Day 2 Lecture Notes: Vectors, Matrices, and Systems

Mathematical Modeling & Computational Projects Camp

Morning Structure (10:00–12:30)

- 10:00–10:30: Lecture Block 1
- 10:35–11:05: Problem Solving Session 1 (30 minutes)
- 11:10–11:40: Lecture Block 2
- 11:40–12:10: Problem Solving Session 2 (shorter) + 12:10–12:30 synthesis/review

Lecture Block 1 (10:00–10:30): Vectors, Matrices, and Systems Setup

Introduction to vectors

A vector is an ordered list of numbers, such as

$$\mathbf{v} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}.$$

We use vectors to keep related quantities together.

Core operations:

- Vector addition: add entries in the same position.
- Scalar multiplication: multiply every entry by the same number.

Quick worked example:

$$\begin{bmatrix} 3 \\ -1 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 \\ 8 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}.$$

Instructor prompt: “If a vector has two entries, what could they represent in real life?”

Introduction to matrices

A matrix is a rectangular array of numbers. Example:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

Matrices can store data and connect inputs to outputs.

Example interpretation:

$$P = \begin{bmatrix} 6 & 9 \\ 4 & 7 \end{bmatrix}$$

where rows are two stores and columns are two products.

Entry notation reminder: a_{ij} means row i, column j.

Matrix dimensions and basic operations

Dimension language:

- A matrix with m rows and n columns is called an $m \times n$ matrix.
- Matrix-vector multiplication $A\mathbf{x}$ is defined when columns of A match entries of $\mathbf{x}$.

Dimension check rule:

- Inside numbers must match.
- Outside numbers give the output size.

Worked example:

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}, \quad A\mathbf{x} = \begin{bmatrix} 2(4) + 1(2) \\ 0(4) + 3(2) \end{bmatrix} = \begin{bmatrix} 10 \\ 6 \end{bmatrix}.$$

Quick check:

$$(2 \times 3)(3 \times 1) \to (2 \times 1) \text{ defined}, \qquad (2 \times 3)(2 \times 2) \text{ not defined}.$$

Systems of equations as models

Many real situations have multiple conditions at once. Example with student tickets s and adult tickets a:

$$\begin{cases} s + a = 120 \\ 5s + 8a = 780 \end{cases}$$

Equation 1 is a count condition; Equation 2 is a revenue condition.

Matrix form of the same model:

$$\begin{bmatrix} 1 & 1 \\ 5 & 8 \end{bmatrix} \begin{bmatrix} s \\ a \end{bmatrix} = \begin{bmatrix} 120 \\ 780 \end{bmatrix}.$$

Problem Solving Session 1 (10:35–11:05): Practice Set A (30 minutes)

Goal: Fluency with vector arithmetic, matrix-vector products, dimension checks, and writing simple systems from context.

Suggested pacing:

- First 10 minutes: Problems 1–4.
- Next 12 minutes: Problems 5–8.
- Last 8 minutes: Problems 9–10 and discussion.

1. Compute $\begin{bmatrix} 3 \\ -1 \end{bmatrix} + \begin{bmatrix} 4 \\ 2 \end{bmatrix}$.
2. Compute $2 \begin{bmatrix} 1 \\ 5 \end{bmatrix} - \begin{bmatrix} 3 \\ 4 \end{bmatrix}$.

3. For $A = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$, compute $A\mathbf{x}$.

4. Let $M = \begin{bmatrix} 1 & 4 & 2 \\ 0 & 3 & 5 \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$. Compute $M\mathbf{u}$.

5. Is AB defined if A is 2×3 and B is 3×2? What is the size of AB?

6. Is BA defined for the same matrices? If yes, what size? If no, explain quickly.

7. Snack stand model (set up only): Let x be the number of granola bars (\$3 each) and y be the number of fruit cups (\$5 each). If 120 total items were sold for \$480 total revenue, write (i) the two equations in x, y, and (ii) the matrix form $A\mathbf{x} = \mathbf{b}$.

8. Write the matrix form $A\mathbf{x} = \mathbf{b}$ for your system in Problem 7.

9. Give one real meaning of a 2-entry vector in school life.

10. Explain why a dimension check should happen before arithmetic.

Instructor checkpoint questions:

- "What does each entry of $A\mathbf{x}$ mean in context?"
- "How can you reject invalid products quickly without doing arithmetic?"

Lecture Block 2 (11:10–11:40): Inverse, Determinant, and Gaussian Elimination

1) Definition of matrix inverse (2x2, conceptual)

For a square matrix A, an inverse is a matrix A^{-1} such that

$$AA^{-1} = A^{-1}A = I,$$

where I is the identity matrix.

For 2x2 work, use

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

when $ad - bc \neq 0$.

Worked example:

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, \quad A^{-1} = \frac{1}{2(1) - 1(1)} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}.$$

Quick check:

$$AA^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.$$

2) Definition of determinant and invertibility (2x2 only)

For

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

the determinant is

$$\det(A) = ad - bc.$$

Interpretation for today:

- If $\det(A) \neq 0$, then A is invertible.
- If $\det(A) = 0$, then A is not invertible.

Worked example:

$$B = \begin{bmatrix} 3 & 6 \\ 1 & 2 \end{bmatrix}, \quad \det(B) = 3 \cdot 2 - 6 \cdot 1 = 0,$$

so B is not invertible.

Short note on solutions: for a 2x2 system $A\mathbf{x} = \mathbf{b}$, invertibility of A is tied to uniqueness. If A is invertible, there is exactly one solution.

3) Solving systems using Gaussian elimination (equation form)

Worked example:

$$\begin{cases} 2x + y = 7 \\ x - y = 2 \end{cases}$$

Add the equations after multiplying the second by 1 (already ready to add):

$$(2x + y) + (x - y) = 7 + 2 \Rightarrow 3x = 9 \Rightarrow x = 3.$$

Substitute into $x - y = 2$:

$$3 - y = 2 \Rightarrow y = 1.$$

Check:

$$2(3) + 1 = 7, \quad 3 - 1 = 2.$$

4) Solving systems using Gaussian elimination (matrix form)

Write the same style of problem as an augmented matrix and row-reduce.

Worked example:

$$\begin{cases} x + 2y = 5 \\ 3x + y = 7 \end{cases} \quad \Longleftrightarrow \quad \left[\begin{array}{cc|c} 1 & 2 & 5 \\ 3 & 1 & 7 \end{array}\right].$$

Use row operations:

$$R_2 \leftarrow R_2 - 3R_1 \Rightarrow \left[\begin{array}{cc|c} 1 & 2 & 5 \\ 0 & -5 & -8 \end{array}\right] \Rightarrow y = \frac{8}{5}.$$

Back-substitute into row 1:

$$x + 2\left(\frac{8}{5}\right) = 5 \Rightarrow x = \frac{9}{5}.$$

Reminder: equation form and matrix form are two views of the same elimination process.

Problem Solving Session 2 (11:40–12:10): Practice Set B (shorter)

Goal: Practice determinant, inverse (2x2 only), and Gaussian elimination in both equation and matrix form.

Suggested pacing:

- First 12 minutes: Problems 1–4.
- Next 10 minutes: Problems 5–7.
- Last 8 minutes: Problems 8–9 and solution share-out.

1. Compute $\det\left(\begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}\right)$ and state whether the matrix is invertible.
2. Compute $\det\left(\begin{bmatrix} 2 & 6 \\ 1 & 3 \end{bmatrix}\right)$ and state whether it is invertible.
3. Find the inverse of $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$.
4. Find the inverse of $\begin{bmatrix} 3 & 2 \\ 5 & 4 \end{bmatrix}$.
5. Solve by elimination in equation form:
$$\begin{cases} x + 2y = 11 \\ 3x - y = 4 \end{cases}$$
6. Solve by elimination in equation form:
$$\begin{cases} 2x + y = 7 \\ x - y = 2 \end{cases}$$
7. Solve in matrix form by row operations:
$$\left[\begin{array}{cc|c} 1 & 1 & 6 \\ 2 & -1 & 3 \end{array}\right].$$
8. Solve in matrix form by row operations:
$$\left[\begin{array}{cc|c} 1 & 2 & 5 \\ 3 & 1 & 7 \end{array}\right].$$
9. Reflection: in one sentence, explain why $\det(A) = 0$ means the 2x2 system may fail to have a unique solution.

Instructor move for fast finishers: ask students to check one solution by plugging it into both original equations.

Synthesis and Review (12:10–12:30)

- Revisit Block 1: vectors, matrices, dimension checks, and systems as models.
- Revisit Block 2: inverse, determinant, and Gaussian elimination in equation/matrix form.
- Common errors: forgetting dimension checks, arithmetic sign errors, and row-operation mistakes.
- Exit check: "How does determinant tell us about invertibility for a 2x2 matrix?"
- Exit check 2: "Why does an invertible coefficient matrix lead to a unique solution?"
- Exit check 3: "What is one advantage of matrix-form elimination over equation-form elimination?"

Python Preview (Afternoon)

Use NumPy arrays for vectors/matrices, compute 2x2 determinants and inverses, and compare manual elimination with `numpy.linalg.solve`.