Day 5 Exercises: Problem Solving Sessions 1 and 2

This handout combines both Day 5 problem-solving sessions.

Session 1

Build and analyze flow-network models, then express max flow as an LP.

1. A school donation network has source s, sink t, and edges $s \to a : 6$, $s \to b : 4$, $a \to c : 4$, $b \to c : 2$, $a \to t : 2$, $c \to t : 6$. Draw the corresponding directed graph with capacities. Then find one feasible flow of value at least 7, or explain why that is impossible.

2. For the same network, write the max-flow LP explicitly: redraw the graph from Problem 1, then list variables, objective function, capacity constraints, and conservation constraints.

3. In a network with edges $s \to x : 5$, $s \to y : 5$, $x \to t : 3$, $y \to t : 4$, $x \to y : 2$, draw the corresponding directed graph, then compute the maximum possible flow value and justify your answer using capacities into t.

4. A student proposes flow values that violate conservation at node c (inflow 6, outflow 4). Explain in one or two sentences why this is not feasible, and give one fix.

5. Real-model question: in a package-delivery network, what might capacities represent? Give two realistic factors that could change capacities during the day.

Session 2

Model assignment problems with bipartite graphs and matching LPs.

1. Let $U = \{S_1, S_2, S_3, S_4\}$ and $V = \{T_1, T_2, T_3, T_4\}$. Allowed edges are S_1–T_1, S_1–T_3, S_2–T_1, S_2–T_2, S_3–T_2, S_3–T_4, S_4–T_3. Draw the corresponding bipartite graph. Then find a maximum matching and state whether a perfect matching exists.

2. Write the matching LP for Problem 1: redraw (or reuse) the graph from Problem 1, define one variable per edge, write the objective, and write all "at most one" constraints for each vertex.

3. Give a real camp example that is naturally bipartite (for example, students and project topics, or mentors and time slots). Define the two sets and what an edge means, and draw the corresponding bipartite graph.

4. Explain why the edge set $\{S_1$–$T_1,\ S_2$–$T_1\}$ is not a matching, draw the small graph containing these two edges, and modify it to become a valid matching.