Deep Learning for Image Analysis

A brief introduction to Magnetic Resonance Imaging (MRI) reconstruction, and tutorial 2

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Overview

MRI data acquisition

Brief recap on the 2D discrete Fourier transform

MRI acceleration and reconstruction

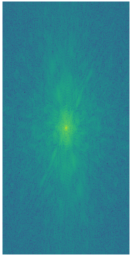
Magnetic Resonance Imaging (MRI)

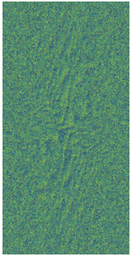
- MRI is a "very powerful **diagnostic tool** for a wide range of disorders, including neurological, musculoskeletal, and oncological diseases" [Zbontar et al., 2018].

Figure: Patient being positioned for MR study of the head and abdomen. By Ptrump16 - Own work, CC BY-SA 4.0, https://commons.wikimedia.org/w/index.php?curid=64157788

- "However, the **long acquisition time** in MRI, which can easily exceed 30 minutes, leads to low patient throughput, problems with patient comfort and compliance, artifacts from patient motion, and high exam costs" [Zbontar et al., 2018].

MRI data acquisition





MRI data acquisition

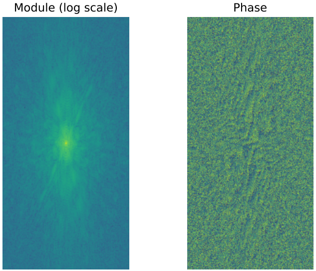
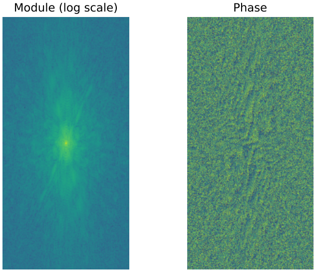


Figure: Raw MRI data in k-space (= Fourier domain)

MRI data acquisition

- Magnetic resonance images are produced from frequency and phase measurements instead of direct, spatially-resolved measurements [Zbontar et al., 2018].
- Each point in the Fourier domain, up to a specified maximum frequency, corresponds to a measurement.



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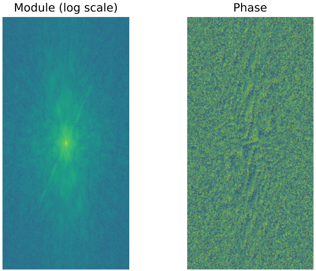
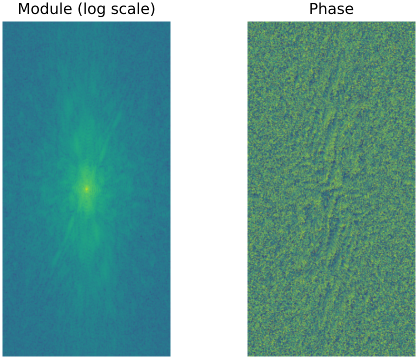


Figure: Raw MRI data in k-space (= Fourier domain)

[Disclaimer] The physics of the acquisition process is beyond the scope of this introduction and my area of expertise.

From k-space to image space



k-space

Inverse Fourier Transform

$\longrightarrow$

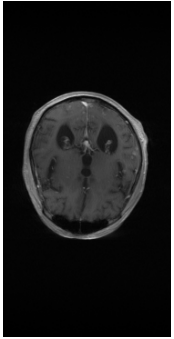


image space

Overview

Brief recap on the 2D Discrete Fourier Transform

- Let us define a grayscale image of $M \times N$ pixels as follows:

$$\begin{aligned} f : \{1, \dots, M\} \times \{1, \dots, N\} &\longrightarrow \mathbb{R} \\ (m, n) &\longmapsto f(m, n) \end{aligned}$$

- The discrete Fourier transform of f is a complex image F of size $M \times N$:

$$\begin{aligned} F : \{1, \dots, M\} \times \{1, \dots, N\} &\longrightarrow \mathbb{C} \\ (u, v) &\longmapsto \sum_{m=1}^{M} \sum_{n=1}^{N} f(m, n) e^{-j2\pi\left(\frac{u(m-1)}{M} + \frac{v(n-1)}{N}\right)} \end{aligned}$$

- The inverse Fourier transform restores the image from the Fourier transform:

$$f(m, n) = \frac{1}{MN} \sum_{u=1}^{M} \sum_{v=1}^{N} F(u, v) e^{j2\pi\left(\frac{(u-1)m}{M} + \frac{(v-1)n}{N}\right)} \tag{1}$$

The Discrete Fourier Transform: a change of basis

- We can write the image as

$$
\begin{aligned}
\begin{bmatrix} f(1,1) & \dots & f(1,N) \\ \vdots & & \vdots \\ f(M,1) & \dots & f(M,N) \end{bmatrix} &= f(1,1) \begin{bmatrix} 1 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \end{bmatrix} + \dots + f(1,N) \begin{bmatrix} 0 & \dots & 1 \\ \vdots & & \vdots \\ 0 & \dots & 0 \end{bmatrix} + \dots \\
&+ f(M,1) \begin{bmatrix} 0 & \dots & 0 \\ \vdots & & \vdots \\ 1 & \dots & 0 \end{bmatrix} + \dots + f(M,N) \begin{bmatrix} 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 1 \end{bmatrix} \\
&= \sum_{m=1}^{M} \sum_{n=1}^{N} f(m,n) E_{m,n}
\end{aligned}
$$

where $(E_{m,n})_{ij} = \begin{cases} 1 & \text{if } i = m \text{ and } j = n, \\ 0 & \text{otherwise.} \end{cases}$ is the canonical basis matrix.

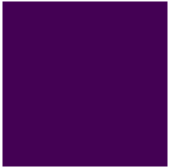
The Discrete Fourier Transform: a change of basis

- Plugging equation 1 into the above, we write the image as

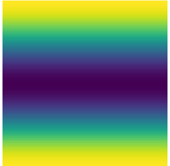
$$
\begin{aligned}
&\frac{1}{MN}\sum_{m=1}^{M}\sum_{n=1}^{N}\sum_{u=1}^{M}\sum_{v=1}^{N} F(u,v)e^{j2\pi(\frac{(u-1)m}{M}+\frac{(v-1)n}{N})}E_{m,n} \\
&= F(1,1)\frac{1}{MN}\sum_{m=1}^{M}\sum_{n=1}^{N}E_{m,n} + \ldots + F(M,N)\frac{1}{MN}\sum_{m=1}^{M}\sum_{n=1}^{N} e^{j2\pi(\frac{(M-1)m}{M}+\frac{(N-1)n}{N})}E_{m,n} \\
&= F(1,1)\frac{1}{MN}\underbrace{\begin{bmatrix}1 & \ldots & 1\\ \vdots & & \vdots \\ 1 & \ldots & 1\end{bmatrix}}_{\mathbf{1}_{M\times N}} + \ldots + F(M,N)\frac{1}{MN}\left[e^{j2\pi(\frac{(M-1)m}{M}+\frac{(N-1)n}{N})}\right]_{\substack{m=1,\ldots,M\\ n=1,\ldots,N}}
\end{aligned}
$$

The Discrete Fourier Transform: a change of basis

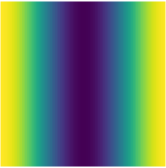
- The image is expressed in a new basis that comprises complex sinusoids on the plane. The image is the sum of those sinusoids weighted by the Fourier coefficients.



(a) $\mathbf{1}_{M \times N}$



(b) $\Re\left(\left[e^{j2\pi\frac{(N-1)n}{N}}\right]_{\substack{m=1,\dots,M\\n=1,\dots,N}}\right)$



(c) $\Re\left(\left[e^{j2\pi\frac{(M-1)m}{M}}\right]_{\substack{m=1,\dots,M\\n=1,\dots,N}}\right)$

(d) $\Re\left(\left[e^{j2\pi(\frac{(M-1)m}{M}+\frac{(N-1)n}{N})}\right]_{\substack{m=1,\dots,M\\n=1,\dots,N}}\right)$

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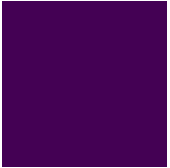
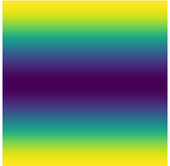
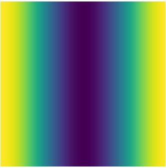


(a) $u = 1, v = 1$



(b) $u = 1, v = N$



(c) $u = M, v = 1$

(d) $u = M, v = N$

A few examples of Fourier transforms

Note: usually, the zero-frequency component is shifted to the center of the spectrum for vizualisation.

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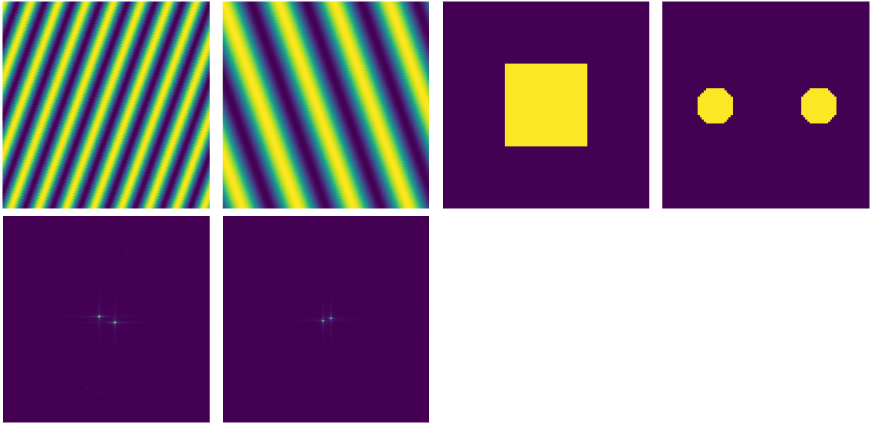

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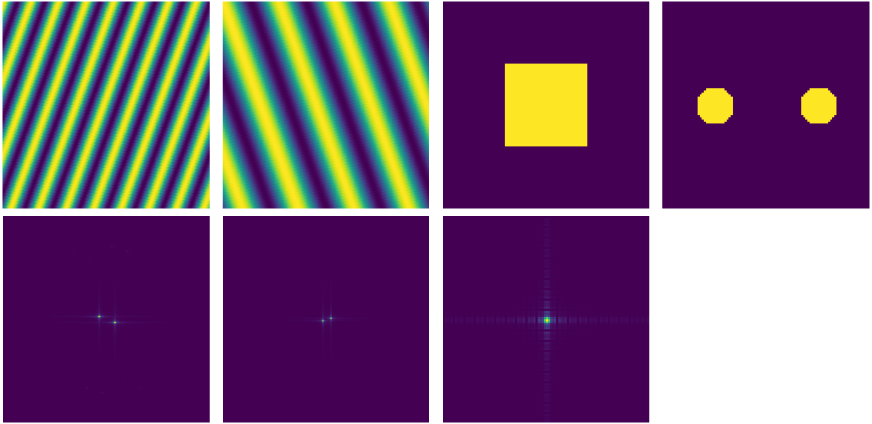

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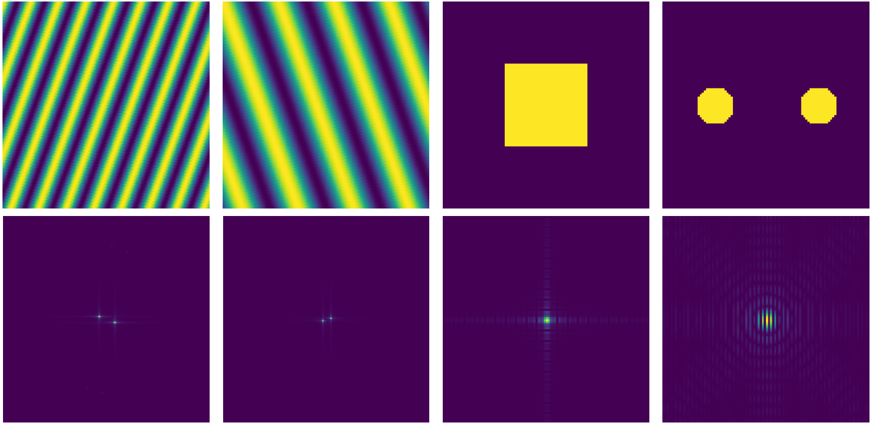

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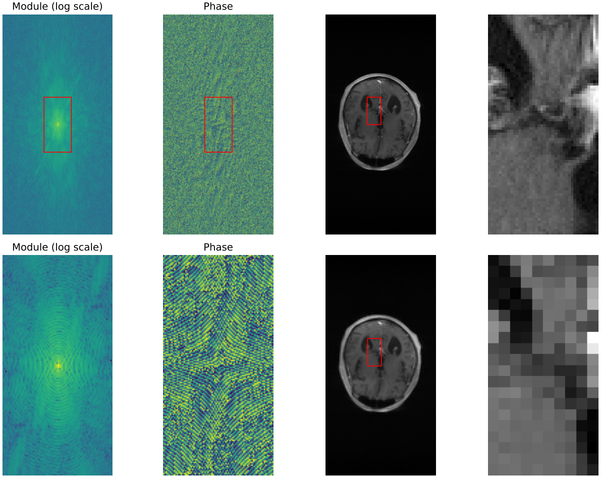
Overview

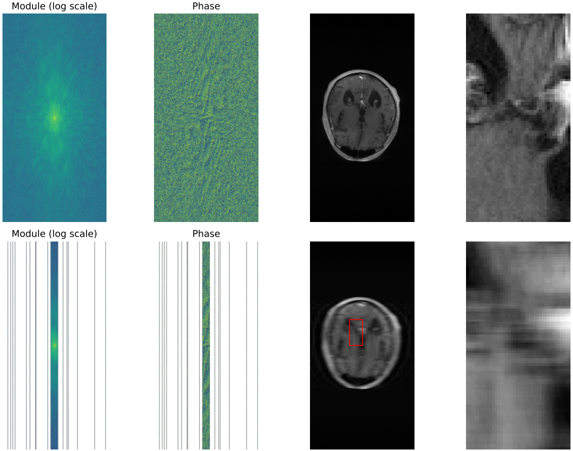
Acceleration of MRI data acqusition

- Naive solution: decrease the maximum frequency.



Acceleration of MRI data acqusition

- Compressed sensing: (partially) random[1] undersampling that creates aliasing artifacts in image space.



[1]See [Snieder and Wakin, 2022] for the reason behind random rather than regular undersampling. The lower frequencies are kept as they contain most of the information.

MRI reconstruction: problem formulation

- Let's denote

— $x \in \mathbb{C}^N$ the fully-sampled MRI to recover,

— $y \in \mathbb{C}^m$ the under-sampled k-space observation,

— $A \in \mathbb{C}^{m \times N}$ the sampling operator (the Fourier transform and the undersampling),

— and $e \in \mathbb{C}^m$ the measurement noise.

MRI reconstruction

$$\text{Given measurements } y = Ax + e, \text{ recover } x. \tag{2}$$

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$$\mathcal{N}(A) := \{x | Ax = 0\} \neq \{0\}$$

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- Therefore, there are infinitely many solutions x that yield the same observation y.

Classical vs Deep Learning based reconstructions

Variational methods

A regularization function $\mathcal{R}$ restricts the set of possible solutions:

$$\arg\min_{x \in \mathbb{C}^N} \|Ax - y\|_2^2 + \lambda \mathcal{R}(x)$$

where $\lambda > 0$ is the regularization coefficient.

Classical vs Deep Learning based reconstructions

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Deep Learning based inversion

Learn a reconstruction map

$$\begin{aligned} \hat{\Psi} : \mathbb{C}^m &\longrightarrow \mathbb{C}^N \\ y &\longmapsto \hat{\Psi}(y) = \hat{x} \end{aligned}$$

from a training set $\mathcal{T} = \{(y_i, x_i)\}_{i=1}^K$ where $y_i = Ax_i + e_i$.

How is Deep Learning doing for MRI reconstruction?

Try for yourself!

https://dl4ia.readthedocs.io/en/latest/tutorials/mri_restoration/mri_restoration.html

Snieder, R. and Wakin, M. B. (2022).
When randomness helps in undersampling.
SIAM Review, 64(4):1062–1080.

Zbontar, J., Knoll, F., Sriram, A., Murrell, T., Huang, Z., Muckley, M. J., Defazio, A., Stern, R., Johnson, P., Bruno, M., et al. (2018).
fastmri: An open dataset and benchmarks for accelerated mri.
arXiv preprint arXiv:1811.08839.