

Unit - 3 - Laplace transformation

25-8-25 Intro - Laplace transformation is a freq domain approach for cont. signals. Converts func of a +ve real variable (t) (time) to a complex function of a complex variable s (freq) where $s = \alpha + j\beta$. It is particularly used in solving linear ordinary diff eq. such as those encountered in the analysis of electric circuits.

Mathematical def:

Laplace transform of a function $f(t)$ is defined for real no.s, $t \geq 0$ is the function $F(s)$ OR $\mathcal{L}[f(t)]$ is defined by $\mathcal{L}[f(t)] = F(s) = \int_0^{\infty} e^{-st} f(t) dt$ where s is complex

func, known as freq parameter

* Properties

$$1) \mathcal{L}[f(t) \pm g(t)] = \int_0^{\infty} e^{-st} [f(t) \pm g(t)] dt = \int_0^{\infty} e^{-st} f(t) dt \pm \int_0^{\infty} e^{-st} g(t) dt$$

$$\mathcal{L}[f(t) \pm g(t)] = \mathcal{L}[f(t)] \pm \mathcal{L}[g(t)]$$

$$2) \mathcal{L}[kf(t)] = k \mathcal{L}[f(t)]$$

$$3) \mathcal{L}[1] = \frac{1}{s}$$

→ We know

$$\mathcal{L}[f(t)] = \int_0^{\infty} e^{-st} f(t) dt$$

$$\mathcal{L}[1] = \int_0^{\infty} e^{-st} (1) dt$$

$$= \left[-\frac{e^{-st}}{s} \right]_0^{\infty}$$

$$= \frac{1}{s} \left[-e^{-\infty} + e^{-0} \right]$$

$$= \frac{1}{s} (1)$$

$$\left[L(1) = \frac{1}{s} \right]$$

$$4. L[e^{at}] = \frac{1}{s-a}, \quad s > a$$

$$= L[e^{at}] = \int_0^{\infty} e^{-st} e^{at} dt$$

$$= \int_0^{\infty} e^{t(a-s)} dt$$

$$= \left[\frac{e^{t(a-s)}}{a-s} \right]_0^{\infty}$$

$$= \frac{1}{a-s} (e^{\infty} - e^0)$$

$$= \frac{-1}{a-s}$$

$$L[e^{at}] = \frac{1}{s-a}$$

$$5. L[\sin at] = \frac{a}{s^2 + a^2}$$

$$6. L[\cos at] = \frac{s}{s^2 + a^2}$$

$$7. L[\sinh at] = \frac{a}{s^2 - a^2}$$

$$8. L[\cosh at] = \frac{s}{s^2 - a^2}$$

Derive 4th

9. If $\mathcal{L}[f(t)] = F(s)$ then,

$$\mathcal{L}[e^{at} f(t)] = F(s-a)$$

Proof: $\mathcal{L}[f(t)] = \int_0^{\infty} e^{-st} f(t) dt$

$$\begin{aligned} \mathcal{L}[e^{at} f(t)] &= \int_0^{\infty} e^{-st} f(t) dt \\ &= \int_0^{\infty} e^{-t(s-a)} f(t) dt \end{aligned}$$

Let, $s-a = p$

$$= \int_0^{\infty} e^{-pt} f(t) dt = F(p)$$

$$= F(s-a)$$

eg. 1 $\mathcal{L}[\sin 3t]$

$$\mathcal{L}[\sin at] = \frac{a}{s^2 + a^2} = \frac{3}{s^2 + 9}$$

eg. 2 $\mathcal{L}[e^{2t} \sin 3t]$

$$\mathcal{L}[\sin 3t] = \frac{3}{s^2 + 9}$$

$$\mathcal{L}[e^{2t} \sin 3t] = \frac{3}{(s-2)^2 + 9}$$

$$= \frac{3}{s^2 - 4s + 4 + 9}$$

$$= \frac{3}{s^2 - 4s + 13}$$

10. $\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}$

(Gamma function)

$$\frac{n!}{s^{n+1}} = \frac{\Gamma(n+1)}{s^{n+1}}, \quad \Gamma = 1, \quad \Gamma_{1/2} = \sqrt{\pi}, \quad \Gamma_n = (n-1)!$$

eg. $\Gamma_4 = \Gamma_{3+1} = 3\Gamma_3$

$$= 3\Gamma_{2+1} = 3 \cdot 2\Gamma_2$$

$$= 3 \cdot 2 \Gamma_{1+1}$$

$$= 3 \cdot 2 \cdot 1 \Gamma_1$$

$$= 3 \cdot 2 \cdot 1 = 6$$

11. $\mathcal{L}[e^{at} t^n] = \frac{n!}{(s-a)^{n+1}}$

Q. Find ~~inverse~~ tan

e.g. $\sqrt{5/2} = \sqrt{3/2 + 1}$

$$= \frac{3}{2} \sqrt{\frac{3}{2}}$$

$$= \frac{3}{2} \cdot \sqrt{\frac{1}{2} + 1}$$

$$= \frac{3}{2} \cdot \frac{1}{2} \sqrt{\frac{1}{2}}$$

$$= \frac{3}{2} \cdot \frac{1}{2} \sqrt{\pi}$$

$$= \frac{3\sqrt{\pi}}{4}$$

e.g. $\sqrt{9/2} = \frac{7}{2} \cdot \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \sqrt{\pi}$

$$\sqrt{11/2} = \frac{9}{2} \cdot \frac{7}{2} \cdot \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \sqrt{\pi}$$

$$\sqrt{5} = 4!$$

$$\sqrt{7} = 6!$$

11. $L[e^{at} t^n] = \frac{n!}{(s-a)^{n+1}}$

12. $L[t^n] = \frac{n!}{s^{n+1}}$

13. $L[e^{at} t^n] = \frac{n!}{(s-a)^{n+1}}$ $s \rightarrow s-a$

Q. Find Laplace transform

1) $L[\omega s^2 t]$

2) $L[t^{7/2} e^{5t}]$

3) $L[(e^{-2t} + e^{3t})^2]$

4) $L[4t]$

5) $L[5t - 7e^{-6t} + t^{5/2}]$

1. $L[\omega s^2 t]$

$L\left[\frac{1 + \omega s^2 t}{2}\right]$

 $\rightarrow$ Property 1.2

$L\left[\frac{1}{2}\right] + L\left[\frac{\omega s^2 t}{2}\right] \rightarrow (3) + (6)$

$\frac{1}{2} L[1] + \frac{1}{2} L[\omega s^2 t]$

$\frac{1}{2} \left(\frac{1}{s}\right) + \frac{1}{2} \left(\frac{s}{s^2 + 4}\right)$

2. $L[t^{7/2} e^{5t}]$

$= \frac{n+1}{(s-a)^{n+1}}$

$= \frac{7/2+1}{(s-5)^{7/2+1}} = \frac{9/2}{(s-5)^{9/2}}$

$= \frac{7/2 \cdot 5/2 \cdot 3/2 \cdot 1/2 \cdot \sqrt{\pi}}{(s-5)^{9/2}}$

$= \frac{105}{16} \left[\frac{\sqrt{\pi}}{(s-5)^{9/2}} \right]$

$$3. \quad L[(e^{-2t} + e^{3t})^2]$$

$$L[e^{-4t} + 2e^t + e^{6t}]$$

$$L[e^{-4t}] + 2L[e^t] + L[e^{6t}]$$

$$\frac{1}{s+4} + 2 \cdot \frac{1}{s-1} + \frac{1}{s-6}$$

$$\frac{1}{s+4} + \frac{2}{s-1} + \frac{1}{s-6}$$

$$4. \quad L[4^t]$$

$$L[e^{t \log(4)}]$$

$$L[e^{t \log(4)}]$$

$$= \frac{1}{s - \log 4}$$

$$s - \log 4$$

$$5. \quad L[5t - 7e^{-6t} + t^{5/2}]$$

$$5L[t] - 7L[e^{-6t}] + L[t^{5/2}]$$

$$5 \left[\frac{1}{s^2} \right] - 7 \left[\frac{1}{s+6} \right] + \frac{7/2}{s^{7/2}}$$

$$5 \left[\frac{1}{s^2} \right] - \frac{7}{s+6} + \frac{15\sqrt{\pi}}{8} \left(\frac{1}{s^{7/2}} \right)$$

$$6. \mathcal{L}[3\cos(4t+7)]$$

$$\mathcal{L}[3(\cos 4t \cdot \cos 7 - \sin 4t \sin 7)]$$

$$= 3\cos 7 \mathcal{L}[\cos 4t] - 3\sin 7 \mathcal{L}[\sin 4t]$$

$$= 3\cos 7 \left(\frac{s}{s^2+16} \right) - 3\sin 7 \left(\frac{4}{s^2+16} \right)$$

$$7. \mathcal{L}[4\sin(2t+3)]$$

$$\mathcal{L}[4\sin 2t \cos 3 + \cos 2t \sin 3]$$

$$= 4\cos 3 \mathcal{L}[\sin 2t] + 4\sin 3 \mathcal{L}[\cos 2t]$$

$$4\cos 3 \left(\frac{2}{s^2+4} \right) + 4\sin 3 \left(\frac{s}{s^2+4} \right)$$

$$8. \mathcal{L}[\cos(t-2)]$$

$$\mathcal{L}[\cos t \cos 2 + \sin t \sin 2]$$

$$\cos 2 \left(\frac{s}{s^2+1} \right) + \sin 2 \left(\frac{1}{s^2+1} \right)$$

$$* \mathcal{L}[f(at)] = \frac{1}{a} \mathcal{L}[f(t)]$$

③ Theorem: On Derivatives

$$L[f'(t)] = sF(s) - f(0)$$

$$= sL[f(t)] - f(0)$$

$$L[f''(t)] = s^2 F(s) - f(0) - f'(0)$$

$$L[f'''(t)] = s^3 F(s) - f(0) - f'(0) - f''(0)$$

④ Theorem: On integrals

$$L\left[\int_0^t f(u) du\right] = \frac{1}{s} F(s)$$

$$= \frac{1}{s} L[f(t)]$$

$$L\left[\int_0^t \int_0^u f(u) du\right] = \frac{1}{s^2} F(s)$$

⑤ Theorem: Multiplication by t

$$L[tf(t)] = (-1)' \frac{d}{ds} F(s)$$

$$= (-1)' \frac{d}{ds} L[f(t)]$$

$$L[t^2 f(t)] = (-1)^2 \frac{d^2}{ds^2} L[f(t)]$$

⑥ Division by t

$$L\left[\frac{f(t)}{t}\right] = \int_s^\infty F(s) ds \rightarrow \text{Provided, the limit should exist}$$

$$L\left[\frac{f(t)}{t^2}\right] = \int_s^\infty \int_s^\infty F(s) ds$$

Integration $\leftrightarrow$ Divide

Multiplication $\leftrightarrow$ Derivation

e.g. Find Laplace transform of following

$$e^{-3t} t^{-1/2}$$

$$L[e^{-3t} t^{-1/2}] = ?$$

$$e^{-at} f(t) \rightarrow \{f(s)\} \rightarrow s \rightarrow s+a$$

$$L[t^{-1/2}] = \frac{\frac{-1/2+1}{s^{-1/2+1}}}{s^{1/2+1}} = \frac{\sqrt{\pi}}{s^{1/2}}$$

$$L[e^{-3t} t^{-1/2}] = \frac{\sqrt{\pi}}{(s+3)^{1/2}} = \frac{\sqrt{\pi}}{\sqrt{s+3}}$$

$$2. \int_0^t \sin 2u \, du$$

We know,

$$\int_0^t f(u) \, du = \frac{1}{s} F^*(s) = \frac{1}{s} L[f(t)] = \frac{1}{s} L[\sin 2u] = \left(\frac{2}{s^2+4} \right) \frac{1}{s}$$

$$3. t \cos at$$

$$L[tf(t)] = (-1)' \frac{dF(s)}{ds}$$

$$(-1)' \frac{d}{ds} L[\cos at]$$

$$(-1)' \frac{d}{ds} \left(\frac{s}{s^2+a^2} \right)$$

$$(-1)' \left[\frac{s^2+a^2(1) - (s)(2s)}{(s^2+a^2)^2} \right]$$

$$= (-1) \left[\frac{-s^2+a^2}{(s^2+a^2)^2} \right] = \frac{s^2-a^2}{(s^2+a^2)^2}$$

4. $t^2 \sin at$

$$L[t^2 \sin at] = ?$$

$$L[t^2 f(t)] = (-1)^2 \frac{d^2}{ds^2} f(s)$$

$$= \frac{d^2}{ds^2} \left(\frac{a}{s^2 + a^2} \right)$$

$$\frac{d}{ds} \left[\frac{-a(2s)}{(s^2 + a^2)^2} \right]$$

$$= -2a \frac{d}{ds} \left(\frac{s}{(s^2 + a^2)^2} \right)$$

$$= -2a \left[\frac{(s^2 + a^2)^2(1) - (s)[2(s^2 + a^2)(2s)]}{(s^2 + a^2)^4} \right]$$

$$= -2a \left[\frac{s^2 + a^2 - 4s^2}{(s^2 + a^2)^3} \right]$$

$$= -2a \left[\frac{-3s^2 + a^2}{(s^2 + a^2)^3} \right]$$

$$= \frac{3as^2 - 2a^3}{(s^2 + a^2)^3}$$

$$\frac{2s^3 - 24s}{(s^2 + 4)}$$

$$L[t^2 \cos 2t]$$

$$(-1)^2 \frac{d^2}{ds^2} \left(\frac{s}{s^2 + 4} \right)$$

$$\frac{d^2}{ds^2} \left(\frac{(s^2 + 4)(1) - (s)(2s)}{(s^2 + 4)^2} \right)$$

$$\frac{d}{ds} \left[\frac{s^2 + 4 - 2s^2}{(s^2 + 4)^2} \right]$$

$$\frac{d}{ds} \left[\frac{-s^2 + 4}{(s^2 + 4)^2} \right]$$

$$\frac{d}{ds} \left[\frac{-(1+s^2)}{(1+s^2)^2} \right]$$

$$\left[\frac{-(1+s^2)}{(1+s^2)^2} \right] \cdot 1 - \left[\frac{2s}{(1+s^2)^2} \right]$$

$$\frac{-(1+s^2)}{(1+s^2)^2} + (0) \cdot 1 - \left[\frac{2s}{(1+s^2)^2} \right]$$

$$\frac{-(1+s^2)}{(1+s^2)^2} + \frac{0}{(1+s^2)^2} =$$

$$\frac{-1-s^2}{(1+s^2)^2}$$

$$\left[\frac{-1-s^2}{(1+s^2)^2} \right] \cos$$

$$6. \frac{1}{t} [1 - \cos t]$$

$$\int_s^\infty f(s) ds$$

$$= \int_s^\infty \left[\frac{1}{s} - \frac{s}{s^2+1} \right] ds$$

$$= \int_s^\infty \frac{1}{s} ds - \int_s^\infty \frac{s}{s^2+1} ds$$

$$[\log s]_s^\infty - \frac{1}{2} \int_s^\infty \frac{2s}{s^2+1} ds$$

$$[\log s]_s^\infty - \frac{1}{2} [\log(s^2+1)]_s^\infty$$

$$[\log \infty - \log s] - \frac{1}{2} [\log \infty - \log(s^2+1)]$$

$$(0 - \log s) - \frac{1}{2} (0) + \frac{1}{2} \log(s^2+1)$$

$$= -\log s + \log(s^2+1)^{1/2}$$

$$\log \sqrt{s^2+1} - \log s$$

$$\log \left[\frac{\sqrt{s^2+1}}{s} \right]$$

$$7. \mathcal{L}[t e^{3t} \sin 2t]$$

We know

$$\mathcal{L}[e^{at} f(t)] = \mathcal{L}[f(t)] \rightarrow s \rightarrow s-a$$

$$\mathcal{L}[e^{3t} t \sin t] = \mathcal{L}[t \sin t] \quad s \rightarrow s-3$$

$$\mathcal{L}[t f(t)] = (-1) \frac{d}{ds} \mathcal{L}[f(t)]$$

$$= (-1) \frac{d}{ds} \mathcal{L}[\sin^2 t]$$

$$= (-1) \frac{d}{ds} \left(\frac{2}{s^2+4} \right)$$

$$= (-2) \left(\frac{-2s}{(s^2+4)^2} \right)$$

$$\mathcal{L}[t \sin t] = \frac{4s}{(s^2+4)^2}$$

$$\mathcal{L}[e^{3t} t \sin t] = \frac{4(s-3)}{[(s-3)^2+4]^2}$$

$$8. \mathcal{L}[e^t t \cos 3t]$$

$$= \mathcal{L}[t \cos 3t]_{s \rightarrow s-1}$$

$$\mathcal{L}[t \cos 3t] = (-1) \frac{d}{ds} \mathcal{L}[\cos 3t]$$

$$(-1) \frac{d}{ds} \left(\frac{s}{s^2+9} \right)$$

$$(-1) \left[\frac{(s^2+9) - (2s)s}{(s^2+9)^2} \right]$$

$$(-1) \left[\frac{9 - s^2}{(s^2+9)^2} \right]$$

$$\left[\frac{s^2 - 9}{(s^2+9)^2} \right]$$

$$\mathcal{L}[e^t t \cos 3t] = \left[\frac{(s-1)^2 - 9}{((s-1)^2+9)^2} \right]$$

$$9. \mathcal{L} \left[\int_0^t \sin t \, dt \right]$$

$$\mathcal{L} \left[\int_0^t f(t) \, dt \right] = \frac{1}{s} F(s) = \frac{1}{s} \mathcal{L}[f(t)]$$

$$= \frac{1}{s} \mathcal{L} \left[\sin t \right]$$

$$= \frac{1}{s} \left[\int_s^\infty \mathcal{L}[\sin t] \, ds \right]$$

$$= \frac{1}{s} \left[\int_s^\infty \frac{1}{s^2+1^2} \, ds \right]$$

$$= \frac{1}{s} \left[\tan^{-1}(s/1) \right]_s^\infty$$

$$\frac{1}{s} [\tan^{-1} \infty - \tan^{-1} s]$$

$$= \frac{1}{s} \left[\frac{\pi}{2} - \tan^{-1} s \right] = \frac{1}{s} (\cot^{-1} s)$$

$$L\left[\frac{e^{-4t} \sin 3t}{t}\right]$$

$$L\left[\frac{e^{-4t} \sin 3t}{t}\right] = L\left[\frac{\sin 3t}{t}\right]_{s \rightarrow s+4}$$

$$= \int_s^\infty L[\sin 3t] ds_{s \rightarrow s+4}$$

$$= \int_s^\infty \frac{3}{s^2+9} ds_{s \rightarrow s+4}$$

$$= 3 \left[\int_s^\infty \frac{1}{s^2+3^2} ds \right]_{s \rightarrow s+4}$$

$$3 \left[\frac{1}{3} \tan^{-1}(s/3) \right]_s^\infty$$

$$\left[\tan^{-1} \infty - \tan^{-1}(s/3) \right]_{s \rightarrow s+4}$$

$$\left[\frac{\pi}{2} - \tan^{-1}(s/3) \right]_{s \rightarrow s+4}$$

$$\left[\cos t^{-1}\left(\frac{s}{3}\right) \right]_{s \rightarrow s+4}$$

$$= \frac{\cos t^{-1}(s+4)}{3}$$

$$11. \quad L \left[t \int_0^t e^{-4t} \sin 3t \, dt \right]$$

$$L \left[t \int_0^t \sin 3t \, dt \right]_{s \rightarrow s+4}$$

$$= \left\{ (-1) \frac{d}{ds} L \left[\int_0^t \sin 3t \, dt \right] \right\}_{s \rightarrow s+4}$$

$$(-1) \frac{d}{ds} \left[\frac{1}{s} L(\sin 3t) \right]_{s \rightarrow s+4}$$

$$(-1) \frac{d}{ds} \left[\frac{1}{s} \left(\frac{3}{s^2+9} \right) \right]_{s \rightarrow s+4}$$

$$(-1) \frac{d}{ds} \left[\frac{3}{s^3+9s} \right]_{s \rightarrow s+4}$$

$$-3 \frac{d}{ds} \left[\frac{1}{s^3+9s} \right]_{s \rightarrow s+4}$$

$$-3 \left[\frac{-1(3s^2+9)}{(s^3+9s)^2} \right]_{s \rightarrow s+4}$$

$$\frac{3(3s^2+9)}{(s^3+9s)^2}$$

$$\left[\frac{9s^2+27}{(s^3+9s)^2} \right]_{s \rightarrow s+4}$$

$$\frac{9(s+4)^2+27}{[(s+4)^3+9(s+4)]^2}$$

Q. Solve the following integrals using Laplace transform

$$L[f(t)] = \int_0^{\infty} e^{-st} f(t) dt$$

$$L[t \sin t] = \int_0^{\infty} e^{-st} t \sin t ds$$

$$L[t \sin t] = (-1) \frac{d}{ds} L[\sin t] \quad \text{where } s=3$$

$$(-1) \frac{d}{ds} \left[\frac{1}{s^2+1} \right]$$

$$(-1) \left[\frac{-1(2s)}{(s^2+1)^2} \right]$$

$$L[t \sin t] = \frac{2s}{(s^2+1)^2}$$

$$\therefore \int_0^{\infty} e^{-st} t \sin t dt = \frac{2s}{(s^2+1)^2}$$

$$\int_0^{\infty} e^{-3t} t \sin t dt = \frac{2(3)}{(9+1)^2} = \left[\frac{6}{100} \right] = \frac{3}{50}$$

2. $\int_0^{\infty} t^2 e^{-t} \sin t \, dt$

$$L[f(t)] = \int_0^{\infty} e^{-st} f(t) \, dt$$

$$L[t^2 \sin t] = \int_0^{\infty} e^{-st} t^2 \sin t \, dt$$

Where $s=1$

$$L[t^2 \sin t] = (-1)^2 \frac{d^2}{ds^2} L[\sin t]$$

$$\frac{d^2}{ds^2} \left[\frac{1}{s^2+1} \right]$$

$$\frac{d}{ds} \left[\frac{-1(2s)}{(s^2+1)^2} \right]$$

$$-2 \frac{d}{ds} \left[\frac{s}{(s^2+1)^2} \right]$$

$$-2 \left[\frac{(s^2+1)^2(1) - s(2(s^2+1))(2s)}{(s^2+1)^4} \right]$$

$$-2 \left[\frac{(s^2+1)^2 - 4s^2(s^2+1)}{(s^2+1)^4} \right]$$

$$\frac{8s^2(s^2+1) - 2(s^2+1)^2}{(s^2+1)^4}$$

$$\frac{8s^2 - 2(s^2+1)}{(s^2+1)^3}$$

Put $s=1$

$$\frac{8-2(2)}{2^3}$$

$$\frac{8-4}{8} = \frac{4}{8} = \left(\frac{1}{2} \right) = \int_0^{\infty} e^{-t} t^2 \sin t \, dt$$

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Formula list

$$1) \int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2 + b^2} (a \cos bx + b \sin bx)$$

$$2) \int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2 + b^2} (a \sin bx - b \cos bx)$$

$$3) \int_0^{\infty} e^{-ax} \cos bx \, dx = \frac{a}{a^2 + b^2}$$

$$4) \int_0^{\infty} e^{-ax} \sin bx \, dx = \frac{b}{a^2 + b^2}$$

Type-3

Obtain Laplace transform of following.

$$1) f(t) = \begin{cases} 2 & 0 \leq t < 1 \\ 0 & t > 1 \end{cases}$$

$$L[f(t)] = \int_0^{\infty} e^{-st} f(t) \, dt$$

$$\int_0^1 e^{-st} (2) \, dt + \int_1^{\infty} e^{-st} (0) \, dt$$

$$= 2 \left(\frac{e^{-st}}{-s} \right)_0^1$$

$$= \frac{-2}{s} [e^{-s} - e^0]$$

$$= \frac{-2}{s} [e^{-s} - 1]$$

$$2. f(t) = \begin{cases} \cos t & , 0 \leq t < 2\pi \\ 0 & , t > 2\pi \end{cases}$$

$$L[f(t)] = \int_0^{\infty} e^{-st} f(t) dt$$

$$= \int_0^{2\pi} e^{-st} (\cos t) dt + \int_{2\pi}^{\infty} e^{-st} (0) dt$$

$$= \left[\frac{e^{-st}}{s^2+1} (-s \cos t + \sin t) \right]_0^{2\pi}$$

~~2. $\frac{e^{-st}}{s^2+1}$~~

$$\left[\frac{e^{-2\pi s}}{s^2+1} (\sin \pi - s \cos \pi) \right] - \left[\frac{1 (\sin 0 - s \cos 0)}{s^2+1} \right]$$

$$\frac{\sin \pi - s \cos \pi}{s^2+1} \left[\frac{e^{-2\pi s}}{s^2+1} - 1 \right]$$

$$\frac{e^{-2\pi s}}{s^2+1} (-s)$$

$$3. f(t) = \begin{cases} 0 & , 0 \leq t < 1 \\ t & , 1 \leq t \leq 2 \\ 0 & , t \geq 2 \end{cases}$$

$$L[f(t)] = \int_0^{\infty} e^{-st} f(t) dt$$

$$= \int_0^1 e^{-st} (0) dt + \int_1^2 e^{-st} (t) dt + \int_2^{\infty} e^{-st} (0) dt$$

$$= \int_1^2 e^{-st} t \cdot dt$$

$$\left[t \left(\frac{e^{-st}}{-s} \right) - (1) \left(\frac{e^{-st}}{-s^2} \right) \right]_1^2$$

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$$\left[-\frac{te^{-st}}{s} + \frac{e^{-st}}{s^2} \right]$$

~~$$\left[-\frac{te^{-st}}{s} + \frac{e^{-st}}{s^2} \right]$$~~

$$\left[-\frac{2e^{-2s}}{s} + \frac{e^{-2s}}{s^2} \right] - \left[-\frac{e^{-s}}{s} + \frac{e^{-s}}{s^2} \right]$$

$$-\frac{2e^{-2s}}{s} + \frac{e^{-2s}}{s^2} + \frac{e^{-s}}{s} - \frac{e^{-s}}{s^2}$$

$$\left[\frac{e^{-s}}{s} - \frac{e^{-s}}{s^2} \right] - \left[\frac{e^{-s}}{s} - \frac{e^{-s}}{s^2} \right]$$

$$\left[\frac{e^{-s}}{s} - \frac{e^{-s}}{s^2} \right] - \left[\frac{e^{-s}}{s} - \frac{e^{-s}}{s^2} \right]$$

$$4. f(t) = \begin{cases} \cos t, & 0 < t < \pi \\ \sin t, & t > \pi \end{cases}$$

$$L[f(t)] = \int_0^{\infty} e^{-st} f(t) dt$$

$$= \int_0^{\pi} e^{-st} \cos t dt + \int_{\pi}^{\infty} e^{-st} \sin t dt$$

$$= \left[\frac{e^{-st}}{s^2+1} (-s \cos t + \sin t) \right]_0^{\pi} + \left[\frac{e^{-st}}{s^2+1} (-s \sin t - \cos t) \right]_{\pi}^{\infty}$$

$$\left\{ \left[\frac{e^{-s\pi}}{s^2+1} (-s \cos \pi + \sin \pi) \right] - \left[\frac{e^0}{s^2+1} (-s \cos 0 + \sin 0) \right] \right\} +$$

$$\left\{ \left[\frac{e^{-s\infty}}{s^2+1} (-s \sin \infty - \cos \infty) \right] - \left[\frac{e^{-s\pi}}{s^2+1} (-s \sin \pi - \cos \pi) \right] \right\}$$

Inverse Laplace

$$1) \mathcal{L}^{-1}[F(s) \pm G(s)] = \mathcal{L}^{-1}[F(s)] \pm \mathcal{L}^{-1}[G(s)]$$

$$2) \mathcal{L}^{-1}[kF(s)] = k \mathcal{L}^{-1}[F(s)]$$

$$3) \mathcal{L}^{-1}\left[\frac{1}{s}\right] = 1$$

$$4) \mathcal{L}^{-1}\left[\frac{1}{s-a}\right] = e^{at}$$

$$5) \mathcal{L}^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{\sin at}{a}$$

$$6) \mathcal{L}^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at$$

$$7) \mathcal{L}^{-1}\left[\frac{1}{s^2-a^2}\right] = \frac{\sinh at}{a}$$

$$8) \mathcal{L}^{-1}\left[\frac{s}{s^2-a^2}\right] = \cosh at$$

$$9) \mathcal{L}^{-1}\left[\frac{1}{s^n}\right] = \frac{t^{n-1}}{(n-1)!}$$

$$Q_1. \mathcal{L}^{-1} \left[\frac{4}{3s-1} \right]$$

$$= 4 \mathcal{L}^{-1} \left[\frac{1}{3s-1} \right]$$

$$\frac{4}{3} \mathcal{L}^{-1} \left[\frac{1}{s-1/3} \right]$$

$$= \frac{4}{3} e^{1/3 t}$$

$$\mathcal{L}^{-1} \left[\frac{1}{s-a} \right] = e^{at}$$

$$2. \mathcal{L}^{-1} \left[\frac{2s+1}{s(s+1)} \right]$$

$$\mathcal{L}^{-1} \left[\frac{s+(s+1)}{s(s+1)} \right]$$

$$\mathcal{L}^{-1} \left[\frac{1}{s+1} \right] + \mathcal{L}^{-1} \left[\frac{1}{s} \right]$$

$$= e^{-t} + 1$$

$$3. \mathcal{L}^{-1} \left[\frac{2s-5}{4s^2+25} \right]$$

$$\mathcal{L}^{-1} \left[\frac{2s-5}{(2s)^2+5^2} \right]$$

$$= \cancel{\frac{2s}{2s^2}} + \cancel{\frac{-5}{2s^2}} = \frac{2s}{(2s)^2+5^2} - \frac{5}{(2s)^2+5^2}$$

$$\mathcal{L}^{-1} \left[\frac{2s}{(2s)^2+5^2} \right] - \mathcal{L}^{-1} \left[\frac{5}{(2s)^2+5^2} \right]$$

$$\frac{2}{4} \mathcal{L}^{-1} \left[\frac{s}{s^2+(5/2)^2} \right] - \frac{5}{4} \mathcal{L}^{-1} \left[\frac{1}{s^2+(5/2)^2} \right]$$

$$\frac{2}{4} \cos \frac{5}{2} t - \frac{5}{4} \sin \frac{5}{2} t$$

$$4. \mathcal{L}^{-1} \left[\frac{s-4}{s^2-4} \right]$$

$$= \mathcal{L}^{-1} \left[\frac{s-4}{s^2-2^2} \right]$$

$$\mathcal{L}^{-1} \left[\frac{s}{s^2-2^2} \right] - 4 \mathcal{L}^{-1} \left[\frac{1}{s^2-2^2} \right]$$

$$\frac{\cos 2t - 4 \sin 2t}{2}$$

$$5. \mathcal{L}^{-1} \left[\frac{a_1}{s} + \frac{a_2}{s^2} + \frac{a_3}{s^3} \right]$$

$$\mathcal{L}^{-1} \left[\frac{a_1}{s} \right] + \mathcal{L}^{-1} \left[\frac{a_2}{s^2} \right] + \mathcal{L}^{-1} \left[\frac{a_3}{s^3} \right]$$

$$a_1 + a_2 \frac{t}{1!} + a_3 \frac{t^2}{2!}$$

$$a_1 + a_2 t + \frac{a_3 t^2}{2}$$

$$6. \mathcal{L}^{-1} \left[\frac{5s+4}{s^3} - \frac{(2s-18)}{s^2+9} + \frac{24-30\sqrt{s}}{s^4} \right]$$

$$\mathcal{L}^{-1} \left[\frac{5s}{s^3} + \frac{4}{s^3} \right] - \frac{2s}{s^2+3^2} + \frac{18}{s^2+3^2} + \frac{24}{s^4} - \frac{30}{s^{7/2}}$$

$$= 5 \mathcal{L}^{-1} \left[\frac{1}{s^2} \right] + 4 \mathcal{L}^{-1} \left[\frac{1}{s^3} \right] - 2 \mathcal{L}^{-1} \left[\frac{s}{s^2+3^2} \right] + 18 \mathcal{L}^{-1} \left[\frac{1}{s^2+3^2} \right] +$$

$$24 \mathcal{L}^{-1} \left[\frac{1}{s^4} \right] - 30 \mathcal{L}^{-1} \left[\frac{1}{s^{7/2}} \right]$$

$$\frac{5t}{1!} + \frac{4t^2}{2!} - 2 \cos 3t + \frac{18 \sin 3t}{3} + \frac{24t^3}{3!} - \frac{30t^{5/2}}{\Gamma(7/2)}$$

$$\frac{5t}{1} + \frac{4t^2}{2} - 2 \cos 3t + 6 \sin 3t + \frac{4t^3}{1} - \frac{30t^{5/2}}{5/2 \cdot 3/2 \cdot 1/2 \cdot \pi}$$

$$\rightarrow \frac{16t^{5/2}}{\Gamma \pi}$$

Theorem - 1

$$\begin{aligned} \mathcal{L}^{-1}[f(s+a)] &= e^{-at} f(t) \\ &= e^{-at} \mathcal{L}^{-1}[F(s)] \end{aligned}$$

$$\begin{aligned} \mathcal{L}^{-1}[f(ks)] &= \frac{1}{k} f(t/k) = \frac{1}{k} \mathcal{L}^{-1}[F(s)] \\ &= \frac{1}{k} \{ \mathcal{L}^{-1}[F(s)] \}_{a \rightarrow t/k} \end{aligned}$$

3 Inverse Laplace transform of derivative

$$\mathcal{L}^{-1}\left[\frac{d}{ds} F(s)\right] = (-1)t f(t)$$

on integrals

$$\mathcal{L}^{-1}\left[\int_s^\infty F(s) ds\right] = f(t)$$

5. Multiplication by powers of s

$$\mathcal{L}^{-1}[sF(s)] = f'(t) \quad \text{if } f(0) = 0$$

$$= \frac{d}{dt} \mathcal{L}^{-1}[F(s)]$$

6 Division by powers of s

$$\mathcal{L}^{-1}\left[\frac{F(s)}{s}\right] = \int_0^t f(u) du$$

Type - 1

$$\mathcal{L}^{-1}[F(s)] = f(t)$$

$$\mathcal{L}^{-1}[F(s+a)] = e^{-at} f(t) \\ = e^{-at} \mathcal{L}^{-1}[F(s)]$$

Q.1 $\mathcal{L}^{-1}\left[\frac{1}{(s+4)^6}\right]$

$$\mathcal{L}^{-1}[F(s+a)] = e^{-at} \mathcal{L}^{-1}[F(s)] \\ = e^{-4t} \mathcal{L}^{-1}\left[\frac{1}{s^6}\right]$$

$$e^{-4t} \left[\frac{t^{6-1}}{\Gamma 6} \right]$$

$$e^{-4t} \frac{t^5}{5!}$$

Q.2. $\mathcal{L}^{-1}\left[\frac{s}{s^2+6s+25}\right]$

$$\mathcal{L}^{-1}\left[\frac{s}{s^2+6s+9+16}\right]$$

$$\mathcal{L}^{-1}\left[\frac{s}{(s+3)^2+4^2}\right]$$

$$\mathcal{L}^{-1}\left[\frac{(s+3)-3}{(s+3)^2+4^2}\right]$$

$$\mathcal{L}^{-1}\left[\frac{s+3}{(s+3)^2+4^2}\right] - 3 \mathcal{L}^{-1}\left[\frac{1}{(s+3)^2+4^2}\right]$$

$$e^{-3t} \mathcal{L}^{-1}\left[\frac{s}{s^2+4^2}\right] - 3 e^{-3t} \mathcal{L}^{-1}\left[\frac{1}{s^2+4^2}\right]$$

$$e^{-3t} \cos 4t - 3 e^{-3t} \sin 4t$$

$$\mathcal{L}^{-1} \left[\frac{s}{(s-3)^3} \right]$$

$$\mathcal{L}^{-1} \left[\frac{(s-3) + 3}{(s-3)^3} \right]$$

$$\mathcal{L}^{-1} \left[\frac{s-3}{(s-3)^3} \right] + 3 \mathcal{L}^{-1} \left[\frac{1}{(s-3)^3} \right]$$

$$\mathcal{L}^{-1} \left[\frac{1}{(s-3)^2} \right] + 3 \mathcal{L}^{-1} \left[\frac{1}{(s-3)^3} \right]$$

$$e^{3t} \mathcal{L}^{-1} \left[\frac{1}{s^2} \right] + 3e^{3t} \mathcal{L}^{-1} \left[\frac{1}{s^3} \right]$$

$$e^{3t} \left[\frac{t^{2-1}}{\Gamma 2} \right] + 3e^{3t} \left[\frac{t^{3-1}}{\Gamma 3} \right]$$

$$e^{3t} \left[\frac{t + 3t^2}{2} \right]$$

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Quick Work

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$$4. \mathcal{L}^{-1} \left[\frac{3s+1}{(s+1)^4} \right]$$

$$\mathcal{L}^{-1} \left[\frac{2s + (s+1)}{(s+1)^4} \right]$$

$$\mathcal{L}^{-1} \left[\frac{2s}{(s+1)^4} \right] + \mathcal{L}^{-1} \left[\frac{1}{(s+1)^3} \right]$$

$$2 \mathcal{L}^{-1} \left[\frac{(s+1)-1}{(s+1)^4} \right] + \mathcal{L}^{-1} \left[\frac{1}{(s+1)^3} \right]$$

$$2 \left[\mathcal{L}^{-1} \left[\frac{s+1}{(s+1)^4} \right] - \mathcal{L}^{-1} \left(\frac{1}{(s+1)^4} \right) \right] + \mathcal{L}^{-1} \left[\frac{1}{(s+1)^3} \right]$$

$$2 \left[\mathcal{L}^{-1} \left(\frac{1}{(s+1)^3} \right) - 2 \mathcal{L}^{-1} \left(\frac{1}{(s+1)^4} \right) \right] + \mathcal{L}^{-1} \left(\frac{1}{(s+1)^3} \right)$$

$$2 e^{-t} \mathcal{L}^{-1} \left(\frac{1}{s^3} \right) - 2 e^{-t} \mathcal{L}^{-1} \left(\frac{1}{s^4} \right) + e^{-t} \mathcal{L}^{-1} \left(\frac{1}{s^3} \right)$$

$$\frac{2e^{-t} t^2}{2} - \frac{2e^{-t} t^3}{6} + \frac{e^{-t} t^2}{2}$$

Type - 2: Change of scale

$$\mathcal{L}^{-1}[F(ks)] = \frac{1}{k} f(t/k)$$

$$\frac{1}{k} \{ \mathcal{L}^{-1}[F(s)] \} \quad s \rightarrow t/k$$

Prove that

$$\mathcal{L}^{-1} \left[\frac{s}{a^2 s^2 + b^2} \right] = \frac{1}{a^2} \cos \left(\frac{bt}{a} \right)$$

$$\mathcal{L}^{-1} \left[\frac{s}{s^2 + b^2} \right] = \cos bt$$

$$\mathcal{L}^{-1} \left[\frac{s}{a^2 (s^2 + (b/a)^2)} \right]$$

$$\frac{1}{a^2} \mathcal{L}^{-1} \left[\frac{s}{s^2 + (b/a)^2} \right]$$

$$\frac{1}{a^2} \cos \frac{bt}{a}$$

Type - 3: $\mathcal{L}^{-1} \left[\frac{d}{ds} F(s) \right] = (-1)t f(t)$

1. Find inverse Laplace of following.

$$\mathcal{L}^{-1} [\omega t^{-1} s]$$

consider

$$\mathcal{L}^{-1} [F(s)] = f(t) = \omega t^{-1} s$$

$$\mathcal{L}^{-1} [\omega t^{-1} s]$$

$$\text{We know } \mathcal{L}^{-1} \left[\frac{d}{ds} F(s) \right] = (-1)t f(t)$$

$$\mathcal{L}^{-1} \left[\frac{d}{ds} \omega t^{-1} s \right] = (-1)t f(t)$$

$$\mathcal{L}^{-1} \left[\frac{-1}{s^2 + 1} \right] = (-1)t f(t)$$

$$\mathcal{L}^{-1} \left[\frac{1}{s^2 + 12} \right] = t f(t)$$

$$\sin t = t f(t)$$

$$f(t) = \frac{\sin t}{t}$$

$$\mathcal{L}^{-1}[F(s)] = \frac{\sin t}{t} = \mathcal{L}^{-1}[\cot^{-1} s]$$

$$2) \mathcal{L}^{-1} \left[\log \left(\frac{s+b}{s+a} \right) \right]$$

$$\mathcal{L}^{-1} [\log(s+b) - \log(s+a)]$$

$$\mathcal{L}^{-1}[F(s)] = f(t)$$

$$F(s) = \log(s+b) - \log(s+a)$$

$$\mathcal{L}^{-1} \left[\frac{d}{ds} F(s) \right] = (-1) t f(t)$$

$$\mathcal{L}^{-1} \left[\frac{d}{ds} (\log s+b - \log(s+a)) \right] = -t f(t)$$

$$\mathcal{L}^{-1} \left[\frac{1}{s+b} - \frac{1}{s+a} \right] = -t f(t)$$

$$\mathcal{L}^{-1} \left[\frac{1}{s+b} \right] - \mathcal{L}^{-1} \left[\frac{1}{s+a} \right] = -t f(t)$$

$$e^{-bt} - e^{-at} = -t f(t)$$

$$\frac{e^{-at} - e^{-bt}}{t} = f(t)$$

$$\frac{e^{-at} - e^{-bt}}{t} = \mathcal{L}^{-1}[F(s)] = \mathcal{L}^{-1} \left[\log \left(\frac{s+b}{s+a} \right) \right]$$

Type - 4

$$\mathcal{L}^{-1} \left[\int_s^{\infty} F(s) ds \right] = \frac{f(t)}{t} = \mathcal{L}^{-1} \left[\frac{F(s)}{t} \right]$$

$$\mathcal{L}^{-1} \left[\frac{2s}{(s^2-4)^2} \right]$$

$$f(s) = \frac{2s}{(s^2-2^2)^2}$$

$$\mathcal{L}^{-1} \left[\int_s^{\infty} F(s) ds \right] = \frac{f(t)}{t}$$

$$\mathcal{L}^{-1} \left[\int_s^{\infty} \frac{2s}{(s^2-2^2)^2} ds \right]$$

$$s^2 - 4 = t$$

$$2s ds = dt$$

$$\mathcal{L}^{-1} \left[\int_s^{\infty} \frac{1}{t^2} dt \right] = \frac{f(t)}{t}$$

$$\mathcal{L}^{-1} \left[\frac{1}{t} \right] = \frac{f(t)}{t}$$

$$\mathcal{L}^{-1} \left[\frac{-1}{s^2-4} \right] = \frac{f(t)}{t}$$

$$\mathcal{L}^{-1} \left[\frac{1}{s^2-2^2} \right] = \frac{f(t)}{t}$$

$$= \frac{\sin 2t}{2} = \frac{f(t)}{t}$$

$$f(t) = \frac{\sin 2t \cdot t}{2}$$

$$\mathcal{L}^{-1} \left[\frac{2s}{(s^2-4)^2} \right] = \frac{\sin 2t \cdot t}{2}$$

Ques
for

Q2. $\mathcal{L}^{-1} \left[\frac{2s+1}{(s^2+s+1)^2} \right]$

$$F(s) = \frac{2s+1}{(s^2+s+1)^2}$$

By th^m

$$\mathcal{L}^{-1} \left[\int_s^\infty F(s) ds \right] = \frac{F(t)}{t}$$

~~$\mathcal{L}^{-1} \left[\frac{2s+1}{(s^2+s+1)^2} \right]$~~

Put $s^2+s+1=t$

$$(2s+1)ds = dt$$

$$\mathcal{L}^{-1} \left[\int_s^\infty \frac{1}{t^2} dt \right] = \frac{F(t)}{t}$$

$$\mathcal{L}^{-1} \left[\left[-\frac{1}{t} \right]_s^\infty \right] = \frac{F(t)}{t}$$

$$\mathcal{L}^{-1} \left[-\frac{1}{s^2+s+1} \right] = \frac{F(t)}{t}$$

$$\mathcal{L}^{-1} \left[-\frac{1}{\infty} - \left(-\frac{1}{s^2+s+1} \right) \right] = \frac{F(t)}{t}$$

$$\mathcal{L}^{-1} \left[\frac{1}{s^2+s+1} \right] = \frac{F(t)}{t}$$

$$\mathcal{L}^{-1} \left[\frac{1}{s^2+s+\frac{1}{4}-\frac{1}{4}+1} \right] = \frac{F(t)}{t}$$

$$\mathcal{L}^{-1} \left[\frac{1}{(s+\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2} \right] = \frac{F(t)}{t}$$

$$e^{-t/2} \mathcal{L}^{-1} \left[\frac{1}{(s+\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2} \right] = \frac{F(t)}{t}$$

$$e^{-t/2} \left(\frac{\sin \sqrt{3} \cdot t/2}{\sqrt{3}/2} \right) = \frac{F(t)}{t}$$

$$F(t) = e^{-t/2} \cdot t \cdot \frac{\sin \sqrt{3} t/2}{\sqrt{3}/2}$$

Theorem 5

$$\mathcal{L}^{-1}[sF(s)] = f'(t) = \frac{d}{dt} \mathcal{L}^{-1}[F(s)]$$

Q1. $\mathcal{L}^{-1}\left[\frac{s}{s^2+4}\right]$

$$F(s) = \frac{1}{s^2+2^2}$$

$$\mathcal{L}^{-1}[s \cdot F(s)] = \frac{d}{dt} \mathcal{L}^{-1}[F(s)]$$

$$\mathcal{L}^{-1}\left[s \cdot \frac{1}{s^2+2^2}\right] = \frac{d}{dt} \mathcal{L}^{-1}\left[\frac{1}{s^2+2^2}\right]$$

$$= \cos(2t) \cdot \frac{2}{2}$$

$$\mathcal{L}^{-1}\left[\frac{s}{s^2+2^2}\right] = \cos 2t$$

Q2. $\mathcal{L}^{-1}\left[\frac{s}{s^2-9}\right]$

$$\mathcal{L}^{-1}[s \cdot F(s)] = \frac{d}{dt} \mathcal{L}^{-1}[F(s)]$$

$$\mathcal{L}^{-1}\left[s \cdot \frac{1}{s^2-3^2}\right] = \frac{d}{dt} \mathcal{L}^{-1}\left[\frac{1}{s^2-3^2}\right]$$

$$= \frac{d}{dt} \left[\frac{\sinh 3t}{3} \right]$$

$$\cosh 3t \cdot \frac{3}{3}$$

$$\mathcal{L}^{-1}\left[\frac{s}{s^2-9}\right] = \cosh 3t$$

Theorem-6

$$\mathcal{L}^{-1} \left[\frac{1}{s} \cdot F(s) \right] = \int_0^t \mathcal{L}^{-1} F(s) dt$$

$$Q1. \mathcal{L}^{-1} \left[\frac{1}{s(s+2)} \right]$$

$$F(s) = \frac{1}{s+2}$$

$$\mathcal{L}^{-1} \left[\frac{1}{s} F(s) \right] = \int_0^t \mathcal{L}^{-1} F(s) \cdot dt$$

$$\mathcal{L}^{-1} \left[\frac{1}{(s+2)s} \right] = \int_0^t \mathcal{L}^{-1} \left[\frac{1}{s+2} \right] dt$$

$$= \int_0^t e^{-2t} \cdot dt$$

$$= \left[\frac{e^{-2t}}{-2} \right]_0^t$$

$$\frac{e^{-2t}}{-2} + \frac{1}{2} = \frac{1}{2} [1 - e^{-2t}]$$

$$Q2. \mathcal{L}^{-1} \left[\frac{1}{s(s^2+4)} \right]$$

$$\mathcal{L}^{-1} \left[\frac{1}{s} F(s) \right] = \int_0^t \mathcal{L}^{-1} F(s) dt$$

$$= \int_0^t \mathcal{L}^{-1} \left[\frac{1}{s^2+2^2} \right] dt$$

$$\int_0^t \frac{\sin 2t}{2} dt$$

$$\frac{1}{2} \left[-\frac{\cos 2t}{2} \right]_0^t$$

$$-\frac{1}{4} [\cos 2t - 1]$$

$$\frac{1}{4} [1 - \cos 2t]$$

Partial Fraction

a $\frac{k}{(n-\alpha)(n-\beta)} = \frac{A}{n-\alpha} + \frac{B}{n-\beta}$

b $\frac{an+b}{(n-\alpha)(n-\beta)} = \frac{A}{n-\alpha} + \frac{B}{n-\beta}$

c $\frac{an^2+bn+c}{(n-\alpha)(n-\beta)(n-\gamma)} = \frac{A}{n-\alpha} + \frac{B}{n-\beta} + \frac{C}{n-\gamma}$

d $\frac{an^2+bn+c}{(n-\alpha)(n-\beta)(n-\gamma)} = \frac{A}{(n-\alpha)} + \frac{B}{(n-\alpha)^2} + \frac{C}{n-\beta}$

e $\frac{an+b}{(n-\alpha)(n^2-\beta)} = \frac{A}{n-\alpha} + \frac{Bs+C}{n^2+\beta}$

Q1. Find inverse using P.F

i) $L^{-1} \left[\frac{3s+7}{s^2+(-2s)-3} \right]$

$\frac{3s+7}{s^2-2s-3} = \frac{3s+7}{(s-3)(s+1)} = \frac{A}{s-3} + \frac{B}{s+1} \quad \text{--- (1)}$

$3s+7 = A(s+1) + B(s-3)$

$3s+7 = s(A+B) + (A-3B)$

$3 = A+B$

$7 = A-3B$

ACB

$7 = 3-3B-3B$

$A = 3-B$

$B = -1$

$A = 4$

$\frac{3s+7}{(s-3)(s+1)} = \frac{4}{s-3} - \frac{1}{s+1}$

Take L^{-1} on b.s

$L^{-1} \left[\frac{3s+7}{(s-3)(s+1)} \right] = 4e^{3t} - e^{-t}$

2. $\mathcal{L}^{-1} \left[\frac{11s^2 - 2s + 5}{(s-2)(2s-1)(s+1)} \right]$

$$\frac{11s^2 - 2s + 5}{(s-2)(2s-1)(s+1)} = \frac{A}{s-2} + \frac{B}{2s-1} + \frac{C}{s+1}$$

$$11s^2 - 2s + 5 = A(2s-1)(s+1) + B(s-2)(s+1) + C(s-2)(2s-1) \quad \text{--- (1)}$$

Roots = 2, $\frac{1}{2}$, -1

Put $s = 2$

$$A = 5$$

Put $s = \frac{1}{2}$

$$B = -3$$

Put $s = -1$

$$C = 2$$

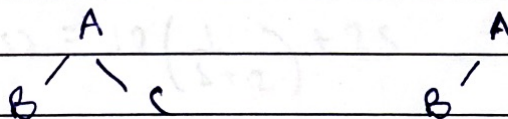
Put A, B, C in (1)

$$\frac{11s^2 - 2s + 5}{(s-2)(2s-1)(s+1)} = \frac{5}{s-2} - \frac{3}{2s-1} + \frac{2}{s+1}$$

Take $\mathcal{L}^{-1}$ on b.s

$$\mathcal{L}^{-1} \left[\frac{11s^2 - 2s + 5}{(s-2)(2s-1)(s+1)} \right] = 5e^{2t} - \frac{3}{2} \mathcal{L}^{-1} \left[\frac{1}{s - \frac{1}{2}} \right] + 2e^{-t}$$

$$5e^{2t} - \frac{3}{2} e^{t/2} + 2e^{-t}$$



3. $\mathcal{L}^{-1} \left[\frac{3s+1}{(s-1)(s^2+1)} \right]$

$$\frac{3s+1}{(s-1)(s^2+1)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+1}$$

$$3s+1 = A(s^2+1) + (Bs+C)(s-1)$$

Put $s = 1$

$$3+1 = A(2) + B(0)$$

$$A = 2$$

$$3s+1 = As^2 + A + Bs^2 - Bs + Cs - C$$

$$3s+1 = (A+B)s^2 + (C-B)s + (A-C)$$

$$A+B=0$$

$$C-B=3$$

$$A-C=1$$

$$2+B=0$$

$$C+2=3$$

$$A-1=1$$

$$B=-2$$

$$C=1$$

$$A=2$$

$$\frac{3s+1}{(s-1)(s^2+1)} = \frac{2}{s-1} + \frac{(-2)s+1}{s^2+1}$$

$$\mathcal{L}^{-1} \left[\frac{3s+1}{(s-1)(s^2+1)} \right] = 2\mathcal{L}^{-1} \left[\frac{1}{s-1} \right] - 2\mathcal{L}^{-1} \left[\frac{s}{s^2+1} \right] + \mathcal{L}^{-1} \left[\frac{1}{s^2+1} \right]$$

$$= 2e^t - 2\cos t + \sin t$$

Applications.

If $L[f(t)] = F(s)$ then $L^{-1}[F(s)] = f(t)$

~~we know~~ we know

$$L[f'(t)] = sF(s) - f(0) \quad \text{where } Y(s) = L[y(t)]$$

Similarly

$$L[y'(s)] = sY(s) - y(0)$$

$$L[y''(s)] = s^2Y(s) - sy(0) - y'(0)$$

$$L[y'''(s)] = s^3Y(s) - s^2y(0) - sy'(0) - y''(0)$$

Q. Find sol of diff eq

$$y'' - 3y' + 2y = 12e^{-2t}, \quad y(0) = 2, \quad y'(0) = 5$$

take L on b.s

$$L[y''] - 3L[y'] + 2L[y] = 12L[e^{-2t}]$$

$$[s^2Y(s) - sy(0) - y'(0)] - 3[sY(s) - y(0)] + 2[Y(s)] = 12\left(\frac{1}{s+2}\right)$$

$$s^2Y(s) - 2s - 6 - 3[sY(s) - 2] + 2Y(s) = 12\left(\frac{1}{s+2}\right)$$

$$s^2Y(s) - 2s - 6 - 3sY(s) + 6 + 2Y(s) = 12\left(\frac{1}{s+2}\right)$$

$$s^2Y(s) - 3sY(s) + 2Y(s) - 2s = 12\left(\frac{1}{s+2}\right)$$

$$(s^2 - 3s + 2)Y(s) = 12\left(\frac{1}{s+2}\right) + 2s$$

$$(s^2 - 3s + 2)Y(s) = \frac{12 + 2s^2 + 4s}{s^2 + 2}$$

$$Y(s) = \frac{2s^2 + 4s + 12}{(s+2)(s^2 - 3s + 2)}$$

$$L[y(t)] = \frac{2s^2 + 4s + 12}{(s+2)(s-2)(s-1)}$$

$$y(t) = L^{-1}\left[\frac{2s^2 + 4s + 12}{(s+2)(s-2)(s-1)}\right]$$

$$\frac{2s^2 + 4s + 12}{(s+2)(s-2)(s-1)} = \frac{A}{s+2} + \frac{B}{s-2} + \frac{C}{s-1}$$

$$2s^2 + 4s + 12 = A(s-2)(s-1) + B(s+2)(s-1) + C(s+2)(s-2)$$

Roots are: 2, -2, 1

Put $s = 2$

$B = 7$

Put $s = -2$

$A = 1$

Put $s = 1$

$C = -6$

$$\frac{2s^2 + 4s + 12}{(s+2)(s-2)(s-1)} = \frac{1}{s+2} + \frac{7}{s-2} - \frac{6}{s-1}$$

$$\mathcal{L}^{-1} \left[\frac{2s^2 + 4s + 12}{(s+2)(s-2)(s-1)} \right] = e^{-2t} + 7e^{2t} - 6e^t$$

2. $y'' + y = t$, $y(0) = 1$, $y'(0) = 2$

take L on b.s

$$L[y''] + L[y] = L[t]$$

$$s^2 Y(s) - sy(0) - y'(0) + Y(s) = \frac{1}{s^2}$$

$$s^2 Y(s) - s + 2 + Y(s) = \frac{1}{s^2}$$

$$(s^2 + 1) Y(s) = \frac{1}{s^2} + s - 2$$

$$(s^2 + 1) Y(s) = \frac{1 + s^3 - 2s^2}{s^2}$$

$$(s^2 + 1) Y(s) = \frac{s^3 - 2s^2 + 1}{s^2}$$

$$Y(s) = \frac{s^3 - 2s^2 + 1}{s^2(s^2 + 1)}$$

$$L[y(t)] = \frac{s^3 - 2s^2 + 1}{s^2(s^2 + 1)} = \frac{s^2(s - 2) + 1}{s^2(s^2 + 1)}$$

$$= \frac{s - 2}{s^2 + 1} + \frac{1}{s^2(s^2 + 1)}$$

$$= \frac{s}{s^2 + 1} - \frac{2}{s^2 + 1} + \frac{1}{s^2(s^2 + 1)}$$

$$= \frac{s}{s^2 + 1} - \frac{2}{s^2 + 1} + \frac{1}{s^2} - \frac{1}{s^2 + 1}$$

L^{-1} on b.s

$$y(t) = L^{-1} \dots$$

$$= \cos t - 2 \sin t + t - \sin t$$

$$\cos t - 3 \sin t + t$$

Q $\frac{dy}{dt} + 3y(t) + 2 \int_0^t y(t) dt = t$, $y(0) = 0$

take L on both

$$L[y'] + 3L[y] + 2L\left[\int_0^t y(t) dt\right] = L[t]$$

$$sY(s) - y(0) + 3Y(s) + 2 \frac{1}{s} Y(s) = \frac{1}{s^2}$$

$$\left(s + \frac{2}{s} + 3\right) Y(s) = \frac{1}{s^2}$$

$$Y(s) = \frac{1}{s^2(s^2 + 3s + 2)}$$

$$L[y(t)] = \frac{s}{s^2(s^2 + 3s + 2)}$$

$$L[y(t)] = \frac{1}{s(s+1)(s+2)}$$

take L^{-1} on bs

$$y(t) = L^{-1}\left[\frac{1}{s(s+1)(s+2)}\right]$$

consider

$$\frac{1}{s(s+1)(s+2)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2} \quad \text{--- (1)}$$

$$1 = A(s+1)(s+2) + B(s)(s+2) + C(s)(s+1)$$

Roots $s = 0, -1, -2$

Put $s = 0$

$$A = \frac{1}{2}$$

Put $s = -1$

$$B = 1$$

Put $s = -2$

$$C = \frac{1}{2}$$

Put in (1)

$$\frac{1}{s(s+1)(s+2)} = \frac{1}{2s} - \frac{1}{s+1} + \frac{1}{2(s+2)}$$

$$y(t) = \frac{1}{2} \mathcal{L}^{-1} \left[\frac{1}{s} \right] - \mathcal{L}^{-1} \left[\frac{1}{s+1} \right] + \frac{1}{2} \mathcal{L}^{-1} \left[\frac{1}{s+2} \right]$$

$$= \frac{1}{2} - e^{-t} + \frac{1}{2} e^{-2t}$$

Q. $y' + 2y + \int_0^t y(t) dt = \sin t, y(0) = 1$

$$\mathcal{L}[y'] + 2\mathcal{L}[y] + \mathcal{L} \left[\int_0^t y(t) dt \right] = \mathcal{L}[\sin t]$$

$$sY(s) - y(0) + 2Y(s) + \frac{1}{s} Y(s) = \frac{1}{s^2 + 1}$$

$$(s + 2 + \frac{1}{s}) Y(s) - 1 = \frac{1}{s^2 + 1}$$

$$\left(\frac{s^2 + 2s + 1}{s} \right) Y(s) = \frac{1}{s^2 + 1} + 1$$

$$Y(s) = \frac{s}{(s^2 + 1)(s^2 + 2s + 1)} + \frac{s}{s^2 + 2s + 1}$$

$$= \frac{s}{(s^2 + 1)(s+1)^2} + \frac{s}{(s+1)^2}$$

$$(s^2 + 2s + 1) Y(s) = \frac{s^2 + 2}{s^2 + 1}$$

$$(s+1)^2 Y(s) = \frac{s(s^2 + 2)}{s^2 + 1}$$

$$(s^2 + 2s + 1) Y(s) = \frac{s(s^2 + 2)}{s^2 + 1}$$

$$(s+1)^2 Y(s) = \frac{s(s^2 + 2)}{s^2 + 1}$$

$$Y(s) = \frac{s^3 + 2s}{(s^2 + 1)(s+1)^2}$$

Laplace transform of some special function

Type-1: Periodic function

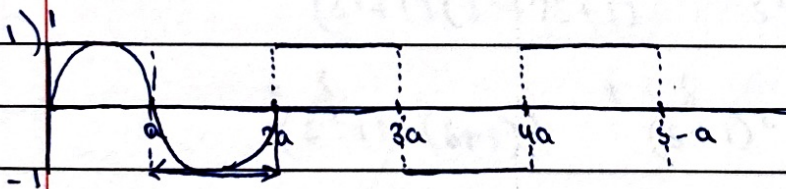
A function $f(t)$ for $t > 0$ is periodic with a $T > 0$ if $f(t+T) = f(t)$, $\forall t > 0$

From above def., it follows that if m is integer, $f(t+mT) = f(t)$, $\forall t > 0$ i.e. $2T, 3T, 4T, \dots$ at the periods of $f(t)$

Th^m 1: If func. is periodic with a period T then Laplace of

$$L[f(t)] = \frac{1}{1-e^{-sT}} \left[\int_0^T e^{-st} f(t) dt \right]$$

Q1. Find Laplace transform of square or rectangle shown in figure



$$f(t) = 1, \quad 0 < t < a$$

$$= -1, \quad a < t < 2a$$

Here, $T = 2a$

By th^m

$$L[f(t)] = \frac{1}{1-e^{-sT}} \left[\int_0^T e^{-st} f(t) dt \right]$$

$$= \frac{1}{1-e^{-2as}} \left[\int_0^{2a} e^{-st} f(t) dt \right]$$

$$\frac{1}{1-e^{-2as}} \left[\int_0^a e^{-st} (1) dt + \int_a^{2a} e^{-st} (-1) dt \right]$$

$$\frac{1}{1-e^{-2as}} \left[\left(\frac{e^{-st}}{-s} \right)_0^a + \left(\frac{e^{-st}}{s} \right)_a^{2a} \right]$$

$$\frac{1}{1-e^{-2as}} \left[\left(\frac{e^{-as}}{-s} \right) + \left(\frac{e^{-2as}}{s} - \frac{e^{-as}}{s} \right) \right]$$

2) $f(t) = 4-t^2$, $0 \leq t < 2$ Function is periodic with period $T=2$

→ Since $T=2$, By the m.

$$L[f(t)] = \frac{1}{1-e^{-sT}} \left[\int_0^T e^{-st} f(t) dt \right]$$

$$= \frac{1}{1-e^{-2s}} \left[\int_0^2 e^{-st} f(t) dt \right]$$

$$\frac{1}{1-e^{-2s}} \left[\int_0^2 e^{-st} (4-t^2) dt \right]$$

$$\frac{1}{1-e^{-2s}} \left[(4-t^2) \left(\frac{e^{-st}}{-s} \right) - (-2t) \left(\frac{e^{-st}}{s^2} \right) + (-2) \left(\frac{e^{-st}}{s^3} \right) \right]_0^2$$

$$\frac{1}{1-e^{-2s}} \left[0 + \frac{4e^{-2s}}{s^2} + \frac{2e^{-2s}}{s^3} - \left(-\frac{4}{s} + 0 + \frac{2}{s^3} \right) \right]$$

$$\frac{1}{1-e^{-2s}} \left[\frac{e^{-2s}}{s^2} \left(4 + \frac{2}{s} \right) - \frac{2}{s^3} + \frac{4}{s} \right]$$

3 $f(t) = 3t \quad 0 < t < 2$ Function periodic
 $= 6 \quad 2 < t < 4 \quad T = 4$

$$1[f(t)] = \frac{1}{1-e^{-sT}} \left[\int_0^T e^{-st} f(t) dt \right]$$

$$\frac{1}{1-e^{-4s}} \left[\int_0^2 e^{-st} 3t dt + \int_2^4 e^{-st} 6 dt \right]$$

$$\frac{1}{1-e^{-4s}} \left\{ \left[3t \left(\frac{e^{-st}}{-s} \right) - (3) \left(\frac{e^{-st}}{s^2} \right) \right]_0^2 + \left[\frac{6e^{-st}}{-s} \right]_2^4 \right\}$$

$$\frac{1}{1-e^{-4s}} \left[\frac{6e^{-2s}}{-s} - \frac{3e^{-2s}}{s^2} + \frac{3}{s^2} - \frac{6e^{-4s}}{s} + \frac{6e^{-2s}}{s} \right]$$

4 Find Laplace

$$\left[\frac{1}{s} \right]$$

$$\left[\frac{1}{s} \right]$$

$$\left[\frac{1}{s} \right]$$

$$\left[\frac{1}{s} \right]$$

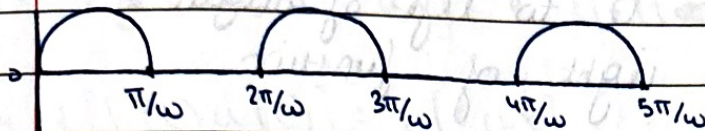
$$\left[\frac{1}{s} \right]$$

$$\left[\frac{1}{s} \right]$$

u. Find Laplace transform for half wave rectified defined for

$$f(t) = \begin{cases} \sin \omega t & 0 < t < \pi/\omega \\ 0 & \pi/\omega < t < 2\pi/\omega \end{cases}$$

Also draw graph



$$T = \frac{2\pi}{\omega}$$

$$L[f(t)] = \frac{1}{1 - e^{-sT}} \left[\int_0^T e^{-st} f(t) dt \right]$$

$$\frac{1}{1 - e^{-s \frac{2\pi}{\omega}}} \left[\int_0^{\pi/\omega} e^{-st} \sin \omega t dt \right] \dots \int e^{-ax} \sin bx =$$

$$\frac{1}{1 - e^{-\frac{2\pi s}{\omega}}} \left[\frac{e^{-st}}{s^2 + \omega^2} (-s \sin \omega t - \omega \cos \omega t) \right]_0^{\pi/\omega}$$

$$\frac{1}{1 - e^{-\frac{2\pi s}{\omega}}} \left[\frac{e^{-\frac{s\pi}{\omega}}}{s^2 + \omega^2} (-s \sin \omega \frac{\pi}{\omega} - \omega \cos \omega \frac{\pi}{\omega}) - \frac{1}{s^2 + \omega^2} (-s \sin 0 + \omega \cos 0) \right]$$

$$\frac{1}{1 - e^{-\frac{2\pi s}{\omega}}} \left[\frac{e^{-\frac{s\pi}{\omega}}}{s^2 + \omega^2} (\omega) + \frac{\omega}{s^2 + \omega^2} \right]$$

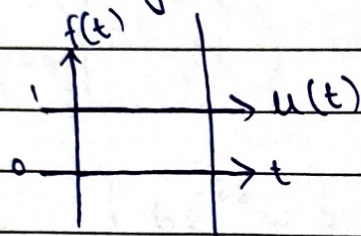
$$\frac{1}{1 - e^{-\frac{2\pi s}{\omega}}} \left[\frac{\omega}{s^2 + \omega^2} (e^{-\frac{s\pi}{\omega}} + 1) \right]$$

Type: 2: Unit Step function (Heaviside function)
The unit step function $u(t)$ is defined as

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$

The unit step function is a curve or function which has value 0 to left of origin & equal to 1 to right of function

Graph of unit step function:



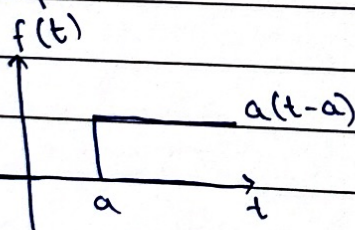
Displaced unit step funct.

In many elec ~~systems~~ circuits, at reg interval $T=0$,

Such func are described using displaced unit step function.

$$u(t-a) = \begin{cases} 0 & t < a \\ 1 & t \geq a \end{cases}$$

Graph:



Transforms using step function.

$$1) \mathcal{L}[u(t)] = \frac{1}{s}$$

$$2) \mathcal{L}[u(t-a)] = \frac{e^{-as}}{s}$$

$$3) \mathcal{L}[f(t-a)u(t-a)] = e^{-as} \mathcal{L}[f(t)]$$

$$4) \mathcal{L}[f(t)u(t)] = \mathcal{L}[f(t)]$$

$$5) \mathcal{L}[f(t)u(t-a)] = e^{-as} \mathcal{L}[f(t+a)]$$

$$6) \text{ If } f(t) = \begin{cases} f_1(t) & 0 < t < a \\ f_2(t) & t > a \end{cases}$$

Then $f(t)$ is expressed in terms of unit step function as

$$f(t) = f_1(t)u(t-0) + f_2(t)u(t-a)$$

In general

$$f(t) = f_1(t) \quad 0 < t < a$$

$$f_2(t) \quad a < t < b$$

$$f_3(t) \quad b < t < c$$

$$f_4(t) \quad \dots$$

$\vdots$

then

$$\mathcal{L}[f(t)] = f_1(t)u(t-0) + (f_2-f_1)u(t-a) + (f_3-f_2)u(t-b) + (f_4-f_3)u(t-c) \dots$$

$$1) \mathcal{L}[t^2 u(t-2)]$$

$$\mathcal{L}[f(t) u(t-a)]$$

$$f(t) = t^2, \quad u(t-a) = u(t-2)$$

$$f(t+2) = (t+2)^2 \\ = t^2 + 4t + 4$$

We know

$$\mathcal{L}[f(t) u(t-a)] = e^{-as} \mathcal{L}[f(t+a)]$$

$$e^{-2s} \mathcal{L}[f(t+2)]$$

$$e^{-2s} \mathcal{L}[t^2 + 4t + 4]$$

$$e^{-2s} \left[\frac{2 \times 2}{s^3} + \frac{4}{s^2} + \frac{4}{s} \right]$$

$$2) \mathcal{L}[(1+2t-3t^2) u(t-2)]$$

$$\mathcal{L}[f(t) u(t-a)]$$

$$f(t) = 1+2t-3t^2$$

$$u(t-a) = u(t-2)$$

$$f(t+2) = 1+2(t+2)-3(t+2)^2$$

$$a=2$$

$$1+2t+4-3t^2-12t-12$$

$$f(t+2) = -3t^2 - 10t - 7$$

We know

$$\mathcal{L}[f(t) u(t-a)] = e^{-as} \mathcal{L}[f(t+a)]$$

$$e^{-2s} \mathcal{L}[f(t+2)]$$

$$e^{-2s} \mathcal{L}[-3t^2 - 10t - 7]$$

$$e^{-2s} \left[\frac{-3 \times 2}{s^3} - \frac{10}{s^2} - \frac{7}{s} \right]$$

3. $\mathcal{L}[(t-\pi)u(t-\pi)]$

$$\mathcal{L}[f(t-a)u(t-a)]$$

$$f(t-a) = t-\pi$$

$$u(t-a) = u(t-\pi)$$

$$f(t) = t$$

$$a = \pi$$

We know

$$\mathcal{L}[f(t-a)u(t-a)] = e^{-as} \mathcal{L}[f(t)]$$

$$= e^{-\pi s} \mathcal{L}[t]$$

$$= \frac{e^{-\pi s}}{s^2}$$

4. $\mathcal{L}[\sin t \cdot u(t-4)]$

$$\mathcal{L}[f(t)u(t-a)]$$

$$f(t) = \sin t$$

$$u(t-a) = u(t-4)$$

$$f(t+4) = \sin(t+4)$$

$$a = 4$$

$$\sin t \cos 4 + \cos t \sin 4$$

We know that

$$\mathcal{L}[f(t)u(t-a)] = e^{-as} \mathcal{L}[f(t+a)]$$

$$= e^{-4s} \mathcal{L}[\sin(t+4)]$$

$$= e^{-4s} \mathcal{L}[\sin t \cos 4 + \cos t \sin 4]$$

$$= e^{-4s} [\cos 4 \mathcal{L}(\sin t) + \sin 4 \mathcal{L}(\cos t)]$$

$$= e^{-4s} \left[\cos 4 \left(\frac{1}{s^2+1} \right) + \sin 4 \left(\frac{s}{s^2+1} \right) \right]$$

Q Express in terms of func or unit step func & find Laplace transform

$$1) f(t) = t^2, 0 < t < 1$$

$$4t - t^2, t > 1$$

Can be expressed in terms of unit step func as

$$f(t) = f_1(t)u(t-a) + (f_2t - f_1(t))u(t-a)$$

$$f(t) = t^2u(t-0) + (4t - t^2)u(t-1)$$

Take L on bs

$$L[f(t)] = L[t^2u(t)] + L[(4t - t^2)u(t-1)] \quad \text{--- (1)}$$

Now,

$$L[t^2u(t)]$$

$$f(t) = t^2, \quad u(t-a) = u(t-0)$$

$$a = 0$$

We know

$$L[f(t)u(t)] = L[f(t)]$$

$$= L[t^2]$$

$$L[f(t)u(t)] = \frac{2}{s^3} \quad \text{--- (2)}$$

$$2) L[(4t - t^2)u(t-1)]$$

$$f(t) = 4t - t^2$$

$$f(t+1) = 4(t+1) - (t+1)^2$$

$$4t + 4 - t^2 - 2t - 1$$

$$4t + 4 - t^2 - 2t - 1$$

$$-t^2 + 2t + 3$$

We know

$$L[f(t)u(t-a)] = e^{-as} L[f(t+a)]$$

$$e^{-s} L[-t^2 + 2t + 3]$$

$$e^{-s} [L(-t^2) + 2L[t] + 3L(1)]$$

$$e^{-s} \left[-\frac{2}{s^3} + \frac{2}{s^2} + \frac{3}{s} \right] \quad \text{--- (3)}$$

Put (2) & (3) in (1)

$$L[f(t)] = \frac{2}{s^3} + e^{-s} \left[-\frac{2}{s^3} + \frac{2}{s^2} + \frac{3}{s} \right]$$

$$2. f(t) = \begin{cases} \cos t & 0 < t < \pi \\ \sin t & t > \pi \end{cases}$$

$$f(t) = f_1(t)u(t-a) + [f_2(t) - f_1(t)]u(t-a)$$

$$f(t) = f_1(t)u(t-a) + [f_2(t) - f_1(t)]u(t-a)$$

$$= \cos t u(t) + (\sin t - \cos t)u(t-\pi)$$

$$L[f(t)] = L[\cos t u(t)] + L[(\sin t - \cos t)u(t-\pi)] \quad \text{--- (1)}$$

$$L[\cos t u(t-0)]$$

$$f(t) = \cos t \quad u(t-a) = u(t-0), \quad a=0$$

$$L[f(t) \cdot u(t)] = L[f(t)]$$

$$L[\cos t]$$

$$= \frac{s}{s^2+1} \quad \text{--- (2)}$$

$$\mathcal{L}\{(\sin t - \cos t)u(t-\pi)\}$$

$$f(t) = \sin t - \cos t$$

$$u(t-a) = u(t-\pi)$$

$$a = \pi$$

$$f(t+\pi) = \sin(t+\pi) - \cos(t+\pi)$$

$$[\sin t \cos \pi + \cos t \sin \pi] - [\cos t \cos \pi - \sin t \sin \pi]$$

$$-\sin t + \cos t$$

$$\mathcal{L}\{(\sin t - \cos t)u(t-\pi)\} = e^{-\pi s} \mathcal{L}\{f(t+\pi)\}$$

$$e^{-\pi s} [\mathcal{L}\{-\sin t\} + \mathcal{L}\{\cos t\}]$$

$$e^{-\pi s} \left[\frac{-1}{s^2+1} + \frac{s}{s^2+1} \right] \quad \text{--- (3)}$$

Put (2) & (3) in (1)

$$\mathcal{L}\{f(t)\} = \frac{s}{s^2+1} + e^{-\pi s} \left[\frac{-1}{s^2+1} + \frac{s}{s^2+1} \right]$$

Type 3: ~~Dirac~~ Dirac delta func.

Relation bet unit step func & dirac delta func
is $\delta(t-a) = u'(t-a)$

Properties

1) $L[\delta(t-a)] = e^{-as}$

2) case: if $a=0$ then

$$L[\delta(t)] = 1$$

3) $L[\delta(t-a)] = L[u'(t-a)] = e^{-as}$

4) $L[\delta'(t-a)] = L[u''(t-a)] = se^{-as}$

5) Shifting property of $\delta(t-a)$

One of the imp property is shifting func.

$$\int_{-\infty}^{\infty} f(t) \delta(t-a) dt = f(a)$$

Provided, $f(t)$ is cont. func.

6. $L[f(t) \delta(t-a)] = e^{-sa} f(a)$

7. $L[f(t) \delta(t)] = f(0)$

find L of following

$$1) \mathcal{L}[\sin 2t \delta(t-a)]$$

$$\mathcal{L}[f(t) \delta(t-a)] = e^{-as} f(a)$$

$$a=2$$

$$e^{-2s} f(2)$$

$$e^{-2s} \sin 4$$

$$f(t) = \sin 2t$$

$$f(a) = \sin 2(a)$$

$$f\left(\frac{2}{2}\right) = \sin 4$$

$$2. \mathcal{L}[t^4 \delta(t-5)]$$

$$f(t) = t^4$$

$$\delta(t-a) = \delta(t-5)$$

$$a=5$$

$$f(5) = 5^4$$

$$= 625$$

$$\mathcal{L}[f(t) \delta(t-a)] = e^{-as} f(a)$$

$$e^{-5s} f(5)$$

$$e^{-5s} (625)$$

$$3. \mathcal{L}[t^2 u(t-2) - \cos t \delta(t-4)]$$

$$\mathcal{L}[t^2 u(t-2)] = \mathcal{L}[\cos t \delta(t-4)] \quad - (1)$$

Now

$$\mathcal{L}[t^2 u(t-2)]$$

$$f(t) = t^2$$

$$u(t-a) = u(t-2)$$

$$a=2$$

$$f(t+2) = (t+2)^2$$

$$= t^2 + 4t + 4$$

$$\mathcal{L}[f(t) u(t-a)] = e^{-as} \mathcal{L}[f(t+a)]$$

$$e^{-2s} \mathcal{L}[f(t+2)]$$

$$e^{-2s} \mathcal{L}[t^2 + 4t + 4]$$

$$e^{-2s} [\mathcal{L}(t^2) + 4\mathcal{L}(t) + 4\mathcal{L}(1)]$$

$$e^{-2s} \left[\frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s} \right] \quad - (2)$$

$$L[\cos ht \delta(t-4)]$$

$$f(t) = \cos ht$$

$$f(4) = \cos h 4$$

$$\delta(t-a) = \delta(t-4)$$

$$a = 4$$

$$L[f(t) \delta(t-a)] = e^{-as} f(a)$$

$$= e^{-4s} f(4)$$

$$e^{-4s} \cos h 4 \quad \text{--- (3)}$$

Put 2 & (3) in (1)

$$e^{-2s} \left(\frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s} \right) - e^{-4s} \cos h 4$$

$$4. L[t u(t-4) - t^3 \delta(t-2)]$$

$$L[t u(t-4)] - L[t^3 \delta(t-2)] \quad \text{--- (1)}$$

Now $L[t u(t-4)]$

$$f(t) = t$$

$$u(t-a) = u(t-4)$$

$$f(t+4) = t+4$$

$$a = 4$$

We know

$$L[f(t) u(t-a)] = e^{-as} L[f(t+a)]$$

$$= e^{-4s} L[f(t+4)]$$

$$e^{-4s} L[t+4]$$

$$e^{-4s} \left(\frac{1}{s^2} + \frac{4}{s} \right) \quad \text{--- (2)}$$

$$L[t^3 \delta(t-2)]$$

$$f(t) = t^3, \quad \delta(t-a) = \delta(t-2)$$

$$f(2) = 2^3$$

$$a = 2$$

$$= 8$$

We know

$$L[f(t) \delta(t-a)] = e^{-as} f(a)$$

$$= e^{-2s} f(2)$$

$$= 8e^{-2s} \quad \text{--- (3)}$$

Put (2) & (3) in (1): $L[f(t)] = e^{-4s} \left(\frac{1}{s^2} + \frac{4}{s} \right) - 8e^{-2s}$

$$① \int_{-\infty}^{\infty} e^{-3t} \delta(t-4) dt$$

We know

$$\int_{-\infty}^{\infty} f(t) \delta(t-a) dt = f(a) \quad \text{--- (1)}$$

$$f(t) = e^{-3t}$$

$$f(a) = e^{-3a} \rightarrow f(4) = e^{-12}$$

$$\int_{-\infty}^{\infty} e^{-3t} \delta(t-4) dt = e^{-12}$$

$$② \int_{-\infty}^{\infty} \cos 2t \delta(t-\pi/3) dt$$

We know that

$$\int_{-\infty}^{\infty} \cos 2t \delta(t-\pi/3) dt = f(a)$$

$$= f(\pi/3)$$

$$\cos 2(\pi/3)$$

$$= -\frac{1}{2}$$

$$③ \int_{-\infty}^{\infty} t^m (\log t)^m \delta(t-3) dt$$

$$= f(a)$$

$$= f(3)$$

$$= 3^m (\log 3)^m$$