Day 3 Exercises: Iteration, Solvers, and Eigenvalues

This handout combines both Day 3 problem-solving sessions.

Session 1: Iteration and Solver Steps

This session practices repeated matrix updates and the first steps of Jacobi and Gauss–Seidel. Focus on careful arithmetic and on explaining what each update means.

For Problems 1–4, use the repeated-update rule $\mathbf{x}_{k+1} = A\mathbf{x}_k$.

1. For $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $\mathbf{x}_0 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, compute $\mathbf{x}_1$.

2. Continue Problem 1 to compute $\mathbf{x}_2$.

3. For $A = \begin{bmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{bmatrix}$ and $\mathbf{x}_0 = \begin{bmatrix} 100 \\ 50 \end{bmatrix}$, compute one step.

4. Explain one sentence interpretation of entry 0.2 in row 2, column 1.

For Problems 5–9, the goal is to solve the linear system

$$A\mathbf{x} = \mathbf{b}$$

by writing and using Jacobi or Gauss–Seidel update formulas.

5. For $B = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, write the Jacobi update formulas for $x_1^{(k+1)}$ and $x_2^{(k+1)}$.

6. Using Problem 5 and $\mathbf{x}^{(0)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, compute Jacobi $\mathbf{x}^{(1)}$.

7. For the same system, write the Gauss–Seidel update formulas.

8. Using Problem 7 and $\mathbf{x}^{(0)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, compute Gauss–Seidel $\mathbf{x}^{(1)}$.

Session 2: Eigenvalues and Convergence Intuition

This session focuses on the eigenvector test and on qualitative reasoning about repeated updates.

1. Check whether $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is an eigenvector of $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$.

2. Check whether $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector of $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$.

3. For $A = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$, find eigenvalue for $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

4. For the same A, what happens to the first component after many iterations if that direction is present?

5. Suppose an eigenvalue is 0.6. What does that suggest about repeated updates in that direction?

6. If $\rho(G) = 0.8$ for an iteration matrix, what do you predict about convergence speed versus a method with $\rho(G) = 0.2$?

7. In one sentence, explain why a repeated matrix process can settle down when the important eigenvalues are small in magnitude.