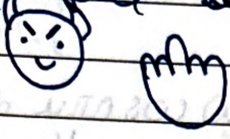
UNIT-3 Fourier



Time to frequency

It's useful in solving boundary value problems. like heat, wave etc

compleх exponential form

If $f(x)$ is periodic func of period $2L$ defined in $-L \leq x \leq L$ & satisfies dirichlet condition ($f(x)$ must have finite max & mini, max finite no. of discontinuities & $f(x)$)

Then $f(x)$ is represented is

$$\left\{ f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right) \right\} ***$$

Where $a_0 = \frac{1}{L} \int_{-L}^{L} f(u)\, du$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(u) \cos \frac{n\pi u}{L}\, du$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(u) \sin \frac{n\pi u}{L}\, du$$

using :- $\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$, $\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{-i}{2}(e^{i\theta} - e^{-i\theta})$

$\theta = \frac{n\pi x}{L}$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} + \sum_{n=1}^{\infty} \left[a_n \left(\frac{e^{\frac{in\pi x}{L}} + e^{-\frac{in\pi x}{L}}}{2} \right) + b_n \left(\frac{e^{\frac{in\pi x}{L}} - e^{-\frac{in\pi x}{L}}}{2} \right) \right.$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[\left(\frac{a_n - ib_n}{2} \right) e^{\frac{in\pi x}{L}} + \left(\frac{a_n + ib_n}{2} \right) e^{-\frac{in\pi x}{L}} \right]$$

$$f(x) = c_0 + \sum_{n=1}^{\infty} \left[c_n e^{\frac{in\pi x}{L}} + c_n e^{-\frac{in\pi x}{L}} \right]$$

$$\left\{ f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{L}} \right\} \text{—(1)} \quad **$$

Quick Work
Page No.:
Date:
M T W T F S S

when $c_n = \frac{a_n - ib_n}{2}$

$$= \frac{1}{2}\left[\frac{1}{L}\int_{-L}^{L} f(u)\cos\frac{n\pi u}{L}\,du - i\frac{1}{L}\int_{-L}^{L} f(u)\sin\frac{n\pi u}{L}\,du\right]$$

$$= \frac{1}{2L}\left[\int_{-L}^{L} f(u)\left(\cos\frac{n\pi u}{L} - i\sin\frac{n\pi u}{L}\right)du\right]$$

$$\left\{c_n = \frac{1}{2L}\left[\int_{-L}^{L} f(u)e^{\frac{-in\pi u}{L}}\,du\right]\right\} **$$

$$c_{-n} = \frac{1}{2L}\left[\int_{-L}^{L} f(u)e^{\frac{in\pi u}{L}}\,du\right] \quad -(2)$$

$\therefore$ complex form of fourier is obtained by (1) & (2)

\# Fourier Integral

We will now consider the limiting form of fourier series for periodic func of period $2L$ where $L \to \infty$

We know

$$\boxed{f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{L}}} \quad -(1)$$

where

$$c_n = \frac{1}{2L}\left[\int_{-L}^{L} f(u)e^{\frac{-in\pi u}{L}}\,du\right] \quad -(2)$$

Put (2) in (1)

$$f(x) = \sum_{n=-\infty}^{\infty} \left[\frac{1}{2L} \left(\int_{-L}^{L} f(u) e^{-\frac{in\pi u}{L}} du \right) e^{\frac{in\pi x}{L}} \right]$$

$$= \sum_{n=-\infty}^{\infty} \frac{1}{2L} \left[\int_{-L}^{L} f(u) e^{-\frac{in\pi}{L}(u-x)} du \right]$$

Let $\left[\lambda = \frac{n\pi}{L}\right]$, $\Delta\lambda = (n+1)\frac{\pi}{L} - \frac{n\pi}{L}$

$$= \frac{n\pi}{L} + \frac{\pi}{L} - \frac{n\pi}{L}$$

$$\Delta\lambda = \frac{\pi}{L}$$

$$= \sum_{n=-\infty}^{\infty} \frac{1}{2\pi} \left(\frac{\pi}{L} \right) \left[\int_{-L}^{L} f(u) e^{-\frac{in\pi}{L}(u-x)} du \right]$$

$$f(x) = \sum_{n=-\infty}^{\infty} \frac{1}{2\pi} \left[\int_{-L}^{L} f(u) e^{-i\lambda(u-x)} du \right] \Delta\lambda$$

$$= \lim_{L\to\infty} \sum_{n=-\infty}^{\infty} \frac{1}{2\pi} \left[\int_{-L}^{L} f(u) e^{i\lambda(u-x)} du \right] \Delta\lambda$$

$$\lim_{\Delta\lambda \to 0} \sum_{n=-\infty}^{\infty} \frac{1}{2\pi} \left[\int_{-L}^{L} f(u) e^{-i\lambda(u-x)} du \right] \Delta\lambda$$

$$\lim \sum f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[f(u) e^{-i\lambda(u-x)} du \, d\lambda \right]$$

FOURIER INTEGRAL

Quick Work
Page No.:
Date:
M T W T F S S

Equivalent forms of fourier integral

Various forms

We know

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u)\, e^{-i\lambda(u-x)}\, du\, d\lambda$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u)\left[\cos\lambda(u-x) - i\sin\lambda(u-x)\right] du\, d\lambda$$

$$= \frac{1}{2\pi}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u)\cos\lambda(u-x)\, du\, d\lambda - i\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u)\sin\lambda(u-x)\, du\, d\lambda\right]$$

Since $\sin\lambda(u-x)$ is odd func of u

$$\therefore \int_{\lambda=-\infty}^{\infty} \sin\lambda(u-x)\, d\lambda = 0$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u)\cos\lambda(u-x)\, du\, d\lambda$$

$* \cos\lambda(u-x)$

We know

$$\int_{\lambda=-\infty}^{\infty} \cos\lambda(u-x)\, d\lambda \rightarrow 2\int_{0}^{\infty}\cos\lambda(u-x)\, d\lambda$$

$$f(x) = \frac{1}{\pi}\int_{\lambda=0}^{\infty}\int_{u=-\infty}^{\infty} f(u)\cos\lambda(u-x)\, du\, d\lambda$$

Quick Work
Page No.:
Date:
M T W T F S S

$$f(x) = \frac{1}{\pi}\int_{\lambda=0}^{\infty}\int_{u=-\infty}^{\infty} f(u)\left[\cos\lambda u\cos\lambda x + \sin\lambda u\sin\lambda x\right] du\, d\lambda \quad -(4)$$

This is equivalent form of fourier integral

Case 1: if $f(x)$ is an even func then

① $f(u)\sin\lambda u =$ odd func. $\rightarrow \int_{-\infty}^{\infty} f(u)\sin\lambda u = 0$

$$\therefore f(x) = \frac{1}{\pi}\int_{\lambda=0}^{\infty}\int_{u=-\infty}^{\infty} f(u)\cos\lambda u\cos\lambda x\, du\, d\lambda$$

② Since it is even func,

$\therefore$ we have

$$\left\{ f(x) = \frac{2}{\pi}\int_{0}^{\infty}\int_{0}^{\infty} f(u)\cos\lambda u \cos\lambda x\, du\, d\lambda \right\} ** \quad -(5)$$

⑤ is fourier cosine integral of $f(x)$

Case 2: if $f(x)$ is an odd func then

$f(u)\cos\lambda u \rightarrow$ odd

$$\int_{-\infty}^{\infty} f(u)\cos\lambda u = 0$$

$$\therefore f(x) = \frac{1}{\pi}\int_{\lambda=0}^{\infty}\int_{u=-\infty}^{\infty} f(u)\sin\lambda u\sin\lambda x\, du\, d\lambda$$

$$\int_{-\infty}^{\infty} f(u)\sin\lambda x = \text{even} = 2\int_{0}^{\infty} f(u)\sin\lambda u\, du$$

$$\left\{ f(x) = \frac{2}{\pi}\int_{0}^{\infty}\int_{0}^{\infty} f(u)\sin\lambda u\sin\lambda x\, du\, d\lambda \right\} ** \quad -(6)$$

⑥ is fourier sine integral representation

Fourier sine & cosine transform

Eq 5 can be written as

$$f(x) = \frac{2}{\pi}\int_0^\infty \left[\int_0^\infty f(u)\cos\lambda u\,du\right]\cos\lambda x\,d\lambda$$

$$F_c(\lambda) = \int_0^\infty f(u)\cos\lambda u\,du$$

Inverse fourier cosine transform

$$IF_c(\lambda) = f(x) = \frac{2}{\pi}\int_0^\infty F_c(\lambda)\cos\lambda x\,d\lambda$$

$-\infty < x < \infty$

$$f(x) = \frac{2}{\pi}\int_0^\infty \underbrace{\left[\int_0^\infty f(u)\sin\lambda u\,du\right]}_{\text{fourier sine transform}}\sin\lambda x\,d\lambda$$

$$F_s(\lambda) = \int_0^\infty f(u)\sin\lambda u\,du$$

$$IF_s(\lambda) = f(x) = \frac{2}{\pi}\int_0^\infty F_s(\lambda)\sin\lambda x\,d\lambda$$

* If $f(x)$ is even func. defined in $0 < x < \infty$ then we use fourier cosine T. eq^n 7 or 8

If $f(x)$ is an odd func defined in $0 < x < \infty$ then we use fourier sine T. ie eq 9 & 10

If $f(x)$ is neither E or O then we use generalised fourier transform i.e 11 or 12

Name of transformation	Interval	Transformation	Inverse
1. Fourier	$-\infty < x < \infty$	$F(\lambda) = \int_{-\infty}^{\infty} f(u) e^{-i\lambda u} du$	$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\lambda) e^{i\lambda x} d\lambda$
2. Fourier cosine (even)	$-\infty < x < \infty$	$F_c(\lambda) = \int_0^{\infty} f(u) \cos\lambda u \, du$	$f(x) = \frac{2}{\pi} \int_0^{\infty} F_c(\lambda) \cos\lambda x \, d\lambda$
3. Fourier sine (odd)	$-\infty < x < \infty$	$F_s(\lambda) = \int_0^{\infty} f(u) \sin\lambda u \, du$	$f(x) = \frac{2}{\pi} \int_0^{\infty} F_s(\lambda) \sin\lambda x \, d\lambda$
4. Fourier cosine	$0 < x < \infty$	$F_c(\lambda) = \int_0^{\infty} f(u) \cos\lambda u \, du$	$f(x) = \frac{2}{\pi} \int_0^{\infty} F_c(\lambda) \cos\lambda x \, d\lambda$
5. Fourier sine	$0 < x < \infty$	$F_s(\lambda) = \int_0^{\infty} f(u) \sin\lambda u \, du$	$f(x) = \frac{2}{\pi} \int_0^{\infty} F_s(\lambda) \sin\lambda x \, d\lambda$

Pre-requisites

1) $\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$

2) $\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$

3) $\int_0^{\infty} e^{-ax} \cos bx \, dx = \frac{a}{a^2+b^2}$

4) $\int_0^{\infty} e^{-ax} \sin bx \, dx = \frac{b}{a^2+b^2}$

5) $\int_0^{2\pi} \cos(mx)\cos(nx) \, dx = 0$ where $m \neq n$
$= \pi$, $m = n$

7) $e^{ix} = \cos x + i\sin x$
$e^{-ix} = \cos x - i\sin x$

Quick Work
Page No.:
Date:
M T W T F S

8. $\int_0^\infty \cancel{\frac{\sin ax}{x}} \frac{\sin ax}{a}\, dx = \pi/2, \quad a \text{ +ve}$

$= -\pi/2, \quad a \text{ -ve}$

9. $\sin x = \frac{e^{ix} - e^{-ix}}{2}$

10. $\cos x = \frac{e^{ix} + e^{-ix}}{2}$

Type 1: Fourier Integral Rep.

Q1. Find the fourier transform & also give the fourier integral representation.

i) $f(x) = 1 \quad 0 \le x \le 1$

$= 0 \quad x < 0, \; x > 1$

i. The F.I.T is given by

$$f(\lambda) = \int_{-\infty}^{\infty} f(u) e^{-i\lambda u}\, du$$

$$= \int_0^1 (1) e^{-i\lambda u}\, du$$

$$= \left[\frac{-e^{-i\lambda u}}{i\lambda} \right]_0^1$$

$$= \frac{-1}{i\lambda} \left[e^{-i\lambda} - e^0 \right]$$

$$f(\lambda) = \frac{i}{\lambda} \left[e^{-i\lambda} - 1 \right]$$

ii. F.I.R : $f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(u) e^{-i\lambda u}\, du \right] e^{-i\lambda x}\, d\lambda$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\lambda) e^{-i\lambda x}\, d\lambda$$

$\therefore f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{i}{\lambda}(e^{i\lambda} - 1) e^{i\lambda x} \, d\lambda$

2] Find F.I.R.

$f(x) = 1, \quad |x| \leq 1$

$\quad = 0, \quad |x| > 1$

and evaluate $\int_0^{\infty} \frac{\sin\lambda \cos\lambda x}{\lambda} d\lambda$ & $\int_0^{\infty} \frac{\sin\lambda}{\lambda} d\lambda$

$\therefore f(x)$ is an even func. $\therefore$ we use $F_c(\lambda)$

i $F_c(\lambda) = \int_0^{\infty} f(u) \cos\lambda u \, du$

$= \int_0^{1} (1) \cos\lambda u \, du$

$= \left[\frac{\sin\lambda u}{\lambda}\right]_0^1$

$F_c(\lambda) = \frac{\sin\lambda}{\lambda}$

ii The fourier cosine integral rep is given by

$f(x) = \frac{2}{\pi} \int_0^{\infty} \left[\int_0^{\infty} f(u) \cos\lambda u \, du\right] \cos\lambda x \, d\lambda$

$f(x) = \frac{2}{\pi} \int_0^{\infty} \frac{\sin\lambda}{\lambda} \cos\lambda x \, d\lambda$

(A) ~~f(x)~~ $f(x) \frac{\pi}{2} = \int_0^{\infty} \frac{\sin\lambda \cos\lambda x}{\lambda} d\lambda$

$(1) \frac{\pi}{2} = \int_0^{\infty} \frac{\sin\lambda \cos\lambda x}{\lambda} d\lambda, \quad |x| \leq 1$

$\int_0^{\infty} \frac{\sin\lambda \cos\lambda x}{\lambda} d\lambda = 0, \quad |x| > 1$

iii We know that;

$$\int_0^\infty \frac{\sin\lambda \cos\lambda x}{\lambda}\, d\lambda = f(x)\frac{\pi}{2}$$

Put $x = 0$

$$\int_0^\infty \frac{\sin\lambda}{\lambda}\, d\lambda = f(0)\frac{\pi}{2}$$

$$\therefore \int_0^\infty \frac{\sin\lambda}{\lambda}\, d\lambda = \frac{\pi}{2}$$

By considering fourier sine & cosine transformation of:

$f(x) = e^{-mx}$ $(m > 0)$

1) Prove that $\displaystyle\int_0^\infty \frac{\lambda}{\lambda^2+m^2} \sin\lambda x\, d\lambda = \frac{\pi}{2} e^{-mx}$

2) Prove that $\displaystyle\int_0^\infty \frac{1}{\lambda^2+m^2} \cos\lambda x\, d\lambda = \frac{\pi}{2m} e^{-mx}$

Fourier sine trans is given by

$$F_s(\lambda) = \int_0^\infty f(u) \sin\lambda u\, du$$

$$= \int_0^\infty e^{-mu} \sin\lambda u\, du$$

$$F_s(\lambda) = \frac{\lambda}{\lambda^2+m^2}$$

By fourier sine integral

$$f(x) = \frac{2}{\pi}\int_0^\infty F_s(\lambda) \sin\lambda x\, d\lambda$$

$$e^{-mx} = \frac{2}{\pi}\int_0^\infty \frac{\lambda}{\lambda^2+m^2} \sin\lambda x\, d\lambda$$

Quick Work
Page No.:
Date:
M T W T F S S

$$\boxed{\int_0^\infty \frac{\lambda}{\lambda^2+m^2}\sin\lambda x \, d\lambda = \frac{\pi}{2}e^{-mx}}$$

fourier cosine trans is given by

$$F_c(\lambda) = \int_0^\infty f(u)\cos\lambda u \, du$$

$$= \int_0^\infty e^{-mu}\cos\lambda u \, du$$

$$F_c(\lambda) = \frac{m}{\lambda^2+m^2}$$

By fourier cosine integral

$$f(x) = \frac{2}{\pi}\int_0^\infty F_c(\lambda)\cos\lambda x \, d\lambda$$

$$e^{-mx} = \frac{2m}{\pi}\int_0^\infty \frac{1}{\lambda^2+m^2}\cos\lambda x \, d\lambda$$

$$\boxed{\frac{\pi}{2m}e^{-mx} = \int_0^\infty \frac{1}{\lambda^2+m^2}\cos\lambda x \, d\lambda}$$

4. Show that the fourier transform of $f(x) = e^{-|x|}$ is $\frac{2}{1+\lambda^2}$

* To check if the fun. is even or odd Put $x=-x$ if func is 0 then odd.

Put $x=-1$

$f(-x) = e^{-|-x|} = e^{-|x|}$ = even

Hence we use fourier cosine transform

$$F_c(\lambda) = \int_0^\infty f(u)\cos\lambda u \, du$$

$$= \int_0^\infty e^{-|u|}\cos\lambda u \, du$$

Q5. Find the fourier sine transform of

$$f(x) = \begin{cases} x, & 0 \le x \le a \\ 0, & x > a \end{cases}$$

By fourier sine transform

$$F_s(\lambda) = \int_0^\infty f(u) \sin \lambda u \, du$$

$$= \int_0^a u \sin \lambda u \, du + \int 0$$

$$= \left[u\left(\frac{-\cos \lambda u}{\lambda}\right) - \left(\frac{\sin \lambda u}{\lambda^2}\right) \right]_0^a$$

$$F_s(\lambda) = \frac{\sin \lambda a}{\lambda^2} - \frac{a \cos \lambda a}{\lambda}$$

Quick Work

Page No.:

Date:

M T W T F S S

6. $f(x) = \begin{cases} 0 & , x < 0 \\ e^{-x} & x > 0 \\ 1/2 & x = 0 \end{cases}$

By using fourier trans

$$F(\lambda) = \int_{-\infty}^{\infty} f(u) e^{-i\lambda u} \, du$$

$$= \int_{0}^{\infty} e^{-u} e^{-i\lambda u} \, du$$

$$= \int_{0}^{\infty} e^{-u(1+i\lambda)} \, du$$

$$= \left[\frac{e^{-u(1+i\lambda)}}{-(1+i\lambda)} \right]_0^{\infty}$$

$$= \frac{1}{1+i\lambda}$$

$$= \frac{1-i\lambda}{(1+i\lambda)(1-i\lambda)}$$

$$= \frac{1-i\lambda}{1+\lambda^2}$$

Now Fourier Integral

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\lambda) e^{i\lambda x} \, d\lambda$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1-i\lambda}{1+\lambda^2} (\cos\lambda x + i\sin\lambda x) \, d\lambda$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{1+\lambda^2} (\cos\lambda x + i\sin\lambda x - i\lambda\cos\lambda x + \lambda\sin\lambda x) \, d\lambda$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{1+\lambda^2} \underbrace{(\cos\lambda x + \lambda\sin\lambda x)}_{\substack{\text{Even} \\ f(x)}} d\lambda + i\frac{1}{2\pi} \int_{-\infty}^{\infty} \underbrace{\frac{(\sin\lambda x - \lambda\cos\lambda x)}{1+\lambda^2} d\lambda}_{\substack{\text{Odd} \\ g(x) = 0}}$$

$$= \frac{2}{2\pi}\int_0^{\infty}\frac{1}{1+\lambda^2}(\cos\lambda x+\lambda\sin\lambda x)\,d\lambda+0$$

$$f(x)=\frac{1}{\pi}\int_0^{\infty}\frac{1}{1+\lambda^2}(\cos\lambda x+\lambda\sin\lambda x)\,d\lambda$$

Q. find fourier integral cosine function

$$f(x)=\begin{cases}x^2, & 0<x<a\\ 0, & x>a\end{cases}$$

fourier cosine transform

$$F_c(\lambda)=\int_0^{\infty}f(u)\cos\lambda u\,du$$

$$=\int_0^{a}u^2\cos\lambda u\,du$$

$$\left[u^2\left(\frac{\sin\lambda u}{\lambda}\right)-(2u)\left(-\frac{\cos\lambda u}{\lambda^2}\right)+2\left(-\frac{\sin\lambda u}{\lambda^3}\right)\right]_0^a$$

$$F_c(\lambda)=\frac{a^2\sin\lambda a}{\lambda}+\frac{2a\cos\lambda a}{\lambda^2}-\frac{2\sin\lambda a}{\lambda^3}$$

Now, FCIR,

$$f(x)=\frac{2}{\pi}\int_0^{\infty}F_c(\lambda)\cos\lambda x\,d\lambda$$

$$f(x)=\frac{2}{\pi}\int_0^{\infty}\left(\frac{a^2\sin\lambda a}{\lambda}+\frac{2a\cos\lambda a}{\lambda^2}-\frac{2\sin\lambda a}{\lambda^3}\right)\cos\lambda x\,d\lambda$$

$\lambda u + u$

~~$\lambda u(+1$~~

~~$\lambda($~~

Q. using fourier integral representation show that

$$\int_0^\infty \frac{\lambda^3}{\lambda^4+4} \sin \lambda x \, d\lambda = \frac{\pi}{2} e^{-x} \cos x, \; x>0$$

By using fourier sin transf,

$$F_s(\lambda) = \int_0^\infty f(u) \sin \lambda x \, du$$

$$= \frac{\pi}{2} \int_0^\infty e^{-u} \cos u \sin \lambda x \, du$$

$a=1, b=\lambda$

$$\frac{\pi}{4} \int_0^\infty e^{-u} [\sin(\lambda+1)u + \sin(\lambda-1)u] \, du$$

$$= \frac{\pi}{4} \left[\int_0^\infty e^{-u} \sin(\lambda+1)u \, du + \int_0^\infty e^{-u} \sin(\lambda-1)u \, du \right]$$

$a=1, b=\lambda+1$ $\quad$ $a=1, b=\lambda-1$

$$= \frac{\pi}{4} \left[\frac{\lambda+1}{(\lambda+1)^2+1} + \frac{\lambda-1}{(\lambda-1)^2+1} \right]$$

$$= \frac{\pi}{4} \left[\frac{\lambda+1}{\lambda^2+2\lambda+2} + \frac{\lambda-1}{\lambda^2-2\lambda+2} \right]$$

$$\frac{\pi}{4} \left[\frac{2\lambda^3}{\lambda^4+4} \right]$$

$$F_s(\lambda) = \frac{\pi}{2} \left(\frac{\lambda^3}{\lambda^4+4} \right) \quad \text{--(1)}$$

By using fourier integral sine rep.
FSIR

$$f(x) = \frac{2}{\pi} \int_0^\infty F_s(\lambda) \sin \lambda x \, d\lambda$$

$$= \frac{2}{\pi} \int_0^\infty \frac{\pi}{2} \left(\frac{\lambda^3}{\lambda^4+4} \right) \sin \lambda x \, d\lambda$$

$$\frac{\pi}{2} e^{-x} \cos x = \int_0^\infty \frac{\lambda^3}{\lambda^4+4} \sin \lambda x \, d\lambda$$

Put value of four trans in inverse and don't solve further.

Type-2 Problems on fourier transform

Q1. Find fourier sine & cosine transform of following function

1) $f(x) = \begin{cases} x & , \quad 0 \le x \le 1 \\ 2-x & , \quad 1 \le x \le 2 \\ 0 & , \quad x > 2 \end{cases}$

~~$F_c(\lambda)$~~

$$F_c(\lambda) = \int_0^\infty f(u)\cos\lambda u \, du$$

$$= \int_0^1 u\cos\lambda u \, du + \int_1^2 (2-u)\cos\lambda u \, du + 0$$

$$= \left[u\left(\frac{\sin\lambda u}{\lambda}\right) - (1)\left(-\frac{\cos\lambda u}{\lambda^2}\right)\right]_0^1 + \left[(2-u)\left(\frac{\sin\lambda u}{\lambda}\right) - (-1)\left(-\frac{\cos\lambda u}{\lambda^2}\right)\right]_1^2$$

$$\left[\frac{\sin\lambda}{\lambda} + \frac{\cos\lambda}{\lambda^2} - \frac{1}{\lambda^2}\right] + \left[-\frac{\cos 2\lambda}{\lambda^2} - \frac{\sin\lambda}{\lambda} + \frac{\cos\lambda}{\lambda^2}\right]$$

$$F_c(\lambda) = \frac{2\cos\lambda}{\lambda^2} - \frac{1}{\lambda^2} - \frac{\cos 2\lambda}{\lambda^2}$$

Similarly:

$$F_s(\lambda) = \int_0^\infty f(u)\sin\lambda u \, du$$

$$= \int_0^1 u\sin\lambda u \, du + \int_1^2 (2-u)\sin\lambda u \, du + 0$$

$$\left[u\left(\frac{-\cos\lambda u}{\lambda}\right) + (1)\left(\frac{\sin\lambda u}{\lambda^2}\right)\right]_0^1 + \left[(2-u)\left(\frac{-\cos\lambda u}{\lambda}\right) - (-1)\left(-\frac{\sin\lambda u}{\lambda^2}\right)\right]_1^2$$

PSW

$$\left(-\frac{\cos\lambda}{\lambda}-\frac{\sin\lambda}{\lambda^2}\right)+\left[\left(-\frac{\sin 2\lambda}{\lambda^2}\right)-\left(-\frac{\cos\lambda}{\lambda}-\frac{\sin\lambda}{\lambda^2}\right)\right]$$

$$\left[-\frac{\cos\lambda}{\lambda}-\frac{\sin\lambda}{\lambda^2}\right]+\left[-\frac{\sin 2\lambda}{\lambda^2}+\frac{\cos\lambda}{\lambda}+\frac{\sin\lambda}{\lambda^2}\right]$$

$$\cancel{-\frac{\cos\lambda}{\lambda}}\cancel{-\frac{\sin\lambda}{\lambda^2}}-\frac{\sin 2\lambda}{\lambda^2}\cancel{+\frac{\cos\lambda}{\lambda}}\cancel{+\frac{\sin\lambda}{\lambda^2}}$$

$$-\frac{\sin 2\lambda}{\lambda^2}$$

2) $f(x)=\begin{cases}\cos x, & 0<x<a\\ 0 & x>a\end{cases}$

$$F_c(\lambda)=\int_0^\infty f(u)\cos\lambda u\,du$$

$$=\int_0^a \cos u\cos\lambda u\,du$$

$$=\frac{1}{2}\int_0^a\left[\cos(1+\lambda)u+\cos(1-\lambda)u\right]du$$

$$=\frac{1}{2}\left[\frac{\sin(1+\lambda)u}{1+\lambda}+\frac{\sin(1-\lambda)u}{(1-\lambda)}\right]_0^a$$

$$F_c(\lambda)=\frac{1}{2}\left[\frac{\sin(1+\lambda)a}{1+\lambda}+\frac{\sin(1-\lambda)a}{1-\lambda}\right]$$

$$\sin a\,\&\,\sin b=-\frac{1}{2}\left[\cos(a+b)-\cos(a-b)\right]$$

Quick Work
Page No.:
Date:
M T W T F S S

3. $f(x) = \begin{cases} 1-x^2, & |x| \leq 1 \\ 0, & |x| > 1 \end{cases}$

$$\int_0^\infty \left[\frac{x\cos x - \sin x}{x^3}\right] \cos\frac{x}{2}\, dx$$

Hence evaluate integral

$f(x) = 1 - x^2$

FCT.

$$F_c(\lambda) = \int_0^\infty f(u)\cos\lambda u\, du$$

$$= \int_0^1 (1-u^2)\cos\lambda u\, du$$

$$= \left[(1-u^2)\left(\frac{\sin\lambda u}{\lambda}\right) - (-2u)\left(-\frac{\cos\lambda u}{\lambda^2}\right) + (-2)\left(-\frac{\sin\lambda u}{\lambda^3}\right)\right]_0^1$$

$$= \left[\frac{-2\cos\lambda}{\lambda^2} + \frac{2\sin\lambda}{\lambda^3}\right]$$

$$F_c(\lambda) = 2\left[\frac{\sin\lambda - \lambda\cos\lambda}{\lambda^3}\right]$$

Now FCIR:

$$f(x) = -\frac{2\times 2}{\pi}\int_0^\infty \left[\frac{\lambda\cos\lambda - \sin\lambda}{\lambda^3}\right]\cos\lambda x\, d\lambda$$

$$\int_0^\infty \left[\frac{\lambda\cos\lambda - \sin\lambda}{\lambda^3}\right]\cos\lambda x\, d\lambda = -f(x)\frac{\pi}{4}$$

Quick Work

Page No.:

Date:

M T W T F S S

Put $x = \frac{1}{2}$

$$\int_0^\infty \left(\frac{\lambda\cos\lambda - \sin\lambda}{\lambda^3}\right)\cos\frac{\lambda}{2}\,d\lambda = -f(1/2)\frac{\pi}{4}$$

$f(x) = 1 - x^2$

$f(1/2) = 1 - 1/4 = 3/4$

$$\int_0^\infty \left(\frac{\lambda\cos\lambda - \sin\lambda}{\lambda^3}\right)\cos\frac{\lambda}{2}\,d\lambda = -\frac{3}{4}\left(\frac{\pi}{4}\right)$$

$$= -\frac{3\pi}{16}, \quad |x| \le 1$$

and $= 0, \quad |x| > 1$

Type: 3 : Problems on inverse fourier transform

Q. Find inverse transform who's fourier cosine transform is

1) $\frac{\sin a\lambda}{\lambda}$

$$F_c(\lambda) = \frac{\sin a\lambda}{\lambda}$$

To find inverse

IFCT

$$f(x) = \frac{2}{\pi}\int_0^\infty F_c(\lambda)\cos\lambda x\,d\lambda$$

$$= \frac{2}{\pi}\int_0^\infty \frac{\sin a\lambda}{\lambda}\cos\lambda x\,d\lambda$$

$$= \frac{\cancel{2}\,1}{\pi}\int_0^\infty \frac{1}{\lambda}\left[\sin(a+x)\lambda + \sin(a-x)\lambda\right]d\lambda$$

Quick Work
Page No.:
Date:
M T W T F S S

$$= \frac{1}{\pi}\left[\int_0^{\infty} \frac{\sin(a+x)\lambda}{\lambda}\, d\lambda + \int_0^{\infty} \frac{\sin(a-x)\lambda}{\lambda}\, d\lambda\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{2} + \frac{\pi}{2}\right] \quad (a+x)>0,\ (a-x)>0$$

$(a+x)>0,\ (a-x)<0$

$$= 1$$

$$= \frac{1}{\pi}\left[\frac{\pi}{2} - \frac{\pi}{2}\right]$$

$$= 0$$

Q. Solve foll integral eq.

2) $\int_0^{\infty} f(x)\sin\lambda x\, dx = \begin{cases} 1-\lambda, & 0 \le \lambda \le 1 \\ 0, & \lambda \ge 1 \end{cases}$

We know that

$$F_s(\lambda) = \int_0^{\infty} f(u)\sin\lambda u\, du$$

$F_s(\lambda) = 1-\lambda, \quad 0 \le \lambda \le 1$

$= 0, \quad \lambda \ge 1$

Now IFST

$$f(x) = \frac{2}{\pi}\int_0^{\infty} F_s(\lambda)\sin\lambda x\, d\lambda$$

$$= \frac{2}{\pi}\int_0^{1} (1-\lambda)\sin\lambda x\, d\lambda$$

$$= \frac{2}{\pi}\left[(1-\lambda)\left(\frac{-\cos\lambda x}{x}\right) - (-1)\left(\frac{-\sin\lambda x}{x^2}\right)\right]_0^1$$

$$\frac{2}{\pi}\left[-\frac{\sin x}{x^2} + \frac{1}{x}\right]$$

$$\frac{2}{\pi}\left[\frac{x - \sin x}{x^2}\right]$$

Q. Solve the integral eq $\int_0^\infty f(x)\cos\lambda x\,dx = \begin{cases} 1-\lambda & 0\le\lambda\le 1 \\ 0, & \lambda \ge 1 \end{cases}$

Show that $\int_0^\infty \frac{\sin^2 z}{z^2}\,dz = \frac{\pi}{2}$

$$F_c(\lambda) = \int_0^\infty f(u)\cos\lambda u\,du \quad \text{---} (1)$$

$$F_c(\lambda) = 1-\lambda, \quad 0\le\lambda\le 1$$
$$= 0, \quad \lambda \ge 1$$

Now,
IFCT

$$f(x) = \frac{2}{\pi}\int_0^\infty F_c(\lambda)\cos\lambda x\,d\lambda$$

$$= \frac{2}{\pi}\int_0^1 (1-\lambda)\cos\lambda x\,d\lambda$$

$$= \frac{2}{\pi}\left[(1-\lambda)\left(\frac{\sin\lambda x}{x}\right) - (-1)\left(\frac{-\cos\lambda x}{x^2}\right)\right]_0^1$$

$$= \frac{2}{\pi}\left[-\frac{\cos x}{x^2} + \frac{1}{x^2}\right]$$

$$f(x) = \frac{2}{\pi}\left[\frac{1-\cos x}{x^2}\right]$$

then $f(u) = \frac{2}{\pi}\left[\frac{1-\cos u}{u^2}\right]$

Put $f(u)$ in eq (1)

$$F_c(\lambda) = \int_0^\infty \frac{2}{\pi}\left\{\left[\frac{1-\cos u}{u^2}\right]\cos\lambda u \, du\right.$$

$$= \frac{4}{\pi}\int_0^\infty \frac{\sin^2 u/2}{u^2} \cos\lambda u \, du$$

Put $\lambda = 0$ because $\cos 0 = 1$

$$F_c(0) = \frac{4}{\pi}\int_0^\infty \frac{\sin^2 u/2}{u^2} \, du$$

Put $u/2 = z$

$\therefore u = 2z$

$du = 2dz$

$$F_c(0) = \frac{4}{\pi}\int_0^\infty \frac{\sin^2 z \, 2dz}{4z^2}$$

$$F_c(0) = \frac{2}{\pi}\int_0^\infty \frac{\sin^2 z}{z^2} dz$$

$$\frac{\pi}{2} F_c(0) = \int_0^\infty \frac{\sin^2 z}{z^2} \, dz$$

$$\int_0^\infty \frac{\sin^2 z}{z^2} dz = \frac{\pi}{2} F_c(0)$$

$F_c(\lambda) = 1-\lambda, \quad 0 \le \lambda < 1$

$F_c(0) = 1-0$

$= 1 \qquad 0 \le \lambda < 1$

$\therefore \quad F_c(0) = 0, \quad \lambda \geq 1$

Quick Work

Page No.:

Date:

M T W T F S S

2. $\int_0^\infty \frac{\sin^2 dz}{z^2} = \frac{\pi}{2}, \quad 0 \le \lambda < 1$

$\int_0^\infty \frac{\sin^2 dz}{z^2} = 0, \quad \lambda \ge 1$

Q. Solve the foll. integral

4. $\int_0^\infty f(x) \sin \lambda x \, d\lambda = \begin{cases} 1, & 0 \le \lambda < 1 \\ 2, & 1 \le \lambda < 2 \\ 0, & \lambda \ge 2 \end{cases}$

~~$F_s(\lambda) = \int_0^\infty f(u) \sin \lambda u \, du$~~

$F_s(\lambda) = \int_0^\infty f(u) \sin \lambda u \, du \quad -(1)$

Now IFST

$f(x) = \frac{2}{\pi} \int_0^\infty F_s(\lambda) \sin \lambda x \, d\lambda$

$f(x) = \frac{2}{\pi} \int_0^1 (1) \sin \lambda x \, d\lambda + \frac{2}{\pi} \int_1^2 (2) \sin \lambda x \, d\lambda + \int 0$

$\frac{2}{\pi} \left[\left(\frac{-\cos \lambda x}{x} \right)_0^1 + \left(\frac{-2\cos \lambda x}{x} \right)_1^2 \right]$

$\frac{2}{\pi} \left[\frac{-\cos x}{x} + \frac{1}{x} - \frac{2\cos 2x}{x} + \frac{2\cos x}{x} \right]$

$f(x) = \frac{2}{\pi} \left[\frac{\cos x - 2\cos 2x}{x} + \frac{1}{x} \right]$

Q. Find inverse transform if

$F_c(\lambda) = e^{-\lambda}, \quad \lambda > 0$

$F_c(\lambda) = \int_0^\infty f(u) \cos \lambda u \, du$

Now IFCT

$f(x) = \frac{2}{\pi} \int_0^\infty F_c(\lambda) \cos \lambda x \, d\lambda$

$f(x) = \frac{2}{\pi} \int_0^\infty e^{-\lambda} \cos \lambda x \, d\lambda$

$a = 1, \; b = x$

$= \frac{2}{\pi} \left[\frac{1}{1+x^2} \right]$