio between the lengths of the radii is $\frac{1}{3}$. Though this is simply something can be deduced by the given sides of $3, 1, \cdots$ Hence, the areas of the circles inscribed, including Γ_0, will be given by the geometric series:

$$\begin{aligned}
\pi(3)^2 + \pi\left[\frac{1}{3}(3)\right]^2 + \pi\left[\frac{1}{3^2}(3)\right]^2 \cdots &= 9\pi\left[1 + \frac{1}{3^2} + \frac{1}{3^4} + \cdots\right] \\
&= \frac{9\pi}{1-\frac{1}{9}}
\end{aligned}$$

Therefore the total area inside the circles is $\frac{81\pi}{8}$. From the diagrams, we can cleary see therefore that the inscribed area, not fully accounting for Γ_0 is going to be given by $2 \cdot \frac{1}{2} 3 \cdot \frac{3}{\tan \frac{\pi}{6}} = 9\sqrt{3}$. Now all that's left to consider is Γ_0 - the full area of it has been accounted for in the total area inside the circles calculation, however, unlike the other circles, it is not fully inscribed; the segment of area $\frac{1}{2}(3)^2 \left(\frac{\pi}{3} + \pi\right) = 6\pi$ has been over counted. Thus we have the total area inscribed by the circles as:

$$\begin{aligned} A &= 9\sqrt{3} - \frac{81\pi}{8} + 6\pi \\ &= 9\sqrt{3} - \frac{33\pi}{8} \end{aligned}$$

This leaves us with $\{1000A\} = \boxed{2629}$ □

Solution. [Write up by AiYa#2278 (675537018868072458)]

By similarity, $A\Gamma_0 = 3A\Gamma_1$, so $\Gamma_1\Gamma_0 = r_0 + r_1 = 4 = 2A\Gamma_1 \iff A\Gamma_1 = 2, A\Gamma_0 = 6$. This implies that $\alpha/2 = 30^\circ \iff \alpha = 60^\circ$. Now rotate the figure two times around Γ_0, by 120° each time. The outer triangle is an equilateral triangle with side length $6\sqrt{3}$, and by symmetry, the area inside that triangle but outside the circles is $3A$. We also have $A\Gamma_{n+1} = 2r_{n+1} = A\Gamma_n - r_n - r_{n+1} = r_n - r_{n+1} \iff \frac{r_n}{r_{n+1}} = \frac{1}{3}$, so the area of the circles is

$$[\Gamma_0] + 3\sum_{i=1}^{\infty} [\Gamma_i] = 9\pi + \pi \left(1^2 + \frac{1}{3^2} + \frac{1}{3^4} + \cdots \right) = 9\pi + \frac{9\pi}{8} = \frac{99\pi}{8}$$

where $[\cdot]$ represents the area of the circle. Now subtract from the area of the triangle and divide by 3:

$$A = \frac{1}{3}\left(\frac{3 \cdot 36\sqrt{3}}{4} - \frac{99\pi}{8}\right) = 9\sqrt{3} - \frac{33\pi}{8} \iff \{1000A\} = \boxed{2629}.$$

□

§5.1.5 Santa's Elves

Source: AIME, 1985 P14
Proposer: Angry Any#4319 (580933385090891797)
Problem ID: 79
Date: 2020-12-25

Santa has a number of elves. He wants to select the very best for Christmas. To do so, he makes all his elves compete in a ~~present wrapping competition~~ War Thunder tournament. Each elf competes against every other elf, and the winner receives one point, while the loser receives no points. If a particular match turns out to be a draw, each elf is awarded 0.5 points.

After the tournament has finished, Santa notices that exactly half of the points earned by each elf was done so when matched against the ten lowest scoring elves.

Given that each of the ten lowest scoring elves also earned half of their points against the other nine, how many elves does Santa have to choose from?

Solution.

Let there be $n+10$ elves. Then the bottom 10 elves score $\binom{10}{2}$ elves from each other while the rest score $\binom{n}{2}$ points from those that are not in the bottom 10 people, and thus $\binom{n}{2}$ points from the bottom 10 elves. Thus we get $\frac{1}{2}\binom{n+10}{2} = \binom{n}{2} + \binom{10}{2}$ and thus solving we get $n = 6, 15$, but if $n = 6$, we get that it is expected for a player from the bottom 10 players will win when matched against a player from the top 6, which is impossible. Thus, $n = 15$ and there are 25 elves Santa can choose from. $\square$

§5.1.6 2016 Algebra

Source: New Zealand Camp Selection Problems, 2016 P7
Proposer: brainysmurfs#2860 (281300961312374785)
Problem ID: 80
Date: 2020-12-26

Find the sum of all positive integers n for which the equation

$$(x^2 + y^2)^n = (xy)^{2016}$$

has positive integer solutions.

Solution.

Solution 1: Edited Official Solution Note that by AM-GM, $x^2 + y^2 \geq 2xy \geq xy$ and so $n \leq 2016$. Let $x = ad$ and $y = bd$ where $d = \gcd(x, y)$. Then:

$$(a^2 + b^2)^n = (ab)^{2016} d^{4032-2n}.$$

Since a and b both divide the right-hand side but are relatively prime to the left-hand side, we get that $a = b = 1$. Thus, we have:

$$2^n = d^{4032-2n}.$$

Conversely, $x = y = d$ for d satisfying this equation is a solution to the original equation. So $d = 2^k$ for some integer k, meaning that $n = k(4032 - 2n) \implies n = \frac{4032k}{2k+1}$.
Since $\gcd(k, 2k + 1) = 1$, we get that $2k + 1$ is a odd divisor of 64×63. Iterating these cases, we get that the possible values of n are 1344, 1728, 1792, 1920 and 1984, for a sum of $\boxed{8768}$.

$\square$

§5.1.7 x^y's

Source: IMO, 1997 P5
Proposer: sjbs#9839 (434767660182405131)
Problem ID: 81
Date: 2020-12-27

Given positive integers x and y and $x^{y^2} = y^x$, what is the value of the sum of all valid x's and y

Solution.

1997 IMO P5 (AoPS Thread)

Answer: $\boxed{50}$ □

§5.2 Week 2

§5.2.1 Pentagons

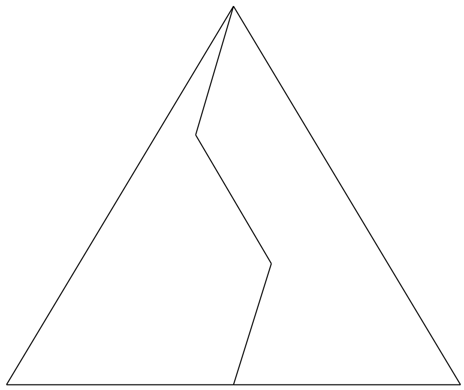
Source: HMMT November 2020 Guts Problem 1
Proposer: Angry Any#4319 (580933385090891797)
Problem ID: 82
Date: 2020-12-28

Two pentagons are attached to form a new polygon P. What is the minimum number of sides P can have?

Note: The two pentagons are not necessarily regular.

Solution.

The answer is $\boxed{3}$ as shown in the following diagram. Anything less will result in a shape that is not a polygon.



□

§5.2.2 Digits in a String

Source: Folklore
Proposer: sjbs#9839 (434767660182405131)
Problem ID: 83
Date: 2020-12-29

If we write the numbers 99999 down to 1 in the following string:

$$999999999899997\ldots10987654321$$

What is the 42069^{th} digit multiplied by the 42070^{th} digit?

Note: In the given string, we consider the first 9 on the left to be the first digit and 1 to be the last digit.

Solution.

Observe that each number in the string can be split up into their respective numbers by separating them by 5 for all digits greater than 9999, 4 for all digits greater than 999, and less than 100000, and so on.

$$\underbrace{99999}_{5}\underbrace{99998}_{5}\underbrace{99997}_{5}\ldots\underbrace{10}_{2}\cdots$$

Since there are $(99999 - 9999) \cdot 5 = 450000$ digits from five-digit numbers contained within the string, we know that the 42069^{th} digit will be within a five-digit number. Since 42069 is one less than a multiple of 5, we can deduce that it will be the 4th digit in a five-digit number.
We note that the $5n-4$th to $5n$th digit belong to the number $100000-n$ (by either Engineer's or mathematical induction).
Thus these two digits are the fourth and fifth digits of 91586 respectively, meaning their product is $8 \cdot 6 = \boxed{48}$. □

§5.2.3 Relatively Prime Function

Source: Original
Proposer: TaesPadhihary#8557 (665057968194060291)
Problem ID: 84
Date: 2020-12-30

There are N tuples of integers a, b, c, d satisfying $1 \leq a, b, c, d \leq 101$ and exactly 3 out of a, b, c, d are relatively prime to 2020.

What is the sum of the (not necessarily distinct) prime factors of N?

Solution.

Note that there are 40 elements of $\{1, 2, 3, \cdots, 101\}$ which are relatively prime to 2020, and 61 elements which are not. We have 4 ways to choose which element out of a, b, c, d is not relatively prime to 2020, and so $N = 4 \times 61 \times 40^3 = 2^2 \times 61 \times 2^6 \times 5^3$.
Our answer is thus $4 + 61 + 12 + 15 = \boxed{92}$. $\square$

§5.2.4 Intersecting Circles

Source: RMO Maharashtra and Goa, 2019 P6
Proposer: TaesPadhihary#8557 (665057968194060291)
Problem ID: 85
Date: 2020-12-31

Let k be a positive real number. In the Cartesian coordinate plane, let S be the set of all points of the form $(x, x^2 + k)$ where $x \in \mathbb{R}$. Let C be the set of all circles whose center lies in S, and which are tangent to X-axis. Find the minimum value of k such that any two circles in C have at least one point of intersection.
If $k = \frac{m}{n}$ where $\gcd(m, n) = 1$ and m, n are positive integers, find $m + n$.

Solution.

AoPS Solution

Answer: $\boxed{5}$ □

§5.2.5 Discord Ping Fight

Source: Original
Proposer: Charge#3766(481250375786037258)
Problem ID: 86
Date: 2021-1-01

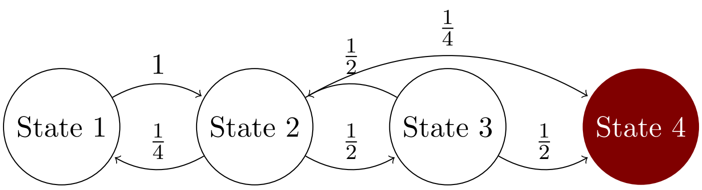
In a discord ping fight, 2 immature friends create a discord server with 5 channels numbered 1 through 5. Then player 1 and player 2 both select a (not necessarily distinct) odd numbered channel. Each round the two players ping in their channel.

After each round, they move to channel $n+1$ or $n-1$ where n is their current channel with equal probability, and start another round. If the expected number of rounds before both end up on the same channel and make up to each other can be expressed as $\frac{a}{b}$ in lowest terms, find ab.

Note: If one of the friends is on channel 1 or 5, then the next turn they will be on channel 2 and 4 respectively.

Solution. [Write up by Charge#3766(481250375786037258)]

Note that the expected value of rounds it will take if the players start on channel 1 and 3 is the same as if they started on 3 and 5. Then we just have if the players are on 1 and 5 to be a separate case. Let our first state be when players are on 1 and 5, the second state be when there are players on 2 and 4, the third state be when one is on 1 and 3 or 3 and 5, and the last state be the end state. We now draw this diagram (forgive me if it looks terrible).



So we basically just solve for the expected value from each state and go from there.
Setting up a system of equations where S_n denotes the expected value to reach state 4 from n, we have

$$S_1 = 1 + S_2$$
$$S_2 = 1 + \frac{1}{4}S_1 + \frac{1}{2}S_3 + \frac{1}{4}S_4$$
$$S_3 = 1 + \frac{1}{2}S_2 + \frac{1}{2}S_4$$
$$S_4 = 0$$

So we have $S_1 = \frac{9}{2}, S_2 = \frac{7}{2}, S_3 = \frac{11}{4}$. Now we do some simple casework.
Case 1: It starts on state 1. This happens with probability $\frac{2}{9}$ so the expected number of rounds minus the one at the end from state 1 is 1.
Case 2: It starts on state 3. This happens with probability $\frac{4}{9}$ so the expected number of rounds minus the one at the end from state 3 is $\frac{11}{9}$.
So by linearity of expectation, the expected value of rounds before the end is $\frac{20}{9}$. So $ab = \boxed{180}$

□

§5.2.6 Falling Cards

Source: New Zealand Monthly Maths Workshop, December 2020, Problem 6
Proposer: brainysmurfs#2860 (281300961312374785)
Problem ID: 87
Date: 2021-1-02

Point One Nine throws a standard pack of cards into the air such that each card is equally likely to land face up or face down and lands independently of other cards. The total value of the cards which landed face up is then calculated.

Suppose the probability that the total is divisible by 13 is $\frac{m}{n}$ with $m, n \in \mathbb{Z}^+, \gcd(m, n) = 1$. Calculate the largest integer value of x such that $2^x \leq n$.

Note: Values of cards are assigned as follows: Ace = 1, 2 = 2, . . ., Jack = 11, Queen = 12, King = 13.

Solution.

Note that we may discard the Kings, since they represent values which are 0 (mod 13). Further, note that 2 is a generator in mod 13. In this way we establish a bijection between the values of the cards $1, 2, 3, \ldots, 12, 1, 2, 3, \ldots, 12, \ldots, 12$ (4 sets of 1 to 12) and $2^0, 2^1, \ldots, 2^{47}$ in mod 13.
Each number has a unique binary representation and since each configuration of cards is equally likely, each number from 0 to $2^{48} - 1$ is equally likely. By Fermat's Little Theorem, since $2^{12} \equiv 1 \pmod{13}$, we get $13 | 2^{48} - 1$.
Thus out of those binary numbers $\frac{2^{48}-1}{13} + 1$ are divisible by 13, and so our probability is

$$\frac{2^{48} + 12}{13 \cdot 2^{48}} = \frac{2^{46} + 3}{13 \cdot 2^{46}}.$$

But since $2^{46} \equiv 2^{-2} \equiv -3 \pmod{13}$, we get that the numerator is divisible by 13, and so $n = 2^{46}$. From here we get that our answer is $\boxed{46}$.

$\square$

§5.2.7 Computing the Area of a Triangle

Source: HMMT Feb 2019 Geometry P8
Proposer: Angry Any#4319 (580933385090891797)
Problem ID: 88
Date: 2021-1-03

In triangle ABC with $AB < AC$, let H be the orthocenter and O be the circumcenter. Given that the midpoint of OH lies on BC, $BC = 1$, and the perimeter of ABC is 6, compute the area of ABC.

If this area is $\frac{m}{n}$ for coprime positive integers m and n, find $m + n$.

Solution.

Official Solution

Answer: $\boxed{13}$ $\square$

§6 Third Week of School (Season 6)

§6.1 Week 1

§6.1.1 Maximising Square Factors

Source: Senior Mathematical Challenge, 2015 Q18
Proposer: sjbs#9839 (434767660182405131)
Problem ID: 89
Date: 2021-01-04

What is the largest integer k for which k^2 is a factor of $11!$?

Solution.

Let us write the factorisation of 10! in a table, so we can see the indexes of the powers.

	7	5	3	2
2	0	0	0	1
3	0	0	1	0
4	0	0	0	2
5	0	1	0	0
6	0	0	1	1
7	1	0	0	0
8	0	0	0	3
9	0	0	2	0
10	0	1	0	1

We want to find k^2, where k is maximised. That means we want to add as many of the indexes together such that they all end up even. From inspecting the table, we see that they all add up to an even number so long as we don't include the 7. Thus $k^2 = \frac{10!}{7} = 5^2 \cdot 3^4 \cdot 2^8$, this gives an answer of $k = \boxed{720}$ □

§6.1.2 Njoy's Balls

Source: Original Problem
Proposer: TaesPadhihary#8557 (665057968194060291)
Problem ID: 91
Date: 2021-01-05

NJOY has a box of 200 balls, 100 of which are blue, 98 of which are red, and 2 of which are green.

At each stage he randomly selects one ball from the box. If it is blue, he wins; if it is red, he loses, and if it is green, he replaces the ball and draws another one with the same rules.

If the probability that he wins is $\frac{m}{n}$ where $\gcd(m, n) = 1$ and m, n are both positive integers, find $100m + n$.

Solution.

Let P be the probability of a win. Note that $P = P(\text{blue}) + P(\text{green}) * P$, since after he draws and replaces a green ball the state returns to the starting state. Solving this with the provided values, we get

$$P = \frac{1}{2} + \frac{1}{100}P$$

and hence $\frac{99}{100}P = \frac{1}{2}$ and so $P = \frac{50}{99}$, giving us $100m + n = 5000 + 99 = \boxed{5099}$. $\square$

Solution. [Solution by AiYa#2278 (675537018868072458)]

The last ball NJOY picks will be blue or red, and the probability that he picks that final ball does not depend on his history of picking greens. Therefore, the green balls are irrelevant to the problem and the probability he picks a blue is $\frac{100}{100+98}$ giving us our answer of $\boxed{5099}$. $\square$

§6.1.3 Aiya's Function

Source: Original Problem
Proposer: AiYa#2278 (675537018868072458)
Problem ID: 90
Date: 2021-01-06

Let $f : [0, 1] \to \mathbb{R}$ be a function with the following properties:

$$f(x) + f\left(\tfrac{1}{2}\right) f(1 - x) = 2f\left(\tfrac{1}{2}\right)$$

- $2f(x) = f(3x)$
- f is non-decreasing

Then the sum of the possible values of $f\left(\frac{4}{5}\right) + 1$ can be written in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. What is the value of $100m + n$?

Solution. [Solution by AiYa#2278 (675537018868072458)]

First, find some basic values: $2f(0) = f(0) \iff f(0) = 0$ and $f(1/2) + f(1/2)^2 = 2f(1/2) \iff f(1/2) = 0, 1$. Now if $f(1/2) = 0$ note that $f(1) = 0$; since f is nondecreasing f is the zero function. Now for $f(1/2) = 1$: $f(x) + f(1 - x) = 2 \iff f(1) = 2$. It's fruitless to get $4/5$ from 0, 1, and $1/2$ by repeated operations of $1 - x$ and $x/3$; however note that $f(1/3) = 1 \iff f(2/3) = 1$ and since f is nondecreasing all values of x between $1/3$ and $2/3$ inclusive will have $f(x) = 1$. Then finding $f(3/5)$ is easy: it's just 1, so $f(1/5) = 1/2 \iff f(4/5) = 3/2$. Our possible values of $g(4/5)$ are 1 and $5/2$. This gives us an answer of $\boxed{702}$ □

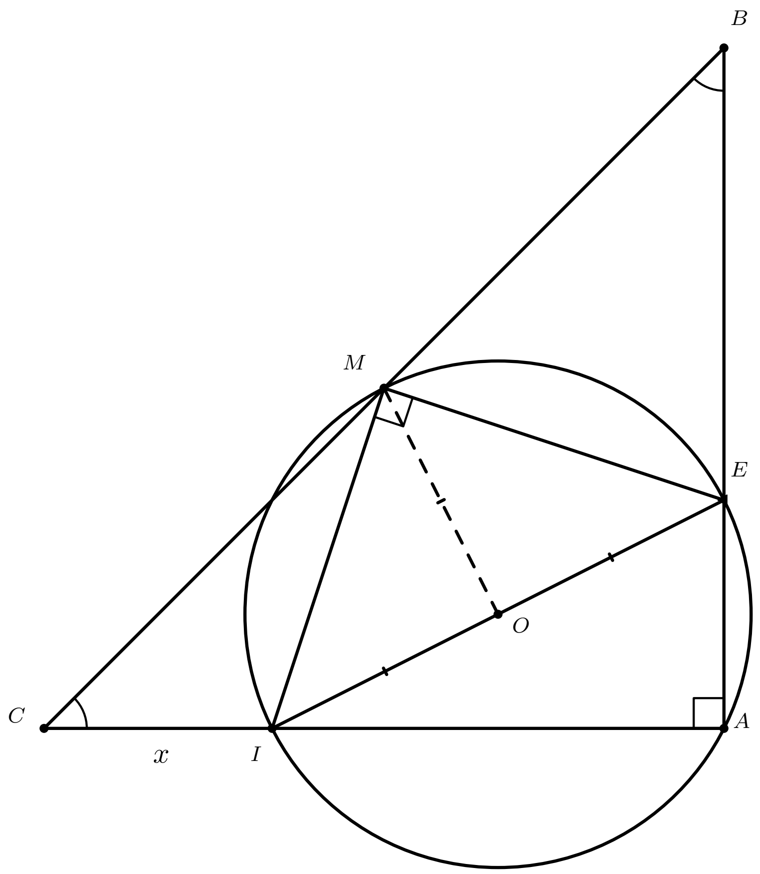
§6.1.4 The AIME Cyclic

Source: 2018 AMC 12A, Q20 of 25
Proposer: Angry Any#4319 (580933385090891797)
Problem ID: 92
Date: 2021-01-07

Triangle ABC is an isosceles right triangle with $AB = AC = 3$. Let M be the midpoint of hypotenuse $\overline{BC}$. Points I and E lie on sides $\overline{AC}$ and $\overline{AB}$, respectively, so that $AI > AE$ and $AIME$ is a cyclic quadrilateral. Given that triangle EMI has area 2, the length CI can be written as $\frac{a-\sqrt{b}}{c}$, where a, b, and c are positive integers and b is not divisible by the square of any prime. What is the value of $10000a + 100b + c$?

Solution.

Let $IM = a$, since $IM = EM$, we have $a^2 = 4$. Further, as $CM = \frac{1}{2}\sqrt{3^2+3^2} = \frac{3\sqrt{2}}{2}$, and given $\triangle ABC$ being isosceles we have $\angle BCA = \frac{\pi}{4}$. Therefore $a^2 = x^2 + \frac{9}{2} - 3x$, and so $x^2 - 3x + \frac{9}{2} = 4 \iff x = \frac{3\pm\sqrt{7}}{2}$ (We take the negative as $AI > AE$). Hence, by the problem statement we have our answer as $\boxed{30702}$



□

§6.1.5 Prime Floors

Source: NIMO AoPS Thread
Proposer: Angry Any#4319 (580933385090891797)
Problem ID: 93
Date:2021-01-08

Let $p = 10^9 + 7$ be a prime. Find the remainder when

$$\left\lfloor \frac{1^p}{p} \right\rfloor + \left\lfloor \frac{2^p}{p} \right\rfloor + \left\lfloor \frac{3^p}{p} \right\rfloor + \dots + \left\lfloor \frac{(p-3)^p}{p} \right\rfloor + \left\lfloor \frac{(p-2)^p}{p} \right\rfloor$$

is divided by p

Solution.

We will prove the general case where p is an odd prime.
Let p be an odd prime. we have $n^{p-1} \equiv 1 \pmod p$ and thus $n^p \equiv n \pmod p$ for all positive integers $n < p$. Thus $\lfloor \frac{n^p}{p} \rfloor = \frac{n^p - n}{p}$ and the required sum is equivalent to

$$\frac{2^p + 3^p + ... + (p-2)^p}{p} - \frac{2 + 3 + 4 + ... + (p-2)}{p}$$

Now we claim that $\frac{n^p + (p-n)^p}{p} \equiv 0 \pmod p$ for all integer n, which is indeed true since by the binomial theorem, all terms of $(p-n)^p$ are $0 \pmod{p^2}$ except $-n^p$. Therefore $n^p + (p-n)^p \equiv 0 \pmod{p^2}$ and $\frac{n^p + (p-n)^p}{p} \equiv 0 \pmod p$. This implies that the first sum is actually just $0 \pmod p$ and the desired answer stems solely from the second sum.
The second sum can be grouped into $\frac{p-3}{2}$ pairs that add up to p, and thus the answer is $-\frac{p-3}{2} = \frac{p+3}{2} = \boxed{500000005} \pmod{10^9 + 7}$. □

§6.1.6 Nested Periodic Functions

Source: USA EGMO TST #2, 2020 P6
Proposer: sjbs#9839 (434767660182405131)
Problem ID: 94
Date: 2021-01-09

For a polynomial $P(x)$ with integer coefficients such that for each positive integer m,

$$P^m(0) \equiv 0 \mod 2020 \iff m \equiv 0 \mod N$$

Where $N \in \{1, 2, \ldots, 2019\}$. What is the largest such value of N?

(Here we denote P^m to mean the function P applied m times, so $P^1(0) = P(0)$, $P^2(0) = P(P(0))$, and so forth.)

Solution.

Art of Problem Solving write ups

Answer: $\boxed{1980}$ □

§6.1.7 Braniy's Party

Source: China Team Selection Test, 2015 Day 1 P3
Proposer: sjbs#9839 (434767660182405131)
Problem ID: 95
Date: 2021-01-10

Brainy, being the socialite he is, has invited a large number of friends to a little soirée. Brainy will be feeding his friends Chinese Noodle Soup, made by Tony himself.

Wanting to impress Asuka, and Zero-Two, both of whom will be at the party, he asks Tony to provide the party with many different flavours of soup - enough so that for each flavour, there are 260 bowls of soup.

When Brainy's friends arrive at the party, each of them take exatly two soups, each of different flavours.

Given that for any 179 of Brainy's friends, there will at least be two who have at least one flavour of Chinese Noodle Soup in common, and assuming Brainy has invited as many people as possible to the party, how many people are in attendance?

(A four-function calculator may be used)

Solution.

Art of Problem Solving write ups

Answer: $\boxed{69420}$ □

§6.2 Week 2

§6.2.1 Negative Sum

Source: Original Problem
Proposer: Angry Any#4319 (580933385090891797)
Problem ID: 96
Date: 2021-01-11

There is an ordered tuple of n positive real numbers $\{a_1, a_2, a_3, ..., a_n\}$ such that when any element is replaced by itself multiplied by -2021, the sum of the elements of the ordered tuple becomes negative. What is the maximum value of n?

For example, $\{1, 3, 5, 7\}$ satisfies this property, but $\{1, 4000\}$ does not.

Solution.

Note that $n = 2021$ is trivially possible through the construction $\{1, 1, 1, 1, ..., 1\}$.
Now we prove that a higher number is not possible. WLOG, let $a_1 \geq a_2 \geq ... \geq a_n$. Then, $a_1 + a_2 + a_3 + ... + a_{n-1} - 2021a_n \geq (n-1)a_n - 2021a_n = (n-2022)a_n$. Thus, the condition cannot hold true if $n - 2022 \geq 0$ or $n \geq 2022$, which concludes the proof.

$\square$

§6.2.2 Doubling Digit Sum

Source: Australian Intermediate Mathematical Olympiad, 1999 Q10
Proposer: brainysmurfs#2860 (281300961312374785)
Problem ID: 97
Date: 2021-01-12

N is the smallest positive integer such that the sum of the digits of N is 18 and the sum of the digits of $2N$ is 27. Find N.

Solution.

We iterate through cases of $2N$.
The only 3 digit number with a digit sum of 27 is 999 but $2N = 999$ does not work since 999 is not even.
$2N = 1998$ is the next smallest even number with digit sum 27 and $N = 999$ does not work.
$2N = 2898$ is the next smallest even number with digit sum 27 and it does work, giving $N = \boxed{1449}$. $\square$

§6.2.3 Tony Vs. Wang

Source: British Mathematical Olympiad, Round 1, 2018 P1
Proposer: sjbs#9839 (434767660182405131)
Problem ID: 98
Date: 2021-01-13

Tony divides 365 by each of $1, 2, 3, \cdots 365$ in turn, writing down a list of the 365 remainders. Then Wang divides 366 by each of $1, 2, 3, \cdots 366$ in turn writing down a list of the 366 remainders. What is the difference between the two summed lists?

Solution.

Remainder for remainder, the numbers in Wang's list will always be one greater than those in Tony's list - with exception to the cases where the remainder is 0, in which case, for a number n dividing 366, Tony will have a remainder of $n-1$. Since $366 = 2 \cdot 3 \cdot 61$, there are 8 remainders in Wang's list that are 0. Now we wish to find the $n-1$ remainders. Well, we have $n \in \{1, 2, 3, 6, 61, 122, 183\}$ so we must have the remainders for Tony, when Wang's remainder is 0, being $\{0, 1, 2, 5, 60, 121, 182\}$. To summarise these results, we have:

$$T + (0 + 1 + 2 + 5 + 60 + 121 + 182) = W + 365 - 8 + 1$$
$$\therefore \boxed{13} = W - T$$

□

§6.2.4 Equilateral Decagon

Source: OTIS excerpts P118
Proposer: Angry Any#4319 (580933385090891797)
Problem ID:
Date:

Let $ABCDEZYXWV$ be an equilateral decagon with interior angles $\angle A = \angle V = \angle E = \angle Z = \angle C = 90°$, $\angle W = \angle Y = 135°$, $\angle B = \angle D = 225°$, and $\angle X = 270°$. Find the sum of all $1 < n < 100$ such that $ABCDEZYXWV$ can be partition into n congruent polygons.

Solution.

Cut along horizontally such that the vertical distance for each line is constant. This shows that all n is possible. Thus, the answer is $\frac{99 \cdot 100}{2} - 1 = 4949$. $\square$

§6.2.5 Triangles on a Cubic

Source: 2017 AMC 12B, Q23
Proposer: Angry Any#4319 (580933385090891797)
Problem ID:
Date:

The graph of $y = f(x)$, where $f(x)$ is a polynomial of degree 3, contains points $A(2, 4)$, $B(3, 9)$, and $C(4, 16)$. Lines AB, AC, and BC intersect the graph again at points D, E, and F, respectively, and the sum of the x-coordinates of D, E, and F is 24. What is $f(0)$?

Solution.

Art of Problem Solving write-ups □

§6.2.6 Jumping Frogs

Source: International Mathematics Competition for University Students, 2018 P8
Proposer: sjbs#9839 (434767660182405131)
Problem ID:
Date:

Let $\Omega = \left\{(x, y, z) \in \mathbb{Z}^3 : y + 1 \geq x \geq y \geq z \geq 0\right\}$. A frog moves along the points of Ω by jumps of length 1. Determine the number of paths the frog can take to reach $(12, 12, 12)$ starting from $(0, 0, 0)$ in exactly 36 jumps.

Solution.

(Art of Problem Solving write ups)

Answer: $\boxed{50067108}$ $\square$

§6.2.7 Minimising Ratios

Source: China Team Selection Test, 2008 Quiz 1 P1
Proposer: sjbs#9839 (434767660182405131)
Problem ID:
Date:

Suppose a point P is a random point chosen inside a triangle XYZ. Extend the segment XP until it intersects with the circumcircle of PYZ, call the point of intersection which isn't P, X_1. Similarly, for points Y and Z construct points Y_1 and Z_1. What is the smallest value of

$$\left(1+2\frac{XP}{PX_1}\right)\left(1+2\frac{YP}{PY_1}\right)\left(1+2\frac{ZP}{PZ_1}\right)$$

Solution.

Art of Problem Solving write ups

Answer: $\boxed{8}$ □

§7 PSC's Adventure (Season 7)

§7.1 Week 1

§7.1.1 The Jungle Polygon

Source: Folklore
Proposer: sjbs#9839 (434767660182405131)
Problem ID: 103
Date: 2021-01-18

The Problem Selection Committee ventures into a jungle where they find a regular polygon drawn on the ground. Brainysmurfs takes out his infinite precision protractor and measures one of the interior angles to be 179 degrees. How many sides does the polygon have?

Solution.

The formula for the sum of the interior angles is given by $180(n-2)$, and thus we have $180(n-2) = 179n \iff n = \boxed{360}$ $\square$

§7.1.2 Uphill and downhill

Source: Original Problem
Proposer: TaesPadhihary#8557 (665057968194060291)
Problem ID: 104
Date: 2021-01-19

After Brainy discovered a map hidden within the regular polygon, he is able to navigate the PSC through the Jungle. It takes Brainy and the team 4 hours to navigate out of the jungle, moving 2 kilometres per hour walking up inclined surfaces, 6 kilometres an hour when walking down, and 3 kilometres an hour on flat surfaces.

Upon reaching the edge of the jungle Brainy realises that they accidentally left .19 back where they started. Given that it took Brainy and the PSC 6 hours to return to where they started, to rescue .19, what is the total distance Brainy travelled (in kilometres)?

Solution. [Solution by TaesPadhihary#8557 (665057968194060291)]

Let the two "other" points be C and D, going from left to right (so that the labels when reading from left to right are A, C, D, B). We let $AC = x, CD = y, DB = z$. We have $\frac{x}{2}+\frac{y}{3}+\frac{z}{6} = 4$ and $\frac{x}{6}+\frac{y}{3}+\frac{z}{2} = 6$. Multiplying both equations by 6 gives $3x + 2y + z = 24$ and $x + 2y + 3z = 36$. Adding them gives $4x + 4y + 4z = 60$, so our desired value, is $\boxed{30}$. □

Solution. [Solution by brainysmurfs#2860 (281300961312374785)]

Note that the harmonic mean of 2 and 6 kph is 3kph - in particular, since in total there's the same amount of uphill and downhill, we get that the average speed is 3 kph and thus the PSC travelled $\boxed{30}$km.

□

§7.1.3 Just a Beauty!

Source: CMC Mock ARML 2020 i7
Proposer: TaesPadhihary#8557 (665057968194060291)
Problem ID: 105
Date: 2021-01-20

The PSC leave the jungle and find themselves on a long winding road, where they encounter a toll gate. The toll keeper, Joe, asks for an amount of smackaroo's equal to

$$31\left(1+\frac{30}{2}\left(1+\frac{29}{3}\left(1+\frac{28}{4}\left(\cdots\left(1+\frac{17}{15}\right)\cdots\right)\right)\right)\right).$$

How many smackaroo's should the PSC pay Joe?

(A four-function calculator may be used)

Solution.

Official Solutions Booklet p.9
Answer: $\boxed{9223372036854775806}$ □

§7.1.4 Probability of Intervals

Source: One Hundred Problems - 4th Edition Q93
Proposer: HoboSas#3200 (310725130097786880)
Problem ID: 106
Date: 2021-01-21

After paying Joe what is due, the PSC team find themselves at the gates of a destitute underground research facility. Upon exploring the ransacked and decaying tomb they find themselves in a laboratory experimenting with teleportation. The PSC try to get it working:

.19, Brainy, and Yuchan enter the teleporter where they are each positioned randomly along a 1m strip.

For the teleporter to work, .19 and Brainy must be less than half a meter apart, and .19 and Yuchan must also be less than half a meter apart.

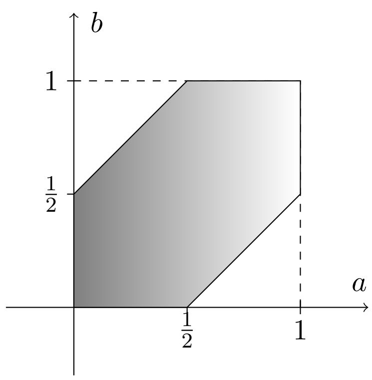
If the probability that the teleporter works is $\frac{m}{n}$, find $100m + n$

Solution. [Solution by HoboSas#3200 (310725130097786880)]

The problem statement can be reduced to this:

Let a, b and c be real numbers randomly chosen from the interval $[0, 1]$, if the probability of $|a - b| < \frac{1}{2}$ and $|b - c| < \frac{1}{2}$ can be expressed as $\frac{m}{n}$, find $100m + n$.

I will pursue a geometrical approach using a 3D space with coordinates (a,b,c), such that for every triplet (a,b,c) there exist one and only point enclosed in the cube whose vertices are (0,0,0) (1,0,0) (1,1,0) (0,1,1) and so on. The probability we are seeking can be seen as ratio between volumes, in particular the volume of the intersection of the two solids $|a - b| < \frac{1}{2}$ and $|b - c| < \frac{1}{2}$, over the volume of the cube with side 1. Let's focus on finding what the solid $|a - b| < \frac{1}{2}$ looks like: if we consider the plane c=0, with some trivial analytic algebra we can draw the following polygon: Introducing the 3rd dimension, $|a - b| < \frac{1}{2}$ turns out to be an



hexagonal prism (the base is shown above) whose height is 1 (along the c-axis). Similarly $|b - c| < \frac{1}{2}$ is the same solid, but rotated 90 degrees around the line parallel to the b-axis and going through the point($\frac{1}{2}$,$\frac{1}{2}$,0). Let's now compute the intersection between those solids, which turns out to be the sum of few smaller solids, in particular: 2 cubes with side $\frac{1}{2}$, 2 square pyramids with base length and height $\frac{1}{2}$ and 4 triangular prism, having height $\frac{1}{2}$ and an isosceles right triangle with side $\frac{1}{2}$ as base. Finally the probability is

$$p = \frac{2 \cdot \frac{1}{8} + 2 \cdot \frac{1}{24} + 4 \cdot \frac{1}{8}}{1} = \frac{7}{12}$$

which leads to 712 as final answer. □

§7.1.5 Dividing Functions

Source: Original Problem
Proposer: ChristopherPi#8528 (696497464621924394)
Problem ID: 106
Date: 2021-01-22

Find the smallest integer $n > 1$ for which there exist positive integers $a_1, a_2, \ldots, a_n$such that

$$(a_1)^2 + \ldots + (a_n)^2 \big| \left[(a_1 + \cdots a_n)^2 - 1)\right]$$

Solution. [Solution by ChristopherPi#8528 (696497464621924394)]

Note that squares are congruent to themselves (mod 2), so

$$(a_1)^2 + \ldots + (a_n)^2 = a_1 + \ldots + a_n = (a_1 + \ldots + a_n)^2 \pmod 2$$

Since $(a_1)^2 + \ldots + (a_n)^2$ divides $(a_1 + \ldots + a_n)^2 - 1$ which has a different parity, we must have the former odd and the latter even, so $(a_1 + \ldots + a_n)$ is odd. As all odd squares are $1 \mod 8$, $(a_1 + \ldots + a_n)^2 - 1$ must be divisible by 8, and since $(a_1)^2 + \ldots + (a_n)^2$ is odd, we must have

$$\frac{(a_1 + \ldots + a_n)^2 - 1}{(a_1)^2 + \ldots + (a_n)^2} \geq 8$$

Using Cauchy-Schwartz on the sequences $(a_1, \ldots, a_n)$ and $(1, 1, ..., 1)$ gives

$$n((a_1)^2 + ... + (a_n)^2 \geq (a_1 + \ldots + a_n)^2) > ((a_1 + \ldots + a_n)^2 - 1)$$

, which is a direct contradiction if n is less than 9. Therefore we must have $n \geq 9$. It is easy to see $n = 9$ works, with an example construction being $(1, 1, 1, 1, 1, 1, 1, 2, 2)$ which works as $7 * (1^2) + 2 * (2^2) = 15$ and $(7 * 1 + 2 * 2)^2 - 1 = 120$, with $120/15 = 8$ being an integer.

$\boxed{9}$ $\square$

§7.1.6 Paper Monster

Source: Online Math Open2013 P29
Proposer: Angry Any#4319 (580933385090891797)
Problem ID: 107
Date: 2021-01-23

After the TST the PSC meets Chrispi.

Chrispi has 255 sheets of paper, each labeled with a unique nonempty subset of $1, 2, 3, 4, 5, 6, 7, 8$. Each minute, he chooses one sheet of paper uniformly at random out of the sheets of paper not yet eaten.

Then, he eats that sheet of paper, and all remaining sheets of paper that are labeled with a subset of that sheet of paper (for example, if he chooses the sheet of paper labeled with $1, 2$, he eats that sheet of paper as well as the sheets of paper with 1 and 2).

The expected value of the number of minutes that Chrispi eats a sheet of paper before all sheets of paper are gone can be expressed in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $100m + n$. difficulty

Solution.

Art of Problem Solving write-ups
Answer: $\boxed{20508}$

§7.1.7 Tan's Optimisation

Source: Cambridge Handout
Proposer: tanoshii#3160 (300065144333926400)
Problem ID: 108
Date: 2021-01-24

After Brainy and Yuchan placed joint first at the IMO, both perfect scoring, they decide to take the day off before going on a journey to replace .19 and add more people to the problem solving committee. As part of the recruitment process, they ask the following question to the applicants:

Given that the maximum value of

$$\frac{x_1x_2 + x_2x_3 + \cdots + x_{20}x_{21} + x_{21}x_1}{x_1^2 + \cdots + x_{21}^2}$$

is M for $x_1 + \cdots + x_{21} = 0$, find $\lfloor 1000M \rfloor$

What answer should the applicants put down?

(A scientific calculator may be used)

Solution. [Solution by tanoshii#3160 (300065144333926400)]

We will prove it for a given n, here $n = 21$.

Let ω be the principle nth root of unity.

Let

$$y_k = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} x_i \omega^k.$$

Note that

$$|y_1|^2 + |y_2|^2 + \cdots + |y_k|^2 = |x_1|^2 + |x_2|^2 + \cdots + |x_k|^2$$

and

$$x_1x_2 + \cdots + x_nx_1 = \frac{1}{n} \sum_{k=1}^{n} (\omega^k + \omega^{-k}) y_k^2$$

Since $x_1 + \cdots + x_n = 0$, we get $y_n = 0$, so the sum is maximised when y_{n-1} or y_1 is maximal since $k = 1, n-1$ are the values where $\omega^k + \omega^{-k}$ is the biggest and that gives $2\cos(\frac{2\pi}{n})$. $\square$

§7.2 Week 2

§7.2.1 A Digital Product equal to Half the Number

Source: Folklore
Proposer: sjbs#9839 (434767660182405131)
Problem ID: 110
Date: 2021-01-25

AiYa claims that he has found the smallest two-digit positive integer with the property that when divided by two, it is equal to the product of its digits.

Assuming AiYa is correct, what number has he found?

(If needed, a four-function calculator may be used)

Solution.

Let the integer have digits x and y. We require:

$$\begin{aligned} xy &= \frac{10x+y}{2} \\ 2xy - 10x - y &= 0 \\ 2xy - 10x - y + 5 &= 5 \\ (2x-1)(y-5) &= 5 \end{aligned}$$

Since 5 is prime, it has factors of either 1 or 5. Hence we have either $y - 5 = 5$ or $y - 5 = 1$. We don't need to consider negatives as it's clear x would not be positive (and non-zero). We see that $y = 6$, as in the other case, $y > 9$. THis means that $2x - 1 = 5$, giving $x = 3$. So the answer is $\boxed{36}$ $\square$

§7.2.2 Unique Odd Numbers

Source: Folklore
Proposer: Kiesh#0917 (544960202101751838)
Problem ID: 112
Date: 2021-01-26

Call a number unique if each of its digits are unqiue (no two are the same). How many odd integers in the interval $[3 \cdot 10^4, 8 \cdot 10^4]$ are unique?

(A four-function calculator may be used)

Solution.

There are 5 ways to choose the first digit. If the first digit is odd, then we must consider the final digit too. In the even cases, there are 2 numbers to choose from for the first digit, then last digit we have 5 odd digits to choose from. Then for the other 3 digits, we have 7! ways to pick digits, so there are $2 \cdot 5 \cdot \frac{8!}{5!}$ ways when the first digit is even. When it is odd, however, there is one less odd digit to choose from for the last digit. There are 3 ways to choose the first digit and 4 ways to choose the last digit. As before there are $\frac{8!}{5!}$ ways to choose the middle three digits. This gives us a total of $3 \cdot 4 \cdot \frac{8!}{5!}$ when the first digit is odd.
Thus in total, there are $2 \cdot 5 \cdot \frac{8!}{5!} + 3 \cdot 4 \cdot \frac{8!}{5!} = \boxed{7392}$ □

§7.2.3 Maximising $a^4 + b^4$

Source: Folklore
Proposer: sjbs#9839 (434767660182405131)
Problem ID: 113
Date: 2021-01-27

Returning home from Matteddy's demonstration, the PSC are trapped in a vault by NJOY, trying to join the PSC by coercion and brute force. On the wall are two haikus. The first reads:

The sum of the squares
of a and b minus their
product fears 15

They take this to mean that the given expression of a and b is at most 15, before reading the second haiku:

The unfearing sum
of their fourth powers is on
the door of freedom

which they take to mean that the door labeled with the maximal value of $a^4 + b^4$ is their exit.

After some thought, they scan the walls for they want, climb up to it and walk out, to the rising moon. They've escaped.

What door did they leave through?

Solution.

Notice that $a^4 + b^4 = (a^2 + b^2)^2 - 2(ab)^2$. Thus, $a^4 + b^4 \leq (2 + ab)^2 - 2(ab)^2$ - this is simply a quadratic in ab.

Therefore, we have, by *the method of maximising a quadratic of your choosing*

$$a^4 + b^4 \leq 450 - (ab - 2)^2$$

So our answer is $\boxed{450}$. □

§7.2.4 Minimising Perimeter

Source: China Mathematical Competition (Extra Test), 2003 P2
Proposer: sjbs#9839 (434767660182405131)
Problem ID: 114
Date: 2021-01-28

Find the **minimum** perimeter of a triangle having integer sides $a > b > c > 0$, such that

$$\frac{3^a}{10^4} - \left\lfloor \frac{3^a}{10^4} \right\rfloor = \frac{3^b}{10^4} - \left\lfloor \frac{3^b}{10^4} \right\rfloor = \frac{3^c}{10^4} - \left\lfloor \frac{3^c}{10^4} \right\rfloor,$$

where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x, e.g. $\lfloor \pi \rfloor = 3$, and so on.

Solution.

Note that this condition implies $l \equiv m \equiv n \pmod{(\text{ord}_{10000}(3))}$. Let us calculate $\text{ord}_{10000}(3)$.

By LTE, $\nu_2(3^n - 1) = \nu_2(3-1) + \nu_2(n) + \nu_2(3+1) - 1 = \nu_2(n) + 2$, and so if $3^n \equiv 1 \pmod{2}^4$ we must have $\nu_2(n) \geq 2$, implying that $n \geq 4$ and so $\text{ord}_{2^4}(3) = 4$.

By LTE again, note that $\nu_5(81^n - 1) = \nu_5(81-1) + \nu_5(n) \geq 4$. Hence we must have $\nu_5(n) \geq 3$ which implies $n \geq 125$. Since $81 = 3^4$ we get that $\text{ord}_{5^4}(3) = 500$.

Now $\text{ord}_{10^4}(3) = \text{lcm}(\text{ord}_{2^4}(3), \text{ord}_{5^4}(3)) = 500$.

Hence for our minimum we must have $l = n + 1000, m = n + 500$. Since they are the sides of a triangle $l < m + n \implies n + 1000 < 2n + 500 \implies n > 500$. So $n = 501$ and so the smallest value of $l + m + n$ is $1501 + 1001 + 501 = \boxed{3003}$. □

§7.2.5 Binary Blocks

Source: Harvard-MIT Math Tournament, 2015 C5
Proposer: Angry Any#4319 (580933385090891797)
Problem ID: 116
Date: 2021-01-29

One night while sleeping, Brainy has a vision that the entirety of the new PSC has been found, and that they must now return to the jungle to find their mysterious advisor who will lead them out of this dimension.

The next day, Brainy picks up everybody, but is stopped at Yuchan's house by MODSbot, who plans to trap the PSC in this dimension.

To distract it, Brainy orders MODSbot to write out every integer from 1 to 256 in binary, with a space between each. It takes MODSbot 1 minute to write each unbroken block of 1s and a negligible amount of time to write 0s and spaces.

How long in minutes does Brainy have to collect the rest of the PSC and escape the city?

Solution.

Define $g(0) = 0$. Call a digit of a number represented in binary "good" if it is 1 and the preceding digit is 0. Then $g(0) + g(1) + g(2) + ... + g(255)$ is $256E(X_8)$ where $E(X_8)$ is the expected value of the number of "good" digits given that the number is less than 2^8.
By linearity of expectation, the expected number of good digits is $E(X_8) = E(G_1) + E(G_2) + ... + E(G_8)$ where G_i is defined as

$$\begin{cases} 1 & \text{if the } i\text{th digit from the back is good} \\ 0 & \text{otherwise} \end{cases}$$

For example, G_3 for 7 would be 1 but G_2 for 7 would be 0 since there is already a 1 before that. Then we can find $E(G_i) = \frac{1}{4}$ for all $i = 1, 2, 3, 4, 5, 6, 7$ since its just $\frac{1}{2}$ chance that the ith digit is 1 multiplied by $\frac{1}{2}$chance that the preceding digit is 0. However, $E(G_8) = \frac{1}{2}$ since the preceding digit is guaranteed to be 0, from which we can find $E(X_8) = \frac{1}{2} + 7\frac{1}{4} = \frac{9}{4}$
Thus $g(0) + g(1) + g(2) + \ldots + g(255) = 256 \cdot \frac{9}{4}$ and $g(1) + g(2) + \ldots + g(255) + g(256) = \boxed{577}$. □

Solution. [Solution by RishiNandha Vanchi#3379 (562608039224410112)]

Let's say the bot's done printing until $n-1$ digits. Now let us append a digit.
We see that the bot will have to print all the numbers it has printed again so that one set of it can be appended with 0 and the other with 1. This takes exactly twice the amount of time taken for $(n-1)$ digits.

Whenever the ending digit was a 0 and the appended digit a 1. The bot takes a minute more to print it. It can be seen from some combinatorics that no. of numbers of $(n-1)$ digit numbers ending with 0 is exactly $\binom{2}{1}^{n-2} \cdot 1$ and one resulting number appended with 1 for each of those.

Thus the recurrence is:

$$t_n = 2t_{n-1} + 2^{n-2}$$

1 to 256 would be all 8 digit strings of 1 or 0 along with the number 100000000. Thus the time Bot takes is:

$$t_8 + 1$$

It can be seen that:

$$t_n = 2^k t_{n-k} + k \cdot 2^{n-2} \ ; \ t_1 = 1$$

$$t_8 + 1 = \boxed{577}$$

□

§7.2.6 Funkey Triangles

Source: Original Problem
Proposer: Matteddy#0482 (329956567132930048)
Problem ID: 117
Date: 2021-01-30

Having successfully gathered the PSC and escaped the scheming MODSbot, Brainy manages to lead the group to the jungle. Not far into the jungle, they meet Tan, who tells them that he is their mysterious advisor. He also says that he needs to take them to the last remaining population of wild triangles.

Some of the triangles are a deep shade of red; Tan explains that a triangle $\triangle ABC$ is red if given its incenter I, points B_1 and C_1 the intersections of BI and CI with AC and AB and P, Q the intersections of line PQ with (ABC), $\angle PIQ$ attains the minimal possible value over all acute triangles. Each red triangle has upon its face the value of $\cos(\angle BAC)$.

If the product of all numbers written on any triangle in the population for which $AB = AC$ can be written in the form $a - b\sqrt{c}$, where a, b, c are positive integers with c squarefree, find $10000a + 100b + c$.

Solution. [Solution by Matteddy#0482 (329956567132930048)]

Let I_b and I_c be the excenters opposite B and C respectively. Since IAI_cB is cyclic, we have $C_1A \cdot C_1B = C_1I \cdot C_1I_c$, which implies C_1 lies on the radical axis of (ABC) and (I_bII_c), and so does B_1. This means that B_1C_1 is the radical axis of those circles, which implies that P and Q are their intersection points. Since (ABC) is the Feuerbach circle of $\triangle I_bII_C$, the radius of (I_bII_C) is twice that of (ABC) for all triangles, which means that the minimal value of $\angle PIQ$ happens when PQ is maximised, so when it is a diameter, and in this case it's easy to see $\angle PIQ = 150°$. Now consider the case when $\triangle ABC$ is funky and isosceles, and let O be the circumcenter, M be the midpoint of arc BC in (ABC) and suppose the radius of (ABC) is 1. Then $2Rsin(\angle MAB) = MB = MI = R \pm OI = R(1 \pm tan(15°))$, which gives the possible values of $\cos(\angle BAC)$ to be $\sqrt{3} - 1$ and $3\sqrt{3} - 5$, and their product is $14 - 8\sqrt{3}$. This therefore gives us an answer of $\boxed{336}$ □

§7.2.7 A Lotta Touples!

Source: China Team Selection Test4 2017, P3
Proposer: Matteddy#0482 (329956567132930048)
Problem ID: 118
Date: 2021-01-31

The numbers of ordered touples $(x_1, ..., x_{100})$ which satisfy:

$x_1, ..., x_{100} \in \{1, 2, .., 2017\}$;

b) $2017 | x_1 + ... + x_{100}$;

c) $2017 | x_1^2 + ... + x_{100}^2$

can be expressed as a product of its prime factors, $p_1^{\alpha} p_2^{\beta} \ldots$, find $p_1\alpha + p_2\beta + \cdots$

Solution.

Art of Problem Solving write-ups
Answer: $\boxed{197666}$ □

§8 A New Chapter (Season 8)

§8.1 Week 1

§8.1.1 Smallest Non-Factor

Source: Senior Mathematical Challenge1999 Q9
Proposer: sjbs#9839 (434767660182405131)
Problem ID: 119
Date: 2021-02-01
Difficulty: Beginner

Let $N = 50!$. What is the smallest positive integer which does not divide N?

Solution.

The question can be rephrased as '*what is the smallest prime number greater than 50*'. To which the answer is $\boxed{53}$ $\square$

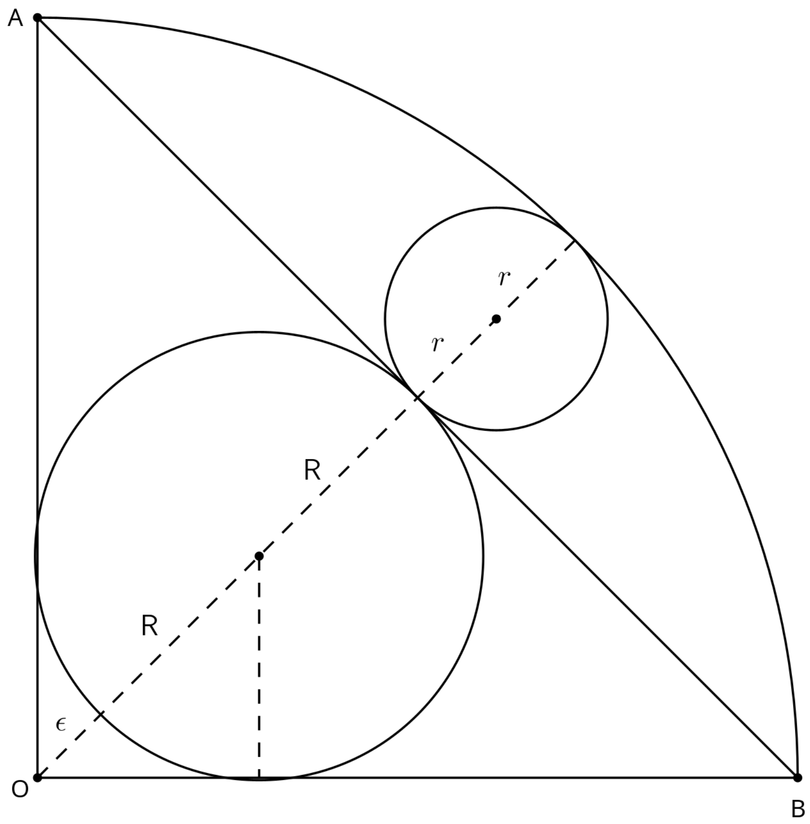
§8.1.2 Product of Radii

Source: Senior Mathematical Challenge2003 Q24
Proposer: sjbs#9839 (434767660182405131)
Problem ID: 120
Date: 2021-02-02
Difficulty: Beginner

Let AOB be an isosceles right-angled triangle drawn in a quadrant of a circle of radius unit 1. The largest possible circle drawn in the minor segment cut by the line AB has radius r. The radius of the inscribed circle of the triangle AOB is R. Given that the value of Rr can be written in the form $\frac{a-b\sqrt{c}}{d}$, where a, b, c, d are positive integers and c is square-free.

What is the value of $a^2 + b^2 + c^2 + d^2$?

Solution.



Observe that $\varepsilon + 2R + 2r = 1$ and $\varepsilon + 2R = \frac{1}{\sqrt{2}}$, therefore we have $r = \frac{\sqrt{2}-1}{2\sqrt{2}}$. Then by the sine rule, $\frac{\sin(45)}{R} = \frac{\sin(90)}{R+\varepsilon} \Rightarrow R = \frac{1}{\sqrt{2}+2}$, this gives $Rr = \frac{3-2\sqrt{2}}{4}$, hence our answer is $3^2 + 2^2 + 2^2 + 4^2 = \boxed{33}$ □

§8.1.3 Two-to-One Functions

Source: Folklore
Proposer: sjbs#9839 (434767660182405131)
Problem ID: 121
Date: 2021-02-03
Difficulty: Easy

Define a function $f : \{1, 2, \dots, 12\} \to \{1, 2, \dots, 6\}$ such that for every $y \in \{1, 2, \dots, 6\}$ there exists exactly two elements $x_1, x_2 \in \{1, 2, \dots, 12\}$ such that $f(x_1) = f(x_2) = y$. How many such functions are there which map $\{1, 2, \dots, 12\}$ to $\{1, 2, \dots, 6\}$ with the described property?

(A four-function calculator may be used)

Solution.

This well known property can be derived simply by considering how we can choose two elements in $\{1, 2, \dots, 12\}$ and enumerating that. Let the number of such functions be F, then we have:

$$\begin{aligned} F &= \prod_{i=1}^{12} \binom{2i}{2} \\ &= \frac{12!}{2^6} \\ &= \boxed{7484400} \text{ such functions} \end{aligned}$$

□

§8.1.4 Modular Powers

Source: New Zealand Mathematical Olympiad Round 1, 2019 Q4 (Adapted)
Proposer: brainysmurfs#2860 (281300961312374785)
Problem ID: 122
Date: 2021-02-04
Difficulty: Medium

Find the remainder when $122^{2020} - 102^{2020} - 21^{2020}$ is divided by 2020.

Solution.

Let $X = 122^{2020} - 102^{2020} - 21^{2020}$.
Note that $2020 = 20 \times 101$. In particular, considering the expression mod 20 we get $X = 2^{2020} - 2^{2020} - 1^{2020} \equiv -1 \pmod{20}$, and considering it mod 101 we get $X = 21^{2020} - 1^{2020} - 21^{2020} \equiv -1 \pmod{101}$.
In particular, by Chinese Remainder Theorem we get $X \equiv -1 \pmod{2020}$ which means that the remainder on division by 2020 of X is $\boxed{2019}$.
Note: The choice of 2020 in the exponent is not special. □

§8.1.5 Australian Nim

Source: AMO 2020 P2
Proposer: ChristopherPi#8528 (696497464621924394)
Problem ID: 123
Date: 2021-02-05
Difficulty: Medium

sjbs and Brainy are playing a game. First, they'll use a random number generator to generate three random positive integers. Then, three piles of stones will magically appear before them; the number of stones in each pile are the numbers generated previously. Then they'll take turns, with sjbs going first. On a player's turn, they can pick one pile and split it into either two or three nonempty piles, throwing the rest of the stones in the other two piles away. The game ends when a player can't make a move. Suppose both players play perfectly like the smart people they are - then if the probability that sjbs wins is m/n, with $gcd(m,n) = 1$ and m, n positive integers, find $m + n$.

Solution.

Call a pile *perilous* if it contains a number of stones equal to $3k + 1$ where k is a nonnegative integer and *safe* otherwise. I claim that a player can force a win, if at their turn they have at least one safe pile in front of them, and lose otherwise. Note that from a safe pile, the turn player can always give the other player either 2 or 3 perilous piles: simply consider splitting a pile of stones equal to $3k + 2$ into two piles, one of 1 stone and 1 of $3k + 1$ stones, which are both perilous, or a pile equal to $3k$ stones into three piles, two of 1 stone and 1 of $3k - 2 = 3(k - 1) + 1$ stones which are all perilous. Also note that from a perilous pile, it is impossible to leave only perilous piles behind, since two or three perilous piles imply that the original split pile was safe.

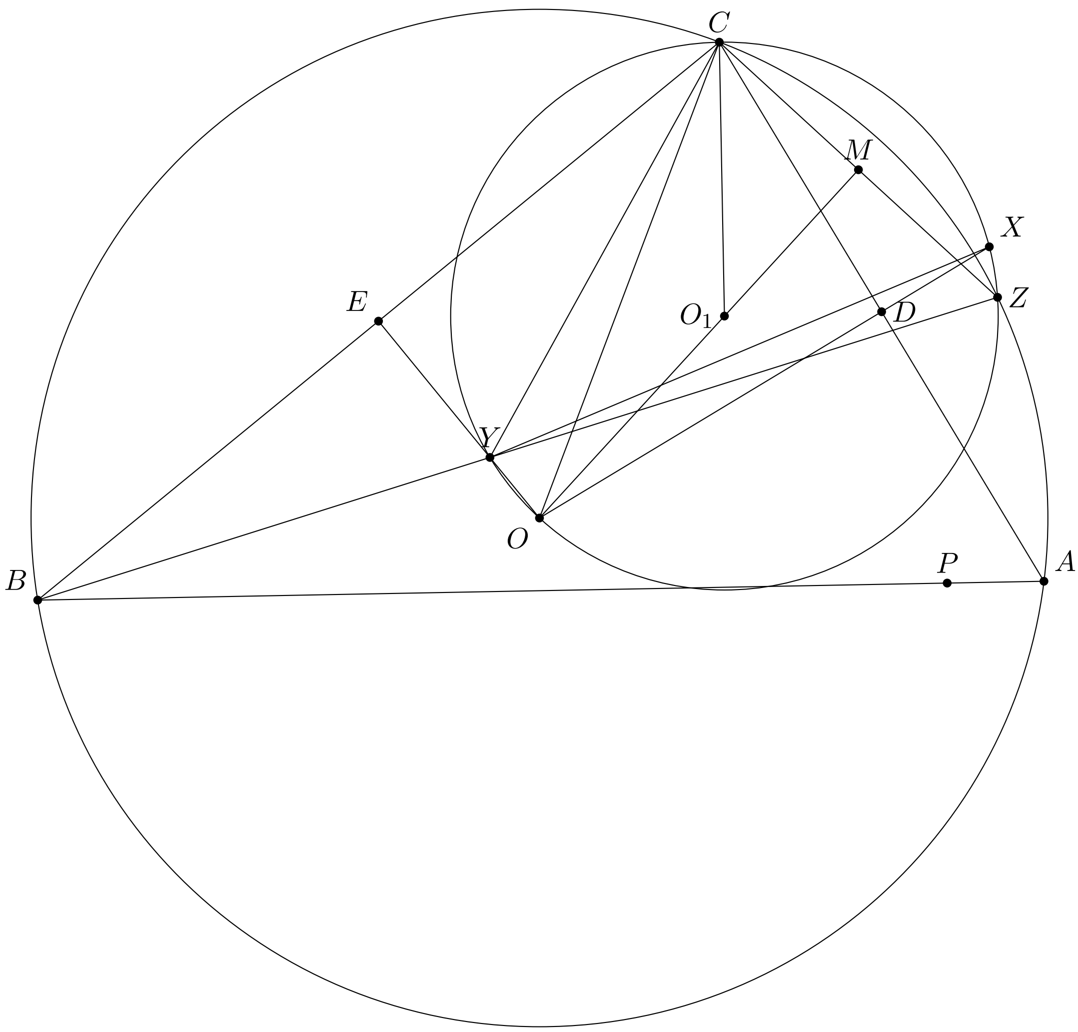
So if a player has at least one safe pile in front of them on their turn, they may make two or three perilous piles; then their opponent must return them at least one safe pile and they can repeat this; noting that the total number of stones in game is strictly decreasing so the game must end and that the losing state (all piles with 1 stone) is all perilous piles shows that they will win. And clearly if a player only has perilous piles before them then they must give their opponent at least one safe pile and thus their opponent wins. Thus sjbs wins if at least one of the three piles generated at the start is safe, and brainy wins if they are all perilous. The probability of brainy's win is thus $\frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{27}$ so the probability sjbs wins is $1 - \frac{1}{27} = \frac{26}{27}$ and so the answer is $26 \cdot 100 + 27 = \boxed{2627}$. □

§8.1.6 Another Geo Config

Source: Italian Team Competition Final 2020
Proposer: Matteddy#0482 (329956567132930048)
Problem ID: 124
Date: 2021-02-06
Difficulty: Challenging

For a triangle ABC, let P be on AB, and X be the circumcenter of APC, Y be the circumcenter of BPC, and Z be the intersection of AX and BY. Given that $AB = 91$, $BC = 104$, and $CA = 65$, what is the length of CZ?

Solution.



By simple angle chasing $\triangle CXA$ and $\triangle CYB$ are similar, we have $\angle ZAC = \angle XAC = \angle YBC = \angle ZBC$, so Z is on $\odot(ABC)$. Standard calculations on $\triangle ACX$ and $\triangle ABC$ give $\sin(\angle ZAC) = \frac{3\sqrt{3}}{14}$ and $R_{\odot(ABC)} = \frac{91\sqrt{3}}{3}$, so $CZ = 2R_{\odot(ABC)}\sin(\angle ZAC) = 39$ $\square$

§8.1.7 Product of Root Differences

Source: Original
Proposer: tanoshii#3160 (300065144333926400)
Problem ID: 125
Date: 2021-02-07
Difficulty: Challenging

Let $\alpha_1, \alpha_2, \ldots, \alpha_{2021}$ be the roots of $x^{2021} + 20x^2 + 21$. Find

$$\prod_{1 \le i < j \le 2021} (\alpha_i - \alpha_j)$$

Solution. [Write up by epicxtroll#6007 (300008472978653184)]

We solve for general $f(x) = x^n + ax^2 + b$, where $n \equiv 1 \pmod 4$. Let P be our product, we have

$$P = \prod_{i<j} (\alpha_i - \alpha_j)^2 = \prod_{i \neq j} (\alpha_i - \alpha_j)$$

since we flipped the signs of $\sum_{k=1}^{n-1} k \equiv 0 \pmod 2$ factors. Fix i; our product becomes

$$P = \prod_{i=1}^{n} \prod_{j \neq i} (\alpha_i - \alpha_j) = \prod_{i=1}^{n} \frac{\prod_{j=1}^{n} (\alpha_i - \alpha_j)}{\alpha_i - \alpha_i} = \prod_{i=1}^{n} \frac{f(\alpha_i)}{\alpha_i - \alpha_i}$$

after writing $x^n + ax^2 + b = \prod_{i=1}^{n} (x - \alpha_i)$. Although division by zero is undefined, since polynomials are continuous functions we can use L'Hopital's Rule to write the product as a limit, then use calculus to get

$$P = \prod_{i=1}^{n} \lim_{x \to \alpha_i} \frac{f(x)}{x - \alpha_i} = \prod_{i=1}^{n} n\alpha_i^{n-1} + 2a\alpha_i = -bn^n \prod_{i=1}^{n} \left(\alpha_i^{n-2} + \frac{2a}{n} \right).$$

Let ω be a primitive $(n-2)^{\text{nd}}$ root of unity and $r = \left(\frac{2a}{n}\right)^{\frac{1}{n-2}}$. The roots of $x^{n-2} + r^{n-2}$ are $-r\omega^k$ where $0 \le k < n-2$, so we can rewrite our product as

$$P = -bn^n \prod_{i=1}^{n} \prod_{j=0}^{n-2} \left(\alpha_i + r\omega^j\right) = -bn^n \prod_{j=0}^{n-2} \prod_{i=1}^{n} \left(\alpha_i + r\omega^j\right) = bn^n \prod_{j=0}^{n-2} \prod_{i=1}^{n} \left(-r\omega^j - \alpha_i\right) = bn^n \prod_{j=0}^{n-2} f\left(-r\omega^j\right).$$

The rest is just computation.

$$\begin{aligned} P &= bn^n \prod_{j=0}^{n-2} \left(-r^n\omega^{jn} + ar^2\omega^{2j} + b\right) = bn^n \prod_{j=0}^{n-2} \left[r^2\omega^j \left(-r^{n-2} + a\right) + b\right] \\ &= bn^2 \prod_{j=0}^{n-2} \left[r^2\omega^j a(n-2) + bn\right] = bn^2 \left[r^{2(n-2)} a^{n-2} (n-2)^{n-2} + bn^{n-2}\right] \\ &= 4a^n b(n-2)^{n-2} + bn^n \\ &\equiv \boxed{821} \pmod{10000} \end{aligned}$$

□

Solution. [Write up by AiYa#2278 (675537018868072458)]

Recall the fundamental theorem of symmetric polynomials: *every symmetric polynomial of n variables can be written as a function of the n-variable elementary symmetric polynomials.* Since $P = \prod_{i \neq j} (\alpha_i - \alpha_j)$ is symmetric in n variables and all elementary symmetric polynomials in n variables are zero except for $e_{n-2}(\alpha) = -a$ and $e_n(\alpha) = -b$ (where $e_k(\alpha)$ is the k-th elementary symmetric sum), P can be written as a sum of $(-a)^x(-b)^y$. P has degree $n(n-1)$, $-a$ has degree $n-2$, and $-b$ has degree n so we solve $(n-2)x + ny = n(n-1)$ to find our suitable powers of $-a$ and $-b$. Solving, we get $(x, y) = (n, 1); (0, n-1)$ so $P = Qa^n b + Rb^{n-1}$. It remains to choose easy a, b to find P, Q; an obvious choice is $(a, b) = (0, 1)$ and we have

$$P = \prod_{i=1}^{n} n\alpha_i^{n-1} = n^n \iff R = n^n.$$

To find Q, we plug in $(a, b) = (1, 1)$ and since $\alpha^n + \alpha^2 + 1 = 0 \iff \alpha^{n-2} = -1 - \frac{1}{\alpha^2}$ we have

$$P = -\prod_{i=1}^{n} \left(n\alpha_i^{n-2} + 2\right) = -\prod_{i=1}^{n} \left[(2-n) - \frac{n}{\alpha_i^2}\right] = -\sum_{k=0}^{n} (2-n)^{n-k}(-n)^k e_k\left(\frac{1}{\alpha^2}\right).$$

We first calculate $e_k\left(\frac{1}{\alpha}\right)$; from substituting $x \to \frac{1}{x}$ into $f(x)$ we get $e_2\left(\frac{1}{\alpha}\right) = 1, e_n\left(\frac{1}{\alpha}\right) = -1$, $e_k = 0$ otherwise except for $e_0 = 1$ as convention. We solve for $e_1\left(\frac{1}{\alpha^2}\right)$ as follows:

$$e_1^2\left(\frac{1}{\alpha}\right) = e_1\left(\frac{1}{\alpha^2}\right) + 2e_2\left(\frac{1}{\alpha}\right) \iff e_1\left(\frac{1}{\alpha^2}\right) = -2$$

and similarly

$$e_2^2\left(\frac{1}{\alpha}\right) = e_2\left(\frac{1}{\alpha^2}\right) + 2e_4\left(\frac{1}{\alpha}\right) + 2e_1\left(\frac{1}{\alpha^2}\right) e_2\left(\frac{1}{\alpha}\right) = 1.$$

For larger values of k, note that

$$e_k^2\left(\frac{1}{\alpha}\right) = e_k\left(\frac{1}{\alpha^2}\right) + 2\sum_{j=0}^{k} e_j\left(\frac{1}{\alpha^2}\right) e_{2(k-j)}\left(\frac{1}{\alpha}\right)$$

so all of them evaluate to zero except for $e_n\left(\frac{1}{\alpha^2}\right) = 1$. The rest is computation.

$$\begin{aligned}
P &= (n-2)^n - 2n(n-2)^{n-1} + n^2(n-2)^{n-2} + n^n \\
&= (n-2)^{n-2}\left(n^2 - 4n + 4 - 2n^2 + 4n + n^2\right) + n^n \\
&= 4(n-2)^{n-2} + n^n \iff P = 4(n-2)^{n-2} \\
P &= 4a^n b(n-2)^{n-2} + b^{n-1}n^n \\
&\equiv \boxed{821} \pmod{10000}
\end{aligned}$$

□

§8.2 Week 2

§8.2.1 Counting Intersections

Source: Gray Kangaroo 2003 Q24
Proposer: sjbs#9839 (434767660182405131)
Problem ID: 126
Date: 2021-02-08
Difficulty: Beginner

Rui draws 10 points on a large piece of paper, making sure that no three points are in a straight line. He then draws a segment joining each pair of points. If Orlo draws a straight line across Rui's diagram, without going through any of Rui's original points, what is the greatest possilbe number of lines that can be crossed?

Solution.

I claim the answer is $\boxed{\frac{420^2}{4} = 44100}$.

For a line drawn across Rui's construction splitting the points so that there are n on one side and $420 - n$ on the other, clearly there will be $n(420 - n)$ intersections. I.e. $\frac{420^2}{4} - \left(n - \frac{420}{2}\right)^2$. Thus the number of intersections is maximised when $n = 210$.

□

§8.2.2 The Race

Source: UK Senior Kangaroo, 2013 Q20
Proposer: sjbs#9839 (434767660182405131)
Problem ID: 127
Date: 2021-02-09
Difficulty: Beginner

Zella and Samantha stand at either end of a straight track. They then run at a constant (not different) speets to the other end of the track, turn and run back to their origional end at the same seed they ran before. On their first leg, they pass each other 20m from one end of the track. When they are both on their return leg, they pass each other for a second time 10m from the other end of the track. How many meters long is the track?

Solution.

Let the distance of the track be d, and the speeds of the two runners be u and v respectively. And let t_i be the time for which the runners are at a particular position. Then on the first leg we must have $20 = ut_1$ and $d - 20 = vt_1$, While on the second leg, $d + 10 = ut_2$ and $2d - 10 = vt_2$. Therefore we have

$$\frac{ut_1}{vt_2} = \frac{ut_2}{vt_2} \Rightarrow \frac{20}{d-20} = \frac{d+10}{2d-10} \tag{1}$$

$$d(d-50) = 0 \tag{2}$$

Thus we must have $d = \boxed{50}$ $\square$

§8.2.3 Remainder of Sums of Squares

Source: British Matematical Olympiad, Round 1, 1998 P2
Proposer: brainysmurfs#2860 (281300961312374785)
Problem ID: 128
Date: 2021-02-10

Let $a_1 = 19,\ a_2 = 98$. For $n \geq 1$, define a_{n+2} to be the remainder of $a_n + a_{n+1}$ when it is divided by 100. What is the remainder when

$$a_1^2 + a_2^2 + \cdots + a_{1998}^2$$

is divided by 8?

Solution.

Consider the remainders of a_i mod 4: the first six are $3, 2, 1, 3, 0, 3$ and it repeats every six. Since $6|1998$ it sufices to find the remainder when $a_1^2 + a_2^2 + \cdots + a_6^2$ is divided by 8; since odd squares are 1 mod 8, 2^2 is 4, and 0^2 is 0 we have $4 \cdot 1^2 + 2^2 + 0^2 \equiv \boxed{0} \pmod 8$. $\square$

§8.2.4 Another 255 Subsets

Source: 2017 HMMT General #8
Proposer: AiYa#2278 (675537018868072458)
Problem ID: 129
Date: 2021-02-11
Difficulty: Medium

Reimu has a collection of the 255 nonempty subsets of $\{1, 2, 3, 4, 5, 6, 7, 8\}$. Each minute, she takes two subsets chosen uniformly at random from the collection, and replaces them with either their union or their intersection, with each being equally likely. (The collection can contain repeated sets.) After 254 minutes, she is left with one set. The expected size of this subset can be expressed in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $100m + n$.

Solution. Official Solutions File

Answer: $\boxed{102655}$

A formulaic proof of the invariance of the expected size of all remaining subsets is as follows. Let $S_1, S_2, \cdots S_n$ be the sets in question, where a merge consists of taking two sets and replacing them with either their union or their intersection. If sets S_i, S_j are merged then the resulting set would be either $S_i \cap S_j$ or $S_i \cup S_j$. Summing over all $\binom{n}{2}$ pairs of i, j, our expected average size of sets after merging is

$$\frac{\binom{n}{2}2S - \sum_{i \neq j} (|S_i \cap S_j| + |S_i \cup S_j|)}{2\binom{n}{2}} = \frac{n(n-1)S - (n-1)S}{n(n-1)} = \frac{S}{n}$$

where $S = \sum_{i=1}^{n} |S_i|$.

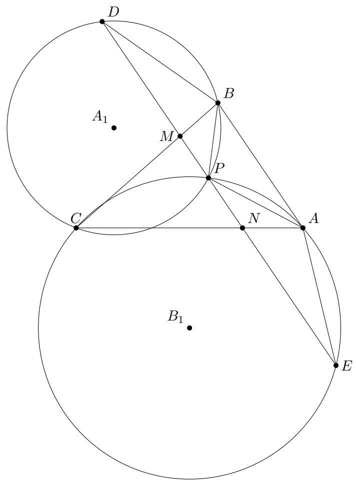
□

§8.2.5 4-5-6 Triangle

Source: AMOC December 2020 Camp Exam 2 (A/G) P4
Proposer: ChristopherPi#8528 (696497464621924394)
Problem ID: 130
Date: 2021-02-12
Difficulty: Hard

Let ABC be a triangle with side lengths AB = 4, BC = 5, CA = 6. Suppose P is a point inside ABC such that the circumcenters B' and A' of triangles ACP and BCP respectively lie outside ABC, with A, P, A' collinear and B, P, B' collinear. The line through P parallel to AB meets the circumcircles of ACP and BCP at E and D respectively, where D, E, P are distinct. Suppose $DE^2 - AP = (a - sqrt(b))/c$ where a, b, c are integers and b is squarefree. Find a + 69b + 420c.

Solution.



We claim P is the incenter of $\triangle ABC$. By arc lengths, $\angle PA'B = 2\angle BCP$ and since $A'P = A'B$ we have $\angle A'PB = 90 - \angle BCP \iff \angle APB = 90 + \angle BCP$. However, doing this to the circle centered at B' gives $\angle APB = 90 + \angle ACP \iff \angle ACP = \angle BCP = \frac{\angle C}{2}$ and $\angle APB = 90 + \frac{\angle C}{2}$. The locus of all points such that $\angle APB = 90 + \frac{C}{2}$ is (AIB), which the C-angle bisector intersects twice, once at I and once outside of $\triangle ABC$; since P is specified to be inside $\triangle ABC$ we conclude $P \equiv I$.

Since $\angle AEP = \angle ACP = \frac{\angle C}{2}$ and $\angle CAP = \angle APE = \frac{\angle A}{2}$ we have $\triangle EAP \cong \triangle CPA$ by AAS; similarly $\triangle CPB \cong \triangle DBP$ so $DE = EP + DP = AC + AB = 11$. Also, the tangents to the incircle from A have length $s - a = \frac{5}{2}$ and $[ABC] = rs \iff r = \frac{\sqrt{7}}{2}$ so $AP = 2\sqrt{2}$. $\boxed{1210202}$ □

§8.2.6 "What's a signed integer?"

Source: Original Problem
Proposer: Constan#6792
Problem ID: 131
Date: 2021-02-13
Difficulty: Challenging

For all pairs (n, q) where n is a positive integer, q is a rational non-integer, and $n^q - q$ is an integer, find $\lfloor n + q \rfloor = x$. The sum of all possible values of x is the answer. (For example, if we have $(1, 1)$ and $(2, 2)$, then the answer is $2 + 4 = 6$)

Solution. [Solution by Constan#6792]

Let's see that if n^q is rational and q is positive, n^q is integer. To do this, suppose it is rational and not integer where $(a, b) = (c, d) = 1$. $n^{\frac{a}{b}} = \frac{c}{d}$ $d^b.n^a = c^b$ Now we consider the exponent of a certain prime in the factorization of d, c, n which are e_d, e_c, e_n respectively. $b.e_d + a.e_n = b.e_c$ $a.e_n \equiv 0$ (b) $e_n \equiv 0$ (b) So n is a b -th power. $d^b.k^{ab} = c^b$ $d.k^a = c$ $k^a = \frac{c}{d}$ Absurd, because a perfect power must be a whole number.
Now if q is positive there are no solutions because if n^q and $n^q - q$ are integers, q would be.
Then $n = -\frac{a}{b}$ with a and b co-prime positive integers and $\frac{1}{n^{\frac{a}{b}}} + \frac{a}{b}$ is an integer.
Now if $\frac{n_1}{d_1} + \frac{n_2}{d_2} = \frac{n_1 d_2 + n_2 d_1}{d_1 d_2}$ is an integer where $(n_1, d_1) = (n_2, d_2) = 1$ we have to: $d_1 | n_1 d_2 \rightarrow d_1 | d_2$. $d_2 | n_2 d_1$ $\rightarrow d_2 | d_1$. So $d_1 = d_2$.
That taking this to what we have tells us that $n^{\frac{a}{b}} = b$ y $n^a = b^b$ Where with a very similar reasoning as before, b is a a -th power. $b = k^a$ $n^a = k^{ak^a}$ $n = k^{k^a}$
$\frac{1}{(k^{k^a})^{\frac{a}{k^a}}} + \frac{a}{k^a} = \frac{a+1}{k^a}$
Where: $k^a \leq a + 1$ If $k \geq 3$ $3^a \leq k^a \leq a + 1$ has no solutions. Then $k \leq 2$ where if $k = 1$, $\frac{a}{b}$ would be an integer, so $k = 2$. $2^a \leq a + 1$ Where the only solution is $a = 1$ that gives us the triple: $(k^{k^a}, k^a, a) = (4, 2, 1)$. Thus $(n, q) = (4, -\frac{1}{2})$ the only solution. So the answer is 4. □

§8.2.7 Diagonals in a Convex 1001-gon

Source: Sharygin 2019 Finals Grade 8 P8
Proposer: TaesPadhihary#8557 (665057968194060291)
Problem ID: 132
Date: 2021-02-14
Difficulty: Challenging

What is the least positive integer k such that, in every convex 1001-gon, the sum of any k diagonals is greater than or equal to the sum of the remaining diagonals?

Solution. Offical Solution, page 6 P8

Answer: $\boxed{499000}$ $\square$

§10 CCCC After Math (Season 10)

§10.1 Week 1

§10.1.1 EpicXtroll

Source: Original
Proposer: epicxtroll#6007 (300008472978653184)
Problem ID: 133
Date: 2021-02-15
Difficulty: Beginner

How many possibilities are there for the total score among all participants at the end of a standard QoTD season? (Assume that no rounding occurs. Recall that a standard QoTD season has 14 questions.)

Solution.

The answer is $\boxed{15}$: if $0 \leq N \leq 14$ is the number of questions solved (by anyone), then the total score is $1000N$. □

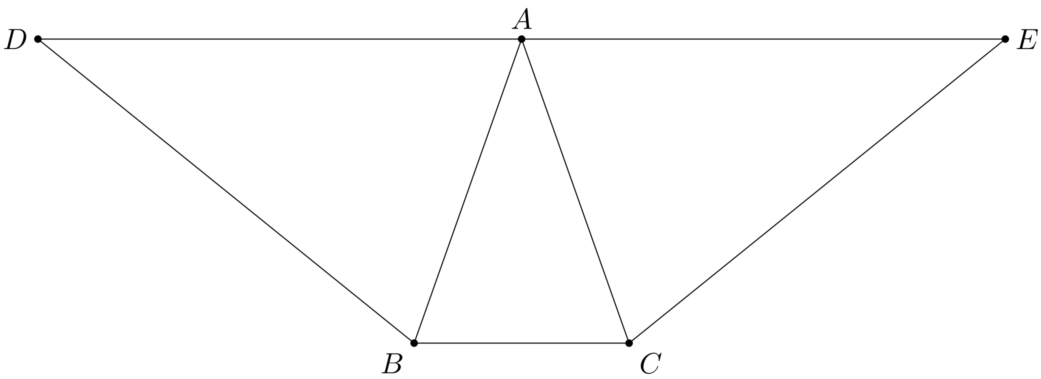
§10.1.2 Similar Triangles

Source: Original
Proposer: Matteddy#0482 (329956567132930048)
Problem ID: 134
Date: 2021-02-16
Difficulty: Beginner

Let ABC be a triangle with $AB = AC$ and $BC = 10$. Construct D and E outside ABC closer to B and C respectively such that $\triangle DAB \sim \triangle ABC \sim \triangle EAC$.

If $DE = 45$, then what is AB?

Solution.



Note that $\angle DAB = \angle ACB = \angle ABC$ and hence $AD \parallel BC$. Similarly $AE \parallel BC$ and so A is the midpoint of DE, and A, D, E are collinear.

So $AD = 22.5$ and hence $AB = BC * \sqrt{\frac{AD}{BC}}$ by similarity, which is $\boxed{15}$. $\square$

§10.1.3 Factorial Sums and Divisors

Source: Folklore
Proposer: sjbs#9839 (434767660182405131)
Problem ID: 135
Date: 2021-02-17
Difficulty: Easy

What is the largest prime number which divides

$$0! + 1! \times 1 + 2! \times 2 + 3! \times 3 + \cdots + 159! \times 159 + 160! \times 160$$

Solution.

We proceed by using the identity $n! + n! \times n = (n+1)!$ and iterating:

$$0! + 1! \times 1 + \cdots \tag{3}$$
$$2! + 2! \times 2 + \cdots \tag{4}$$
$$3! + 3! \times 3 + \cdots \tag{5}$$
$$\cdots \tag{6}$$
$$161! \tag{7}$$

As neither 161 nor 159 are prime, the largest prime divisor of the sum is $\boxed{157}$ □

Solution.

Note that $(n+1)! - 1$ counts the non-identity permutations of $(1, 2, \cdots, n+1)$. Consider a permutation where $(1, 2, \cdots n-k)$ are fixed points but $n-k+1$ is not a fixed point. So $n-k+1$ can go in the $n-k+2^{\text{nd}}$ to $n+1^{\text{st}}$ spots, which is a total of k possibilites; the remainder of the k terms can be permuted willy-nilly. Thus, there $k \cdot k!$ such possibilities, and summing over k from 1 to n (since $k = n+1$ gives the identity permutation) yields the identity. $\boxed{157}$ □

§10.1.4 Brainy's Passcode

Source: Italian team competition
Proposer: HoboSas#3200 (310725130097786880)
Problem ID: 136
Date: 2021-02-18
Difficulty: Medium

Brainy forgot the numerical unlock code of his Nokia once again... This time he remembers that it is 69-digits long and the leading digit is greater or equal than the sum of all the remaining digits. Find the total number of possible codes.

A computational aid may be used to calculate the final answer.

Solution.

Let n be the number of digits of the code, in our case $n = 69$ and let k be the leading digit, clearly $k \in \{1, 2, \ldots, 9\}$. Let i be the sum of the remaining digits, where $i \in \{1, 2, \ldots, k\}$. Using the stars and bars method (also called sticks and stones or Tonys and Wangs), the number of possible codes (with k and i fixed) is

$$\binom{(n-1)-1+i}{(n-1)-1}$$

Cycling this for all possible values of k and i we get the following expression

$$\sum_{k=1}^{9}\sum_{i=0}^{k}\binom{n-2+i}{n-2}$$

Using the Hockey-stick identity 2 times in a row we get

$$\begin{aligned}\sum_{k=1}^{9}\sum_{i=0}^{k}\binom{n-2+i}{n-2} &= \sum_{k=1}^{9}\binom{k+n-1}{k}\\ &= \binom{n+9}{9}-1\end{aligned}$$

Plugging in $n = 69$ we get

$$\binom{78}{9} - 1 = \boxed{182364632449}.$$

□

Solution. [Write up by AiYa#2278 (675537018868072458)]

Let the k^{th} digit from the left be d_k. Our constraints become $9 \le d_1 \le \sum_{k=2}^{69} d_k$; let $y = 9 - d_1$ and $x = d_1 - \sum_{k=2}^{69} d_k$. Our contraints become

$$9 = d_1 + y = x + y + \sum_{k=2}^{69} d_k$$

so it suffices to count the ways to distribute 9 objects among 70 people. Letting 69 "sticks" divide the 9 "stones" into 70 parts, we see that this is $\binom{78}{9}$ then subtract 1 because a string of 69 zeros is invalid. $\boxed{182364632449}$ □

§10.1.5 Production Loop

Source: Original/Folklore
Proposer: AiYa#2278 (675537018868072458)
Problem ID: 137
Date: 2021-02-19
Difficulty: Hard (CN4)

Let $f : \mathbb{N} \to \mathbb{N}$ be a function defined as follows:

$$f(n) = 2n + 1 - 2^{\lfloor \log_2 n \rfloor + 1}$$

and let $f^a(n) = f\left(f^{a-1}(n)\right)$. Let $t(n)$ be the smallest positive integer such that there exists a positive integer N such that $f^{t(n)}(n) = f^{t(n)+N}(n)$. Determine the remainder when $\sum_{n=2^{2020}}^{2^{2021}} f^{t(n)}(n)$ is divided by 1009.

(A scientific calculator can be used)

Solution.

Consider what f does to n in binary: $2n+1$ concatenates a 1 to the end of n, while $2^{\lfloor \log_2 n \rfloor + 1}$ subtracts the concatenation of 0 to the end of n from $2n+1$. This can be seen as cycling the leading 1 of n to the end of n; for instance $f(6 = 101_2) = 11_2 = 3$. After applying f enough times, we're left with the number of ones present in n. We now count the number of integers $2^{2020} \le n < 2^{2021}$ which have k ones present in their binary representation. All these numbers are 2021 digits long, so we want to distribute $k-1$ ones among 2020 digits; there are $\binom{2020}{k-1}$ ways to do this and each such n contributes $2^k - 1$ to the sum. Remembering that $f^{t(n)}(n) = 1$ when $n = 2^{2021}$, it remains to sum

$$\begin{aligned}
1 + \sum_{k=1}^{2021} \binom{2020}{k-1}\left(2^k - 1\right) &= 1 + 2\sum_{k=1}^{2021} \binom{2020}{k-1} 2^{k-1} - \sum_{k=1}^{2021} \binom{2020}{k-1} \\
&= 1 + 2 \cdot 3^{2020} - 2^{2020} \\
&\equiv 1 + 2 \cdot 81 - 16 \pmod{1009} \\
&= \boxed{147}
\end{aligned}$$

where we have used $(1+2)^n = \sum_{k=0}^{n} \binom{n}{k} 2^k = 3^k$ to help simplify. □

§10.1.6 Weird polynomial

Source: NYCMT, 2020 P9 of 10
Proposer: TaesPadhihary#8557 (665057968194060291)
Problem ID: 138
Date: 2021-02-20

Let $p = 2053$ be a prime. For positive integers x let

$$f(x) = x^{400} + x^{326} + x^{200} + x^{126} + 2$$

and let $g(x)$ denote the unique integer $0 \le y \le p-1$ such that $y^{1759} - x$ is divisible by p. Compute the remainder when

$$\sum_{i=1}^{2052} g(f(i))$$

is divided by 2053.

Solution.

To find y, we want to raise $y^{1759} \equiv x \pmod p$ to a power k such that $1759k \equiv 1 \pmod{p-1}$; solving we get $k \equiv 7 \pmod{p-1}$ and so $y \equiv x^7 \pmod p$. It remains to sum $f(i)^7 \pmod p$. Since $g, g^2, \cdots, g^{p-1}$ where g is a generator is a permutation of the nonzero residues mod p we find that

$$\sum_{i=1}^{p-1} i^k = \sum_{i=1}^{p-1} g^{ik} = g^k \left(\frac{g^{(p-1)k} - 1}{g^k - 1} \right) \equiv 0 \pmod p$$

unless $p-1 | k$ in which case each of the $p-1$ terms is 1 for a sum of $-1 \pmod p$. Since the maximum coefficient in $f(x)^7$ is $7 \cdot 400 = 2800$ it suffices to find the coefficient of x^{2052} and the constant term. Using multinomial expansion, this is equivalent to finding nonnegative integers (a,b,c,d,e) such that $400a + 326b + 200c + 126d + 0e = 2052 \iff 200a + 163b + 100c + 63d = 1026$ and $a+b+c+d+e = 7$. Notice $a > 1$ since $200 \cdot 1 + 163 \cdot 6 < 1026$ and $a < 5$ since $200 \cdot 5 = 1000$ and no combination of $160, 100, 63$ will make that sum to 1026. Also, reducing mod 9 yields $2a + b + c \equiv 0 \pmod 9 \iff a + 7 \equiv d + e \pmod 9$. When $a = 2$, we are forced $d, e = 0$ so $163b + 100c = 626 \iff (a,b,c,d,e) = (2,2,3,0,0)$. When $a = 3$, we are forced $d = 0, 1$ so either $163b + 100c = 426, 363 \iff (a,b,c,d,e) = (3,2,1,0,1), (3,1,2,1,0)$. When $a = 4$, we have $163b + 100c + 63d = 226 \iff (a,b,c,d,e) = (4,1,0,1,1), (4,0,1,2,0)$. Remembering the constant term is 2^7, it remains to sum

$$-\binom{7}{2,2,3,0,0} - 2\binom{7}{3,2,1,0,1} - \binom{7}{3,1,2,1,0} - 2\binom{7}{4,1,0,1,1} - \binom{7}{4,0,1,2,0} - 2^7 \equiv \boxed{1983} \pmod{2053}.$$

□

§10.1.7 Weird triangle

Source: Original
Proposer: tanoshii#3160 (300065144333926400)
Problem ID: 139
Date: 2021-02-21
Difficulty: Challenging (G7)

Let ABC be a $13-14-15$ triangle, with $AC = 14$. Points X, Y satisfy

$$CX - BX = BX - AX = 1 = AY - BY = BY - CY.$$

The value of XY can be written in the form $\frac{a\sqrt{b}}{c}$, where b is not divisible by the square of any prime and a and c are relatively prime positive integers. Find $10000a + 100b + c$.

Solution.

Let I be the incenter of $\triangle ABC$. Construct three circles, each centered at one of A, B, C. Note that these three circles touch each other at the three intouch points of $\triangle ABC$. Thus their radii are 6, 7 and 8. WLOG assume AB = 13. Let W be the center of the unique circle externally tangent to our three circles at A, B, C, and Z the center of the unique circle internally tangent to our three circles at A, B, C. By definition, since the radius of the circle at C is 8, the radius of the circle at B is 7 and the radius of the circle at A is 6, W and Z must be precisely X and Y. Now consider an inversion with respect to the incircle. Let the intersection of the line AI and the circle at A further from I be D, and the one closer to I be E. Then if the radius of the circle at A is R, and the inradius is r, $DI \cdot EI = (AI - R)(AI + R) = AI^2 - R^2 = R^2 + r^2 - R^2 = r^2$, so by definition D and E swap under our inversion. Thus the circle at (A) must map to itself under our inversion, and so do the circles at (B) and (C). Then since inversion preserves tangency, the circle at X externally tangent to our three circles must swap with the circle at Y internally tangent to our three circles.

Now suppose a line through I meets the circle at X at M, N and the circle at Y at P, Q. Then since the image of M must lie on the ray MI, and also lie on the circle at Y since M lies on the circle at X, WLOG M and P swap, and then so do N and Q. Thus $IM \cdot IP = r^2 = IN \cdot IQ$, and by power of a point $IM \cdot IN = {R_x}^2 - IX^2$ and $IN \cdot IQ = {R_y}^2 - IY^2$ where R_x and R_y are the radii of the circles at X and Y. Since $IM \cdot IP$ is independent of M and P we find that I is the insimilicenter of the circles at X and Y and therefore lies on the line segment XY. Thus if we set $IX = cR_x$ and $IY = cR_y$ then $r^4 = {R_x}^2{R_y}^2(1-c^2)^2$; by direct calculation (the inradius formula for r and Descartes' Kissing Circles Theorem for R_x and R_y) we obtain four values for c. However, two of these values are negative and a third is greater than 1 (we know that clearly IX and IY are smaller than R_x and R_y by simply drawing the diagram in this case), so the only acceptable value of c is $c = \frac{\sqrt{37}}{42}$. Then by direct calculation we obtain that $XY = IX + IY = cR_x + cR_y = \frac{672\sqrt{37}}{1727}$ so the answer is $\boxed{6725427}$ as required. □

Solution.

An elementary, computationally feasible analytic solution is still possible using Cartesian coordinates. From the given condition, if we let $BX = d$ then $AX = d-1, CX = d+1$. Consider circles $\omega_A, \omega_B, \omega_C$ centered at A, B, C with radii $d-1, d, d+1$ respectively. Let $A = (-5, 0), B = (0, 12), C = (9, 0)$. Then the radical axis of ω_A and ω_C is given by

$$(x+5)^2 + y^2 - (d-1)^2 = (x-9)^2 + y^2 - (d+1)^2 \iff 7x = 14 - d$$

and the radical axis of ω_A and ω_B is given by

$$(x+5)^2 + y^2 - (d-1)^2 = x^2 + (y-12)^2 - d^2 \iff 5x + 12y = 60 - d$$

so the radical center of $\omega_A, \omega_B, \omega_C$ satisfies

$$(x, y) = \left(\frac{14-d}{7}, \frac{350-2d}{84}\right).$$

This point must be on ω_B, so plugging in and simplifying we have

$$\left(\frac{14-d}{7}\right)^2 + \left(\frac{350-2d}{84} - 12\right)^2 = d^2 \iff \frac{1727}{1764}d^2 + \frac{25}{126}d - \frac{2353}{36} = 0.$$

Solving this quadratic would be hell, but we don't need to. Let the two roots be r, s with r positive and s negative. Consider a point P with $PA = |s-1|, PB = |s|, PC = |s+1|$; since $s < -1$ we have $PA = -s+1, PB = -s, PC = -s-1$. This satisfies $AP - BP = BP - CP = 1$, so $P \equiv Y$! Thus, it remains to find

$$XY = \sqrt{\frac{(r-s)^2}{7^2} + \frac{(r-s)^2}{42^2}} = \frac{r-s}{7}\sqrt{\frac{37}{36}}$$

and now the miraculous

$$(r-s)^2 = (r+s)^2 - 4rs = \frac{1764^2}{1727^2}\left(\frac{25^2}{126^2} + \frac{2353}{9} \cdot \frac{1727}{1764}\right) = 256 \cdot \frac{1764^2}{1727^2} \iff r - s = 16 \cdot \frac{1764}{1727}$$

so $XY = \frac{672\sqrt{37}}{1727} \iff \boxed{6725427}$.

□

§10.2 Week 2

§10.2.1 Real Square Roots

Source: Purple Comet! Math Meet, High School, 2020 P4
Proposer: TaesPadhihary#8557 (665057968194060291)
Problem ID: 140
Date: 2021-02-22
Difficulty: Beginner

Let $\mathcal{S}$ be the set of integers n such that

$$\sqrt{\frac{(2021-k)^2}{2021-k^2}}$$

is a real number.

Compute the sum of all elements in $\mathcal{S}$.

Solution.

The numerator is always real, so we seek k such that $2021 - k^2 > 0$; however if k satisfies this inequality so does $-k$. Then observe $k = 2021$ makes the numerator 0, so it is real too. $\boxed{2021}$. □

§10.2.2 "What's a positive integer?"

Source: Original Problem
Proposer: TaesPadhihary#8557 (665057968194060291)
Problem ID: 141
Date: 2021-02-23
Difficulty: Beginner

Let n be a positive integer. Find the number of positive divisors of 69420^{69} which are of the form $4n+1$.

Solution. [Solution by TaesPadhihary#8557 (665057968194060291)]

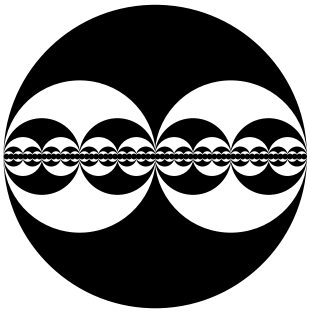
Note that $69420^{69} = 2^{138} \cdot 3^{69} \cdot 5^{69} \cdot 13^{69} \cdot 89^{69}$. From these, only 3 is a prime which is equivalent to $3 \pmod 4$ and others, i.e. $5, 13, 89$ are all $1 \pmod 4$. Therefore, the number of ways to choose a divisor of the form $4n+1$ is $35 \times 70 \times 70 \times 70$[5]. But, since n is positive integer, so 1 can not be a chosen divisor. Hence, the desired answer is $35 \cdot 70^3 - 1 = \boxed{12004999}$. □

[5] 35 ways being for the even powers of 3 and 70 otherwise

§10.2.3 Converging Tangent Circles

Source: Original Problem
Proposer: HoboSas#3200 (310725130097786880)
Problem ID: 142
Date: 2021-02-24
Difficulty: Easy

In the following diagram every circle has two internally tangent circles also tangent to each other. The radius of the biggest circle is $\frac{1}{\sqrt{\pi}}$. If the area of the black region can be expressed as $\frac{a}{b}$, compute $100a + b$.



Solution. [Solution by HoboSas#3200 (310725130097786880)]

Let r_n and A_n be the radius and the area of the nth circle (in descending order). From construction we have $r_{n+1} = \frac{r_n}{2}$ which leads to $r_n = \frac{r_0}{2^n}$, where $r_0 = \frac{1}{\sqrt{\pi}}$. Looking at the desired area A_ζ as difference of black and white circles areas, we get the following:

$$A_\zeta = \sum_{n=0}^{\infty}(-2)^n A_n \tag{8}$$

$$= \sum_{n=0}^{\infty}(-2)^n (r_n)^2 \pi \tag{9}$$

$$= (r_0)^2 \pi \sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n \tag{10}$$

$$= \left(\frac{1}{\sqrt{\pi}}\right)^2 \pi \frac{1}{1-(-\frac{1}{2})} \tag{11}$$

$$= \frac{2}{3} \tag{12}$$

Which means $\boxed{203}$ is the solution. □

Solution. [Solution by flame#6784 (185229437787176962)]

Let B be the total black area; the total white area is then $1 - B$. Looking at the two biggest white circles, we see that each of them is similar to the big circle by a factor of $\frac{1}{2}$ with colors inverted; thus the total black area inside each of the two white circles is $\frac{1-B}{4}$. Since the black area outside the two big white circles is $\frac{1}{2}$ we have

$$B = \frac{1}{2} + \frac{1-B}{2} \iff B = \frac{2}{3} \iff \boxed{203}.$$

□

§10.2.4 Maze Escape

Source: 2018 HMMTNov Guts 18
Proposer: AiYa#2278 (675537018868072458)
Problem ID: 143
Date: 2021-02-25
Difficulty: Medium (C3)

MODSbot has trapped Brainy at the southwest entrance of a two-by-fifteen grid of cells. To escape, Brainy has to enter into the grid from the southwest cell and exit through the northeast cell. However, MODSbot can fill any subset of the 20 cells with cement, effectively blocking off those cells. Out of all 2^{30} possible grid configuations, in how many of them can Brainy escape?

Solution.

Follow the official solution to get the recursion for a_n and b_n. Iterating 15 indices of a and b is inefficient; for efficient computation we let $c_n = a_n + b_n$ and note that $c_{n+1} = 2c_n + c_{n-1}$; either directly recursing or finding a closed form yields $c_{15} = 665857$. By inspection, we have $a_n = b_n - 1$ so $a_n = \boxed{332928}$. □

§10.2.5 Primes divide powers of 5

Source: China National Olympiad 2009, P2
Proposer: Matteddy#0482 (329956567132930048)
Problem ID: 144
Date: 2021-02-26
Difficulty: Hard

Suppose $\mathcal{S}$ is the set of all pairs $(p, q), p \leq q$ such that

$$pq \mid 5^p + 5^q.$$

Find

$$\sum_{(p,q)\in\mathcal{S}} (1000p + q).$$

Solution.

AoPS Solution

Answer: $\boxed{14326}$ □

§10.2.6 tRiViaL

Source: Original Problem
Proposer: QuantumSigma#9996
Problem ID: 145
Date: 2021-02-27
Difficulty: Challenging

Consider an integer-valued function $f(x, y)$ defined on the lattice points of the $x - y$-plane. The value of f at each point is average its 4 neighbours. The difference in f is less than 70000 for any pair of lattice points a distance of exactly 2021 apart. If $f(a, b + c) - f(a, b) = 1780$ for some integers (a, b, c), find the sum of all possible $|c|$.

Solution. [Solution by QuantumSigma#9996] 3738. Let F be the set of functions $g : Z^2 \to Z$ fulfilling the 'average of neighbours' criteria.

1. For any $g \in F$, g is either constant or unbounded (Klarijōn#0829, modified). Proof: Assume there exist 2 adjacent points (A, B) such that $g(A) < g(B)$. Then there exists point C adjacent to B such that $g(B) < g(C)$. Repeating ad infinitum, there exists an infinite strictly monotonic chain of points. Thus g is unbounded.

2. Define P and Q:

$$Pf(x, y) = f(x + 2021, y) - f(x, y)$$
$$Qf(x, y) = f(x, y + 2021) - f(x, y)$$

   Then for any $g \in F, P\{g\}, Q\{g\} \in F$ (trivial)

3. For any $f \in F$ such that $P\{f\}, Q\{f\}$ are bounded, f is linear. Proof: By (1) and (2), Pf, Qf are constant. Let $h = 2021f - Pfx - Qfy$, then $h \in F$. Since h is doubly periodic (trivial), it is bounded, and thus constant. Then $f = \frac{Pfx+Qfy+h}{2021}$.

It follows that $|c|$ is a factor of 1780. The problem is then trivial. □

§10.2.7 Lamps in a Circle

Source: Olympic Revenge, 2013 P5
Proposer: ChristopherPi#8528 (696497464621924394)
Problem ID: 146
Date: 2021-02-28
Difficulty: Challenging

Suppose n lamps are arranged in a circle, numbered from 1 to n clockwise, with $0 \geq L \geq n$ lamps turned on. A testing procedure consists of performing the following operations:

for each lamp turned on, we take its number i; the i lamps following it clockwise receive a signal

- After a signal has been sent for each lamp turned on, every lamp which received an odd number of signals will have its state changed; those receiving an even number of signals will not.

Let T be the set of all 2^n possible initial configurations. We define a function t from T to T, where if X represents an initial configuration of lamps, $t(X)$ is the configuration obtained after applying the testing procedure to X. Suppose S is the sum of all integers $n < 2^{2015}$ such that the function t is bijective. Find the smallest positive integer c such that 2^c is greater than the sum of every element of S.

Solution. (Art of Problem Solving write ups) □

§11 err¶¶: ¶¶ ¶¶¶ ¶¶¶¶¶¶¶¶ ¶¶¶¶ ¶¶ ¶¶¶ ¶¶¶¶ ¶¶¶¶¶. (Season 11)

§11.1 Week 1

§11.1.1 Integral AM-GM

Source: Original
Proposer: tanoshii#3160 (300065144333926400) and iparis#5018 (269634843468496906)
Problem ID: 147
Date: 2021-03-01
Difficulty: Beginner

p, q are positive integers such that $pq = 2021^{2021}$. What is the minimal number of digits in $p+q$?

Solution.

By AM-GM the minimal value of $p+q$ occurs when p, q are close; an obvious close pair is $(p,q) = \left(43^{1010}, 47^{1010}\right)$ and the minimal possible value of $p+q$ (over positive reals) occurs at $p = q = 2021^{1010.5}$. The number of digits in a number n is $\lfloor \log_{10}(n) \rfloor + 1$; noticing this we see that both our obvious guess and the absolute minimum have $\boxed{3341}$ digits. □

§11.1.2 d20 but not D20

Source: Original
Proposer: tanoshii#3160 (300065144333926400) and iparis#5018 (269634843468496906)
Problem ID: 148
Date: 2021-03-02
Difficulty: Beginner

21 people, including IcosahedralDice, each roll an icosahedral die, which have faces numbered 1 through 20 and each face is equally likely to land face-up. Given that IcosahedralDice rolled the (strictly) largest number, find the expected value of his roll.

Solution.

Let IcosahedralDice roll r. The probability that a person rolls between 1 and $r-1$ inclusive is $\frac{r-1}{20}$ so the probability that all 20 other people rolled less than r is $\left(\frac{r-1}{20}\right)^{20}$; thus the probability IcosahedralDice's roll is strictly greater than everyone else's is $\sum_{r=1}^{20}\left(\frac{r-1}{20}\right)^{20}$. The expected value is then

$$E = \frac{\sum_{r=1}^{20} r \cdot \left(\frac{r-1}{20}\right)^{20}}{\sum_{r=1}^{20}\left(\frac{r-1}{20}\right)^{20}} \iff \lfloor 100000E \rfloor = \boxed{1952901}.$$

□

§11.1.3 Is it better to walk or run in the rain?

Source: Original
Proposer: tanoshii#3160 (300065144333926400) and iparis#5018 (269634843468496906)
Problem ID: 149
Date: 2021-03-03
Difficulty: Easy

Rain is falling at an angle of 2.021 (radians) measured anticlockwise from the positive x-axis, with velocity 2021. A unit sphere rolls from the origin to $(2021, 0)$ with constant speed v.

Given that v minimizes the amount of rain that falls on the ball, find $\lfloor 10^2 v \rfloor$, where $\lfloor x \rfloor$ is the greatest integer less than or equal tos x.

Note that if you run too fast, you end up running into *the rain.*

Solution.

Change the frame of reference such that the rain is still relative to you; this can be done by moving the entire plane with velocity $(2021\cos\theta, 2021\sin\theta)$ where $\theta = 2.021$, the angle at which rain falls. This means that your destination point will be on the line with slope $\tan\theta$ passing through $(2021, 0$ and your new velocity vector is $(v + 2021\cos\theta, 2021\sin\theta)$. To minimize the amount of rain that falls on you, you should run through as little rain as possible, which means your path should be perpendicular to the destination line; this is achieved when your velocity vector has direction $\tan\left(\theta - \frac{\pi}{2}\right)$. Solving, we get

$$\tan\left(\theta - \frac{\pi}{2}\right) = \frac{2021\sin\theta}{v + 2021\cos\theta} \iff v = 4644.3870 \iff \boxed{464438}.$$

□

§11.1.4 Don't over think this

Source: Original
Proposer: tanoshii#3160 (300065144333926400) and iparis#5018 (269634843468496906)
Problem ID: 150
Date: 2021-03-04
Difficulty: Medium

Let N be the number of ways to put the numbers $2, 4, 8, \cdots, 2^{420}$ into a 20-by-21 grid such that the sums of rows are increasing from top to bottom and the sums of columns are increasing from left to right. How many digits does N have?

Solution.

We solve for a general m-by-n grid. Note that permuting rows leaves the sums of columns intact, and vice versa. For each of the $(mn)!$ ways to fill the grid, we must divide by $m!$ to correct for the ordering of the rows and $n!$ for the columns; as we established before these events are indepent so our answer is $\frac{(mn)!}{m!n!}$; in the case of $(m, n) = (20, 21)$ our answer is $\left\lfloor \log_{10} \left(\frac{420!}{20!21!} \right) \right\rfloor + 1 = \boxed{883}$. □

§11.1.5 Genfun is Not Fun

Source: Original
Proposer: tanoshii#3160 (300065144333926400) and iparis#5018 (269634843468496906)
Problem ID: 151
Date: 2021-03-05
Difficulty: Hard

Suppose $a_0 = 0$, $a_1 = 1$ and for $n \geq 1$,

$$a_n = n\left(a_{n-1} + 3a_{n+1}\right).$$

If $S = \sum_{n\geq 0} a_n$ then find the 2021^{st} to 2025^{th} digits of S. (For instance, the third to fifth digits of π is 415.)

Solution.

We let $f(x) = \sum_{n\geq 0} a_n x^n$ be a power series; our desired sum S is just $f(1)$. To get something of the form $na_n x^n$ we can differentiate to get $f'(x) = \sum_{n\geq 0}(n+1)a_{n+1}x^n$ where $a_k = 0$ if $k \leq 0$ for convenience. Now, if we can construct a function $F(x)$ using a linear combination of $f(x)$ and $f'(x)$ such that the coefficient of x^n in $F(x)$ is $n\left(a_{n-1} + 3a_{n+1}\right)$ then we know $F(x) = f(x)$. We can shift indices by multiplying or dividing by powers of x accordingly; we use this to try and turn $3f'(x) = \sum_{n\geq 0} 3(n+1)a^{n+1}x^n$ into $\sum_{n\geq 0} 3na_{n+1}x^n$ so we need to subtract some shifted form of $f(x)$. Indeed, $\frac{f(x)}{x} = \sum_{n\geq 0} a_{n+1}x^n$ so $f'(x) - \frac{f(x)}{x} = \sum_{n\geq 0} na_{n+1}x^n$; similarly we have $x^2 f'(x) + xf(x) = \sum_{n\geq 0} na_{n-1}x^n$. So, we have

$$F(x) = 3f'(x) - \frac{3f(x)}{x} + x^2 f'(x) + xf(x) = \sum_{n\geq 0}\left(a_{n-1} + 3a_{n+1}\right)x^n = \sum_{n\geq 0} a_n x^n = f(x)$$

which is a separable differential equation; chucking this into WolframAlpha we get

$$f(x) = \frac{Cx}{x+3}\exp\left(\frac{\arctan\left(\frac{x}{\sqrt{3}}\right)}{\sqrt{3}}\right)$$

and since $\sum_{n\geq 0} a_{n+1}x^n = \frac{f(x)}{x}$ is 0 when $x = 0$ we have $C = 3$ and

$$f(1) = \frac{3}{4}\exp\left(\frac{\pi\sqrt{3}}{18}\right)$$

and using WolframAlpha our required 5-digit number is $\boxed{64744}$. $\square$

§11.1.6 Harmonic Sum

Source: Original
Proposer: tanoshii#3160 (300065144333926400)and iparis#5018 (269634843468496906)
Problem ID: 152
Date: 2021-03-06
Difficulty: Challenging

Let S be the set of positive integers which when written in base 2021^{2021} have no identical consecutive digits. Find the first five digits of

$$\sum_{n \in S} \frac{1}{n}.$$

Solution.

Call an integer *distinctive* if it is in S. Let $S(d,n)$ be the set of distinctive positive integers which when written in base $b = 2021^{2021}$ are n digits long and have leading digit d and let $T(d,n)$ be the sum of the reciprocals of the integers in S. We will attempt to bound $T(d,n)$ as follows, since if the leading digit of an integer is d then the second digit cannot be d:

$$T(d,n) = \sum_{k \in S(d,n)} \frac{1}{k} = \sum_{k=1, k \neq d}^{b-1} T(d, n-1) = \sum_{k=0, k \neq d}^{b-1} \sum_{h \in S(k,n-1)} \frac{1}{h}$$

and note that h takes the form of $d \cdot b^{n-1} + k \cdot b^{n-2} + O\left(b^{n-3}\right)$. An upper bound would be to let $O\left(b^{n-3}\right) = 0$ and count $T(d, n-1)$; since there are $(b-1)^{n-2}$ $n-1$-digit distinctive numbers with a fixed leading digit this gives the bound

$$T(d,n) < \sum_{k=0}^{b-1} \frac{(b-1)^{n-2}}{d \cdot b^{n-1} + k \cdot b^{n-2}} = \sum_{k=0}^{b-1} b^{2-n} \frac{(b-1)^{n-2}}{db+k} = \frac{(b-1)^{n-2}}{b^{n-2}} \left(H_{db+b-1} - H_{db}\right);$$

letting $T(n)$ be the sum of $T(d,n)$ over all d we have

$$T(n) < \frac{(b-1)^{n-2}}{b^{n-2}} \left(H_{b^2} - H_b\right)$$

or

$$\sum_{n \in S} \frac{1}{n} < \sum_{n \geq 1} \frac{(b-1)^{n-2}}{b^{n-2}} \left(H_{b^2} - H_b\right) \approx \sum_{n \geq 1} \left(\frac{b-1}{b}\right)^{n-2} \ln b = \frac{b^2 \ln b}{b-1}$$

whose first five digits are 54518.

By AM-HM inequality, if two distinct positive integers a, b have average m then $\frac{1}{a} + \frac{1}{b} > \frac{2}{m}$; noting that $n-1$-digit distinctive numbers that start with d come in pairs where the last $n-2$ digits of each pair add up to $b^{n-1} - 1$ (try subtracting any four-digit base-10 distinctive integer from 9999 and see what I mean) we can bound

$$T(d,n) > \sum_{k=0}^{b-1} \frac{2(b-1)^{n-2}}{2d \cdot b^{n-1} + (2k+1)b^{n-1} - 1} - \frac{2(b-1)^{n-2}}{2d \cdot b^{n-1} + d \cdot b^{n-2}}$$

and since b is so large we take ignore the -1 term and get

$$T(d,n) > \sum_{k=0}^{b-1} \frac{2(b-1)^{n-2}}{b^{n-1}} \cdot \frac{1}{2bd + 2k + 1} - \frac{2(b-1)^{n-2}}{b^{n-2}} \cdot \frac{1}{2bd+1}.$$

Now summing over d, the first harmonic sum turns into

$$\frac{1}{2b+1} + \frac{1}{2b+3} + \cdot + \frac{1}{2b^2+1} = \log\left(2b^2+1\right) - \log(2b) - \frac{1}{2}\left[\log\left(b^2\right) - \log(b+1)\right]$$

and the second harmonic sum is

$$\frac{1}{2b+1} + \frac{1}{4b+1} + \cdots + \frac{1}{2b^2-b+1} < \frac{1}{2b}\log(b-1) = O\left(\frac{1}{b}\right)$$

which we can safely ignore (feel free to plug it in if you want; it's just extra terms). This is a bound sharp enough to also give $\boxed{54518}$ as its first five digits, and we are done. □

§11.1.7 Lonely Tetrahedron

Source: Original
Proposer: tanoshii#3160 (300065144333926400)and iparis#5018 (269634843468496906)
Problem ID: 153
Date: 2021-03-07
Difficulty: Challenging

For a point P on the surface of a regular tetrahedron with side length 1, define the *isolation* of P to be the smallest d such that an ant crawling on the surface of the tetrahedron, starting at P, can reach any other point on the tetrahedron by traveling at most d units. Find the total length of the locus of points whose isolation is $\frac{21}{20}$.

Solution.

□

§11.2 Week 2

§11.2.1 Don't over think this part 2

Source: Original
Proposer: tanoshii#3160 (300065144333926400)and iparis#5018 (269634843468496906)
Problem ID: 154
Date: 2021-03-08
Difficulty: Beginner

Find the number of real solutions to $\left\{x^{21}\right\} = \left\{x^{20}\right\}$, where $20 \le x \le 21$.

Solution.

If the fractional parts of x^{20} and x^{21} are equal, then $x^{21} - x^{20}$ must be an integer. Since $p(x) = x^{21} - x^{20}$ is continuous and strictly increasing when $20 \le x \le 21$, it remains to find the number of integers between $p(21)$ and $p(20)$ inclusive, which is $p(21) - p(20) + 1 = \boxed{21^{21} - 21^{20} - 20^{21} + 20^{20} + 1}$. $\square$

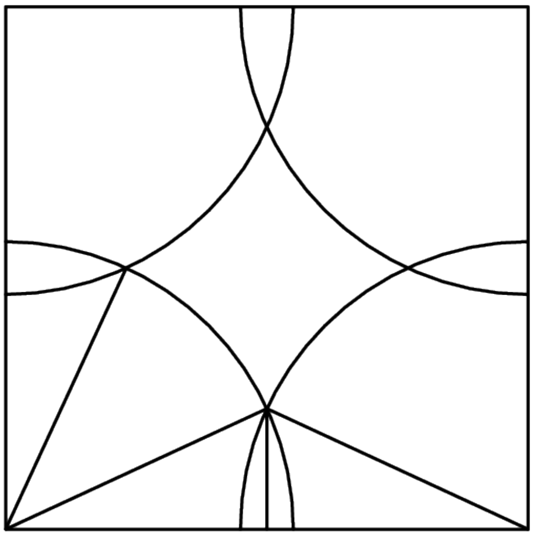
§11.2.2 Lattice Points on a Circle

Source: Original
Proposer: tanoshii#3160 (300065144333926400)and iparis#5018 (269634843468496906)
Problem ID: 155
Date: 2021-03-09
Difficulty: Beginner

A circle with area $\frac{20}{21}$ is placed randomly in the plane. What is the probability that the circle will contain a lattice point? (Formally, if p_M is the probability that a circle with center chosen randomly in $[-M, M]^2$ contains a lattice point, find $\lim_{M\to\infty} p_M$.)

Solution.

By symmetry, it suffices to consider the probability when the circle's center is chosen randomly inside a unit square. Draw four quartercircular arcs, one at each vertex; such a circle's center must be contained in the union of those arcs in order for it to contain a lattice point (otherwise a lattice point would more more than r away from the circle's center). Shown below is the diagram.



Since the arcs overlap, we split the area into four isosceles triangles and four arcs. The base angle of the triangle is $\arccos\left(\frac{1}{2r}\right)$ and the height of the triangle is $\sqrt{r^2 - \frac{1}{4}}$ so the angle of the arc is $\frac{\pi}{2} - 2\arccos\left(\frac{1}{2r}\right)$. Multiplying by four and plugging in $r = \sqrt{\frac{20}{21\pi}}$ we get

$$P = \boxed{2\sqrt{\frac{20}{21\pi} - \frac{1}{4}} + \frac{20}{21}\left(1 - \frac{4\arccos\left(\frac{\sqrt{21\pi}}{2\sqrt{20}}\right)}{\pi}\right)}.$$

□

§11.2.3 WA is Required

Source: Original
Proposer: tanoshii#3160 (300065144333926400)and iparis#5018 (269634843468496906)
Problem ID: 156
Date: 2021-03-10
Difficulty: Easy

Find the number of digits in the smallest positive integer x such that there exists a positive integer y such that

$$2020x^{20^{21}} = 2021y^{21^{20}}.$$

Solution.

Since $2020 = 2^2 \cdot 5 \cdot 101$ and $2021 = 43 \cdot 47$ we split x, y into their prime factors; for x to be minimal it should only have the necessary prime factors $2, 5, 43, 47, 101$. Let $x = 2^{x_1} \cdot 5^{x_2} \cdot 43^{x_3} \cdot 47^{x_4} \cdot 101^{x_5}$ and $y == 2^{y_1} \cdot 5^{y_2} \cdot 43^{y_3} \cdot 47^{y_4} \cdot 101^{y_5}$; we now can solve for the exponent of each prime independently. Looking at the first equation we get $2 + 20^{21}x_1 = 21^{20}y_1$, or the equivalent $2 + 20^{21}x_1 \equiv 0 \pmod{21^{20}}$. Let $I(a)$ be the modular inverse of a mod 21^{20}; then we have $x_1 \equiv -2I\left(20^{21}\right) \pmod{21^{20}}$. Solving similar equations for x_2 through x_5 we get $x_2 \equiv -I\left(20^{21}\right) \pmod{21^{20}}, x_3 = x_4 = I\left(20^{21}\right) \pmod{21^{20}}$. Putting this together, the number of digits is

$$\left\lfloor \left[21^{20} - 2I\left(20^{21}\right)\right] \log_{10}(2) + \left[21^{20} - I\left(20^{21}\right)\right] \log_{10}(505) + I\left(20^{21}\right) \log_{10}(2021) \right\rfloor + 1$$

$$= \boxed{835858264863808864064926340}.$$

□

§11.2.4 Hard

Source: Original
Proposer: tanoshii#3160 (300065144333926400)and iparis#5018 (269634843468496906)
Problem ID: 157
Date: 2021-03-11
Difficulty: Hard

Let n be a power of 10 greater than 10^{100}. A regular n-gon is inscribed in a unit circle, and a subset S is chosen uniformly at random from the 2^n subsets of the polygon's vertices. If A is the expected area of the convex hull of S, find $\pi - A$.

Solution.

By linearity of expectation, finding $\pi - A$ is the same as finding the expected area outside the convex hull inside the circle. So, we find the expected area of a circular segment (the little cap thingy formed when you subtract the triangle off from the circular arc). Label the points 1 through n counterclockwise. Consider a circular segment that reaches from 1 to k. The probability of getting such a circular segment from your point selection is 2^{-k}, since the endpoints of the segment (vertices 1 and k) need to be included while the $k-2$ middle segments must be excluded. The area of such a circular segment is $\frac{(k-1)\pi}{n} - \frac{1}{2}\sin\left(\frac{2\pi(k-1)}{n}\right)$ (yes, this covers segments with an obtuse angle as well - why?). Summing, we see that k can range from 2 to n; however we need to correct for the edge cases where we select one or zero points, which contributes an expected area of $\frac{(n+1)\pi}{2^n}$ (the whole circle). Since there are n of each of the circular segments, our sum is

$$n\sum_{k=2}^{n} 2^{-k}\left[\frac{(k-1)\pi}{n} - \frac{1}{2}\sin\left(\frac{2\pi(k-1)}{n}\right)\right] + \frac{(n+1)\pi}{2^n} = \frac{n}{2}\sum_{k=1}^{n-1} 2^{-k}\left[\frac{k\pi}{n} - \frac{1}{2}\sin\left(\frac{2k\pi}{n}\right)\right] + \frac{(n+1)\pi}{2^n}$$

and since n is huge we can approximate $n-1$ with ∞ and ignore the edge case (since 2^n grows way faster than n). The first sum of $k2^{-k}$ is an arithno-geometric series and evaluates to 2; the second sum is the imaginary part of the sum of complex numbers

$$\sum_{k=1}^{\infty}\left(\frac{z}{2}\right)^k = \frac{z}{2}\cdot\frac{1}{1-\frac{z}{2}} = \frac{z}{2-z}$$

where $z = \cos(a) + i\sin(a)$ and $a = \frac{2\pi k}{n}$. Simplifying further, we have

$$\frac{z}{2-z} = \frac{\cos(a)+i\sin(a)}{2-\cos(a)-i\sin(a)} = \frac{2\cos(a)+2i\sin(a)}{5-4\cos(a)}$$

and scaling by a factor of 2 we have

$$\sum_{k=1}^{\infty} 2^{-k}\frac{1}{2}\sin\left(\frac{2k\pi}{n}\right) = \frac{\sin(a)}{5-4\cos(a)}$$

so putting it all together the expected value of $\pi - A$ is

$$\pi - \frac{n\sin\left(\frac{2\pi}{n}\right)}{10-8\cos\left(\frac{2\pi}{n}\right)}.$$

Since n is huge, we use the first two terms of the Taylor series to approximate our trigonometric functions; doing this we get $\boxed{\frac{26\pi^3}{3n^2}}$. □

Solution. [Write-up by Boldizsár#3106 (694967174921715853)]

n is huge. Without loss of generality, one can assume that S has at least 3 elements. We try to compute the expected area of one slice, i. e. the part of a circular slice outside the polygon. Because of linearity of expectance, n times this is $\pi - A$. Assume that, on the left of this circular sector, the k-th vertex is the first one in S and on the right, the l-th one. So the minimum case is when $k = l = 0$ when two neighboring vertices are both in S. The probability of one choice of k, l happening is $\frac{1}{2^{k+1}2^{l+1}}$ The whole circular sector our slice lies in consists of $k + l + 1$ slices, so it has the area $\frac{\pi(k+l+1)}{n}$, the corresponding triangle has area $\frac{\sin(\frac{2\pi(k+l+1)}{n})}{2}$. As n is huge, we can sum to infinity without problems and we are interested in

$$n\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{\frac{\pi(k+l+1)}{n}-\frac{\sin(\frac{2\pi(k+l+1)}{n})}{2}}{2^{k+1}2^{l+1}(k+l+1)}.$$

Now, again, n is huge so we can use the "fundamental theorem of engineering refined" without any concerns, i. e. that $\sin(x) = x - \frac{x^3}{6}$, so we have:

$$\frac{n}{24}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{\left(\frac{2\pi(k+l+1)}{n}\right)^3}{2^{k+l+1}(k+l+1)} = \frac{\pi^3}{3n^2}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{(k+l+1)^2}{2^{k+l+1}} =$$

$$= \frac{\pi^3}{3n^2}\sum_{m=0}^{\infty}\sum_{k=0}^{m-1}\frac{m^2}{2^m} = \frac{\pi^3}{3n^2}\sum_{m=0}^{\infty}\frac{m^3}{2^m}.$$

So we ask WolframAlpha and we are surprised that there comes a nice answer, namely 26, a random integer. Where the 26 comes from is left to the reader. Finally, we are left with $\frac{26\pi^3}{3n^2}$. As we all know, the PSC consists of nice people, so they set n and hence n^2 to a power of 10, so we now understand why they asked for the 2021st to 2030th significant figures, namely because this is just the 2021th to 2030th digits of $\frac{26\pi^3}{3}$, which is, according to WolframAlpha, $\boxed{8481609952}$. □

§12 Return to Normalcy (Season 12)

§12.1 Week 1

§12.1.1 Quadrilateral Perimeter

Source: Folklore
Proposer: flame#6784 (185229437787176962)
Problem ID: 161
Date: 2021-03-29
Difficulty: Beginner

Let $ABCD$ be a convex quadrilateral satisfying

$$AB = 1, BC = 6, CD = 18, AC \perp BD.$$

Find its perimeter.

Solution.

Let E be the intersection of the diagonals of $ABCD$. Then $AB^2 + CD^2 = AE^2 + EB^2 + CE^2 + ED^2$ and $BC^2 + DA^2 = BE^2 + EC^2 + DE^2 + EA^2$ by the Pythagorean Theorem. This implies that $AB^2 + CD^2 = BC^2 + DA^2$, or $1^2 + 18^2 = 6^2 + x^2$. Solving yields $x = 17$, so the answer is $1 + 18 + 6 + 17 = \boxed{42}$. □

§12.1.2 Involution Convolution

Source: Australian Senior Mathematical Challenge, 2019 Q30
Proposer: brainysmurfs#2860 (281300961312374785)
Problem ID: 162
Date: 2021-03-30
Difficulty: Easy

A function f, defined on the set of positive integers, has $f(1) = 2$ and $f(2) = 3$. Also $f(f(f(n))) = n+2$ if n is even and $f(f(f(n))) = n + 4$ if n is odd. What is $f(777)$?

Solution. Observe that $777 = 4 \times 194 + 1$. Thus

$$\begin{aligned} f(777) &= f(f^{194\times 3}(1)) \\ &= f^{194\times 3}(f(1)) \\ &= f^{194\times 3}(f(2)) \\ &= 2 + 194 \times 2 \\ &= \boxed{390} \end{aligned}$$

□

§12.1.3 Coordinates without Bash

Source: Original Problem
Proposer: Denial#4301 (118831126239248397)
Problem ID: 163
Date: 2021-03-31
Difficulty: Medium

A circle of radius 1 is centred at $(1, 1)$. Points A and B have co-ordinates $(a, 0)$ and $(0, b)$, with $a, b > 2$. If AB is tangent to the circle, what is the sum of all possible integer values of a and b?

Solution. [Solution by Denial#4301 (118831126239248397)]

Let O, C, D and E have coordinates $(1, 1), (0, 0), (0, 1)$ and $(1, 0)$ respectively, and let the point of tangency of AB with the circle be F.
Note since O and D have the same y co-ordinate OD is parallel to the x-axis and hence perpendicular to the y-axis. Since D is also on the y-axis, BC is tangent to the circle at D. Similarly, O and E have the same x co-ordinate, so OE is parallel to the y-axis and hence perpendicular to the x-axis. Since E is also on the x-axis, AC is tangent to the circle at E.
Since the tangents to a circle from a point have equal length, $BF = BD$ and $FA = AE$. Summing, $BF + FA = BD + AE$, so $BA = (BC - CD) + (AC - CE) = (b - 1) + (a - 1) = a + b - 2$, giving $AB^2 = (a + b - 2)^2$. However, since $\angle ACB = 90°$, by Pythagoras' theorem, $AB^2 = a^2 + b^2$, and so we have

$$a^2 + b^2 = (a + b - 2)^2$$

$$a^2 + b^2 = a^2 + b^2 + 4 + 2ab - 4a - 4b$$

$$0 = 2ab - 4a - 4b + 4$$

$$ab - 2a - 2b + 2 = 0$$

$$ab - 2a - 2b + 4 = 2$$

$$(a - 2)(b - 2) = 2$$

However, $a, b > 2$, so if a and b are integers, then $a - 2$ and $b - 2$ are positive integers and hence must both be positive factors of 2. Since 2 is primes, its only factorisation is as $2 = 2 \times 1$, so either $a - 2 = 2$ and $b - 2 = 1$ or $a - 2 = 1$ and $b - 2 = 2$. Hence the possible integer values of a and b are $(a, b) = (3, 4)$ or $(4, 3)$. Giving a final answer of $\boxed{14}$ □

§12.1.4 April Fool's

Source: brain
Proposer: brainysmurfs#2860 (281300961312374785)
Problem ID: 164
Date: 2021-04-01
Difficulty: Random

How many official solves will this Sunday's problem get?

Solution.

Answer: $\boxed{21}$ Five people managed to guess 21. □

§12.1.5 FE

Source: 2011 Kosovo TST #5
Proposer: AiYa#2278 (675537018868072458)
Problem ID: 165
Date: 2021-04-02
Difficulty: Hard

There exist functions $f : \mathbb{R} \setminus \{1, -1\} \to \mathbb{R}$ such that the following holds:

$$f\left(\frac{x-3}{1+x}\right) + f\left(\frac{x+3}{1-x}\right) = x$$

Then the average value of

$$\sum_{k=2}^{100} \frac{f(k)}{k}$$

can be expressed in the form $\frac{m}{n}$, where m, n are relatively prime integers and n is positive. Find $100m + n$.

(For example, if the solutions to that FE were $f(x) \equiv 0$ and $f(x) \equiv -x^2$ then the average value of that sum would be $-\frac{5049}{2}$ and your answer would be $2 - 504900 = -504898$.)

(A four-function calculator may be used.)

Solution.

The unique function can be found using AoPS solutions; then note that $\frac{f(x)}{x}$ can be decomposed into partial fractions and so the sum telescopes. Answer: $\boxed{\text{-13243675}}$ □

§12.1.6 So this is where American problem quality went

Source: 2012 AIME II #15
Proposer: AiYa#2278 (675537018868072458)
Problem ID: 166
Date: 2021-04-03
Difficulty: Challenging

Triangle PQR is inscribed in circle Γ with $PQ = 5$, $QR = 7$, and $RP = 3$. The bisector of angle P meets side $\overline{QR}$ at X and circle Γ at a second point Y. Let γ be the circle with diameter $\overline{XY}$. Circles Γ and γ meet at Y and a second point Z. Then $PZ^2 = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $100m + n$.

Solution.

AoPS Solutions Answer: $\boxed{90019}$ □

§12.1.7 640 Generators of Pain

Source: 2015 PUMAC Number Theory A7
Proposer: AiYa#2278 (675537018868072458)
Problem ID: 167
Date: 2021-04-04
Difficulty: Challenging

For positive integers n, let $s(n)$ be the sum of the n-th powers of the primitive roots mod 1601. Find the number of positive integers $n \le 2021$ such that $s(n)$ is divisible by 1601.

Solution.

First, some lemmas about orders and generators (skip if you think people should know this already):

Lemma 1. The order of g^n, where g is a generator mod prime p, is $d = \frac{p-1}{(n,p-1)}$ where (a,b) is the greatest common divisor function.
Proof. Since $g^{nd} \equiv 1 \pmod p$ we have $p-1|nd$ so $\frac{p-1}{(n,p-1)}|d$. Since d is defined to be minimal we have $d = \frac{p-1}{(n,p-1)}$.

Lemma 2. The number of residues mod p with order d is $\varphi(d)$, where φ is Euler's totient function.
Proof. We want to find the number of positive integers k such that d is the least positive integer for which $\left(g^{k\frac{p-1}{d}}\right)^d \equiv 1 \pmod p$. For that to be satisfied, k and d cannot share any common factors; otherwise $\left(g^{k\frac{p-1}{d}}\right)^{\frac{d}{(k,d)}} \equiv 1 \pmod p$. Also, since d is the order only $k < d$ generate unique residues. The number of positive integers k less than d such that $(d,k) = 1$ is $\varphi(d)$.

Let $p = 1601$ since 1601 is prime. The order of g^n mod p is $d = \frac{p-1}{(n,p-1)}$; also each of the $\varphi(p-1)$ terms in $s(n)$ can take only one of $\varphi\left(\frac{p-1}{(n,p-1)}\right)$ residues with order d mod p. By symmetry, each residue of order d is represented the same amount of times in $s(n)$. Now let $r(d)$ be the sum of the residues with order d mod p, so

$$s(n) = \frac{\varphi(p-1)}{\varphi\left(\frac{p-1}{(n,p-1)}\right)} r(d).$$

Notice that the ratio of totient functions is less than p, so if $p|s(n)$ then $p|r(d)$.
Consider a residue a with order d; then $p|a^d - 1 = (a-1)\left(a^{d-1} + a^{d-2} + \cdots + 1\right)$ and since $a < p$ we have $a^{d-1} + a^{d-2} + \cdots + 1 \equiv 0 \pmod p$. Remembering $\sum_{d|n} \varphi(d) = n$ we conjecture that that sum is $\sum_{e|d} r(e) \equiv 0 \pmod p$; indeed we can treat a like the ""generator"" of the residue class of order d mod p and follow the steps of Lemma 2.

Claim. $r(d) \equiv 0 \pmod p$ if and only if d is not squarefree.

As with any divisor sum, we try products of primes: $r(1) = 1$, $r(p) = -1, r(pq) + r(p) + r(q) + 1 \equiv 0 \iff r(pq) = 1$ and induct on k to get $r(p_1 p_2 \cdots p_k) = (-1)^k$. Notice $r\left(p^2\right) + r(p) + r(1) \equiv 0 \iff r\left(p^2\right) \equiv 0$; a similar induction argument yields $r(d) \equiv 0$ if d is not squarefree.
We are homeward-bound: for $d = \frac{p-1}{(n,p-1)} = \frac{1600}{(n,1600)}$ to be squarefree we must have $2^5 \cdot 5 = 160|n$. We have $2021 - \left\lfloor \frac{2021}{160} \right\rfloor = \boxed{2009}$ such n. □

Solution. [Solution by Boldizsár#3106 (694967174921715853)]

n-th powers of roots is something that polynomials can handle very well. The definition of order, $g^{p-1} \equiv 1 \pmod p \iff g^{p-1}-1 \equiv 0 \pmod p$, motivates us to look at the polynomial $x^{p-1}-1$. Which integers g satisfy $g^{p-1}-1 \equiv 0 \pmod p$ but not $g^d - 1 \equiv 0 \pmod p$ for all $d < p$? We claim that the polynomial in question is $\Phi_{p-1}(x)$, the $p-1$-th cyclotomic polynomial. (If you don't know what a cyclotomic polynomial is, google.) While it is easy to compute $\Phi_p(x)$, computing $\Phi_{p-1}(x) = \Phi_{1600}(x)$ will be harder. Using $\Phi_n(x) = \prod_{d|n} \Phi_d(x)$ we can get

$$x^{p^k} - 1 = \left(x^{p^{k-1}} - 1\right) \Phi_{p^k}(x) = \left[\left(x^{p^{k-1}}\right)^p - 1\right] \iff \Phi_{p^k}(x) = \sum_{h=0}^{p-1} x^{hp^{k-1}}.$$

We now turn our attention to $\Phi_{p^k n}(x)$ for $(p,n)=1$. Note that $\Phi_n\left(x^{p^k}\right)$ has roots of unity $\exp\left(\frac{\tau m}{p^k n}\right)$ where $\left(m, p^k n\right) = 1, p, \cdots, p^k$. However, we want only $\left(m, p^k n\right) = 1$; note then, that $\Phi_n\left(x^{p^{k-1}}\right)$ has roots of unity $\exp\left(\frac{\tau m}{p^{k-1} n}\right) = \exp\left(\frac{\tau p m}{p^k n}\right)$ where $\left(m, p^{k-1} n\right) = 1, p, \cdots, p^{k-1}$ so $\left(pm, p^k n\right) = p, p^2, \cdots, p^k$ and so

$$\Phi_{p^k n}(x) = \frac{\Phi_n\left(x^{p^k}\right)}{\Phi_n\left(x^{p^{k-1}}\right)}.$$

Thus, we have $\Phi_{25}(x) = 1 + x^5 + x^{10} + x^{15} + x^{20} = \frac{x^{25}-1}{x^5-1}$ and so

$$\Phi_{1600}(x) = \Phi_{2^6 \cdot 25}(x) = \frac{\Phi_{25}\left(x^{64}\right)}{\Phi_{25}\left(x^{32}\right)} = \frac{x^{800}+1}{x^{160}+1} = 1 - x^{160} + x^{320} - x^{480} + x^{640}$$

where we have omitted two difference of squares factorizations since $2^6 = 2^5 \cdot 2$ and so $x^{2^6} = \left(x^{2^5}\right)^2$.

Newton's Sums. Let $r_1, r_2, \cdots, r_n$ be the roots of polynomial $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0 x^0$, and let $s_k = \sum_{i=1}^n r_i^k$. Then $\sum_{i=0}^n a_{n-i} s_{k-i} = 0$ if $k \geq n$ and $(-1)^k k a_{n-k} + \sum_{i=0}^{k-1} (-1)^i a_{n-i} s_{k-i} = 0$ if $k < n$.

Using Newton's sums, it becomes clear that the only nonzero terms of $s(n)$ happen when $160|n$; our answer is thus $\boxed{2009}$. $\square$

Solution. [Solution by Hopeless#9949 (399043449443057704)]

Let g be a known generator mod p. Note that if $(k, p-1) = 1$ then g^k is also a generator mod p. We have $p-1 = 1600 = 2^6 \cdot 5^2$; the integers not coprime to $p-1$ are either multiples of 2, multiples of 5, or multiples of 10. Now observe that

$$\sum_{a=1}^{1600} g^a$$

sums over all powers of g; however since we only want coprime powers we need to subtract off powers that are multiples of 2 or 5. Using inclusion-exclusion, this is

$$s(n) = \sum_{a=1}^{1600} g^{an} - \sum_{b=1}^{800} g^{bn} - \sum_{c=1}^{320} g^{cn} + \sum_{d=1}^{160} g^{dn}$$

which can be rewritten and re-summed as

$$\frac{g^{1601n}-1}{g^n-1} - \frac{g^{1602n}-1}{g^{2n}-1} - \frac{g^{1605n}-1}{g^{5n}-1} + \frac{g^{1610n}-1}{g^{10n}-1} \equiv 0 \pmod{p}$$

with a small catch: if any of $g^n, g^{2n}, g^{5n}, g^{10n}$ is 1 the geometric series sum represntation fails, and it is easy to see that $s(n)$ will not be 0 mod p in that case. Thus, the cases where $s(n) \not\equiv 0 \pmod{p}$ is when $160|n$; our answer is then $\boxed{2009}$. $\square$

§12.2 Week 2

§12.2.1 Even Numbers with Restrictions

Source: AMC10, 2011 Q13
Proposer: sjbs#9839 (434767660182405131)
Problem ID: 168
Date: 2021-04-05

How many even integers are there between 200 and 700 whose digits are all different and come from the set $\{1, 2, 5, 7, 8, 9\}$

Solution.

AoPS Solutions Answer: $\boxed{12}$ □

§12.2.2 Enjoyable

Source: Canadian Open Mathematics Challenge 2018
Proposer: TaesPadhihary#8557 (665057968194060291)
Problem ID: 169
Date: 2021-04-06
Difficulty: Beginner

Five *distinct* integers e, n, j, o, y are such that

$$e \cdot n \cdot j \cdot o \cdot y = 775.$$

Find $e + n + j + o + y$.

Solution.

Note that $775 = 5^2 \cdot 31$. In particular, this means that the only way to factor into 5 distinct integers is to have $\{e, n, j, o, y\} = \{1, -1, 5, -5, 31\}$, giving us a sum of $1 - 1 + 5 - 5 + 31 = \boxed{31}$. □

§12.2.3 Isosceles Tetrahedron

Source: IMTS Year 1991 R4 Problem 5 (modified)
Proposer: flame#6784 (185229437787176962)
Problem ID: 170
Date: 2021-04-07
Difficulty: Easy

Paper $\triangle XYZ$ has $XY = 69$, $YZ = 91$, $ZX = 100$. Let A, B, C be the midpoints of $\overline{XY}$, $\overline{YZ}$, $\overline{ZX}$ respectively. The triangle is folded about AB, BC, and CA to form a tetrahedron $\mathcal{T}$. What is the volume of $\mathcal{T}$?

Solution.

AoPS Solution using Pythagorean Theorem

We will prove that in an isosceles tetrahedron (opposite edges have the same length) with edge lengths a, b, c, the volume of such a tetrahedron is

$$V = \sqrt{\frac{\left(a^2+b^2-c^2\right)\left(a^2-b^2+c^2\right)\left(-a^2+b^2+c^2\right)}{72}}.$$

First, note that every "acute" tetrahedron can be inscribed in a parallepiped prism, with each edge being a diagonal of each face (see here for a picture). This is because there is exactly one pair of parallel planes that pass through each pair of opposite edges (since opposite edges are skew lines, so you can't just rotate the planes wrt the axis of one edge). Also note that since opposite edges are skew, this means that the opposite edges take up the two different diagonals of their inscribed parallelogram face. Since opposite edges are equal in an isosceles tetrahedron, this implies that the parallelogram has equal diagonals, implying that it's a rectangle, and the circumscribed parallepiped is actually a rectangular prism. Letting x, y, z represent the edges of the prism, we have

$$x^2+y^2=a^2, y^2+z^2=b^2, z^2+x^2=c^2 \iff \left(x^2,y^2,z^2\right)=\left(\frac{-a^2+b^2+c^2}{2},\frac{a^2-b^2+c^2}{2},\frac{a^2+b^2-c^2}{2}\right)$$

and now we claim that the volume of the prism is three times the volume of the tetrahedron; indeed the space outside the inscribed tetrahedron but inside the prism is composed of three congruent right-angle-cornered tetrahedra, each with volume $\frac{xyz}{6}$. Thus

$$V = \frac{xyz}{3} = \sqrt{\frac{\left(a^2+b^2-c^2\right)\left(a^2-b^2+c^2\right)\left(-a^2+b^2+c^2\right)}{72}}.$$

Answer: $\boxed{7605}$ □

§12.2.4 Sums of powers of 2 but weird

Source: 2018 PUMAC Algebra A5
Proposer: Angry Any#4319 (580933385090891797)
Problem ID: 171
Date: 2021-04-08
Difficulty: Medium

For $k \in \{0, 1, \ldots, 9\}$, let $\varepsilon_k \in \{-1, 1\}$. Find the minimum possible value of

$$\sum_{i=1}^{9} \sum_{j=0}^{i-1} \varepsilon_i \varepsilon_j 2^{i+j}.$$

Solution.

Solution PDF Answer: $\boxed{174762}$ $\square$

§12.2.5 "D4 I swear"

Source: 2012 Winter OMO 43
Proposer: AiYa#2278 (675537018868072458)
Problem ID: 172
Date: 2021-04-09
Difficulty: Hard

Let $p = 2017$. An integer n is uniformly and randomly selected between 1 and $(p^p - 1)!$ inclusive. The probability that $n^n - 1$ is divisible by p can be written in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $100m + n$.

Solution.

Fix an order d; we find the probability that $d|n$ and n has order d mod p. Since the first probability involves n's behavior mod d and the second n's behavior mod p and $\gcd(d, p-1) = 1$, these events are independent. Since our interval is from 1 to $(p^p - 1)!$, each residue mod p and mod $p-1$ (and thus mod d) appears the same amount of times. Thus, our first event happens with probability $\frac{1}{d}$ and our second with probability $\frac{\varphi(d)}{p}$, where we have used the fact that there are exactly $\varphi(d)$ residues with order d mod p. It remains to sum this probability over all possible values of d, or,

$$P = \frac{1}{p} \sum_{d|p-1} \frac{\varphi(d)}{d} = \frac{1}{p} s(p-1),$$

where $s(n) = \sum_{d|n} \frac{\varphi(d)}{d}$. As with all divisor sums, we test with $n = p, p^2, pq$: $s(p) = 1 + \frac{p-1}{p} = 2 - \frac{1}{p}$, $s\left(p^2\right) = s(p) + \frac{p(p-1)}{p^2} = 3 - \frac{2}{p}$, $s(pq) = 1 + \frac{p-1}{p} + \frac{q-1}{q} + \frac{(p-1)(q-1)}{pq} = \left(2 - \frac{1}{p}\right)\left(2 - \frac{1}{q}\right)$; we conjecture that $s(n)$ is multiplicative. Indeed, letting $n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$, two induction arguments, one on k and one on e, yields

$$s(n) = \prod_{i=1}^{k} e_i + 1 - \frac{e_i}{p_i}$$

so $s(p-1) = s(2016) = \left(6 - \frac{5}{2}\right)\left(3 - \frac{2}{3}\right)\left(2 - \frac{1}{7}\right) = \frac{91}{6}$ so $P = \frac{91}{6p} \iff \boxed{21202}$. $\square$

§12.2.6 "change the point distribution up so weekends don't carry people"

Source: Unknown
Proposer: AiYa#2278 (675537018868072458)
Problem ID: 173
Date: 2021-04-10
Difficulty: Challenging

2021 coplanar lines intersect at $2021 \cdot 1010$ distinct points $K_1, K_2, \cdots K_{2021 \cdot 1010}$. Let P be a point in the plane not on any of the lines. We call an intersection point K *k-reachable* if segment PK intersects at most k lines, excluding the intersection point at K. If 1999420 points are k-reachable, find the largest possible value of k.

Solution.

First note that no two of these 2021 lines are parallel and no three of them are concurrent, otherwise the number of intersection points would be less than $2021 \cdot 1010$. After some playing around with small cases of n, we claim that the minimal number of k-reachable points among n such lines is $\binom{k+2}{2}$. The base case $k = 0$ for arbitrary n is trivial. Assume the condition holds for $k-1$ and $n-1$. Consider the line with minimal distance to P. Label its intersection points $P_1, P_2, \cdots, P_n$ where $KP_1 < KP_2 < \cdots < KP_n$. Among these n points, at least k of them are k-reachable (consider the worst-case scenario where P is on the left of all the intersection points). Now remove this line; among the remaining points there are at least $\binom{k+1}{2}$ $(k-1)$-reachable points. When we add the line back, those $(k-1)$-reachable points become k-reachable points (any point Q in that set either had PQ cross that deleted line or not). It remains to bound $\binom{k+2}{2} \approx (k+2)^2/2$. Noticing $1999420 \approx 1999200 = 1400 \cdot 1428 \approx 1414^2$ and $\sqrt{2} = 1.414$, we have $2 \cdot 1414^2 \approx 2000^2 \iff 1999420 > \binom{2000}{2}$. Our answer is $\boxed{1998}$. □

§12.2.7 Notice me, sqing-senpai

Source: 2019 China Mathematical Olympiad P1, Modified
Proposer: AiYa#2278 (675537018868072458)
Problem ID: 174
Date: 202-04-11
Difficulty: Challenging

Let $(a_1, a_2, a_3, a_4, a_5, a_6)$ be reals no less than -2020 that sum to 2021. Let M and m be the maximum and minimum, respectively, of the function

$$f(a_1, a_2, a_3, a_4, a_5, a_6) = \prod_{k=1}^{6} (a_k + a_{k+1}),$$

where indices are taken modulo 6, and let N and n be the number of ordered pairs which maximize and minimize f, respectively. Find $\lfloor \ln(-Mm) \rfloor + N + n$.

(A scientific calculator may be used.)

Solution.

We switch variables to (a, b, c, d, e, f) for sake of legibility. WLOG a is maximal; therefore $a + b, f + a \geq 0$. We solve for the minimum first, as that is less painful. Negative values are possible, which correspond to either one or three negative factors. If there is only one negative factor, then there are at least two positive variables less than 2021; flipping their signs and increasing the value of the greatest two variables accordinaly produces a product greater in absolute value, so we want three negative factors. Let $a + b, b + c, c + a < 0$ (the factor adjacency really does matter as we'll see in a bit). Applying AM-GM twice, once to the positive factors and once to the negative, we have

$$f \leq \left(\frac{-a - 2b - 2c - d}{3}\right)^3 \left(\frac{d + 2e + 2f + a}{3}\right)^3$$

and since $d + 2e + 2f + a = 2 \cdot 2021 - (a + 2b + 2c + d)$ we see that the maximum occurrs when $a + 2b + 2c + d$ is minimized; this happens at $a = b = c = d = -2020$ and $e = f = \frac{10101}{2}$. Note that the variables with coefficient 2 must be different variables; in other words, the three factors we chose to be negative matters. In esssence, the two positive terms have to be next to each other, so $n = 6$.

Now for the maximum. By the same argument, having all positive factors does not yield a maximal configuration; therefore there are either two or four negative factors. Applying AM-GM in the same fashion (once to the positive factors and once to the negative factors) we get that the equality case is one of four and that the maximal configuration happens when all the variables in the negative terms are equal to -2020. Thus, four negative factors is optimal; this happens at $a = b = c = d = e = -2020$ and $f = 12121$ and $N = 6$.

Our answer is

$$\left\lfloor \ln\left(2^5 \cdot 2020^6 \cdot 6061^2 \cdot 10101^3\right) \right\rfloor + 6 + 6 = \boxed{113}.$$

□

§13 Quality Uncontrol (Season 13)

§13.1 Week 1

§13.1.1 Choi's Balls

Source: Folklore
Proposer: sjbs#9839 (434767660182405131)
Problem ID: 175
Date: 2021-04-19
Difficulty: Beginner

Choi has three types of counters, coloured red, blue and green which she must place on a numberline consisting of the numbers 1,2 and 3.

In how many ways can she arrange the counters such that no red counter is directly to the left of a blue counter?

Solution.

There are 3^3 ways of arranging the counters without restrictions and $2 \cdot 3 = 6$ ways to pick the red counter being next to the blue counter, thus we have $3^3 - 6 = \boxed{21}$ total ways.

□

§13.1.2 Biscriminant

Source: Folklore
Proposer: sjbs#9839 (434767660182405131)
Problem ID: 176
Date: 2021-04-20
Difficulty: Beginner

Given that for $f(x) = x^2 + bx + 7$ and $f(x) \neq -18$, the possible values of b can be written in the form $x < b < y$. What is the value of $x^2 + y^2$?

Solution.

We want to find the values of be such that $x^2+bx+7 = -18$ has no real solutions. By taking the discriminant, we see that we require $b^2 - 4 \cdot 25 < 0 \iff -10 < b < 10$. Therefore our answer is $\boxed{200}$.

$\square$

§13.1.3 IMOK Maclaurin2012 P6

Source: 2012 IMOK Maclaurin Problem 6
Proposer: alchemyst_ttk#7268 (531054379852365824)
Problem ID: 177
Date: 2021-04-21
Difficulty: Easy

Three different positive integers have the property that each of them divides the sum of the other two. Let S be a possible set of three such integers. What is the smallest possible integer in S such that the sum of the three integers in S is greater than 2021?

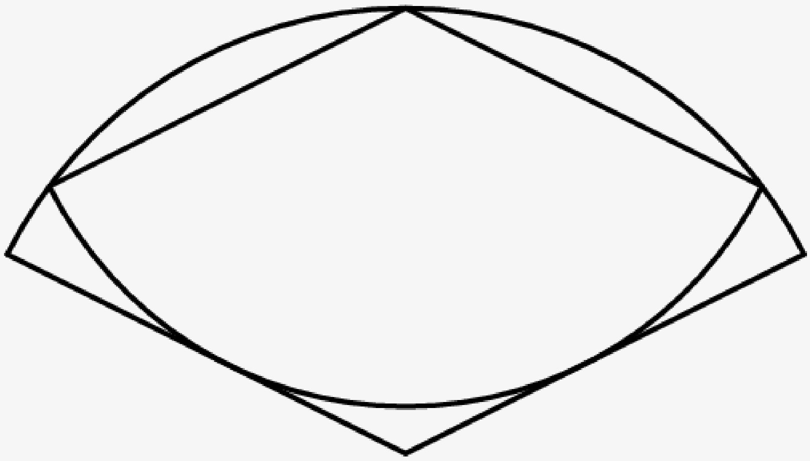
Solution.

Let the three integers be a, b, c where $0 < a < b < c$. This means that $a + b < 2c$ and because $c|a + b \to a + b = c$. We also know that $b|a + c \to b|2a + b \to b|2a$. $b > a$ meaning that $b = 2a$ and this gives us the solutions $(a, 2a, 3a)$. This gives that the sum of the integers in S is $6a$ and the smallest multiple of 6 greater than 2021 is 2022. This gives that a is $\frac{2022}{6} = \boxed{337}$ which is the answer. □

§13.1.4 NICE Geometry

Source: 2021 NICE Problem 10
Proposer: Angry Any#4319 (580933385090891797)
Problem ID: 178
Date: 2021-04-22
Difficulty: Medium

A circular sector of angle $\theta < 180°$ is inscribed inside another sector of the same angle, shown below. If $\cos(\theta) = -\frac{m}{n}$, where m, n are relatively prime positive integers, find $100m + n$.



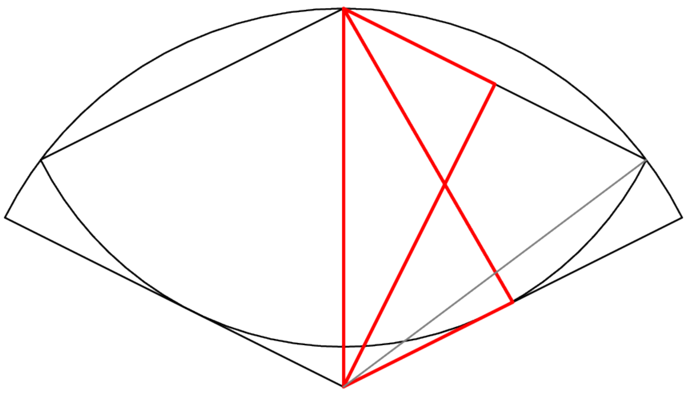
Solution.

Let $r < R$ be the two radii of the sectors. Notice that the two red triangles shown below are both right triangles (one's a tangent, the other's the symmetry axis of an isosceles triangle) and one of their acute angles is $\frac{\theta}{2}$ in both triangles; since they share the same hypotenuse the two are congruent. Then by Pythagorean Theorem we have

$$r^2 + \left(\frac{r}{2}\right)^2 = R^2 = \frac{5r^2}{4} \iff \frac{r}{R} = \frac{2}{\sqrt{5}} = \sin\left(\frac{\theta}{2}\right)$$

so by double angle formula we have

$$\cos(\theta) = 1 - 2\sin^2\left(\frac{\theta}{2}\right) = -\frac{3}{5} \iff \boxed{305}.$$



□

§13.1.5 Graph Theory in Disguise

Source: CMC Mock ARML 2019 P10
Proposer: TaesPadhihary#8557 (665057968194060291)
Problem ID: 179
Date: 2021-04-23
Difficulty: Hard

Tony and Wang play a game in which Tony chooses a list of 6 real numbers and Wang devises 6 questions, each of the form "What is the sum of the x^{th} and y^{th} number in your list?" where $1 \leq x < y \leq 6$. Without regard to order, compute the number of ways Wang can choose his questions so that he can determine Tony's numbers.

Solution. [Solution by TaesPadhihary#8557 (665057968194060291)]

We can consider the the game as a graph with 6 vertices, representing Tony's numbers, and 6 edges,representing Wang's questions. Note that we can't have a cycle with even length since then the corresponding system of equations wouldn't be linearly independent. For example, the system of equations $x + y = c_1$, $y + z = c_2$, $z + w = c_3$, $w + x = c_4$ is not linearly independent, since the sum of the 1st and 3rd equations is equal to the sum of the 2nd and 4th; hence one equation is wasted, yet 6 independent equations are required to find 6 variables. Since the largest tree in a graph with 6 vertices consists of 5 edges, a cycle must exist. We distinguish three cases, depending on the length of the largest cycle:

1. We have a 5-cycle. There are 6 ways to choose which number is left out of the 5-cycle, 12 ways to order the elements of the 5-cycle, and 5 ways to connect the number left out to the cycle, for a total of 360 possibilities.

2. There are two 3-cycles. As the cycles are disjoint, there are $\frac{\binom{6}{3}}{2} = 10$ such possibilities.

3. There is only one 3-cycle. We choose the cycle C in $\binom{6}{3} = 20$ ways, and then choose the paths to connect the other three vertices $(X, Y, Z)$to C, where each of these points must have exactly 1 path. If $X \to C$, $Y \to C$ and $Z \to C$,there are $3^3 = 27$ ways; if$X \to Y \to C$,and$Z \to C$(and permutations), there are $6 \times 3^2 = 54$ ways; if $X \to Z, Y \to Z$and $Z \to C$(and permutations), there are $3 \cdot 3 = 9$ ways; finally, if $X \to Y \to Z \to C$ (and permutations), there are $6 \to 3 = 18$ ways, for a total of $20 \to (27 + 54 + 9 + 18) = 2160$ possibilities.Thus, Wang has a total of $360 + 10 + 2160 = \boxed{2530}$.

□

§13.1.6 What the H-E-double hockey sticks?

Source: old italian contest
Proposer: Matteddy#0482 (329956567132930048)
Problem ID: 180
Date: 2021-04-24
Difficulty: Challenging

There are 2021 people working in a 2021-floor building, one on each floor. On day zero, each worker writes a zero on their sheet of paper. On day $k > 0$, the worker on floor $f > 1$ writes on their sheet of paper the sum of the numbers of the worker on floor $j - 1$ had written in the days up to $k - 1$. The worker on floor 1, meanwhile, writes k^2. If N is the number the worker on floor 2021 writes on day 10000, find the number of digits in $\nu_2(N) + \nu_3(N) + \nu_5(N)$.

(A scientific calculator may be used.)

Solution.

We denote worker f as the worker who works on floor f. It's clear that on day k, worker 2 writes

$$1^2 + 2^2 + \cdots + (k-1)^2 = \frac{k(k-1)(2k-1)}{6}$$

using the well-known sum of squares formula, but it's hard to see where it goes from here. Instead, write k^2 as a sum of binomial coefficients. We know that $2\binom{k}{2} = k(k-1) = k^2 - k$ and $\binom{k}{1} = k$, so we have

$$k^2 = 2\binom{k}{2} + \binom{k}{1} \iff 1^2+2^2+\cdots+(k-1)^2 = 2\left[\binom{2}{2} + \binom{3}{2} + \cdots + \binom{k-1}{2}\right] + \left[\binom{1}{1} + \binom{2}{1} + \cdots + \binom{k-1}{1}\right]$$

which sums to $2\binom{k}{3} + \binom{k}{2}$ after applying the hockey stick identity. Use the hockey stick identity again and again to show that on day k, worker f writes down the number

$$2\binom{k}{f+1} + \binom{k}{f} = \binom{k+1}{f+1} + \binom{k}{f+1} = \binom{k}{f+1} \cdot \frac{2k - f + 1}{k - f}.$$

Substituting $f = 2021, k = 10000$ (and doing either Legendre or Krummer's) yields $\boxed{25}$. □

§13.1.7 "What's an mgf?"

Source: 2013 HMMT C9
Proposer: Angry Any#4319 (580933385090891797)
Problem ID: 181
Date: 2021-04-25
Difficulty: Challenging

Given a permutation π of $\{1, 2, \cdots, 2021\}$, let $p(\pi)$ be the number of fixed points of π. If P is the set of all permutations π, then let

$$N = \sum_{\pi \in P} p(\pi)^4.$$

Find $\nu_2(N) + \nu_3(N) + \nu_5(N)$.

(A scientific calculator may be used.)

Solution.

Contest Solutions Answer: $\boxed{3523}$ □

§13.2 Week 2

§13.2.1 It's Okay :)

Source: Adapted from a FTW problem.
Proposer: ==#5400 (658718130326994954)
Problem ID: 182
Date: 2021-04-26
Difficulty: Beginner

A two-digit positive integer, n, is doubled and then added to 2 to obtain N. Then, the digits of n are reversed to form a new positive integer m. Given that $N = m$, find n.

Solution.

(Sorry, I don't know how to use LaTeX.)
Answer: $\boxed{25}$
It is not too difficult to find the answer through educated guessing and checking, but below is a solution that does not require guessing.
Let x be the tens digit of n and let y be the ones digit of n.
Then the problem can be translated into: $2(10x + y) + 2 = 10y + x$
Simplifying the equation, we get: $19x = 8y - 2$
We can easily deduce that: $19x = 8y - 2 = 38$
Therefore: $x = 2andy = 5$
Finally: $n = 10x + y = 25$ □

§13.2.2 Points on a Line

Source: Online Mathematics Open Spring 2020 Problem 1
Proposer: TaesPadhihary#8557 (665057968194060291)
Problem ID: 183
Date: 2021-04-27
Difficulty: Beginner

Let ℓ be a line and let points A, B, C lie on ℓ so that $AB = 7$ and $BC = 5$. Let m be the line through A perpendicular to ℓ. Let P lie on m. Compute the smallest possible value of $PB + PC$.

Solution.

Clearly, the minimum occurs when $A = P$. If C is between A and B then the distance $PB + PC$ is lower than if C is not between A and B, giving us an answer of $\boxed{9}$. □

§13.2.3 Diamond Operator

Source: PUMaC 2017 A1
Proposer: Angry Any#4319 (580933385090891797)
Problem ID: 184
Date: 2021-04-28
Difficulty: Easy

Let $a \diamond b = ab - 4(a+b) + 20$. Evaluate

$$1 \diamond (2 \diamond (3 \diamond (\cdots (2020 \diamond 2021) \cdots)))$$

Solution.

Official Solution Answer: $\boxed{4}$ □

§13.2.4 Weird Wilson

Source: Baltic Way [Unknown]
Proposer: epicxtroll#6007 (300008472978653184)
Problem ID: 185
Date: 2021-04-29
Difficulty: Medium

Find the smallest prime factor of $712! + 1$.

Solution.

Unfortunately, 713 is not prime; neither is 715 nor 717. 719 is prime, so by Wilson's Theorem we know that $718! \equiv 1 \pmod{719}$. Now we rewrite $718!$ as follows:

$$718! = 718 \cdot 717 \cdot 716 \cdot 715 \cdot 714 \cdot 713 \cdot 712! \equiv (-1)(-2)(-3)(-4)(-5)(-6) \cdot 712! \equiv 712! \equiv 1 \pmod{719}$$

so $\boxed{719} | 712! + 1$. $\square$

§13.2.5 Get the CRUX of this inequality!

Source: CRUX Mathematicorum
Proposer: TaesPadhihary#8557 (665057968194060291)
Problem ID: 186
Date: 2021-04-30
Difficulty: Hard

Six positive real numbers $r_1, r_2, r_3, r_4, r_5, r_6$ are such that

$$3 \cdot \left(\sum_{i=1}^{6} (-1)^i r_i \right) = \sum_{i=1}^{6} r_i^2 = 18.$$

If M is the least possible constant such that $r_1 r_2 r_3 r_4 r_5 r_6 \leq M$, find the value of $\lfloor 100M \rfloor$.

Solution.

Source: **CRUX Mathematicorum (go to Page 44-45)** Answer: $\boxed{100}$ □

§13.2.6 Don't overthink this

Source: Original (?)
Proposer: ChristopherPi#8528 (696497464621924394)
Problem ID: 187
Date: 2021-05-01
Difficulty: Challenging

Points A, B, C, D are located at $(-3,0),(1,0),(6,0),(7,0)$ in the coordinate plane. If the locus of points P such that the circumcircles of $\triangle ABP$ and $\triangle CDP$ are tangent has perimeter p, find $\lfloor 100p \rfloor$.

Solution.

Let X have coordinates (x, y), and Y be the intersection of the radical axis of the two circumcircles and segment $\overline{BC}$. Let $BY = b$; then $CY = 5 - b$. By Power of a Point, we have

$$XY^2 = BY \cdot AY = CY \cdot DY = b(4+b) = (5-b)(6-b) \iff b = 2, XY^2 = 12$$

so X is always a distance of $\sqrt{12}$ from the point $Y = (3,0)$; this is a circle with radius $\sqrt{12}$ and so our answer is $\boxed{2176}$. □

§13.2.7 The Party, The Politburo, and the PSC

Source: IZhO 2021/5
Proposer: Circumrectangular Hyperbola#8766 (446065841172250638)
Problem ID: 188
Date: 2021-05-02
Difficulty: Challenging

On a party with 102 guests, hosts Tony and Wang play a game (the hosts are not regarded as guests). There are 102 chairs arranged in a circle; initially, all guests hang around those chairs. The hosts take turns alternately. By a turn, a host orders any standing guest to sit on an unoccupied chair c. If some chair adjacent to c is already occupied, the same host orders one guest on such chair to stand up (if both chairs adjacent to c are occupied, the host chooses exactly one of them). All orders are carried out immediately. Tony makes the first move; her goal is to fulfill, after some move of hers, that at least k chairs are occupied. Determine the largest k for which Tony can reach the goal, regardless of Wang's play.

Solution.

We claim the answer is $\boxed{35}$. Note the number of chairs does not decrease, given that every empty chair must be next to a full chair, we have at least 34 chairs. In addition, when there are 33 chairs, every third chair is occupied.
If Wang forces Tony into such a configuration, say Tony placed a guest on chair x and Wang on y. However, if y is adjacent to x, Tony could have easily have chosen y instead and wins. Otherwise, she could have placed it on $x \pm 1$, as there were no guests there, and now not every third chair is occupied. Thus, Wang never reaches a position with 34 chairs such that no more guests can be placed for Tony, and Tony can happily place her last guest.
To show that Wang can win otherwise, we show inductively that Wang can hold a strategic position:
Claim. Wang can make sure that in the sets $S_2, S_3, \ldots, S_{34}$ where $S_k = \{3k+1, 3k+2, 3k+3\}$, none of $3k+1$ and $3k+3$ is occupied. Furthermore, there is at most two occupied chairs in S_1.
Proof. The proof is by induction.
Base Case. No moves are made, in which case the claim holds.
Induction Hypothesis. Assume that Wang holds such a position. We claim after 2 moves, Wang can hold this position again. *Induction Step.* If Alice's move holds the position, Wang can simply put a person somewhere in S_1 (as chairs 4 and 102 are unoccupied, we can once again only have at most 2 people in S_1). If it doesn't, she could not have operated on S_1, and thus placed a guest in some of $S_2, \ldots, S_{99}$. However, if she operates on S_k, Wang places a person in the middle of S_k and removes the mischief caused by Alice. Thus Wang can hold his position. □

§14 Mityushikha Bay (Season 14)

§14.1 Week 1

§14.1.1 Rectangle Rotation

Source: Original
Proposer: RandomGuy#8751 (821212688633823235)
Problem ID: 189
Date: 2021-05-17
Difficulty: Beginner

A 35-by-42 rectangle is rotated around one of its vertices. If A is the area the rectangle sweeps out, find $\lfloor 100A \rfloor$.

(A scientific calculator may be used.)

Solution.

Noticably after a rectangle is rotated around one of its vertex, it will then form a circle.
The radius of the circle is a diagonal because the furthest point from one of its vertex is a diagonal and in a few rotation, it will cover an area of circle with the radius of the diagonal length.
Firstly, calculate the radius of the circle by calculating the rectangle diagonal
$r = \sqrt{a^2 + b^2}$
$r = \sqrt{35^2 + 42^2}$
$r = \sqrt{2989}$
so the area of the circle of $2989\pi \iff \boxed{939022}$.

Remark. 3.1415926 and $\frac{355}{113}$ are the least "precise" approximations of π needed to get the correct answer. □

§14.1.2 NJOY Troll (again)

Source: Original
Proposer: TaesPadhihary#8557 (665057968194060291)
Problem ID: 190
Date: 2021-05-18
Difficulty: Beginner

If the minimum of $(20\cos x + 21)^2 + 2021$ is M, find $\lfloor 100M \rfloor$.

Solution.

The expression inside the square has a minimum of 1, achieved when $\cos x = -1$. Answer: $\boxed{202200}$ $\square$

§14.1.3 You Just Got Vector'ed

Source: Sixth Term Examination Paper I, 2013 Q3
Proposer: sjbs#9839 (434767660182405131)
Problem ID: 191
Date: 2021-05-19
Difficulty: Easy

Given for any two points X and Y, with position vectors $\mathbf{x}$ and $\mathbf{y}$ respectively, $X * Y$ is defined to be the point Z along the line $\overleftrightarrow{XY}$ such that

$$\frac{XZ}{ZY} = \frac{\lambda}{1-\lambda},$$

where λ is a fixed number.

The points $P_1, P_2, \ldots$ are defined by $P_1 = X * Y$ and, for $n \geq 2$, $P_n = P_{n-1} * Y$. Given that X and Y are distinct and that $0 < \lambda < 1$, the ratio in which P_n divides the line segment XY can be expressed in the form $1 : A$. If $\lambda = 0.75$ and $n = 69$, find $\lfloor 10^{12} A \rfloor$.

(A scientific calculator may be used.)

Solution.

We have

$$\begin{aligned} &P_1 = X * Y \\ &P_2 = (X * Y) * Y \\ &\ldots \\ &P_n = (X * Y) * Y) * Y)\ldots * Y \end{aligned}$$

Where P_n has n Y's. Substituting $X * Y = \lambda\mathbf{x} + (1-\lambda)\mathbf{y}$ and going through a number of iterations of $*Y$, we guess $P_n = \lambda^n + (1-\lambda^n)\mathbf{y}$.

We now proceed by induction on n. Given we have already verified the case of $n = 1$, let $n = k,\ k \in \mathbb{N}$ and assume $P_k = \lambda^k\mathbf{x} + (1-\lambda^k)\mathbf{y}$. Then for $n = k+1$, we have $P_{k+1} = P_k * Y$, and so we have $P_{k+1} = \lambda^{k+1}\mathbf{x} + \lambda(1-\lambda^k)\mathbf{y} + (1-\lambda)\mathbf{y}$. This simplifies to $P_{k+1} = \lambda^{k+1}\mathbf{x} + (1-\lambda^{k+1})\mathbf{y}$, as required. Therefore:

$$P_n = \begin{pmatrix} \lambda^n \\ 1-\lambda^n \end{pmatrix}$$

Now it's simply a case of

$$XY : P_n = \begin{pmatrix} 1 \\ 1 \end{pmatrix} : \begin{pmatrix} \lambda^n \\ 1-\lambda^n \end{pmatrix} \Rightarrow XY : P_n = 1 : \frac{\lambda^n}{1-\lambda^n}$$

Hence when $\lambda = 0.75$ and $n = 69$, we have $\{10^{12} \cdot A\} = \boxed{2}394$ □

§14.1.4 Answer Extraction

Source: Original
Proposer: Charge#3766(481250375786037258)
Problem ID: 192
Date: 2021-05-20
Difficulty: Medium

On a QoTD, the PSC has a probability $\frac{m}{n}$ that they need to decide how to make people submit. Sjbs thinks that people should submit $m+n$. Brainy however, thinks that the people should submit $100m+n$. The rest of the PSC come up with an even better idea of people submitting $420m+69n$.

If the PSC decides every single answer will be 420 in a season, the number of possible ways they can choose m and n and a way of formatting their answer ($420m+69n$, $100m+n$, or $m+n$) in a 14 day season can be written as a^b where a and b are positive integers and b is maximized. Compute $420a+69b$.

(The PSC can use a specific $\frac{m}{n}$ more than once.)

Solution.

We can do some casework.
Case 1: The PSC choose to use the $420m+69n=420$ system Clearly only $m=1, n=0$ works, however this will not fit into the constraints.

Case 2: The PSC choose to use the $100m+n=420$ system Only $(0,420),(1,320),(2,220),(3,120),(4,20)$ work, and of these, only $(1,320)$ works.

Case 3: The PSC choose to use the $m+n=420$ system. For this to work we must have $\gcd(m,n) = \gcd(m, 420-m) = \gcd(m,420) = 1$. So to calculate the number of m possible, we do $\varphi(420) = 420\cdot\frac{1}{2}\frac{2}{3}\frac{4}{5}\frac{6}{7} = 96$. Since m must also be greater than n we divide by 2 to get 48 ways.

So our answer is $49^{14} = 7^{28} \iff \boxed{4872}$. □

§14.1.5 Not a Fan

Source: Baltic Way 2020 Problem 18
Proposer: Circumrectangular Hyperbola#8766 (446065841172250638)
Problem ID: 193
Date: 2021-05-21
Difficulty: Hard

Let $n \geq 1$ be a positive integer. We say that an integer k is a *fan* of n if $0 \leq k \leq n-1$ and there exist integers $x, y, z \in \mathbb{Z}$ such that

$$\begin{aligned} x^2 + y^2 + z^2 &\equiv 0 \pmod{n}; \\ xyz &\equiv k \pmod{n}. \end{aligned}$$

Let $f(n)$ be the number of fans of n. Determine $f(2020)$.

Solution.

We note that if $4 \mid x^2+y^2+z^2$, then clearly $x \equiv y \equiv z \equiv 0 \pmod 2$, so $xyz \equiv 0 \pmod 4$. Similarly, if $5 \mid x^2+y^2+z^2$, then as $x^2 \in \{0,1,4\}$, one of x, y, z is divisible by 5. Now, we claim we can find x, y, z such that $x^2+y^2+z^2 \equiv 0 \pmod{101}$ and $xyz \not\equiv 0 \pmod{101}$. We can let $\alpha = \sqrt{-1} \pmod{101}$ which has to exist as $101 \equiv 1 \pmod 4$. Thus if $101 \nmid z$,

$$(x - \alpha y)(x + \alpha y) = x^2 - (\alpha y)^2 = -z^2$$

We have $101 - 1 = 100$ choices for $x + \alpha y$, and only two values of $x + \alpha y$ force $x + \alpha y = \pm(x - \alpha y)$, implying that we have a solution such that $0 \nmid x, y, z$ and

$$101 \mid x^2 + y^2 + z^2$$

Note that $3 \nmid 101 - 1$, so thus taking this solution and multiplying by an arbitrary number w, we see that (xw, yw, zw) works and thus spans all possible values of $k \pmod{101}$. Thus, there are 101 *fans* of 101, but only 1 of *20*, so there are $\boxed{101}$ of 2020. □

§14.1.6 you're welcome, brainy

Source: 2019 Turkey Math Olympiad, P3
Proposer: Alp Eren#5330 (691708663462887464)
Problem ID: 194
Date: 2021-05-22
Difficulty: Challenging

There are 2021 crewmates in the universe, and some of these crewmates are members of different crews (crewmates can belong to multiple crews at once). Each crew has a security council consisting of 12 crewmates who are members of that particular crew. An *emergency meeting* (for a particular crew) can be realized only when each participant (of the meeting) is a member of that crew, and moreover, each of the 12 crewmates forming the security council (for that crew) are present among the participants. It is known that each subset of at least 12 crewmates can realize an emergency meeting for exactly one crew. If the total number of different crews with exactly 29 crewmates is N, find $\nu_2(N) + \nu_3(N) + \nu_5(N)$.

Solution.

AoPS Solutions

The answer is $\binom{2003}{11} \iff \boxed{4}$. □

§14.1.7 RDS-1

Source: 2007 Sharygin Correspondence Round 18
Proposer: AiYa#2278 (675537018868072458)
Problem ID: 195
Date: 2021-05-23
Difficulty: Challenging

O, G are points in the plane such that $OG = 20.21$. Let S be the set of vertices of all triangles with positive area that have circumcentre O and centre of mass G, and T the set of points in the plane that are not in S. If the area of T is U, find $\lfloor 100U \rfloor$.

(A scientific calculator may be used.)

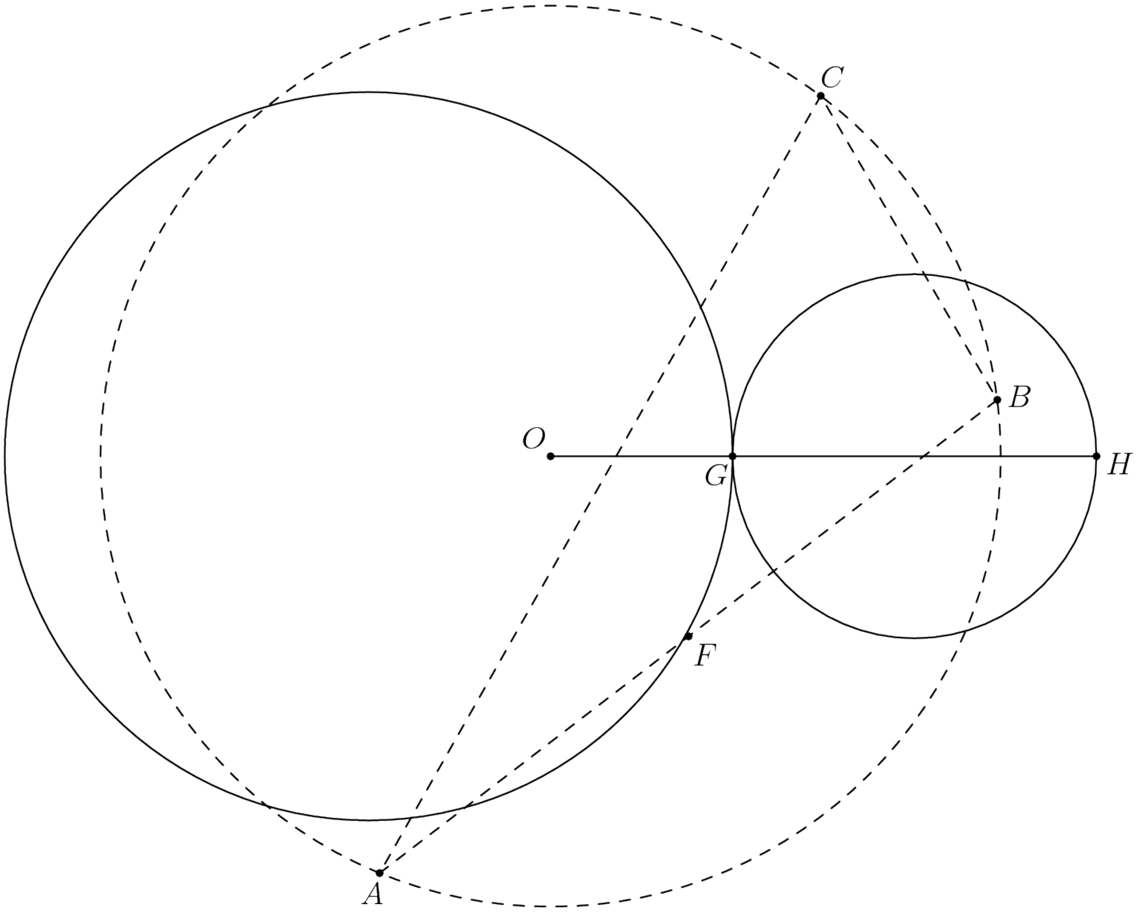
Solution.

The official solution is in Russian. I don't understand Russian.

The main motivation here is the Euler Line and the points on it.
Euler Line. O, G, H, circumcentre, centre of mass, and orthocentre of a triangle, are collinear with $2OG = GH$ or all three coincident. Consider the homothety that sends each vertex of $\triangle ABC$ to the midpoint of its opposite side, thus sending $\triangle ABC$ to its medial triangle; this homothety is then centered at G (why?). Then, H is mapped to the orthocentre of the medial triangle, but this point is the circumcentre of $\triangle ABC$.

Lemma. Let F be the midpoint of $\overline{AB}$. Then, $CH = 2OF$. This is closely related with the proof of the existence of the nine-point-circle. Consider the homothety centered at N_9 that sends O to H; this homothety has scale -1. Then, F is sent to the midpoint of $\overline{CH}$ since the foot of the C-altitude is on the nine-point-circle, implying that F and the midpoint of $\overline{CH}$ are diametrically opposite. We are done.

But note that $OF < R$, so we have the constraint $CH = 2OF < 2R = 2OC$. The locus for which $\frac{CH}{OC} \le 2$ is a union of Apollonian circles forming a disk with the boundary circle being the Apollonian circle where $\frac{CH}{OC} = 2$. Two diametrically opposite points on this circle are G and G', where G' is on the Euler line and $G'O = 60.63$, making the disk have radius 40.42. To construct such $\triangle ABC$, let F be the point on $\overrightarrow{CG}$ such that $CF > CG$ and $CG = 2GF$. Then draw a line at F perpendicular to $\overline{OF}$ and a circle centered at O with radius OC; the points A, B are the intersections of circle and line.



We are missing one detail: what if A, B, C are all collinear? Then, since $\overline{FO} \parallel \overline{CH}$ we have $\angle CFO = \angle GCH$, but since A, B, C collinear we have $\angle CFO = 90$, or, C is on the circle with diameter $\overline{GH}$, which in our case has radius 20.21 and does not intersect the aforementioned disk.

Our answer is $100\pi\left(40.42^2 + 20.21^2\right) \iff \boxed{641582}$.

Remark. RDS-1 was the first Soviet nuke. $\square$

§14.2 Week 2

§14.2.1 All About that Base

Source: Komal C1336
Proposer: flame#6784 (185229437787176962)
Problem ID: 196
Date: 2021-05-24
Difficulty: Beginner

Let a, b, and c denote the digits 0, 1, and 2, in some order. If

$$aaa_3 \times bb_3 = acbab_3,$$

evaluate $bcba_{10} \times bcbb_{10}$.

(A 4-function calculator may be used.)

Solution.

If a or b are 0, then $aaa_3 \times bb_3 = 0_3$, which is clearly false. Hence $c = 0$. Looking at the units digits, $a_3 \times b_3 = b_3$, so $a = 1$ and $b = 2$. (It can be checked that $111_3 \times 22_3 = 10212_3$.) We seek $2021 \times 2022 = \boxed{4086462}$. $\square$

§14.2.2 Number of QoTD Seasons

Source: Original
Proposer: cooked#8292 (573431880091828224)
Problem ID: 197
Date: 2021-05-25
Difficulty: Beginner

A QoTD season, lasting 14 days, has 3 alg, 3 combi, 3 geo and 3 NT problems, as well as 2 other problems in any of alg, combi, geo or NT, If N is the number of distinct possible QoTD topic sequences, find $\nu_2(N) + \nu_3(N) + \nu_5(N)$. (Each question belongs to exactly one subject, so no CG or anything.)

Solution.

Without loss of generality, we can consider two cases- the case that both problems are the same topic and the case that both problems are in different topics.
Consider the case the 2 problems are in the same topic. Without loss of generality, let us call the 4 topics A, B, C and D, and let us assume the 2 problems are in topic A, so that there are 5 A problems, 3 B problems, 3 C problems and 3 D problems. Now, since this is permutations of like objects, we can see the number of topic sequences using A, B, C and D is

$$\frac{14!}{5! \times 3! \times 3! \times 3!}$$

, which is 3363360. We can also see that there are 4! ways to map Alg, Combi, Geo and NT to the topics A, B, C and D, so there are actually 3363360

$$\times$$

4! topic sequences, which is 80720640.
Now consider the case the two problems are in different topics. Without loss of generality, let us call the 4 topics A, B, C and D, and let us assume the two problems are in topic A and B respectively, so that there are 4 A problems, 4 B problems, 3 C problems and 3 D problems. Like before, we can see the number of topic sequences using A, B, C and D is

$$\frac{14!}{4! \times 4! \times 3! \times 3!}$$

is 4204200. Like before, there are 4! ways to map Alg, Combi, Geo and NT to the topics A, B, C, and D, so there are actually 4204200

$$\times$$

4! topic sequences, which is 100900800.
Add to get

$$\frac{14! \cdot 4!}{3!^3 \cdot 4!}\left(\frac{1}{5} + \frac{1}{4}\right) = 2^6 \cdot 3^4 \cdot \cdot 5 \cdot 7^2 \cdot 11 \cdot 13 \iff \boxed{11}.$$

□

§14.2.3 Square and Point

Source: Original
Proposer: RandomGuy#8751 (821212688633823235)
Problem ID: 198
Date: 2021-05-26
Difficulty: Beginner

There exist a square $ABCD$ with its side length 5 and there also exist a random point P on the same plane such that $AP = 1$.
Let M be the longest distance possible between P and C . Let m be the shortest distance possible between P and C.
Find $\lfloor 1000M + 1000m \rfloor$.

(A scientific calculator may be used.)

Solution.

We notice that the distance CP could be maximum and minimum if C, A, and P are collinear. This can be proved by drawing a circle around A. Then , we continue by finding M and m
M $= \sqrt{5^2 + 5^2} + 1 = 5\sqrt{2} + 1$
m $= \sqrt{5^2 + 5^2} - 1 = 5\sqrt{2} - 1$
Hence, $M + m = 5\sqrt{2} + 1 + 5\sqrt{2} - 1 = 10\sqrt{2} = \sqrt{200} \iff \boxed{14142}$.

$\square$

§14.2.4 2012 HMMT A6

Source: Harvard-MIT Mathematics Tournament2012 February A6
Proposer: Angry Any#4319 (580933385090891797)
Problem ID: 199
Date: 2021-05-27
Difficulty: Medium

Let $a_0 = -2$, $b_0 = 1$, and for $n \geq 0$, let

$$a_{n+1} = a_n + b_n + \sqrt{a_n^2 + b_n^2}$$

$$b_{n+1} = a_n + b_n - \sqrt{a_n^2 + b_n^2}$$

If a_{2020} can be represented in the form $a^x\sqrt{b} - c^y$, where a, b, c are integers such that a, c are minimized and c is square free. Find $x + y$.

Solution. By adding the two equations and multiplying them respectively, we get

$$a_{n+1} + b_{n+1} = 2(a_n + b_n)$$

$$a_{n+1}b_{n+1} = (a_n + b_n)^2 - (a_n^2 + b_n^2) = 2a_nb_n$$

Thus, we get that

$$a_{2020} + b_{2020} = -2^{2020}$$

$$a_{2020}b_{2020} = -2^{2021}$$

Either by vieta's or substitution, we get a_{2020}, b_{2020} are the roots of the quadratic

$$x^2 + 2^{2020}x - 2^{2021}$$

where a_{2020} is the larger root since the positive sqrt was added. Solving the quadratic gives that

$$a_{2020} = 2^{1010}\sqrt{2^{2018} + 2} - 2^{2019}$$

Thus, the answer is $\boxed{3029}$ □

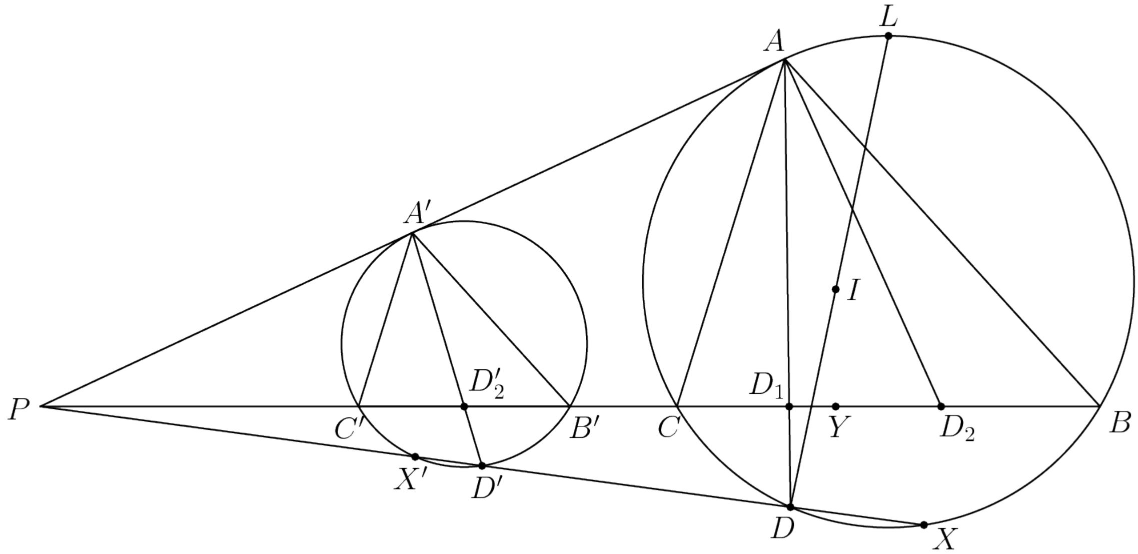
§14.2.5 [under construction]

§14.2.6 Projective Pain

Source: Unknown
Proposer: ChristopherPi#8528 (696497464621924394)
Problem ID: 201
Date: 2021-05-22
Difficulty: Challenging

$\triangle ABC$ has circumcircle Γ and incentre I, with sidelengths $AB = 44, BC = 37, CA = 15$. Let L be the midpoint of the arc $\widehat{BC}$ containing A. $\overleftrightarrow{LI}$ meets Γ again at D, and the tangent to Γ at A meets $\overleftrightarrow{BC}$ at P. $\overleftrightarrow{PD}$ meets Γ again at X. If Y is the point on $\overline{BC}$ such that $\angle BAX = \angle CAY$, the ratio $\frac{BY}{CY}$ can be expressed in the form $\frac{m}{n}$, where m, n are relatively prime positive integers. Find $100m + n$.

Solution.



We claim that for a general cevian ℓ through A, Y will be the reflection across the midpoint M of $\overline{BC}$ of the intersection of the isogonal line of ℓ and $\overline{BC}$. Consider the following transformation that sends points on $\overline{BC}$ to points on $\overline{BC}$: take the projections of points wrt A onto Γ, invert $\triangle ABC$ and those points wrt P, project points back onto $\overline{BC}$ wrt A, then invert the points back onto the original $\triangle ABC$. Let $\overline{AD}$ intersect $\overline{BC}$ at D_1, and let D_2 be the intersection of the isogonal line of $\overline{AD}$ and $\overline{BC}$. Applying this transformation to $\overline{AD}$ and $\overline{AX}$, we let D', X', B', C', D_2' be the images of D, X, B, C, D_2 under inversion, and Y' the intersection of $\overline{AD'}$ and $\overline{B'C'}$; by angle chase we have $\overline{AX'}$ isogonal to $\overline{AD}$ and $\overline{AD'}$ isogonal to $\overline{AX}$. By inversion distance formula we find

$$\frac{BY}{CY} = \frac{CD_2}{BD_2} = \frac{PB}{PC};$$

furthermore since $\overline{PA}$ is a tangent we have $\overline{A'C'} \parallel \overline{AB}, \overline{A'B'} \parallel \overline{AC}$ and $\overline{AD'} \parallel \overline{AX}, \overline{AX'} \parallel \overline{AD}$ so by similar triangles and inversion distance formula we have

$$\frac{C'D_2'}{A'D_2'} = \frac{BD_1}{AD_1}, \frac{B'D_2'}{A'D_2'} = \frac{CD_1}{AD_1} \iff \frac{C'D_2'}{B'D_2'} = \frac{PB}{PC} = \frac{BD_1}{AD_1} = \frac{CD_2}{BD_2}$$

and we are done.

In the case of this problem, it is well-known (in EGMO; invert at A) that D is the touchpoint of the A-mixtilinear incircle and Γ, and so the isogonal line pases through the A-excircle touchpoint, so Y is actually the incircle touchpoint on $\overline{BC}$ so our answer is $\frac{s-b}{s-c} = \frac{33}{4} \iff \boxed{3304}$. □

§14.2.7 AN602

Source: 2021 Bundeswettbewerb Mathematik R1P4, Modified
Proposer: Boldizsár#3106 (694967174921715853)
Problem ID: 202
Date: 2021-05-23
Difficulty: Challenging

Consider a pyramid with a regular n-gon as a base. Draw a line segment in either blue or red between any two points among the n vertices of the base and the apex, except for the edges of the base, which are not drawn at all. Find the smallest integer n with the following property: There exist 4 points among the aforementioned $n+1$ points such that all six segments connecting those four points are drawn in the same color.

Solution.

The solution is $n = \boxed{33}$. We first show that $n = 32$ does not work, i. e. that we can draw the pyramid with a 32-gon as a base such that there is no complete quadrilateral in a single color. By $R(4,4) = 18$, there exists a complete graph G with 17 vertices, the edges drawn in red and blue that contains no K_4 (the complete graph with four vertices) drawn in only one color. Now, organise the 32 vertices of the base into pairs, each pair consisting of two neighboring vertices. We have 17 *units* now, 16 pairs of neighboring base vertices and one apex. Now, match every unit with one vertex of G and connect two units in the color in which the edge connecting the correcpoding vertices of G is drawn. Connect two vertices of the pyramid in the color in which the units they are in are connected. As there are no edges drawn within units (because those are neighboring vertices of the base), our construction will indeed work. Now show that for $n = 33$, there indeed is a K_4 as a subgraph drawn in a single color. So consider the pyramid with the regular 33-gon as a base. WLOG, 17 vertices are connected with the apex in red. But among those 17 vertices, there are 9 from which any two are connected. But $R(4,3) = 9$, so those 9 vertices contain either a red K_3 or a blue K_4, where in the first case the pyramid contains a red K_4, as desired.

This problem is almost impossible without the knowledge of Ramsey numbers, but it can be done; the theory required to discover isn't all that much and you can always search up $R(3,4)$ and $R(4,4)$.

2-colour Ramsey numbers. $R(b,w)$ is the smallest positive integer n such that any edge colouring in black and white of the complete graph on n vertices K_n will always contain either an induced subgraph K_b of all black edges or an induced subgraph K_w of all white edges. For example, $R(3,3) = 6$. To see this, fix one vertex v; by pigeonhole at least three of its edges will be of one color, WLOG black. Let the other vertices of those black edges be x, y, z respectively. If none of the edges formed by connecting any of x, y, z together are black, then x, y, z is K_3; if any of them are then v and those two vertices forming the other black edge is K_3. To prove $R(3,3) \neq 5$ we can construct a counterexample: take K_5 as a pentagon; the perimeter of the pentagon is coloured black while all the interior diagonals are coloured white. Convince yourself that this works.

Ramsey's Theorem. $R(b,w) \le R(b,w-1) + R(b-1,w)$. We proceed by induction on $b+w$. First, convince yourself that $R(b,2) = b$; this is our base case. Now consider the complete graph on $R(b,w-1)+R(b-1,w)$ vertices with edges coloured in black or white. Fix one vertex v, and let B be the set of all vertices whose edges with v are black, and W the set of all vertices whose edges with b are white. Since $R(b,w-1)+R(b-1,w) = |B|+|W|+1$ we have either $|B| \ge R(b-1,w)$ or $|W| \ge R(b,w-1)$; WLOG the first case is true. If we have K_w with edges all white we're done; otherwise we have K_{b-1} with edges all black; the add in vertex v and all the black edges connected to v, which are all from B; thus we have K_b all black edges and we're done. If both $R(b,w-1)$ and $R(b-1,w)$ are even, the inequality strengthens to $R(b,w) < R(b,w-1) + R(b-1,w)$. Consider the complete graph on $R(b,w-1) + R(b-1,w) - 1$ vertices with edges coloured in black or white. Order the vertices, and let b_i be the *black-degree* of vertex v_i; that is, the number of black edges

emananting from v_i. By handshaking lemma there are an even number of vertices with odd degree; since $R(b, w-1) + R(b-1, w) - 1$ is odd there must exist at least one vertex with even degree; WLOG it's v_1. Then $|B| = b_i, |W| = R(b, w-1) + R(b-1, w) - 2 - b_i$ are both even, so by parity it is impossible for both $|B| < R(b-1, w) - 1$ and $|W| < R(b, w-1)$ and we are done.
R(3, 4) is 9. Since $R(3,3) = 6$ and $R(2,4) = 4$ are both even we conclude $R(3,4) < 10$. We can colour a regularly octoganal K_8 as follows: all perimeter edges are black, and all edges connecting diametrically opposite vertices (vertices 4 apart from another) are black; everything else is white. This way, the only way to get from vertex to vertex along black edges is by moving along by 1 or 4 vertices; no combination of three ones and fours will be zero mod 8, so there is no black K_3; similarly, the only way to get from vertex to vertex along white edges is by moving along by 2 or 3 vertices, and the only 4-tuplet of twos and threes that is zero mod 8 are $(2, 2, 2, 2)$ which clearly doesn't work since edges joining antipodeal vertices are coloured black.

R(4, 4) is 18. We know that $R(4,4) \leq 2R(3,4) = 18$. We can colour a regularly 17-gonal K_{17} as follows: all edges that connect vertices that are a power of two apart are black and everything else is white. The only combination of $\{1, 2, 4, 8\}$ that is zero mod 17 is $(8, 4, 4, 1)$ and that doesn't work because a pair of vertices are a distance of twelve apart, and that edge would be coloured white. Similarly, the only combination of $\{3, 5, 6, 7\}$ that is zero mod 17 is $(3, 3, 5, 6)$ and that doesn't work be a pair of vertices are a distance of eight apart, and that edge would be coloured black.

Remark. Mityushikha Bay was the testing site for AN602 (nicknamed the Tsar Bomba), along with many other Russian nukes. □

§15 SALT-1 (Season 15)

§15.1 Week 1

§15.1.1 "What are distinct integers?"

Source: 2018 Fall OMO 1
Proposer: TaesPadhihary#8557 (665057968194060291)
Problem ID: 204
Date: 2021-06-07
Difficulty: Beginner

Call two integers *primely* if 1 is the greatest integer dividing both of them. Find the minimum possible sum of 2021 nonnegative, primely integers.

Solution.

One zero and 2020 ones. Answer: $\boxed{2020}$ □

§15.1.2 Troll Pigeonhole

Source: Original Problem
Proposer: Ada Virus#6266 (439054981966856202)
Problem ID: 205
Date: 2021-06-08
Difficulty: Beginner

In the following fraction every letter represents a different decimal digit.

$$\frac{B \cdot L \cdot U \cdot E \cdot B \cdot E \cdot R \cdot R \cdot Y}{I \cdot C \cdot E \cdot C \cdot R \cdot E \cdot A \cdot M}$$

If the sum of all possible real values this expression can take is S, find $\lfloor 100S \rfloor$.

Solution.

There are exactly 10 different letters (A, B, C, E, I, L, M, R, U, Y), so one of the letters must be 0. 0 divided by anything (except 0 which will make it undefined) is 0 so the answer is $\boxed{0}$. □

§15.1.3 WA Can't Solve This

Source: 2019 Pui Ching Contest
Proposer: QuantumSigma#5000 (374937087301910528)
Problem ID: 206
Date: 2021-06-09
Difficulty: Easy

$$\frac{69}{420} + \frac{69 \cdot 70}{420 \cdot 421} + \frac{69 \cdot 70 \cdot 71}{420 \cdot 421 \cdot 422} + \cdots = \frac{m}{n},$$

where m, n are relatively prime positive integers. Find $100m + n$.

Solution.

$$\begin{aligned}
&\frac{69}{420} + \frac{69 \cdot 70}{420 \cdot 421} + \frac{69 \cdot 70 \cdot 71}{420 \cdot 421 \cdot 422} + \ldots \\
=&\frac{1}{350}\left[\frac{69 \cdot (420-70)}{420} + \frac{69 \cdot 70 \cdot (421-71)}{420 \cdot 421}\right. \\
&\left. + \frac{69 \cdot 70 \cdot 71 \cdot (422-72)}{420 \cdot 421 \cdot 422} + \ldots\right] \\
=&\frac{1}{350}\left[\frac{69 \cdot 420}{420} - \frac{69 \cdot 70}{420} + \frac{69 \cdot 70 \cdot 421}{420 \cdot 421} - \frac{69 \cdot 70 \cdot 71}{420 \cdot 421}\right. \\
&\left. + \frac{69 \cdot 70 \cdot 71 \cdot 422}{420 \cdot 421 \cdot 422} - \frac{69 \cdot 70 \cdot 71 \cdot 72}{420 \cdot 421 \cdot 422} + \ldots\right] \\
=&\frac{69}{350}
\end{aligned} \tag{13}$$

Thus, the answer is $\boxed{7250}$. □

§15.1.4 Prime Product

Source: HMMT-Feb-2018-AN4
Proposer: Angry Any#4319 (580933385090891797)
Problem ID: 207
Date: 2021-06-10
Difficulty: Medium

Distinct prime numbers p, q, r satisfy the equation

$$2pqr + 50pq = 7pqr + 55pr = 8pqr + 12qr = A$$

for some positive integer A. What is A?

Solution. Comparing the first two equations, we get

$$\begin{aligned} 50pq - 55pr = 5pqr &\iff \\ 10q - 11r = qr &\iff \\ (q + 11)(r - 10) = -110 & \end{aligned}$$

Since q, r are positive integers, $(r - 10)$ must be negative and $(11 + q)$ must be a divisor of 110 larger than 11. A quick check gives that q must be 11, which gives that $r = 5$. Plugging these values into the original equation gives $p = 3$ and thus $A = \boxed{1980}$.

□

§15.1.5 The Rare Non-Horrible States Problem

Source: David Patrick
Proposer: AiYa#2278 (675537018868072458)
Problem ID: 208
Date: 2021-06-11
Difficulty: Hard

> We play a game. The pot starts at \$0. On every turn, you flip a fair coin. If you flip heads, I add \$100 to the pot. If you flip tails, I take all of the money out of the pot, and you are assessed a "strike." You can stop the game before any flip and collect the contents of the pot, but if you get 3 strikes, the game is over and you win nothing. Find, with proof,the expected value of your winnings if you follow an optimal strategy.

Solution.

We work backwards. If we have only one strike left, when is it advantageous to flip again? Suppose there are d dollars in the pot. Then, the expected pot value of our next flip is $\frac{1}{2}(d+100)$, so we want to solve $\frac{1}{2}(d+100) > d \iff d < 100$. However, the only value for which $d < 100$ is $d = 0$, so, with only one strike remaining, the best strategy is to flip once and end the game, no matter what the result; our expected earnings is \$50.

What about with two strikes? Suppose there are d dollars in the pot. This time, the expected pot value is $\frac{1}{2}(d+100) + \frac{1}{2} \cdot 50$, since if it's heads we add \$100 and if we lose we play as if we have one strike left, which optimal expected earnings is \$50 as covered before. Solving $d + 100 + 50 > 2d$ we have $d < 150$. Now, we can split into cases: the result of our first coin, and the result of our second (if the first coin is heads). Our expected earnings are then $\frac{1}{2} \cdot 50 + \frac{1}{2}\left(\frac{1}{2} \cdot 200 + \frac{1}{2} \cdot 50\right) = 87.5$.

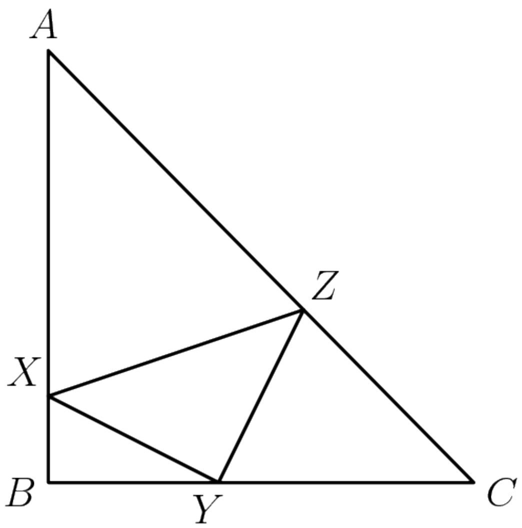
For three strikes, it's exactly the same: solve $\frac{1}{2}(d+100) + \frac{1}{2} \cdot 87.5 > d \iff d < 187.5$; then our expected earnings are $\frac{1}{2} \cdot 87.5 + \frac{1}{2}\left(\frac{1}{2} \cdot 200 + \frac{1}{2} \cdot 87.5\right) = 115.625 \iff \boxed{11562}$. $\square$

§15.1.6 Minimal Area, Minimal Difficulty for a Saturday

Source: Original (?)
Proposer: Matteddy#0482 (329956567132930048)
Problem ID: 209
Date: 2021-06-12
Difficulty: Hard

Right isosceles $\triangle ABC$ has area 1200. X, Y, Z are points on sides $\overline{AB}, \overline{BC}, \overline{CA}$ respectively such that $\triangle XYZ$ is also right and isosceles. Find the minimal area of $\triangle XYZ$.

Solution.



Let $\angle B$ be the right angle. If Z is the right angle of $\triangle XYZ$, then we leave it as an exercise to the reader to show that the minimal configuration happens when $\triangle XYZ$ is the medial triangle of $\triangle ABC$. However, there is another case: the right angle is on one of the legs. Let Y be the right angle; then let $\angle XYB = a$. Through some angle chasing, we find $\triangle AXZ \sim \triangle CZY$ with ratio $\sqrt{2}$. Letting s be the leg length of $\triangle ABC$ and x the leg length of $\triangle XYZ$ we have

$$BY = x\cos(a) \iff YC = s - x\cos(a) \iff AZ = \sqrt{2}(s - x\cos(a))$$

and

$$BX = x\sin(a) \iff XA = s - x\sin(a) \iff CZ = \frac{s - x\sin(a)}{\sqrt{2}}$$

and so

$$AZ + CZ = AC = s\sqrt{2} \iff s = x(2\cos(a) + \sin(a)).$$

The maximum of $2\cos(a)+\sin(a)$ is $\sqrt{5}$ (use Cauchy-Schwarz or properties of combined sinusodial functions) so our minimum x is $\frac{s}{\sqrt{5}}$ and our minimum area is $\frac{[ABC]}{5} = \boxed{240}$. □

§15.1.7 Detente

Source: 2016 CMIMC N10
Proposer: AiYa#2278 (675537018868072458)
Problem ID: 210
Date: 2021-06-13
Difficulty: Challenging

Let
$$S = \sum_{n=1}^{\infty} \operatorname{lcm}[n, 69]^{-2}.$$
Find $\lfloor 10^9 S \rfloor$.

(A scientific calculator may be used.)

Hint. $\sum_{n=1}^{\infty} \operatorname{lcm}[n, 1]^{-2} = \frac{\pi^2}{6}$.

Solution.

Solution Answer: $\boxed{1303995}$ □

§15.2 Week 2

§15.2.1 Senior Math; What Challenge?

Source: 1998 SMC
Proposer: sjbs#9839 (434767660182405131)
Problem ID: 211
Date: 2021-06-14
Difficulty: Beginner

A cube is inscribed in a sphere of diameter 1. What is the surface area of the cube?

Solution.

This means that the cube has space diagonal 1, so $3s^2 = 1 \iff s^2 = \frac{1}{3} \iff SA = 6s^2$. Answer: $\boxed{4}$ $\square$

§15.2.2 Prisoner's Stupid Dilemma

Source: Original
Proposer: brainysmurfs#2860 (281300961312374785)
Problem ID: 212
Date: 2021-06-15
Difficulty: Beginner

There are 100 prisoners numbered 1 to 100, and 100 boxes also labelled 1 to 100. The boxes are kept inside a room and the numbers 1 to 100 are randomly placed inside each box (one number in each box, and each number appears exactly once). In sequence, each prisoner enters the room and is allowed to look inside 50 boxes. If they find their number, they are released and allowed to communicate with the next prisoners. Otherwise, all the prisoners are executed. If the prisoners play optimally, the probability they are all released is p. Find $\lfloor 1000p \rfloor$.

(A scientific calculator may be used.)

Solution.

Suppose Brainy is the first prisoner. He clearly has no better strategy than random guessing, getting his number with probability $\frac{1}{2}$. If he does get his number, then he should remember the numbers in the remaining 49 boxes that he opened and tell that to the remaining 99 prisoners. Then, each prisoner will either know which box to open (because Brainy opened the box with their number in it) or they will know 50 boxes that don't contain their number (Brainy doesn't mention their number, so of the 50 boxes he opened none of them contain their number). But this means that they can all successfully escape! Our answer is $\boxed{500}$. $\square$

§15.2.3 Coordinates without Bash

Source: Original Problem
Proposer: nexal#3868 (381673088216989742)
Problem ID: 213
Date: 2021-06-16
Difficulty: Easy

There is a circle that goes through the points $(2+\sqrt{2}, 1-\sqrt{7})$ and $(5, 1)$ with radius 3. Let (a, b) be the circle's center. Given $a+b$ is a positive integer, find $a^b + b^a$.

Solution.

Notice that $\sqrt{2}^2 + \sqrt{7}^2 = 3^2$, so we try and see that a circle centered at $(2, 1)$ does go through $(2+\sqrt{2}, 1-\sqrt{7})$. Using the distance formula, we see it also goes through $(5, 1)$. The other and only possible circle has its center reflected across the midpoint of the two given points. Since one point has lattice coordinates and the other is irrational, the midpoint must have irrational coordinates. Therefore, this other circle's center has irrational coordinates. Since 2 and 7 are relatively prime, it is impossible for the sum of the this center's coordinates to be an integer. Therefore the circle centered at $(2, 1)$ is the only possible circle, giving us the answer of $2^1 + 1^2 = \boxed{3}$.□ □

§15.2.4 WA Can't Solve This, Either

Source: 2013 HMMT A7
Proposer: Angry Any#4319 (580933385090891797)
Problem ID: 214
Date: 2021-06-17
Difficulty: Medium

$$\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\cdots\sum_{k_8=0}^{\infty}\frac{k_1+k_2+\cdots+k_8}{5^{k_1+k_2+\cdots+k_8}}=\frac{m}{n},$$

where m, n are relatively prime positive integers. Find the sum of positive divisors of mn.

(A scientific calculator may be used.)

Solution.

Solution Answer: $\boxed{19199495335}$ □

§15.2.5 WA Can't Solve This, Either

Source: USAMTS Y19R1P3
Proposer: AiYa#2278 (675537018868072458)
Problem ID: 215
Date: 2021-06-18
Difficulty: Hard

Let S be the set of all ordered triples of positive integers (a, b, c) such that

$$\arctan\frac{1}{a} + \arctan\frac{1}{b} + \arctan\frac{1}{c} = \frac{\pi}{4},$$

where the range of $\arctan(x)$ is $[-\pi/2, \pi/2]$. Find

$$\sum_{(a,b,c)\in S} abc.$$

Solution.

Solution Answer: $\boxed{1293}$ □

§15.2.6 cHriS geO

Source: 2020 Iran MO R3G4
Proposer: ChristopherPi#8528 (696497464621924394)
Problem ID: 216
Date: 2021-06-19
Difficulty: Challenging

$\triangle PQR$ has $PQ = 6, QR = 7, RP = 8$, incentre I, and circumcentre O. The external angle bisector of P meets $\overleftrightarrow{QR}$ at S, and I_Q is the Q-excentre. The point T is chosen along $\overleftrightarrow{IQ}$ such that $QT = 2IQ$ and $IT > IQ$. Let F be the point on the circumcircle of $\triangle DI_QT$ such that DF is a diameter. Then the perimeter of $\triangle OIF$ can be expressed in the form $\frac{a+b\sqrt{c}}{d}$, where $\gcd(a, b, d) = 1$, a, b are integers, and c, d are positive integers. Find $1000a + 100b + 10c + d$.

Solution.

Solution so O, I, F are collinear; using $OI^2 = R^2 - 2Rr$ we get $IF = \frac{16\sqrt{15}}{15} \iff \boxed{3530}$. $\square$

§15.2.7 Treaty Violation

Source: chinese guy
Proposer: AiYa#2278 (675537018868072458)
Problem ID: 217
Date: 2021-06-20
Difficulty: Challenging

Let $f : \mathbb{R}^{2020} \to \mathbb{R}$ be the function

$$f(a_1, a_2, \cdots, a_{2020}) = \frac{a_1^2 + a_2^2 + \cdots + a_{2020}^2 + (a_1 + a_2 + \cdots + a_{2020})^2}{\sqrt{a_1^4 + a_2^4 + \cdots + a_{2020}^4 + (a_1 + a_2 + \cdots + a_{2020})^4}}.$$

Then the maximum of f is M, and the minimum of f is m. Find $\lfloor 10^9 M + 10^6 m \rfloor$.

(A scientific calculator may be used.)

Solution.

In fact, f has the range

$$\left[\sqrt{\frac{m(m-1)}{m^2-3m+3}}, \sqrt{\frac{m\left(m^2-1\right)}{m^2+3}}\right]$$

where $m = n+1$, giving us an answer of $\boxed{44956512067}$. The condition is equivalent to $a_1 + \cdots + a_m = 0$, where m is odd. To find the minimum, WLOG a_m be the largest term in absolute value (positive or negative) then use C-S on

$$a_1^2 + \cdots + a_{m-1}^2 = (a_1 + \cdots + a_{m-1})^2 - 2\sum_{i<j} a_i a_j = a_m^2 - 2\sum_{i<j} a_i a_j$$

to arrive at an answer. The end result is that we should maximize a_m^2 while minimizing $a_1^2 + \cdots a_{m-1}^2$.

Unfortunately, the maximum is not so easy. Intuitively from QM-AM, the maximum should happen when the a's are close together; in fact, the equality case happens when there are $\lfloor m/2 \rfloor$ equal terms of one sign and $\lceil m/2 \rceil$ equal terms of the other. The proof of this is absolute hell and involves 4 different inequalities (one that you have to make up yourself) and smoothing/mixing variables, so just trust that the fakesolve works. $\square$

§16 Quality Control (Season 16)

§16.1 Week 1

§16.1.1 The Rare non-Troll NJOY Problem

Source: Unknown
Proposer: TaesPadhihary#8557 (665057968194060291)
Problem ID: 218
Date: 2021-06-28
Difficulty: Beginner

If $n^2\%$ of n is 80, find n.

Solution.

We have
$$\frac{n^2}{100} \cdot n = 80 \iff n = \boxed{20}.$$

□

§16.1.2 The Common Troll NJOY Problem

Source: 2017 Fall OMO #4
Proposer: TaesPadhihary#8557 (665057968194060291)
Problem ID: 219
Date: 2021-06-29
Difficulty: Beginner

Chris draws a line segment between each of the points

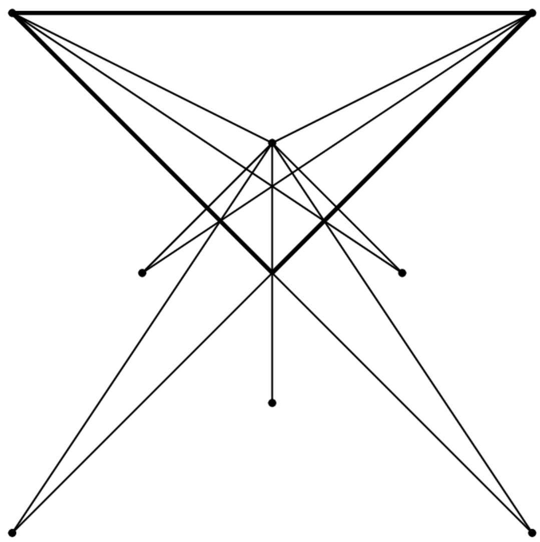
$$A(4,4), B(-4,4), C(-4,-4), D(4,-4), E(2,0), F(0,2), G(-2,0), H(0,-2)$$

on a sheet of paper large enough so that none of the line segments touch the edge of the paper. He wants to separate all the regions obtained by the line segments and later on eat them.

What is the maximum number of regions of paper that Chris can eat?

Solution.

Note that the diagram is diagonally symmetric, so we need only to look at the intersections in the interior of $\triangle AOB$, where O is the origin. Count that there are 15 intersections, then multiply by 4 to get 60. However, the troll is that Chris is cutting out sections on a piece of paper, so the outside of square $ABCD$ is also a region and our answer is $\boxed{61}$.



□

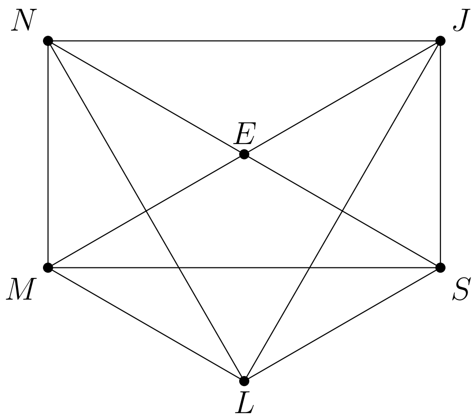
§16.1.3 Parallelogram PTSD

Source: Italian Team Competition
Proposer: HoboSas#3200 (310725130097786880)
Problem ID: 220
Date: 2021-06-30
Difficulty: Medium

Diagonals of parallelogram $NJSM$ interescts at E, angle bisectors of $\angle MNE$ and $\angle EJS$ intersects at L. Given that $MESL$ is a parallelogram and $\overline{NM} = 35$, compute $\overline{NJ}^2$.

Solution.

Note $NS \parallel ML$ so $\angle NLM = \angle LNE = \angle MNL$ which makes $\triangle MNL$ isosceles, hence $\overline{MN} = \overline{ML} = \overline{ES}$. With a similar reasoning we get $\triangle JSL$ isoceles, with $\overline{JS} = \overline{SL} = \overline{EM}$, but $\overline{MN} = \overline{JS}$ so $\overline{EM} = \overline{ES}$, but recalling E is the midpoint of both diagonals, $\overline{NS} = \overline{JM}$, making $NJSM$ a rectangle. Now $\overline{MJ} = 2 \cdot \overline{MN}$, which gives $\overline{NJ} = \sqrt{3} \cdot \overline{NM}$, so finally $\overline{NJ}^2 = 3 \cdot \overline{NM}^2 = \boxed{3675}$. □



§16.1.4 WA Could Solve This and you Still Failed

Source: Original
Proposer: AiYa#2278 (675537018868072458)
Problem ID: 221
Date: 2021-07-01
Difficulty: Medium

A real number a is selected uniformly at random between -50 and 50 inclusive. The probability that $20x - \left|21x - \left|x + \frac{20a}{21}\right|\right| = 42|x-1|$ has real solutions can be expressed in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $100m + n$.

Solution.

Let the LHS be $f(x)$. The graph of f looks like a bunch of lines, and f is continuous (why?). The maximum absolute value of the slope of a line that's part of f can be is 42, which means that if f crosses the RHS then $f(1) \geq 0$, since $x = 1$ is where the minimum of $42|x-1|$ is attained; if $f(1) < 0$ then f will never cross $42|x-1|$. This is because for f to cross $42|x-1|$ for $x < 1$ when $f(1) < 0$, f would have to somehow go from being above the V-shaped absolute value graph of $42|x-1|$ to below it at $x = 1$, impossible since no part of f has slope of absolute value greater than 42; analogously we see that f cannot cross $42|x-1|$ when $x > 1$. Solve to get $-\frac{441}{10} \leq a \leq -\frac{21}{10}$ and $0 \leq a \leq 42$. Our probability is $\frac{21}{25} \iff \boxed{2125}$. $\square$

§16.1.5 Surprising System

Source: AoPS Mock AIME
Proposer: AiYa#2278 (675537018868072458)
Problem ID: 222
Date: 2021-07-02
Difficulty: Hard

p, q, r are reals such that the following system holds:

$$(p+q)(q+r) = -1$$
$$(p-q)^2\left(1+(p+q)^2\right) = 20$$
$$(q-r)^2\left(1+(q+r)^2\right) = 21$$

Find $(r-p)^2\left(1+(r+p)^2\right)$.

Solution.

Consider the three points $P = \left(p, p^2\right), Q = \left(q, q^2\right), R = \left(r, r^2\right)$. Then $\overleftrightarrow{PQ}$ has slope $p+q$ and $\overleftrightarrow{QR}$ has slope $q+r$, so $\overleftrightarrow{PQ} \perp \overleftrightarrow{QR}$. Then, using distance formula, we have $PQ = (p-q)^2 + \left(p^2-q^2\right)^2 = 20$ and likewise for QR, so $(r-p)^2\left(1+(r+p)^2\right)^2 = RP^2 = PQ^2 + QR^2$ from Pythagorean Theorem. Our answer is $\boxed{41}$. $\square$

§16.1.6 Really Weird NT

Source: National High School Math League (China) 2004 Test 2 Question 3, altered greatly
Proposer: watanuki-taiga-2#1569 (769480637245554688)
Problem ID: 223
Date: 2021-07-03
Difficulty: Challenging

Let m and $n \geq 4$ be positive integers. If any k-element subset of $\{m, m+1, \cdots, m+n-1\}$ has at least 3 mutually coprime elements, and the minimum value of k is 2021, calculate the sum of all possible values of n.

Solution.

Let $f(n)$ defined for integers $n \geq 4$ be the minimum value required such that any $f(n)$-element subset of $\{m, m+1, \cdots, m+n-1\}$ has at least 3 mutually coprime elements. We wish to solve for $f(n) = 2021$.
First, it is trivial that the number of elements in $\{2, 3, 4, \cdots, n+1\}$ which are either a multiple of 2 or 3 is given by $\lfloor \frac{n+1}{2} \rfloor + \lfloor \frac{n+1}{3} \rfloor - \lfloor \frac{n+1}{6} \rfloor$, where $\lfloor x \rfloor$ is the greatest integer less than or equal to x. Therefore $f(n) \geq \lfloor \frac{n+1}{2} \rfloor + \lfloor \frac{n+1}{3} \rfloor - \lfloor \frac{n+1}{6} \rfloor + 1$. We show that the RHS is indeed equal to $f(n)$.
We will show that $f(4) = 4$, $f(5) = 5$, $f(6) = 5$, $f(7) = 6$, $f(8) = 7$, $f(9) = 8$ and then proceed with induction.
Step 1. When $n = 4$, let the set be $\{m, m+1, m+2, m+3\}$. One of the triples $(m, m+1, m+2)$ and $(m+1, m+2, m+3)$ must all be mutually coprime. Therefore it is true for $n = 4$. It follows that this is also true for $n = 5$.
When $n = 6$, let the set be $\{m, m+1, m+2, m+3, m+4, m+5\}$. Let us choose three numbers from x_1, x_2, x_3, x_4, x_5. If three of them are odd, then they are mutually coprime. Else there are three even numbers, let them be x_1, x_2 and x_3. Since $|x_i - x_j| \in \{2, 4\} (1 \leq i < j \leq 3)$, therefore at most one is divisible by 3, at most one is divisible by 5, so there must exist one that is neither divisible by 3 nor 5, WLOG let it be x_3. Then x_3, x_4, x_5 are coprime. It follows that this is true for $n = 7$, $n = 8$ and $n = 9$.
Step 2. If the proposition is true for $n = t (t \geq 4)$, then when $n = t+6$, we have $f(t+6) - f(t) = 4$. Note that $\{m, m+1, \cdots, m+t+5\} = \{m, m+1, m+2, m+3, m+4, m+5\} \cup \{m+6, m+7, \cdots, m+t+5\}$. We choose $f(t+6) = f(t) + 4$ elements from it.
If we choose at least 5 elements from $\{m, m+1, \cdots, m+5\}$, then from the conclusion of $n = 6$, we are done. Otherwise we must have chosen $f(t)$ elements from $\{m+6, m+7, \cdots, m+t+5\}$, which is true from the induction hypothesis.
Therefore the proposition is true for $n = t+6$.
Therefore, $f(n) = \lfloor \frac{n+1}{2} \rfloor + \lfloor \frac{n+1}{3} \rfloor - \lfloor \frac{n+1}{6} \rfloor + 1$. Note that this can be rewritten as:

$$f(n) = \begin{cases} 4t+1, & (n = 6t, t \in \mathbb{Z}^+), \\ 4t+2, & (n = 6t+1, t \in \mathbb{Z}^+), \\ 4t+3, & (n = 6t+2, t \in \mathbb{Z}^+), \\ 4t+4, & (n = 6t+3, t \in \mathbb{Z}^+), \\ 4t+4, & (n = 6t+4, t \in \mathbb{Z}^+), \\ 4t+5, & (n = 6t+5, t \in \mathbb{Z}^+) \end{cases}$$

Finally we solve $f(n) = 2021$ to get $n = 3029$ or 3030. Therefore the answer is $3029 + 3030 = \boxed{6059}$. □

§16.1.7 You just Lost The Game

Source: 2021 Canadian MO #5
Proposer: AiYa#2278 (675537018868072458)
Problem ID: 224
Date: 2021-07-04
Difficulty: Challenging

Tony and Wang play a game. A triple (p, s, c) of nonnegative integers which sum to 42069 is written on the board. Both of them *move*, with Tony going first.

The player who moves chooses a positive integer n and selects one of the three integers on the board. The player then adds n to that integer and subtracts n from each of the other two. A player loses if, right after they move, at least one of the integers on the board becomes negative.

Find the number of triples (p, s, c) for which Wang has a winning strategy.

Solution.

Solution Answer: $\boxed{2187}$ □

§16.2 Week 2

§16.2.1 Honestly, Too Much of a Beginner Problem

Source: Folklore
Proposer: sjbs#9839 (434767660182405131)
Problem ID: 225
Date: 2021-07-05
Difficulty: Beginner

Find the sum of all integers x such that

$$3^{2x} + 5^{2x} = 34 \cdot 15^{x-1}$$

Solution.

We have a quadratic in 3^x and also 5^x. Taking it as a quadratic in 3^x gives $\left(3^x - \frac{17}{15} \cdot 5^x\right)^2 - \left(\frac{15^2-17^2}{15^2}\right) 5^{2x} = 0$, this gives $3^x = \frac{1}{3} \cdot 5^{x+1}$ or $3^x = 3 \cdot 5^{x-1}$. Thus we see the only possible values of x are ± 1. So the answer is $\boxed{0}$ $\square$

§16.2.2 Someone Leaked the Answer and you Still Failed

Source: Original Problem
Proposer: epicxtroll#6007 (300008472978653184)
Problem ID: 226
Date: 2021-07-06
Difficulty: Beginner

Consider 2021 people with ages 20, 40, 60, $\cdots$, 40420. Each person's father (except the 40420-year-old) is the person in the group who is 20 years older than them.

Let N be the number of ways to arrange any 2021 people in a queue such that no person is somewhere behind someone who is their son. Find N.

Solution.

The troll here is realizing that girls exist. If all 2021 people are male, then there is exactly one way to arrange them all, that is, in reversed age order. However, the 20-year-old need not be male; in this case they can go anywhere in the line for a total of 2021 placements. Our answer is $\boxed{2022}$. $\square$

§16.2.3 Non-Chris Geo

Source: Mandelbrot (Year Unknown)
Proposer: Charge#3766(481250375786037258)
Problem ID: 227
Date: 2020-07-07
Difficulty: Easy

Points B' and C' are chosen on side $\overline{BC}$ of $\triangle ABC$ such that $BB' = 1$, $B'C' = 24$, and $CC' = 3$. Given that the circle with diameter $\overline{B'C'}$ is tangent to $\overline{AB}$ and $\overline{AC}$, the perimeter of $\triangle ABC$ is p. Find $\lfloor 100p \rfloor$.

Solution.

Let the circle touch $\overline{AB}$ at P and $\overline{AC}$ at Q. Then, by Power of Point we have $BP^2 = BB' \cdot BC' \iff BP = 5$ and similarly $CQ = 9$. Then let $AP = d = AQ$ and the midpoint of $\overline{B'C'}$ be M; by angle bisector theorem we have

$$\frac{5+d}{9+d} = \frac{13}{15} \iff d = 21$$

so our perimeter is $\boxed{84}$. □

§16.2.4 WA could Solve This and you Failed Again!

Source: 2020 STEMS AP1
Proposer: TaesPadhihary#8557 (665057968194060291)
Problem ID: 228
Date: 2021-07-08
Difficulty: Medium

Given four positive reals b_1, b_2, b_3, b_4 such that $b_1^2 + b_3^2 = b_2^2 + b_4^2$. Positive reals a_1, a_2, a_3, a_4 satisfy $a_i^2 + a_{i+1}^2 = b_i^2$ for $i = 1, 2, 3, 4$ where indices are taken mod 4 (in particular, $a_5 = a_1$). Express the maximum value that $(a_1 + a_3)(a_2 + a_4)$ can take in terms of b_1, b_2, b_3, b_4.

Solution.

We claim the answer is $b_1 b_3 + b_2 b_4 \iff \boxed{1584}$.

Consider quadrilateral $ABCD$ such that $\overline{AC} \perp \overline{BD}$, $\overline{AB} \perp \overline{DA}$, and $\overline{BC} \perp \overline{CD}$; let $AB = b_1, BC = b_2, CD = b_3, DA = b_4$. Then let $\overline{AC}$ intersect $\overline{BD}$ at E, and let $AE = a_1, BE = a_2, CE = a_3, DE = a_4$; the problem constraints are now satisfied and it suffices to maximize $AC \cdot BD$. But this is precisely Ptolemy's Inequality, with $AB \cdot CD + BC \cdot AD = b_1 b_3 + b_2 b_4 \geq AC \cdot BD$ with equality iff $ABCD$ is cyclic.

Remark. Can you figure out what role the constraint $b_1^2 + b_3^2 = b_2^2 + b_4^2$ plays here? □

§16.2.5 d20(20) but not D20

Source: 2021 Mathcamp Application Problem 2e
Proposer: Angry Any#4319 (580933385090891797)
Problem ID: 229
Date: 2021-07-09
Difficulty: Hard

IcosahedralDice has a shapeshifting die. It originally has N sides; however, when rolled, it changes shape to have a number of sides equal to the number rolled. (For example, if he originally has an icosahedral dice and rolls a 12, his die will shapeshift into having 12 sides.)

Brainy and IcosahedralDice are playing a game: the first person to roll one of the numbers $1, 2, \cdots, k$ loses. If IcosahedralDice rolls first, what is the probability he loses?

Solution.

Let p_n denote the probability that person to roll loses when rolling an $n > k$-sided die. IcosahedralDice rolls anything in the range $[1, k]$ with probability $\frac{k}{n}$, instantly losing. He rolls h, where $k < h \leq n$, with probability $\frac{1}{n}$; in this case Brainy will lose with probability p_h, so IcosahedralDice loses with probability $1 - p_h$. Our recursion is then

$$p_n = 1 - \frac{1}{n}\left(p_{k+1} + p_{k+2} + \cdots + p_n\right)$$

which rearranges to

$$(n+1)p_n + (p_{k+1} + p_{k+2} + \cdots + p_{n-1}) = n.$$

Shifting $n \to n+1$ and subtracting the two recursions we get

$$(n+2)p_{n+1} - np_n = 1.$$

This is almost a telescoping product, except the RHS is 1 and not 0. Noticing that the coefficients of p differ by 2, and wanting to get rid of the 1, we shift the opposite of p by $\frac{1}{2}$; that is, let $q_n = \frac{1}{2} - p_n$. Our recursion becomes

$$1 + nq_n - (n+2)q_{n+1} = 1 \iff q_{n+1} = \frac{n}{n+2}q_n \iff q_n = \frac{n-1}{n+1}q_{n-1}$$

and since

$$p_{k+1} = \frac{k}{k+1} + \frac{1}{k+1}\left(1 - p_{k+1}\right) \iff q_{k+1} = \frac{1}{2} - \frac{k+1}{k+2} = -\frac{k}{2(k+2)}$$

we have

$$q_n = -\frac{(n-1)(n-2)\cdots k}{(n+1)n\cdots(k+3)\cdot 2(k+2)} = -\frac{k(k+1)}{2n(n+1)}$$

so our answer is

$$p_N = \frac{1}{2} + \frac{k(k+1)}{2N(N+1)} \iff \boxed{622}.$$

□

§16.2.6 That's Not LTE

Source: Original
Proposer: AiYa#2278 (675537018868072458)
Problem ID: 231
Date: 2021-07-10
Difficulty: Hard

For all nonnegative integers, n, the integer

$$p(n) = \sum_{k=0}^{3n+1} \binom{6n+2}{2k} 3^k$$

can be written in the form $q(n) \cdot 2^{3n+2} + r(n)$, where $q(n)$ is an integer and $r(n)$ is a **positive integer** between 1 and 2^{3n+2} inclusive. Find the last three digits of $\sum_{n=0}^{100} r(n)$.

Solution.

Rewrite our sum as $p(n) = \sum_{k=0}^{3n+1} \binom{6n+2}{2k} \sqrt{3}^{2k}$. This looks like a binomial expansion, with the caveat that only the even binomial coefficients are summed. Using the Binomial Theorem, we see that $\left(1+\sqrt{3}\right)^{6n+2} = \sum_{j=0}^{6n+2} \binom{6n+2}{j} \sqrt{3}^j$ and $\left(1-\sqrt{3}\right)^{6n+2} = \sum_{j=0}^{6n+2} \binom{6n+2}{j} (-1)^j \sqrt{3}^j$; adding the two gives us

$$p(n) = \frac{\left(1+\sqrt{3}\right)^{6n+2} + \left(1-\sqrt{3}\right)^{6n+2}}{2}.$$

Seeing a sum of non-integral terms, LTE fails; we seek a recursion for $p(n)$. Let $x^2 + bx + c$ be the monic polynomial with roots $\left(1 \pm \sqrt{3}\right)^6$. By Vieta, this polynomial is $x^2 - 416x + 64$. Let r, s be the roots; we have

$$r^2 - 416r + 64 = 0 \iff r^{n+2} = 416r^{n+1} - 64r^n$$

which implies

$$r^{n+2} + s^{n+2} = 416\left(r^{n+1} + s^{n+1}\right) - 64\left(r^n + s^n\right)$$

so

$$p(n) = 416p(n-1) - 64p(n-2) = 32\left[13p(n-1) - 2p(n-2)\right].$$

After calculating $p(0) = 4, p(1) = 1552 = 2^4 \cdot 97$ we recurse, finding $p(2) = 2^8 \cdot a, p(3) = 2^{10} \cdot a$ where a is a varying odd integer. A quick induction proves that $p(n) = 2^{3n+2} \cdot a$ if n is even and $2^{3n+1} \cdot a$ if n is odd. Now we sum:

$$\sum_{n=0}^{100} r(n) = \sum_{j=0}^{50} 2^{6j+2} + \sum_{k=0}^{49} 2^{6k+4} = \frac{2^{308} + 2^{304} - 2^4 - 2^2}{63}$$

. That is 4 mod 8; using $\varphi(125) = 100$ and Euler's Theorem we get that the numerator is $2^8 + 2^4 - 2^4 - 2^2 \equiv 2 \pmod{125}$. To find the inverse of 63 mod 125, note that $125 - 2 \cdot 63 = -1 \iff 1/63 \equiv 2 \pmod{125}$ so our sum is $4 \pmod{125}$. Our answer is $\boxed{4}$. □

§16.2.7 Genfun is Not Fun, v2

Source: Folklore
Proposer: tanoshii#3160 (300065144333926400)
Problem ID: 232
Date: 2021-07-11
Difficulty: Challenging

A recursive sequence is defined as follows:

$$a_0 = 1, a_n = \sum_{k=0}^{n-1} \frac{2^{k+1} a_{n-k-1}}{k!n}.$$

Find $\sum_{k=0}^{\infty} a_k$.

Solution.

□

§17 Testsolver's Adventure (Season 17)

§17.1 Week 1

§17.1.1 Indian Remainder Theorem

Source: Unknown
Proposer: TaesPadhihary#8557 (665057968194060291)
Problem ID: 232
Date: 2021-07-19
Difficulty: Beginner

Find the smallest positive integer that leaves a remainder of 3 when divided by each of 4, 5, 6, 7.

Solution.

It's $\boxed{3}$. □

§17.1.2 The Rare non-NJOY Tuesday

Source: Turtle Mock AMC
Proposer: flame#6784 (185229437787176962)
Problem ID: 233
Date: 2021-07-20
Difficulty: Beginner

$\triangle ABC$ has circumcenter O and an area of 360. If $[AOB] = 72, [COA] = 192$, find $[BOC]$.

Solution.

The trick is that $\triangle ABC$ is obtuse, so O lies outside the triangle. This is because in an acute triangle $\triangle XYZ$, O will always be on one side of a median, so the maximum of $[XOY]$ is $[XYZ]/2$. We also know that $\triangle BOC$ cannot be the triangle completely outside $\triangle ABC$ since then $[AOB] + [COA] = [BOCA] > [ABC]$, contradiction. If $\triangle COA$ were the one outside, then we would have $[AOB] + [BOC] - [COA] = [ABC] \iff [BOC] = 480$. But this is impossible, since the area of the portion of $\triangle BOC$ that's outside of $\triangle ABC$ would be strictly greater than $[BOC] - [ABC] = 120 > [COA]$, contradiction since that portion is only part of $\triangle COA$. Thus, we have $[BOC] + [COA] - [AOB] = [ABC] \iff [BOC] = \boxed{240}$. □

§17.1.3 Periodic Function

Source: Folklore
Proposer: sjbs#9839 (434767660182405131)
Problem ID: 234
Date: 2021-07-21
Difficulty: Easy

Find the sum of all integers n for which the function $f : \mathbb{R} \to \mathbb{R}$, such that

$$f(x) = \cos(nx)\sin\left(\frac{5x}{n}\right)$$

for $x \in \mathbb{R}$ is periodic with minimal period 3π

Solution.

We require $f(x + 3\pi) = f(x)$. Observe:

$$\begin{aligned} f(x+3\pi) &= \cos(nx + 3n\pi)\sin\left(\frac{5x}{n} + \frac{15\pi}{n}\right) \\ &= (-1)^n \cos(nx)\sin\left(\frac{5x}{n} + \frac{15\pi}{n}\right) \\ \sin\left(\frac{5x}{n}\right) &= (-1)^n \sin\left(\frac{5x}{n} + \frac{15\pi}{n}\right) \end{aligned}$$

Clearly, for equality to hold, n cannot be even. Thus we consider the case when n is odd.

$$\begin{aligned} \sin\left(\frac{5x}{n} + \frac{15\pi}{n}\right) - \sin\left(\frac{5x}{n}\right) &= 0 \\ 2\sin\left(\frac{\frac{5x}{n} + \frac{15\pi}{n} + \frac{5x}{n}}{2}\right)\cos\left(\frac{\frac{5x}{n} + \frac{15\pi}{n} - \frac{5x}{n}}{2}\right) &= 0 \\ \sin\left(\frac{5x}{n} + \frac{15\pi}{n}\right)\cos\left(\frac{15\pi}{2n}\right) &= 0 \end{aligned}$$

So, the possible values of n are $1, 3, 5, 15$. However, if $n = 1, 5$ then $f(x) = \cos(5x)\sin(x)$ or $\cos(x)\sin(5x)$, both of which have periods π. To see this, note

$$\sin(5x)\cos(x) = \frac{1}{2}\left[\sin(5x + x) + \sin(5x - x)\right] = \frac{\sin(6x) + \sin(4x)}{2}$$

and the periods of $\sin(6x)$ and $\sin(4x)$ are $\frac{\pi}{3}$ and $\frac{\pi}{2}$, respectively, and $\left[\frac{\pi}{3}, \frac{\pi}{2}\right] = \pi$. Our answer is $\boxed{18}$. □

§17.1.4 I don't feel like putting the flavourtexted version up here

Source: 2011 Purple Comet High School 28
Proposer: AiYa#2278 (675537018868072458)
Problem ID: 235
Date: 2021-07-22
Difficulty: Medium

A chain of three circles, each with radius 3, are each externally tangent to their neighbors in the chain and internally tangent to a circle of radius 30. Two circles, each with radius 2, are each externally tangent to two circles in the chain. The distance between the two circles with radius 2 can be expressed in the form $\frac{p\sqrt{q}-r}{s}$, where q is squarefree, p, q, r, s are positive integers, and $\gcd(p, r, s) = 1$. Find $1000p + 100q + 10r + s$.

Solution.

Let O be the centre of the large circle, A be the centre of the central cicle of radius 3, B be the centre of the right circle of radius 3, and C the centre of the circle of radius 2 tangent to A, B. The midpoint M of $\overline{AB}$ is also the point of tangency of the two circles since they're congruent. Let H be the foot of the perpendicular from C to $\overline{OA}$. Then from right triangles, our desired length is $10\sin\angle CAO$. But from right $\triangle OAM$ and $\triangle CAH$ and angle subtraction identity we have

$$\sin\angle CAO = \sin\left(\angle OAM - \angle CAM\right) = \frac{4\sqrt{5}}{9}\cdot\frac{3}{5} - \frac{1}{9}\cdot\frac{4}{5}$$
$$10\sin\angle CAO = \frac{2\left(12\sqrt{5}-4\right)}{9} = \frac{24\sqrt{5}-8}{9} \iff \boxed{24589}.$$

□

§17.1.5 Wang is at the Center of Everything that Happens to Me

Source: 2019 EMLO #2
Proposer: TaesPadhihary#8557 (665057968194060291)
Problem ID: 236
Date: 2021-07-23
Difficulty: Hard

Wang aims to locate the centroid of Tony's convex n-gon. He has no standard ruler or compass, But, Tony gives him a device which can dissect the segment between two given points into k equal parts. What is the least k for which Wang can fulfill his aim?

Solution.

AoPS Solutions □

§17.1.7 MODS Logo

Source: Original
Proposer: Matteddy#0482 (329956567132930048)
Problem ID: 238
Date: 2021-07-25
Difficulty: Challenging

The new MODS logo is an infinitely extended M; that is, a non-self-intersecting polyline made of a ray, two segments, and a ray, in that order (the two outside "legs" of the M extend infinitely but the two segments that make up the middle "valley" don't). What is the maximum number of regions the plane can be partitioned into by 200 MODS logos?

Solution.

The formula in general is $8N^2 - 7N + 1$. Answer: $\boxed{318601}$

We would like to convert this problem into graph form, to use Euler's characteristic formula on. To get rid of the infinite condition, we draw a circle with a sufficiently large enough radius such that the circle intersects only the infinitely long lines, "cutting off" the infinitely long lines there and letting those intersection points with the circle, as well as all other intersection points, be vertices. (For example, for $N = 1$ we have $V = E = 5$.)

Our aim is to minimize $V - E$, or, maximize $E - V$. Suppose there are already N MODS logos; we now add one more. For each new intersection added, we turn two intersecting segments into four; since we added a vertex and two edges $E - V$ goes up by 1. To maximize the number of intersections, we let each of the four segments of the new logo intersect all $4N$ original segments (this is made rigorous by something something lines are continuous) for a total change of $16N$. Now, the new MODS logo itself adds four edges (each line in the M) and five vertices (three for the M's arches and two for the intersection for the big circle) for a total difference of $16N - 1$. When $N = 1$ we have a maximum of two regions; induction finishes it off. $\square$

§17.2 Week 2

§17.2.1 The Difficulty of an IMO Training Camp

Source: 2010 NZ Camp
Proposer: AiYa#2278 (675537018868072458)
Problem ID: 239
Date: 2021-07-26
Difficulty: Beginner

Circles ω_1 of radius 20 and ω_2 of radius 21 are externally tangent at P. Points A and B, both distinct from P, lie on ω_1 and ω_2, respectively, such that $AB = 42$ and A, P, B are collinear. Find $\lfloor 100PA \rfloor$.

(A scientific calculator may be used.)

Solution.

Connect the center of ω_1 to A and P and the center of ω_2 to B and P. Then, we have two isosceles triangles with equal base angles (the angles at P are vertical angles) so

$$\frac{AP}{BP} = \frac{r_1}{r_2} = \frac{20}{21} \iff AP = 42 \cdot \frac{20}{41} \iff \boxed{2048}.$$

□

§17.2.2 The Rare non-NJOY Tuesday

Source: Original Problem
Proposer: epicxtroll#6007 (300008472978653184)
Problem ID: 240
Date: 2021-07-27
Difficulty: Beginner

For an integer n, let $d(n)$ be the sum of integer divisors of n. Find the sum of all $1 < n < 100$ such that n maximizes $d(n)$.

Solution.

If $d|n$ then $-d|n$, so $d(n) = 0$ for all n. Our answer is $\boxed{4949}$. □

§17.2.3 Partial Fractions is only a Partial Solution

Source: Mandelbrot (Year Unknown)
Proposer: Charge#3766(481250375786037258)
Problem ID: 241
Date: 2020-07-28
Difficulty: Easy

Find the sum of all real solutions to

$$\sum_{k=-30}^{14} \frac{4x}{(x+k)(x+k+1)} = 9.$$

Solution.

We proceed with partial fraction decomposition. Note that

$$\frac{1}{x+k} - \frac{1}{x+k+1} = \frac{1}{(x+k)(x+k+1)}$$

so our sum is

$$4x\left(\frac{1}{x-30} - \frac{1}{x+15}\right) = 9 = \frac{4x \cdot 45}{(x-30)(x+15)}$$

and the quadratic boils down to

$$(x-30)(x+15) = 20x \iff x^2 - 35x - 450 = 0 = (x-45)(x+10)$$

but $x = -10$ makes the denominator of $\frac{4x}{(x+10)(x+11)}$ zero and so is not a valid solution. Our answer is $\boxed{45}$. □

§17.2.4 Who had the Courage to Coordbash This?

Source: Unknown
Proposer: TaesPadhihary#8557 (665057968194060291)
Problem ID: 242
Date: 2021-07-29
Difficulty: Medium

$\triangle ABC$ has orthocentre H, circumcentre O, and foot of the A-altitude T. Given that $\triangle ABC$ is isosceles with area 2021, $[ABC]$ can be expressed in the form $\frac{a\sqrt{b}}{c}$, where a, c are relatively prime positive integers and b is squarefree. Find $10000a + 100b + c$.

Solution.

Through angle chasing, we find that $\angle BHC = 180 - A$ so the reflection H' of H over T lies on the circumcircle of $\triangle ABC$. Let M be the midpoint of $\overline{BC}$ and $2s$ be the sidelength of $\triangle ABC$; then $TM = s\sqrt{3}, BT = BM - s\sqrt{3}, CT = BM + s\sqrt{3}$ (WLOG BT is the shorter side). Using Power of a Point we have

$$BT \cdot CT = H'T \cdot AT = BM^2 - 3s^2 = 2sAT.$$

From Law of Cosines on $\triangle HOA$ we have

$$R^2 = BM^2 + s^2 = (AT - 2s)^2 + (2s)^2 + (2s)(AT - 2s) \iff BM^2 = AT^2 - 2sAT + 3s^2$$

so we are left with

$$2sAT = AT^2 - 2sAT \iff AT = 4s \iff BM = s\sqrt{11} \iff [ABC] = \frac{8084\sqrt{33}}{3} \iff \boxed{80843303}.$$

□

§17.2.5 "not hard enough for a weekend"

Source: 2021 OTIE #15
Proposer: TaesPadhihary#8557 (665057968194060291)
Problem ID: 243
Date: 2021-07-30
Difficulty: Hard

How many positive integers $1 \le n \le 2020^4 - 17 \cdot 2020^2 + 17$ satisfy $n^{2n} \equiv 1 \pmod{2021}$, and of those, how many satisfy $n^n \equiv -1 \pmod{2021}$?

Solution.

Note that modulo 2021 has period 2021, and by Euler's Theorem modular exponentiation has period $\varphi(2021) = 1932$; in other words, the sequence $n^n \pmod{2021}$ has period $2021 \cdot 1932$ (or at least the n coprime to 2021, which is what really matters). That upper bound has a nice form. Replacing 2020 with x, that factors to

$$x^4 - 17x^2 + 17 = x^4 - 17x^2 + 16 + 1 = (x+1)(x-1)(x+4)(x-4) + 1;$$

it's not hard to see that those four factors are divisible by $2021 \cdot 1932$.

Lemma. For a given prime p and multiplicative order d, there are exactly $\varphi(d)$ residues with order d. This can be seen by writing residues in the form of g^k, where g is a generator mod p; then d is the smallest positive integer such that $g^{kd} \equiv 1 \pmod p \iff (p-1)|kd \iff h(p-1) = kd$ for some integer h. Note that $(h,d) = 1$; otherwise dividing through by (h,d) would yield a smaller $d' = d/(h,d)$ such that $(p-1)|kd'$. We have $k = \frac{h(p-1)}{d}$, and since (h,d) and $h \ge d$ (since $k < p$), there are $\varphi(h)$ such pairs in total.

We work mod p, where p is one of $43, 47 \equiv 3 \pmod 4$. Fix an order d; we find the probability that $d|2n$ and $\mathrm{ord}_p(n) = d$. Since the first concerns n's behavior mod $p-1$ (since $d|(p-1)$) and the second concerns n's behavior mod p and $(p, p-1) = 1$ these events are independent. The second is much easier to deal with: there are $\varphi(d)$ residues with order d, and p total residues for a probability of $\frac{\varphi(d)}{p}$. We now case on the parity of d. If d is odd, then $d|2n \iff d|n$ which happens with probability $\frac{1}{d}$; if d is even then write $d = 2d'$ and $d|2n \iff d'|n$ which happens with probability $\frac{1}{d'}$. Since $(43, 47) = 1$ the behavior of n^n mod each of those are independent. With $(x-1)(x+1)(x-4)(x+4)$ evaluating out to $2021 \cdot 1932 \cdot 2019 \cdot 2112$ and $1^n \equiv 1 \pmod{2021}$ our total number of n such that $n^{2n} \equiv 1 \pmod{2021}$ is

$$2019 \cdot 2112 \cdot 1932 \left(\sum_{d|46, 2\nmid d} \frac{2\varphi(d)}{d} \right) \left(\sum_{d|42, 2\nmid d} \frac{2\varphi(d)}{d} \right) + 1 = \boxed{199561190401}.$$

Now we claim that there is an $8-1$ correspondence between integers n such that $n^{2n} \equiv 1 \pmod{2021}$ and n such that $n^n \equiv -1 \pmod{2021}$. We must have $d|2n$ but $d \nmid n$, implying that d is even and n is odd. Now, as we have shown earlier, there are an equal amount of n such that $n^{2n} \equiv 1 \pmod{2021}$ with even and odd orders. We claim that for any given order d there are equally many odd and even n such that $n^{2n} \equiv 1 \pmod{2021}$. Consider a residue r with order d. We case on the parity of d: if d is odd then the all such n such that both $n \equiv r \pmod p$ (guaranteeing the same order) and $d|n$ (guaranteeing $n^{2n} \equiv 1 \pmod p$) are of the form $n \equiv r \pmod{\mathrm{lcm}[p,d]}$. But $\mathrm{lcm}[p,d]$ is clearly odd so $\frac{2021 \cdot 1932}{\mathrm{lcm}[p,d]}$ is even, meaning that there are the same amount of even and odd such n. If d is even then write it as $d = 2d'$; note that we only need $d|2n \iff d'|n$ and the proof proceeds as above. Thus, there are three conditions: $2|\,\mathrm{ord}_{43}(n), 2|\,\mathrm{ord}_{47}(n), 2|n$. Thus, there are $\frac{199561190400}{8} = \boxed{24945148800}$ such n.

□

§17.2.6 Fat Tan

Source: Original
Proposer: tanoshii#3160 (300065144333926400)
Problem ID: 244
Date: 2021-07-31
Difficulty: Challenging

2021 distinct rooms are arranged in a circle, each with 4 doorways: two leading to adjacent rooms, one leading to the courtyard, and one leading to the outside. However, these doors are one-way doors (so you can go from Room A to the courtyard but not from the courtyard to Room A through the same door). Find the number of ways to orient the doors such that one can get from any room to any other room.

Solution.

Solve for general n. We convert this into graph format, with $n+2$ vertices, representing each of n rooms, the courtyard, and the outside; and $3n$ edges, representing the one-way doors; and $2n$ faces. The end result should look like the graph G of a neural network with one input node, one output node, and one hidden layer containing n nodes, with the addition that all n hidden nodes are connected in a (undirected) cycle; another visualization is the gluing of two n-gonal pyramids together at their bases. Vertices u and v are connected by a directed edge $u \to v$ iff one can walk from u to v through exactly one door (again, the courtyard and the outside count as rooms).

G isn't all too useful. The condition of G being strongly conncted is that G must have a cycle decomposition such that each cycle must be strongly connected, but that's hardly useful, either. Note that G is planar (curve the edge between the topmost and bottommost hidden nodes around eithe one of the start or end nodes, then draw the rest of the edges as common sense dictates); this means it has a dual graph G'. We direct the edges of said dual graph as follows: let u be an edge of the original graph, and u' the edge of the dual graph that intersects u. Then, rotate u counterclockwise until it coincides with the vertices of u'; that is the direction of u'.

We claim that G is strongly connected iff G' contains no directed cycles. Suppose that G' contained a cycle; this cycle "surrounds" some subset S of vertices in G. Now, because the edges in the cycle go either all clockwise or all counterclockwise, this means that all the edges going from S to $G \setminus S$ either are directed into S or out of it. Conversely, if there exists a subset S of vertices in G such that it is impossible to transverse from S to $G \setminus S$, then all the edges from S to $G \setminus S$ either point all in or all out of S, meaning that there is a cycle in G'. (The motivation here was from the board game *weiqi*; think of black stones as vertices in S and white stones the edges separating S from $G \setminus S$.)

Let $A(G)$ be the number of ways to direct the edges of G such that there are no directed cycles in G afterwards. We claim that $A(G) = (-1)^V \chi(G, -1)$, where V is the number of vertices in G and $\chi(G, n)$ is the chromatic polynomial of G. ($\chi(G, n)$ counts the number of ways to colour the vertices of G with n colours such that no two adjacent vertices share a colour.) More generally, if $A(G, n)$ is the number of ways to:

1. label the vertices of G each with an integer between 1 and n inclusive such that if $u \to v$ is a directed edge, then the label assigned to u is no less than the label assigned to v (akin to "ordering" the colours such that some colours are worth more than others)
2. direct the edges of G such that G is acyclic

then $A(G, n) = (-1)^V \chi(G, -n)$. Recall that $\chi(G, n)$ satisfies the following recurrence: $\chi(G, n) = \chi(G \setminus e, n) - \chi(G/e, n)$, where e is an edge, $G \setminus e$ is the graph of G but with e removed, and G/e is the graph of G with the end vertices of e shrunk down into one vertex. Since the values of $A(G, n)$ and $\chi(G, -n)$ are the same at "trivial" graphs (like the one-vertex graph, and combinations of two disconnected graphs) it suffices to prove the recurrence $A(G,n) = A(G \setminus e, n) + A(G/e, n)$. Consider the graph $G \setminus e$. If this graph contains a broken cycle such that adding e would complete the cycle, then adding e in the opposite direction constitutes a valid orienting of G; however in exactly $A(G/e, n)$ of these graphs there is no broken cycle that would be completed by adding e, so in these graphs both directions of e work. □

Solution. [Write-up by QuantumSigma#5000 (374937087301910528)]

The regions and doorways can be modelled as faces and directed edges of a prism, where one can go from region A to B if the corresponding edge points clockwise w.r.t. to face A.

Suppose there is a directed cycle, then one can only cross from one side of it to the other, thus not fulfilling the criteria of being able to reach any region from any other region. On the contrary, if there are no cycles, then for any nonempty strict subset S of faces, one can always find a face $\notin S$ reachable from some face $\in S$. By selecting a set $\{F\}$ of any face and continuously adding reachable faces to it, one can conclude that any face can be reached from any other face. Thus the answer is the number of ways to direct the edges of a 2021-sided prism to form an acyclic graph.

Let $f(G)$ be the number of ways to direct the edges of an undirected graph G to form an acyclic graph. Let $G\backslash e$ and G/e denote the removal and contraction (removal of edge and merging of 2 vertices) of edge e respectively. Note that $f(G) = 0$ if G contains a loop (edge connecting vertex to itself), and the removal of an edge doesn't change the value of f if there is another edge connecting the same 2 vertices. For some directed graph G with non-loop edge e, suppose reversing the direction of e changes whether G is acyclic, then $G\backslash e$ is acyclic and G/e is cyclic. Suppose G is acyclic/cyclic regardless of the orientation of e, then both G/e and $G\backslash e$ are acyclic/cyclic. Thus we have $f(G) = f(G/e) + f(G\backslash e)$.

Let Q_s denote an undirected graph in the shape of an $s-1$ by 1 grid of squares, containing vertices (x,y) for all pairs of integers $1 \le x \le s$, $0 \le y \le 1$. Let $G+e$ denote the addition of edge e and $G[A,B]$ the merging of vertices A, B. Observe $f(Q_1) = 2$ and

$$\begin{aligned} f(Q_{n+1}) &= f\left(Q_{n+1}\backslash\overline{(n+1,0)(n+1,1)}\right) + f\left(Q_{n+1}/\overline{(n+1,0)(n+1,1)}\right) \\ &= 4f(Q_n) + (2f(Q_n) + f(Q_n)) \\ &= 7f(Q_n) \end{aligned}$$

Therefore, $f(Q_n) = 2 \cdot 7^{n-1}$. Notice $f(Q_2[(1,0),(2,0)]) = 0$, $f(Q_2[(1,0),(2,1)]) = 4$, and

$$\begin{aligned} &f(Q_{n+1}[(1,0),(n+1,0)]) \\ &= f\left(Q_n + \overline{(1,0)(n,0)} + \overline{(n,1)(n+1,1)}\right) + f\left(Q_n + \overline{(1,0)(n,0)} + \overline{(1,0)(n,1)}\right) \\ &= 3f\left(Q_n + \overline{(1,0)(n,0)}\right) + f(Q_n[(1,0),(n,1)]) \\ &= 6 \cdot 7^{n-1} + 3f(Q_n[(1,0),(n,0)]) + f(Q_n[(1,0),(n,1)]) \end{aligned}$$

$$\begin{aligned} &f(Q_{n+1}[(1,0),(n+1,1)]) \\ &= f\left(Q_n + \overline{(1,0)(n,1)} + \overline{(n,0)(n+1,0)}\right) + f\left(Q_n + \overline{(1,0)(n,0)} + \overline{(1,0)(n,1)}\right) \\ &= 3f\left(Q_n + \overline{(1,0)(n,1)}\right) + f(Q_n[(1,0),(n,0)]) \\ &= 6 \cdot 7^{n-1} + 3f(Q_n[(1,0),(n,1)]) + f(Q_n[(1,0),(n,0)]) \end{aligned}$$

Thus, $f(Q_n[(1,0),(n,0)]) = \frac{1}{28}(8 \cdot 7^n - 7 \cdot 2^n(3 \cdot 2^n + 2))$. Finally, $f\left(Q_2 + \overline{(1,0)(2,0)} + \overline{(1,1)(2,1)}\right) = 14$ and

$$\begin{aligned}
& f\left(Q_n + \overline{(1,0)(n,0)} + \overline{(1,1)(n,1)}\right) \\
&= f\left(Q_n + \overline{(1,0)(n,0)}\right) + f\left(Q_n + \overline{(1,0)(n,0)}[(1,1),(n,1)]\right) \\
&= f(Q_n) + 2f(Q_n[(1,0),(n,0)]) + f\left(Q_{n-1} + \overline{(1,0)(n-1,0)} + \overline{(1,1)(n-1,1)}\right) \\
&= 2 \cdot 7^{n-1} + \frac{1}{14}(8 \cdot 7^n - 7 \cdot 2^n(3 \cdot 2^n + 2)) + f\left(Q_{n-1} + \overline{(1,0)(n-1,0)} + \overline{(1,1)(n-1,1)}\right)
\end{aligned}$$

Therefore, $f\left(Q_n + \overline{(1,0)(n,0)} + \overline{(1,1)(n,1)}\right) = 7^n - 2^{2n+1} - 2^{n+1} + 5$. It follows that the answer is the last 10 digits of $7^{2021} - 2^{4043} - 2^{2022} + 5$, which is $\boxed{7972067500}$. $\square$

§17.2.7 I had brain(y) drain after solving W2P6

Source: italian team competition
Proposer: Matteddy#0482 (329956567132930048)
Problem ID:
Date:
Difficulty: Hard

Big Brainy Chess is played on a four-dimensional n-by-n-by-n-by-n board. A Brainy Knight moves as follows: it moves two cells in any of the eight cardinal directions, then moves one cell in a direction perpendicular to the first one. What is the probability that a pair of randomly chosen cells can have a knight placed in one of them which can reach the other in a single move?

(A scientific calculator may be used.)

Solution. [Write-up by QuantumSigma#5000 (374937087301910528)]

A Brainy Knight, just like a regular knight, moves to a square that is distance $\sqrt{5}$ away from its current position, consisting of two translations of $\pm 1, \pm 2$ in two different directions. If board boundaries are not considered, there are 48 moves a Brainy Knight can make, corresponding to the number of 4-tuples of integers with two zeros, one of them either ± 1 and the last one either ± 2. Each Brainy Knight will have coordinates $(w, x, y, z) \in \{0, 1, \cdots, n-1\}^4$. Let a be the number of w, x, y, z that are either 0 or $n-1$, and b the number of w, x, y, z that are either 1 or $n-2$. There are a number of illegal moves a Brainy Knight can make with these restrictions:

1. one translation illegally affects an $a-$coordinate while the other is normal
2. one translation illegally affects a $b-$coordinate while the other is normal
3. both an $a-$ and a $b-$coordinate are illegally affected
4. two $a-$coordinates are illegally affected.

(Think why two $b-$coordinates getting illegally affected is not a case here.) In the first case, there are a possible illegal coordinates that cannot have a translation of -1 or -2, while the $4-a-b$ normal coordinates can have any translation; on the other hand if the other translation affects a $b-$coordinate then the pairs $(a, b) : (-2, -1); (-2, 1); (-1, 2)$ are all fine; lastly if the other translation affects an $a-$coordinate then the pairs $(a_1, a_2) : (-1, 2); (-2, 1)$ are fine too, for a total of $a[16 - 4a - 4b + 3b + 2(a-1)] = a(14 - 2a - b)$ pairs. Similarly, in the second case there are b possible illegal coordinates that cannot have a translaiton of -2, while the $3-a$ normal coordinates can have ± 1 and the $a-$-coordinates can have the pair $(a, b) : (1, -2)$ for a total of $b(6-a)$ pairs. The third count is simple, with only the pair $(a, b) : (-1, -2)$ illegally affecting both for a total of ab. The fourth count is $a(a-1)$, since both translations must be negative. In total, there are

$$
\begin{aligned}
14a - 2a^2 - ab + 6b - ab + ab + a^2 - a &= 12a - 2a(a-1) - ab + 6b + a(a-1) \\
&\iff 48 - 12a - 6b + ab + a(a-1)
\end{aligned}
$$

squares a Brainy Knight can go to. On the other hand, there are two choices for the value of $a-$ and $b-$coordinates, $n-4$ choices for the value of the normal coordinates, and $\binom{4}{a}\binom{4-a}{b}$ ways to choose in which dimensions the $a-$ and $b-$coordinates go. So there are a total of

$$
\sum_{a+b \leq 4} 2^{a+b}(n-4)^{4-a-b}\binom{4}{a}\binom{4-a}{b}[48 - 12a - 6a + a(a+b-1)]
$$

pairs a Brainy Knight's jump away from each other. □

§18 OpenPOTD Games (Season 18)

§18.1 Week 1

§18.1.1 Don't overthink this pt.3

Source: Unknown
Proposer: TaesPadhihary#8557 (665057968194060291)
Problem ID: 246
Date: 2021-08-02
Difficulty: Beginner

Given that a, b, c are the roots of $P(x) = x^3 - x^2 - a^2x + 2021$, find $(45 - a)(45 - b)(45 - c)$.

Solution.

Plugging $x = a$ into the polynomial, we get

$a^2 = 2021 \iff P(x) = x^2 - x^2 - 2021x + 2021 = (x-a)(x-b)(x-c) \iff P(45) = (45-a)(45-b)(45-c) = \boxed{176}$.

□

§18.1.2 You overthought this, didn't you?

Source: Unknown
Proposer: TaesPadhihary#8557 (665057968194060291)
Problem ID: 247
Date: 2021-08-03
Difficulty: Beginner

There are three eight-digit positive integers, two of which are 24678051 and 88593477, such that each are equal to the sum of the eighth powers of their digits. Find the third.

Solution.

It's $\boxed{24678050}$, since that's one less than 24678051 and 0^8 is one less than 1^8. □

§18.1.3 Oh no, I drew the short...ball?

Source: Mandelbrot (Year Unknown)
Proposer: Charge#3766(481250375786037258)
Problem ID: 248
Date: 2021-08-04
Difficulty: Easy

In the MODS Lottery, 50 balls are drawn at random from $n > 100$ balls numbered 1 through n. If the probability that no pair of consecutive numbers is drawn is equal to the probability that exactly 1 pair is drawn, find n.

Solution. [Write up by Charge#3766(481250375786037258)]

Since the probabilities are equal, we have that the number of ways each is possible to be equal.
We first count the number of drawings that no pair of consecutive numbers are drawn. We can add 0 and $n+1$ to the list for ease. So we have some $a_0 + a_1 + \cdots + a_{50} = n - 50$ where the plus signs represent the numbers drawn. But since all a_1 to a_{49} need to be positive, we subtract 49 from $n - 50$. So we are left with $a_0 + a_1' + \cdots + a_{49}' + a_{50} = n - 49 - 50$. There are $\binom{n-49}{50}$ ways to do this.
For the second case, there are $n - 49$ "slots" that we can put our numbers in. We can choose one of these "slots" and put 2 of our 50 balls in them. Then we have $n - 50$ "slots" remaining for our 48 balls. So we have $(n-49) \cdot \binom{n-50}{48}$.
Then setting the two equal we have

$$\frac{(n-49)!}{50!(n-99)!} = \frac{(n-49)!}{48!(n-98)!}$$

$$50 \cdot 49 = n - 98$$

$$n = \boxed{2548}$$

□

§18.1.4 Finally, a Non-Guessable FE

Source: 2009 Turkey TST #1
Proposer: AiYa#2278 (675537018868072458)
Problem ID: 249
Date: 2021-08-05
Difficulty: Medium

Find all functions $f : \mathbb{Q}^+ \to \mathbb{Z}$ such that

$$f\left(\frac{1}{x}\right) = f(x) \quad \text{and} \quad (x+1)f(x-1) = xf(x)$$

for all rationals $x > 1$.

Solution.

We claim that the only solution is $f\left(\frac{m}{n}\right) = \frac{f(1)(m+n)}{2}$, where $m \geq n$ are relatively prime positive integers and $f(1)$ is even. We proceed by strong induction on $m+n$; it is trivial when $m+n=2 \iff m,n=1$. Then, rearrange to note that $f(x) = \frac{(x+1)f(x-1)}{x}$ and then $x-1 = \frac{m-n}{n}$. $\square$

§18.1.5 Chris, but Hardly Geo

Source: Unknown
Proposer: ChristopherPi#8528 (696497464621924394)
Problem ID: 250
Date: 2021-08-06
Difficulty: Hard

Two congruent 2021-gons are placed such that a pair of their edges touch each other. A pencil is attached to one of the vertices, and that 2021-gon is rotated around the other. Each time their two edges perfectly meet again, the pencil dots the page, and a line is drawn connecting the two most recent dots. Find the area of this shape the pencil traces out.

Solution.

Killed by complex shoelace. Sad :c □

§18.1.6 answer extraction why

Source: Original
Proposer: TaesPadhihary#8557 (665057968194060291)
Problem ID: 251
Date: 2021-08-07
Difficulty: Challenging

Find all ordered triples (a, b, c) of positive integers such that $a + b + c = 2020$ and there exist nonzero reals x, y, z such that

$$x^2 + y^2 = axy, y^2 + z^2 + byz, z^2 + x^2 = czx.$$

Solution.

If $a+b+c = N$, the valid (a, b, c) are such that $abc = (N+2)^2$. I can't find my scratch work, so this'll have to suffice for now. Answer: $\square$

Alternatively you can use some obscure trig identity from *Problems from the Book*. $\square$

§22 Season '22 (Season 22)

§22.1 Week 1

§22.1.1 Don't overthink this pt.4

Source: Unknown
Proposer: TaesPadhihary#8557 (665057968194060291)
Problem ID: 290
Date: 2022-01-03
Difficulty: Beginner

How many ways are there to arrange 20 white balls and 22 black balls in a line such that no two black balls are next to each other?

Solution.

$\boxed{0}$ ways. Place balls from left to right: every time we place a black ball, a white must follow right after. But there are only 20 whites, so there must exist a black-black ball pair. □

§22.1.2 No Trig Identities necessary

Source: 2016 HMMT Guts #3
Proposer: AiYa#2278 (675537018868072458)
Problem ID: 291
Date: 2022-01-04
Difficulty: Beginner

Regular octagon $QUESTION$ is inscribed in a unit circle. Diagonals $\overline{UO}$ and $\overline{EN}$ meet at D. Find DT.

Solution.

Note that $\overline{UO}$ and $\overline{EN}$ are bisectors of $\angle QOE$ and $\angle QEO$ respectively, so D is the incentre of $\triangle QEO$. Using $A = rs$ we get

$$DA = r = \frac{\sqrt{2}^2}{2+2\sqrt{2}} = \sqrt{2}-1 \iff DT = DA + AT = \boxed{\sqrt{2}}.$$

□

§22.1.3 Pascal's Fractions

Source: USAMTS Y30R3P2
Proposer: AiYa#2278 (675537018868072458)
Problem ID: 292
Date: 2022-01-05
Difficulty: Easy

Lizzie writes a list of fractions as follows. First, she writes $\frac{1}{1}$, the only fraction whose numerator and denominator add to 2. Then she writes the two fractions whose numerator and denominator add to 3, in increasing order of denominator. Then she writes the three fractions whose numerator and denominator sum to 4 in increasing order of denominator. She continues in this way until she has written all the fractions whose numerator and denominator sum to at most 1000. So Lizzie's list looks like:

$$\frac{1}{1}, \frac{2}{1}, \frac{1}{2}, \frac{3}{1}, \frac{2}{2}, \frac{1}{3}, \ldots, \frac{1}{999}.$$

Let p_k be the product of the first k fractions in Lizzie's list. Find, with proof, the value of $p_1 + p_2 + \cdots + p_{499500}$.

Solution.

Solution. In general, if this list of fractions continues indefinitely, and the sum goes to T_n instead of 499500, the answer is $\boxed{2^{n+1} - (n+2)}$.

□

§22.1.4 Closet

Source: 1996 Korean Math Olympiad #3
Proposer: A Hong Kong Math Lover#7962 (823884866671476736)
Problem ID: 293
Date: 2022-01-06
Difficulty: Medium

n is a positive integer satisfying $\lfloor\sqrt{n}\rfloor = 69$. If

$$S = \sum_{k=1}^{n} \left\lfloor\sqrt{k}\right\rfloor,$$

find n such that the last three digits of S is closest (in absolute value) to 420.

Solution.

We separate the sum up into two parts: where $k < 69^2$ and $k \geq 69^2$. When $k^2 < 69$, the sum is fixed:

$$\sum_{k=1}^{69^2-1} \left\lfloor\sqrt{k}\right\rfloor = \sum_{k=1}^{68} k\left[(k+1)^2 - k^2\right] = \sum_{k=1}^{68} 2k^2 + k,$$

where the second expression comes from the fact that for each k, there are $(k+1)^2 - k^2$ integers h, namely $k^2, k^2+1, \cdots, (k+1)^2 - 1$, such that $\left\lfloor\sqrt{h}\right\rfloor = k$. The last sum can be easily evaluated with well-known formulas to be $614 \pmod{1000}$. Now, suppose $n = 69^2 - 1 + m$, so there m integers such that $k \geq 69^2$, so $S \equiv 69m + 614 \pmod{1000}$. It would be real nice if $S \equiv 420 \pmod{1000}$ directly, which implies $m = 374$. However, there are only $70^2 - 69^2 = 139$ integers with floor of square root 69, so 374 is way too large. Instead, note that $69 \cdot 29 \equiv 1 \pmod{1000}$, so decreasing m by 69 decreases m by 1. Thus, if we keep decreasing m by 29 until $m \leq 139$ we'll get a suitable S. Indeed, $374 - 29 \cdot 9 = 113 < 139$ is the best we can do. Our answer is then $69^2 - 1 + 113 = \boxed{4873}$.

Remark. We found $69^{-1} \mod 1000$, i.e. the integer a such that $69a \equiv 1 \pmod{1000}$. Why didn't we find a such that $69a \equiv -1 \pmod{1000}$? Then, decrementing m by a would increase S by 1, achieving the same result. So why not?

□

§22.1.5 Object Class: Safe

Source: Unknown
Proposer: ChristopherPi#8528 (696497464621924394)
Problem ID: 294
Date: 2022-01-07
Difficulty: Hard

You're trying to open a peculiar lock. The lock consists of a circular ring of n equally-spaced indistinguishable buttons, which you can see, that covers a ring of n pins, which you cannot see. Each button covers a pin, and pushing a button toggles the pin underneath it between *engaged* and *disengaged*. The lock opens when all n pins are disengaged.

You can attempt to open the lock by simultaneously pressing any number of buttons. If all pins are disengaged as a result, the lock opens. Otherwise, the ring of pins rotates to an arbitrary position; the buttons might cover different pins, but the state of each pin is retained.

As complicated as this lock may seem, for certain values of n it is always possible to open such a lock, no matter the initial state of the lock. Find all such values of n, and find the most number of attempts you have to make to guarantee that the lock will open.

Solution.

n must be a $\boxed{\text{power of 2}}$. The worst-case scenario requires $\boxed{2^n - 1}$ attempts.

Experiment with small values of n first. $n = 1$ is trivial: just press the button. For $n = 2$, the lock could start with either all pins engaged, or only one. To rule out the all pins engaged case, press both buttons; if you fail then that means there is exactly one pin engaged. In this case, press one button; if you still fail then that means you engaged a disengaged pin (otherwise you would've unlocked it). Thus, pressing both buttons guarantees an unlock. As you can see, the worst-case scenario required $2^2 - 1 = 3$ attempts, which goes through all possible locked pin states.

After trying with $n = 3$ for a while, you'll probably fail to find a way to solve the cases where either one or two pins are engaged. Indeed, $n = 3$ is impossible. Note that since n is odd, for every attempt there exists a pair of adjacent pins that are either both toggled or both not toggled. (If not, it means you were able to push every other button, implying that n is even.) If, for every attempt you make, this pair has one pin engaged and one pin disengaged, you will never be able to open the lock, since there's no way to match the pin states of that pair. The following is such a scenario: such a mismatched pair exists, and due to bad luck you always engage both pins of that pair during each attempt you make. This argument extends to all n such that n has an odd factor d: choose d equally spaced pins, and define adjacency by ignoring all unchosen pins.

The same idea of pin-pairing gets us an algorithm for unlocking $n = 2^k$. We've already seen the algorithm for $n = 2^1$, and can extend it to $n = 2^2$ as follows. Imagine separating the pins into two pairs, then matching the state of pins in each pair. Then, the lock reduces down to the $n = 2$ case. However, we have no way of knowing whether each pair is matched or not, so we have to go through all $2^{n/2} = 4$ states of pair-matchedness, trying the $n = 2$ algorithm each time. To match or unmatch a pair, push exactly one button of the pair; thus, going through all pair-matchedness states is the same as unlocking the $n = 2$ lock, since for the $n = 4$ case there are two pins we need to toggle to go through all pair-matchedness states and the $n = 2$ unlocking algorithm goes through all pin states. But we can't just pair up adjacent buttons, as those would get confused under rotation. Instead, pair up diametrically opposite buttons, as those stay invariant under rotation. The algorithm is as follows:

- Pair up diametrically opposite buttons.
- Apply the $n = 2$ unlocking algorithm.
- Apply the first move in the $n = 2$ unlocking algorithm.
- Apply the $n = 2$ unlocking algorithm.
- Apply the second move in the $n = 2$ unlocking algorithm.
- Apply the $n = 2$ unlocking algorithm.
- Apply the third move in the $n = 2$ unlocking algorithm.
- Apply the $n = 2$ unlocking algorithm.

This algorithm can be extended; for $n = 2^k$ replace 2 with 2^{k-1} and the same applies. Note that this algorithm has a worst-case of $2^n - 1$. Indeed, this is the best we can do: since we'll never be able to tell the exact state of any pin, the best we can do is systematically iterate through every possible pin state, of which there are $2^n - 1$. $\square$

§22.1.6 10$ to anyone who can gf this

Source: 2010 USA TST #8
Proposer: AiYa#2278 (675537018868072458)
Problem ID: 295
Date: 2022-01-08
Difficulty: Challenging

Let S be the sum of all ordered n-tuples $(a_1, a_2, \cdots, a_n)$ such that $a_1 + a_2 + \cdots + a_n = m$ for $m \geq n$. Find a "nice" sum (so, one simple summation) for

$$\sum_S \prod_{k=1}^{n} k^{a_k}.$$

Solution.

Consider the number of functions $f : \{1, 2, \cdots, m\} \to \{1, 2, \cdots, n\}$ such that for every $1 \leq p \leq n$ there exists a $q \in [1, m]$ such that $f(q) = p$; such functions are called *surjections*. We claim our sum counts all such surjections. Let $s_k = a_1 + a_2 + \cdots + a_k$, and $(\sigma_1, \sigma_2, \cdots, \sigma_n)$ be a permutation of $(1, 2, \cdots, n)$. We impose the following restrictions: for each $1 \leq k < n$, the output of q such that $s_{k-1} < q < s_k$ must be restricted to $\{\sigma_1, \sigma_2, \cdots, \sigma_k\}$ (treat $s_0 = 1$); and for each $1 \leq k \leq n$ we must have $f(s_k) = \sigma_k$. There are a total of $\prod_{k=1}^{n} k^{a_k - 1}$ such surjections; now multiply through by $n!$ to account for all possible permutations σ. Indeed, each surjection corresponds to a n-tuple $(a_1, a_2, \cdots, a_n)$. List out the sequence $(f(1), f(2), \cdots, f(m))$, and read it from left to right:

1. Let $j, k = 1$.
2. Look at $f(k)$.
3. If $f(k) \neq f(h)$ for all $h < k$, let $s_j = k$ and increment j by 1.
4. Increment k by 1.
5. If $k \leq n$ return to step 2.

It remains to count the number of such surjections. There are n^m total functions. We count the number of functions with range of size no more than $n - 1$. There are $\binom{n}{n-1}(n-1)^m$ functions with a codomain of size $n - 1$. However, subtracting off just that would be overcounting functions with a codomain of size less than $n - 1$; for $1 \leq k < n$ there are $\binom{n}{n-k}(n-k)^m$ functions with a codomain of size $n - k$. Using inclusion-exclusion, our sum is

$$\boxed{\sum_{k=0}^{n-1} (-1)^k \binom{n}{k} (n-k)^m}$$

Remark. The answer extraction asked for the first and last few digits of the sum. We actually only need WolframAlpha to get the first few digits, since $n!$ is a factor of each term of the sum, so the sum will have lots of leading zeros at the end. □

§22.1.7 Spectral Snakes

Source: Original
Proposer: Cryolite.#3854 (413599141537513473)
Problem ID: 296
Date: 2022-01-09
Difficulty: Challenging

A sequence is defined to be k-*snakelike* if it consists of integers 1 through k inclusive and the absolute difference between consecutive terms is at most 1. Let $N(n,k)$ be the number of k-snakelike sequence of length n. Find

$$\lim_{n\to\infty} N(n,k)^{1/n}.$$

Solution.

Let $N(n,k,d)$ be the number of k-snakelike sequences of length n with first term d. We get the following recurrences:

$$\begin{aligned}
N(n,k,1) &= N(n-1,k,1) + N(n-1,k,2)\\
N(n,k,2) &= N(n-1,k,1) + N(n-1,k,2) + N(n-1,k,3)\\
&\cdots\\
N(n,k,k-1) &= N(n-1,k,k-2) + N(n-1,k,k-1) + N(n-1,k,k)\\
N(n,k,k) &= N(n-1,k,k-1) + N(n-1,k,k)
\end{aligned}$$

Let $\vec{v}_n$ represent the column vector $(N(n,k,1), N(n,k,2), \cdots, N(n,k,k))$. Since this is a system of linear recurrences, we can express it in the form $\vec{v}_{n+1} = M\vec{v}_n$ for some k-by-k matrix M. In our case, the matrix is

$$\begin{bmatrix}
1 & 1 & 0 & 0 & \cdots & 0\\
1 & 1 & 1 & 0 & \cdots & 0\\
 & \ddots & & & & \\
0 & \cdots & 0 & 1 & 1 & 1\\
0 & \cdots & 0 & 0 & 1 & 1
\end{bmatrix}$$

where the main diagonal and the two diagonals directly above and below it are all 1's, and zeros everywhere else. By spectral theorem, M is diagonalizable. Since $N(n,k) = \sum_{j=1}^{k} N(n,k,j) = \vec{v}_n \cdot \vec{1}$ (where $\vec{1}$ is the appropriately-sized all-ones vector) we have

$$\lim_{n\to\infty} N(n,k)^{1/n} = \left(M^{n-1}\vec{v}_1 \cdot \vec{1}\right)^{1/n} = \left(PD^nP^{-1}\vec{v}_1 \cdot \vec{1}\right)^{1/n}.$$

That last expression evaluates to some linear combination of the n-th powers of the eigenvalues of M; as the limit goes to infinity only the largest eigenvalue contributes, with all coefficients going to 1 as $1/n$ approaches 0.

It remains to find the largest eigenvalue of M. The way we did answer extraction, using an iterative algorithm probably would've worked if we allowed programming languages. Doing a quick online search also yields the complete spectrum for path graphs, as $M - I$ is the adjacency matrix for the path graph, so M's eigenvalues are simply the path's plus 1. Our answer is then $\boxed{1 + 2\cos\left(\frac{\pi}{k+1}\right)}$.

For the sake of completion, and to provide some motivation for that PDF's pretty confusing explanations, we find the spectra of both C_n and P_n, the cycle and path graphs, respectively. Observe $A\left(C_n\right)\vec{v}=\lambda\vec{v}:$ if $\vec{v}=\left(v_1,v_2,\cdots,v_n\right)$ then we have the following system:

$$\begin{cases} v_n+v_2 & =\lambda v_1 \\ v_1+v_3 & =\lambda v_2 \\ \quad\cdots & \\ v_{n-1}+v_1 & =\lambda v_n \end{cases}$$

Adding this all up gives $\lambda\vec{v}\cdot\vec{1}=2\vec{v}\cdot\vec{1}$, so either $\lambda=2$ or $\vec{v}\cdot\vec{1}=0$. The former case is boring, so we focus on the latter. Noting the symmetry of the adjacency matrix of C_n and the fact that C_n can be represented as a regular n-gon, we do a bit of wishful thinking: we conjecture that the v_i's are the n-th roots of unity; indeed, $\vec{v}=\left(\omega^0,\omega^1,\cdots,\omega^{n-1}\right)$ is an eigenvector of C_n where ω is a primitive n-th root of unity. Multiplying through we get $\lambda=\omega+\omega^{-1}$. The remaining eigenvalues follow from replacing ω with $\omega^k, 0<k<n$.

We did C_n first because P_n is C_n, just with the top-right and bottom-left 1's flipped to 0's. Setting up the same system of equations and adding, we get $\lambda\vec{v}\cdot\vec{1}=2\vec{v}\cdot\vec{1}-\left(v_1+v_n\right)$. Now, this is where we do a **lot** of wishful thinking: we wish for $v_1+v_n=0$, so that the v_i's are some sort of roots of unity. After all, they should be; P_n and C_n are very similar. To that end, we try using eigenvectors of C_m (for some $m\neq n$) to try and guess at an eigenvector of P_n. We notice that simple roots of unity won't work: after multiplying by P_n, the resultant vector will have one term in v_1, v_n but two terms everywhere else. So, we try linear combinations of two eigenvectors of C_m.

Note that $\vec{v}=\left(\omega^1,\cdots,\omega^n\right)$ and $\vec{u}=\left(\omega^{-1},\cdots,\omega^{-n}\right)$ correspond to the same eigenvalue, $\omega+\omega^{-1}$, where ω is now a primitive m-th root of unity. A linear combination $\vec{w}$ of these two vectors will have two terms in w_1, w_n but four everywhere else, with $\left(\lambda\vec{w}\right)_i=\left(\omega+\omega^{-1}\right)w_i$. To check out w_1, we recall the difference of squares $\omega^2-\omega^{-2}=\left(\omega+\omega^{-1}\right)\left(\omega-\omega^{-1}\right)$ so $\vec{w}=\vec{v}-\vec{u}$ seems like a good guess. Indeed, all entries except for w_n check out immediately. Now $w_n=\omega^{n+1}-\omega^{-(n+1)}+\omega^{n-1}-\omega^{-(n-1)}$ and I'd like the first two terms to cancel each other out, implying $\omega^{2(n+1)}=1$. If $\omega^{n+1}=1$, we would have eigenvalues $2\cos\left(\frac{2\pi k}{n+1}\right), k\neq 0$. However, the sum of these eigenvalues is -2, whereas the trace of P_n (or any graph for that matter), which is also the sum of eigenvalues, is 0, contradiction. Thus $\omega^{n+1}=-1$ and our eigenvalues are $2\cos\left(\frac{\pi k}{n+1}\right), 0<k\leq n$.

A less guesswork-based approach to find the spectrum of P_n is also possible. Simply consider the cofactor expansion of $\det\left(\lambda I-P_n\right)=p_n(\lambda)$, the characteristic polynomial of P_n. Doing cofactor expansion on the first row, we see that $C_{1,1}$ is simply $p_{n-1}(\lambda)$, while $C_{1,2}$ is the determinant of a matrix with the first column being a pivot column (although the first entry is a -1 not a 1); doing cofactor expansion on the first column gives us p_{n-2}. Putting it together, we have the recurrence

$$p_n(\lambda)=\lambda p_{n-1}(\lambda)-p_{n-2}\lambda.$$

For example, we have $p_2(\lambda)=\lambda^2-1$ and $p_3(\lambda)=\lambda^3-2\lambda$. The recurrence, and $p_2(\lambda)$ and $p_3(\lambda)$, remind us of the Chebyshev recurrence; indeed, letting $\lambda=2x$ we obtain $p_n(\lambda)=U_n(x)$, where $U(x)$ are the Chebyshev polynomials of the second kind. Then, the roots of $p_n(\lambda)$ are two times the roots of $U_n(x)$, which are easily found. $\square$

§22.2 Week 2

§22.2.1 i feel like we've done the exact same trick before

Source: Original
Proposer: TaesPadhihary#8557 (665057968194060291)
Problem ID: 297
Date: 2022-01-10
Difficulty: Beginner

Find the smallest integer n such that $n^6 \leq 42^9$.

Solution.

We find $n = \pm 42\sqrt{42}$. To approximate $\sqrt{42}$, we could brute-force, using $65^2 = 4225$ and working our way down from $n = 274$ to find $n = 272$ as the largest integer solution. Since $n^6 = (-n)^6$, our answer is $\boxed{-272}$.

However, a better way to approximate is to use the binomial theorem:

$$42^{1/2} = (36+6)^{1/2} = \sum_{k=0}^{\infty} \binom{1/2}{k} 36^{1/2-k} 6^k = \sum_{k=0}^{\infty} \binom{1/2}{k} 6^{1-k}.$$

Just the first three terms gives us $\sqrt{42} \approx 6 + 1/2 - 1/48 = 6 + 23/48$, so

$$42\sqrt{42} \approx 42 \cdot 6 + \frac{23 \cdot 7}{8} = 252 + \frac{23 \cdot 8 - 23}{8} = 272 + \frac{1}{8}.$$

□