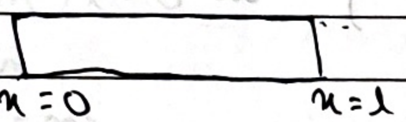
Unit 3 2 Partial DE

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Consider a rod of length l



i The heat flows throught the end is negligible
ii heat from a soure has finite temp
iii Temp is given by $(x, t) \rightarrow$ temp distribution
iv System is governed by $\left[\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}\right]$ — ①

conditions
i Boundary condition :-
$u(0,t) = 0$, $u(l,t) = 0, \forall t$

ii Initial conditions
$u(x,0) = f(x)$ — ②

Let, $u = X(x)T(t)$ be the sol of eq ①
diff eq w.r.to t $u = XT$ — ②
$\cancel{\frac{\partial u}{\partial t}} = \cancel{\frac{\partial X}{\partial t}}$ $\frac{\partial u}{\partial t} = X\frac{\partial T}{\partial t} = X\frac{\partial T}{\partial t} = XT'$

Diff eq w.r. to x twice
$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 X}{\partial x^2} T = \frac{\partial^2 X}{\partial x^2} T = X''T$

Put above values in ①
$XT' = c^2 X'' T$
Seperate variable
$\frac{X''}{X} = \frac{1}{c^2}\frac{T'}{T} = k$ (say) — ③

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* Where $k=0$

$\frac{X''}{X} = k$	$\frac{1}{c^2} = \frac{T'}{T} = k$
$X'' = kX$	$T' = c^2 Tk$
$X'' - kX = 0$, $D = \frac{d}{dx}$	$T' - c^2 Tk = 0$, $D = \frac{d}{dt}$
$D^2 = 0$	$T' = 0$
$D = 0, 0$	$D = 0$
	$D = 0$
$CF = (c_1 + c_2 x)e^{0x}$	$CF = c_3 e^{0t}$
$CF = c_1 + c_2 x$	$CF = c_3$
$PI = 0$	$PI = 0$
$GS = CF + PI$	$GS = CF + PI$
$X = c_1 + c_2 x$	$T = c_3$

$u(x,t) = XT = (c_1 + c_2 x)\, c_3$ — (4)

Where k is +ve. $k = m^2$

from eq (3)

$X'' = kX$	$T' = c^2 Tk$
$X'' - kX = 0$	$T' - c^2 Tk = 0$
$X'' - m^2 X = 0$	$T' - c^2 m^2 T = 0$
$(D^2 - m^2)X = 0$	$(D - c^2 m^2)T = 0$
Aux	Aux
$D^2 - m^2 = 0$	$D - c^2 m^2 = 0$
$D^2 = m^2$	$D = c^2 m^2$
$D = \pm m$	
$CF = c_1 e^{mx} + c_2 e^{-mx}$	$CF = c_3 e^{c^2 m^2 t}$
$PI = 0$	$PI = 0$
$GS = c_1 e^{mx} + c_2 e^{-mx}$	$GS = c_3 e^{c^2 m^2 t}$

$u(x,t) = XT$

$u(x,t) = (c_1 e^{mx} + c_2 e^{-mx})\, c_3 e^{c^2 m^2 t}$ — (5)

at $t \to \infty$
$e^{\infty} = \infty$
$\therefore u \to \infty$
case fails

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where k is $-ve$, $k = -m^2$

from eq ③

$X'' = kX$

$X'' - kX = 0$

$X'' + m^2 X = 0$

$(D^2 + m^2)X = 0$

Aux

$D^2 = -m^2$

$D = \pm mi$

$CF = c_1 \cos mx + c_2 \sin mx$

$PI = 0$

$GS = c_1 \cos mx + c_2 \sin mx$

$X = c_1 \cos mx + c_2 \sin mx$

$T' = c^2 T k$

$T' - c^2 T k = 0$

$T' + c^2 m^2 T = 0$

$(D + c^2 m^2) T = 0$

Aux

$D = -c^2 m^2$

$CF = c_3 e^{-c^2 m^2 t}$

$PI = 0$

$GS = c_3 e^{-c^2 m^2 t}$

$T = c_3 e^{-c^2 m^2 t}$

$u(x,t) = XT$

$u(x,t) = (c_1 \cos mx + c_2 \sin mx) c_3 e^{-c^2 m^2 t}$ — ⑥

Remark We know that

$t = \infty$, $u \to 0$

Most appropriate GS is eq ⑥

$u(x,t) = (c_1 \cos mx + c_2 \sin mx) c_3 e^{-c^2 m^2 t}$

$$\left\{ u(x,t) = (c_4 \cos mx + c_5 \sin mx) e^{-c^2 m^2 t} \right\} \quad — ⑦$$

* IMP EQ. *

* Boundary conditions

① $u(0,t) = 0$

Put $x = 0$ in eq ⑦

$u(0,t) = (c_4 \cos 0 + c_5 \sin 0) e^{-c^2 m^2 t}$

$0 = c_4 e^{-c^2 m^2 t}$

$\therefore c_4 = 0$ OR $\underbrace{e^{-c^2 m^2 t} \neq 0}_{\text{eq } \Leftarrow \text{ exponencial cannot be } 0}$

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$$\boxed{u(x,t) = (c_5 \sin mx)\, e^{-c^2 m^2 t}} \quad -8$$

(2) $u(l,t) = 0$

Put $x = l$ in eq (8)

$$u(l,t) = (c_5 \sin ml)\, e^{-c^2 m^2 t}$$
$$0 = (c_5 \sin ml)(e^{-c^2 m^2 t})$$

$$\therefore \underbrace{c_5 \sin ml = 0}_{} \quad \text{OR} \quad e^{-c^2 m^2 t} \neq 0$$

$c_5 \neq 0$ OR $\sin ml = 0$

$\sin ml = 0$

eq 8 $\rightarrow$ $$\boxed{u(x,t) = \left(c_5 \sin \frac{n\pi}{l} x\right) e^{-c^2\left(\frac{n^2\pi^2}{l^2}\right)t}} \quad -(9),\ n = 1, 2 \ldots \infty$$

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} b_n \left(\sin \frac{n\pi x}{l}\right)\left(e^{-\frac{c^2 n^2 \pi^2 t}{l^2}}\right)} \quad -(10)$$

* Initial condition :-

$u(x,0) = f(x)$

Put, $t = 0$ in (10)

$$u(x,0) = \sum_{n=1}^{\infty} b_n \left(\sin \frac{n\pi x}{l}\right) e^0$$

$$\boxed{f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}}$$

where $$\boxed{b_n = \frac{2}{l}\int_0^l f(x) \sin \frac{n\pi x}{l}\, dx}$$ ~~for~~ fourier sine series

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Q1 Solve eq.

$\frac{\partial u}{\partial t} = k^2 \frac{\partial^2 u}{\partial x^2}$

1) u is finite for all t
2) $u(0,t) = 0$
3) $u(l,t) = 0$
4) $u(x,0) = u_0 \quad 0 \leq x \leq l$

Note :- u_0 is a const.

Given eq is a heat eq.

i.e. $\frac{\partial u}{\partial t} = k^2 \frac{\partial^2 u}{\partial x^2}$ —①

General eq:

$u(x,t) = (C_4 \cos mx + C_5 \sin mx) e^{-k^2 m^2 t}$ —② $\quad 0 \leq x \leq l$

① u is finite for all t

② $u(0,t) = 0$

Put $x = 0$ in ②

$u(0,t) = (C_4 \cos 0 + C_5 \sin 0) e^{-k^2 m^2 t}$

$u(0,t) = C_4 e^{-k^2 m^2 t}$

$0 = (C_4)(e^{-k^2 m^2 t})$

$C_4 = 0$ OR $e^{-k^2 m^2 t} \neq 0$

$[C_4 = 0]$

Put $C_4 = 0$ in ②

$u(x,t) = (0 \cos mx + C_5 \sin mx) e^{-k^2 m^2 t}$

$u(x,t) = (C_5 \sin mx) e^{-k^2 m^2 t}$ —③

③ $u(l,t) = 0$

Put $x = l$ in ②

$u(l,t) = (C_5 \sin ml) e^{-k^2 m^2 t}$

$0 = (C_5 \sin ml) e^{-k^2 m^2 t}$

$C_5 \sin ml = 0$ OR $e^{-k^2 m^2 t} = 0$

$C_5 \neq 0$, $\sin ml = 0$, $e^{-k^2 m^2 t} \neq 0$

$\sin ml = 0$

$\sin ml = \sin n\pi \quad , \quad n = 1, 2, 3$

$ml = n\pi$

$m = \frac{n\pi}{l} \quad , \quad n = 1, 2, 3 \ldots$

Put in (3)

$$\left[u(l,t) = \left(C_5 \sin \frac{n\pi x}{l} \right) e^{-K^2\left(\frac{n^2\pi^2}{l^2}\right)t} \right] \quad \cancel{\text{(4)}} \quad \text{where } n = 1, 2, 3 \ldots$$

$$\left[u(x,t) = \sum_{n=1}^{\infty} b_n \left(\sin \frac{n\pi x}{l} \right) e^{-K^2\left(\frac{n^2\pi^2}{l^2}\right)t} \right] \quad \text{— (4)}$$

(4) $u(x,0) = u_0$

Put $t = 0$ in (4)

$$u(x,l) = \sum_{n=1}^{\infty} b_n \left(\sin \frac{n\pi x}{l} \right) e^{0}$$

Here b_n = Half Range of fourier sin series.

$$b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} \, dx$$

$$= \frac{2}{l} \int_0^l u_0 \sin \frac{n\pi x}{l} \, dx$$

$$= \frac{2}{l} u_0 \int_0^l \sin \frac{n\pi x}{l} \, dx$$

$$= \frac{2}{l} u_0 \left[\frac{-\cos \frac{n\pi x}{l}}{(n\pi)/l} \right]_0^l$$

$\cos n\pi = -1$

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$$= \frac{2}{n\pi} u_0 \left[-\frac{\cos n\pi l}{l} + \cos 0 \right]$$

$$\frac{2}{n\pi} u_0 \left[-\cos n\pi + 1 \right]$$

$$\frac{2}{n\pi} u_0 \left[1 - \cos n\pi \right]$$

$$\frac{2}{n\pi} u_0 \left[1 - (-1) \right]$$

$$\frac{2}{n\pi} u_0 (2)$$

$$b_n = \frac{4}{n\pi} u_0$$

$n =$

$b_n = 0$

$$u(x,t) = \sum_{n=1}^{\infty} \frac{2}{n\pi} u_0 [1 - (-1)] \qquad 0 \le x \le l$$

Q. Solve heat eq

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

① ~~x~~ $u(x,t)$ is finite

② $u(0,t) = 0$

③ $u(l,t) = 0$

④ $u(x,0) = \frac{2x}{l}$

Given eq is finite

① $u(x,t) = (c_4 \cos mx + c_5 \sin mx) e^{-c^2 m^2 t}$ — ② $0 \le x \le l$

② $u(0,t) = 0$

Put $x = 0$ in ②

$u(0,t) = (c_4 \ast) e^{-c^2 m^2 t}$

$c_4 = 0$ OR $e^{-c^2 m^2 t} \ne 0$

$\therefore c_4 = 0$ — Put in ②

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$$u(x,t) = (c_5 \sin mx) e^{-c^2m^2t} \quad -③$$

③ $u(l,t) = 0$

Put $x = l$ in ③

$$u(l,t) = (c_5 \sin ml) e^{-c^2m^2t}$$

$$0 = (c_5 \sin ml) e^{-c^2m^2t}$$

$$c_5 \neq 0 \;, \; \sin ml = 0 \;, \; e^{-c^2m^2t} \neq 0$$

$$\sin ml = 0 \;,$$

$$\sin ml = \sin n\pi \quad , n = 1, 2, 3 \ldots$$

$$ml = n\pi$$

$$m = \frac{n\pi}{l}$$

Put in ③

$$u(x,t) = \left(c_5 \sin \frac{n\pi}{l}\right) e^{-c^2\frac{n^2\pi^2}{l^2}t}$$

$$\left[u(x,t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{l} x \; e^{-c^2\left(\frac{n^2\pi^2}{l^2}\right)t}\right] \quad -④$$

④ $u(x,0) = \frac{2x}{l}$

Put $t = 0$ in ④

$$u(x,0) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)(e^0)$$

$$\frac{2x}{l} = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

Half rang fourier sin series

$$b_n = \frac{2}{l}\int f(x) \sin \frac{n\pi x}{l}\, dx$$

$$= \frac{2}{l}\int_0^l f(x) \sin \frac{n\pi x}{l}\, dx$$

$$= \frac{2}{l}\int_0^l \frac{2x}{l} \sin \frac{n\pi x}{l}\, dx$$

$$= \frac{4}{l^2}\int_0^l x \sin \frac{n\pi x}{l}\, dx$$

Note $\int uv = uv_1 - u'v_2 + u''v_3$

suffix = 1, 2 ..

eg' $\int x^3 \sin x$

$$= \underbrace{x^3(-\cos x)}_{uv'} - \underbrace{(3x^2)(-\sin x)}_{u'v'} + \underbrace{(6x)(\cos x)}_{u''v''}$$

$$b_n = \frac{4}{l^2}\int_0^l x \sin \frac{n\pi x}{l}\, dx$$

$$= \frac{4}{l^2}\left[(x)\left(\frac{-\cos \frac{n\pi x}{l}}{(n\pi x)/l}\right) - (1)\left(\frac{-\sin \frac{n\pi x}{l}}{(n^2\pi^2)/l^2}\right) + (0)\right]_0^l$$

$$= \frac{4}{l^2}\left[\frac{-l}{n\pi} x \cos \frac{n\pi x}{l} + \frac{l^2}{n^2\pi^2} \sin \frac{n\pi x}{l}\right]_0^l$$

$$= \frac{4}{l^2}\left[\frac{-l}{n\pi} x \cos \frac{n\pi x}{l} + \frac{l^2}{n^2\pi^2} \sin \frac{n\pi x}{l}\right]_0^l$$

$$\frac{4}{l^2}\left\{\left[\frac{-l}{n\pi}\, l\cos\frac{n\pi l}{l} + \frac{l^2}{\pi^2 n^2}\sin\frac{n\pi l}{l}\right] - 0\right\}$$

$$-\frac{4}{n\pi}\cos n\pi + \frac{4}{n^2\pi^2}\sin n\pi$$

$$= (-1)\frac{4}{n\pi}(-1)^n + 0$$

$$\boxed{b_n = \frac{4l(-1)^{n+1}}{n\pi}}$$

4. A homo rod of conducting material of length 2cm has its equal ends capped at 0°C & initially the temp is $u(x,0) = x$, $0 \le x \le 1$

$= 2-x$, $1 \le x \le 2$

$u(x,0) = x$, $0 \le x \le 1$

$= 2-x$, $1 \le x \le 2$

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad -①$$

with conditions

a $u(0,t) = 0$

b $u(2,t) = 0$

c $u(x,0) = x$ $\quad 0 \le x \le 1$

$= 2-x$ $\quad 1 \le x \le 2$

d u is finite for all t.

Most general eq ~~is~~ sol -

$u(x,t) = (c_4 \cos mx + c_5 \sin mx) e^{-c^2 m^2 t}$ $\quad -②$

$0 \le x \le 2$

a) $u(0,t) = 0$

Put $x=0$ in eq ②

$u(0,t) = c_4 e^{-c^2 m^2 t}$

$0 = c_4$

Put in eq ②

$u(x,t) = (c_5 \sin mx) e^{-c^2 m^2 t}$ $\quad -③$

b. $u(2,t) = 0$

Put $x=2$ in eq ③

$u(2,t) = (c_5 \sin 2m) e^{-c^2 m^2 t}$

$0 = (c_5 \sin 2m) e^{-c^2 m^2 t}$

$\sin 2m = 0$

$\sin 2m = \sin n\pi \quad , \quad n = 1, 2, \ldots$

$2m = n\pi$

$m = \frac{n\pi}{2} \quad , \quad n = 1, 2, 3, \ldots$

Put $m = \frac{n\pi}{2}$ in (3)

$$u(x,t) = \left[C_5 \sin \frac{n\pi}{2} x \right] e^{-c^2 \frac{n^2\pi^2}{4} t}$$

$$\left[u(x,t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{2} \, e^{-\frac{c^2 n^2 \pi^2 t}{4}} \right] \quad -(4)$$

d $u(x,0) = x \quad , \quad 0 \leq x \leq 1$

$\quad = 2 - x \quad , \quad 1 \leq x \leq 2$

Put $t = 0$ in (4)

$$u(x,0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{2}$$

Half range fourier sine series

$$b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} \, dx$$

$$= \frac{2}{2} \int_0^2 f(x) \sin \frac{n\pi x}{2} \, dx$$

$$= \underbrace{\int_0^1 x \sin \frac{n\pi}{2} x \, dx}_{*} + \underbrace{\int_1^2 (2-x) \sin \frac{n\pi x}{2} \, dx}_{**}$$

* $\int_0^1 x \sin \frac{n\pi}{2} x \, dx$

$$\left[x \left(\frac{-\cos \frac{n\pi x}{2}}{n\pi/2} \right) - (1) \left(\frac{-\sin \frac{n\pi x}{2}}{n^2\pi^2/4} \right) \right]_0^1$$

$$\left(\frac{-2}{n\pi} \cos \frac{n\pi}{2} + \frac{4}{n^2\pi^2} \sin \frac{n\pi}{2} \right) - (0 - 0)]$$

$$= \left[\frac{-2}{n\pi} \cos \frac{n\pi}{2} + \frac{4}{n^2\pi^2} \sin \frac{n\pi}{2} \right]$$

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** $\int_1^2 (2-x)\sin\frac{n\pi x}{2}\,dx$

$$\left[\frac{(2-x)\left(-\cos\frac{n\pi x}{2}\right)}{n\pi/2} - (-1)\frac{\left(-\sin\frac{n\pi x}{2}\right)}{n^2\pi^2/4}\right]_1^2$$

$$\left[\left(0 - \frac{4\sin n\pi}{n^2\pi^2}\right) - \left(\frac{2}{n\pi}\cos\frac{n\pi}{2} - \frac{4}{n^2\pi^2}\sin\frac{n\pi}{2}\right)\right]$$

$$= \cancel{-\frac{4}{n^2\pi^2}\sin n\pi + \frac{2}{n\pi}\cos\frac{n\pi}{2} + \frac{4}{n^2}\sin\frac{n\pi}{2}}$$

$$\frac{-4}{n^2\pi^2}\cancel{\sin n\pi} + \frac{2}{n\pi}\cos\frac{n\pi}{2} + \frac{4}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$$\left(\frac{2}{n\pi}\cos\frac{n\pi}{2} + \frac{4}{n^2\pi^2}\sin\frac{n\pi}{2}\right)$$

combine * & **

$$\cancel{-\frac{2}{n\pi}\cos\frac{n\pi}{2}} + \frac{4}{n^2\pi^2}\sin\frac{n\pi}{2} + \cancel{\frac{2}{n\pi}\cos\frac{n\pi}{2}} + \frac{4}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$$\frac{4}{n^2\pi^2}\sin\frac{n\pi}{2} + \frac{4}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$$\boxed{b_n = \frac{8}{n^2\pi^2}\sin\frac{n\pi}{2}}$$

$$\therefore u(x,t) = \sum_{n=1}^{\infty}\left(\frac{8}{n^2\pi^2}\sin\frac{n\pi}{2}\right)\sin\frac{n\pi x}{2}\, e^{-c^2 m^2 t}$$

8. $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$

1) $u(0,t) = 0$

2) $u(20,t) = 0$

3 $u(x,0) = \sin \frac{\pi x}{20}$, $0 \le x \le 20$

1) $u(0,t) = 0$

$c_4 = 0$

2) $u(20,t) = 0$

$m = \frac{n\pi}{l}$

$m = \frac{n\pi}{20}$ - $n = 1, 2, 3$

$u(x,t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{20} e^{-c^2 m^2 t}$ —(4)

$u(x,0) = \sin \frac{\pi x}{20}$

$t = 0$ in (4)

$u(x,0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{20}$

$\sin \frac{\pi x}{20} = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{20}$

$\sin \frac{\pi x}{20} = \sin \frac{\pi x}{20} + \sin \frac{2\pi x}{20} + \sin \frac{3\pi x}{20} \ldots$

comp both sides

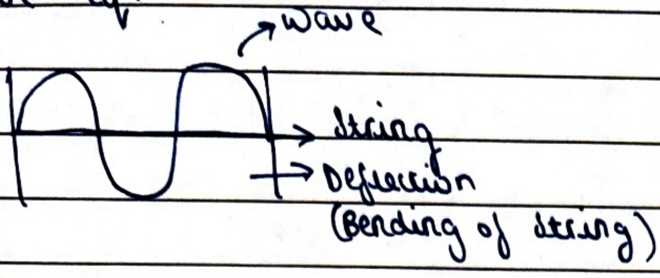
$\left[b_1 = \sin \frac{\pi x}{20}\right]$, $b_2 = 0$, $b_3 = 0$

$\sin \frac{\pi x}{20} = b_1 \sin \frac{\pi x}{20}$

$\therefore b_1 = 1$, $b_2 = 0$, $b_3 = 0$

$$\left[u(x,t) = \left(\sin \frac{\pi x}{20}\right) e^{-c^2 m^2 t}, \quad 0 \le x \le 20\right]$$

* Wave eq.



$u(x,t)$ depends on distance & time

The vibration of an elastic string is generated by the wave eq.

$$\frac{\delta^2 u}{\delta t^2} = c^2 \frac{\delta^2 u}{\delta x^2} \quad \text{—(1)}$$

1) $u(0,t) = 0$ 2) $u(l,t) = 0$

3) $\left(\frac{\delta u}{\delta t}\right)_{t=0} = f(x)$ 4) $u(x,0) = g(x)$

Let,

$u(x,t) = f(x)\,g(t)$ —(2)

be the solⁿ of the D.E of eqⁿ (1)

Diff eq (2) w.r.t. t twice

$$\frac{\delta^2 u}{\delta t^2} = f(x)\frac{\delta^2 g}{\delta t^2} = f(x)\frac{\delta^2 g}{\delta t^2} = f(x)\,g''(t)$$

Diff. eq (2) w.r.t. x twice

$$\frac{\delta^2 u}{\delta x^2} = \frac{\delta^2 f}{\delta x^2}\,g(t) = g(t)\,f''(x)$$

$\rightarrow$ $fg'' = c^2 f'' g$

$$\frac{g''}{c^2 g} = \frac{f''}{f} = k \text{ (say)}$$

Case ③ :- $k = -ve = -m^2$

$\frac{f''}{f} = \frac{g''}{c^2 g} = -m^2$	$\frac{f''}{f} = -m^2$
$\frac{g''}{c^2 g} = -m^2$	$f'' = -m^2 f$
$g'' = -m^2 c^2 g$ $g'' + m^2 c^2 g = 0$	$f'' + m^2 f = 0$
$D = \pm icm$	$D = \pm im$
$CF = c_3 \cos cmt + c_4 \sin cmt$	$CF = c_1 \cos mx + c_2 \sin mx$
$\therefore PI = 0$	$PI = 0$
$y = c_3 \cos cmt + c_4 \sin cmt$	$y = c_1 \cos mx + c_2 \sin mx$
$g(t)$	$f(x)$

$u(x,t) = (c_1 \cos mx + c_2 \sin mx)(c_3 \cos cmt + c_4 \sin cmt)$ — ③

This is the most accurate / general sol

i $u(0,t) = 0$

Put $x = 0$ in ③

$u(0,t) = (c_1)(c_3 \cos cmt + c_4 \sin cmt) = 0$

$[c_1 = 0]$

Put $c_1 = 0$ in ③

$u(x,t) = (c_2 \sin mx)(c_3 \cos cmt + c_4 \sin cmt)$ — ④

ii $u(l,t)=0$

Put $x=l$

$u(l,t)=(c_2 \sin ml)(c_3 \cos(mt)+c_4 \sin(mt))$

$c_2 \sin ml = 0$

$\sin ml = 0$

$\sin ml = \sin n\pi$

$ml = n\pi$

$m=\frac{n\pi}{l}$, $n=1,2,3 \ldots$

Put $m=\frac{n\pi}{l}$ in (4)

$u(x,t)=\left(c_2 \sin \frac{n\pi x}{l}\right)\left(c_3 \cos \frac{(mt)}{l} + c_4 \sin \frac{(mt)}{l}\right)$

$u(x,t)=$

Solve the ~~heat~~ wave eq.

Q.1) $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ — ①

i $u(0,t)=0$ iv $u(x,0) = u_0 \sin^3\left(\frac{\pi x}{l}\right)$

ii $u(l,t)=0$

iii $\left(\frac{\partial u}{\partial t}\right)_{t=0} = 0$

Most general sol

$u(x,t) = (c_1 \cos mx + c_2 \sin mx)(c_3 \cos cmt + c_4 \sin cmt)$ — ②

$0 \le x \le l$

i $u(0,t)=0$

$u(0,t) = (c_1)(c_3 \cos cmt + c_4 \sin cmt)$

$0 = (c_1)(c_3 \cos cmt + c_4 \sin cmt)$

$c_1 = 0$

Put $c_1 = 0$ in ②

$u(x,t) = (c_2 \sin mx)(c_3 \cos cmt + c_4 \sin cmt)$ — ③

ii $u(l,t)=0$

$u(l,t) = (c_2 \sin ml)(c_3 \cos cmt + c_4 \sin cmt) = 0$

$\sin ml = 0$

$\sin ml = \sin n\pi$

$ml = n\pi$

$m = \frac{n\pi}{l}$ — Sub in ③

$u(x,t) = \left(c_2 \sin \frac{n\pi x}{l}\right)\left(c_3 \cos cmt + c_4 \sin cmt\right)$, $n=1,2,3$ — ④

$u(x,t) = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{l}\left(A_n \cos cmt + B_n \sin cmt\right)$ — ⑤

iii Diff eq. ⑤ w.r.t t

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$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{l}\left(-cmA_n\sin cmt + cm B_n\cos cmt\right)$$

$$= \sum_{n=1}^{\infty} \sin\frac{n\pi x}{l}\left(-\frac{n\pi c}{l}A_n\sin\frac{cn\pi t}{l} + \frac{cn\pi}{l}B_n\cos\frac{cn\pi t}{l}\right)$$

$$\left(\frac{\partial u}{\partial t}\right)_{t=0} = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{l}\left(0 + \frac{n\pi c}{l}B_n\right)$$

$$0 = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{l}\left(\frac{n\pi}{l}cB_n\right) \quad \text{— ⑥} \qquad \therefore \boxed{B_n = 0}$$

iv $u(x,0) = u_0\sin^3\left(\frac{\pi x}{l}\right)$

Put $t=0$ in ⑤

$$u(x,0) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{l}\left(A_n\cos 0 + B_n\sin 0\right)$$

$$u_0\sin^3\frac{\pi x}{l} = \sum_{n=1}^{\infty} A_n\sin\frac{n\pi x}{l}$$

On equating

$$u_0\left(\frac{3}{4}\sin\frac{\pi x}{l} - \frac{1}{4}\sin\frac{3\pi x}{l}\right) = A_1\sin\frac{\pi x}{l} + \cancel{A_2\sin\frac{2\pi x}{l}} + A_3\sin\frac{3\pi x}{l} \dots$$

$A_1 = \frac{3}{4}u_0 \qquad A_3 = \frac{-1}{4}u_0$

$A_2 = 0 \qquad A_4 = A_5 \dots = 0$

Sub in ⑤

$$u(x,t) = \sin\frac{n\pi x}{l}\left(\frac{3}{4}u_0\cos\left(\frac{c\pi t}{l}\right)\right) - \sin\frac{3\pi x}{l}\left(\frac{1}{4}u_0\cos\left(\frac{3\pi ct}{l}\right)\right)$$

Q2. $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ — (1)

i $u(0,t)=0$ iii $\left(\frac{\partial u}{\partial t}\right)_{t=0} = 0$ iv $u(x,0) = 0.01(\pi - x)$

ii $u(\pi,t)=0$

General sol:

$u(x,t) = (c_1 \cos mx + c_2 \sin mx)(c_3 \cos cmt + c_4 \sin cmt)$ — (2)

$0 \le x \le \pi$

Put $x=0$

i $u(0,t) = (c_1)(c_3 \cos cmt + c_4 \sin cmt)$

$0 = c_1$

Put in (2)

$u(x,t) = (c_2 \sin mx)(c_3 \cos cmt + c_4 \sin cmt)$ — (3)

ii $u(\pi,t)=0$ Put $x=\pi$ in (3)

$u(\pi,t) = (c_2 \sin m\pi)(c_3 \cos cmt + c_4 \sin cmt) = 0$

$\sin m\pi = 0$

$\sin m\pi = \sin n\pi$

$m = n$

Put $m=n$ in (3)

~~$u(x,t) = (c_2 \sin nx)(c_3 \cos (\pi t + c_4 \sin (\pi t)$~~

$u(x,t) = (c_2 \sin nx)(c_3 \cos cnt + c_4 \sin cnt)$

$n = 1, 2 \ldots$

$u(x,t) = \sum_{n=1}^{\infty} \sin nx \left(A_n \cos cnt + B_n \sin cnt \right)$ — (4)

Diff (4) w.r.t t

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$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} \sin nx \left(-cn A_n \sin cnt + B_n cn \cos cnt\right)$$

$$\left(\frac{\partial u}{\partial t}\right)_{t=0} = \sum_{n=1}^{\infty} \sin nx \left(0 + cn B_n\right)$$

$$0 = \sum_{n=1}^{\infty} (\sin nx)(cn B_n)$$

$$\therefore \boxed{B_n = 0}$$

Put $B_n = 0$ in (4)

$$u(x,t) = \sum_{n=1}^{\infty} \sin nx \left(A_n \cos cnt\right) \quad \text{— (5)}$$

iii) $u(x,0) = 0.01(\pi - x)$

Put $t=0$ in (5)

$$u(x,0) = \sum_{n=1}^{\infty} \sin nx \,(A_n)$$

$$0.01(\pi - x) = \sum_{n=1}^{\infty} A_n \sin nx$$

~~On~~ Half Range.

$$A_n = \frac{2}{l}\int_0^l f(x) \sin \frac{n\pi x}{l}\, dx$$

$$A_n = \frac{2}{\pi}\int_0^{\pi} 0.01(\pi - x) \sin nx \, dx$$

$$= \frac{0.02}{\pi}\int_0^{\pi} \underset{u}{(\pi - x)} \underset{v}{\sin nx}\, dx$$

$$= \frac{0.02}{\pi}\left[(\pi - x)\left(-\frac{\cos nx}{n}\right) - (0-1)\left(-\frac{\sin nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{0.02}{\pi}\left[(0-0) - \left(\frac{-\pi}{n}\right)\right]$$

$$= \frac{0.02}{\pi}\left(\frac{\pi}{n}\right) = \frac{0.02}{n}$$

$A_n = \frac{0.02}{n}$. Sub. A_n in (5)

$$u(x,t) = \sum_{n=1}^{\infty} \sin nx \left(\frac{0.02}{n} \cos(nt) \right) \quad — (6)$$