Day 3 Lecture Notes: Iteration, Solvers, and Eigenvalues

Mathematical Modeling & Computational Projects Camp

Lecture Block 1 (10:00–10:30): Iterative Models

If

$$\mathbf{x}_{k+1} = A\mathbf{x}_k,$$

then the same matrix A is applied over and over. The vector $\mathbf{x}_k$ is the state at step k, and $\mathbf{x}_0$ is the starting vector.

This idea appears in two places in our camp:

- repeated processes, where we study how a system changes step by step;
- solving linear systems, where we use a repeated rule to improve an estimate.

Core vocabulary:

- **State vector $\mathbf{x}_k$**: the current values in the system.
- **Update matrix A**: the rule that tells us the next step.
- **Initial vector $\mathbf{x}_0$**: where the process begins.

Worked example 1:

$$A = \begin{bmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{bmatrix}, \qquad \mathbf{x}_0 = \begin{bmatrix} 100 \\ 50 \end{bmatrix}.$$

Compute one step:

$$\mathbf{x}_1 = A\mathbf{x}_0 = \begin{bmatrix} 0.7(100) + 0.2(50) \\ 0.3(100) + 0.8(50) \end{bmatrix} = \begin{bmatrix} 80 \\ 70 \end{bmatrix}.$$

Interpretation: the update rebalances the two entries.

Worked example 2 (second step):

$$\mathbf{x}_2 = A\mathbf{x}_1 = \begin{bmatrix} 0.7(80) + 0.2(70) \\ 0.3(80) + 0.8(70) \end{bmatrix} = \begin{bmatrix} 70 \\ 80 \end{bmatrix}.$$

Observation to discuss: repeated updates can move toward a balance, bounce back and forth, or drift away depending on the matrix.

Iterative Methods for Solving Systems

Split $B = D + L + U$, where D is diagonal, L is strictly lower triangular, and U is strictly upper triangular.

Jacobi method:

$$\mathbf{x}^{(k+1)} = D^{-1}\big(\mathbf{b} - (L+U)\mathbf{x}^{(k)}\big).$$

Each new entry is computed from the old vector only.

Gauss–Seidel method:

$$\mathbf{x}^{(k+1)} = (D+L)^{-1}\big(\mathbf{b} - U\mathbf{x}^{(k)}\big).$$

This method uses fresh values as soon as they are available.

For a 2×2 system

$$b_{11}x_1 + b_{12}x_2 = c_1,$$
$$b_{21}x_1 + b_{22}x_2 = c_2,$$

the Jacobi updates are

$$x_1^{(k+1)} = \frac{c_1 - b_{12}x_2^{(k)}}{b_{11}}, \qquad x_2^{(k+1)} = \frac{c_2 - b_{21}x_1^{(k)}}{b_{22}}.$$

For Gauss–Seidel, the second update uses the new value of $x_1^{(k+1)}$.

Practical note: a common sufficient condition for convergence is strict diagonal dominance of B.

Worked Examples: Jacobi vs. Gauss–Seidel

Example A:

$$B = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 10 \\ 6 \end{bmatrix}, \qquad \mathbf{x}^{(0)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The system is

$$2x_1 + x_2 = 10, \qquad 3x_2 = 6.$$

Jacobi and Gauss–Seidel give the same updates here because the second equation does not depend on x_1:

$$x_1^{(k+1)} = \frac{10 - x_2^{(k)}}{2}, \qquad x_2^{(k+1)} = 2.$$

From $\mathbf{x}^{(0)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$,

$$\mathbf{x}^{(1)} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}, \qquad \mathbf{x}^{(2)} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}.$$

Example B:

$$B = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \qquad \mathbf{x}^{(0)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Jacobi:

$$x_1^{(k+1)} = \frac{3 - x_2^{(k)}}{2}, \qquad x_2^{(k+1)} = 2 - x_1^{(k)}.$$

Iteration values:

$$\mathbf{x}^{(1)} = \begin{bmatrix} \frac{3}{2} \\ 2 \end{bmatrix}, \qquad \mathbf{x}^{(2)} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}.$$

Gauss–Seidel:

$$x_1^{(k+1)} = \frac{3 - x_2^{(k)}}{2}, \qquad x_2^{(k+1)} = 2 - x_1^{(k+1)}.$$

Iteration values:

$$\mathbf{x}^{(1)} = \begin{bmatrix} \frac{3}{2} \\ \frac{1}{2} \end{bmatrix}, \qquad \mathbf{x}^{(2)} = \begin{bmatrix} \frac{5}{4} \\ \frac{3}{4} \end{bmatrix}.$$

Exact solution is

$$\mathbf{x}^* = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

and both methods move toward it; Gauss–Seidel often converges faster because it uses updated values within each step.

Problem Solving Session 1 (10:35–11:05): Practice Set A (30 minutes)

Goal: Compute iterative updates accurately, set up Jacobi/Gauss–Seidel steps, and interpret convergence clues.

For Problems 1–4, use the repeated-update rule $\mathbf{x}_{k+1} = A\mathbf{x}_k$.

1. For $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $\mathbf{x}_0 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, compute $\mathbf{x}_1$.
2. Continue Problem 1 to compute $\mathbf{x}_2$.
3. For $A = \begin{bmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{bmatrix}$ and $\mathbf{x}_0 = \begin{bmatrix} 100 \\ 50 \end{bmatrix}$, compute one step.
4. Explain one sentence interpretation of entry 0.2 in row 2, column 1.

For Problems 5–8, the goal is to solve the linear system

$$A\mathbf{x} = \mathbf{b}$$

by writing and using Jacobi or Gauss–Seidel update formulas.

5. For $B = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, write the Jacobi update formulas for $x_1^{(k+1)}$ and $x_2^{(k+1)}$.
6. Using Problem 5 and $\mathbf{x}^{(0)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, compute Jacobi $\mathbf{x}^{(1)}$.
7. For the same system, write the Gauss–Seidel update formulas.
8. Using Problem 7 and $\mathbf{x}^{(0)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, compute Gauss–Seidel $\mathbf{x}^{(1)}$.

Instructor checkpoint: ask one team to interpret each entry of A in words before any multiplication.

Lecture Block 2 (11:10–11:40): Eigenvalues and Convergence Intuition

A nonzero vector $\mathbf{v}$ is an eigenvector of A if

$$A\mathbf{v} = \lambda\mathbf{v}.$$

Interpretation: direction stays the same, size scales by λ.

If $|\lambda| < 1$, that direction shrinks over repeated updates. If $|\lambda| > 1$, that direction grows.

Geometric intuition for students:

- Most vectors change direction under A.
- Eigenvectors are special directions that do not turn.
- Eigenvalue tells how fast stretching/shrinking happens.

Simple sketch (vector keeps direction):

```
same line before and after update


    v
   /
  /      Av (longer or shorter, same direction)
 /_________>
```

Rubber-band analogy: Imagine a rubber band drawn along one direction on a board. After applying the transformation, that special direction acts like a straight stretch or squeeze of the band: it gets longer or shorter, but it does not rotate away from that line.

Worked check example:

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}.$$

This matrix is not diagonal. Its characteristic polynomial is

$$\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} = (3-\lambda)^2 - 1 = 0,$$

so eigenvalues are $\lambda_1 = 4$ and $\lambda_2 = 2$ (distinct and real).

Choose

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Check:

$$A\mathbf{v}_1 = \begin{bmatrix} 4 \\ 4 \end{bmatrix} = 4\mathbf{v}_1, \qquad A\mathbf{v}_2 = \begin{bmatrix} 2 \\ -2 \end{bmatrix} = 2\mathbf{v}_2.$$

Connection to long-term behavior: in many models, the dominant eigenvalue helps explain which pattern survives after many repeated updates.

Convergence Analysis for Iterative Methods

Many iterative algorithms can be written as

$$\mathbf{x}^{(k+1)} = G\mathbf{x}^{(k)} + \mathbf{c},$$

where G is the iteration matrix.

Error dynamics follow

$$\mathbf{e}^{(k+1)} = G\mathbf{e}^{(k)}, \qquad \mathbf{e}^{(k)} = \mathbf{x}^{(k)} - \mathbf{x}^{*}.$$

So convergence is controlled by eigenvalues of G.

Key criterion:

$$\rho(G) < 1 \quad \Longrightarrow \quad \mathbf{x}^{(k)} \to \mathbf{x}^{*},$$

where $\rho(G)$ is the spectral radius (largest absolute eigenvalue).

Interpretation for students:

- If all error-directions shrink ($|\lambda| < 1$), the method converges.
- If some direction has $|\lambda| > 1$, error grows in that direction.
- If $|\lambda| \approx 1$, convergence can be very slow.

For system solvers:

- Jacobi: $G_J = -D^{-1}(L + U)$.
- Gauss–Seidel: $G_{GS} = -(D + L)^{-1}U$.

Small Application: Markov Chains

A Markov chain also uses repeated matrix multiplication:

$$\mathbf{p}_{k+1} = P\mathbf{p}_k,$$

where $\mathbf{p}_k$ is a probability vector and P is a special matrix with nonnegative entries.

The main idea is the same as before: if repeated updates shrink differences, the system settles into a stable pattern. We will only use this as a quick example of where eigenvalue ideas appear.

Problem Solving Session 2 (11:40–12:10): Practice Set B (shorter)

Goal: Identify eigenvectors/eigenvalues and connect them to iteration behavior.

1. Check whether $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is an eigenvector of $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$.
2. Check whether $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector of $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$.
3. For $A = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$, find eigenvalue for $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.
4. For the same A, what happens to the first component after many iterations if that direction is present?

5. Suppose an eigenvalue is 0.6. What does that suggest about repeated updates in that direction?

6. If $\rho(G) = 0.8$ for an iteration matrix, what do you predict about convergence speed versus a method with $\rho(G) = 0.2$?

7. In one sentence, explain why a repeated matrix process can settle down when the important eigenvalues are small in magnitude.

Synthesis and Review (12:10–12:30)

- Iteration step: $\mathbf{x}_{k+1} = A\mathbf{x}_k$ means keep applying the same rule.
- Jacobi uses only old values; Gauss–Seidel uses new values right away.
- Eigenvalues help us guess whether repeated updates shrink, grow, or stay about the same.
- Short application note: the same repeated-update idea appears in Markov chains.
- Exit check: "What does $A\mathbf{x}$ mean in words?"
- Exit check 2: "What does $|\lambda| < 1$ tell us about repeated updates?"
- Exit check 3: "How does $\rho(G)$ help predict whether an iterative solver converges?"
- Exit check 4: "Why can two methods start from the same vector but end up with different intermediate steps?"