

ON ERDŐS PROBLEM #1044

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ABSTRACT. We resolve Problem #1044 on Bloom's *Erdős Problems* website. For monic polynomials $f(z) = \prod_{j=1}^n (z - z_j)$ with $|z_j| \leq 1$, writing $\Omega(f) = \{z \in \mathbb{C} : |f(z)| < 1\}$ and letting $\Lambda(f)$ be the maximum boundary length among the connected components of $\Omega(f)$, we prove that $\inf_f \Lambda(f) = 2$. We also discuss a variant of the problem and record partial results.

1. INTRODUCTION

In [2, Problem 11], Erdős, Herzog, and Piranian posed the following question, which also appears as Problem #1044 on Bloom's Erdős Problems website [1].

Problem 1.1 ([2]). *Let $f(z) = \prod_{i=1}^n (z - z_i) \in \mathbb{C}[x]$ where $|z_i| \leq 1$ for all i . If $\Lambda(f)$ is the maximum of the lengths of the boundaries of the connected components of*

$$\{z : |f(z)| < 1\}$$

then determine the infimum of $\Lambda(f)$.

1.1. Setup. Throughout, let

$$f(z) = \prod_{i=1}^n (z - z_i), \quad |z_i| \leq 1,$$

and set

$$\Omega(f) := \{z \in \mathbb{C} : |f(z)| < 1\}.$$

Write the connected components of $\Omega(f)$ as $\Omega_1, \dots, \Omega_m$, and define

$$\Lambda(f) := \max_{1 \leq j \leq m} \text{length}(\partial\Omega_j),$$

where length denotes the arc-length (equivalently, the one-dimensional Hausdorff measure) of the boundary.

We will use two infima, corresponding to whether the degree is allowed to vary. Let

$$\lambda := \inf_f \Lambda(f)$$

denote the infimum taken over all degrees $n \geq 1$ and all monic polynomials f whose zeros lie in $\overline{\mathbb{D}}$, and for each fixed $n \geq 1$ let

$$\lambda_n := \inf_{\deg f = n} \Lambda(f)$$

denote the analogous infimum restricted to monic polynomials of degree n with zeros in $\overline{\mathbb{D}}$.

Accordingly, one may consider the following two versions of Problem 1.1 (Version 1 is the formulation in [2]):

Problem 1.2 (Version 1). *Determine $\lambda = \inf_f \Lambda(f)$.*

Problem 1.3 (Version 2). *For each fixed integer $n \geq 1$, determine $\lambda_n = \inf_{\deg f = n} \Lambda(f)$.*

In this note we settle Problem 1.2, i.e. the original formulation of Erdős, Herzog, and Piranian [2]. We also investigate Problem 1.3: we determine λ_n for $n = 1, 2$, and discuss the case $n = 3$.

Organisation of the note. In Section 2 we prove that $\inf_f \Lambda(f) \leq 2$. In Sections 3 and 4 we prove the matching lower bound $\inf_f \Lambda(f) \geq 2$, and hence resolve Problem 1.2. In Section 5 we discuss Problem 1.3 and some partial progress. Finally, Appendix A contains an unfinished argument toward Problem 1.3.

2. THE MODEL POLYNOMIAL $f(z) = z^n - 1$

Let $B(x, y)$ and $\Gamma(s)$ denote the classical Beta and Gamma functions, defined for $x, y > 0$ and $\Re(s) > 0$ by

$$B(x, y) := \int_0^1 t^{x-1} (1-t)^{y-1} dt, \quad \Gamma(s) := \int_0^\infty t^{s-1} e^{-t} dt,$$

and related by the identity $B(x, y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$.

The following explicit formula for the lemniscate of $z^n - 1$ is classical and may be found, for instance, in [6, Eq. (1.1)].

Proposition 2.1. *Let $f_n(z) = z^n - 1$ with $n \geq 1$. Then $\Omega(f_n)$ has exactly n congruent connected components, and each component has boundary length*

$$\Lambda(z^n - 1) = \frac{2^{1/n}}{n} B\left(\frac{1}{2}, \frac{1}{2n}\right) := L_n.$$

As a direct consequence of Proposition 2.1, we obtain the following upper bound for $\inf_f \Lambda(f)$.

Theorem 2.2. $\inf_f \Lambda(f) \leq 2$.

Proof. Let $f_n(z) = z^n - 1$ with $n \geq 1$. Then by Proposition 2.1, we know that

$$\inf_f \Lambda(f) \leq \inf_n \Lambda(f_n) = \inf_n \frac{2^{1/n}}{n} B\left(\frac{1}{2}, \frac{1}{2n}\right) = 2. \quad \square$$

3. A SPECIAL CASE

For a set $E \subset \mathbb{C}$ write

$$\text{diam}(E) := \sup\{|z - w| : z, w \in E\}.$$

We first establish a lemma.

Lemma 3.1. *Let $\Gamma \subset \mathbb{C}$ be the image of a rectifiable closed curve $\gamma : [0, L] \rightarrow \mathbb{C}$ with $\gamma(0) = \gamma(L)$. Then*

$$\text{length}(\Gamma) \geq 2 \text{diam}(\Gamma).$$

Proof. Choose $s, t \in [0, L]$ such that $|\gamma(s) - \gamma(t)| = \text{diam}(\Gamma)$. Let $\Gamma_1 = \gamma([s, t])$ and $\Gamma_2 = \gamma([t, L]) \cup \gamma([0, s])$. Then $\Gamma = \Gamma_1 \cup \Gamma_2$ and each Γ_j is a rectifiable curve joining $\gamma(s)$ to $\gamma(t)$, hence

$$\text{length}(\Gamma_j) \geq |\gamma(s) - \gamma(t)| = \text{diam}(\Gamma) \quad (j = 1, 2).$$

Therefore $\text{length}(\Gamma) = \text{length}(\Gamma_1) + \text{length}(\Gamma_2) \geq 2 \text{diam}(\Gamma)$. $\square$

Theorem 3.2. *Let f be a monic polynomial of degree $n \geq 1$ of the form*

$$f(z) = \prod_{k=1}^n (z - z_k), \quad |z_k| = 1 \quad (1 \leq k \leq n).$$

Then $\Lambda(f) \geq 2$.

Proof. Since $|z_k| = 1$ for all k , we have

$$|f(0)| = \prod_{k=1}^n |z_k| = 1,$$

so $0 \in \{|f| = 1\}$.

Because $f(0) \neq 0$, the function $u(z) := \log |f(z)|$ is harmonic in a neighbourhood of 0 and satisfies $u(0) = 0$. If there were $r > 0$ such that $u \geq 0$ on the disc $D(0, r)$, then u would attain an interior minimum at 0; by the minimum principle for harmonic functions, u would be constant on $D(0, r)$, hence $|f| \equiv 1$ on $D(0, r)$, forcing f to be constant, a contradiction. Therefore every neighbourhood of 0 contains points where $u < 0$, i.e. points where $|f| < 1$, and hence

$$0 \in \partial\Omega(f).$$

Let Ω be the connected component of $\Omega(f)$ whose boundary contains 0; then $0 \in \partial\Omega$.

We claim that Ω contains at least one zero of f . Indeed, if Ω contained no zeros, then $u = \log |f|$ would be harmonic on Ω and extend continuously to $\partial\Omega$ with boundary values $u|_{\partial\Omega} = 0$ (since $\partial\Omega \subset \{|f| = 1\}$). The maximum principle would give $u \equiv 0$ on Ω , contradicting $|f| < 1$ on Ω . Thus Ω contains some zero z_j with $|z_j| = 1$.

Since Ω is open and contains z_j on the unit circle, it cannot be contained in the open unit disc $\{|z| < 1\}$. Hence

$$\max_{z \in \overline{\Omega}} |z| \geq 1.$$

Because $\overline{\Omega}$ is compact and Ω is open, the maximum of $|z|$ over $\overline{\Omega}$ is attained at some point

$$w \in \partial\Omega \quad \text{with} \quad |w| \geq 1.$$

Together with $0 \in \partial\Omega$, this yields

$$\text{diam}(\partial\Omega) \geq |w - 0| = |w| \geq 1.$$

Note that $\partial\Omega \subset \{z : |f(z)| = 1\} = \{(x, y) : |f(x + iy)|^2 - 1 = 0\}$ is contained in a compact real algebraic curve; each compact connected component of a real algebraic curve is a finite union of real-analytic arcs, hence has finite length (in particular, it is rectifiable). Now, Lemma 3.1 gives

$$\text{length}(\partial\Omega) \geq 2 \text{diam}(\partial\Omega) \geq 2.$$

Since $\Lambda(f)$ is the maximum of $\text{length}(\partial\Omega)$ over all components of $\Omega(f)$, we conclude $\Lambda(f) \geq 2$. $\square$

4. SOLUTION TO PROBLEM 1.2

Lemma 4.1. *Every connected component U of $\Omega(f) := \{z \in \mathbb{C} : |f(z)| < 1\}$ contains at least one zero of f . In particular, $\Omega(f)$ has at most n connected components.*

Proof. Since f is a non-constant polynomial, $|f(z)| \rightarrow \infty$ as $|z| \rightarrow \infty$, hence $\Omega(f)$ (and thus each component U) is bounded.

Assume for contradiction that a component U contains no zero of f . Then $g := 1/f$ is holomorphic on U . Moreover, for every $\zeta \in \partial U$ we have $|f(\zeta)| = 1$: indeed $|f| \leq 1$ on $\overline{U}$ by continuity, and if $|f(\zeta)| < 1$ then $\zeta \in \Omega(f)$, so ζ would lie in the (open) component U , contradicting $\zeta \in \partial U$. Thus $|g| = 1$ on ∂U , while $|g| > 1$ on U .

The harmonic function $u := \log |g|$ satisfies $u > 0$ on U and $u = 0$ on ∂U . By the maximum principle for harmonic functions on bounded domains, this is impossible unless $u \equiv 0$, a contradiction. Hence U contains a zero of f .

Finally, distinct components contain disjoint sets of zeros, so the number of components is at most the number of zeros of f , which is $\leq n$. $\square$

Lemma 4.2. *If $U \subset \mathbb{C}$ is a bounded open set, then $\text{diam}(\overline{U}) = \text{diam}(\partial U)$.*

Proof. Since $\overline{U}$ is compact, there exist $a, b \in \overline{U}$ with $|a - b| = \text{diam}(\overline{U})$. If, say, $a \in U$, then there exists $\varepsilon > 0$ such that $B(a, \varepsilon) \subset U$. Moving a a small amount in the direction away from b produces a point $a' \in U$ with $|a' - b| > |a - b|$, contradicting maximality. Thus $a, b \in \partial U$, and $\text{diam}(\overline{U}) \leq \text{diam}(\partial U)$. The reverse inequality is trivial. $\square$

We are now ready to prove the following lower bound for $\inf_f \Lambda(f)$.

Theorem 4.3. $\inf_f \Lambda(f) \geq 2$.

Proof. We split into two cases.

Case 1: $|f(0)| < 1$. Then $0 \in \Omega(f)$. By Lemma 4.1, let U_0 be the connected component of $\Omega(f)$ containing 0. Let $r := \max_k |z_k| \leq 1$. Pommerenke [5, Theorem 3] implies that

$$\text{diam}(U_0) > 2 - r^2 \geq 1.$$

By Lemma 4.2,

$$\text{diam}(\partial U_0) = \text{diam}(\overline{U_0}) \geq \text{diam}(U_0) > 1.$$

By Lemma 3.1,

$$\text{length}(\partial U_0) \geq 2 \text{diam}(\partial U_0) > 2,$$

and therefore $\Lambda(f) \geq \text{length}(\partial U_0) \geq 2$.

Case 2: $|f(0)| = 1$. Since $|f(0)| = \prod_{k=1}^n |z_k|$ and each $|z_k| \leq 1$, we have $|z_k| = 1$ for all k . Then we apply Theorem 3.2 to conclude the proof of this case.

In both cases we prove that $\Lambda(f) \geq 2$. Taking the infimum over all admissible f gives $\inf_f \Lambda(f) \geq 2$. $\square$

Solution to Problem 1.2. By Theorem 2.2 and Theorem 4.3, we know that $\inf_f \Lambda(f) = 2$. $\square$

5. TOWARDS PROBLEM 1.3

Extensive numerical evidence strongly suggests that, for each fixed degree n , the polynomial $z^n - 1$ minimises $\Lambda(f)$ among all monic polynomials of degree n whose zeros lie in the closed unit disk. In this section we solve Problem 1.3 rigorously for the cases $n = 1, 2$, showing that in each of these degrees the infimum of $\Lambda_n(f)$ is indeed attained by $f(z) = z^n - 1$ (up to rotation).

5.1. The infimum for $n = 1$.

Theorem 5.1. *Among all monic linear polynomials $f(z) = z - z_1$ with $|z_1| \leq 1$,*

$$\inf \Lambda_1(f) = 2\pi.$$

Proof. We have $\Omega(f) = \{|z - z_1| < 1\}$, a disk of radius 1, so its (only) boundary component has length 2π . $\square$

5.2. The infimum for $n = 2$.

We will use the following known fact from [4].

Lemma 5.2 ([4]). *Let p be a monic polynomial and suppose $\Omega(p) = \{|p| < 1\}$ is connected. Then the lemniscate $\{|p| = 1\}$ is a single boundary curve, and*

$$\text{length}(\{|p| = 1\}) \geq 2\pi.$$

Theorem 5.3. *Let $f(z) = (z - z_1)(z - z_2)$ with $|z_1|, |z_2| \leq 1$. Then*

$$\Lambda_2(f) \geq \Lambda(z^2 - 1) = \frac{\sqrt{2}}{2} B\left(\frac{1}{2}, \frac{1}{4}\right).$$

Moreover, equality holds if and only if $\{z_1, z_2\} = \{e^{i\theta}, -e^{i\theta}\}$ for some $\theta \in \mathbb{R}$.

Proof. Write

$$m := \frac{z_1 + z_2}{2}, \quad d := \frac{z_1 - z_2}{2}.$$

With the translation $w = z - m$, we have

$$f(z) = (z - z_1)(z - z_2) = (w - d)(w + d) = w^2 - d^2.$$

Translation preserves boundary lengths of components, hence $\Lambda(f) = \Lambda(w^2 - d^2)$. By a further rotation we may assume $d \geq 0$ is real. Since $|z_1 - z_2| \leq |z_1| + |z_2| \leq 2$, we have $0 \leq d \leq 1$.

Thus it suffices to consider

$$p_d(w) := w^2 - d^2, \quad 0 \leq d \leq 1.$$

Case $d < 1$: Put $u = w^2$. Then

$$\Omega(p_d) = \{w : |w^2 - d^2| < 1\} = \{w : u = w^2 \in D(d^2, 1)\},$$

where $D(d^2, 1) = \{u : |u - d^2| < 1\}$ is a disk. Since $d < 1$, we have $0 \in D(d^2, 1)$. As $D(d^2, 1)$ is convex, for any $u \in D(d^2, 1)$ the segment $(1 - t)u$ lies in $D(d^2, 1)$. Thus for any $w \in \Omega(p_d)$ the path $w_t = \sqrt{1 - t} w$ satisfies $w_t^2 = (1 - t)w^2 \in D(d^2, 1)$, hence $w_t \in \Omega(p_d)$ for all $t \in [0, 1]$. Therefore $\Omega(p_d)$ is path-connected, hence connected. By Lemma 5.2, we know that

$$\Lambda(p_d) = \text{length}(\{|p_d| = 1\}) \geq 2\pi \quad \text{for all } d < 1.$$

Case $d = 1$: When $d = 1$, $p_1(w) = w^2 - 1$. By Proposition 2.1 with $n = 2$, each component boundary has length

$$L_2 = \frac{\sqrt{2}}{2} B\left(\frac{1}{2}, \frac{1}{4}\right) \approx 3.708149 < 2\pi.$$

Thus among all $d \in [0, 1]$, the smallest possible value of $\Lambda(p_d)$ is attained at $d = 1$ and equals L_2 . Therefore, for all admissible quadratic f ,

$$\Lambda(f) \geq \Lambda(z^2 - 1),$$

and equality forces $d = 1$, i.e. $|z_1 - z_2| = 2$. Under the constraints $|z_1|, |z_2| \leq 1$, this implies $|z_1| = |z_2| = 1$ and $z_2 = -z_1$. $\square$

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APPENDIX A. A POSSIBLE APPROACH TO THE INFIMUM FOR $n = 3$

In this appendix we present an unfinished proof attempt to Problem 1.3 for the case $n = 3$. The statements highlighted in red indicate the steps that we do not yet know how to justify.

Theorem A.1. *Let $f(z) = \prod_{k=1}^3 (z - z_k)$ with $|z_k| \leq 1$. Then*

$$\Lambda_3(f) \geq \Lambda(z^3 - 1) = \frac{2^{1/3}}{3} B\left(\frac{1}{2}, \frac{1}{6}\right).$$

Proof. We begin with several elementary claims.

Claim 1. *If U is a connected component of $\Omega(f)$, then U contains at least one zero of f .*

Proof. First note that $\Omega(f) = \{|f| < 1\}$ is bounded: since f is monic of degree 3, we have $|f(z)| \rightarrow \infty$ as $|z| \rightarrow \infty$, hence $\{|f| < 1\}$ is contained in a large disk. Thus each component U is bounded. Moreover $\partial U \subset \{|f| = 1\}$ implies $f \neq 0$ on ∂U , so $1/f$ extends continuously to $\overline{U}$.

If U contained no zeros, then $1/f$ is holomorphic on U and satisfies $|1/f| > 1$ on U . On the other hand, $\partial U \subset \{|f| = 1\}$ (by continuity), hence $|1/f| = 1$ on ∂U . This contradicts the maximum modulus principle applied to $1/f$. $\square$

Claim 2. For any $z_1, z_2, z_3 \in \overline{\mathbb{D}}$,

$$\prod_{1 \leq i < j \leq 3} |z_i - z_j| \leq 3\sqrt{3},$$

with equality if and only if $|z_1| = |z_2| = |z_3| = 1$ and $\{z_1, z_2, z_3\}$ is a rotated copy of $\{1, \omega, \omega^2\}$, where $\omega = e^{2\pi i/3}$.

Proof. Consider the Vandermonde matrix

$$V = \begin{pmatrix} 1 & z_1 & z_1^2 \\ 1 & z_2 & z_2^2 \\ 1 & z_3 & z_3^2 \end{pmatrix}.$$

Then $|\det V| = \prod_{i < j} |z_j - z_i|$. Let $r_k = (1, z_k, z_k^2)$ be the k th row. Since $|z_k| \leq 1$,

$$\|r_k\|_2^2 = 1 + |z_k|^2 + |z_k|^4 \leq 3, \quad \text{so} \quad \|r_k\|_2 \leq \sqrt{3}.$$

By Hadamard's inequality,

$$|\det V| \leq \prod_{k=1}^3 \|r_k\|_2 \leq (\sqrt{3})^3 = 3\sqrt{3}.$$

For equality, we must have $\|r_k\|_2 = \sqrt{3}$ for each k (hence $|z_k| = 1$), and the rows must be pairwise orthogonal. If $i \neq j$, orthogonality gives

$$0 = \langle r_i, r_j \rangle = 1 + z_i \overline{z_j} + z_i^2 \overline{z_j^2} = 1 + t + t^2, \quad t := z_i \overline{z_j} \in \mathbb{T},$$

so $t \in \{\omega, \omega^2\}$. Thus $z_i/z_j \in \{\omega, \omega^2\}$ for all $i \neq j$, i.e. the points form a regular triangle on $|z| = 1$. $\square$

Claim 3. Let ξ_1, ξ_2 be the (possibly equal) critical points of f (i.e. zeros of f'), and let

$$\Delta(f) := \prod_{1 \leq i < j \leq 3} (z_i - z_j)^2$$

be the discriminant in terms of the roots. Then

$$f(\xi_1) f(\xi_2) = -\frac{\Delta(f)}{27}.$$

Proof. We use the resultant $\text{Res}(P, Q)$ of two polynomials P, Q . If $\deg P = n$ and $\deg Q = m$, with leading coefficients a_n and b_m , and if $\{\alpha_i\}$ and $\{\beta_j\}$ are the roots of P and Q respectively (counted with multiplicity), then one has the standard product identities

$$\text{Res}(P, Q) = a_n^m \prod_{i=1}^n Q(\alpha_i) = b_m^n \prod_{j=1}^m P(\beta_j).$$

Apply this with $P = f$ and $Q = f'$. Since f is monic of degree 3 and $\deg f' = 2$, we get

$$\text{Res}(f, f') = \prod_{k=1}^3 f'(z_k). \tag{A.1}$$

On the other hand, $f'(z_k) = \prod_{j \neq k} (z_k - z_j)$, hence

$$\prod_{k=1}^3 f'(z_k) = \prod_{k=1}^3 \prod_{j \neq k} (z_k - z_j) = \prod_{1 \leq i < j \leq 3} (z_i - z_j)(z_j - z_i) = - \prod_{1 \leq i < j \leq 3} (z_i - z_j)^2 = -\Delta(f).$$

Therefore $\text{Res}(f, f') = -\Delta(f)$.

Now use the second product identity for resultants. The leading coefficient of f' is 3, so

$$\text{Res}(f, f') = 3^3 \prod_{f'(\xi)=0} f(\xi) = 27 f(\xi_1) f(\xi_2), \quad (\text{A.2})$$

where the product runs over the roots of f' counted with multiplicity. Combining $\text{Res}(f, f') = -\Delta(f)$ with (A.2) yields

$$27 f(\xi_1) f(\xi_2) = -\Delta(f),$$

as claimed. $\square$

We now proceed by cases on the number m of connected components of $\Omega(f)$. By Claim 1, each component contains at least one zero of f , so $m \in \{1, 2, 3\}$.

Case 1: $m = 1$ (connected). By Lemma 5.2, we obtain $\Lambda(f) \geq 2\pi$.

Case 2: $m = 2$ (two components). Numerical evidence suggests that in this case one always has $\Lambda(f) \geq 3.06$, and moreover that the polynomial which minimizes $\Lambda(f)$ should be $(z+1)^2(z-1)$. At present, however, I do not know how to prove this rigorously.

Case 3: $m = 3$ (three components). Then each component contains exactly one zero, and all zeros are simple. Let ξ_1, ξ_2 be the critical points of f . By [3, Lemma 2.5], we know that

$$|f(\xi_1)| \geq 1, \quad |f(\xi_2)| \geq 1. \quad (\text{A.3})$$

Now by Claim 2 and the definition of $\Delta(f)$,

$$|\Delta(f)| = \prod_{i < j} |z_i - z_j|^2 \leq (3\sqrt{3})^2 = 27.$$

By Claim 3,

$$|f(\xi_1) f(\xi_2)| = \frac{|\Delta(f)|}{27} \leq 1.$$

Together with (A.3), this forces $|f(\xi_1)| = |f(\xi_2)| = 1$ and $|\Delta(f)| = 27$. Hence equality holds in Claim 2, so the zeros are the vertices of a regular triangle on $|z| = 1$. Therefore $f(z) = z^3 - e^{i\theta}$ for some $\theta \in \mathbb{R}$. By Proposition 2.1, clearly we have $\Lambda(f) = L_3 \approx 3.05991$.

Putting the cases together: in Cases 1 and 2 we have $\Lambda(f) \geq 3.06$, while in Case 3 necessarily $\Lambda(f) = L_3$. Therefore, $\inf \Lambda(f) = L_3$, with equality exactly in the rotated $z^3 - 1$ family. $\square$