QuiverCombinatoricsTools

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CONTENTS

QuiverCombinatoricsTools is a SageMath package that adds combinatorial functions to *QuiverTools* to calculate symplectic leaves of quiver varieties, available here https://github.com/QuiverCombinatoricsTools/QuiverCombinatoricsTools. This page can be read as a pdf. It adds on to the *QuiverTools* package written by Pieter Belmans, Hans Franzen and Gianni Petrella, as seen here https://sage.quiver.tools/ and here https://github.com/QuiverTools/QuiverTools, so consult their documentation when needed.

To install it, make sure you have both *QuiverTools* and *QuiverCombinatoricsTools*

```
sage --pip install git+https://github.com/QuiverTools/QuiverTools.git
sage --pip install git+https://github.com/QuiverCombinatoricsTools/
↪QuiverCombinatoricsTools.git
```

and then you can simply run

```
from quiver import *
from quivercombinatorics import *
```

to get started. You can also run it online here through binder

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We were supervised by Gwyn Bellamy (University of Glasgow), as part of an AGQ computing project.

How to cite QuiverCombinatoricsTools

If you have used this code in any way, please consider citing it in the following way

```
@software{quivercombinatoricstools,
   author = {Caba, Tudor-Ioan and Lam, Mia and Roth, Emanuel},
   title = {QuiverCombinatoricsTools},
   url = {https://quivercombinatoricstools.github.io},
}
```

CHAPTER

ONE

GENERATING QUIVERS

Here are some functions that help generate quivers to test examples.

quivercombinatorics.quivercombinatorics.**quiver_from_cartan_matrix**(*C*)

Returns the quiver Q given by a Cartan matrix C

INPUT:

- `C` – a square matrix with entries in $\mathbb{Z}$

OUTPUT: The quiver Q given by a Cartan matrix C

EXAMPLE:

```
sage: from quivercombinatorics import *
sage: C = [[2, -1, 0], [-1, 2, -2], [0, -2, 2]]
sage: quiver_from_cartan_matrix(C)
a quiver with 3 vertices and 3 arrows
```

quivercombinatorics.quivercombinatorics.**random_quiver**(*vertices*, *max_arrows_per_edge*)

Returns a randomly generated quiver

INPUT:

- `vertices` – number of vertices you want in the quiver, in $\mathbb{N}_0$
- `max_arrows_per_edge` – maximum of number of arrows you want in the quiver, in $\mathbb{N}_0$

OUTPUT: A randomly generated quiver with the prescribed vertices and maximum number of edges

EXAMPLES:

```
sage: from quivercombinatorics import *
sage: random_quiver(3, 2)
a quiver with 3 vertices and 6 arrows

sage: from quivercombinatorics import *
sage: random_quiver(46, 25)
a quiver with 46 vertices and 26388 arrows
```

We notate quivers by Q, with vertices in Q_0 and edges in Q_1.

CHAPTER

TWO

CONSTRUCTING Σ_λ

We follow William Crawley-Boevey in this paper. Given a quiver Q with n vertices, and $\lambda \in \mathbb{Z}^n$, Σ_λ is the set of $\alpha \in \mathbb{N}^n$ such that α is a positive root of Q, $\alpha \cdot \lambda = 0$, and

$$p(\alpha) > \sum_{t=1}^{r} p(\beta^{(t)})$$

for any decomposition $\alpha = \beta^{(1)} + \cdots + \beta^{(r)}$ with $r \geq 2$ and $\beta^{(t)}$ a positive root of Q with $\lambda \cdot \beta^{(t)} = 0$ for all t.

`Quiver.`**`p_function`**`(x)`

Outputs the function $p(x) = 1 - \frac{1}{2}(x, x)$, where (x, x) is the symmetrized Euler form. Note that $p(x) \geq 0$ if x is a root, and $p(x) = 0$ if and only if x is a real root

INPUT:

- `x` – an element of $\mathbb{Z}Q_0$

OUTPUT: $1 - \frac{1}{2}(x, x)$

EXAMPLE:

```
sage: from quivercombinatorics import *
sage: Q = Quiver([[0, 1], [1, 0]])
sage: Q.p_function((2, 3))
0.0
```

We write R_λ^+ to denote the set of positive roots α with $\alpha \cdot \lambda = 0$ and $\mathbb{N}R_\lambda^+$ for the set of sums of elements of R_λ^+. Using Theorem 5.6 of Crawley-Boevey's paper.

Theorem (Crawley-Boevey, 2001)

For $\alpha \in \mathbb{N}^n$, then $\alpha \in \Sigma_\lambda$ if and only if $0 \neq \alpha \in \mathbb{N}R_\lambda^+$ and $(\beta, \alpha - \beta) \leq -2$ whenever $\beta \in \mathbb{N}^n$ and $0 < \alpha < \beta$.

`Quiver.`**`R_lambda_plus`**`(l, v)`

Returns a list of elements of R_λ^+, the set of positive roots α with $\alpha \cdot \lambda = 0$, up to the upper bound v

INPUT:

- `l` – an element of $\mathbb{Z}Q_0$
- `v` – an element of $\mathbb{N}Q_0$

OUTPUT: A list of elements of R_λ^+, where `l` is λ, the set of positive roots α with $\alpha \cdot \lambda = 0$, up to the upper bound v

EXAMPLE:

```
sage: from quivercombinatorics import *
sage: Q = Quiver([[0, 1], [1, 0]])
sage: Q.R_lambda_plus((1, -1), (5, 5))
[(1, 1), (2, 2), (3, 3), (4, 4)]
```

We also need the helper function `N_set`.

`quivercombinatorics.quivercombinatorics.`**`N_set`**`(`S`, `v`)`

For a list of vectors S in $\mathbb{Z}Q_0$, returns all possible sums of vectors in S as a list, up to the upper bound v. Though not directly about quivers, this is a helper function to define Σ_λ

INPUT:

- `S` – a list of vectors S in $\mathbb{Z}Q_0$
- `v` – vector in $\mathbb{Z}Q_0$

OUTPUT: A list of all possible sums of vectors in S as a list, up to the upper bound v

EXAMPLE:

```
sage: from quivercombinatorics import *
sage: N_set([[0, 1]], [0, 3])
[(0, 1), (0, 2), (0, 3)]
```

With these functions, we can define Σ_λ.

`Quiver.`**`sigma_lambda`**`(`l`, `v`)`

Returns a list of elements of Σ_λ, up to the upper bound v

INPUT:

- `l` – an element of $\mathbb{Z}Q_0$
- `v` – an element of $\mathbb{N}Q_0$

OUTPUT: A list of elements of Σ_λ, where `l` is λ, up to the upper bound v

EXAMPLE:

```
sage: from quivercombinatorics import *
sage: Q = Quiver([[0, 1], [1, 0]])
sage: Q.sigma_lambda((1, -1), (5, 5))
[(1, 1)]
```

CHAPTER

THREE

CB-DECOMPOSITIONS

A *representation type* of x of a quiver Q is a sequence $\tau = (\beta^{(1)}, n_1; \ldots; \beta^{(k)}, n_k)$ with $\beta^{(i)} \in \Sigma_\lambda$ and $n_i \in \mathbb{N}$ such that

$$x = \sum_{i=1}^{k} n_i \beta^{(i)}$$

We allow $\beta^{(i)}$ to occur multiple times if it is an imaginary root, i.e. $p(\beta^{(i)}) > 0$. The *CB-decomposition* of x is defined to be the unique representation type of x for which

$$p(\tau) := \sum_{i=1}^{k} p(\beta^{(i)})$$

is maximal. In order to construct CB-decompositions (called canonical decompositions in this paper, but are *not* the same as `canonical_decomposition` from *QuiverTools*). We first need to find all representation types of x, up to a bound v, for which we need the following helper functions.

`quivercombinatorics.quivercombinatorics.`**`vector_decomposition`**`(`x, S`)`

For a list of vectors S in $\mathbb{Z}Q_0$, returns all possible sums of vectors in S as a list that sum up to x. This is a helper function in order to define CB-decompositions

INPUT:

- `x` – a vector in $\mathbb{Z}Q_0$
- `S` – a list of vectors S in $\mathbb{Z}Q_0$

OUTPUT: All possible sums of vectors in S as a list that sum up to x

EXAMPLE:

```
sage: from quivercombinatorics import *
sage: vector_decomposition((4, 5), [(0, 1), (1, 0), (1, 1)])
[[[(0, 1), 1], [(1, 1), 4]],
[[(0, 1), 2], [(1, 0), 1], [(1, 1), 3]],
[[(0, 1), 3], [(1, 0), 2], [(1, 1), 2]],
[[(0, 1), 4], [(1, 0), 3], [(1, 1), 1]],
[[(0, 1), 5], [(1, 0), 4]]]
```

`quivercombinatorics.quivercombinatorics.`**`small_decomposition`**`(`v, n`)`

For a vector v in $\mathbb{Z}Q_0$, it returns all possible partitions of the representation type $[v, n]$

INPUT:

- `v` – a vector in $\mathbb{Z}Q_0$
- `n` – a natural number in $\mathbb{N}$

OUTPUT: All possible partitions of the representation type $[v, n]$

EXAMPLE:

```
sage: from quivercombinatorics import *
sage: small_decomposition((1, 1), 3)
[[[(1, 1), 1], [(1, 1), 1], [(1, 1), 1]],
[[(1, 1), 2], [(1, 1), 1]],
[[(1, 1), 3]]]
```

With these functions, we can determine all representation types.

`Quiver.`**`all_representation_types`**(l, x)

Returns a list of representation types of a quiver, with respect to x

INPUT:

- `l` – an element of $\mathbb{Z}Q_0$
- `x` – an element of $\mathbb{N}Q_0$

OUTPUT: A list of representation types of a quiver, with respect to x, and where `l` is λ. Each representation type is stored as a list whose elements are 2-tuples, the first element is in $\mathbb{N}Q_0$, and the second is in $\mathbb{N}$

EXAMPLES:

```
sage: from quivercombinatorics import *
sage: Q = Quiver([[0, 1], [1, 0]])
sage: Q.all_representation_types((1, -1), (5, 5))
[[[(1, 1), 1], [(1, 1), 1], [(1, 1), 1], [(1, 1), 1], [(1, 1), 1]],
[[(1, 1), 1], [(1, 1), 1], [(1, 1), 1], [(1, 1), 2]],
[[(1, 1), 1], [(1, 1), 1], [(1, 1), 3]],
[[(1, 1), 1], [(1, 1), 2], [(1, 1), 2]],
[[(1, 1), 1], [(1, 1), 4]],
[[(1, 1), 2], [(1, 1), 3]],
[[(1, 1), 5]]]

sage: from quivercombinatorics import *
sage: C = [[2, -1, 0], [-1, 2, -2], [0, -2, 2]]
sage: Q = quiver_from_cartan_matrix(C)
sage: Q.all_representation_types((0, 0, 0), (2, 4, 3))
[[[(0, 0, 1), 1],
[(0, 1, 0), 2],
[(0, 1, 1), 1],
[(0, 1, 1), 1],
[(1, 0, 0), 2]],
[[(0, 0, 1), 1], [(0, 1, 0), 2], [(0, 1, 1), 2], [(1, 0, 0), 2]],
[[(0, 0, 1), 2], [(0, 1, 0), 3], [(0, 1, 1), 1], [(1, 0, 0), 2]],
[[(0, 0, 1), 3], [(0, 1, 0), 4], [(1, 0, 0), 2]],
[[(0, 1, 0), 1],
[(0, 1, 1), 1],
[(0, 1, 1), 1],
[(0, 1, 1), 1],
[(1, 0, 0), 2]],
[[(0, 1, 0), 1], [(0, 1, 1), 1], [(0, 1, 1), 2], [(1, 0, 0), 2]],
[[(0, 1, 0), 1], [(0, 1, 1), 3], [(1, 0, 0), 2]],
[[(2, 4, 3), 1]]]
```

Representation types of x correspond to a symplectic leaf of the quiver variety $M_0(Q,x)$, and the symplectic leaf dimension is $2p$ applied to the representation type, where p is the `p_function`.

`Quiver.`**`symplectic_leaf_dimension`**`(`*`tau`*`)`

Returns the dimension of the symplectic leaf corresponding to the representation type τ

INPUT:

- `tau` – a representation type, i.e., a list whose elements are 2-tuples, the first element is in $\mathbb{N}Q_0$, and the second is in $\mathbb{N}$

OUTPUT: The corresponding symplectic leaf dimension, i.e., $2p$ applied to every element of τ

EXAMPLE:

```
sage: from quivercombinatorics import *
sage: Q = Quiver([[0, 1], [1, 0]])
sage: Q.symplectic_leaf_dimension([[(1, 1), 2], [(1, 1), 3]])
4
```

`Quiver.`**`CB_decomposition`**`(`l, x`)`

Returns the CB decomposition, which is the representation type maximizing p

INPUT:

- `l` – an element of $\mathbb{Z}Q_0$
- `x` – an element of $\mathbb{N}Q_0$

OUTPUT: The CB decomposition with respect to x, where `l` is λ

EXAMPLE:

```
sage: from quivercombinatorics import *
sage: Q = Quiver([[0, 1], [1, 0]])
sage: Q.CB_decomposition((1, -1), (5, 5))
[[(1, 1), 1], [(1, 1), 1], [(1, 1), 1], [(1, 1), 1], [(1, 1), 1]]
```

Knowing the CB-decomposition, it is then easy to calculate the dimension of the quiver variety $M_0(Q,x)$, as it gives the largest (open) symplectic leaf inside $M_0(Q,x)$.

`Quiver.`**`quiver_variety_dimension`**`(`l, x`)`

Returns the dimension of the quiver variety

INPUT:

- `l` – an element of $\mathbb{Z}Q_0$
- `x` – an element of $\mathbb{N}Q_0$

OUTPUT: The dimension of the corresponding quiver variety

EXAMPLE:

```
sage: from quivercombinatorics import *
sage: Q = Quiver([[0, 1], [1, 0]])
sage: Q.quiver_variety_dimension((1, -1), (5, 5))
10
```

It's often important to know the codimension 2 leaves of the quiver variety $M_0(Q,x)$, so we code this here.

Quiver.**codimension_two_leaves**(l, x)

Returns the codimension 2 leaves of the quiver variety

INPUT:

- l – an element of $\mathbb{Z}Q_0$
- x – an element of $\mathbb{N}Q_0$

OUTPUT: The representation types of the codimension 2 leaves of the quiver variety

EXAMPLE:

```
sage: from quivercombinatorics import *
sage: Q = Quiver([[0, 1], [1, 0]])
sage: Q.codimension_two_leaves((0, 0), (5, 5))
[[[(0, 1), 1],
[(1, 0), 1],
[(1, 1), 1],
[(1, 1), 1],
[(1, 1), 1],
[(1, 1), 1]],
[[(1, 1), 1], [(1, 1), 1], [(1, 1), 1], [(1, 1), 2]]]
```

CHAPTER

FOUR

ext-QUIVERS

Let $\tau = (\beta^{(1)}, n_1; \ldots; \beta^{(k)}, n_k)$ be a representation type of Q. The ext-*quiver* $\tilde{Q}$ associated to the symplectic leaf is the quiver with k vertices where the number of edges between vertex i and vertex j is $-(\beta^{(i)}, \beta^{(j)})$ and the number of loops at vertex i is $p(\beta^{(i)})$. The corresponding dimension vector of $\tilde{Q}$ is $\mathbf{n} = (n_1, \ldots, n_k)$, and we take $\tilde{\lambda} = 0$.

`Quiver.`**`ext_quiver`**`(`*`tau`*`)`

Given a representation type, returns the ext-quiver

INPUT:

- `tau` – a representation type, i.e., a list whose elements are 2-tuples, the first element is in $\mathbb{N}Q_0$, and the second is in $\mathbb{N}$

OUTPUT: The ext-quiver

EXAMPLE:

```
sage: from quivercombinatorics import *
sage: Q = Quiver([[0, 1], [1, 0]])
sage: tau = [[(1, 1), 2], [(1, 1), 1], [(1, 1), 1], [(1, 1), 1]]
sage: Q.ext_quiver(tau)
a quiver with 4 vertices and 4 arrows
```

`quivercombinatorics.quivercombinatorics.`**`ext_dimension_vector`**`(`*`tau`*`)`

For a representation type τ, returns the associated dimension vector

INPUT:

- `tau` – a representation type, i.e., a list whose elements are 2-tuples, the first element is in $\mathbb{N}Q_0$, and the second is in $\mathbb{N}$

OUTPUT: a vector in $\mathbb{N}Q_0$

EXAMPLE:

```
sage: from quivercombinatorics import *
sage: tau = [[(1, 1), 2], [(1, 1), 1], [(1, 1), 1], [(1, 1), 1]]
sage: ext_dimension_vector(tau)
(2, 1, 1, 1)
```

CHAPTER

FIVE

CLASSIFICATION OF MINIMAL DEGENERATIONS OF SYMPLECTIC LEAVES

For a quiver Q, a positive root α is *minimal imaginary* if it is imaginary and given any positive root β, $\alpha > \beta$ implies β is real.

`Quiver.`**`is_minimal_imaginary_root`**`(`x`)`

Tests whether a vector is a vector is a minimal imaginary root

INPUT:

- `x` – a vector in $\mathbb{Z}Q_0$

OUTPUT: `True` if `x` is a minimal imaginary root, and `False` otherwise

EXAMPLE:

```
sage: from quivercombinatorics import *
sage: Q = KroneckerQuiver(2)
sage: Q.is_minimal_imaginary_root((1, 1))
True
```

`Quiver.`**`all_minimal_imaginary_positive_roots`**`(`v`)`

Returns all minimal imaginary positive roots up to a bound v

INPUT:

- `v` – a vector in $\mathbb{N}Q_0$

OUTPUT: A list of all minimal imaginary positive roots up to a bound v

EXAMPLE:

```
sage: from quivercombinatorics import *
sage: Q = KroneckerQuiver(2)
sage: Q.all_minimal_imaginary_positive_roots((3, 3))
[(1, 1)]
```

Given a quiver Q, a *subquiver* $T = (T_0, T_1)$ is specified by a subset $T_0 \subset Q_0$ of the vertices. Then, for $i, j \in T_0 \subset Q_0$ the number of edges between vertex i and j in T equals the number of edges between vertex i and j in Q. In particular, if $i \in T_0$ then the number of loops at i in T is the same as the number of loops at i in Q. The support of a dimension vector v is the subset $T_0 \subset Q_0$ of all vertices i such that $v_i \neq 0$. We think of the support of v as a subquiver of Q.

We say that the subquiver T is of *affine ADE type* if it is one of the graphs appearing in the classification of simply-laced affine Dynkin diagrams $\tilde{A}_l$, $\tilde{D}_l$ or $\tilde{E}_6$, $\tilde{E}_7$, $\tilde{E}_8$. Then there is a unique imaginary root δ, with $p(\delta) = 1$, whose support is T. The key result is the classification of minimal imaginary roots is the following (Bellamy-Schedler 2026).

Theorem (Bellamy-Schedler, 2026)

If M is a closed leaf in $M_0(Q,x)$ and $M \subset \bar{L}$ is a minimal degeneration, then $\bar{L} \cong M \times S$, where S is isomorphic to one of the following isolated singularities:

1. A Kleinian singularity $(\mathbb{C}^2/\Gamma, 0)$.
2. The type A minimal nilpotent orbit closure $(\mathcal{O}_{\min}, 0)$ in $\mathfrak{sl}_r(\mathbb{C})$.
3. $(\mathbb{C}^{2g}/\mathbb{Z}_2, 0)$.
4. $\mathrm{Spec}(\mathbb{C}[x \mid \deg x \geq 2])$.

We define:

$$\tilde{\tau}(\beta, n) := (\beta, n; e_1, \alpha_1 - n\beta_1; \ldots; e_r, \alpha_r - n\beta_r)$$

In each of the above cases, the leaf L from this theorem corresponds to representation types:

1. $\tilde{\tau}(\delta, n)$ for δ a minimal imaginary root on an affine ADE subquiver.
2. $\tilde{\tau}((1,1), n)$ for a two-vertex subquiver with $t \geq 3$ edges between the vertices and no loops at the vertices.
3. $$\tau = (e_i, a; e_i, a; e_1, \alpha_1; \ldots; e_{i-1}, \alpha_{i-1}; e_{i+1}, \alpha_{i+1}; \ldots)$$
   where i is a vertex with $g \geq 1$ loops and $2a = \alpha_i$.
4. $$\tau = (e_i, a; e_i, b; e_1, \alpha_1; \ldots; e_{i-1}, \alpha_{i-1}; e_{i+1}, \alpha_{i+1}; \ldots)$$
   where i is a vertex with $g \geq 1$ loops and $0 < a \neq b < \alpha_i$ with $a + b = \alpha_i$.

We assign the following labels to each subminimal representation type (i.e., representation types corresponding to a minimal degeneration):

1. A_l, D_l or E_6, E_7, E_8 for affine Dynkin diagram of type $\tilde{A}_l, \tilde{D}_l$ or $\tilde{E}_6, \tilde{E}_7, \tilde{E}_8$.
2. a_{r-1}.
3. c_g.
4. m_g.

`Quiver.all_subminimal_representation_types(v)`

Returns all subminimal representation types up to a bound v, and classifies them by affine Dynkin type, or a_{r-1}, c_g, m_g

INPUT:

- `v` – a vector in $\mathbb{N}Q_0$

OUTPUT: A list of 2-tuples, the first element is a subminimal representation type up to a bound v, the second element is its corresponding affine Dynkin type, or a_{r-1}, c_g, m_g

EXAMPLES:

```
sage: from quivercombinatorics import *
sage: Q = CyclicQuiver(3)
sage: Q.all_subminimal_representation_types((3, 3, 3))
[([[(0, 0, 1), 2], [(0, 1, 0), 2], [(1, 0, 0), 2], [(1, 1, 1), 1]], 'A_{2}'),
([[(0, 0, 1), 1], [(0, 1, 0), 1], [(1, 0, 0), 1], [(1, 1, 1), 2]], 'A_{2}'),
([[(1, 1, 1), 3]], 'A_{2}')]

sage: from quivercombinatorics import *
sage: Q = LoopQuiver(3)
sage: Q.all_subminimal_representation_types((5))
[([[(1), 1], [(1), 4]], 'm_{3}'), ([[(1), 2], [(1), 3]], 'm_{3}')]

sage: from quivercombinatorics import *
sage: A = [[0, 1, 0, 3], [1, 0, 0, 0], [0, 0, 2, 0], [0, 0, 0, 0]]
sage: Q = Quiver(A)
sage: Q.all_subminimal_representation_types((1, 1, 4, 1))
[([[(0, 0, 0, 1), 1], [(0, 0, 1, 0), 4], [(1, 1, 0, 0), 1]], 'A_{1}'),
([[(0, 0, 1, 0), 4], [(0, 1, 0, 0), 1], [(1, 0, 0, 1), 1]], 'a_{2}'),
([[(0, 0, 0, 1), 1],
[(0, 0, 1, 0), 1],
[(0, 0, 1, 0), 3],
[(0, 1, 0, 0), 1],
[(1, 0, 0, 0), 1]],
'm_{2}'),
([[(0, 0, 0, 1), 1],
[(0, 0, 1, 0), 2],
[(0, 0, 1, 0), 2],
[(0, 1, 0, 0), 1],
[(1, 0, 0, 0), 1]],
'c_{2}')]
```

CHAPTER

SIX

PLOTTING MINIMAL DEGENERATIONS OF SYMPLECTIC LEAVES IN HASSE DIAGRAMS

To compute these Hasse diagrams, make sure you have `dot2tex` and `graphviz` installed.

```
sage --pip install dot2tex
sage --pip install graphviz
```

We visualize the poset of symplectic leaves (equivalently of representation types) as a Hasse diagram: The vertices in this diagram are the representation types and the minimal degenerations, represented by an edge connecting two representation types, correspond to the situation where $L_1 \leq L_2$ (equivalently, $\tau_1 \leq \tau_2$) but there does not exist L_3 such that $L_1 < L_3 < L_2$. We wish to compute the Hasse diagram, for which there are two methods.

Method 1

Traverse the Hasse diagram from the bottom, starting with the lowest leaves. When we reach a leaf L_i, we compute the ext-quiver associated to L_i. On this ext-quiver, we compute subminimal representation types of $\mathbf{n}$. For each such representation type $\tilde{\tau} = (\gamma^{(1)}, m_1; \ldots; \gamma^{(r)}, m_r)$, define:

$$D(\tilde{\tau}) := \left(D(\gamma^{(1)}), m_1; \ldots; D(\gamma^{(r)}), m_r\right)$$

By (Bellamy-Schedler 2026), it is known that $D(\tilde{\tau})$ is a representation type for v.

In this way, we have a rule that assigns to each subminimal representation type of $\tilde{Q}$ a representation type $D(\tilde{\tau}(\alpha))$ of v. It is known that $\tau < D(\tilde{\tau}(\alpha))$ is a minimal degeneration and they all occur in this way. Proceeding in this way allows one to build the Hasse diagram from the bottom up.

For Method 1, we need the following helper functions.

`quivercombinatorics.quivercombinatorics.`**`D_map`**(*m*, *tau*)

Evaluates a map $D : \mathbb{Z}^k \to \mathbb{Z}^n$ from dimension vectors of the ext-quiver to dimension vectors of the original quiver by $D(m_1, \ldots, m_k) := \sum_{i=1}^{k} m_i \beta^{(i)}$

INPUT:

- `m` – a vector in $\mathbb{Z}\,\mathrm{ext}(Q)_0$
- `tau` – a representation type of the original quiver, i.e., a list whose elements are 2-tuples, the first element is in $\mathbb{N}Q_0$, and the second is in $\mathbb{N}$

OUTPUT: a vector in $\mathbb{Z}Q_0$

EXAMPLE:

```
sage: from quivercombinatorics import *
sage: tau = [[(1, 1), 2], [(1, 1), 1], [(1, 1), 1], [(1, 1), 1]]
sage: D_map((1, 2, 8, -3), tau)
(8, 8)
```

quivercombinatorics.quivercombinatorics.**D_lifting**(*tau*, *L*)

Applies D_map to a representation type of the ext-quiver

INPUT:

- L – a representation type of the ext-quiver, i.e., a list whose elements are 2-tuples, the first element is in $\mathbb{N}\text{ext}(Q)_0$, and the second is in $\mathbb{N}$
- tau – a representation type, i.e., a list whose elements are 2-tuples, the first element is in $\mathbb{N}Q_0$, and the second is in $\mathbb{N}$

OUTPUT: a representation type of the ext-quiver

EXAMPLE:

```
sage: from quivercombinatorics import *
sage: tau = [[(1, 1), 2], [(1, 1), 1], [(1, 1), 1], [(1, 1), 1]]
sage: D_lifting([[(1, 2, 8, -3), 3]], tau)
[[(8, 8), 3]]
```

Quiver.**minimal_degenerations**(*L*)

Returns all minimal degenerations of a a symplectic leaf, corresponding to the representation type L

INPUT:

- L – a representation type, i.e., a list whose elements are 2-tuples, the first element is in $\mathbb{N}Q_0$, and the second is in $\mathbb{N}$

OUTPUT: A list of representation types

EXAMPLES:

```
sage: from quivercombinatorics import *
sage: A = [[0, 1, 0, 3], [1, 0, 0, 0], [0, 0, 2, 0], [0, 0, 0, 0]]
sage: Q = Quiver(A)
sage: Q.minimal_degenerations([[(0, 0, 0, 1), 1], [(0, 0, 1, 0), 4], [(1, 1, 0,␣
↪0), 1]])
[([[(0, 0, 1, 0), 4], [(1, 1, 0, 1), 1]], '$a_{2}(1)$'),
([[(0, 0, 0, 1), 1], [(0, 0, 1, 0), 1], [(0, 0, 1, 0), 3], [(1, 1, 0, 0), 1]],
'$m_{2}(1)$'),
([[(0, 0, 0, 1), 1], [(0, 0, 1, 0), 2], [(0, 0, 1, 0), 2], [(1, 1, 0, 0), 1]],
'$c_{2}(1)$')]
```

The second method to compute this Hasse diagram is as follows.

Method 2

Consider the larger set $\mathcal{D}$ of all decompositions of a given dimension vector v. Here, a decomposition is simply a way of writing v as a sum of smaller dimension vectors with multiplicities:

$$\left(n_1, \beta^{(1)}; \ldots; n_k, \beta^{(k)}\right),$$

where the $\beta^{(i)}$ are dimension vectors satisfying $\beta^{(i)} \leq v$. Following this paper, we say that:

$$\varepsilon = (p_1, \gamma^{(1)}; \ldots; p_l, \gamma^{(l)}) > \rho = (m_1, \alpha^{(1)}; \ldots; m_r, \alpha^{(r)})$$

is a *direct successor* (ε is the successor of ρ) if either:

1. $l = r + 1$ and ε can be ordered such that for all $1 \leq i \leq r - 1$ we have $(m_i, \alpha^{(i)}) = (p_i, \gamma^{(i)})$ and $m_r = p_r = p_{r+1}, \gamma^{(r)} + \gamma^{(r+1)} = \alpha^{(r)}$.
2. $l = r - 1$ and ε can be ordered such that for all $1 \leq i \leq r - 2$ we have $(m_i, \alpha^{(i)}) = (p_i, \gamma^{(i)})$ and $\alpha^{(r)} = \alpha^{(r-1)} = \gamma^{(r-1)}, m_{r-1} + m_r = p_{r-1}$.

The symplectic leaves form a subset of this set of decompositions, and by restriction, they inherit a sub-poset structure. It is shown in (Bellamy-Schedler, 2026) that this is the partial ordering given by leaf closure. We need the following helper functions for Method 2.

`Quiver.`**`all_decompositions`**`(`*v*`)`

Constructs all decompositions of a given dimension vector v

INPUT:

- `v` – an element of $\mathbb{N}Q_0$

OUTPUT: A list of decompositions of *v*

EXAMPLE:

```
sage: from quivercombinatorics import *
sage: A = [[0, 1, 0, 3], [1, 0, 0, 0], [0, 0, 2, 0], [0, 0, 0, 0]]
sage: Q = Quiver(A)
sage: Q.all_decompositions((1, 0, 2, 0))
[[[(0, 0, 1, 0), 1], [(0, 0, 1, 0), 1], [(1, 0, 0, 0), 1]],
[[(0, 0, 1, 0), 1], [(1, 0, 1, 0), 1]],
[[(0, 0, 1, 0), 2], [(1, 0, 0, 0), 1]],
[[(0, 0, 2, 0), 1], [(1, 0, 0, 0), 1]],
[[(1, 0, 2, 0), 1]]]
```

`quivercombinatorics.quivercombinatorics.`**`is_direct_successor`**`(`*epsilon*, *rho*`)`

Verifies whether `epsilon` is the direct successor of `rho`

INPUT:

- `epsilon` – an element of $\mathbb{N}^n$
- `rho` – an element of $\mathbb{N}^n$

OUTPUT: `True` if `epsilon` is the direct successor of `rho`, and `False` otherwise

EXAMPLE:

```
sage: from quivercombinatorics import *
sage: rho = [((1, 0), 2), ((1, 0), 3)]
sage: epsilon = [((1, 0), 5)]
sage: is_direct_successor(epsilon, rho)
True
```

We are now able to obtain and plot the Hasse diagram for Methods 1 and 2. Method 1 seems to be much faster, so it is enabled by default.

Quiver.**get_Hasse_diagram**(*l*, *v*, *method=1*)

Applies Method 1 or Method 2 to obtain data for the Hasse diagram of minimal degenerations

INPUT:

- `l` – an element of $\mathbb{Z}Q_0$
- `v` – an element of $\mathbb{N}Q_0$
- `method` – When set to `1`, Method 1 will be used to obtain the Hasse diagram of minimal degenerations. When set to `2`, Method 2 will be used to obtain the Hasse diagram of minimal degenerations.

OUTPUT:

- `leaves_poset` – the underlying poset of the Hasse diagram of minimal degenerations
- `all_leaves` – a list of all symplectic leaves, with respect to their representation types, so a list of representation types, i.e., a list whose elements are 2-tuples, the first element is in $\mathbb{N}Q_0$, and the second is in $\mathbb{N}$
- `dimensions` – a list of dimensions of all the symplectic leaves, using `symplectic_leaf_dimension`
- `edge_labels` – a list of 3-tuples, the first two entries of each tuple define the edge, and the last entry is the edge label of the corresponding minimal degeneration, classified by `all_subminimal_representation_types`

EXAMPLE:

```
sage: from quivercombinatorics import *
sage: C = [[2, -1, 0], [-1, 2, -2], [0, -2, 2]]
sage: Q = quiver_from_cartan_matrix(C)
sage: Q.get_Hasse_diagram((0, 0, 0), (2, 4, 3))
(Finite poset containing 8 elements,
[[[(0, 0, 1), 1],
[(0, 1, 0), 2],
[(0, 1, 1), 1],
[(0, 1, 1), 1],
[(1, 0, 0), 2]],
[[(0, 0, 1), 1], [(0, 1, 0), 2], [(0, 1, 1), 2], [(1, 0, 0), 2]],
[[(0, 0, 1), 2], [(0, 1, 0), 3], [(0, 1, 1), 1], [(1, 0, 0), 2]],
[[(0, 0, 1), 3], [(0, 1, 0), 4], [(1, 0, 0), 2]],
[[(0, 1, 0), 1],
[(0, 1, 1), 1],
[(0, 1, 1), 1],
[(0, 1, 1), 1],
[(1, 0, 0), 2]],
[[(0, 1, 0), 1], [(0, 1, 1), 1], [(0, 1, 1), 2], [(1, 0, 0), 2]],
[[(0, 1, 0), 1], [(0, 1, 1), 3], [(1, 0, 0), 2]],
[[(2, 4, 3), 1]]],
[4, 2, 2, 0, 6, 4, 2, 8],
[(0, 4, 'A_{1}(1)'),
(1, 5, 'A_{1}(1)'),
(1, 0, 'c_{1}(1)'),
(2, 0, 'A_{1}(1)'),
(2, 5, 'A_{1}(1)'),
(3, 2, 'A_{1}(1)'),
(3, 1, 'A_{1}(1)'),
(3, 6, 'A_{1}(1)'),
```

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```
(4, 7, 'D_{4}(1)'),
(5, 4, 'c_{1}(1)'),
(6, 5, 'm_{1}(1)')])
```

Quiver.**plot_Hasse_diagram**(*l*, *v*, *method=1*, *format='tikz'*, *filename='output'*)

Applies Method 1 or Method 2 to plot the Hasse diagram for minimal degenerations

INPUT:

- `l` – an element of $\mathbb{Z}Q_0$
- `v` – an element of $\mathbb{N}Q_0$
- `method` – When set to `1`, Method 1 will be used to obtain the Hasse diagram of minimal degenerations. When set to `2`, Method 2 will be used to obtain the Hasse diagram of minimal degenerations.
- `format` – Takes values "tikz", "dot", and "sage", depending on whether you want a `.tex` file with *tikzpicture*, a `.dot` file, or a sage Hasse diagram output
- `filename` – filename of output

OUTPUT:

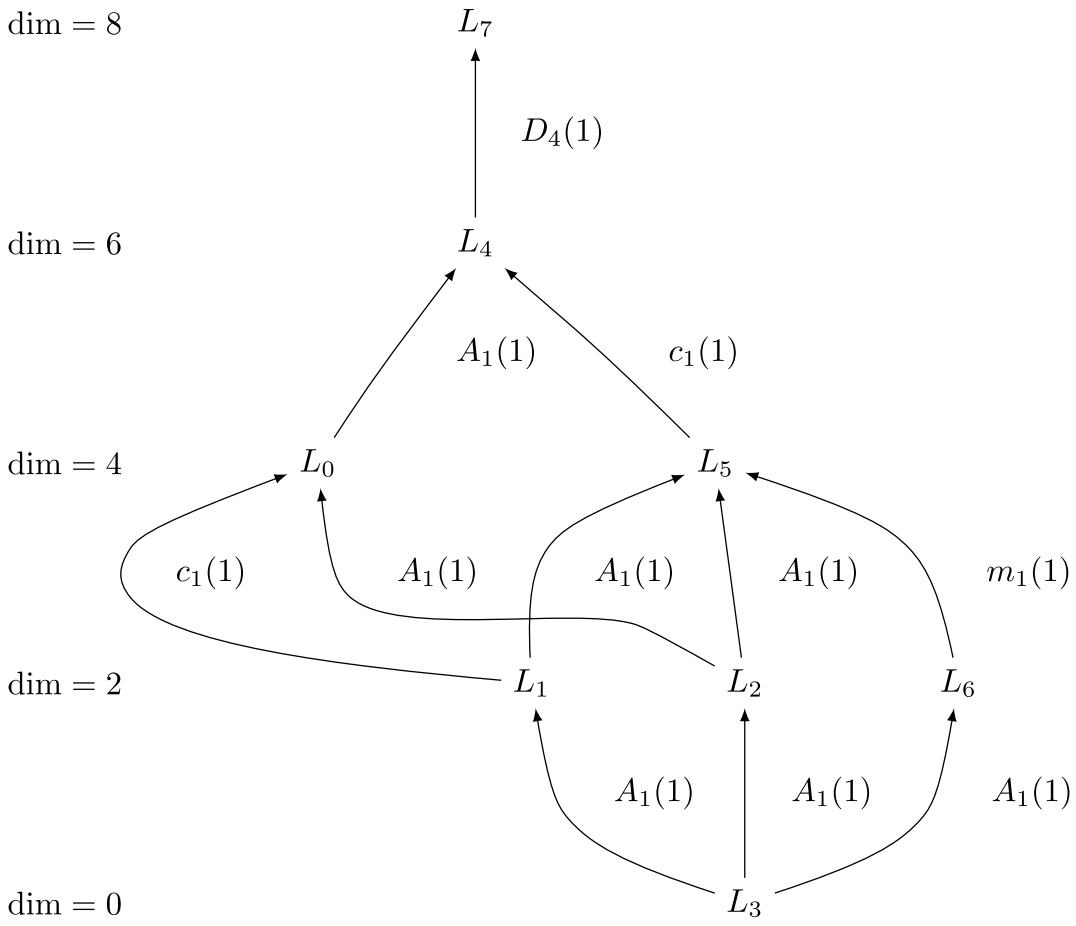
- the Hasse diagram for minimal degenerations

EXAMPLE:

```
sage: C = [[2, -1, 0], [-1, 2, -2], [0, -2, 2]]
sage: Q = quiver_from_cartan_matrix(C)
sage: Q.plot_Hasse_diagram((0, 0, 0), (2, 4, 3))
\documentclass[tikz]{standalone}
\begin{document}

\begin{tikzpicture}[>=latex,line join=bevel,]
%%
\begin{scope}
    \pgfsetstrokecolor{black}
---
77 lines not printed (5057 characters in total).
Use print to see the full content.
---
    \draw (221.5bp,185.5bp) node {$c_{1}(1)$};
    \draw [->] (6) ..controls (293.59bp,110.99bp) and (289.95bp,119.79bp)  ..␣
↪(284.0bp,126.0bp) .. controls (277.86bp,132.41bp) and (269.81bp,137.25bp)  ..␣
↪(5);
    \draw (320.5bp,118.5bp) node {$m_{1}(1)$};
%
\end{tikzpicture}
\end{document}
```

The example above should export the following diagram:


dim = 8
L_7
$D_4(1)$
dim = 6
L_4
$A_1(1)$
$c_1(1)$
dim = 4
L_0
L_5
$c_1(1)$
$A_1(1)$
$A_1(1)$
$A_1(1)$
$m_1(1)$
dim = 2
L_1
L_2
L_6
$A_1(1)$
$A_1(1)$
$A_1(1)$
dim = 0
L_3

INDEX