Optimal Transport Networks in Spatial Equilibrium*

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Supplementary Material (Authors' website only)

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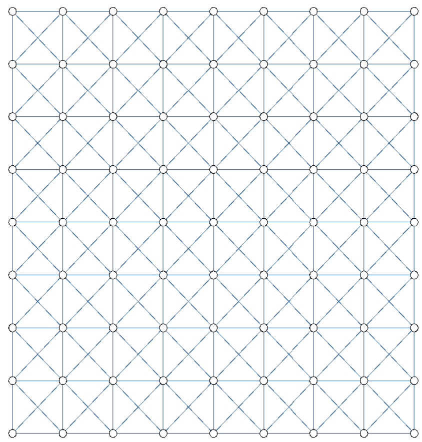
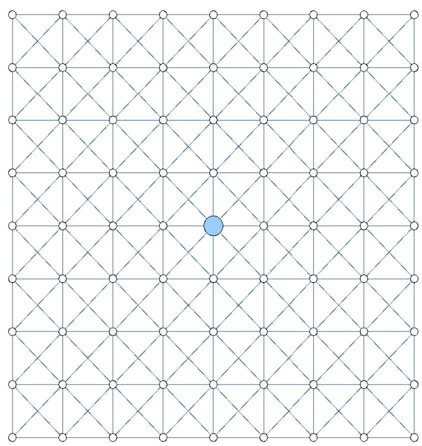
A Appendix to Section 4 (Illustrative Examples)

Figure A.3: A Simple Underlying Geography

(a) Population



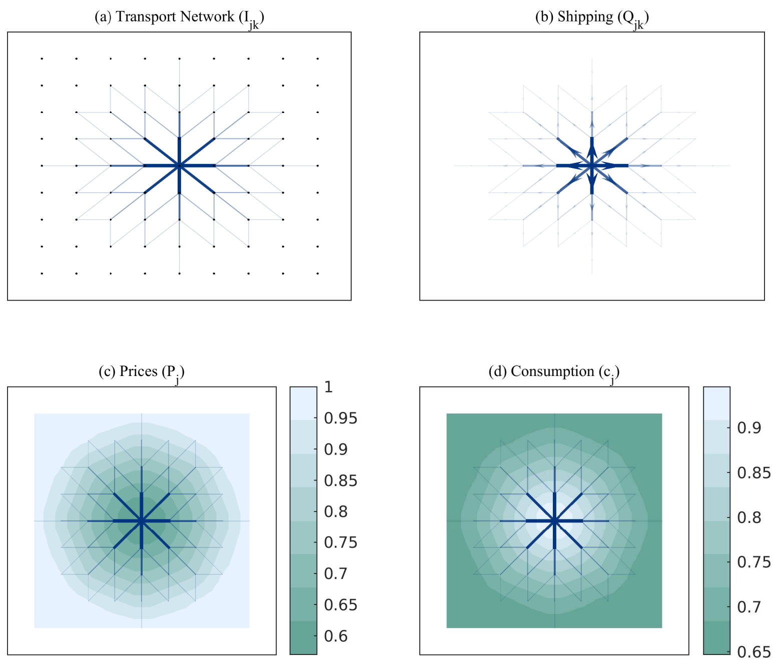
(b) Productivity



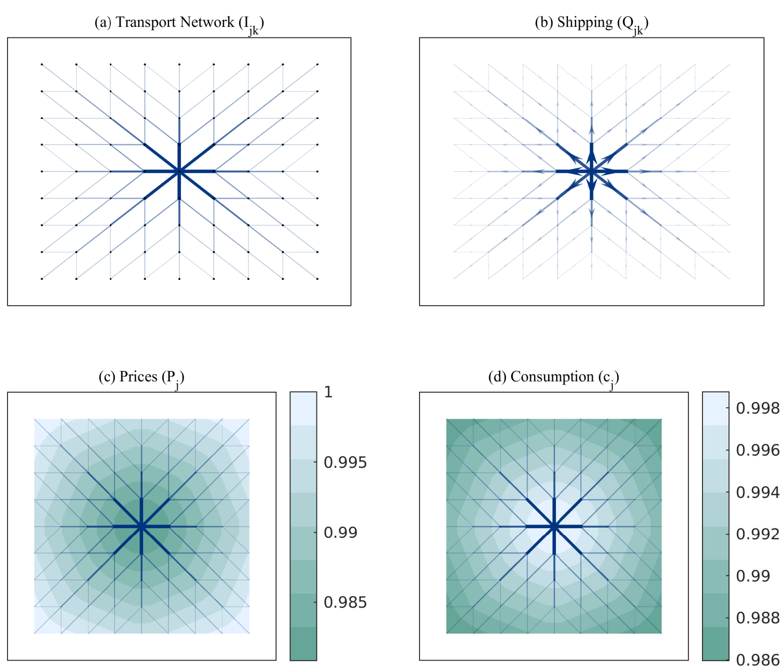
Notes: On panel (a), each circle represents a location. The links represent the underlying network, i.e., links upon which the transport network may be built. Population and housing are uniform across space, normalized to 1. On panel (b), the size of the circles represent the productivity of each location.

Figure A.4: The Optimal Network for $K = 1$ and $K = 100$

(a) K=1



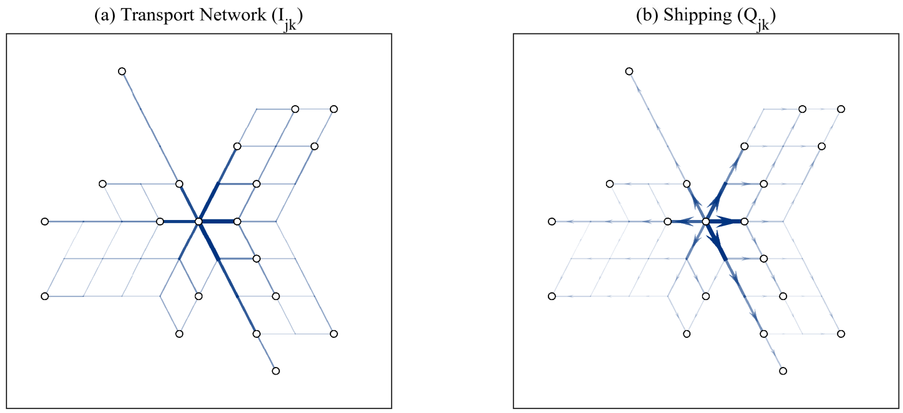
(b) K=100



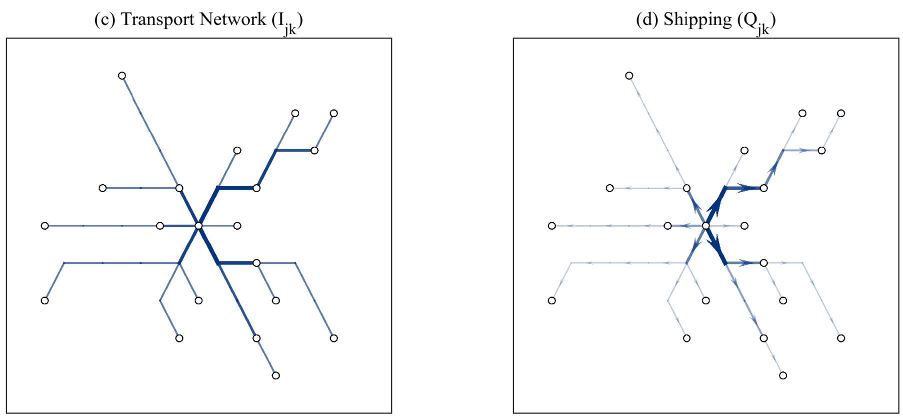
Notes: On each panel, the thickness and color of the segments reflects the level of infrastructure built on a given link. Thicker and darker colors represent more infrastructure. On the bottom panels, the heat map represents the level of prices and consumption, normalized to 1 at the center. Lighter color represents higher values for prices and consumption. Prices and consumption levels are linearly interpolated across space to obtain smooth contour plots.

Figure A.5: Optimal Network with Randomly Located Cities

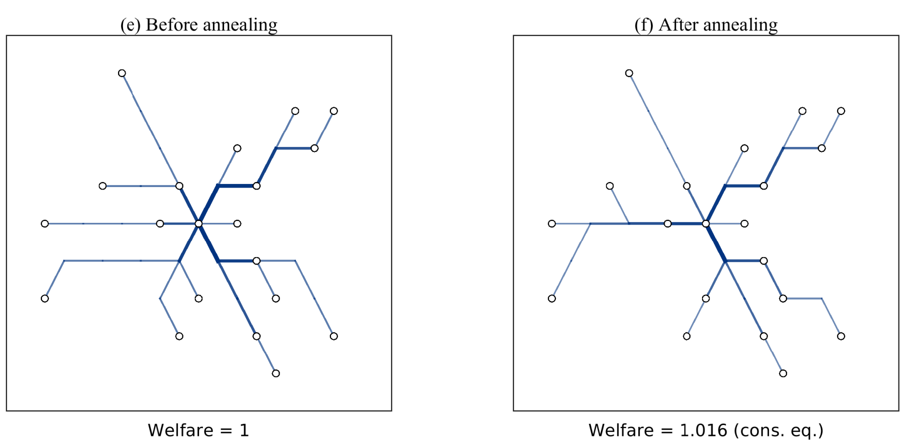
(a) Convex Case: $\gamma = \beta = 1$



(b) Non-Convex Case: $\gamma = 2 > \beta = 1$

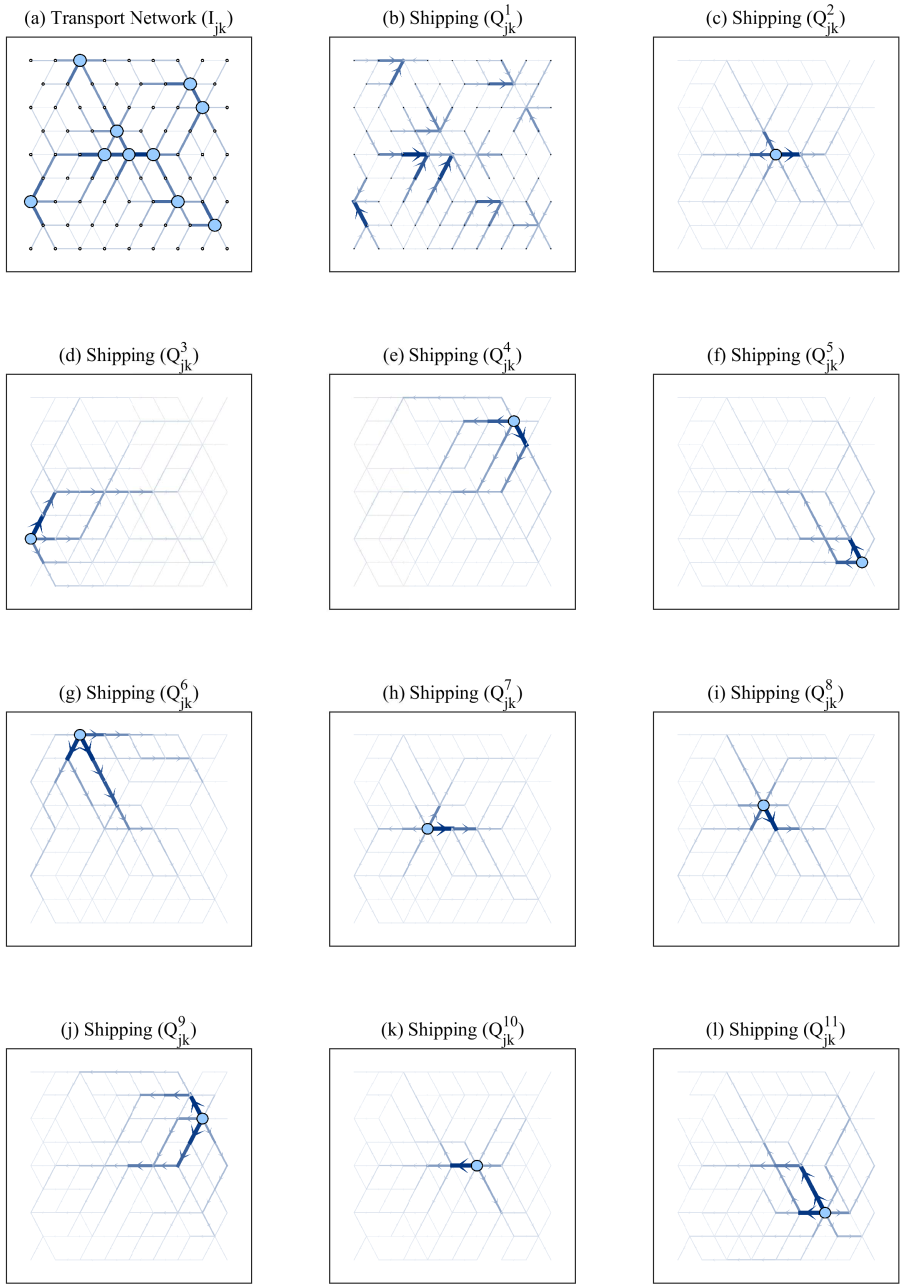


(c) Optimal Network Before and After Annealing Refinement in Non-Convex Case



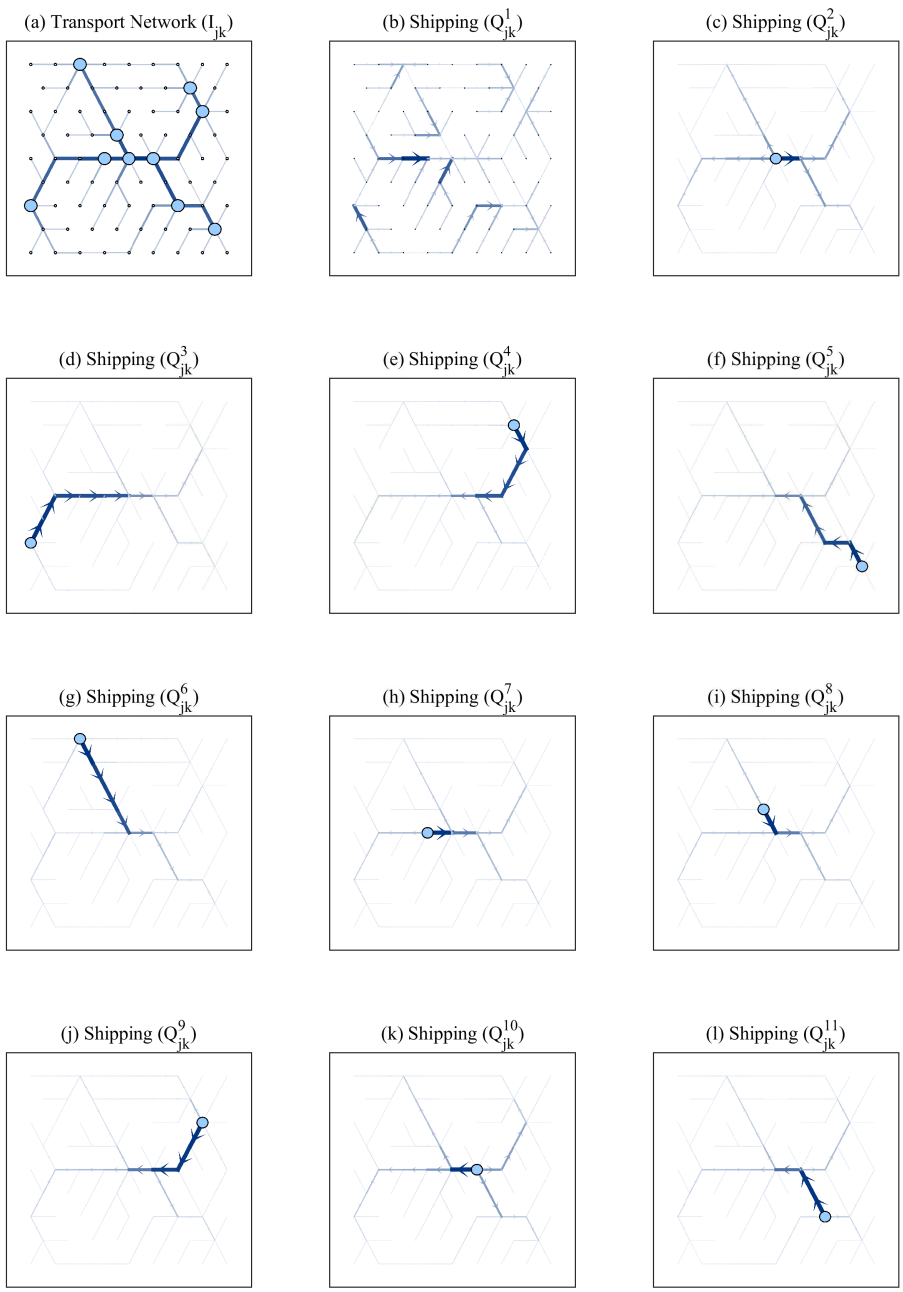
Notes: On each panel, the thickness and color of the segments reflects the level of infrastructure built or the shipment sent on a given link. Thicker and darker colors represent higher infrastructure or quantity.

Figure A.6: Optimal Network with 10+1 Goods, Convex Case ($\beta = \gamma = 1$), Labor Mobility



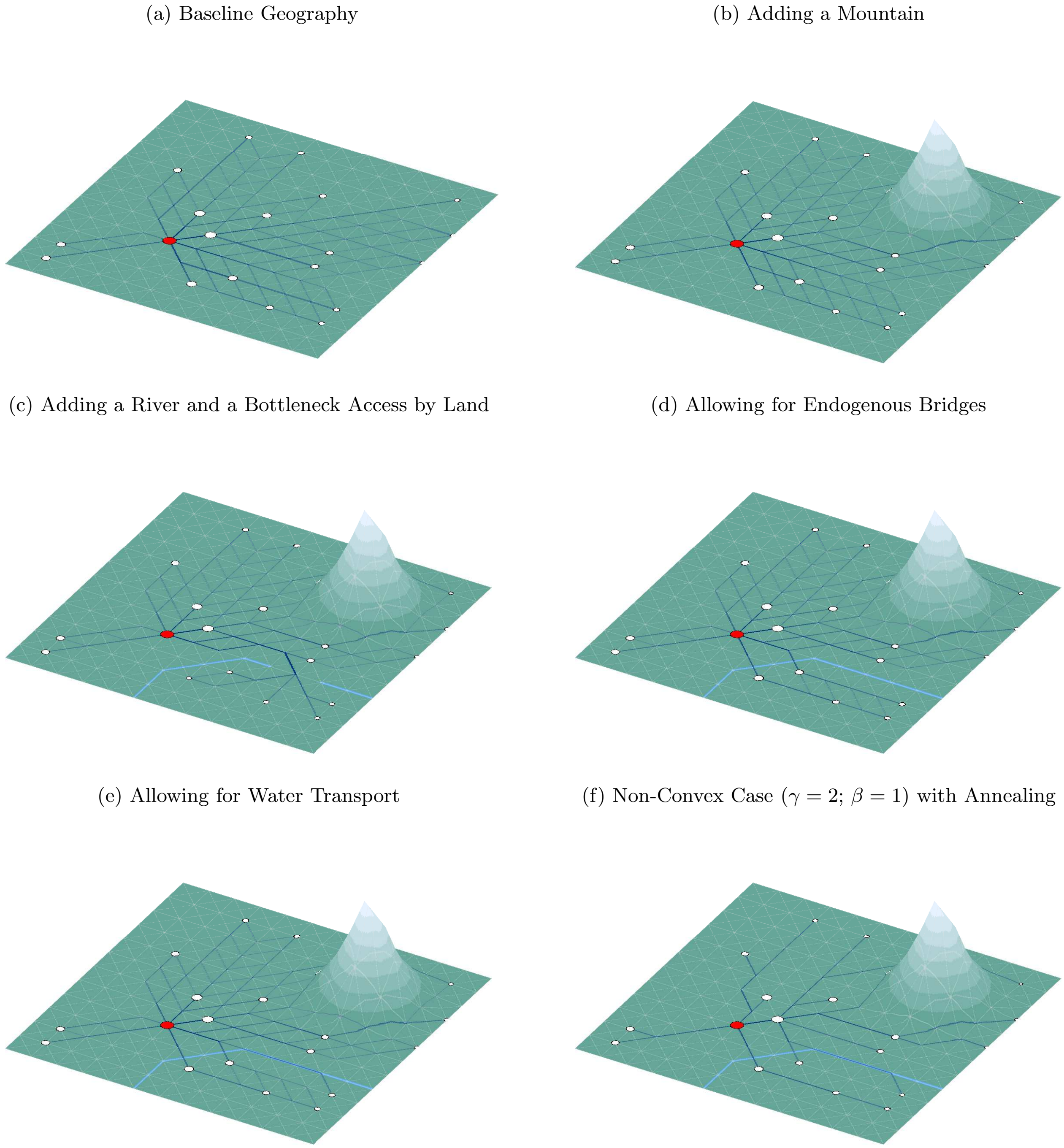
Notes: On panel (a), the thickness and color of the segments reflects the level of infrastructure built on a given link, and the size of each circle is the population share. On the other panels, the segments represent the quantity shipped through each link and the circles represent the location of producers.

Figure A.7: Optimal Network with 10+1 Goods, Nonconvex Case ($\beta = 1$,$\gamma = 2$), Labor Mobility



Notes: On panel (a), the thickness and color of the segments reflects the level of infrastructure built on a given link, and the size of each circle is the population share. On the other panels, the segments represent the quantity shipped through each link and the circles represent the location of producers. This figure represents a local optimum.

Figure A.8: The Optimal Transport Network under Alternative Building Costs



(a) Baseline Geography

(b) Adding a Mountain

(c) Adding a River and a Bottleneck Access by Land

(d) Allowing for Endogenous Bridges

(e) Allowing for Water Transport

(f) Non-Convex Case ($\gamma = 2$; $\beta = 1$) with Annealing

Notes: The thickness and color of the segments reflect the level of infrastructure built on a given link. Thicker and darker colors represent more infrastructure and quantities. The circles represent the 20 cities randomly allocated across spaces. The larger red circle represents the city with the highest productivity. The different panels vary in the parametrization of the cost of building infrastructure. In panel (a), it is only a function of Euclidean distance. In panel (b), we add a mountain and assume that the cost also depend on difference in elevation. In panel (c), we add a river with a natural land crossing and assume that the cost of building along or across the river is infinite. In panel (d) there is no natural land crossing but allow for construction of bridges. In panel (e) we additionally allow for investment in water transport. Panel (d) makes the assumptions as Panel (e) but assumes increasing returns to network building.

B Numerical Implementation

In this section, we provide a more detailed explanation of the numerical algorithms we use to solve the model. A Matlab toolbox implementing our model with detailed documentation and a few examples is available on the authors' websites.[46]

B.1 Resolution method

Convex case and duality approach

As explained in section 3.6, our preferred approach to solve the model relies on solving the dual Lagrangian problem of the planner. We provide, here, a simple example of how to solve the joint optimal allocation and transport problem taking the infrastructure network $\{I_{jk}\}$ as given. This example can easily be generalized to the full problem, including the network design problem, in the convex case, but is also part of our resolution method for the nonconvex case. We focus on the case studied in the quantitative part of the paper, in which: i) we use the log-linear specification of transport costs, $\tau_{jk}^n = \delta_{jk}^\tau \left(Q_{jk}^n\right)^\beta I_{jk}^{-\gamma}$ (own-good congestion, $\chi = 0$) or $\tau_{jk}^n = m^n \delta_{jk}^\tau \left(Q_{jk}\right)^\beta I_{jk}^{-\gamma}$ with $Q_{jk} = \sum_{n=1}^N m^n Q_{jk}^n$ (cross-good congestion, $\chi = 1$); ii) labor is the sole production factor, $F_j^n\left(L_j^n\right) = z_j^n \left(L_j^n\right)^a$; and iii) C^T is a CES aggregator with elasticity of substitution σ. We consider the case with immobile labor.[47]

We write the Lagrangian of the problem

$$\begin{aligned}
\mathscr{L} = & \sum_j \omega_j L_j U\left(c_j, h_j\right) - \sum_j P_j^D \left[c_j L_j + \chi \sum_{k \in \mathcal{N}(j)} \delta_{jk}^\tau \left(\sum_{n=1}^N m^n Q_{jk}^n\right)^{1+\beta} I_{jk}^{-\gamma} - \left(\sum_n \left(D_j^n\right)^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}}\right] \\
& - \sum_j \sum_n P_j^n \left[D_j^n + \sum_{k \in \mathcal{N}(j)} \left(Q_{jk}^n + (1-\chi)\,\delta_{jk}^\tau \left(Q_{jk}^n\right)^{1+\beta} I_{jk}^{-\gamma}\right) - z_j^n \left(L_j^n\right)^a - \sum_{i \in \mathcal{N}(j)} Q_{ij}^n\right] \\
& - \sum_j W_j \left[\sum_n L_j^n - L_j\right] + \sum_{j,k,n} \zeta_{jkn}^Q Q_{jk}^n + \sum_{j,n} \zeta_{jn}^L L_j^n + \sum_{j,n} \zeta_{jn}^C D_j^n + \sum_j \zeta_j^c c_j .
\end{aligned}$$

Recall that the dual problem consists of solving

$$\inf_{\lambda \geq \mathbf{0}} \sup_{\mathbf{x}} \mathscr{L}\left(\mathbf{x}, \boldsymbol{\lambda}\right).$$

We start by expressing our control variables $\mathbf{x} = \left(c_j, D_j^n, Q_{jk}^n, L_j^n\right)$ as functions of the Lagrange multipliers $\boldsymbol{\lambda} = \left(P_j^D, P_j^n, W_j, \zeta_{jkn}^Q, \zeta_{jn}^L, \zeta_{jn}^C, \zeta_j^c\right)$. Using the optimality conditions, one obtains the following expressions:

$$c_j = U_c^{-1}\left(\omega_j^{-1}\left(\sum_{n'}\left(P_j^{n'}\right)^{1-\sigma}\right)^{\frac{1}{1-\sigma}}, h_j\right)$$

$$D_j^n = \left[\frac{P_j^n}{\left(\sum_{n'}\left(P_j^{n'}\right)^{1-\sigma}\right)^{\frac{1}{1-\sigma}}}\right]^{-\sigma}\left(c_j L_j + \chi \sum_{k \in \mathcal{N}(j)} \delta_{jk}^\tau \left(\sum_{n=1}^N m^n Q_{jk}^n\right)^{1+\beta} I_{jk}^{-\gamma}\right)$$

$$L_j^n = \frac{\left(P_j^n z_j^n\right)^{\frac{1}{1-a}}}{\sum_{n'}\left(P_j^{n'} z_j^{n'}\right)^{\frac{1}{1-a}}} L_j .$$

[46] Latest version available at https://sites.google.com/site/edouardschaal/OptimalTransportNetworkToolbox.zip.

[47] In the mobile labor case, we can only show that the planner's problem is a quasi-convex optimization problem. Hence, a duality gap may exist. We therefore adopt a (slower) primal approach in that case.

In the case of own-good congestion ($\chi = 0$), the flows can be easily inverted,

$$Q_{jk}^n = \left[\frac{1}{1+\beta}\frac{I_{jk}^\gamma}{\delta_{jk}^\tau}\max\left(\frac{P_k^n}{P_j^n} - 1, 0\right)\right]^{\frac{1}{\beta}},$$

but in the case of cross-good congestion, one first needs to invert the aggregate flows $Q_{jk} = \sum_{n=1}^N m^n Q_{jk}^n$, which is sufficient to evaluate the Lagrangian at any point,[48]

$$Q_{jk} = \max_n \left(\frac{P_k^n - P_j^n}{(1+\beta) P_j^D m^n \delta_{jk}^\tau I_{jk}^{-\gamma}}\right)^{\frac{1}{\beta}}.$$

As these expressions illustrate, we have been able to eliminate a large number of the multipliers directly, so that the only remaining Lagrange multipliers are $\boldsymbol{\lambda} = \left(P_j^n\right)_{j,n}$. We may now compute the inner part of the saddle-point problem:[49]

$$\begin{aligned}\mathscr{L}\left(\mathbf{x}\left(\boldsymbol{\lambda}\right), \boldsymbol{\lambda}\right) = &\sum_j \omega_j L_j U\left(c_j\left(\boldsymbol{\lambda}\right), h_j\right) - \sum P_j^D \left(c_j\left(\boldsymbol{\lambda}\right) L_j + \chi \sum_{k\in\mathcal{N}(j)} \delta_{jk}^\tau Q_{jk}^{1+\beta}\left(\boldsymbol{\lambda}\right) I_{jk}^{-\gamma}\right) \\ &- \sum_j \sum_n P_j^n \left[D_j^n\left(\boldsymbol{\lambda}\right) + \sum_{k\in\mathcal{N}(j)} \left(Q_{jk}^n\left(\boldsymbol{\lambda}\right) + (1-\chi)\delta_{jk}^\tau \left(Q_{jk}^n\left(\boldsymbol{\lambda}\right)\right)^{1+\beta} I_{jk}^{-\gamma}\right) - z_j^n \left(L_j^n\left(\boldsymbol{\lambda}\right)\right)^a - \sum_{i\in\mathcal{N}(j)} Q_{ij}^n\left(\boldsymbol{\lambda}\right)\right].\end{aligned}$$

The dual problem then consists of the simple unconstrained, convex minimization problem in $J \times N$ unknowns:[50]

$$\min_{\boldsymbol{\lambda} \geqslant 0} \mathscr{L}\left(\mathbf{x}\left(\boldsymbol{\lambda}\right), \boldsymbol{\lambda}\right).$$

This problem can be readily fed into numerical optimization software. Faster convergence can be achieved by providing the software with an analytical gradient and hessian. Note that, as a direct implication of the envelope theorem, the gradient of the dual problem is simply the vector of constraints:

$$\nabla \mathscr{L}\left(\mathbf{x}\left(\boldsymbol{\lambda}\right), \boldsymbol{\lambda}\right) = -\begin{pmatrix} \vdots \\ C_j^n\left(\boldsymbol{\lambda}\right) + \sum_{k\in\mathcal{N}(j)} \left(Q_{jk}^n\left(\boldsymbol{\lambda}\right) + \delta_{jk}^\tau \left(Q_{jk}^n\left(\boldsymbol{\lambda}\right)\right)^{1+\beta} I_{jk}^{-\gamma}\right) - z_j^n \left(L_j^n\left(\boldsymbol{\lambda}\right)\right)^a - \sum_{i\in\mathcal{N}(j)} Q_{ij}^n\left(\boldsymbol{\lambda}\right) \\ \vdots \end{pmatrix}.$$

Nonconvex cases

When the conditions for convexity fail to obtain, the full planner's problem is not a convex optimization problem. It is, however, easy to find local optima by using the following iterative procedure. We then search for a global maximum using a simulated annealing method that we describe below.

Finding Local Optima Despite the failure of global convexity for the full planner's problem, the joint optimal allocation and transport problems, taking the network as given, is always convex as long as $\beta \geqslant 0$. We thus use our duality approach to solve for $\left(c_j, D_j^n, Q_{jk}^n, L_j^n\right)$ for a given level of infrastructure I_{jk}, and then iterate on the (necessary) first order conditions that characterize the optimal network. The procedure can be summarized in pseudo-code as follows.

[48] The specific values Q_{jk}^n can be recovered at the end of the optimization by inverting the linear system (for given multipliers $\boldsymbol{\lambda}$) which corresponds to the balanced-flows constraints, the constraints $Q_{jk}\left(\boldsymbol{\lambda}\right) = \sum_{n=1}^N m^n Q_{jk}^n$ and the complementary slackness conditions, $\zeta_{jkn}^Q Q_{jk}^n = 0$.

[49] Note that, due to complementary slackness, we can drop the constraints that correspond to all the Lagrange multipliers that we were able to solve by hand. As a result, only the balanced flows constraints remain.

[50] Dual problems are always convex, by construction, even when the primal problem is not.

1. Let $l := 1$. Guess some initial level of infrastructure $\left\{I_{jk}^{(1)}\right\}$ that satisfies the network building constraint.

2. Given the network $\left\{I_{jk}^{(l)}\right\}$, solve for $\left(c_j, D_j^n, Q_{jk}^n, L_j^n\right)$ using a duality approach.

3. Given the flows Q_{jk}^n and the prices P_j^n, get a new guess $I_{jk}^{(l+1)} = \left[\frac{\gamma}{P_K}\frac{\delta_{jk}^\tau}{\delta_{jk}^I}\left(P_j^D Q_{jk}^{1+\beta}\right)\right]^{\frac{1}{1+\gamma}}$ for $\chi = 1$ (or $I_{jk}^{(l+1)} = \left[\frac{\gamma}{P_K}\frac{\delta_{jk}^\tau}{\delta_{jk}^I}\left(\sum_n P_j^n \left(Q_{jk}^n\right)^{1+\beta}\right)\right]^{\frac{1}{1+\gamma}}$ for $\chi = 0$) and set P_K such that $\sum \delta_{jk}^I I_{jk}^{(l+1)} = K$.

4. If $\sum_{j,k}\left|I_{jk}^{(l+1)} - I_{jk}^{(l)}\right| \leqslant \varepsilon$, then we have converged to a potential candidate for a local optimum. If not, set $l := l+1$ and go back to (2).

Simulated Annealing In the absence of global convexity results, the above iterative procedure is likely to end up in a local extremum. Unfortunately, there exists to our knowledge few global optimization methods that would guarantee convergence to a global maximum in a reasonable amount of time.[51] We opt for the simple but widely used heuristic method of simulated annealing, which is a very popular probabilistic method to search for the global optimum of high dimensional problems such as, for instance, the traveling salesman problem. Simulated annealing can be described as follows:

1. Let $l := 1$. Set the initial network $\left\{I_{jk}^{(1)}\right\}$ to a local optimum from the previous section and compute its welfare $v^{(1)}$. Set the initial "temperature" T of the system to some number.

2. Draw a new candidate network $\left\{\hat{I}_{jk}\right\}$ by perturbing $\left\{I_{jk}^{(l)}\right\}$ (see below). (Optional: deepen the network.) Compute the corresponding optimal allocation and transport $\left\{c_j, D_j^n, Q_{jk}^n, L_j^n\right\}$. Compute associated welfare $\hat{v}$.

3. Accept the new network, i.e., set $I_{jk}^{(l+1)} = \hat{v}$ and $v^{(l+1)} = \hat{v}$ with probability $\min\left[\exp\left(\left(\hat{v} - v^{(l)}\right)/T\right), 1\right]$, if not keep the same network, $\left\{I_{jk}^{(l+1)}\right\} = \left\{I_{jk}^{(l)}\right\}$ and $v^{(l+1)} = v^{(l)}$.

4. Stop when $T < T_{min}$. Otherwise set $l := l+1$ and $T := \rho_T T$ and return to (2),

where $\rho_T < 1$ controls the speed of convergence. Note that we allow to "deepen" the network in step (2), meaning that we additionally apply the iterative procedure from the previous section for a pre-specified number of iterations so that the candidate network $\left\{\hat{I}_{jk}\right\}$ is more likely to be a local optimum.

Drawing Candidate Networks The performance of the simulated annealing depends on how new candidate networks are drawn. Because of the complex network structure, purely random perturbations are likely to be rejected and the algorithm may easily fail to improve the initial network. We propose the two different perturbation methods that have consistently produced the best results throughout our simulations.

Algorithm #1 "Rebranching" This algorithm exploits the structure of the problem to make educated guesses for the candidate networks. The algorithm builds on the idea that, under increasing returns, a welfare improvement can be achieved by directly connecting locations to more central locations. Since a lower price level indicates that a location has higher relative availability of goods produced anywhere in the economy, we use the price level as a proxy for centrality. We thus construct candidate networks where random locations are better connected to their lowest-price neighbors. The algorithm can be described as follows:

1. Given an initial network $I_{jk}^{(l)}$, draw a random set of locations $I \subset \mathcal{J}$ for "random rebranching" or set $I = \mathcal{J}$ for "deterministic rebranching".

[51] Techniques such as the branch-and-bound method are guaranteed to converge to the global optimum, but remain heavy to implement and computationally intensive.

2. For each $j \in I$, identify $m(j)$ as the neighbor with the lowest price index for the bundle of tradable goods, $m(j) = \underset{k \in \mathcal{N}(j)}{\operatorname{argmin}} P_k^D$, and $n(j)$ as the "parent" with the highest level of infrastructure, $n(j) = \underset{k \in \mathcal{N}(j) | P_k^n \leqslant P_j^n}{\operatorname{argmax}} I_{kj}$.

3. For each $j \in I$, define the candidate network $I_{jk}^{(l+1)}$ by switching the infrastructure levels of $m(j)$ and $n(j)$:

$$I_{kj}^{(l+1)} = \begin{cases} I_{m(j)j}^{(l)} & \text{if } k = n(j), j \in I \\ I_{n(j)j}^{(l)} & \text{if } k = m(j), j \in I \\ I_{kj}^{(l)} & \text{if } j \notin I \text{ or } (j \in I \text{ and } k \notin \{m(j), n(j)\}), \end{cases}$$

which, by construction, satisfies the network building constraint.

Algorithm #2 "Hybrid Alder-FS" This algorithm attempts to implement the spirit of the algorithm proposed in Alder (2019), while exploiting the continuity of infrastructure investments and differentiability of the problem. The algorithm can be described as follows:

1. Compute the welfare gradient with respect to infrastructure investments and some given price of asphalt P_K. Delete the 5% links that correspond to the lowest elements of the gradient.

2. Compute the new economic allocation given the new network.

3. Compute the new welfare gradient and add a predefined quantity of asphalt to the link that has the highest element in the welfare gradient among inactive links.

4. Rescale the network so that the total quantity of asphalt is used and make sure that the network is connected (otherwise add the most beneficial link).

B.2 Numerical Evaluation of the Nonconvex Algorithm

We evaluate our simulated annealing approach in the nonconvex case in a range of random economies for which one can compute or approximate the global optimum.

Brute Force Approach An important caveat when running this performance evaluation is that infrastructure investments are continuous, so that the space of feasible networks is infinite. There is, unfortunately, no readily available brute force algorithm that guarantees finding the global optimum in this case. We explore every discrete combination of bilateral links (on/off), which we then use as initial conditions to iterate on the first-order conditions that characterize optimal infrastructure investment in our model. This brute force algorithm can only be used for small economies with a limited number of locations, as the number of combinations $2^{n(n-1)/2}$ explodes rapidly.

Alder (2019) We also compare the performance of our algorithm to Alder (2019), who proposes another heuristic algorithm in a spatial economic model. Since the Alder algorithm is discrete in nature (locations are connected or not, and there is no intensive aspect to infrastructure investments), there is no obvious way to perform the comparison. We propose two approaches.

In the first approach, which we refer to as "Alder", we implement a simple version of the Alder (2019) algorithm in our model. Specifically, we restrict infrastructure investments to be discrete (0 or I), where I is constant across links and chosen such that the asphalt budget constraint is satisfied with equality. Under this constraint, we implement every step of the Alder algorithm, that is: 1) start from the full network, 2) compute welfare by removing each link one by one, 3) remove the 5% least profitable links, 4) add the most profitable link, 5) make sure that the network in connected, and iterate over (2)-(5) until no further step improves welfare. As a final step, we use the outcome of the Alder algorithm as initial condition in the first-order condition iteration from our model, making sure that welfare keeps improving along the way. As a result we obtain a continuous version of the optimal network that is comparable to ours.

A caveat in the first approach is that the first part of the algorithm is constrained by the discreteness of investments. We thus devise a new algorithm that combines the appealing aspects of Alder (2019) with our continuous-investment approach which relies on differentiation and on the simulated annealing to escape local optima. We refer to this second approach as "Hybrid Alder-FS". To be more precise, we modify our simulated annealing algorithm described in the previous section so that the stage in which we perturb the network is done in the spirit of the Alder algorithm. Specifically, the network is perturbed in the following way: 1) compute the welfare gradient with respect to infrastructure investment, 2) remove the 5% links with the lowest welfare gradient, 3) recompute the welfare and welfare gradient in the network after removal, 4) add the link that has the highest welfare gradient among non-existing links, and 5) make sure that the network is connected. The rest of the algorithm is kept identical to ours. In particular, it iterates on the first-order conditions associated with infrastructure investment before accepting the perturbation. In that sense, the intensive dimension of infrastructure investment is preserved throughout.

Note finally that the original Alder (2019) algorithm is designed to maximize aggregate income net of the dollar value of infrastructure investments. In this comparison, we change the objective function so that it maximizes welfare instead, subject to a fixed asphalt budget.

Simulated annealing There are many ways one can perturb the network in each loop of the simulated annealing method. To explore the role of the perturbation stage, we experiment with various ways to perturb the network: 1) purely random perturbations, 2) deterministic rebranching, and 3) random rebranching. "Rebranching" refers to the algorithm that we described in the previous section. In the deterministic version, all nodes are selected for rebranching every loop (i.e., reconnected to their best neighbors), while they are picked at random in the random version. This comparison allows us to evaluate the importance of randomness in the perturbation process as a way to escape local optima.

Small random economies

To perform the algorithm comparison, we draw random geographies for a given number of replications N_{reps}. Specifically, for each geography, we draw n locations from a uniform distribution over $[0,1] \times [0,1]$. The underlying network is fully connected. We parametrize the transport cost parameters δ^{τ}_{jk} and network building costs δ^{I}_{jk} to be the euclidean distance between each pair of locations. To ease comparison with Alder (2019), who uses a gravity model without labor mobility, we study an Armington version of our model in which each location produces its variety with fixed labor. The rest of the model is otherwise identical to the one described in section 3. In particular, there is congestion in the transport technology, that is $\beta > 0$ using technology (10). We focus exclusively on nonconvex cases with $\gamma > \beta$.[52]

Table A.7 below reports the performance of the various numerical approaches for several numbers of locations and replications. Due to the computation time of the brute force algorithm, we must restrict ourselves to 6 locations at most.

Before starting the individual comparison of each algorithm, it should be noted that the welfare losses are overall quite small with numbers in the magnitude of 10^{-4} in consumption equivalent. This suggests that, all things considered, each of these algorithms perform quite well for small economies. Looking at their individual performance, we find that the simple Alder algorithm is the least effective method with a welfare loss of about 0.02-0.03% in consumption equivalent compared to the brute force algorithm in most cases. A potential reason is that the discreteness constraint heavily distorts the search for the optimum in an economy with continuous investments. Turning to the other algorithms, our findings suggest that the random version of our "rebranching" algorithm and the Hybrid Alder-FS algorithm get the best results with average welfare losses of about 0.004% or less. The Hybrid

[52] In the convex case $\beta \geq \gamma$, a simple iteration over our model's first-order conditions or descent along the gradient suffice to find the global optimum quickly, so the algorithm comparison is irrelevant.

Alder-FS algorithm does slightly better in our simulations with 5 locations, but the random rebranching algorithm has the edge in the 6 locations case. In terms of computation time, the Hybrid algorithm requires slightly more time than the FS algorithm due to an additional evaluation of the equilibrium during the updating process. The Alder algorithm is the fastest algorithm given the small amount of locations, but the difference quickly reverts when we increase their number, as we show next.

Large random economies

We extend our comparison to economies with more locations. Unfortunately, we can no longer use the brute force algorithm, but we can still compare the performance between the other algorithms. In this last exercise, we focus on the version of our model used for the European road exercise in Section 5. Namely, we consider $w \times h$ rectangular networks with Moore neighborhood (each location is connected to its immediate horizontal, vertical and diagonal neighbors). Since we study economies with a large number of locations, we limit the number of traded goods to 2. As we did in the European exercise, we assume that one of the goods is a homogeneous "agricultural" good that can be produced in any location. The other good is a differentiated variety that can only be produced in a number of locations N_{cities} drawn uniformly among all locations. The rest of the model is otherwise identical to the previous section.

The results of this exercise, in Table A.8 below, are reported in terms of the welfare gains with respect to the local optimum that results from iterating over our model's first-order conditions without using annealing. Our findings suggest larger differences between the algorithms as spatial complexity increases. While the Hybrid Alder-FS algorithm seems to perform quite well overall (and definitely better than random perturbations), our "rebranching" algorithm, in its deterministic and random versions, yields the best results in all cases. The random version performs the best with welfare improvements ranging between 0.01% and 0.1%. The simple (discrete) Alder algorithm does not do as well, most likely for the reason mentioned before.

Table A.7: Comparison of the welfare losses for small economies

#locations	N_{reps}	Alder	Simulated annealing			
			Random	Rebranching (deterministic)	Rebranching (random)	Hybrid Alder-FS
4	100	-0.0360%	-0.0036%	-0.0036%	-0.0020%	-0.0020%
5	100	-0.1281%	-0.0046%	-0.0047%	-0.0043%	-0.0032%
6	10	-0.0216%	-0.0016%	-0.0019%	0.0004%	-0.0019%

Notes: This table reports welfare in percentage consumption equivalent for the optimal network resulting from each algorithm compared to the brute force algorithm. The parameters chosen for the simulation are $\beta = 1$, $\gamma = 2$, $K = 1$, $\sigma = 5$, uniform amenities, productivity and population, no cross-good congestion, Cobb-Douglas utility with share $\alpha = 0.5$ on traded goods.

Table A.8: Comparison of the welfare gains for large economies

				Simulated annealing			
#locations	N_{cities}	N_{reps}	Alder	Random	Rebranching (deterministic)	Rebranching (random)	Hybrid Alder-FS
25	10	100	-0.0272%	0.0000%	0.0122%	0.0153%	0.0035%
64	20	100	-0.0133%	0.0000%	0.0378%	0.0431%	0.0017%
64	40	100	-0.0068%	0.0003%	0.0057%	0.0061%	0.0012%
100	20	100	-0.0033%	0.0011%	0.0858%	0.0954%	0.0024%
100	40	100	-0.0062%	0.0015%	0.0259%	0.0303%	0.0018%
100	60	100	-0.0048%	0.0005%	0.0042%	0.0061%	0.0008%

Notes: This table reports the welfare gain in percentage consumption equivalent compared to the outcome of our approach in non-convex cases without simulated annealing, simply obtained by iterating over the first-order conditions of our model starting from a full network. The parameters chosen for the simulation are $\beta = 1$, $\gamma = 2$, $K = 1$, $\sigma = 5$, uniform amenities, productivity and population, no cross-good congestion, and Cobb-Douglas utility with share $\alpha = 0.5$ on traded goods.

C Intuition for Global Convexity Condition

We show that the infrastructure investment and shipping decisions can be viewed as the outcome of a game between a decentralized planning agency and a shipping company. Due to the complementarity between infrastructure and trade, an investment in infrastructure in a link leads to an increase in commodity flows. These additional flows in turn encourage further investments. The convexity condition on the transport technology determines whether this "race" is stable or divergent. When the transport technology is convex, each additional increase in investment or flow becomes smaller until convergence to a non-trivial interior point.

For clarity, we focus on the single good case. We focus on a link (o, d) from origin o to destination d. We first consider the problem of a shipping company who takes prices and infrastructure as given and fully internalizes the congestion externality. From section 3 that this is equivalent to having the shipping company not internalize the congestion externality while facing an appropriately defined Pigouvian tax. The problem faced by the shipping company is

$$\max_{Q \geq 0} P_d Q - P_o Q \left(1 + \tau \left(Q, I\right)\right),$$

where P_i is the price in location $i = o, d$. The first-order condition is

$$P_d - P_o - P_o \frac{\partial \left(Q\tau\right)}{\partial Q} \leq 0 \text{ with equality if } Q > 0. \tag{A.9}$$

This first-order condition is identical to (8) from the full planning problem. Denote $Q^* \left(I\right)$ the optimal amount of commodity flow Q that solves equation (A.9).[53] Consider now the problem of a myopic decentralized agency. It is myopic in the sense that it takes the trade flow Q as given and does not internalize the fact that Q responds to infrastructure in equilibrium, and decentralized in the sense that it solely invests in link (o, d), taking as given the price P_K of asphalt (that is, the Lagrange multiplier of the asphalt budget constraint in the full planning problem) and the price of commodities. The objective of the planning agency is to minimize the sum of the market value of transport costs and the cost of asphalt. We write the problem as follows

$$\min_{I \geq 0} P_o Q \tau \left(Q, I\right) + P_K \delta^I I,$$

where δ^I is the marginal cost of investing infrastructure in the link. We obtain the following first-order condition,

$$P_o \frac{\partial \left(Q\tau\right)}{\partial I} + P_K \geq 0 \text{ with equality if } I > 0. \tag{A.10}$$

This condition is identical to equation (9) in the main text. We denote $I^* \left(Q\right)$ the optimal infrastructure investment that solves equation (A.10).[54]

Figure A.9 shows the two best response functions $Q^* \left(I\right)$ and $I^* \left(Q\right)$ for the convex and nonconvex cases using the functional form $\tau \left(Q, I\right) = \delta^\tau Q^\beta I^{-\gamma}$.[55] As the figure illustrates, there are two equilibria: the null equilibrium and an interior equilibrium with $I > 0$ and $Q > 0$. The reason why the game features two equilibria even in the convex case is that the planning agency is myopic and takes the commodity flow Q as given. In the full model and in the convex case, the social planner internalizes the response in Q and never chooses the null equilibrium under the Inada conditions.

The curve $Q^* \left(I\right)$ crosses $I^* \left(Q\right)$ from above in the convex case $(\beta \geq \gamma)$ and from below otherwise. In the convex case, only the interior equilibrium is stable, as a small increase (resp. decrease) in infrastructure prompts an increase

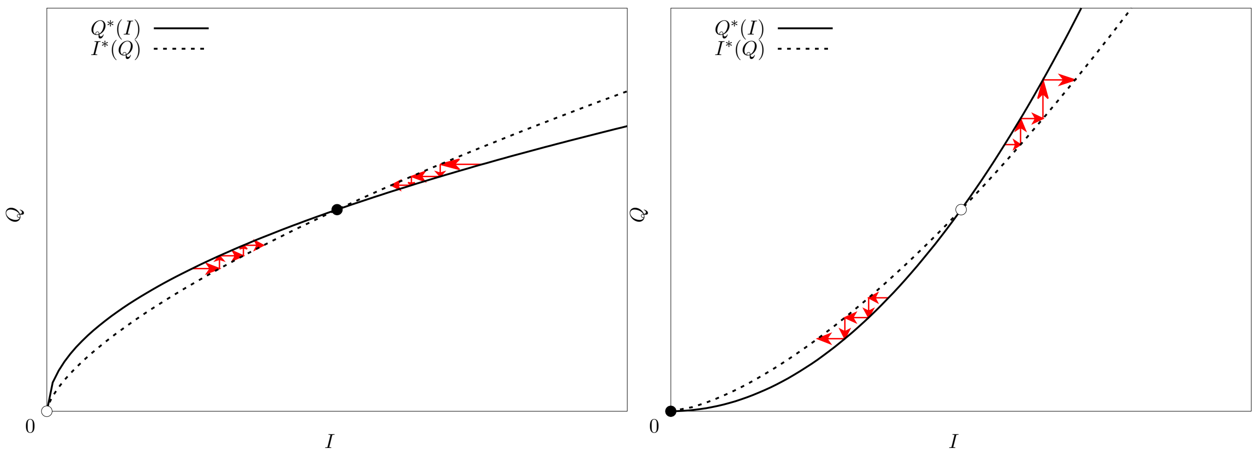
[53] Under the requirement that $Q\tau$ is convex in Q and satisfies some Inada conditions $\lim_{Q \to 0} \frac{\partial(Q\tau)}{\partial Q} = 0$ and $\lim_{Q \to \infty} \frac{\partial(Q\tau)}{\partial Q} = \infty$, the solution $Q^* \left(I\right)$ exists and is unique.

[54] Under the assumption that $Q\tau$ is convex in I and satisfies the Inada conditions $\lim_{I \to 0} \frac{\partial(Q\tau)}{\partial I} = -\infty$ and $\lim_{I \to \infty} \frac{\partial(Q\tau)}{\partial I} = 0$, $I^* \left(Q\right)$ exists and is unique.

[55] In this case we obtain: $Q^* \left(I\right) = \left(\frac{1}{1+\beta} \frac{1}{\delta^\tau} \frac{P_d - P_o}{P_o}\right)^{\frac{1}{\beta}} I^{\frac{\gamma}{\beta}}$ and $I^* \left(Q\right) = \left(\frac{\gamma}{P_K} \frac{\delta^\tau}{\delta^I} P_o\right)^{\frac{1}{1+\gamma}} Q^{\frac{\beta+1}{\gamma+1}}$.

Figure A.9: Nash equilibria of the infrastructure investment game

(a) Convex case $\beta \geq \gamma$ (b) Nonconvex case $\gamma > \beta$



Note: Stable equilibria indicated by a black dot, unstable equilibria by a white dot.

(resp. decrease) in Q, but the response is limited due to the strong congestion forces. In turn, the moderate increase in flows justifies reducing (resp. increasing) the level of infrastructure back to the interior equilibrium. In the nonconvex case, the interior equilibrium is no longer stable but the null equilibrium is.

This discussion shows that the convexity property on the transport technology controls the complementarities between the planning agency and the private economy. In the nonconvex case, complementarities are strong. The convex case corresponds to weak complementarities: a deviation by one player is accompanied by a deviation by the other player in the same direction (they remain complements) but the response is too weak to push the economy away from the interior equilibrium. The interior equilibrium is stable when $Q^*(I)$ crosses $(I^*)^{-1}(I)$ from above, that is

$$\frac{\partial Q^*}{\partial I} \leq \left(\frac{\partial I^*}{\partial Q}\right)^{-1}. \tag{A.11}$$

Differentiating the first-order conditions (A.9) and (A.10), we have $\frac{\partial Q^*}{\partial I} = -\frac{\partial^2(Q\tau)}{\partial Q \partial I} / \frac{\partial^2(Q\tau)}{(\partial Q)^2}$ and $\frac{\partial I^*}{\partial Q} = -\frac{\partial^2(Q\tau)}{\partial Q \partial I} / \frac{\partial^2(Q\tau)}{(\partial I)^2}$. The stability condition (A.11) is thus equivalent to

$$\frac{\frac{\partial^2(Q\tau)}{\partial Q \partial I}}{\frac{\partial^2(Q\tau)}{(\partial Q)^2}} \cdot \frac{\frac{\partial^2(Q\tau)}{\partial Q \partial I}}{\frac{\partial^2(Q\tau)}{(\partial I)^2}} \leq 1. \tag{A.12}$$

In turn, (A.12) is equivalent to $\det(\text{Hess}(Q\tau)) \geq 0$, which along with the maintained assumption that $Q\tau$ is convex in each individual argument, $\frac{\partial^2(Q\tau)}{(\partial Q)^2} \geq 0$ and $\frac{\partial^2(Q\tau)}{(\partial I)^2} \geq 0$, is equivalent to requiring that the total transport costs $Q\tau(Q, I)$ is jointly convex over Q and I.

D Dealing with Inefficiencies

We show how to extend our approach to deal with cases with externalities in which Pigouvian taxes are not available or incorrectly set. We provide an example with partially internalized congestion in transport as well as externalities in production and amenities.

D.1 Environment

We modify the environment by introducing agglomeration externalities or spillovers in production

$$Y_j^n = F\left(L_j^n, \mathbf{V}_j^n, \mathbf{X}_j^n; \overline{L}_j\right)$$

where $\overline{L}_j$ captures the production spillovers. $\overline{L}_j$ is taken as given by firms and equals total employment in the location,

$$\overline{L}_j = \sum_n L_j^n.$$

We continue assuming that $F_j^n(\cdot)$ has constant returns to scale in its first three arguments, and is increasing and concave in each. Similarly, we allow for externalities in amenities by assuming that utility is given by

$$U\left(c_j, h_j; \overline{L}_j\right).$$

Finally, we assume that the congestion externalities in transport are only partially internalized by shipping companies. We focus on the convex case studied in our numerical exercise with mobile labor and cross-good congestion, but the analysis can be easily applied to the other cases. Specifically, we assume that the transport technology is given by

$$\tau_{jk}^n\left(Q_{jk}, I_{jk}; \overline{Q}_{jk}\right) = m^n \delta_{jk}^\tau \left(\overline{Q}_{jk}\right)^{\beta-\tilde{\beta}} \left(Q_{jk}\right)^{\tilde{\beta}} I_{jk}^{-\gamma},\ \beta \geq \gamma,\ \beta \geq \tilde{\beta} \geq 0,$$

where $Q_{jk} = \sum_{n=1}^N m^n Q_{jk}^n$. In this specification, the total congestion externalities amount to a total exponent of β, but shipping companies only internalize them up to exponent $\tilde{\beta}$, taking the aggregate flow $\overline{Q}_{jk}$ as given.[56] In equilibrium, the perceived flow on a given link $\overline{Q}_{jk}$ is required to satisfy $\overline{Q}_{jk} = Q_{jk}$.

D.2 Fictitious planning approach

Solving for the economic allocation $\left\{c_j^n, L_j^n, ...\right\}$ and trade flows $\left\{Q_{jk}^n\right\}$ given an infrastructure network $\{I_{jk}\}$ is an important step in the optimization. The method described in the main text involves solving for the allocation and flows jointly using a planning approach. Unfortunately, the presence of externalities does not allow us to invoke the First Welfare Theorem directly. We nonetheless propose an approach in which a "fictitious planner" does not internalize the effect of her decisions on $\overline{L}_j$ and $\overline{Q}_{jk}$.

The fictitious planning problem for our economy can be characterized as the solution of a fixed-point problem. We first define the problem faced by the fictitious planner given $\left\{\overline{L}_j, \overline{Q}_{jk}\right\}$:

$$W\left(\left\{\overline{L}_j, \overline{Q}_{jk}\right\}_{j\in\mathcal{J}, k\in\mathcal{N}(j)}; \{I_{jk}\}\right) = \max_{\substack{u, c_j, h_j, L_j, D_j^n, \\ L_j^n, \mathbf{V}_j^n, \mathbf{X}_j^n, Q_{jk}, Q_{jk}^n}} u \tag{A.13}$$

subject to (i) availability of traded commodities,

$$c_j L_j + \sum_{k\in\mathcal{N}(j)} \tau_{jk}\left(Q_{jk}, I_{jk}; \overline{Q}_{jk}\right) Q_{jk} \leq D_j\left(D_j^1, \ldots, D_j^N\right) \text{ for all } j,$$

where $Q_{jk} = \sum_{n=1}^N m^n Q_{jk}^n$, and non-traded commodities,

$$h_j L_j \leq H_j \text{ for all } j;$$

[56] This example is akin to having congestion taxes set to a fraction $\tilde{\beta}/\beta$ of the efficient Pigouvian taxes relative to unit transport costs.

(ii) the balanced-flows constraint,

$$D_j^n + \sum_{n'} X_j^{nn'} + \sum_{k \in \mathcal{N}(j)} Q_{jk}^n \leq F_j^n \left(L_j^n, \mathbf{V}_j^n, \mathbf{X}_j^n; \overline{L}_j \right) + \sum_{i \in \mathcal{N}(j)} Q_{ij}^n \text{ for all } j, n;$$

(iii) local factor market clearing,

$$\sum L_j^n \leq L_j \text{ for all } j;$$

and

$$\sum_n V_j^{mn} \leq V_j^n \text{ for all } j, m;$$

(iv) free labor mobility,

$$L_j u \leq L_j U \left(c_j, h_j; \overline{L}_j \right) \text{ for all } j;$$

(v) aggregate labor market clearing,

$$\sum_j L_j = L;$$

and (vi) non-negativity constraints on consumption, flows and factors.

For a given $\overline{L}_j$ and $\overline{Q}_{jk}$, the above fictitious planning problem is a well-behaved convex problem under the same conditions as in the main text and can be efficiently solved using a convex solver. Imposing that $\overline{L}_j = L_j$ and $\overline{Q}_{jk} = Q_{jk}$ in equilibrium, the first-order conditions of this planning problem are identical to that of the competitive equilibrium. We are now ready to define a solution to the fictitious planning problem.

Definition 4. An allocation $\Omega = \left\{ u, c_j, h_j, L_j, D_j^n, L_j^n, \mathbf{V}_j^n, \mathbf{X}_j^n, Q_{jk}, Q_{jk}^n \right\}_{j \in \mathcal{J}, k \in \mathcal{N}(j), 1 \leq n \leq N}$ is a solution to the fictitious planning problem of this economy for a given infrastructure network $\{I_{jk}\}$ if it is a solution of the problem (A.13),

$$\Omega \in \text{argmax } W \left(\left\{ \overline{L}_j, \overline{Q}_{jk} \right\}; \{I_{jk}\} \right)$$

and satisfies

$$\overline{L}_j = L_j \text{ for all } j,$$
$$\overline{Q}_{jk} = Q_{jk} \text{ for all } j, k \in \mathcal{N}(j).$$

A fictitious planning allocation is thus the solution of a fixed point problem over $\overline{L}_j$ and $\overline{Q}_{jk}$, which can be solved using an iterative procedure.[57] This solution yields the competitive equilibrium of an economy with spillovers. In cases where uniqueness of the competitive equilibrium is guaranteed, there is an equivalence between the competitive equilibrium and the fictitious planning economy. In cases when uniqueness is not guaranteed, the fictitious planning approach remains a tractable way to compute candidate equilibria.

D.3 Infrastructure gradient

After solving for the economic allocation and trade flows given a certain infrastructure network $\{I_{jk}\}$, we are now ready to compute the welfare gradient with respect to infrastructure investments from the point of view of an overall planner who internalizes that the allocation corresponds to the solution to the fictitious planning problem and is therefore not generically efficient. To achieve this, we use the fact the competitive equilibrium is locally characterized as the solution to the fictitious planner's first-order conditions. We then differentiate these first-order conditions to evaluate the local impact of infrastructure changes.

We consider the case studied in our numerical exercise from section 5 with (i) labor as the only factor of production, (ii) D_j is a CES aggregator with elasticity of substitution σ, (iii) a unique good produced in each location (there is

[57]In practice, we start with a guess on $\left\{ \overline{L}_j, \overline{Q}_{jk} \right\}$, solve problem (A.13), update our guess and repeat until convergence. We do not have any proof guaranteeing the convergence of this procedure, but our experience has shown that it converges most of the time for reasonable externality parameters.

at most a unique $n(j)$ such that $z_j^{n(j)} > 0$ for each j) and the production function is given by $Y_j^n = z_j^n L_j^n \overline{L}_j^{\varepsilon}$ where ε governs the production externality and (iv) partially internalized transport externality ($0 \leq \tilde{\beta} < \beta$). In that case, the fictitious planning problem described in (A.13) simplifies to

$$\max_{\Omega=\left(u,c_j,h_j,L_j,D_j^n,Q_{jk}^n\right)} u$$

subject to

$$L_j u \leqslant L_j U\left(c_j, h_j; \overline{L}_j\right), \forall j \quad [\times \omega_j] \tag{A.14}$$

$$c_j L_j + \sum_{k\in\mathcal{N}(j)} \delta_{jk}^{\tau} \overline{Q}_{jk}^{\overline{\beta}-\beta} \left(\sum_n m^n Q_{jk}^n\right)^{1+\beta} I_{jk}^{-\gamma} \leqslant \left(\sum_n \left(D_j^n\right)^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}}, \forall j \quad \left[\times P_j^D\right] \tag{A.15}$$

$$h_j L_j \leqslant H_j, \forall j \quad \left[\times P_j^H\right] \tag{A.16}$$

$$D_j^n + \sum_{k\in\mathcal{N}(j)} Q_{jk}^n \leqslant z_j^n L_j \overline{L}_j^{\varepsilon} + \sum_{k\in\mathcal{N}(j)} Q_{kj}^n, \forall (j,n) \quad \left[\times P_j^n\right] \tag{A.17}$$

$$\sum_j L_j = 1 \quad [\times W] \tag{A.18}$$

$$Q_{jk}^n \geqslant 0, L_j \geqslant 0 \quad \left[\times \nu_{jk}^n, \xi_j\right]$$

where the variables in brackets denote the associated Lagrange multipliers. Constraints (A.14)-(A.18) will bind in equilibrium and define a system of $2 + 3J + J \times N$ equations that we summarize as

$$G_{cons}\left(\Omega; \overline{Q}_{jk}, \overline{L}_j, I_{jk}\right) = 0.$$

The first-order conditions of the above problem are

$$\begin{aligned}
[u] &\quad 1 - \sum_j \omega_j L_j = 0 \\
[c_j] &\quad \omega_j L_j U_c\left(c_j, h_j; \overline{L}_j\right) - P_j^D L_j = 0 \\
[h_j] &\quad \omega_j L_j U_h\left(c_j, h_j; \overline{L}_j\right) - P_j^H L_j = 0 \\
[L_j] &\quad -\omega_j\left(u - U\left(c_j, h_j\right)\right) - P_j^D c_j - P_j^H h_j + P_j^{n(j)} z_j^n \overline{L}_j^{\varepsilon} - W + \xi_j = 0 \\
[D_{jn}] &\quad P_j^D D_j^{\frac{1}{\sigma}} \left(D_j^n\right)^{-\frac{1}{\sigma}} - P_j^n = 0 \\
\left[Q_{jk}^n\right] &\quad -P_j^D m^n (1+\beta) \delta_{jk}^{\tau} \overline{Q}_{jk}^{\overline{\beta}-\beta} Q_{jk}^{\beta} I_{jk}^{-\gamma} - P_j^n + P_k^n + \nu_{jk}^n = 0
\end{aligned}$$

along with the complementary slackness conditions

$$\nu_{jk}^n Q_{jk}^n = 0$$

$$\xi_j L_j = 0.$$

Together with the complementary slackness conditions, the first-order conditions define a system of $1 + 4J + J \times N + 4 \times nlinks \times N$ equations, which we denote by

$$G_{FOC}\left(\Omega; \overline{Q}_{jk}, \overline{L}_j, I_{jk}\right) = 0.$$

Imposing $\overline{Q}_{jk} = Q_{jk}$ and $\overline{L}_j = L_j$, the constraints and first-order conditions jointly define the system

$$G\left(\Omega; Q_{jk}, I_{jk}\right) = \begin{pmatrix} G_{cons}\left(\Omega; Q_{jk}, I_{jk}\right) \\ G_{FOC}\left(\Omega; Q_{jk}, I_{jk}\right) \end{pmatrix} = 0$$

of $2 + 7J + 2J \times N + 4 \times nlinks \times N$ equations in $2 + 4J + 2J \times N + 4 \times nlinks \times N$ unknowns (more precisely, the allocation u, c_j, h_j, L_j, D_j^n, Q_{jk}^n and the multipliers ω_j, P_j^D, P_j^H, P_j^n, W, ν_{jk}^n, ξ_j).

By differentiating our equilibrium conditions around a solution to the fictitious planning problem described above, we can now compute how each element of the allocation Ω is affected by a change in infrastructure. Using the implicit function theorem, we obtain

$$J_I(\Omega) = -\left[J_\Omega(G)\right]^{-1} J_I(G),$$

where $J_X(F)$ denotes the Jacobian of some function F with respect to some vector of variables X. Since welfare u is one of the variables in the allocation Ω, the gradient of welfare with respect to a change in infrastructure I_{jk} assuming a shadow value P_K for asphalt/concrete is simply given by

$$\frac{du}{dI_{jk}} - \delta_{jk}^I P_K.$$

With this welfare gradient, it is possible to optimize over the infrastructure network using a simple gradient-descent algorithm. There are, however, two new caveats in comparison to the efficient case covered in the main text: i) the computations can be quite time-consuming since the fictitious planning problem needs to be solved at every step of the optimization, ii) Proposition 1 on the convexity of the overall planning problem no longer applies and the resulting network has no guarantee to be the global optimum.

D.4 Numerical comparison

To quantitatively evaluate the importance of inefficiencies, we return to the benchmark case of Spain in our main quantitative exercise.[58] We put aside agglomeration externalities in production and consumption to focus on partially internalized congestion in transport as the only source of inefficiency. The model parameters are those estimated in the calibration and we simply vary the perceived congestion parameter $\tilde{\beta}$ while keeping the overall level of congestion β constant. We consider three cases: $\tilde{\beta} = 0.25\beta$, 0.5β and 0.75β,which correspond to having the Pigouvian taxes set to 25%, 50% and 75% of what would be their optimal level relative to transport costs.

Table A.9 reports some summary statistics. Unsurprisingly, overall welfare resulting from the fictitious planner is lower compared to the efficient planner the less shipping companies internalize congestion (lower $\tilde{\beta}$). The difference is nonetheless quite small with welfare differences in the order of 10^{-4} in consumption equivalent terms. Similarly, we find that the resulting equilibrium trade flows Q_{jk} and infrastructure gradients are almost identical, with correlations close to 1 between the efficient and inefficient allocations.

$\tilde{\beta}$	Welfare	corr($Q_{jk}^{\text{efficient}}$,$Q_{jk}^{\text{inefficient}}$)	corr($\frac{dW}{dI_{jk}}^{\text{efficient}}$,$\frac{dW}{dI_{jk}}^{\text{inefficient}}$)
Efficient	0	1	1
0.75β	-0.0057%	0.9999	0.9951
0.5β	-0.0235%	0.9998	0.9902
0.25β	-0.0545%	0.9996	0.9862

Notes: dW/dI_{jk} denotes the welfare gradient. Welfare is stated in percentage deviation from the efficient benchmark in consumption equivalent.

Table A.9: Statistical comparison between efficient and inefficient allocations

Since shipping companies fail to internalize their impact on congestion, we find that the inefficient allocation tends to feature larger trade flows the less the externality is internalized in comparison to the efficient one (Figure A.10). The left panel of Figure A.12 shows that the difference in flows are especially important around Madrid, Barcelona and Valencia. The discrepancies between the welfare gradients in infrastructure do not show a systematic pattern, as Figure A.11 illustrates. The right panel of Figure A.12 shows nonetheless a tendency for the planner to

[58]More precisely, we consider the case $\beta = 0.13$, $\gamma = 0.10$ with labor mobility and cross-good congestion.