Day 4 Lecture Notes: Linear Programming and Optimization

Mathematical Modeling & Computational Projects Camp

Morning Structure (10:00–12:30)

- 10:00–10:30: Lecture Block 1
- 10:35–11:05: Problem Solving Session 1 (30 minutes)
- 11:10–11:40: Lecture Block 2
- 11:40–12:10: Problem Solving Session 2 (shorter) + 12:10–12:30 synthesis/review

Lecture Block 1 (10:00–10:30): Primal Linear Programming

Optimization means finding the best value (maximum or minimum) while following rules.

In the primal linear program, we start with the choices we control:

- Decision variables: the numbers we choose,
- Objective: the quantity we want to maximize or minimize,
- Constraints: the limits the choices must satisfy.

Example form:

$$\text{maximize } z = ax + by \text{ subject to linear constraints.}$$

Vocabulary focus:

- **Feasible solution**: satisfies every constraint.
- **Feasible region**: all feasible solutions together.
- **Optimal solution**: feasible point with best objective value.

For two-variable problems, the feasible region is usually a polygon on a graph. The corner principle says that, when a best solution exists, it occurs at a corner point of that polygon.

Worked example 1:

- Let x = number of posters and y = number of flyers.
- Suppose profit is $P = 6x + 4y$.
- Resource limits give $2x + y \leq 120$ and $x + 3y \leq 150$.
- We also need $x, y \geq 0$.

Here the variables are the choices, the objective is profit, and the inequalities are the rules. If we graph the constraints, the feasible region is the overlap of all the shaded parts.

To find the best point, we test the corners of the feasible region. For this example, the corners are $(0, 0)$, $(60, 0)$, $(0, 50)$, and the intersection of the two lines. The intersection is $(45, 35)$, so we compare:

$$P(0, 0) = 0, \quad P(60, 0) = 360, \quad P(0, 50) = 200, \quad P(45, 35) = 430.$$

The best corner is $(45, 35)$.

Worked example 2:

- Let s = smoothies and j = juice cups.
- Maximize profit $P = 4s + 5j$.
- Constraints: $2s + j \leq 18$, $s + 2j \leq 16$, and $s, j \geq 0$.

The feasible corners are $(0, 0)$, $(9, 0)$, $(0, 8)$, and the intersection of the two lines. Solving $2s + j = 18$ and $s + 2j = 16$ gives $(s, j) = (20/3, 14/3)$. Checking the objective values shows that the best choice is near the corner where both resources are used well.

Message to emphasize: optimization is about the best *valid* choice, not just the biggest number.

Problem Solving Session 1 (10:35–11:05): Practice Set A (30 minutes)

Goal: Translate context to primal LP components and test corner points.

Suggested pacing:

- First 10 minutes: Problems 1–2.
- Next 12 minutes: Problems 3–4.
- Last 8 minutes: Problem 5 and group check.

1. A school club sells posters and flyers. Define two decision variables and write one objective for profit.
2. For the club model, explain what the constraints $2x + y \leq 120$, $x + 3y \leq 150$, and $x, y \geq 0$ mean in context, and say which rule is a resource limit and which rule just keeps the variables realistic.
3. Check whether each point is feasible: $(0, 0)$, $(20, 30)$, and $(40, 10)$.
4. Use $P = 5x + 4y$ to compare the three feasible points in Problem 3.
5. One student claims that $(50, 40)$ is the best point because it has larger coordinates, but that is not enough in LP

Instructor checkpoint questions:

- Which corner gave the best value in your test?
- Why do we still need to check feasibility before comparing objective values?

Lecture Block 2 (11:10–11:40): Dual Linear Programming

Every primal linear program has a matching dual linear program. The dual uses one variable for each constraint in the primal.

High-level idea:

- The primal starts with choices and asks, What is the best answer we can get?
- The dual starts with resources and asks, What is the cheapest way to price those resources?

If the primal is a maximum problem with $\leq$ constraints and nonnegative variables, the dual is a minimum problem with $\geq$ constraints and nonnegative dual variables. That pattern is the main thing to remember.

Interpretation of dual variables: each dual variable acts like a shadow price, or the value of one more unit of a resource. If a time limit becomes one minute larger, the dual variable tells us how much the best profit might increase, approximately.

Worked example side-by-side:

$$\text{Primal: maximize } P = 3x + 2y$$
$$\text{subject to } x + y \leq 100,\ 2x + y \leq 140,\ x, y \geq 0$$

$$\text{Dual: minimize } C = 100u + 140v$$
$$\text{subject to } u + 2v \geq 3,\ u + v \geq 2,\ u, v \geq 0$$

Here u is the shadow price for the first resource and v is the shadow price for the second resource. If we try $u = 1$ and $v = 1$, then the dual cost is $100(1) + 140(1) = 240$. That matches the best primal value from the corner check in Block 1, so the two problems are telling the same story from different directions.

Weak duality intuition: the dual gives an upper bound on the primal profit. So if we find a primal value of 240, any dual plan must cost at least 240. This is why primal and dual answers are so closely connected.

Comparison table:

Primal	**Dual**
Choose the decision variables.	Choose the shadow prices.
Maximize profit or output.	Minimize resource cost.
Constraints describe limits.	Constraints describe fair prices.
Feasible choices form the primal region.	Feasible prices form the dual region.

Why LP is used so widely: logistics and scheduling teams use LP to allocate limited trucks, time slots, staff, and budgets while meeting many constraints at once.

Problem Solving Session 2 (11:40–12:10): Practice Set B (shorter)

Goal: Build dual LPs and compare them with the primal.

Suggested pacing:

- First 10 minutes: Problems 1–2.
- Next 10 minutes: Problems 3–4.
- Last 10 minutes: group discussion and reflection.

1. For the primal $\max P = 2x + y$ with constraints $x + y \leq 10$, $2x + y \leq 14$, and $x, y \geq 0$, write the dual in standard form.
2. In words, what does a dual variable represent in the context of a resource limit?
3. Check whether $u = 1$, $v = \frac{1}{2}$ is feasible for the dual in Problem 1, and compute its dual objective value.
4. Explain why a dual value smaller than a claimed primal value would be impossible.
5. Reflection: describe one way the dual helps you understand the primal answer without drawing a graph.

Instructor move for fast finishers: ask them to solve Problem 1 both by algebra and by checking the corner points of the primal.

Synthesis and Review (12:10–12:30)

- Revisit primal vocabulary: decision variable, objective, constraint, feasible region, and corner point.
- Revisit dual vocabulary: shadow price, dual variable, and resource value.
- Class check: explain how the primal and dual in one example describe the same situation from different sides.
- Exit check: Why must a primal answer and a dual answer stay connected?
- Exit check 2: What does a shadow price tell us in plain language?

Python Preview (Afternoon)

In the afternoon, we will use `scipy.optimize.linprog` to solve both the primal and the dual and compare the answers.