LDE Practice Questions

① $(D^2+2D+1)y = e^{2x}+2$

Solⁿ: A.E.: $D^2+2D+1=0$

$\Rightarrow D=-1,-1$

$$y_c = (C_1x+C_2)e^{-x}$$

$$y_p = \frac{1}{(D+1)^2}\{e^{2x}+2\} = \underset{D\to 2}{\frac{1}{(D+1)^2}\{e^{2x}\}} + \underset{D\to 0}{\frac{2}{(D+1)^2}\{e\}}$$

$$= \frac{1}{3^2}e^{2x} + \frac{2}{1^2}e^{0x}$$

$$= \frac{e^{2x}}{9}+2$$

$$\therefore y = y_c+y_p = (C_1x+C_2)e^{-x} + \frac{e^{2x}}{9}+2$$

② $(D^2-5D+6)y = e^{2x}+2e^{x}+3e^{-x}+2$

Solⁿ: A.E.: $D^2-5D+6=0$

$\Rightarrow D=2,3$

$$y_c = C_1e^{2x}+C_2e^{3x}$$

$$y_p = \frac{1}{(D-2)(D-3)}\{e^{2x}+2e^{x}+3e^{-x}+2e^{0x}\}$$

$$= \underset{D\to 2}{\frac{1}{(D-2)(D-3)}\{e^{2x}\}} + 2\underset{D\to 1}{\frac{1}{(D-2)(D-3)}\{e^{x}\}} + 3\cdot\underset{D\to -1}{\frac{1}{(D-2)(D-3)}\{e^{-x}\}} + 2\cdot\underset{D\to 0}{\frac{1}{(D-2)(D-3)}\{e^{0x}\}}$$

↳ case of failure

$$= \underset{D\to 2}{\frac{x}{2D-5}\{e^{2x}\}} + 2\cdot\frac{1}{(-1)(-2)}e^{x} + 3\cdot\frac{1}{(-3)(-4)}e^{-x} + 2\cdot\frac{1}{(-2)(-3)}e^{0x}$$

$$= -xe^{2x}+e^{x}+\frac{e^{-x}}{4}+\frac{1}{3}$$

$$\therefore y = y_c+y_p = C_1e^{2x}+C_2e^{3x}+\frac{e^{-x}}{4}+e^{x}-xe^{2x}+\frac{1}{3}$$

③ $(D^2+4)y = (1+e^x)^2 + 3 + 2^x$

Sol^n: A.E. : $D^2+4=0$

$\Rightarrow D = \pm 2i$

$y_c = C_1\cos 2x + C_2 \sin 2x$

$$y_p = \frac{1}{D^2+4}\{(1+e^x)^2+3+2^x\} = \frac{1}{D^2+4}\{e^{0x}+2e^x+e^{2x}+3e^{0x}+e^{x\log 2}\}$$

$$= \frac{1}{D^2+4}\{4e^{0x}\} + \frac{1}{D^2+4}\{2e^x\} + \frac{1}{D^2+4}\{e^{2x}\} + \frac{1}{D^2+4}\{e^{x\log 2}\}$$

$D\to 0 \quad D\to 1 \quad D\to 2 \quad D\to \log 2$

$$= 4\cdot\frac{1}{4}e^{0x} + \frac{1}{5}\cdot 2e^x + \frac{1}{8}e^{2x} + \frac{1}{(\log 2)^2+4}2^x$$

$$= 1 + \frac{2e^x}{5} + \frac{e^{2x}}{8} + \frac{2^x}{(\log 2)^2+4}$$

$$y = y_c + y_p = C_1\cos 2x + C_2\sin 2x + \frac{2^x}{(\log 2)^2+4} + \frac{2e^x}{5} + \frac{e^{2x}}{8} + 1$$

④ $(D^2+2D+5)y = \sin 2x + \cos 2x$

Sol^n: A.E. : $D^2+2D+5=0$

$\Rightarrow D = -1\pm 2i$

$y_c = e^{-x}(C_1\cos 2x + C_2\sin 2x)$

$$y_p = \frac{1}{D^2+2D+5}\{\sin 2x\} + \frac{1}{D^2+2D+5}\{\cos 2x\}$$

$D^2\to -4 \quad D^2\to -4$

$$= \frac{1}{2D+1}\{\sin 2x\} + \frac{1}{2D+1}\{\cos 2x\}$$

$$= \frac{1}{2D+1}\times\frac{2D-1}{2D-1}\{\sin 2x\} + \frac{1}{2D+1}\times\frac{2D-1}{2D-1}\{\cos 2x\}$$

$$= \frac{2D-1}{4D^2-1}\{\sin 2x\} + \frac{2D-1}{4D^2-1}\{\cos 2x\}$$

$D^2\to -4 \quad D^2\to -4$

$$= \frac{2D-1}{-17}\{\sin 2x\} + \frac{2D-1}{-17}\{\cos 2x\}$$

$$= \frac{\sin 2x}{17} - \frac{2.2\cos 2x}{17} + \frac{\cos 2x}{17} - \frac{2}{17}\cdot(-2\sin 2x)$$

$$y_p = \frac{\sin 2x}{17} + \frac{4\sin 2x}{17} + \frac{\cos 2x}{17} - \frac{4\cos 2x}{17}$$

$$= \frac{1}{17}(5\sin 2x - 3\cos 2x)$$

$$y = y_c + y_p = e^{-x}(C_1\cos 2x + C_2 \sin 2x) + \frac{5\sin 2x - 3\cos 2x}{17}$$

⑤ $(D^2 - 2D + 5)y = 10\sin x$

Sol^n: A.E: $D^2 - 2D + 5 = 0$

$\Rightarrow D = 1 \pm 2i$

$$y_c = e^{x}(C_1\cos 2x + C_2\sin 2x)$$

$$y_p = \frac{1}{D^2 - 2D + 5}\{10\sin x\} = \frac{10}{-2D-4}\{\sin x\} = -5\cdot\frac{1}{D+2}\{\sin x\}$$

$D^2 \to -1$

$$= -5\cdot\frac{1}{D+2}\times\frac{D-2}{D-2}\{\sin x\} = -5\cdot\frac{D-2}{D^2-2}\{\sin x\}$$

$D^2 \to -1$

$$= \frac{5}{3}(D-2)\{\sin x\} = \frac{5}{3}\left[D\{\sin x\} - 2\sin x\right]$$

$$= \frac{5}{3}\left[\cos x - 2\sin x\right]$$

$$\therefore\ y = y_c + y_p = e^{x}(C_1\cos 2x + C_2\sin 2x) + \frac{5}{3}(\cos x - 2\sin x)$$

⑥ $(D^2 + 2D + 5)y = \cos^2 x$

Sol^n: A.E.: $D^2 + 2D + 5 = 0$

$\Rightarrow D = -1 \pm 2i$

$$y_c = e^{-x}(C_1\cos 2x + C_2\sin 2x)$$

$$y_p = \frac{1}{D^2+2D+5}\{\cos^2 x\} = \frac{1}{D^2+2D+5}\left\{\frac{\cos 2x + 1}{2}\right\}$$

$$= \frac{1}{2}\left[\frac{1}{D^2+2D+5}\{\cos 2x\} + \frac{1}{D^2+2D+5}\{e^{0x}\}\right]$$

$D^2 \to -4$, $D \to 0$

$$= \frac{1}{2}\left[\frac{1}{2D+1}\{\cos 2x\} + \frac{1}{5}e^{0x}\right]$$

$$= \frac{1}{2}\left[\frac{2D-1}{4D^2-1}\{\cos 2x\} + \frac{1}{5}\right] = \frac{1}{2}\left[\frac{2D-1}{-17}\{\cos 2x\} + \frac{1}{5}\right]$$

$D^2 \to -4$

$$= \frac{1}{2}\left[\frac{1}{17}\cos 2x - \frac{2}{17}(-2\sin 2x) + \frac{1}{5}\right] = \frac{1}{2}\left[\frac{\cos 2x}{17} + \frac{4\sin 2x}{17} + \frac{1}{5}\right]$$

$$\therefore y = y_c + y_p = e^{-x}(C_1\cos 2x + C_2\sin 2x) + \frac{\cos 2x}{34} + \frac{2\sin 2x}{17} + \frac{1}{10}$$

⑦ $(D^3+1)y = \cos(2x-1)$ or $\sin(2x+3)$

Sol^n: A.E: $D^3+1=0$

$\Rightarrow D = -1, \frac{1}{2} \pm \frac{\sqrt{3}}{2} i$

$$y_c = C_1 e^{-x} + e^{x/2}\left(C_2\cos\frac{\sqrt{3}}{2}x + C_3\sin\frac{\sqrt{3}}{2}x\right)$$

$$y_p = \frac{1}{D^3+1}\{\cos(2x-1)\} \quad D^2 \to -4$$

$$= \frac{1}{-4D+1}\{\cos(2x-1)\}$$

$$= \frac{1+4D}{1-16D^2}\{\cos(2x-1)\} \quad D^2 \to -4$$

$$= \frac{1+4D}{-63}\{\cos(2x-1)\}$$

$$= -\frac{1}{63}\cos(2x-1) - \frac{4}{63}\{-2\sin(2x-1)\}$$

$$= -\frac{1}{63}[\cos(2x-1) - 8\sin(2x-1)]$$

or

$$y_p = \frac{1}{D^3+1}\{\sin(2x+3)\} \quad D^2 \to -4$$

$$= \frac{1}{-4D+1}\{\sin(2x+3)\}$$

$$= \frac{1+4D}{1-16D^2}\{\sin(2x+3)\} \quad D^2 \to -4$$

$$= \frac{1+4D}{-63}\{\sin(2x+3)\}$$

$$= \frac{-1}{63}\sin(2x+3) - \frac{4}{63}\cdot 2\cos(2x+3)$$

$$= \frac{-1}{63}[\sin(2x+3) + 8\cos(2x+3)]$$

$$\therefore y = y_c + y_p = C_1 e^{-x} + e^{x/2}\left(C_2\cos\frac{\sqrt{3}}{2}x + C_3\sin\frac{\sqrt{3}}{2}x\right) - \frac{1}{63}[\cos(2x-1) - 8\sin(2x-1)]$$

OR

$$y = C_1 e^{-x} + e^{x/2}\left(C_2\cos\frac{\sqrt{3}x}{2} + C_3\sin\frac{\sqrt{3}x}{2}\right) - \frac{1}{63}[\sin(2x+3) + 8\cos(2x+3)]$$

⑧ $(D^3+1)y = \cos^2(x/2)$

Sol^n: A.E.: $D^3+1=0$

$D = -1, \frac{1}{2} \pm \frac{\sqrt{3}}{2} i$

$$y_c = C_1 e^{-x} + e^{x/2}\left(C_2\cos\frac{\sqrt{3}x}{2} + C_3\sin\frac{\sqrt{3}x}{2}\right)$$

$$y_p = \frac{1}{D^3+1}\left\{\cos^2\frac{x}{2}\right\} = \frac{1}{D^3+1}\left\{\frac{\cos x + 1}{2}\right\} = \frac{1}{2}\left[\frac{1}{D^3+1}\{\cos x\} + \frac{1}{D^3+1}\{e^{0x}\}\right] \quad D^2 \to -1,\ D \to 0$$

$$= \frac{1}{2}\left[\frac{1}{1-D}\{\cos x\} + \frac{1}{1}e^{0x}\right] = \frac{1}{2}\left[\frac{D+1}{1-D^2}\{\cos x\} + 1\right] = \frac{1}{2}\left[\frac{-\sin x + \cos x}{2} + 1\right] \quad D^2 \to -1$$

$$\therefore y = y_c + y_p = C_1 e^{-x} + e^{x/2}\left(C_2\cos\frac{\sqrt{3}x}{2} + C_3\sin\frac{\sqrt{3}x}{2}\right) - \frac{\sin x}{4} + \frac{\cos x}{4} + \frac{1}{2}$$

(9) $(D^2+5D+4)y = x^2+7x+9$

Soln: A.E.: $D^2+5D+4=0$

$\Rightarrow D = -1, -4$

$y_c = C_1e^{-x} + C_2e^{-4x}$

~~$y_p = \frac{1}{(D+1)(D+4)}\{x^2+7x+9\}$~~

$$y_p = \frac{1}{D^2+5D+4}\{x^2+7x+9\} = \frac{1}{4}\left[\frac{1}{\left(\frac{D^2+5D}{4}\right)+1}\{x^2+7x+9\}\right]$$

$$= \frac{1}{4}\left[1 - \frac{D^2+5D}{4} + \frac{(D^2+5D)^2}{16} - \frac{(D^2+5D)^3}{64} + \dots\right]\{x^2+7x+9\}$$

$$x^2+7x+9 \xrightarrow{D} 2x+7 \xrightarrow{D^2} 2 \xrightarrow{D^3} 0 \rightarrow \dots$$

$$y_p = \frac{1}{4}\left[1 - \frac{D^2+5D}{4} + \frac{\cancelto{0}{D^4} + \cancelto{0}{10D^3} + 25D^2}{16} - \dots\right]\{x^2+7x+9\}$$

$$= \frac{1}{4}\left[1 + \frac{25D^2-4D^2}{16} - \frac{5D}{4}\right]\{x^2+7x+9\}$$

$$= \frac{1}{4}\left[x^2+7x+9 + \frac{21}{16}D^2\{x^2+7x+9\} - \frac{5}{4}D\{x^2+7x+9\}\right]$$

$$= \frac{1}{4}\left[x^2+7x+9 + \frac{21}{16}\times 2 - \frac{5}{4}(2x+7)\right]$$

$$= \frac{1}{4}\left[x^2+7x+9 + \frac{21}{8} - \frac{5x}{2} - \frac{35}{4}\right]$$

$$= \frac{1}{4}\left[x^2 - \frac{9x}{2} + \frac{23}{8}\right]$$

$$= \frac{x^2}{4} - \frac{9x}{8} + \frac{23}{32}$$

$$\therefore y = y_c + y_p = C_1e^{-x} + C_2e^{-4x} + \frac{x^2}{4} - \frac{9x}{8} + \frac{23}{32}$$

(10) $(D^2-4)y = e^{3x}x^2$

Soln: A.E.: $D^2-4=0 \Rightarrow D = \pm 2$

$\therefore y_c = C_1e^{2x} + C_2e^{-2x}$

$$y_p = \frac{1}{D^2-4}\{e^{3x}x^2\} = e^{3x}\cdot\frac{1}{(D+3)^2-4}\{x^2\}$$

$$y_p = e^{3x} \cdot \frac{1}{D^2+6D+9-4}\{x^2\}$$

$$= \frac{e^{3x}}{5} \cdot \frac{1}{\frac{D^2+6D}{5}+1}\{x^2\} = \frac{e^{3x}}{5}\left[1 - \frac{D^2+6D}{5} + \left(\frac{D^2+6D}{5}\right)^2 - \dots\right]\{x^2\}$$

$$= \frac{e^{3x}}{5}\left[1 - \frac{D^2}{5} - \frac{6D}{5} + \frac{\cancelto{0}{D^4}}{25} + \frac{12\cancelto{0}{D^3}}{25} + \frac{36D^2}{25} - \dots\right]\{x^2\}$$

$$= \frac{e^{3x}}{5}\left[1 + \frac{31D^2}{25} - \frac{6D}{5}\right]\{x^2\}$$

$$= \frac{e^{3x}}{5}\left[x^2 + \frac{31}{25}D^2\{x^2\} - \frac{6}{5}D\{x^2\}\right]$$

$$= \frac{e^{3x}}{5}\left[x^2 + \frac{31}{25}\times 2 - \frac{6}{5}\times 2x\right]$$

$$= \frac{e^{3x}}{5}\left[x^2 - \frac{12x}{5} + \frac{62}{25}\right]$$

$$\therefore y = y_c + y_p = C_1e^{2x} + C_2e^{-2x} + \frac{e^{3x}}{5}\left(x^2 - \frac{12x}{5} + \frac{62}{25}\right)$$

(11) $(D^2+2D+1)y = e^{-x}\log x$

Sol^n: A.E.: $D^2+2D+1=0$

$\Rightarrow D = -1, -1$

$$y_c = (C_1x + C_2)e^{-x}$$

$$y_p = \frac{1}{(D+1)^2}\{e^{-x}\log x\} = e^{-x}\cdot\frac{1}{(D-1+1)^2}\{\log x\}$$

$$= e^{-x}\frac{1}{D^2}\{\log x\} = e^{-x}\cdot\iint \log x\,dx\,dx$$

$$= e^{-x}\int x(\log x + 1)\,dx$$

$$\cancel{= e^{-x}\left[\int x\log x\,dx + \int x\,dx\right]}$$

$$\cancel{= e^{-x}\left[x\cdot x(\log x+1) - 1\cdot\int x(\log x+1)\,dx + \frac{x^2}{2}\right]}$$

$$\cancel{I = \int x\log x\,dx = x^2(\log x+1) - \int x(\log x+1)}$$

$$\cancel{= x^2(\log x+1) - \int x\log x\,dx - \frac{x^2}{2}}$$

$$I = \int x(\log x + 1)\,dx = x^2(\log x+1) - \int x(\log x+1)\,dx + \frac{x^2}{2}$$

$$I = x^2(\log x + 1) - I + \frac{x^2}{2}$$

$$\Rightarrow I = \frac{x^2}{2}(\log x + 1) + \frac{x^2}{4}$$

$$\therefore y_p = e^{-x}\cdot\left(\frac{x^2}{2}(\log x + 1) + \frac{x^2}{4}\right)$$

$$y = (C_1x + C_2)e^{-x} + e^{-x}\frac{x^2}{2}\left[\log x + \frac{3}{2}\right]$$ //

(12) $(D^2-1)y = (1+x^2)e^x$

Sol^n: A.E.: $D^2-1=0$

$\Rightarrow D = \pm 1$

$$y_c = C_1e^x + C_2e^{-x}$$

$$y_p = \frac{1}{D^2-1}\{e^x(1+x^2)\} = e^x\cdot\frac{1}{(D+1)^2-1}\{1+x^2\} = e^x\frac{1}{D^2+2D}\{1+x^2\}$$

$D \to D+1$

$$y_p = e^x \cdot \frac{1}{D^2+2D+1-1}\{1+x^2\} = e^x \cdot \cancel{\frac{1}{(D+1)^2-1}\{1+x^2\}}$$

$$\cancel{= -e^x \frac{1}{1-(D+1)^2}\{1+x^2\}} \qquad \cancel{\frac{1}{1-x} = 1+x+x^2+x^3+x^4+\ldots}$$

$$\cancel{= -e^x\left[1+(D+1)^2+(D+1)^4+\ldots\right]\{1+x^2\}}$$

$$1+x^2 \xrightarrow{D} 2x \xrightarrow{D^2} 2 \xrightarrow{D^3} 0 \xrightarrow{D^4} 0 \rightarrow \ldots$$

$$= e^x \cdot \frac{1}{D(D+2)}\{1+x^2\} = e^x \cdot \frac{1}{D} \cdot \frac{1}{2} \frac{1}{\left(\frac{D}{2}+1\right)}\{1+x^2\}$$

$$= \frac{e^x}{2} \cdot \frac{1}{D} \cdot \frac{1}{1+\frac{D}{2}}\{1+x^2\} = \frac{e^x}{2} \cdot \frac{1}{D} \cdot \left[1 - \frac{D}{2} + \frac{D^2}{4} - \frac{D^3}{8} + \ldots\right]\{1+x^2\}$$

$$= \frac{e^x}{2} \cdot \frac{1}{D} \cdot \left\{1 - \frac{D}{2} + \frac{D^2}{4}\right\}\{1+x^2\} \qquad \left[\because 1+x^2 \xrightarrow{D} 2x \xrightarrow{D^2} 2 \xrightarrow{D^3} 0 \xrightarrow{D^4} 0 \ldots\right]$$

$$= \frac{e^x}{2} \cdot \frac{1}{D}\left\{1+x^2 - \frac{1}{2} D\{1+x^2\} + \frac{1}{4} D^2\{1+x^2\}\right\}$$

$$= \frac{e^x}{2} \cdot \frac{1}{D}\left\{1+x^2 - \frac{1}{2}\cdot 2x + \frac{1}{4}\cdot 2\right\}$$

$$= \frac{e^x}{2} \cdot \frac{1}{D}\left\{1+x^2 - x + \frac{1}{2}\right\} = \frac{e^x}{2} \cdot \frac{1}{D}\left\{x^2 - x + \frac{3}{2}\right\}$$

$$= \frac{e^x}{2}\int\left(x^2 - x + \frac{3}{2}\right)dx = \frac{e^x}{2}\left[\frac{x^3}{3} - \frac{x^2}{2} + \frac{3}{2}x\right]$$

$$y = y_c + y_p = C_1 e^x + C_2 e^{-x} + \frac{e^x}{2}\left[\frac{x^3}{3} - \frac{x^2}{2} + \frac{3x}{2}\right]$$

(13) $(D^2 - 4D + 4)y = x e^{2x} \sin 2x$

Sol^n: A.E.: $D^2 - 4D + 4 = 0 \Rightarrow D = 2, 2$

$$y_c = (C_1 + C_2 x)e^{2x}$$

$$y_p = \frac{1}{(D-2)^2}\{e^{2x} \cdot x \cdot \sin 2x\} = e^{2x} \cdot \frac{1}{(D+2-2)^2}\{x \cdot \sin 2x\} = e^{2x} \cdot \frac{1}{(D)^2}\{x \cdot \sin 2x\}$$

$$~~y_p = e^{2x}\left[x - \frac{2(D-1)}{(D-1)^2}\right]\frac{1}{(D-1)^2}\{\sin 2x\}~~$$

$$~~= e^{2x}\left[x - \frac{2D-2}{D^2-2D+1}\right]\frac{1}{D^2-2D+1}\{\sin 2x\}~~$$

$D^2 \to -4$

$$~~= e^{2x}\left[x - \frac{2D-2}{-2D-3}\right]\frac{1}{-2D-3}\{\sin 2x\}~~$$

$$~~= e^{2x}[~~$$

$$y_p = e^{2x}\left[x - \frac{2D}{D^2}\right]\frac{1}{D^2}\{\sin 2x\}$$

$D^2 \to -4$

$$= e^{2x}\left[x - \frac{2D}{-4}\right]\frac{1}{-4}\{\sin 2x\}$$

$$= -\frac{e^{2x}}{4}\left[x + \frac{D}{2}\right]\{\sin 2x\} = -\frac{e^{2x}}{4}\left[x\sin 2x + \frac{2\cos 2x}{2}\right]$$

$$= -\frac{e^{2x}}{4}\left[x\sin 2x + \cos 2x\right]$$

$$\therefore y = y_c + y_p = (C_1 + C_2 x)e^{2x} - \frac{e^{2x}}{4}\left[x\sin 2x + \cos 2x\right]$$

$$= e^{2x}\left[C_1 + C_2 x - \frac{x\sin 2x}{4} - \frac{\cos 2x}{4}\right]$$

(14) $(D^2-4)y = x\sinh x$

Solⁿ: A.E.: $D^2 - 4 = 0$

$\Rightarrow D = \pm 2$

$\therefore y_c = C_1 e^{2x} + C_2 e^{-2x}$

$$y_p = \frac{1}{D^2-4}\{x\sinh x\} = \left[x - \frac{2D}{D^2-4}\right]\frac{1}{D^2-4}\{\sinh x\}$$

$D^2 \to 1$

$$= \left[x - \frac{2D}{1-4}\right]\frac{1}{1-4}\{\sinh x\} = -\frac{1}{3}\left[x + \frac{2D}{3}\right]\{\sinh x\}$$

$$= -\frac{1}{3}\left[x\sinh x + \frac{2}{3}\cosh x\right]$$

$y = y_c + y_p$

$$= C_1 e^{2x} + C_2 e^{-2x} - \frac{1}{3}\left[x\sinh x + \frac{2}{3}\cosh x\right]$$

(15) $(D^3-4D)y = 2\cosh 2x$

Solⁿ: A.E: $D^3-4D=0$

$\Rightarrow D(D^2-4)=0$

$\Rightarrow D = 0, \pm 2$

$y_c = C_1 e^{2x} + C_2 e^{-2x}$

$y_p = \frac{1}{D(D^2-4)}\{2\cosh 2x\}$

$= 2\frac{1}{D(D^2-4)}\{\cosh 2x\}$ $D^2 \to 4$

case of failure:

$= 2 \cdot \frac{x}{3D^2-4}\{\cosh 2x\}$ $D^2 \to 4$

$= 2 \cdot \frac{x}{8}\cosh 2x = \frac{x\cosh 2x}{4}$

$\therefore y = y_c + y_p = C_1 e^{2x} + C_2 e^{-2x} + \frac{x\cosh 2x}{4}$

(16) $(D^2-4D+3)y = e^{2x}.x$

Solⁿ: A.E.: $D^2-4D+3=0$

$\Rightarrow D = 1, 3$

$\therefore y_c = C_1 e^{x} + C_2 e^{3x}$

$y_p = \frac{1}{D^2-4D+3}\{e^{2x}.x\} = \frac{1}{(D-1)(D-3)}\{e^{2x}.x\}$ $D \to D+2$

$= e^{2x} \cdot \frac{1}{(D+1)(D-1)}\{x\}$

$= e^{2x}\frac{1}{D^2-1}\{x\} = -e^{2x} \cdot \frac{1}{1-D^2}\{x\}$

$= -e^{2x}[1+D^2+D^4+D^6+\ldots]\{x\}$; $x \xrightarrow{D} 1 \xrightarrow{D^2} 0 \xrightarrow{D^3} 0 \to \ldots$

$= -e^{2x}x = -xe^{2x}$

$\therefore y = y_c + y_p = C_1 e^{x} + C_2 e^{3x} - xe^{2x}$

(17) $(D^2+9)y = x\cos 2x$

Sol^n: A.E.: $D^2+9=0$

$\Rightarrow D = \pm 3i$

$y_c = C_1\cos 3x + C_2\sin 3x$

$y_p = \frac{1}{D^2+9}\{x\cos 2x\} = \left[x - \frac{2D}{D^2+9}\right]\frac{1}{D^2+9}\{\cos 2x\}$

$D^2 \to -4$

$= \left[x - \frac{2D}{5}\right]\frac{1}{5}\{\cos 2x\}$

$= \frac{1}{5}\left[x\cos 2x - \frac{2}{5}D\{\cos 2x\}\right]$

$= \frac{1}{5}\left[x\cos 2x + \frac{4}{5}\sin 2x\right]$

$\therefore y = y_c + y_p = C_1\cos 3x + C_2\sin 3x + \frac{1}{5}\left[x\cos 2x + \frac{4}{5}\sin 2x\right]$

(18) $(D^2-6D+9)y = e^{3x}\cdot x^{-2}$

Sol^n: A.E.: $D^2-6D+9=0$

$\Rightarrow D = 3, 3$

$y_c = (C_1+C_2x)e^{3x} = C_1e^{3x} + C_2xe^{3x}$

$y_1 = e^{3x}$; $y_2 = xe^{3x}$

$w = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{3x} & xe^{3x} \\ 3e^{3x} & 3xe^{3x}+e^{3x} \end{vmatrix} = e^{6x}\begin{vmatrix} 1 & x \\ 3 & 3x+1 \end{vmatrix}$

$= e^{6x}(3x+1-3x) = e^{6x}$

$u = -\int \frac{y_2 f(x)}{w}dx = -\int \frac{xe^{3x}\cdot e^{3x}\cdot x^{-2}}{e^{6x}}dx$

$= -\int \frac{dx}{x} = -\log x$

$v = \int \frac{y_1 f(x)}{w}dx = \int \frac{e^{3x}\cdot e^{3x}\cdot x^{-2}}{e^{6x}}dx = \int \frac{dx}{x^2} = -\frac{1}{x}$

$y_p = uy_1 + vy_2 = -e^{3x}\log x - xe^{3x}\frac{1}{x} = -e^{3x}(1+\log x)$

$y = y_c + y_p = (C_1+C_2x)e^{3x} - e^{3x}(1+\log x) = e^{3x}(C_1 + C_2x - \log x - 1)$

(19) $(D^2+1)y = \tan x$

Sol^n: A.E.: $D^2+1=0$

$\Rightarrow D = \pm i$

$y_c = C_1\cos x + C_2 \sin x$

$y_1 = \cos x\ ,\ y_2 = \sin x$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$$

$$u = -\int \frac{y_2 f(x)}{W}\,dx = -\int \frac{\sin x \cdot \tan x}{1}\,dx$$

$$= -\int \frac{\sin^2 x}{\cos x}\,dx = -\int \frac{1-\cos^2 x}{\cos x}\,dx = \int \cos x\,dx - \int \sec x\,dx$$

$$= \sin x - \log(\sec x + \tan x)$$

$$v = \int \frac{y_1 f(x)}{W}\,dx = \int \cos x \cdot \tan x\,dx = \int \sin x\,dx = -\cos x$$

$$y_p = uy_1 + vy_2 = \cos x\left(\sin x - \log(\sec x + \tan x)\right) - \sin x\cos x$$

$$= \sin x\cos x - \cos x\log(\sec x + \tan x) - \sin x\cos x$$

$$= -\cos x \cdot \log(\sec x + \tan x)$$

$$\therefore y = y_c + y_p = C_1\cos x + C_2\sin x - \cos x\log(\sec x + \tan x)$$

$$= [C_1 - \log(\sec x + \tan x)]\cos x + C_2 \sin x$$

(20) $(D^2-2D+2)y = e^x \tan x$

Sol^n: A.E.: $D^2-2D+2=0$

$\Rightarrow D = 1 \pm i$

$y_c = (C_1\cos x + C_2\sin x)e^x$

$$y_p = \frac{1}{D^2-2D+2}\{e^x \tan x\} = e^x \cdot \frac{1}{(D+1)^2-2(D+1)+2}\{\tan x\}$$

$$\cancel{y_p = e^x \frac{1}{D^2+2D+1-2D-2+2}\{\tan x\}}$$

$$\cancel{e^x \cdot \frac{1}{D^2+1}\{\tan x\}}$$

$$\frac{d}{dx} e^x \cos x = e^x(-\sin x) + e^x \cos x = e^x(\cos x - \sin x)$$

$$y_1 = e^x \cos x \; ; \; y_2 = e^x \sin x$$

$$w = \begin{vmatrix} e^x \cos x & e^x \sin x \\ e^x(\cos x - \sin x) & e^x(\cos x + \sin x) \end{vmatrix} = e^{2x}\left[\cos^2 x + \sin x \cos x + \sin^2 x - \sin x \cos x\right]$$

$$= e^{2x} \; \cancel{(\cos x - \sin x)}$$

$$u = -\int \frac{y_2 f(x)}{w}\,dx = -\int \frac{e^x \sin x \cdot e^x \tan x}{e^{2x}\cancel{(\cos x - \sin x)}}\,dx$$

$$= -\int \frac{\sin^2 x}{\cos x}\,dx = -\int \frac{1-\cos^2 x}{\cos x}\,dx = \int \cos x\,dx - \int \sec x\,dx$$

$$= \sin x - \log(\sec x + \tan x)$$

$$v = \int \frac{y_1 f(x)}{w}\,dx = \int \frac{e^x \cos x \cdot e^x \tan x}{e^{2x}}\,dx$$

$$= \int \sin x\,dx = -\cos x$$

$$y_p = u y_1 + v y_2 = e^x \cos x \sin x - e^x \cos x \log(\sec x + \tan x) - e^x \sin x \cos x$$

$$= -e^x \cos x \log(\sec x + \tan x)$$

$$\therefore y = y_c + y_p = e^x(C_1 \cos x + C_2 \sin x) - e^x \cos x \log(\sec x + \tan x)$$

$$\cancel{= e^x[\cos x(C_1 - \log(\sec x + \tan x)) + C_2 \sin x]}$$

$$= e^x[(C_1 - \log(\sec x + \tan x))\cos x + C_2 \sin x]$$

(21) $(D^2+4)y = \sec 2x$

Solⁿ: A.E.: $D^2+4=0$

$\Rightarrow D = \pm 2i$

$y_c = C_1\cos 2x + C_2 \sin 2x$

$y_1 = \cos 2x \; ; \; y_2 = \sin 2x$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2\cos^2 2x + 2\sin^2 2x = 2$$

$$u = -\int \frac{y_2 f(x)}{W}\,dx = -\int \frac{\sin 2x \cdot \sec 2x}{2}\,dx = -\int \frac{\tan 2x}{2}\,dx$$

$$= -\frac{1}{2}\frac{\log(\sec 2x)}{2} = -\frac{\log(\sec 2x)}{4} = \frac{\log(\cos 2x)}{4}$$

$$v = \int \frac{y_1 f(x)}{W}\,dx = \int \frac{\cos 2x \cdot \sec 2x}{2}\,dx = \int \frac{dx}{2} = \frac{x}{2}$$

$$y_p = uy_1 + vy_2 = \cos 2x \cdot \frac{\log(\cos 2x)}{4} + \frac{x}{2}\sin 2x$$

$$\therefore\; y = y_c + y_p = \left(C_1 + \frac{\log(\cos 2x)}{4}\right)\cos 2x + \left(C_2 + \frac{x}{2}\right)\sin 2x$$

(22) $(D^2+1)y = \dfrac{1}{1+\sin x}$

Solⁿ: A.E.: $D^2+1=0$

$\Rightarrow D = \pm i$

$y_c = C_1\cos x + C_2 \sin x$

$y_1 = \cos x, \quad y_2 = \sin x$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$$

$$u = -\int \frac{y_2 f(x)}{W}\,dx = -\int \frac{\sin x}{1+\sin x}\,dx = -\int \frac{\tan x}{\sec x + \tan x}\,dx$$

$$= -\int \frac{\sec x \tan x - \tan^2 x}{\sec^2 x - \tan^2 x}\,dx = -\int \sec x \tan x\,dx + \int \tan^2 x\,dx$$

$$= -\sec x + \int (\sec^2 x - 1)\,dx = -\sec x + \tan x - x$$

$$v = \int \frac{y_1 f(x)}{W}\,dx = \int \frac{\cos x}{1+\sin x}\,dx = \log(1+\sin x)$$

$$y_p = \cos x(-\sec x + \tan x - x) + \sin x(\log(1+\sin x)) = -1 + \sin x - x\cos x + \sin x \log(1+\sin x)$$

$$= \sin x(\log(1+\sin x) + 1) - x\cos x - 1$$

(23) $(D^2-9D+18)y = e^{e^{-3x}}$

Solⁿ: A.E.: $D^2-9D+18=0$

$\Rightarrow D = 3, 6$

$y_c = C_1e^{3x} + C_2e^{6x}$

$$y_p = \frac{1}{(D-6)(D-3)}\left\{e^{e^{-3x}}\right\}$$

$$= \frac{1}{D-6}\left\{e^{3x}\int e^{-3x}e^{e^{-3x}}\,dx\right\}$$

Taking $t = e^{-3x} \Rightarrow dt = -3e^{-3x}\,dx$

$\Rightarrow -\frac{dt}{3} = e^{-3x}\,dx$

$$y_p = \frac{1}{D-6}\left\{-\frac{e^{3x}}{3}\int e^t\,dt\right\}$$

$$= \frac{1}{D-6}\left\{-\frac{e^{3x}}{3}e^t\right\}$$

$$= -\frac{1}{3}\left[e^{6x}\int e^{-6x}e^{3x}e^{e^{-3x}}\,dx\right]$$

$$= -\frac{e^{6x}}{3}\int e^{-3x}e^{e^{-3x}}\,dx$$

$$= -\frac{e^{6x}}{3}\cdot e^{e^{-3x}} = \frac{-e^{(6x+e^{-3x})}}{3}$$

$$y = C_1e^{3x} + C_2e^{6x} - \frac{e^{6x}e^{e^{-3x}}}{3}$$

(24) $(D^2-3D+2)y = \sin(e^{-x})$

Solⁿ: A.E. $D^2-3D+2=0$

$\Rightarrow D = 1, 2$

$y_c = C_1e^x + C_2e^{2x}$

$$y_p = \frac{1}{(D-2)(D-1)}\{\sin(e^{-x})\} = \frac{1}{D-2}\left\{e^x\int e^{-x}\sin(e^{-x})\,dx\right\}$$

$$= \frac{1}{D-2}\{-e^x\cos(e^{-x})\} = -e^{2x}\int e^{-2x}e^x\cos(e^{-x})\,dx$$

$$= -e^{2x}\int e^{-x}\cos(e^{-x})\,dx = -e^{2x}\sin(e^{-x})$$

$y = y_c + y_p$

$= C_1e^x + C_2e^{2x} - e^{2x}\sin(e^{-x})$

(25) $(D^2+D)y = \dfrac{1}{1+e^x}$

Sol^n: A.E. :: $D^2+D=0$

$\Rightarrow D = 0, -1$

$$y_c = C_1e^{0x} + C_2e^{-x} = C_1 + C_2e^{-x}$$

$$y_p = \frac{1}{D(D+1)}\left\{\frac{1}{1+e^x}\right\}$$

$$= \frac{1}{D}\left\{e^{-x}\int\frac{e^{x}}{1+e^x}dx\right\}$$ Taking $1+e^x = t$ $\Rightarrow e^x dx = dt$

$$= \frac{1}{D}\left\{e^{-x}\int\frac{dt}{t}\right\} = \frac{1}{D}\left\{e^{-x}\log t\right\}$$

$$= \frac{1}{D}\left\{e^{-x}\log(1+e^x)\right\}$$

$$= \int e^{-x}\log(1+e^x)\,dx$$ Taking $-e^{-x} = t \Rightarrow e^x = -\frac{1}{t}$ $\Rightarrow e^{-x}dx = dt$

$$= \int\log\left(1-\frac{1}{t}\right)dt$$

$$= \log\left(1-\frac{1}{t}\right)\cdot t - \int\frac{1}{1-\frac{1}{t}}\cdot\frac{1}{t^2}\cdot t\,dt$$

$$= \log\left(1-\frac{1}{t}\right)\cdot t - \int\frac{1}{\left(1-\frac{1}{t}\right)t}dt$$

$$= t\log\left(1-\frac{1}{t}\right) + \int\frac{1}{1-t}dt$$

$$= t\log\left(1-\frac{1}{t}\right) - \log(t)$$

$$= -e^{-x}\log(1+e^x) - \log(e^{-x}+1)$$

$$\therefore y = y_c + y_p = C_1 + C_2e^{-x} - e^{-x}\log(1+e^x) - \log(1+e^{-x})$$

$$= C_1 + [C_2 - \log(1+e^x)]e^{-x} - \log(1+e^{-x})$$

(26) $x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + 4y = \cos(\log x)$

Solⁿ: Cauchy's LDE :

$x = e^z \Rightarrow z = \log x$

$x^2D^2y = D(D-1)y \quad ; D = \frac{d}{dz}$

$xDy = Dy$

$\therefore$ LDE becomes:

$D(D-1)y - Dy + 4y = \cos z$

$\Rightarrow D^2y - Dy - Dy + 4y = \cos z$

$\Rightarrow D^2y - 2Dy + 4y = \cos z$

$\Rightarrow (D^2 - 2D + 4)y = \cos z$

A.E.: $D^2 - 2D + 4 = 0$

$\Rightarrow D = 1 \pm \sqrt{3}i$

$y_c = e^z(C_1\cos\sqrt{3}z + C_2\sin\sqrt{3}z)$

$= x(C_1\cos(\sqrt{3}\log x) + C_2\sin(\sqrt{3}\log x))$

$y_p = \frac{1}{D^2-2D+4}\{\cos z\} = \frac{1}{-2D+3}\{\cos z\}$

$D^2 \to -1$

$= \frac{2D+3}{9-4D^2}\{\cos z\} = \frac{1}{13}(2D+3)\{\cos z\}$

$D^2 \to -1$

$= \frac{1}{13}[2(-\sin z) + 3\cos z]$

$= \frac{1}{13}[3\cos z - 2\sin z] = \frac{1}{13}[3\cos(\log x) - 2\sin(\log x)]$

$\therefore y = y_c + y_p = x[C_1\cos(\log x^{\sqrt{3}}) + C_2\sin(\log x^{\sqrt{3}})] + \frac{1}{13}[3\cos(\log x) - 2\sin(\log x)]$

(27) $(x+2)^2 \frac{d^2y}{dx^2} + 3(x+2)\frac{dy}{dx} + y = 4\sin[\log(x+2)]$

Sol^n: Legendre's LDE :

$x+2 = e^z$

$\Rightarrow z = \log(x+2)$

$\sin(\log(x+2)) = \sin z$

$(x+2)^2 D^2 y = D(D-1)y \quad ; D = \frac{d}{dz}$

$(x+2) Dy = Dy$

$\therefore D(D-1)y + 3Dy + y = 4\sin z$

$\Rightarrow D^2y - Dy + 3Dy + y = 4\sin z$

$\Rightarrow (D^2 + 2D + 1)y = 4\sin z$

A.E : $D^2 + 2D + 1 = 0$

$\Rightarrow D = -1, -1$

$y_c = (C_1 + C_2 z)e^{-z} = [C_1 + C_2\log(x+2)]e^{\log\left(\frac{1}{x+2}\right)} = [C_1 + C_2\log(x+2)]\frac{1}{x+2}$

$y_p = \frac{1}{D^2+2D+1}\{4\sin z\} = \frac{1}{2D}\{4\sin z\} = 2\int \sin z\, dz$
$D^2 \to -1$

$= 2(-\cos z) = -2\cos[\log(x+2)]$

$\therefore y = y_c + y_p = \frac{1}{x+2}[C_1 + C_2\log(x+2)] - 2\cos[\log(x+2)]$

(28) $(4x+1)^2 \frac{d^2y}{dx^2} + 2(4x+1)\frac{dy}{dx} + y = 2x+1$

Sol^n: Legendre's LDE :

$4x+1 = e^z \Rightarrow z = \log(4x+1) \Rightarrow \frac{4x+2}{2} = \frac{e^z+1}{2}$

$(4x+1)^2 D^2 y = 16D(D-1)y$

$(4x+1)Dy = 4Dy$

$\therefore 16D^2y - 16Dy + 8Dy + y = \frac{e^z+1}{2}$

$\Rightarrow 16D^2y - 8Dy + y = \frac{e^z+1}{2}$

A.E : $16D^2 - 8D + 1 = 0$

$\Rightarrow D = \frac{1}{4}, \frac{1}{4}$

$y_c = (C_1 + C_2 z) e^{z/4}$

$= [C_1 + C_2 \log(4x+1)] e^{\frac{1}{4}\log(4x+1)}$

$= [C_1 + C_2 \log(4x+1)] (4x+1)^{1/4}$

$y_p = \frac{1}{16D^2 - 8D + 1} \left\{ \frac{e^z + 1}{2} \right\}$

$= \frac{1}{2} \frac{1}{16D^2 - 8D + 1} \{ e^z + e^{0z} \}$

$= \frac{1}{2} \left[\underset{D \to 1}{\frac{1}{16D^2 - 8D + 1} \{ e^z \}} + \underset{D \to 0}{\frac{1}{16D^2 - 8D + 1} \{ e^{0z} \}} \right]$

$= \frac{1}{2} \left[\frac{e^z}{9} + 1 \right]$

$= \frac{e^z}{18} + \frac{1}{2} = \frac{4x+1}{18} + \frac{1}{2} = \frac{4x+10}{18} = \frac{2x+5}{9}$

$\therefore y = y_c + y_p = [C_1 + C_2 \log(4x+1)](4x+1)^{1/4} + \frac{2x+5}{9}$

(29) $(x^2D^2 - 4xD + 6)y = -x^4 \sin x$

Soln: Cauchy's LDE :

Substituting $x = e^z \Rightarrow z = \log x$

$\Rightarrow -x^4 \sin x = -e^{4z} \sin(e^z)$

$x^2D^2y = D(D-1)y$

$xDy = Dy$

$(D(D-1)y - 4Dy + 6y = -e^{4z} \sin(e^z)$

$\Rightarrow (D^2 - 5D + 6)y = e^{-4z} \sin(e^z)$

$A.E: \; D^2 - 5D + 6 = 0$

$\Rightarrow D = 3, 2$

$y_c = C_1 e^{3z} + C_2 e^{2z} = C_1 e^{3\log x} + C_2 e^{2\log x}$

$= C_1 x^3 + C_2 x^2$

$\cancel{y_p = \frac{1}{D^2-5D+6}\{-e^{4z}\sin(e^z)\}}$

$\cancel{= \{-e^{4z}\frac{1}{(D+4-2)(D+4-3)}\{\sin(e^z)\}}$

$\cancel{= \{-e^{4z}\frac{1}{(D+2)(D+1)}\{\sin(e^z)\}}$

$y_1 = e^{3z} \qquad y_2 = e^{2z}$

$$W = \begin{vmatrix} e^{3z} & e^{2z} \\ 3e^{3z} & 2e^{2z} \end{vmatrix} = 2e^{5z} - 3e^{5z} = -e^{5z}$$

$$u = -\int \frac{y_2 f(z)}{W}\,dz = -\int \frac{e^{2z}(\cancel{-}e^{4z}\sin(e^z))}{\cancel{-}e^{5z}}\,dz$$

$$= -\int e^z \sin(e^z)\,dz = \cos(e^z)$$

$$v = \int \frac{y_1 f(z)}{W}\,dz = \int \frac{e^{3z}(\cancel{-}e^{4z}\sin(e^z))}{\cancel{-}e^{5z}}\,dz$$

$$= \int e^{2z}\sin(e^z)\,dz$$

Take $t = e^z \Rightarrow e^z\,dz = dt$

$$v = \int t\sin t\,dt = t\cdot(-\cos t) - 1\cdot(-\sin t)$$

$$= \sin t - t\cos t = \sin(e^z) - e^z\cos(e^z)$$

$$y_p = e^{3z}\cos(e^z) + e^{2z}\sin(e^z) - e^{2z}\cdot e^z\cos(e^z)$$

$$= e^{2z}\sin(e^z) = x^2\sin x$$

$\therefore\; y = y_c + y_c = C_1 x^3 + (C_2 + \sin x)x^2$ //

(30) $\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} = A + B\log x$

Sol^n: Multiplying by x^2 on both sides :

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = x^2(A + B\log x)$$

$$\Rightarrow x^2D^2y + xDy = x^2(A + B\log x)$$

Cauchy's LDE :

Taking $x = e^z \Rightarrow z = \log x$

$$x^2(A + B\log x) = e^{2z}(A + Bz)$$

$$x^2D^2y = D(D-1)y \quad ; \; D = \frac{d}{dz}$$

$$xDy = Dy$$

$$\therefore D(D-1)y + Dy = e^{2z}(A + Bz)$$

$$\Rightarrow D^2y - \cancel{Dy} + \cancel{Dy} = e^{2z}(A + Bz)$$

$$\Rightarrow D^2y = e^{2z}(A + Bz)$$

A.E. : $D^2 = 0$

$\Rightarrow D = 0, 0$

$$y_c = (C_1 + C_2 z)e^{0z} = C_1 + C_2 z$$

$$= C_1 + C_2 \log x$$

$$y_p = \frac{1}{D^2}\{e^{2z}(A + Bz)\} \cancel{= e^{2z}\frac{1}{(D+2)^2}\{A + Bz\}}$$

$$= \frac{1}{D}\int e^{2z}(A + Bz)\,dz = \frac{1}{D}\left[\frac{Ae^{2z}}{2} + B\int ze^{2z}\,dz\right]$$

$$= \frac{1}{D}\left[A\frac{e^{2z}}{2} + B\left(z\frac{e^{2z}}{2} - 1\cdot\frac{e^{2z}}{4}\right)\right]$$

$$= \int\frac{2Ae^{2z} + 2Bze^{2z} - e^{2z}}{4}\,dz = \int\frac{(2A-1)e^{2z} + 2Bze^{2z}}{4}\,dz$$

$$= \frac{2A-1}{4}\int e^{2z}\,dz + 2B\int ze^{2z}\,dz = \frac{2A-1}{4}\cdot\frac{e^{2z}}{2} + \frac{2Bze^{2z}}{2} - \frac{2Be^{2z}}{4}$$

$$= \frac{(2A-1)}{8}e^{2z} + Bze^{2z} - \frac{Be^{2z}}{2}$$

$$y_p = \frac{2A-1}{8}x^2 + \frac{8Bx^2}{8}\log x - \frac{4Bx^2}{8}$$

$$= \left(\frac{2A-1+8B\log x-4B}{8}\right)x^2$$

$$y = y_c + y_p = C_1 + C_2\log x + \frac{x^2}{8}(2A-4B+8B\log x-1)$$