Master's method

We can solve recurrence relation using a formula denoted by master's method.

$T(n) = a\, T(n/b) + F(n)$ where $n \geq d$ & d is some constant

Then the master theorem can be stated for efficiency analysis as -

if $F(n)$ is $\Theta(n^d)$ where $d \geq 0$ in the recurren relation then

1) $T(n) = \Theta(n^d)$ if $a < b^d$
2) $T(n) = \Theta(n^d \log n)$ if $a = b$
3) $T(n) = \Theta(n^{\log_b a})$ if $a > b^d$

1) Solve the following recurrence relation
$T(n) = 4\, T(n/2) + n$

sol^n - $T(n) = a\, T(n/b) + f(n)$

Now

$f(n) = n$ i.e. n^1 Hence $d = 1$.

$a = 4$ $b = 2$ & $a > b^d$

$4 > 2^1$

$\therefore T(n) = \Theta(n^{\log_b a})$

$= \Theta(n^{\log_2 4})$

$= \Theta(n^2)$ $\because \log_2 4 = 2$.

Another variation of Master theorem is for

$T(n) = a\,T(n/b) + f(n)$ if $n \geq d$.

1) If $f(n) = O(n^{\log_b a - \varepsilon})$ then

$T(n) = \Theta(n^{\log_b a})$

2) If $f(n) = \Theta(n^{\log_b a} \log^k n)$ then

$T(n) = \Theta(n^{\log_b a} \log^{k+1} n)$

3) If $f(n) = \Omega(n^{\log_b a+\varepsilon})$ then

$T(n) = \Theta(f(n))$

1) Solve the following recurrence relation.

$T(n) = 2\,T(n/2) + n \log n$.

Sol^n $f(n) = n \log n$

$a = 2 \quad b = 2 \quad \log_b a = \log_2 2 = 1$.

According to case 2 given in above Master theorem.

$f(n) = \Theta(n^{\log_2 2} \log^1 n)$ i.e. $k = 1$.

then $T(n) = \Theta(\log_b a \log^{k+1} n)$

$= \Theta(n^{\log_2 2} \log^2 n)$

$$T(n) = \Theta(n^1 \log^2 n)$$

$$\therefore T(n) = \Theta(n \log^2 n).$$

2) $T(n) = 8T(n/2) + n^2.$

$a > b^d$

Sol^n Here $f(n) = n^2$

$a = 8 \quad b = 2 \quad \therefore \log_b^a = \log_2^8 = 3.$

Then according to case 1 of above given Master theorem -

$$f(n) = O(n^{\log_b^{a-\epsilon}})$$

$$= O(n^{\log_2^{8-\epsilon}})$$

$$= O(n^{3-\epsilon})$$

if we put $\epsilon = 1$ then

$$O(n^{3-1}) = O(n^2) = f(n).$$

then

$$T(n) = \Theta(n^{\log_b^a})$$

$$= \Theta(n^{\log_2^8})$$

$$\therefore \underline{T(n) = \Theta(n^3)}$$

3) $T(n) = 9\,T(n/3) + n^3$.

sol^n Here $a = 9$ $b = 3$ & $f(n) = n^3$

$\log_b^a = \log_3^9 = 2$.

$$\frac{a}{9} = \frac{3^3}{9}$$

According to case 3 in above master theorem -

As $f(n) = \Omega(n^{\log_3^9 + \varepsilon})$

i.e. $\Omega(n^{2+\varepsilon})$ & we have $f(n) = n^3$.

then to have $f(n) = \Omega(n)$, we must put $\varepsilon = 1$.

$\therefore$ Then $T(n) = \Theta(f(n))$

$T(n) = \Theta(n^3)$.