1) Solve the following recurrence using the substitution method - where n is power of 2.

$$T(n) = \begin{cases} 2 & ; \text{ if } n = 1 \\ 2T(n/2) + n & ; \text{ if } n > 1. \end{cases}$$

sol^n:-

$$T(n) = 2T(n/2) + n \quad \text{— ①}$$

put $n = n/2$ in eq ①

$$T(n/2) = 2T\left(\frac{n/2}{2}\right) + \frac{n}{2}$$

$$= 2T\left(\frac{n}{4}\right) + \frac{n}{2}$$

$$T(n) = 2\left[2T\left(\frac{n}{4}\right) + \frac{n}{2}\right] + n$$

$$= 4T\left(\frac{n}{4}\right) + n + n$$

$$= 4T\left(\frac{n}{4}\right) + 2n.$$

put $n = n/4$ in eq ①

$$T(n/4) = 2T\left(\frac{n/4}{2}\right) + \frac{n}{4}$$

$$= 2T\left(\frac{n}{8}\right) + \frac{n}{4}$$

$$= 8\,T\left(\frac{n}{8}\right) + n + 2n$$

$$= 8\,T\left(\frac{n}{8}\right) + 3n$$

$$= 2^3\,T\left(\frac{n}{2^3}\right) + 3n \quad \text{here } i = 3$$

$$\therefore T(n) = 2^i\,T\left(\frac{n}{2^i}\right) + in \quad \text{for any } \log_2 n \geq i \geq 1$$

put $i = \log_2 n$ where $n = 2^i$

$$\therefore T(n) = 2^{\log_2 n}\,T\left(\frac{n}{2^{\log_2 n}}\right) + n \times \log_2 n$$

$$= n\,T\left(\frac{n}{n}\right) + n \times \log_2 n$$

$$= n \times T(1) + n \times \log_2 n$$

But $T(1) = 2$.

$$\therefore T(n) = 2n + n\log_2 n$$

Thus $T(n) \in \Theta(n \log_2 n)$.

2) solve the following recurrence when n is power of 2.

merge sort

$$T(n) = \begin{cases} a & , n=1 , a = \text{constant} \\ 2T(n/2) + cn & , n>1 , c = \text{constant} \end{cases}$$

n is power of 2. ; $n = 2^k$.

sol^n —

$$T(n) = 2T(n/2) + cn \quad \text{— (1)}$$

put $n = n/2$ in eq (1)

$$T(n/2) = 2T\left(\frac{n/2}{2}\right) + c\frac{n}{2}$$

$$= 2T\left(\frac{n}{4}\right) + c\frac{n}{2}$$

$$T(n) = 2\left[2T\left(\frac{n}{4}\right) + \frac{cn}{2}\right] + cn$$

$$= 4T\left(\frac{n}{4}\right) + cn + cn$$

$$= 4T\left(\frac{n}{4}\right) + 2cn.$$

put $n = n/4$ in eq. (i)

$$T\left(\frac{n}{4}\right) = 2T\left(\frac{n/4}{2}\right) + c\frac{n}{4}$$

$$= 2T\left(\frac{n}{8}\right) + c\frac{n}{4}.$$

$$\therefore T(n) = 4\left[2T\left(\frac{n}{8}\right) + c\frac{n}{4}\right] + 2cn$$

$$= 8T\left(\frac{n}{8}\right) + \underline{cn + 2cn}$$

$$= 8T\left(\frac{n}{8}\right) + 3cn \qquad \therefore \cancel{k=3}$$

$$= 2^3 T\left(\frac{n}{2^3}\right) + 3cn.$$

$$\therefore T(n) = 2^3 T\left(\frac{2^k}{2^3}\right) + 3cn \ldots (n = 2^k)$$

$$= 2^3 T\left(2^{k-3}\right) + kcn \ldots (k = 3)$$

$$\therefore T(n) = 2^k T(1) + kcn$$

$$= nT(1) + kcn$$

$$= na + cn(\log_2^n)$$

$n = 2^k$

$\log n = k \log 2$

$\therefore k = \log_2^n$

&

$T(1) = a$

$\therefore$ it is easy to see that if

$$2^k < n \leq 2^{k+1} \quad \text{then } T(n) \leq T(2^{k+1})$$

$$\therefore T(n) = O(n \log_2^n)$$

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3) solve the following recurrence when n is power of 2:

Bubble sort

$$T(n) = \begin{cases} T(1) & ; \text{ if } n=1 \\ T(n/2)+c & ; \text{ if } n>1. \end{cases}$$

Sol^n — $T(n) = T(n/2) + c$. —— ①

put $n = n/2$ in eq. ①

$\therefore T(n/2) = T\left(\frac{n/2}{2}\right) + c.$

$T\left(\frac{n}{2}\right) = T\left(\frac{n}{4}\right) + c$

$\therefore T(n) = \left[T\left(\frac{n}{4}\right) + c\right] + c$

$= T\left(\frac{n}{4}\right) + 2c$

put $n = n/4$ in eq. ①.

$\therefore T(n) = T\left(\frac{n}{8}\right) + 3c$

$\vdots$

$\therefore T(n) = T\left(\frac{n}{2^i}\right) + ic$

$n = 2^i$

$\log n = i \log 2$

$\therefore i = \log_2^n$

$T(n) = T\left(\frac{2^i}{2^i}\right) + ic$

$T(n) = T(1) + ic$ put $i = \log_2^n$

$\therefore \quad T(n) = T(1) + c \log_2^n$

As $T(1) = \text{constant}$.

$\therefore T(n) \in \Theta(\log_2^n)$.

4) $T(n) = \begin{cases} T(1) & , n = 1 \\ a\,T(n/b) + f(n) & , n > 1. \end{cases}$

where $a = b =$ constants.

$T(1) =$ known.

& $n = b^K$ (n is a power of b)

Sol^n

$$T(n) = a\,T(n/b) + f(n) \quad \text{--- (1)}$$

put $n = n/b$. in eq (1)

$$\therefore T(n/b) = a\,T\left(\frac{n/b}{b}\right) + f\left(\frac{n}{b}\right)$$

$$T\left(\frac{n}{b}\right) = a\,T\left(\frac{n}{b^2}\right) + f\left(\frac{n}{b}\right)$$

$$\therefore T(n) = a\left[a\,T\left(\frac{n}{b^2}\right) + f\left(\frac{n}{b}\right)\right] + f(n)$$

$$= a^2\,T\left(\frac{n}{b^2}\right) + a\,f\left(\frac{n}{b}\right) + f(n)$$

put $n = n/b^2$ in eq. (1) we get.