Day 1 Lecture Notes: Foundations of Modeling, Optimization, and Python Essentials

Mathematical Modeling & Computational Projects Camp

Morning Structure (10:00–12:30)

- 10:00–10:30: Lecture Block 1
- 10:35–11:05: Problem Solving Session 1 (30 minutes)
- 11:10–11:40: Lecture Block 2
- 11:40–12:10: Problem Solving Session 2 (shorter) + 12:10–12:30 synthesis/review

Lecture Block 1 (10:00–10:30): Modeling and Optimization Basics

What is a mathematical model?

A mathematical model is a simplified math description of a real situation. We use a model to answer questions, make predictions, and compare choices.

Instructor prompt: "If two snack prices are possible, how do we decide which one is better?" Expected student idea: we need a rule for measuring "better."

Modeling cycle

1. Ask a clear question.
2. State assumptions.
3. Define variables.
4. Build equations/tables/rules.
5. Analyze and compute.
6. Interpret results in context.
7. Revise if needed.

Quick board example (2–3 minutes):

- Question: How many volunteers do we need to serve 180 attendees in 30 minutes?
- Assumption: one volunteer serves 12 attendees per 30 minutes.
- Variable: v = number of volunteers.

- Model: $12v \geq 180$.
- Compute: $v \geq 15$.
- Interpret: We need at least 15 volunteers if the assumption is reasonable.

What is optimization? (High-school level)

Optimization means finding the best value of something, usually the largest or smallest. Examples:

- Maximize profit.
- Minimize cost.
- Minimize travel time.

Key vocabulary for Day 1:

- **Decision variable**: what we are allowed to choose.
- **Objective**: what we want to maximize or minimize.
- **Constraint**: a limit we must obey (budget, time, capacity, etc.).
- **Feasible choice**: a choice that satisfies all constraints.

A simple optimization workflow:

1. Define the quantity to optimize (objective).
2. List allowed choices.
3. Compute objective for each allowed choice.
4. Choose best value and explain why.

Mini worked example (numerical search):

- A stand can prepare $n \in \{10, 20, 30, 40, 50\}$ snack boxes.
- Profit model: $P(n) = 9n - (60 + 2n) = 7n - 60$.
- Values: $P(10) = 10$, $P(20) = 80$, $P(30) = 150$, $P(40) = 220$, $P(50) = 290$.
- Best tested choice: $n = 50$ boxes.

Discussion note: this is “best among tested choices,” not automatically the global best across all possible n.

Core model forms

- Linear: $y = mx + b$
- Proportional: $y = kx$ (special case with $b = 0$)

Interpretation reminders:

- In $y = mx + b$, slope m is "change in output per one unit of input."
- Intercept b is "starting value when input is zero."
- In proportional models, doubling input doubles output.

Problem Solving Session 1 (10:35–11:05): Practice Set A (30 minutes)

Goal: Build and interpret basic models quickly.

Suggested pacing:

- First 10 minutes: Problems 1–4 (foundations).
- Next 12 minutes: Problems 5–8 (profit and optimization).
- Last 8 minutes: Problems 9–10 (explanation and reflection).

1. A ride costs $C = 2x + 3$. Find cost for $x = 0, 4, 9$ miles.
2. A data plan charges $P = 0.4g$. Find price for $g = 5, 12, 20$ GB.
3. Explain slope and intercept for $y = 5x + 10$ in words.
4. Write a proportional model for paint use if each wall needs 1.8 liters.
5. Revenue is $R = 12n$, cost is $C = 80 + 7n$. Write profit $P(n)$.
6. Evaluate your profit model from Problem 5 at $n = 10, 20, 30$.
7. A model gives outputs $6, 9, 12, 11, 8$ for tested inputs $1, 2, 3, 4, 5$. Which tested input is best?
8. Give one real-world situation where "best" means smallest value, not largest.
9. A club can make either 40, 60, or 80 sandwiches. Profit values are $110, 170, 165$. Which choice is best? What tradeoff might explain why 80 is not best?
10. Write one assumption behind your answer to Problem 9. Explain in one sentence how changing that assumption could change the best decision.

Instructor checkpoint questions:

- "What is the objective in Problems 5–7?"
- "Which choices are feasible in your setup, and why?"

Lecture Block 2 (11:10–11:40): Data, Tables, and Numerical Search

Worked example 1: cost table

If $C = 2x + 3$ and $x = 0, 2, 4, 6$, then costs are $3, 7, 11, 15$. A table helps us see patterns quickly.

Extend the table conversation:

- Add one more value: if $x = 8$, then $C = 19$.
- Ask: does the increase stay constant? Yes, each $+2$ in x gives $+4$ in C.
- Connect to slope: $\Delta C/\Delta x = 4/2 = 2$.

Worked example 2: optimization by testing choices

Suppose profit model is

$$\Pi(n) = 15n - 120$$

for tested values $n = 0, 5, 10, \ldots, 40$. Compute each value and choose the largest profit among tested options.

Teacher walkthrough table:

n	$\Pi(n)$
0	-120
5	-45
10	30
15	105
20	180
25	255
30	330
35	405
40	480

Point out break-even behavior: profit changes from negative to positive between $n = 5$ and $n = 10$.

Worked example 3: compare two plans

Plan A: $C_A = 1.8x + 6$, Plan B: $C_B = 2.2x + 2$. Find where costs are equal:

$$1.8x + 6 = 2.2x + 2 \Rightarrow 4 = 0.4x \Rightarrow x = 10.$$

Interpretation:

- For small x, one plan is cheaper.
- For large x, the other may be cheaper.

Numerical check at two points:

- At $x = 4$: $C_A = 13.2$, $C_B = 10.8$ (Plan B cheaper).
- At $x = 20$: $C_A = 42$, $C_B = 46$ (Plan A cheaper).

Conclusion: intersection gives the switch point in the preferred plan.

Problem Solving Session 2 (11:40–12:10): Practice Set B (shorter)

Goal: Mixed modeling + optimization reasoning.

Suggested pacing:

- First 12 minutes: Problems 1–4.
- Next 10 minutes: Problems 5–7.
- Last 8 minutes: Problems 8–9 and group share.

1. Parking fee model: $C = 5 + 1.5h$. Find C for $h = 3, 7$.

2. Which is proportional: $y = 3x$, $y = 3x + 2$, $y = 2x - 1$?
3. In your own words, what is optimization?
4. A club tests selling $n = 10, 20, 30, 40$ items with profits $30, 90, 120, 100$. Best tested choice?
5. Give one assumption in a snack-stand profit model and explain why it matters.
6. Compare plans: $C_A = 2x + 5$, $C_B = 1.5x + 11$. Find the break-even input x.
7. Use your answer from Problem 6: which plan is cheaper at $x = 2$? at $x = 20$?
8. You have a budget of \$70 and model $C = 2x + 6$. What is the largest whole-number x you can afford?
9. Reflection: write two sentences that separate "math result" from "real-world decision" for one problem above.

Instructor move for fast finishers: ask them to create one original optimization problem with a linear objective and at least one realistic constraint.

Synthesis and Review (12:10–12:30)

- Revisit: model, variable, slope, intercept, objective, best tested value.
- Share one correct solution from Set A and one from Set B.
- Exit check: "What quantity did we optimize today?"
- Exit check 2: "Give one example where maximizing is wrong and minimizing is the right objective."
- Exit check 3: "When do we trust a model less?" (expected: weak assumptions, missing data, unrealistic constraints).

Python Preview (Afternoon)

- Python basics: variables, lists, loops, and functions.
- Build linear models in code.
- Test many inputs and find best tested value.
- Plot data and model outputs.