$\boxed{\text{MA~~..~~ 2101}}$ Analysis 1

Notation
$\mathbb{N}$ $\downarrow$ Natural no.s
$\mathbb{Z} \to$ Zahlen.
$\mathbb{Q} \to$ Rational No.s
$\mathbb{R} \to$ Reals
$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q}$
$\mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$

Algebra of the Real Number Systems

① Properties of Add^n (+) (Axioms)

$\forall x, y, z \in \mathbb{R}$

(A1) $x + y = y + x$ (commutative)

(A2) $(x+y)+z = x+(y+z)$

(A3) $\exists$ an unique element $0 \in \mathbb{R}$

s.t.

$x + 0 = x \quad \forall x \in \mathbb{R}$

→ can also be written for-each

(A4) $\forall x \in \mathbb{R}$, $\exists! y \in \mathbb{R}$ s.t. unique mean !.

$x + y = 0 = y + x$

We denote y by $(-x)$

$\boxed{\text{\# This works. But use foreach in Exam!!}}$

Properties of multiplication $(\cdot)$ (Axioms)

(M1) $x \cdot y = y \cdot x$

(M2) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$

(M3) ~~...~~ $\exists!\ 1 \in \mathbb{R}$

$x \cdot 1 = 1 \cdot x \quad \forall x \in \mathbb{R}$

(M4) $\forall x \in \mathbb{R} \setminus \{0\}$, $\exists!\ y \in \mathbb{R}$ s.t.

$x \cdot y = y \cdot x = 1$.

We denoted by $1/x$

Distributive Property

(A,M1) $x \cdot (y+z) = x \cdot y + x \cdot z \quad \forall x, y, z \in \mathbb{R}$

$\boxed{9 \rightarrow \text{field Axioms}}$ ($\mathbb{R}$ is a field)

Order Properties

Order in $\mathbb{R}$ (Trichotomy of Order)

Given two real numbers x, y then one and exactly one of the following is true:-

(i) $x < y$

(ii) $x = y$

(iii) $x > y$

Properties of Order '<' (Axioms)

(1) if $x < y$ and $y < z$ then $x < z$

(11) if $x > 0$ and $y > 0$ then $xy > 0$

'Provable' Properties of order.

(1) $x < y$ iff $x + z < y + z \quad \forall z \in \mathbb{R}$

(ii) $x < y$ then $-y < -x$

(iii) if $x < y$ and $z > 0$ then $xz < yz$

(iv) if $0 < x < y$ then $0 < \frac{1}{y} < \frac{1}{x}$

(v) $\forall x \in \mathbb{R}, x^2 \geq 0$

Schilts

Let x, y be two real no.s s.t. $x \geq y$ and $y \geq x$.

p.t. $x = y$

Pf Since There are 2 reals it follows trichotomy

(1) $x > y$

Thus $y \ngeq x$ but $y \geq x$ [contradiction]

(11) $x < y$

Thus $y \ngeq x$ but $x \geq y$ [contradiction]

Thus by trichotomy $x = y$.

Defn Upper Bound $\longrightarrow$

Let A be a non-empty subset of $\mathbb{R}$.

A number α is said to be an upper bound of A ($\alpha \in \mathbb{R}$)

if $\forall x \in A, x \leq \alpha$.

($\alpha \in A$ is not necessary)

~~Analysis~~ Assignment

Def^n

~~$\exists$~~ $A \neq \phi$, $A \subseteq \mathbb{R}$ is said bounded above if $\exists$ $\alpha \in \mathbb{R}$ s.t. α is a upper bound of A.

Let $A \neq \phi$, $A \subseteq \mathbb{R}$, let $\in \mathbb{R}$

How to prove α is not an upper bound of A.?

$\Rightarrow$ $\exists x \in A$ s.t. $\alpha < x$.

Def: Lower Bound $\rightarrow$

Let $A \neq \phi$, $A \subseteq \mathbb{R}$. A number $\alpha \in \mathbb{R}$ is said to be lower bound if

$\forall x \in A$, $x \geq \alpha$.

(Sim def^n for bounded below)

Upper and Lower bounds are not unique.

Let A be a bounded above set.

Let α be an upper bound of A

Let $\beta \in \mathbb{R}$ s.t. $\beta \geq \alpha$ $\Rightarrow \beta$ is an upper bound of A.

Pf Since α is an UB, then $x \in A$, $x \leq \alpha$

Again, $\alpha \leq \beta$ then

$\forall x \in A$, $x \leq \alpha \leq \beta$

$\Rightarrow \beta$ is an UB

Def: Maximum

Maximum $\rightarrow$

Let $A \neq \phi$, A number α is said to be the maximum of A if

1. $\alpha \in A$.
2. α is a upper bound of A.

Note $\rightarrow$ Maxima might not always exist. (Supremum might).

Q^n LUB is unique?

Thm

Let $A \neq \phi$. Let α, β be LUB of A Then $\alpha = \beta$. [sup(A) is unique]

Ans

(i) α, β are both UB.

Then

$\alpha \geq \beta$ as β is an LUB

similarly

$\alpha \leq \beta$ as α is an LUB.

$\alpha \geq \beta$ and $\alpha \leq \beta$.

$\Rightarrow \alpha = \beta$ [Proved]

Notation: $lub(A) = \alpha$

$\mathcal{A} = \{B / B \neq \phi, B \subseteq \mathbb{R}\}$, B is bounded above}

$B \in \mathcal{A}$

$B \longmapsto sup(B)$

$\mathcal{A} \longrightarrow \mathbb{R}$

$\mathbb{R} = \{lub(B), B \in \mathcal{A}\}$

$T = \{x / x \in \mathbb{R}, x^2 < 2\}$.

$\alpha = sup(T)$.

Thm Let A be a non empty bounded above.

Then $\alpha = lub(A)$ iff

(I) α is an ub of A.

(ii) Let $\beta < \alpha$, Then $\exists\, x_\beta \in A$ s.t. $\beta < x \leq \alpha$ depends on β.

$\beta \in \mathbb{R}$.

Defn (glb / inf)

Given a non-empty bounded below set A : A real no $\beta \in \mathbb{R}$ is said to be the greatest lower bound (glb / inf)

i) β is a lower bou of A.

(ii) If $\beta' > \beta$ then β' is not a lower bound of A.

(ii) $\Leftrightarrow$ $\forall \ \varepsilon > 0 \ \exists \ x \in A$.

$\beta + \varepsilon > x$.

Uniqueness of g inf

Let $A \neq \phi$, let inf(A), inf(A) be α and β then $\alpha = \beta$.

(i) Since $\alpha = \inf(A)$ and β is another upper bound.

$\alpha \geq \beta$

Since $\beta = \inf(A)$ and α is another upper bound

$\alpha \leq \beta$

$\alpha \geq \beta$ and $\alpha \leq \beta$

$\boxed{\Rightarrow \alpha = \beta.}$

Existence of ~~lubs~~ glb's

Let $A \subseteq \mathbb{R}$ be a non empty set

Define $-A = \{-x : x \in A\}$.

(i) A is bounded above
$\Updownarrow$
$-A$ is bound below

A is not bounded above
$\Updownarrow$
$-A$ is not bounded below

(ii) If A is bound above.
$lub(A) = glb(-A)$
$sup(A) = inf(-A)$.

Thm
Set is in $\mathbb{Z}, \mathbb{Q}$ is neither bounded above nor bounded below [Proof]

Defn
Let $A \subseteq \mathbb{R}$, $A \neq \phi$. A is said to be bounded if A is bounded above and bounded below.

Thm
Let $x \in \mathbb{R}$. Then $\exists$ a unique. $m \in \mathbb{Z}$ s.t. $m \leq x < m+1$

$[x, A)$

Let
$S = \{k \in \mathbb{Z} : k \leq x\}$

S is bounded above as x is an upper bounded from defi

Non - Emptiness

Suppose S is empty.
$\Rightarrow \forall k \in \mathbb{Z}, k > x$
$\Rightarrow x$ is lower bound for $\mathbb{Z}$.

let $m = lub(S)$

Claim $m \leq x < m+1$

Clearly $m \leq x$

Suppose ~~...~~ $x \geq m+1$
$\Rightarrow m+1 \in S$.
Thus $m+1 \leq = lub(S)$.

Claim n is <u>unique</u>

Suppose $m \neq n$.
WLOG ~~rather~~ $n > m$

we have
$m \leq x < m+1$
$n \leq x < n+1$

Since $n > m$, then $n \geq m+1$ as it is an ~~integer~~ integer

Now
$m \leq x < m+1 \leq n$.

And
$x < n$
$x \geq n$ [violation of trichotomy]

This gives a contradiction.

Greatest Integer func.

$f : \mathbb{R} \to \mathbb{Z}$

defined by
$f(x) := [x]$
$[x] \leq x < x+1$

Thm
$a, b \in \mathbb{R}$ and $a < b$. Then $\exists\, q \in \mathbb{Q}$ s.t.
$a < q < b$

Pf

Since $q \in \mathbb{Q} \Rightarrow q = \frac{m}{n}$ s.t. $m, n \in \mathbb{Z}$. $n \neq 0$

using AP property,

$\frac{1}{n_0} > b - a$. $n_0 < \frac{1}{b-a}$

$\Rightarrow$ $a > b - \frac{1}{n_0}$

$a < \frac{m}{n} < b$

$an < m < nb$

choose $m > n_0 a$ s.t. $m \in \mathbb{Z}$

$n_0 a \leq m \leq n_0 a + 1$

$m \leq n_0 a + 1$

$< n_0 a + n_0 (b-a) = n_0 b$

$m < n_0 b$

Thus $n_0 a < m < n_0 b$

Irrationals

A no $\alpha \in \mathbb{R}$, ~~...~~ is said to be rational if $\alpha \notin \mathbb{Q}$

$\frac{m}{n} + \sqrt{2} = \frac{p}{a}$

$\sqrt{2} = \frac{p}{a} - \frac{m}{n}$

Density of irrational no.s

A ~~...~~

Let $a, b \in \mathbb{R}$, $a < b$, Then $\exists\, r \in \mathbb{Q}$ s.t.

$\therefore a < r < b$

By the density of rationals we know, $\exists\, q \in \mathbb{Q}$ s.t. between $a - \sqrt{2}$ and $b - \sqrt{2}$

$a - \sqrt{2} < q < b - \sqrt{2}$

$a < q + \sqrt{2} < b$.

We claim $q + \sqrt{2}$ is irrational

Suppose $q + \sqrt{2}$ is rational, s.t.

$q + \sqrt{2} = r$.

where $q = \frac{k}{y}$, $r = \frac{m}{s}$

$\sqrt{2} = r - q$

$= \frac{ks - ym}{ys}$

Thus $\sqrt{2}$ is rational. contradiction $\square$

Thm

Let p be a prime no let x be a non negative real s.t

$x^2 = p$

Then x is irrational.

Pf Suppose x is ration.

s.t.

$x = \frac{m}{n}$ $m, n \in \mathbb{Z}$, $n \neq 0$. $\gcd(m,n) = 1$

$x^2 = \frac{m^2}{n^2}$

$pn^2 = m^2$

Thus $p \mid m^2$

$m^2 \equiv 0 \pmod p$ since p is prime

$m \equiv 0 \pmod p$.

Thus $p \mid m$

$\Rightarrow m = kp$

$k^2 p^2 = pn^2$

$\Rightarrow p \mid n$ Similarly.

Thus $p \mid m$ and $p \mid n$ Thus $\gcd(m,n) \neq 1$.

Thus contradiction.

Given a ~~contabilityq~~ infinite set $A \subseteq \mathbb{R}$ st.

$$A = \{x \in A \mid x_i \cancel{\ } \}$$

can it be ordered like

$$x_1 < x_2 < x_3 \ldots$$

Thm (Existence of n^{th} root of positive no).

Let $n \in \mathbb{N}$. let $\alpha \geq 0$. Then $\exists$ a unique $x \geq 0$,

s.t. $x^n = \alpha$.

Pf

let $x > 0$

consider the set

$$S = \{y \geq 0 : y^n < \alpha\}$$

CLAIM

S is nonempty and bounded above.

Clearly, $S \neq \Phi$ and $0 \in S$

Assume ~~S is not bounded above~~.

~~let m=α+1~~

~~Since S is not bounded above.~~

~~If~~ Then $\alpha + 1$ is the upper bound.

$$y^n < \alpha < \alpha + 1$$

∴ Then assume $\alpha + 1$ is not the upper bound.

then $\exists\, x \in S$ s.t. $x^n > \alpha + 1$

$$x^n > \alpha + 1 \qquad \alpha > x^n > \alpha + 1$$

$$\Rightarrow \alpha > \alpha + 1 \quad \text{[Contradiction]}$$

CLAIM

$x^n \not> \alpha$

If $x^n > \alpha$ the $\exists\ u \in \mathbb{N}$

$(x - \frac{1}{u})^n > \alpha.$

$(x-\frac{1}{u})^n = x^n + \Sigma(-1)^n \binom{n}{j}(x)^{n-j} \frac{1}{u^j}$

$< x^n + \frac{1}{u} \underbrace{\Sigma(-1)^n \binom{n}{j} x^{n-j}}_{c}$

$x^n + \frac{c}{u}.$

Let $c := \Sigma(-1)^n \binom{n}{j} x^{n-j}$

Then $\exists\ m \in \mathbb{N}$ s.t.

$m > \frac{c}{\alpha - x^n}.$

$\Rightarrow (x - \frac{1}{m})^n > \alpha$

Thus $x - \frac{1}{m} < x$. Thus x is not the least upper bound. [contradiction]

Uniqueness : Suppose $\exists\ x, y \geq 0$ s.t.

$x^n = \alpha$, $y^n = \alpha$, Then we need to show $x = y$

CLAIM

$x \not> y$ and $x \not< y$

Then

if $x > y \Rightarrow x^n > y^n$

Thus contradiction for both

By trichotomy $\underline{x = y}$

Define $x \in \mathbb{R}$.

$|x| = \max\{-x, x\}$

$$= \begin{cases} x & \text{if } x > 0 \\ 0 & x = 0 \\ -x & \text{if } x < 0 \end{cases}$$

Propositions

(i) $|x| = 0 \iff x = 0$

(ii) $|x| \geq 0, \ \forall x \in \mathbb{R}$

(iii) $|cx| = |c|\,|x|, \ c \in \mathbb{R}, x \in \mathbb{R}$

(iv) $|x+y| \leq |x| + |y|$ (Δ ineq)

(v) $\big||x| - |y|\big| \leq |x-y| \quad \forall x, y \in \mathbb{R}.$

Defn

A non empty $A \subseteq \mathbb{R}$ is said to be bounded above and bdd below.

Thm

Let $A \subseteq \mathbb{R}$, $A \neq \phi$. Then A is bdd iff $\exists$ $M > 0$

s.t. $-M \leq x \leq M \quad \forall x \in A.$ — (*)

$(\iff |x| \leq M \ \ \forall x \in A)$

Pf ($\Leftarrow$) If (*) holds true for some $M > 0$.

Then clearly, M is a ub of A and $-M$ is a bound of A. Then A is bdd.

$(\Rightarrow)$ Let A is bdd. Then A is bdd. Then A is bdd. Let α be the upper bound and β be the lower bound.

Aim is to find $M > 0$ s.t. [we need to show the existence of M]

$$[\beta, \alpha] \subseteq [-M, M]$$

Define $M = \max\{|\alpha|, |\beta|\}$

$\alpha \leq M$ and $-\beta \leq M$ $[M \geq |\beta| \geq -\beta$

$\Rightarrow \beta \geq -M$

This implies $A \subseteq [\beta, \alpha] \subseteq [-M, M]$

Cantor Intersection Thm

Let $J_n = [a_n, b_n]$, $n \in \mathbb{N}$ be, $a, b \in \mathbb{R}$ s.t.

$$J_{n+1} \subseteq J_n \quad \forall n \in \mathbb{N}$$

Then $\bigcap_{n=1}^{\infty} J_n \neq \phi$

Pf. Let $A = \bigcup_{n \in \mathbb{N}} \{a_n\}$

$A = \{a_n : n \in \mathbb{N}\}$

Let $x = \sup(A)$ (A is bounded above by b_1)

Since $x = \sup(A)$, then $x \geq a_n \; \forall n \in \mathbb{N}$

Since $b_n \geq b_{n+1}$ and $b_j \geq a_j$ $\quad 3^{n^n} > 3^n$

and b_j is ~~any~~ any upper bound of a_n,

then $a_n \leq x \leq b_j$

$x \in J_n \; \forall n \in \mathbb{N}$

Thus $\bigcap_{n=1}^{\infty} J_n \neq \phi$ as $\forall i \in I$, $x \in J_i$.

SEQUENCES (of Real No's)

Def^n

A seq^n is a map $f: \mathbb{N} \to \mathbb{R}$

* Suppose $f: \mathbb{N} \to \mathbb{R}$

Image set $= \{f(1), f(2), \ldots\}$

$= \{f(n), n \in \mathbb{N}\}$

$f(n) = f_n$

$\{x_n\}$ will be called a sequence (in $\mathbb{R}$)

where $x_n \in \mathbb{R}$ $\forall n \in \mathbb{N}$

Eg: $\{x_n\}$, $x_n = 1$

Aim

* convergence / Existence of limit.
* Boundedness.
* Algebra of limits
* cauchy sequences
* subsequences.

Def^n [convergence of a seq]

Let $\{x_n\}$ be a real sequence and $l \in \mathbb{R}$.

We say that "$\{x_n\}$ converges to l as n tends to infinity" or "$\lim_{n\to\infty} x_n = l$"

$\forall \varepsilon > 0 \; \exists K_\varepsilon \in \mathbb{N}$ s.t.

$|x_n - l| < \varepsilon \quad \forall n \geq K_\varepsilon$

$\Rightarrow x_n \in V_\varepsilon(l) \quad, \forall n \geq K_\varepsilon$

$\Leftrightarrow x_n \in (l-\varepsilon, l+\varepsilon) \quad, \forall n \geq K_\varepsilon$

Def^n [Tail of a Sequence]

Let $K \in \mathbb{N}$

$\{x_K, x_{K+1}, \ldots\}$ is the tail of the given sequence $\{x_n\}$.

Def^n [convergent seq^n]

A seq^n $\{x\}$ is said to convergent if $\exists l \in \mathbb{R}$ s.t. $\lim_{n\to\infty} x_n = l$

Thm (Limit is defined by tail).

Let $\lim_{n\to\infty} x_n = l$. Let $K \in \mathbb{N}$. Let $k \in \mathbb{N}$

Let $\{y_n\}$ be another seq^n such that

$y_n = x_n$

Then $\{y_n\}$ is convergent and

$\lim_{n\to\infty} y_n = l$.

Def

If a sequence is non convergent, then it is called divergent.

* $l \in \mathbb{R}$, $\exists\ \varepsilon > 0$ s.t. $\forall\ N_\varepsilon \in \mathbb{N}$

$\exists\ n > N_\varepsilon$ s.t.

$$|x_n - l| \geq \varepsilon.$$

Thm Uniqueness of Limit

Let $\lim_{n\to\infty} x_n = l$ and $\lim_{n\to\infty} x_n = m$

Then $m = l$

Pf: Let $l \neq m$

Take $0 < \varepsilon < \frac{|l-m|}{2}$ $K = \max\{M, N\}$

$\therefore$ $V_\varepsilon(l) \cap V_\varepsilon(m) = \phi$

Since $\lim_{n\to\infty} x_n = l$ $\exists\ N_\varepsilon \in \mathbb{N}$

$x_n \in V_\varepsilon(l)$

Since $\lim_{n\to\infty} x_n = m$ $\exists\ M_\varepsilon \in \mathbb{N}$

$x_n \in V_\varepsilon(m)$

Define $K = \max\{M_\varepsilon, N_\varepsilon\}$

Then

$x_K \in V_\varepsilon(l)$ and $x_K \in V_\varepsilon(m)$

Thus

$V_\varepsilon(l) \cap V_\varepsilon(m) \neq \phi$

[Contradiction] (Thus $m = l$)

Thm

Let $\{x_n\}$ be a convergent sequence.

Then $\{x_n\}$ is bdd. [Refer to Anirban].

Algebra of limits

1. Let $\{x_n\}$ and $\{y_n\}$ be convergent then

$$\lim_{n\to\infty}\{x_n + y_n\} = \lim_{n\to\infty}\{x_n\} + \lim_{n\to\infty}\{y_n\}$$

2. Let $\{x_n\}$ be convergent. $c \in \mathbb{R}$. Then

$\{cx_n\}$ is cgt. and $\lim_{n\to\infty}(cx_n) = c \lim_{x\to\infty} x_n$

3. Let $\{x_n\}$ and $\{y_n\}$ be ~~convergent~~ cgt.

Then $\{x_n y_n\}$ is cgt.

and

$$\lim_{n\to\infty}(x_n y_n) = \lim_{n\to\infty}(x_n) \lim_{n\to\infty}(y_n)$$

4. Let $\{x_n\}$ be cgt. with

$x_n \neq 0$ and $\lim_{n\to\infty} x_n \neq 0$

$$\lim_{n\to\infty}\left(\frac{1}{x_n}\right) = \frac{1}{\lim_{n\to\infty}(x_n)}$$

Sandwich theorem

Let $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ be sequences in $\mathbb{R}$.

$\forall\, k \in \mathbb{N},\ n \geq K$

$\{x_n \leq y_n \leq z_n$

and

$\lim\limits_{n\to\infty} (x_n) = \lim\limits_{n\to\infty} (z_n)$

Thus

$\lim\limits_{n\to\infty} (x_n) = \lim\limits_{n\to\infty} (z_n) = \lim\limits_{n\to\infty} (y_n)$

Thm

Let $\{x_n\}$ be convergent and $\lim\limits_{x\to\infty} x_n = x$

Then $\{|x_n|\}$ be convergent and

$\lim\limits_{n\to\infty} |x_n| = |x|$ (Prove later)

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converse is true only when $|x_n| = x_n\ \forall n$ or. $x_n \geq 0\ \forall n$. or $|x| = 0$

Cantor's Intersection Thm (NIP)

Let $\{J_n\}$ be a sequence of closed intervals s.t.

$J_{n+1} \subseteq J_n \quad \forall n.$

Then

(i) $\bigcap\limits_{n=1}^{\infty} J_n \neq \{\phi\}$. $|J_n| \to$ length of interval

(ii) If $|J_n| \to \infty$ contains exactly one. $J_n \to b_n - a_n$

Pf (ii)

Let x, y be two pts in $\bigcap\limits_{n=1}^{\infty} J_n$

CLAIM

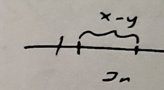
$x = y$.

Pf

$x, y \in \bigcap\limits_{n=1}^{\infty} J_n$

$x, y \in J_n \quad \forall n$

$|x-y| < |J_n| \quad \forall n$



$0 \leq |x-y| \leq |J_n| \quad \forall n.$

Use Sandwich theorem

$\lim\limits_{n\to\infty} 0 = 0.$

$\lim\limits_{n\to\infty} (|J_n|) = 0$ [by func defⁿ]

$\Rightarrow \lim\limits_{n\to\infty} |x-y| = 0$

Since $(x-y)$ is a constant sequence

$\lim\limits_{n\to\infty} (a) = a \Rightarrow \lim\limits_{n\to\infty} (x-y) = (x-y) = 0$

$\Rightarrow x = y$

Thus we can show that it is a singleton set. We do not need to show that ≥ 2 exist contradiction. As if we show 2. and thus recursively prove all.

Cauchy Sequence

Def^n

A sequence $\{x_n\}$ is said to be cauchy if $\forall \varepsilon > 0$, $\exists N_\varepsilon \in \mathbb{N}$ s.t.

$$|x_n - x_m| < \varepsilon \quad \forall\, n, m \geq N_\varepsilon$$

$$\left(|x_n - x_{n+p}| < \varepsilon \quad \forall\, n \geq N_\varepsilon, \forall p \in \mathbb{N}\right)$$

Thm

$\{x_n\}$ is convergent $\iff$ $\{x_n\}$ is cauchy sequence

Pf ($\Rightarrow$)

Let $x_n \Rightarrow l$ as $x_n \to \infty$

$\forall\, \varepsilon/2 > 0 \quad \exists N_\varepsilon \in \mathbb{N}$ s.t.

$$|x_n - l| < \varepsilon/2, \quad \forall\, n \geq N_\varepsilon$$

Then $\forall\, n, m \geq N_\varepsilon$

$$|x_n - l| < \varepsilon/2$$
$$|x_m - l| < \varepsilon/2$$

$$|x_n - x_m| = |x_n - l - (x_m - l)|$$

$$\leq |x_n - l| + |x_n - l| < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

$\Rightarrow |x_n - x_m| < \varepsilon$

($\Leftarrow$) (#Not in proof $\lim_{n\to\infty} x_n = l$)

$E = \{x \mid \exists N \text{ s.t. } x_n > x \ \forall n \geq N\}$

CLAIM

($\text{lub}(E) = l$) where $\lim_{n\to\infty} x_n = l$.

Let $\{x_n\}$ be cauchy

Let $E := \{x \in \mathbb{R} : \exists N \text{ s.t. } x_n > x \ \forall\, n \geq N\}$

CLAIM

E is non empty and bounded above.

Pf

$\forall \varepsilon > 0$. Since $\{x_n\}$ is cauchy. $\exists N_\varepsilon \in \mathbb{N}$. s.t.

$$|x_n - x_m| < \varepsilon \quad \forall\, n, m \geq N_\varepsilon$$

Take $m = N_\varepsilon$.

$$|x_n - x_{N_\varepsilon}| < \varepsilon \quad \forall\, n \geq N_\varepsilon$$

$\Rightarrow x_n \in (x_{N_\varepsilon} - \varepsilon, x_{N_\varepsilon} + \varepsilon)$

$\Rightarrow x_n > x_{N_\varepsilon} - \varepsilon \quad \forall\, n \geq N_\varepsilon$.

Thus $(x_{N_\varepsilon} - \varepsilon) \in E$

Thus E is non empty.

~~$x_n < x_{N_\varepsilon} + \varepsilon$~~

~~Thus x_n is bounded above~~

CLAIM

$x_{N_\varepsilon} + \varepsilon$ is an ub of E

we do this because this is an upper bound for the whole sequence for $\forall n \in \mathbb{N}$, $\boxed{n \geq N}$

Suppose NOT, then $\exists\, x_n \in (x_{N_\varepsilon} - \varepsilon, x_{N_\varepsilon} + \varepsilon)$

$\Rightarrow$ s.t. $x_{N_\varepsilon} + \varepsilon < x$ (*)

Since $x \in E$,
$\exists N \in \mathbb{N}$ s.t.

$x_n > x$

Then

$x_{N_\varepsilon} + \varepsilon < x < x_n \quad \forall n \geq N$ [contradicts $\forall n \geq \max\{N, N_\varepsilon\}$]

Then it contradicts (*).

Then, E is non-empty bounded above set

Let $l = \text{lub}(E)$

CLAIM

$\lim_{n \to \infty} x_n = l$.

Pf $\forall \varepsilon/2 > 0$, since $\{x_n\}$ is cauchy,
$\exists N_\varepsilon \in \mathbb{N}$ s.t.
$|x_n - x_m| < \varepsilon \quad \forall n, m \geq N_\varepsilon \in \mathbb{N}$

we know

$x_{N_\varepsilon} - \varepsilon/2 \in E$, $x_{N_\varepsilon} + \varepsilon/2 \in$ an ub of A.

$\Rightarrow x_{N_\varepsilon} - \varepsilon/2 \leq l \quad [l = \sup(E)]$

Also $\Rightarrow l \leq x_{N_\varepsilon} + \varepsilon/2 \quad$ [As $x_{N_\varepsilon} + \varepsilon/2$ any ub of A]

$x_{N_\varepsilon} - \varepsilon/2 \leq l \leq x_{N_\varepsilon} + \varepsilon/2 \Rightarrow |x_{N_\varepsilon} - l| \leq \varepsilon/2$

Since $\forall\, n, m \geq N_\varepsilon$

$\Rightarrow |x_n - x_{N_\varepsilon}| < \varepsilon/2$ ← From cauchy defn.

again

$|x_n - l| \leq |x_n - x_{N_\varepsilon}| + |x_{N_\varepsilon} - l| < \varepsilon/2 + \varepsilon/2$
$< \varepsilon$

Thus we have
$\forall \varepsilon > 0$
$\exists N_\varepsilon \in \mathbb{N}$ s.t.
$|x_n - l| < \varepsilon \quad \forall n \geq N_\varepsilon$

$\Rightarrow \lim_{n \to \infty} \{x_n\} = x$

Let $\{x_n\}$ be a Cauchy seqn. Then $\{x_n\}$ is convergent in $\mathbb{R}$.

Let $\{x_n\}$ be cauchy. Then $\{x_n\}$ is bounded

Doesn't hold for $\{x_n\}$ in $\mathbb{Q}$ as Completeness axiom doesn't hold.

Metric Space

Let X be a non-empty A metric.

$d: X \times X \longrightarrow [0, \infty)$
$(x, y) \longrightarrow d(x, y)$.

s.t.

(a) $d(x, y) = 0 \iff x = y$
(b) $d(x, y) = d(y, x) \quad \forall x, y \in X$
(c) $d(x, y) \leq d(x, z) + d(z, y)$

Eg:- $X = \mathbb{R}$, $d(x, y) = |x - y|$

Completeness

A set $A \subseteq \mathbb{R}$ is complete if $\forall$ Cauchy seq^n $\{x_n\}$, $x_n \in A$,

$\lim x_n \in A$.

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Monotone Sequence

Def^n

A sequence $\{x_n\}$ is said to be "increasing" if

$x_n \leq x_{n+1} \quad \forall n \in \mathbb{N}$

Def^n

A sequence $\{x_n\}$ is said to be "decreasing" if $x_{n+1} \leq x_n \quad \forall n \in \mathbb{N}$

Def^n A seq^n $\{x_n\}$ is said to be monotone if it's either increasing or decreasing

---

Thm Let $\{x_n\}$ be an increasing sequence and $\{x_n\}$ be bdd above. Then $\{x_n\}$ is convergent and

$$\lim_{n \to \infty} \{x_n\} = \sup(\{x_n : n \in \mathbb{N}\})$$

Thm Let $\{x_n\}$ be a decreasing sequence and $\{x_n\}$ is bdd below. Then $\{x_n\}$ is convergent and

$$\lim_{n \to \infty} \{x_n\} = \inf(\{x_n : n \in \mathbb{N}\})$$

Pf Since $\{x_n\}$ is bdd above, by completeness axioms we have sup $A := \{x_n : n \in \mathbb{N}\}$

Let $l = \sup\{x_n : n \in \mathbb{N}\} =: \sup(A)$

CLAIM

$\lim_{n\to\infty} \{x_n\} = l$

Let $\varepsilon > 0$ sin $l = \sup(A)$

$\exists N \in \mathbb{N}$

s.t.

$l - \varepsilon < x_N$

As $l = \sup(A)$

we have $x_n \le l.$ $\forall n \in \mathbb{N}$

$x_n \le l < l + \varepsilon$

Thus $\forall n \ge N$ we have

$l - \varepsilon < x_n < l + \varepsilon$

$-\varepsilon < x_n - l < l + \varepsilon$

$\Rightarrow |x_n - l| < \varepsilon$ [proved]

Thus we have proved $\{x_n\}$ is convergent to $l = \sup(\{x_n : n \in \mathbb{N}\})$.

Eg Let $r > 0$ let $x_n = r^n$

Case-1 $\therefore 0 < r < 1$

$\frac{x_{n+1}}{x_n} < 1$ as $\frac{x_{n+1}}{x_n} = \frac{r^{n+1}}{r^n} = r < 1$

Thus x_n is decreasing.

$\therefore \{x_n\}$ is decreasing. Since $x_n \ge 0$,

$\therefore \{x_n\}$ is bdd below.

Let $\lim(x_n) = l$.

Note:-

$x_{n+1} = r x_n$ Taking limit on both sides.

$\lim(x_{n+1}) = r \lim(x_n)$

$\Rightarrow l = rl.$

$\Rightarrow (r-1)l = 0$ we know $r \ne 1$

$l = 0$

(ii) $r = 1$ [constant]

(iii) $r > 1$ Divergent as unbounded seq.

Let $0 < r < 1$. Define.

$S_n = 1 + r + \dots r^n$

Note $S_n \le S_{n+1}$ $\forall n \in \mathbb{N}$ (increasing)

we know $0 < r < 1$

then we know r^p $\forall p \in \mathbb{N}$ is convergent.

Then $\forall \varepsilon > 0$ $\exists N_\varepsilon \in \mathbb{N}$ s.t.

$|r^{m-p}| < \varepsilon$.

wlog let $m > p$.

$|x_{m+p} - x_m| < \varepsilon$

$\underbrace{|r^{m+p} + \dots + r^{m+1}|}_{\text{all positive term}}$

$\Rightarrow r^{m+p} + \dots + r^{m+1}$ prove each one known

$$r^{m+p} + \dots + r^{m+1} < \frac{\varepsilon}{m-p} + \frac{\varepsilon}{m-p} \dots \frac{\varepsilon}{m-p}$$

$$= \varepsilon$$

Thus
x_m is Cauchy. Thus S_n is convergent

$$\lim_{n \to \infty} \left(1+\frac{1}{n}\right)^n = e.$$

Let $x_n = \left(1+\frac{1}{n}\right)^n \quad n \in \mathbb{N}$

Then

(i) $\{x_n\}$ is increasing

(ii) $2 \le x_n \le 3$

~~taking limit both side~~:

~~$t = p/q$~~

(iii) e is irrational ...

Subsequence

Let $\{x_n\}$ be a given sequence. Let $\{r_n\}$ be seq. of natural no st

$$r_1 < r_2 < r_3 < \dots$$

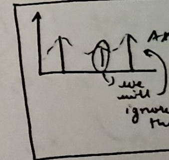
Then $\{x_{r_n}\}_{n \in \mathbb{N}}$ is called a subsequence

Thm Let $\{x_n\}$ be a given sequence. Then $\{x_n\}$ has a ~~montone~~ monotone sequence.

Pf Define the set

$$E := \{ n \in \mathbb{N} \mid x_n \ge x_k \quad \forall k > n \}$$

(x_n will be called a peak if $x_n \ge k$ $\forall k \ge n$)



Case-1 E is finite

Then $\exists N \in \mathbb{N}$ s.t.

$$\forall n \in E \Rightarrow n < N$$

let $n_1 = N$

Claim

$\exists n_2 \in \mathbb{N}$ s.t.

if $n_2 > n_1$ s.t.

$$x_{n_2} \ge x_{n_1}$$

contradiction

If it is not, $\forall n > N$

$$x_n \le x_N$$

$\Rightarrow N \in E$ (contradiction)
contradicts finiteness.

Thus proved

given x_{n_2}, $\exists n_2 \in \mathbb{N}$ s.t.

$n_2 > n_1$ and

$$x_{n_2} > x_{n_1}$$

Similarly for given x_{n_2} we can find n_3 s.t.

$n_3 > n_2$ and

$x_{n_3} > x_{n_2}$

Proceeding this way, we obtain a seq $\{n_r\}$ s.t.

$n_1 < n_2 < n_3 < n_4 < \dots$

and $x_{n_1} < x_{n_2} < x_{n_3} \dots$

This means that $\{x_{n_r}\}$ is an infinite subsequence of E.

Case-2 (E is infinite)

Then $\exists \{n_k\}_{k \in \mathbb{N}}$ s.t.

$n_1 < n_2 < n_3 < \dots$

where $n_k \in E \ \forall k \in \mathbb{N}$

Since $n_1 \in E$, $n_2 > n_1$

then $x_{n_1} \geq x_{n_2}$

Again $n_2 \in E$ and $n_3 > n_2$

$x_{n_2} \geq x_{n_3}$

Similarly $n_k \in E$, $n_{k+1} > n_k$

$x_{n_k} \geq x_{n_{k+1}}$

Therefore $\{x_{n_k}\}$ is a decreasing subsequence.

Thm [Bolzano Weierstrass Theorem]

Let $\{x_n\}$ be a bounded sequence. Then $\{x_n\}$ has a convergent subsequence.

Pf From previous Thm

Any sequence has a monotonic subsequence, And $\{x_n\}$ is bounded. Since $\{x_n\}$ is bounded, the subsequence $\{x_{n_k}\}$(suppose) is bounded.

$\therefore \{x_{n_k}\}$ is a monotonic sequence and bounded just it is convergent.

Consider

Let $x_n = \left(1+\frac{1}{n}\right)^n \longrightarrow (1)$.

(i) $\{x_n\}$ is increasing.

(ii) $\{x_n\}$ is bounded above.

$$x_n = \left(1+\frac{1}{n}\right)^n = \sum_{k=0}^{n} \binom{n}{k} \frac{1}{n^k}$$

$$= 1 + \sum_{k=1}^{n} \binom{n}{k} \frac{1}{n^k}$$

$$= 1 + \sum_{k=1}^{n} \frac{n!}{k!(n-k)!} \frac{1}{n^k}$$

$$= 1 + \sum_{k=1}^{n} \frac{1}{k!} \frac{(n-k+1)(n-k+2)\dots n}{n^k}$$

$$= 1 + \sum_{k=1}^{n} \frac{1}{k!} \underbrace{\left(1-\frac{k-1}{n}\right)\left(1-\frac{k-2}{n}\right)\dots\left(1-\frac{1}{n}\right)}_{\text{not }k}$$

$$x_{n+1} = \left(1+\frac{1}{n+1}\right)^{n+1}$$

$$= 1 + \sum_{k=1}^{n} \frac{(n+1)!}{k!\,(n-k+1)!} + \frac{1}{(n+1)^{n+1}}$$

$$= 1 + \sum_{k=1}^{n} \frac{1}{k!} \underbrace{\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right)\cdots\left(1-\frac{n-1}{n+1}\right)}_{B} + \frac{1}{(n+1)^{n+1}}$$

$$\left(1-\frac{1}{n}\right) \le \left(1-\frac{2}{n}\right)$$

$\vdots$

AMGM Proof

~~To show~~ $$\left(1+\frac{1}{n}\right)^{n} \le \left(1+\frac{1}{n+1}\right)^{n+1}$$

$$\left(\frac{n+1}{n}\right)^{n} \le \left(\frac{n+2}{n+1}\right)^{n+1}$$

$$\Rightarrow \left(\frac{n+2}{n+1}\right) \geq \left(\left(\frac{n+1}{n}\right)\right)^{\frac{n}{n+1}}$$

Let (n) elements to be $\frac{n+1}{n}$ and last element be 1

$$AM = \frac{\frac{n+1}{n} + \cdots + \frac{n+1}{n} + 1}{n+1}$$

$$= \frac{n+2}{n+1}$$

$$GM = \left(\frac{n+1}{n}\right)^{\frac{n}{n+1}}$$

By AM GM inequality

$$\frac{n+2}{n+1} \geq \left(\frac{n+1}{n}\right)^{\frac{n}{n+1}}$$

$$\Rightarrow \left(\frac{n+2}{n+1}\right)^{n+1} \geq \left(\frac{n+1}{n}\right)^{n}$$

$$\Rightarrow \left(1+\frac{1}{n+1}\right)^{n+1} \geq \left(1+\frac{1}{n}\right)^{n}$$

$$\Rightarrow x_{n+1} \geq x_n$$

Thm [Equiv of taylor expansion of 'e']

$$e = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{1}{k!}$$

Pf: Let $y_n := \sum_{k=0}^{n} \frac{1}{k!}$

Note $\{y_n\}$ is an increasing sequence and [bounded] above by 3.

Cauchy First Thm

Let $\{x_n\}$ be a sequence s.t. $x_n \to x$ as $n \to \infty$.

Then
$$\lim \left(\frac{x_1 + \dots + x_n}{n} \right) = x$$

$\boxed{Pf}$ Let $y_n := x_n - x$

Now
$$\frac{x_1 + \dots + x_n}{n} = \frac{y_1 + \dots + y_n}{n} + x$$

CLAIM

$$\lim \left(\frac{y_1 + \dots + y_n}{n} \right) = 0$$

Let $\varepsilon' > 0$, we know $y_n \to 0$.

Thus $\forall n \geq N_{\varepsilon'}$ where $N_{\varepsilon'} \in \mathbb{N}$,

$$|y_n| < \varepsilon'$$

$$\left| \frac{y_1 + \dots + y_n}{n} \right| = \left| \frac{y_1 + \dots + y_{N-1}}{n} + \frac{y_N + \dots + y_n}{n} \right|$$

Since y_n is bounded, $\exists M \in \mathbb{R}^+$ s.t. $|y_n| \leq M \; \forall n$

$$\leq \frac{|y_1| + \dots + |y_{N-1}|}{n} + \frac{|y_N| + \dots + |y_n|}{n}$$

$$\leq \frac{(N-1) M}{n} + \varepsilon' \frac{(n - N + 1)}{n}$$

$$\left|\frac{y_1 + \dots + y_n}{n}\right| \leq \frac{(N-1)M}{n} + \left(1 - \frac{N+1}{n}\right)\varepsilon'$$

$\forall\, n > N+1 \quad \leq \frac{(N-1)}{n} M + \varepsilon'$

$\Rightarrow n \geq N+1$

$\Rightarrow 1 \geq \frac{N+1}{n}$

we can choose n_0 sufficiently large enough s.t.

$$\frac{(N-1)}{n_0} M < \varepsilon'$$

Thus we have

$$\left|\frac{y_1 + \dots + y_n}{n}\right| \leq \varepsilon' + \varepsilon' \quad \forall n > \max\{N, n_0\}$$

$$\leq \varepsilon$$

Thus $\lim \left(\frac{y_1 + \dots + y_n}{n}\right) = 0$

$\Rightarrow \lim \left(\frac{x_1 + \dots + x_n}{n} - x\right) = 0$

$\Rightarrow \lim \left(\frac{x_1 + \dots + x_n}{n}\right) = x$ [From Algebra of limits]

[Nested Interval Property]

Let $J_n := [a_n, b_n]$ be intervals in $\mathbb{R}$ s.t.

$J_{n+1} \subseteq J_n$. Let $|J_n| = |b_n - a_n|$

Then

(i) $\bigcap_n J_n \neq \phi$. (ii) If. $|J_n| \to 0$ then $\exists!\, c$ s.t. $\bigcap_n J_n = \{c\}$

$\boxed{\text{Pf}}$

(i) is already proven.

(ii) we already know $\exists c \in \bigcap_n J_n$.

Let $c, d \in \bigcap_n J_n$, $c \neq d$, WLOG. $c \leq d$.

Since $c, d \in \bigcap_n J_n$

$\forall\, a_n, b_n$

$b_n - a_n \geq d - c \geq 0$

Also we know $\lim (b_n - a_n) = \lim(|J_n|) = 0$ [Given]

By Sandwich theorem, we have

$\lim (d-c) = d-c = 0$

$\Rightarrow c = d$.

Thus $\bigcap_n J_n = c$

[Subsequential limit] $\underline{\text{Defn}}$

Let $\{x_n\}$ be a sequence. A no $l \in \mathbb{R}$ is said to be a subsequential limit of $\{x_n\}$ if $\exists$ a subsequence $\{x_{n_k}\}$ s.t.

$\{x_{n_k}\} \longrightarrow l$

[Random Ass Theorem]

Let $\{x_n\}$ be a seq^n s.t.

$|x_{n+1} - x_n| \leq a_n \quad \forall n \geq 1.$

and $\sum_n a_n$ is convergent. Then $\{x_n\}$ is a cauchy sequence [Trivial]

[Fixed pt Theorem]

Let $\{x_n\}$ be a seq^n s.t.

$|x_{n+1} - x_n| \leq c|x_n - x_{n-1}| \quad \forall n \geq 2$

for some $c \in (0,1)$. Then $\{x_n\}$ is Cauchy. [Easy proof]

$|x_{n+1} - x_n| \leq c|x_n - x_{n-1}|$

$\leq c^2 |x_{n-1} - x_{n-2}|$

$\leq c^{n-1} |x_2 - x_1|$

$\lim (c^{n-1}) = 0$ as $c \in (0,1)$

[Thm] [Uniqueness of fixed pt]

[Banach Fixed pt. Theorem]

Let $f: [0,1] \longrightarrow [0,1]$ be a f^{xn} s.t.

$|f(x) - f(y)| \leq c|x-y| \quad \forall x, y \in [0,1]$

for some $c \in (0,1)$. Then f has a unique fixed pt.

[A no $a \in [0,1]$ is said to be a fixed pt if $\boxed{f(a) = a}$]

$\underline{\text{Defn}}$ [Neighbourhood]

Let $l \in \mathbb{R}$. A neighbourhood of l is an open interval (a,b) s.t. $l \in (a,b)$

$\underline{\text{Defn}}$ [ε-Neighbourhood]

Let $\varepsilon > 0$. Let $l \in \mathbb{R}$. The open interval $(l-\varepsilon, l+\varepsilon)$ is called the ε-Neighbour of l.

$\underline{\text{Prop}^n}$ Let $l \in \mathbb{R}$. Let (a,b) be a nbd of l. Then $\exists \varepsilon > 0$ s.t. $(l-\varepsilon, l+\varepsilon) \subseteq (a,b)$

$\underline{\text{Define}}$ $0 < \varepsilon < \min\{|l-a|, |l-b|\}$

then

$(l-\varepsilon, l+\varepsilon) \subseteq (a,b)$

Thm

Let $\{x_n\}$ be a seq^n s.t. $x_n \to l$ as $n \to \infty$.

Let (a,b) be a nbd of l. Then

(i) (a,b) contain infinitely many elements of x_n

(ii) $(a,b)^c$ contains atmost finitely many elements.

Pf/ If l is a subsequential limit of $\{x_n\}$.

Then $\exists \{x_{n_k}\}$ s.t.

$$x_{n_k} \to l \quad \text{as} \quad k \to \infty$$

$\Rightarrow$ any nbd of l contains infinitely many elements.

(Converse)

Assumption

(a,b) contains infinitely many elements of x_n.

where $(a;b)$ is any nbd of l.

$\Rightarrow$ l is a subsequential limit of x_{n_k}.

Pf Let us consider $(l-1, l+1)$

by assumption $\exists n_1 \in \mathbb{N}$ s.t.

$$x_{n_1} \in (l-1, l+1)$$

$$\Leftrightarrow \quad |x_{n_1} - l| < 1$$

Consider $(l-1/2, l+1/2)$.

CLAIM

$\exists n_2 > n_1$; $x_{n_2} \in (l-1/2, l+1/2)$

If it is NOT true, then

$x_n \in (l-1/2, l+1/2) \quad \forall \; n \leq n_1$

$\Rightarrow$ $V_{1/2}(l)$ contains finitely many elements.

which is a contradiction.

Thus $\exists n_2 > n_1$ s.t.

$$x_{n_2} \in V_{1/2}(l)$$

We continue by taking $\varepsilon = 1/k$, we obtain a subsequence

$\{x_{n_k}\}$ s.t.

$|x_{n_k} - l| < 1/k$.

$\Rightarrow \lim_{k\to\infty} x_{n_k} = l$.

Thus l is a subsequential limit

Thm: [Bolzano-Weierstrass Theorem]

Let $\{x_n\}$ be a bounded seq^n. Then $\{x_n\}$ has a convergent subsequence (i.e. $\{x_n\}$ has a subsequential limit)

Pf

Since $\{x_n\}$ is bdd, $\exists a, b \in \mathbb{R}$ s.t.

$x_n \in [a,b] \; \forall n \in \mathbb{N}$

Let $J_1 = [a,b]$.

Now consider the interval $[a, \frac{a+b}{2}]$ and $[\frac{a+b}{2}, b]$.

Atleast $[a, \frac{a+b}{2}]$ or $[\frac{a+b}{2}, b]$ contains infinitely many elements of x_n [Because x_n contains infinitely many elements].

Let J_2 be the interval that contains infinitely many elements.

Now i) $J_2 \subseteq J_1$

ii) $|J_2| = \frac{1}{2}|J_1| = \frac{1}{2}(b-a)$.

Similarly bisect J_2 and get interval J_4.

we get

(i) $J_3 \subseteq J_2 \subseteq J_1$

(ii) $|J_3| = \frac{1}{2}|J_2| = \frac{1}{2^2}(b-a)$

Continuing this process we get J_n where

(i) $J_n \subseteq J_{n-1}$ $\forall n \in \mathbb{N}$

(ii) $|J_n| = \frac{1}{2}|J_{n-1}| = \frac{1}{2^{n-1}}(b-a)$

$$\lim_{n\to\infty} |J_n| = \lim_{n\to\infty} \frac{1}{2^{n-1}}(b-a) = (b-a)\lim_{n\to\infty}\frac{1}{2^{n-1}}$$

$$= 0$$

By Cantor's Intersection Theorem, we have

$\bigcap_n J_n = \{c\}$

CLAIM

c is a subsequential limit ~~of~~ of x_n

Pf

Every nbd of c contains infinitely many elements of x_n.

~~Let~~ we show $\forall \varepsilon > 0$, $V_\varepsilon(c)$ contains infinitely many elements of x_n.

Since $|J_n| \to 0$ as $n \to \infty$

$\exists N \in \mathbb{N}$

$$|J_N| < \frac{\varepsilon}{5}$$

Say $J_N = [p, q]$ $\Rightarrow x \in [p, q]$

$x \in [p, q]$

$|x - c| \leq |p - q| < \varepsilon/5$

$\Rightarrow |x - c| < \varepsilon/5 < \varepsilon$

$\Rightarrow x \in (c - \varepsilon, c + \varepsilon)$

Thus $[p, q] \subseteq V_\varepsilon(c)$

$\Rightarrow J_N \subseteq V_\varepsilon(c)$

Thus $V_\varepsilon(c)$ contains infinitely many elements of c since J_N contains infinitely many elements

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[Lim sup / Liminf] and [Infinite series] and [Point set Topology] after continuity

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Limits and Continuity of functions

Defn [ε-δ defn]

Let $D \subseteq \mathbb{R}$ and c be a limit pt of D.
Let $f : D \to \mathbb{R}$ be a given fxn. Then a no $l \in \mathbb{R}$ is said to be limit of the function at c if $\forall \varepsilon > 0 \quad \exists \delta > 0$ s.t.

$$|f(x) - l| < \varepsilon \quad \forall x \in D \cap \{x : \underbrace{0 < |x - c|}_{\text{This } > 0 \text{ is done to allow limit for non cont. fxns}} < \delta\}$$

If limit of 'f' at c exists and is l, we denote it by:

$$l := \lim_{x \to c} f(x)$$

Defn [limit is not 'l']

$\exists \varepsilon > 0$ s.t. $\forall \delta > 0 \quad \exists x \in D \cap \{x \mid 0 < |x - c| < \delta\}$

s.t. $|f(x) - l| \geq \varepsilon$

Thm [Sequential defn]

Let $D \subseteq \mathbb{R}$ and c be a limit pt of D.
Let $f : D \to \mathbb{R}$. Then $\lim_{x \to c} f(x) = l$ **iff** for every sequence $x_n \in D \setminus \{c\}$ converges to c, $f(x_n) \to l$.

Pf ($\Rightarrow$) $\lim_{x \to c} f(x) = l$

CLAIM

$\forall x_n \in D \setminus \{c\}$ where $x_n \to c$,

$f(x_n) \to l$.

Let $x_n \in D \setminus \{c\}$ and $x_n \to c$

we have $\lim_{x \to c} f(x) = l$. So for all $\varepsilon > 0$,

we have $\delta > 0$ and $\lim x_n = c$ $\exists N \in \mathbb{N}$

s.t. $0 < |x_n - c| < \delta \quad \forall n \geq N$

Thus $\forall x_n \in V_\delta(c) \setminus \{c\}$

$$|f(x_n) - l| < \varepsilon \quad \forall \quad x_n \in V_\delta(c) \setminus \{c\}$$

$$\Rightarrow |f(x_n) - l| < \varepsilon \quad \forall \quad n \geq N$$

Thus for any arbitrary $x_n \to c$ we have $f(x_n) \to l$.

(Converse)

if for every sequence $x_n \in D \setminus \{c\}$ converges to c, $f(x_n) \to l$. $(\Rightarrow)$ $\lim_{x \to c} (f(x)) = l$.

Suppose $\lim_{x \to c} f(x) \neq l$. Then $\exists \varepsilon > 0$ s.t. $\forall \delta > 0$

$$|f(x) - l| > \varepsilon \quad \forall x \in \{x \in D \mid 0 < |x - c| < \delta\}$$

~~Take δ = 1~~

Let $A_k = \{x \in D \mid 0 < |x-c| < \delta_k\}$

Take $\delta_1 = 1$ $\exists x_1 \in A_1$ s.t. $|f(x_1) - l| > \varepsilon$

$\delta_2 = 1/2$ $\exists x_2 \in A_2$ s.t. $|f(x_2) - l| > \varepsilon$

$\vdots$

$\delta_k = 1/k$ $\exists x_k \in A_k$ s.t. $|f(x_k) - l| > \varepsilon$

Thus $x_n \to c$ but

$f(x_n) \not\to l$. [contradiction]

$\boxed{\text{Thm}}$ limit of a fxn is unique.

Pf let l, m be limits of a fxn.

Thus we have $\forall \varepsilon_1 > 0$, $\exists \delta_1 > 0$ s.t.

$|f(x) - l| < \varepsilon_1$ $\forall x \in D \cap \{x \in \mid 0 < |x-c| \delta_1\}$

$\Rightarrow -\varepsilon_1 < f(x) - l < \varepsilon_1$

we ~~do~~ have $\forall \varepsilon_2 > 0$ $\exists \delta_2 > 0$ s.t.

$|f(x) - m| < \varepsilon_2$ $\forall x \in D \cap \{x \mid 0 < |x-c| < \delta_2\}$

~~$\Rightarrow \varepsilon_1 < f(x) - m < \varepsilon_2$~~

Let $\varepsilon_1 = \varepsilon_2 = \frac{\varepsilon}{2}$

~~$\Rightarrow f(x) < m + \varepsilon_2$~~

~~$f(x) < l + \varepsilon_1$~~

~~$\varepsilon_2 - \varepsilon_1 \to 0$~~

~~$\Rightarrow (l-m) < \varepsilon_2 - \varepsilon_1$~~

$|l - m| = |f(x) - m - (f(x) - l)|$

$\leqslant |f(x) - m| + |f(x) - l| < \varepsilon_1 + \varepsilon_2 = \varepsilon$

$\Rightarrow |l - m| < \varepsilon$ $\forall \varepsilon > 0$

Thus $l = m$ [Uniqueness of limits]

Defn [Right hand Limit]

Let $D \subseteq \mathbb{R}$ and c be a pt s.t. $(c, c+\alpha) \subseteq D$ for some $\alpha > 0$. A no $l \in \mathbb{R}$ is the right limit of f at c if

$\forall \varepsilon > 0$ $\exists \delta > 0$ s.t.

$|f(x) - l| < \varepsilon$ $\forall x \in D \cap (c, c+\delta)$

Defn [Left hand limit]

Let $D \subseteq \mathbb{R}$ and c be a pt s.t. $(c-\delta, c) \subseteq D$ for some $\delta > 0$. A no. $l \in \mathbb{R}$ is the left limit of f at c if

$\forall \varepsilon > 0$ $\exists \delta > 0$ s.t.

$|f(x) - l| < \varepsilon$ $\forall x \in D \cap (c-\delta, c)$

Now if left hand and right hand limit exist ~~but~~ and are the same for a pt, the limit exists.

$\lim_{x \to c^+} (f(x)) = l$ & $\lim_{x \to c^-} (f(x)) = l \Rightarrow \lim_{x \to c} f(x) = l$.

Thm
Let $D \subseteq \mathbb{R}$ and $(c-\delta, c+\delta) \setminus \{c\} \subseteq D$. Then

$$\lim_{x \to c} f(x) = l$$

iff $$\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = l$$

(Pf) $(\Rightarrow)$

$\lim_{x \to c^+} f(x) = l \Rightarrow \lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = l$

$\forall \varepsilon > 0 \quad \exists \delta > 0$ s.t.

$|f(x) - l| < \varepsilon \quad \forall x \in D \cap V_\delta(c) \setminus \{c\}$

- $(c-\delta, c) \subseteq V_\delta(c) \setminus \{c\}$

$\Rightarrow D \cap (c-\delta, c) \subseteq D \cap V_\delta(c) \setminus \{c\}$

$\Rightarrow |f(x) - l| < \varepsilon \quad \forall x \in D \cap V_\delta(c) \setminus \{c\}$.

$\Rightarrow |f(x) - l| < \varepsilon \quad \forall x \in D \cap (c-\delta, c)$

$\Rightarrow \lim_{x \to c^-} (f(x)) = l$.

Similarly

~~$(c, c+\delta)$~~ $(c, c+\delta) \subseteq V_\delta(c) \setminus \{c\}$

$\Rightarrow D \cap (c, c+\delta) \subseteq D \cap V_\delta(c) \setminus \{c\}$

$\Rightarrow |f(x) - l| < \varepsilon \quad \forall x \in D \cap (c, c+\delta)$

$\Rightarrow \lim_{x \to c^+} (f(x)) = l$.

$\Rightarrow \lim_{x \to c^+} f(x) = \lim_{x \to c^-} f(x) = l$.

$(\Leftarrow)$

$\lim_{x \to c^-} f(x) = l$.

$\forall \varepsilon_1 > 0, \exists \delta_1 > 0$ s.t.

$|f(x) - l| < \varepsilon_1 \quad \forall x \in D \cap (c-\delta_1, c)$

$\lim_{x \to c^+} f(x) = l$

$\forall \varepsilon_2 > 0 \quad \exists \delta_2 > 0$ s.t.

$|f(x) - l| < \varepsilon_2 \quad \forall x \in D \cap (c, c+\delta_2)$

Let us take $\delta = \min(\delta_1, \delta_2)$.
and let $\varepsilon_1 = \varepsilon_2 = \varepsilon$

Thus we have

$\forall \varepsilon > 0, \exists \delta > 0$ s.t.

$|f(x) - l| < \varepsilon \quad \forall x \in D \cap (c-\delta, c)$ and

$|f(x) - l| < \varepsilon \quad \forall x \in D \cap (c, c+\delta)$.

$\Rightarrow |f(x) - l| < \varepsilon \quad \forall x \in D \cap V_\delta(c) \setminus \{c\}$

Thus $\lim_{x \to c} f(x) = l$

<u>Algebra of limits</u>

<u>Thm</u>: Let $D \subseteq \mathbb{R}$ and c be a limit pt of D.
Let $f, g : D \to \mathbb{R}$. Assume that $\lim_{x \to c} f(x) = m$
$\lim_{x \to c} g(x) = l$.

Then

(i) $\lim_{x \to c} (f+g)(x)$ exists and

$\lim_{x \to c} (f+g)(x) = l+m$

(ii) $\lim_{x \to c} (fg(x))$ exists and $\lim_{x \to c} (fg(x)) = ml$

(iii) Assume that $\overbrace{g(x) \neq 0 \ \forall\ x \in D}^{\text{Not reqd I Think.}}$,

$\lim_{x \to c} g(x) \neq 0$

then $\lim_{x \to c} \left(\frac{f}{g}\right)(x) = \frac{m}{l}$

Thm [Monotone f^{x^n}]

Let $f:(a,b) \to \mathbb{R}$ be a monotonic increasing f^{x^n}. Then

(i) if $\{f(x) : x \in (a,b)\}$ is bdd. above

then, $\lim_{x \to b^-} f(x)$ exists and

$\lim_{x \to b^-} f(x) = \sup\{f(x) : x \in (a,b)\}$

(ii) if $\{f(x) : x \in (a,b)\}$ is bdd. below, then,

$\lim_{x \to a^+} f(x)$ exists and

$\lim_{x \to a^+} f(x) = \inf\{f(x) : x (a,b)\}$

Pf

(i) Let $\alpha = \sup\{f(x) : x \in (a,b)\}$

let $\varepsilon > 0$ since α is sup $\exists.\ x_0 \in (a,b)$

s.t. $\alpha - \varepsilon < f(x_0) \leq \alpha$

$\forall\ x \in (x_0, b)$

$\alpha-\varepsilon < f(x_0) \leq f(x) \leq \alpha$

$\Rightarrow \alpha - \varepsilon < f(x) \leq \alpha$

choose $\delta > 0$ s.t.

$x_0 < b-\delta < b$

$\Rightarrow \delta < b - x_0$

then $\forall\ x \in (b-\delta, b) \subseteq (x_0, b)$

$\alpha - \varepsilon < f(x) \leq \alpha < \alpha + \varepsilon$

$\Rightarrow \alpha - \varepsilon < f(x) < \alpha + \varepsilon$

$\Rightarrow |f(x) - \alpha| < \varepsilon$

Thus $\lim_{x \to b^-} f(x) = \sup\{f(x) \mid x \in (a,b)\}$

---

similarly for $\inf\{f(x) \mid x \in (a,b)\}$ we have.

$\beta + \varepsilon > f(x) \geq \beta > \beta - \varepsilon$

choose $\delta > 0$ s.t. $a < a+\delta < x_0$

$\Rightarrow |f(x) - \beta| < \varepsilon \quad \forall\ x \in D \cap (a, a+\delta)$

CONTINUOUS FUNCTIONS

Def^n [ε - δ def^n]

Let $f: D \to \mathbb{R}$ be a fx^n. Let $c \in D$. Then fx^n f is continous at c if

$\forall\ \varepsilon > 0 \quad \exists\ \delta > 0$ s.t.

$$|f(x) - f(c)| < \varepsilon \quad \forall\ x \in D \cap V_\delta(c)$$

Def^n [continuous in Domain]

Let $f: D \to \mathbb{R}$. then f is said to be continuous if f is continuous at every pt of D

Remark In the def^n of continuous at c ~~iff~~ where c is a limit pt of D. Then f is continuous at c iff

$$\lim_{x \to c} f(x) = f(c)$$

Def^n [Seq^n def^n]

Let $f: D \to \mathbb{R}$ be a fx^n. Let $c \in D$. Then f is continuous at c if for every seq^n $\{x_n\}$ in D, converging to c, then $f(x_n) \to f(c)$

> Every seq^n where $x_n \in D$
>
> $x_n \to c \implies f(x_n) \to f(c)$

Algebra of Limits

Let $f, g: D \to \mathbb{R}$. Let $c \in D$. Assume that f and g are continuous at c.

(i) $(f+g): D \to \mathbb{R}$ is continuous at c.

(ii) $\alpha \in \mathbb{R}$, then $\alpha f: D \to \mathbb{R}$ defined by

$$(\alpha f)(x) = \alpha f(x).$$

Then αf is cont. at c.

(iii) if $g(x) \neq 0$ and $g(c) \neq 0$

f/g is continuous

Continuous fx^n as a vector space

let $D \subseteq \mathbb{R}$ and $c \in D$.

$\mathcal{F} := \{f: D \to \mathbb{R} : f \text{ is a fx}^n\}$

Then $\mathcal{F}$ is a vector space.

$\mathcal{C} := \{f: D \to \mathbb{R} : f \text{ is continuous at } c\}$

Then $\mathcal{C}$ is a subspace of $\mathcal{F}$.

(iv) $(fg)x = f(x) \cdot g(x)$

fg is continuous at c.

(v) Define $|f|: D \to \mathbb{R}$ by $|f|(x) = |f(x)|$ $x \in D$.

Then $|f|$ is continuous at c.

(vi) Define

$\max\{f,g\} : D \to \mathbb{R}$ by

$$\max\{f,g\}(x) = \max\{f(x), g(x)\}$$

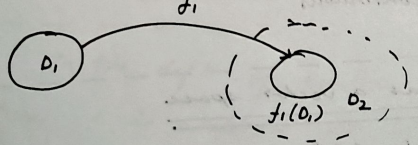
Similarly $\min\{f,g\} : D \to \mathbb{R}$ by

$$\min\{f,g\}(x) = \min\{f(x), g(x)\}$$

Then $\max\{f,g\}$ and $\min\{f,g\}$ are continuous at c.

– [$a_n \to a$, $b_n \to b$ then $\max\{a_n, b_n\} \to \max\{a, b\}$]

(vii) Let $f_1 : D_1 \to \mathbb{R}$, $g_1 : D_2 \to \mathbb{R}$. Assume $f_1(D_1) \subseteq D_2$



Then $g_1 \circ f_1 : D_1 \to \mathbb{R}$ is defined,

$$g_1 \circ f_1(x) = g_1(f_1(x)) \quad x \in D_1$$

Then $g_1 \circ f_1$ is continuous at c if f_1 is cont. at c and g_1 is cont at $f_1(c)$

Fourier Series Relation [Interesting result for fourier coeff]

$f : \mathbb{R} \to \mathbb{R}$, f is periodic

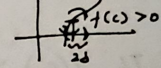
$$\hat{f}(n) = \int_{-1}^{1} f(t) e^{-int} \, dt$$

* $\int_{-1}^{1} |f(x)|^2 dx = \sum_{n \in \mathbb{Z}} |\hat{f}(n)|^2$ [Plancherel Thm]

Neighbourhood propn

Let $f : (a,b) \to \mathbb{R}$. Let $c \in (a,b)$. Assume that f is cont. at c and $f(c) > 0$. Then $\exists\, \delta > 0$ s.t.

$$f(x) > 0 \quad \forall\, x \in (c-\delta, c+\delta) \cap (a,b)$$



Pf Let $\varepsilon > 0$. Since f is cont. at c $\exists\, \delta > 0$ s.t. $\forall\, x \in V_\delta(c) \cap (a,b)$,

$$f(x) \in V_\varepsilon(f(c))$$

Choose $\varepsilon = \frac{f(c)}{2} > 0$

Then for some $\delta > 0$

$\forall\, x \in V_\delta(c) \cap (a,b)$

$$f(x) \in \left(\frac{f(c)}{2}, \frac{3f(c)}{2}\right)$$

$\Rightarrow \forall\, x \in V_\delta(c) \cap (a,b)$,

$$f(x) > 0$$

Thm: [Existence of Root]

Let $f : [a,b] \to \mathbb{R}$ be a continuous fxn. Assume that $f(a) < 0 < f(b)$. Then [$f(a)f(b) < 0$]

$\exists\, c \in (a,b)$ s.t. $f(c) = 0$

Pf [Bisection]

Let $I_1 = [a_1, b_1]$. we bisect the interval I_1

Consider the mid pt

$$c_1 := \frac{a_1 + b_1}{2}$$

If $f(c_1) = 0$, then we are done.

Suppose $f(c_1) \neq 0$. Then

either $f(c_1) > 0$ or

$f(c_1) < 0$

If $f(c_1) > 0$, then $a_2 = a_1$, $b_2 = c_1$

If $f(c_1) < 0$, then $a_2 = c_1$, $b_2 = b_1$

Then we have,

(i) $I_2 \subseteq I_1$, $I_2 = [a_2, b_2]$, $I_1 = [a_1, b_1]$

(ii) $f(a_i) < 0$, $f(b_i) > 0$ $\forall i \in \{1, 2\}$

(iii) $|I_2| = \frac{1}{2}|I_1| = \frac{1}{2}(b-a)$

Again we bisect I_2 and proceed. Then we have a seq^n of $I_n = [a_n, b_n]$ s.t.

(i) $I_n \subseteq I_{n-1} \subseteq \cdots \subseteq I_1$

(ii) $f(a_i) < 0$, $f(b_i) > 0$ $\forall i \in \{1, \ldots, n\}$

(iii) $|I_n| = \frac{1}{2^{n-1}}(b-a)$

$\lim_{n\to\infty} |I_n| = 0$

By Cantor's Intersection theorem, (If $f(c_n) \neq 0$ $\forall n$)

Then $\exists\, c \in [a, b]$ s.t.

$$\{c\} = \bigcap_{n=1}^{\infty} I_n$$

And $\lim a_n = c = \lim b_n$

$\Rightarrow f(a_n) \to f(c)$ and $f(b_n) \to f(c)$

$\Rightarrow f(a_n) \leq 0$ $\forall n$ $f(b_n) \geq 0$

$\Rightarrow \lim_{n\to\infty} f(a_n) \leq 0$ $\forall n$ $\lim_{n\to\infty} f(b_n) \geq 0$

$\Rightarrow f(c) \leq 0$ $\forall n$ $f(c) \geq 0$

$$\Rightarrow \boxed{f(c) = 0}$$

Thm [Intermediate Value Theorem]

Let $f : [a, b] \to \mathbb{R}$ be cont. Let λ be a no. lies in between $f(a)$ & $f(b)$. Then $\exists\, c \in [a, b]$ s.t.

$$f(c) = \lambda.$$

Pf If $f(a) = f(b)$, there is nothing to prove.

Then assume $f(a) \neq f(b)$

Then either (i) $f(a) > f(b)$ or (ii) $f(a) < f(b)$

choose (i) and we prove it

thus $f(a) > \lambda > f(b)$

Let $g : [a, b] \to \mathbb{R}$ be

$g(x) = f(x) - \lambda$

Since f is cont., g is also cont. by algebra of limits.

$g(a) = f(a) - \lambda > 0$

$g(b) = f(b) - \lambda < 0$

we have $g(a)\,g(b) < 0$. By bisection method

$\exists\ c \in [a, b]$ s.t.

$$g(c) = 0$$

$$\Rightarrow g(c) = f(c) - \lambda = 0$$

$$\Rightarrow f(c) = \lambda$$

Similarly we again have for case (ii)

$$f(a) < f(b)$$

Thus we have $f(a) < \lambda < f(b)$

Again we take $g(x) = f(x) - \lambda$.

Similar proof

Defn [Fixed pt]

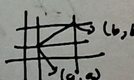
Let $f : D \to \mathbb{R}$ be a fxn. A pt $c \in D$ is said to be a fixed pt of f if

$$f(c) = c$$

Thm [Existence of fixed pt.]

Let $f : [a, b] \longrightarrow [a, b]$ be continuous. Then f has a fixed pt.

$f(x) \in [a, b]$



Pf If $f(a) = a$ or $f(b) = b$, there is nothing to prove.

We assume $f(a) \neq a$ and $f(b) \neq b$

Define $g : [a, b] \longrightarrow \mathbb{R}$ by

$$g(x) = f(x) - x \quad \text{[Again } f(x) \text{ is cont.]}$$

Now $f(a) > a$ and $f(b) < b$

as $f(x) \in [a, b]\ \forall\ x \in [a, b]$

$$g(a) = f(a) - a > 0$$

$$g(b) = f(b) - b < 0$$

$\therefore\ g(a)\,g(b) < 0$

By existence of root $\exists\ c \in [a, b]$ s.t.

$$g(c) = 0$$

$$\Rightarrow f(c) = c$$

Thus existence of fixed pt.

Defn [Bounded fxn]

Let $f : D \to \mathbb{R}$. We say that f is bdd if $\{f(x) : x \in D\}$ is bdd

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Thm [Extreme Value Theorem]

Let $f : [a, b] \longrightarrow \mathbb{R}$ be continuous. Then f is bdd. Then $\exists\ p, q \in [a, b]$ s.t.

$$f(p) = \sup\{f(x) : x \in [a, b]\}$$

$$f(q) = \inf\{f(x) : x \in [a, b]\}$$

More precisely

$f(q) \leq f(x) \leq f(p) \quad \forall\, x \in [a,b]$

Pf Suppose it is unbounded. Then $\exists\, x_n \in [a,b]$ s.t.

$|f(x_n)| > n \quad \forall n \in \mathbb{N} \longrightarrow$ ⊛

Since $[a,b]$ is compact $\exists \{x_{n_k}\}$ s.t. $x_{n_k} \to x$ as $k \to \infty$

and $x \in [a,b]$

Since f is continuous at x.

$f(x_{n_k}) \to f(x)$ as $k \to \infty$

$\Rightarrow \{f(x_{n_k})\}_k$ is bdd.

but ⊛ says its not bdd.

Let $\alpha = \sup \{f(x) \mid x \in [a,b]\}$.

Then $\exists\, x_n \in [a,b]$ s.t.

$f(x_n) \to \alpha$ as $n \to \infty$

Again $x_n \in [a,b]$, $[a,b]$ is compact

$\exists\, x_{n_k} \in [a\, b]$ and $p \in [a,b]$

$x_{n_k} \to p$ as $k \to \infty$

$\Rightarrow f(x_{n_k}) \longrightarrow f(p)$ since f is cont at p

Thus we have $f(p) = \alpha$ as f is cont

Continuity and Monotonicity

Thm-1

Let $J \subseteq \mathbb{R}$ be an interval. Let $f: J \to \mathbb{R}$ be cont and injective. Let $a,b,c \in J$ and $a<b<c$, Then

either $f(a) < f(b) < f()$

$f(a) > f(b) > f(c)$

Pf Let $a<b<c$. Since $a \neq c$.

f injective $f(a) \neq f(c)$

$\Rightarrow$ (i) $f(a) > f(c)$ or (ii) $f(a) < f(c)$

will show

(ii) $\Rightarrow f(a) < f(b) < (c)$

or

(i) $\Rightarrow f(a) > f(b) > f(c)$

Let $f(a) < f(c)$

There are two possibilities,

(i) $f(b) < f(a) < f(c)$

(ii) $f(a) < f(b) < f(c)$

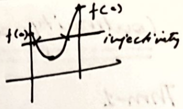
(iii) $f(a) < f(c) < f(b)$

Let (i) holds true

Let $\lambda \in (f(b), f(a))$ $\exists\, p \in (a,b)$ s.t.

$f(p) = \lambda$

Again $\lambda \in (f(b), f(c))$



$\exists\, q \in (b, c)$

$f(q) = \lambda$ (IVT)

$\Rightarrow p \in (a, b)$, $q \in (b, c)$

$\Rightarrow p \neq q)$

but $f(p) = f(q)$ contradicts injectivity

Let ~~us~~ (iii) hold

again take

$\lambda \in (f(a), f(b))$ $\exists\, p \in (a, b)$ s.t.

$f(p) = \lambda$

again take

$\lambda \in (f(c), f(b))$ $\exists\, q \in (b, c)$ s.t.

$f(q) = \lambda$.

Again

$f(p) = f(q)$ but $p \neq q$

[Contradicts Injectivity]

Thus only (ii) is true. $\Rightarrow \underline{f(a) < f(b) < f(c)}$

Again assume $f(a) > f(c)$. Similarly

we can prove $f(a) > f(b) > f(c)$.

[Proved]

$\boxed{\text{Thm - 2}}$

Let $J \subseteq \mathbb{R}$ be an interval and $f : J \to \mathbb{R}$ be cont. and one-one. Then f is strictly monotone.

$\boxed{\text{Pf}}$ Let $a, b \in J$ and $a < b$. Then

wrong pt in known

$f(a) \neq f(b)$

either $f(a) < f(b)$ or $f(a) > f(b)$

Let us take $x \in J \setminus \{a, b\}$

Now if $x < a$

$\Rightarrow x < a < b$

$\Rightarrow f(x) < f(a) < f(b)$ by Thm-1

or

$f(x) > f$

Thm - 3

Let $J \subseteq \mathbb{R}$ be an interval. Let $f: J \to \mathbb{R}$ be monotone ~~and injective~~. If $f(J)$ is an interval. The f is cont.

Pf Let f is strictly increasing. Let $c \in J$ be an int pt. We will p.t. f is cont at c.

Since c is NOT an end pt of J $\exists\, x_1, x_2 \in J$ s.t.

$$x_1 < c < x_2$$

Since f is strictly increasing

$$f(x_1) < f(c) < f(x_2)$$

Now $\exists\, \eta > 0$ s.t.

$$f(x_1) < f(c) - \eta < f(c) < f(c) + \eta < f(x_2)$$

Since $f(J)$ is an interval, $\exists\, p_1, p_2 \in J$ s.t.

$$f(p_1) = f(c) - \eta \quad , \quad f(p_2) = f(c) + \eta$$

where $p_1 < c < p_2$

since f is strictly monotone.

Thus $\exists\, \delta_1 > 0$ s.t.

$$p_1 < c - \delta_1 < c < c + \delta_1 < p_2$$

$$\delta_1 = \frac{1}{2} \min \{ |c - p_1| , |p_2 - c| \}$$

Now since 'f' is strictly increasing,

$\forall x \in (c-\delta_1, c+\delta_1)$

$$f(c)-\eta < f(x) < f(c)+\eta$$

Let $\varepsilon > 0$

Case (i) $\eta = \varepsilon$ (ii) $\eta > \varepsilon$ (iii) $\eta < \varepsilon$

(i) $f(x) \in (f(c)-\eta, f(c)+\eta)$

$\Rightarrow f(x)$ is continuous.

Case (ii) $\eta > \varepsilon$

$$f(c)-\eta < f(c)-\varepsilon < f(c) < f(c)+\varepsilon < f(c)+\eta$$

$\Rightarrow f(x) \in (f(c)-\varepsilon, f(c)+\varepsilon)$

$\Rightarrow f(x)$ is cont.

Case (iii) $\eta < \varepsilon$ $V_{\varepsilon_\eta}(f(c)) \subseteq V_\varepsilon(f(c))$

$f(x) \in (f(c)-\varepsilon, f(c)\ \varepsilon)$

$\Rightarrow f(x)$ is cont.

Concludes the pf that if ~~S~~ I is an open interval f is continuous.

& Now let us consider that I is a closed interval.

Defⁿ [Uniform Continuity]

A f^{x^n} $f: J \to \mathbb{R}$ is uniform cont. if

$\forall\ \varepsilon > 0 \quad \exists\ \delta > 0$ s.t. $x, y \in J$

$$|x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$$

OR

$$|f(x) - f(y)| < \varepsilon \quad \forall\ x, y \in J$$
$$|x-y| < \delta$$

Note: Uniform continuity $\Rightarrow$ Continuity

Defⁿ [Seqⁿ defⁿ]

A f^{x^n} $f: J \to \mathbb{R}$ is UC iff for every pair of seqⁿ $x_n, y_n \in J$ s.t.

$$|x_n - y_n| \to 0 \quad \text{then} \quad |f(x_n) - f(y_n)| \to 0$$

Defⁿ [Lipschitz Cont.]

A f^{x^n} $f: J \to \mathbb{R}$ is Lipschitz cont. if $\exists\ c > 0$ s.t.

$$|f(x) - f(y)| \leq c|x-y| \quad \forall\ x, y \in J.$$

Thm

'f' is Lipschitz iff.

$$\sup\left\{\frac{|f(x) - f(y)|}{|x-y|} : x, y \in J,\ x \neq c\right\} \text{ exists.}$$

Thm

'f' is Lipschitz $\Rightarrow$ f is UC $\Rightarrow$ f is cont.

[Thm] [continuity and compactness]

Let $K \subseteq \mathbb{R}$ be a compact set and $f: K \to \mathbb{R}$ be continuous. Then f is UC.

Pf suppose NOT UC, then $\exists\ \varepsilon > 0$ s.t.

$$|x_n - y_n| < 1/n \quad \text{but} \quad |f(x_n) - f(y_n)| \geq \varepsilon$$

for some $x_n, y_n \in [a, b]$ $\forall n$

By bolzano - weierstrass,

$\exists\ x_{n_k}$ s.t.

$$x_{n_k} \to x. \quad x \in [a, b] \text{ (compactness).}$$

$$|y_{n_k} - x| = |y_{n_k} - x_{n_k} + x_{n_k} - x|$$
$$\leq \underbrace{|x_{n_k} - y_{n_k}|}_{0} + \underbrace{|x_{n_k} - x|}_{0} < \varepsilon$$

$\Rightarrow y_{n_k} \to x$

Since f is continuous, $f(x_{n_k}) \to f(x)$ and $f(y_{n_k}) \to f(x)$. [contradiction]

as $\{f(x_{n_k}) - f(y_{n_k})\} \to 0$

Thm

Let $f: J \to \mathbb{R}$. Let $J_1 \subseteq J$. Then the restriction ~~f x~~. $f|_{J_1} : J_1 \to \mathbb{R}$,

defined by $f|_{J_1}(x) = f(x)$, $\forall x \in J_1$ is UC

Thm

A continuous ~~f~~ $f : (a, b) \to \mathbb{R}$ is UC iff

$\lim_{x \to a^+} f(x)$ and $\lim_{x \to b^-} f(x)$ exists

Pf. Let both the limits exists

Define $g : [a, b] \to \mathbb{R}$ by.

$$g(x) = \begin{cases} f(x) & \text{if } x \in (a, b) \\ \lim_{x \to a^+} f(x) & x = a \\ \lim_{x \to b^-} f(x) & x = b \end{cases}$$

Note $g : [a, b] \to \mathbb{R}$ is conts

$\Rightarrow g$ is UC

$\Rightarrow f$ is UC.

$[\Rightarrow]$

We assume 'f' is UC will prove

$\lim_{x \to a^+} f(x)$ exists.

CLAIM We need to find $l \in \mathbb{R}$ s.t $\forall$ $x_n \to a$ $f(x_n) \to l$.

It is enough to prove.

i) $x_n \to a \Rightarrow \{f(x_n)\}$ is cgt

ii) $x_n \to a$ & $y_n \to a$

$\Rightarrow \lim f(x_n) = \lim f(y_n) = a$

Pf(i) x_n cauchy

$\Rightarrow f(x_n)$ cauchy

$\{x_n\}$ is cgt

$\Rightarrow \{x_n\}$ is Cauchy

$\Rightarrow \{f(x_n)\}$ is Cauchy

$\Rightarrow \{f(x_n)\}$ is cgt

Pf(ii)

$x_n \to a$ & $y_n \to a$

$\Rightarrow |x_n - y_n| \to 0$

$\Rightarrow |f(x_n) - f(y_n)| \to 0$ as f is uniform cont.

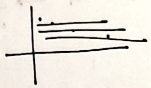
$\Rightarrow \lim f(x_n) = \lim f(y_n)$

Thus $\lim_{x \to a^+} f(x)$ exists

Lim sup and Lim inf

Let $\{x_n\}$ be a bdd. sequence.

Define $A_n := \sup\{x_k : k \geq n\}$



Then $A_1 \geq A_2 \geq A_3 \geq \cdots$

Define $a_n := \inf\{x_k : k \geq n\}$

Then $a_1 \leq a_2 \leq a_3 \leq \cdots$

It is trivial that $A_n \geq a_n \ \forall n \in \mathbb{N}$

$a_1 \leq a_2 \leq a_3 \cdots \leq a_n \leq \cdots \leq A_n \leq \cdots \leq A_2 \leq A_1$

$\{A_n\}$ is decreasing and bdd. below.

Thus we have $\lim_{n\to\infty} A_n$ exists.

We define

$\limsup_{n\to\infty} x_n := \lim_{n\to\infty} A_n$ [limit superior]

and $\boxed{\limsup x_n \geq \liminf x_n}$

and

$\liminf_{n\to\infty} x_n := \lim_{n\to\infty} a_n$

↳ similar reasoning that a_n is bdd above and increasing. Thus limit exists.

Observe

$\limsup x_n = \lim A_n = \inf\{A_n : n \in \mathbb{N}\}$

$\liminf x_n = \lim a_n = \sup\{a_n : n \in \mathbb{N}\}$

Theorem [Subsequential limit & limsup & liminf]

Let x_n be a bdd sequence.

$S := \{l \in \mathbb{R} : l \text{ is a subsequential limit of } x_n\}$

i> $S \neq \phi$

ii> $\inf S$ and $\sup S$ exists.

iii> $\limsup x_n = \sup S$; $\liminf x_n = \inf S$.

Pf By Bolzano Weierstrass Theorem, $S \neq \phi$.

and since $S \subseteq [-M, M]$ as the sequence is bounded, S is also bounded. [Trivial].

Thm. [LIMSUP]

Let $\{x_n\}$ be a bdd. seqn.

Let $\limsup(x_n) = l$. Then

$\forall \varepsilon > 0, \exists N \in \mathbb{N}$ s.t

(i) $x_n < l + \varepsilon \quad \forall n \geq N$

(ii) $x_n > l - \varepsilon$ for infinitely many elements

The main p.t. of difference between the Limsup and liminf is p.t. is true for only infinitely many elements. That is is not true for maybe finitely or maybe infinitely many elements. [[$\forall n \geq N$]]

Thm [LIMINF]

Let $\{x_n\}$ be a bdd. seq^n.

Let $\liminf(x_n) = m$. Then $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ s.t.

(i) $x_n > m - \varepsilon \quad \forall\, n \geq N$

(ii) $x_n < m + \varepsilon$ for infinitely many elements.

Proofs

[LIMSUP]

(ii) Note $l - \varepsilon < A_n \ \forall n$

Thus $l-\varepsilon$ is not an ub of $\{x_k : k \geq n\} \ \forall n$

Define $S_n := \{x_k : k \geq n\}$

$\sup(S_n) = A_n$

Since $l - \varepsilon < A_1 = \sup(S_1)$

$\Rightarrow l - \varepsilon$ is not an ub.

$\Rightarrow \exists n_1 \in \mathbb{N}$ s.t. $x_{n_1} > l - \varepsilon$

Let us take $n_1 + 1$ and S_{n_1+1}

Again $l-\varepsilon$ is not an ub of S_{n_1+1}

[We do this as we need $\underline{\underline{n_2 > n_1}}$]

Again $l-\varepsilon$ is not an UB of S_{n_1+1}

$\exists n_1 \in \mathbb{N}$

s.t. $n_2 \geq n_1$ s.t.

$x_{n_2} > l - \varepsilon$

We recursively do it for infinitely many n

Therefore $\exists \{n_k\}$ s.t.

(i) $n_1 < n_2 < n_3 \dots$

(ii) $x_{n_k} > l - \varepsilon$.

Thus we have constructed a subseq^n wh $x_n > l - \varepsilon$. [infinitely many elements].

Ex [PT converse]

Thm

Let $\{x_n\}$ be a bdd. sequence. Then $\{x_n\}$ is convergent iff $\limsup(x_n) = \liminf(x_n)$

[Trivial]

Thm

Let x_n be bdd. seq. Consider

$S = \{ p \mid p$ is a subsequential limit of $x_n\}$

i) S is bdd.

ii) $\sup(S) = \limsup(x_n)$

iii) $\inf(S) = \liminf(x_n)$

Infinite Series

Given a seq of real no.s $\{a_n\}$, we want to see, when $\sum_{n=1}^{\infty} a_n < \infty$ i.e. the infinite series "converges".

PARTIAL SUM

we define the n^{th} partial sum S_n of a_n to be

$$S_n = \sum_{k=1}^{n} a_k$$

Defn

We say that $\sum_{n=1}^{\infty} a_n$ is cgt if the sequence S_n converges.

Defn

We say the series $\sum_{n=1}^{\infty} a_n$ is ~~conv~~ absolutely convergent is the sequence of partial sums

$$M_n = \sum_{k=1}^{n} |a_k| \quad \text{converges.}$$

Prop

Let $\{a_n\}$ be given seqn. Then $\sum_{n=1}^{\infty} a_n$ is convergent iff for all $\varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t.

$$\left| \sum_{p=1}^{\infty} a_{n+p} \right| < \varepsilon \quad \forall n \geq N$$

Pf $\{S_n\}$ is cgt $\Leftrightarrow$ $\{S_n\}$ is cauchy.

Prop

If $\sum_{n=1}^{\infty} a_n < \infty$. Then

$$\lim_{n \to \infty} a_n = 0$$

Pf we know that series is cauchy, so let us take $m = n-1$ and $n \in \mathbb{N}$.
Thus for a given $\varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t.

$$|S_n - S_{n-1}| < \varepsilon \quad \forall n \geq N$$

$$\Rightarrow \left| \sum_{a=1}^{n} a_n - \sum_{a=1}^{n-1} \right| < \varepsilon$$

$$\Rightarrow |a_n| < \varepsilon$$

[Converse does not hold. As $\frac{1}{n}$ goes to 0 but $\sum_{n=1}^{\infty} \frac{1}{n}$ famously doesn't converge]

$$\Rightarrow \lim_{n \to \infty} a_n = 0$$

Thm

Let $\{a_n\}$ be a seqn. of non-negative real no.s. Then

$\sum_{n=1}^{\infty} a_n$ is cgt $\Leftrightarrow$ $\{S_n\}$ is bdd. above.

Pf Since $a_n \geq 0$ $\forall n \in \mathbb{N}$, S_n is increasing.
Thus $\{S_n\}$ is cgt iff $\{S_n\}$ is bdd. above.

Def^n [~~Cgt~~ Not Cgt. / Divgt series]

We say a series is divergent if it not cgt.

Some Notes

1. **Harmonic Series**

$a_n = \frac{1}{n}$, $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

$S_2 = 1 + \frac{1}{2}$

$S_{2^2} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \geq 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 2$

$$\boxed{S_{2^k} \geq 1 + \frac{k}{2}} \quad \forall k \in \mathbb{N}$$

A subseq^n is NOT bdd above (S_{2^k})

Thus S_n is NOT bdd above.

---

2. p-series : $p \in (0, \infty)$

$\sum_{n=1}^{\infty} \frac{1}{n^p}$ is cgt, if $p > 1$

Pf. $S_n = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \dots$

$S_1 = 1$

$S_2 = 1 + \frac{1}{2^p}$

$S_3 = 1 + \frac{1}{2^p} + \frac{1}{3^p} \leq 1 + \frac{1}{2^p} + \frac{1}{2^p} = 1 + \frac{1}{2^{p-1}}$

$S_n < 1 + \frac{1}{2^{p-1}} + \frac{1}{4^{p-1}} + \frac{1}{8^{p-1}} + \dots < 1 + \sum_{n=1}^{\infty} \left(\frac{1}{2^{p-1}}\right)^n$

$\Rightarrow \{S_n\}$ cgt as $\left(\frac{1}{2^{p-1}}\right)^n$ is a geometric series. with $2^{p-1} > 1$

(*) $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is divgt. if $p < 1$.

Just use comparison test defined later.

Series Tests

1. **Comparison Test**

Let $\{a_n\}$ & $\{b_n\}$ be 2 non-negative sequences and assume that

$$a_n \leq b_n \quad \forall n \in \mathbb{N}$$

Then

(i) $\sum b_n$ is convergent $\Rightarrow$ $\sum a_n$ is ~~div~~ convergent

(ii) $\sum a_n$ is divergent $\Rightarrow$ $\sum b_n$ is divergent.

Pf (i) $S_n = \sum_{k=1}^{n} a_k$, $t_n = \sum_{k=1}^{n} b_k$

We know $\sum_{k=1}^{\infty} b_k$ is cgt.

$\Rightarrow \lim_{n \to \infty} t_n$ exists.

Since t_n is increasing, $\lim_{n\to\infty} t_n = \sup\{t_n / n \in \mathbb{N}\}$

we have

$$0 \leq S_n \leq t_n \leq t = \lim(t_n)$$

$\Rightarrow$ S_n is bdd above.

1. we have S_n increasing, ~~then~~ S_n is bdd above.

Thus $\lim(S_n)$ exists

$\Rightarrow$ $\sum a_n$ converges.

(ii) Contrapositive of (i)

---

2. <u>Ratio Test</u>

Let $\{a_n\}$ be a seqn of ~~non-negative~~ positive reals s.t.

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = r$$

(i) If $0 \leq r < 1$ then $\sum a_n$ is cgt.

(ii) If $r > 1$ then $\sum a_n$ divergent

(iii) If $r = 1$, then test is inconclusive.

Pf i) Since $0 \leq r < 1$, choose $\varepsilon > 0$ s.t. $r + \varepsilon < 1$.

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = r$, $\exists N \in \mathbb{N}$ s.t.

$$\left| \frac{a_{n+1}}{a_n} - r \right| < \varepsilon \quad \forall n \geq N$$

$$\frac{a_{n+1}}{a_n} - r < \varepsilon$$

$$\Rightarrow \frac{a_{n+1}}{a_n} < r + \varepsilon$$

$\Rightarrow a_{n+1} < (r+\varepsilon) a_n$ Let $c := r + \varepsilon$

$\Rightarrow a_{n+1} < c a_n < c^2 a_{n-1} < \dots < c^{n-N+1} a_N$

$\forall n \geq N$

$$a_n \leq c^{n-N} a_N = c^n \left(\frac{a_N}{c^N} \right)$$

Since $c < 1$

The series $c^n \left(\frac{a_N}{c^N} \right)$ is cgt.

Thus by comparison test $\sum a_n$ is cgt.

(ii) Choose $\varepsilon > 0$ s.t.

$$r - \varepsilon > 1$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = r$$

$$\Rightarrow \left| \frac{a_{n+1}}{a_n} - r \right| < \varepsilon$$

$$\Rightarrow \frac{a_{n+1}}{a_n} > r - \varepsilon$$

Let $c := r - \varepsilon$

$\Rightarrow$ ~~$a_{n+1} > (r-\varepsilon) a_n > (r-\varepsilon)^2 a_n$~~

$$\frac{a_{n+1}}{a_n} > r-\varepsilon > 1.$$

$\Rightarrow a_{n+1} > a_n > \dots > a_N$

$\Rightarrow \sum a_{n+1} > \sum a_N$

Now a_N is positive, so $\sum a_N$ is divgt.

Again by comparison test.

$\sum a_{n+1}$ is ~~convergent~~. divergent

(iii) $r=1$ Take $\frac{1}{n}$ and $\frac{1}{n^2}$

$\lim_{n\to\infty}\left(\frac{n}{n+1}\right) = 1$ $\frac{1}{n}$ diverges..

$\lim_{n\to\infty}\left(\frac{n^2}{(n+1)^2}\right) = 1$ $\frac{1}{n^2}$ converges.

---

3. <u>Cauchy's Root Test</u>

Let $\{a_n\}$ be a seqn of positive reals s.t.

$\lim_{n\to\infty}(a_n)^{1/n}$ exists and equals to r

<u>Then</u> (i) $\sum a_n$ is cgt if $r<1$.

(ii) $\sum a_n$ is divergent if $r>1$.

(iii) $r=1$, test is inconclusive.

[Proof next page]

Pf (i) If $r<1$ choose $\varepsilon>0$ st.

$$r+\varepsilon<1$$

$\Rightarrow \lim_{n\to\infty}(a_n)^{1/n} = r$

$\Rightarrow \exists N\in\mathbb{N}$ s.t.

$|(a_n)^{1/n} - r| < \varepsilon \quad \forall n \geq N$

$\Rightarrow (a_n)^{1/n} < r+\varepsilon$ Let $c := r+\varepsilon$

$\Rightarrow (a_n)^{1/n} < c$

$\Rightarrow a_n < c^n$

c^n is a infinite geometric progression with ratio <1.

Thus by comparison test, $\sum_{n=1}^{\infty} a_n$ is convergent.

(ii) If $r>1$, we choose $\varepsilon>0$ s.t.

$r-\varepsilon>1$

$\lim_{n\to\infty}(a_n)^{1/n} = r$

$\Rightarrow (a_n)^{1/n} > (r-\varepsilon) > 1$

$\Rightarrow a_n > 1$ Again by comparison test, $\sum a_n$ is divergent

4. Integral Test

Let $f : [1,\infty) \to (0,\infty)$ be a decreasing func.

Define $a_n = f(n)$ $\quad n \in \mathbb{N}$

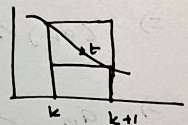
$$b_n = \int_1^n f(t)\,dt$$

Then

$\sum a_n$ converges $\Leftrightarrow$ $\sum b_n$ converges.

Pf

for $t \in [k, k+1]$



$$f(k+1) \le f(t) \le f(k)$$

$$\int_k^{k+1} f(k+1)\,dt \le \int_k^{k+1} f(t)\,dt \le \int_k^{k+1} f(k)\,dt$$

$$\Rightarrow f(k+1) \le \int_k^{k+1} f(t)\,dt \le f(k)$$

$$\Rightarrow \sum_{k=1}^{n-1} f(k+1) \le \sum_{k=1}^{n-1} \int_k^{k+1} f(t) \le \sum_{k=1}^{n-1} f(k)$$

$$\Rightarrow \sum_{k=1}^{n-1} f(k+1) \le \int_1^n f(t)\,dt \le \sum_{k=1}^{n-1} f(k)$$

$$\Rightarrow \lim_{n\to\infty} \sum_{k=1}^{n-1} f(k+1) \le \int_1^{\infty} f(t)\,dt \overbrace{\le \lim_{n\to\infty} \sum_{k=1}^{n-1} f(k)}^{2}$$

By inequality (1) ($\Rightarrow$) is proved, and by (2) ($\Leftarrow$) is proved.

---

5. Cauchy Condensation Test

If (a_n) is a decreasing non-negative sequence, then

1) $\sum a_n$ cgt $\Leftrightarrow$ $\sum 2^n a_{2^n}$ cgt.

($\Leftarrow$)

Given $a_1 \ge a_2 \ge a_3 \ge \dots$

$\Rightarrow a_1,$

$a_2 + a_3 \le 2a_2$

$a_4 + a_5 + a_6 + a_7 \le 4a_4 = 2^2 a_{2^2}$

$\Rightarrow \sum a_n \le \sum 2^k a_{2^k}$

Since $\sum 2^k a_{2^k}$ is cgt, by comparison test, $\sum a_n$ is cgt.

($\Rightarrow$) We will prove the contrapositive.

$a_3 + a_4 \ge 2a_4 = 2a_{2^2}$

$\Rightarrow a_5 + a_6 + \dots + a_8 \ge 4a_8 = 2^2 a_{2^3}$

$\sum a_n \ge \sum 2^{k-1} a_{2^k}$ since $2^{k-1} a_{2^k}$ diverges $\Rightarrow \sum a_n$ diverges.

TOPOLOGY IN $\mathbb{R}$

Def^n [Neighbourhood]

Let $p \in \mathbb{R}$ and $\delta > 0$. Then the set

$$[V_\delta(p)]\ N_\delta(p) := (p-\delta, p+\delta)$$
$$= \{x \in \mathbb{R} : |x-p| < \delta\}$$

is called the δ-neighbourhood of p.

Def^n [Interior pt]

A pt p is an interior pt of E $(\subseteq \mathbb{R})$ if $\exists\, \delta > 0$ s.t. $N_\delta(p) \subseteq E$

Def^n

Let $E \subseteq \mathbb{R}$. Define

$$Int(E) = \{x \in E : x \text{ is an int. pt. of } E\}$$

Remark

If p is an interior pt. of E then $p \in E$

Quite obviously as $(p-\delta, p+\delta) \subseteq E$ for some δ. $\Rightarrow p \in E$

Def^n [Open Set]

A set E is an open set if every pt of E is an interior pt of E i.e.

$$Int(E) = E$$

$Int(E) \subseteq E$ [Always]

$E \subseteq Int(E)$ [Cond^n for Open Set]

$\Rightarrow \boxed{Int(E) = E}$

Eg: (i) $\mathbb{R}$ (ii) ϕ (iii) (a, b)

Def^n [Limit Pt]

Let $E \subseteq \mathbb{R}$. A pt. $x \in \mathbb{R}$ is said to be a limit pt if $\forall\ \varepsilon > 0$

$$V_\varepsilon(x) \setminus \{x\} \cap E \neq \phi$$

Note

$E' = \{x \in \mathbb{R} : x \text{ is a limit pt of } E\}$

↳ called the derived set

Def^n [Closed Set]

A set E is closed if it contains all the limit pts i.e. $E' \subseteq E$

Thm

Let x be a limit pt of E. Then for any $\varepsilon > 0$, $N_\varepsilon(x)$ contains infinitely many pts of E.

Pf: Suppose $\exists\, \varepsilon > 0$ s.t. $N_\varepsilon(x)$ contains finitely many pts

$\alpha_1, \dots, \alpha_k$ of E s.t. $x \neq \alpha_i \; \forall i$

$0 < \delta < \min\{|x-\alpha_1|, \dots, |x-\alpha_k|\}$

Then $\exists\, \delta > 0$ and

$V_\delta(x) \setminus \{x\} \cap E = \phi$

Therefore x is NOT a limit pt of E.

Def^n [Isolated pt]

Analysis II

Differentiability at a pt

Def^n

Let ~~f~~ J be an interval and $c \in J$. Let $f : J \to \mathbb{R}$
We say f is differentiable at c if

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} \quad \text{exists}$$

This is denoted by

$$f'(c) \;/\; \left.\frac{df}{dx}\right|_{x=c}$$

Geometrically, $f'(c)$ denotes the slope of the ~~[illegible]~~ tangent at $(c, f(c))$ to the curve $\{(x, f(x)) : x \in J\}$

~~[illegible]~~ **Def^n**

f is diff at c.

if $\exists\, \alpha \in \mathbb{R}$ s.t.

$$\lim_{x \to c} \frac{f(x) - f(c) - \alpha(x-c)}{x-c} = 0$$

ε-δ def^n

f is differentiable at c $\exists\, \alpha \in \mathbb{R}$ s.t.
for $\varepsilon > 0$ $\exists\, \delta > 0$ s.t.

$$|f(x) - f(c) - \alpha(x-c)| < \varepsilon|(x-c)| \quad \forall\, x \in J \cap V_\delta(c) \setminus \{c\}$$

Thm

Let J be an interval and $c \in \mathbb{R}$. A function $f: J \to \mathbb{R}$ is differentiable at $c \in \mathbb{R}$

iff $\exists f_1 : J \to \mathbb{R}$ satisfying

(i) $f(x) = f(c) + f_1(x)(x-c) \quad \forall x \in J$.

(ii) $f_1(x)$ is cont. at c.

Pf ($\Rightarrow$) Assume f is differentiable at c. Then $f'(c)$ exists.

Define

$$f_1(x) := \begin{cases} \dfrac{f(x)-f(c)}{x-c} & x \neq c \\ f'(c) & x = c \end{cases}$$

Then f and f_1 satisfy (i)

(ii) is trivial from here.

($\Leftarrow$) Suppose $f_1(x)$ exists.

Since f_1 is cont at c

$\forall \varepsilon > 0, \exists \delta > 0$ s.t.

$$|f_1(x) - f_1(c)| < \varepsilon \quad \forall x \in J \cap V_\delta(c) \setminus \{c\}$$

$$\Rightarrow \left| \frac{f(x)-f(c)}{x-c} - f_1(c) \right| < \varepsilon.$$

Thus $\displaystyle\lim_{x \to c} \frac{f(x)-f(c)}{x-c}$ exists. Thus f is differentiable at c.

Corollary

If f is diff at c then f is cont at c.

Remark

Let $f: J \to \mathbb{R}$ be diff at $c \in J$.

Let $\delta > 0$ be s.t. $(c-\delta, c+\delta) \subset J$. Now consider the function $f|_{(c-\delta, c+\delta)}$, this is also differentiable. [Local Property]

Algebra of differentiability

Let $f, g : J \to \mathbb{R}$, $c \in J$. Assume that f, g are diff at c. Then

i) $(f+g)'(c) = f'(c) + g'(c)$

(ii) for some $\alpha \in \mathbb{R}$

$$(\alpha f)'(c) = \alpha f'(c)$$

(iii) $(fg)'(c) = f'(c) \cdot g(c) + g'(c) f(c)$

(iv) $g(x) \neq 0$ in $(c-\delta, c+\delta)$,

$\left(\frac{f}{g}\right)'(x) = (f'(x) g(x) - f(x) g'(x)) / (g(x))^2$

(v) Let ~~f(c)~~ $f(c) \neq 0$. Then $\left(\frac{1}{f}\right)$ is diff at c and $\left(\frac{1}{f}\right)'(c) = -\frac{f'(c)}{(f(c))^2}$

(x^{-1})

$(f(x))^{-1}$

Pf Rest are easy use theorem

(v) Since f is diff. at c, then f is cont at c. Since $f(c) \neq 0$. $\exists \delta > 0$ s.t. $f(x) \neq 0$. $\forall x \in (c-\delta, c+\delta)$

Now $\frac{1}{f} : (c-\delta, c+\delta) \to \mathbb{R}$ is well defined.

Now, since f is diff $\exists f_1 : J \to \mathbb{R}$ s.t.

$$f(x) = f(c) + f_1(x)(x-c)$$

and f_1 is cont a c

$$f_1(c) = f'(c)$$

we want to write

$$\frac{1}{f(x)} = \frac{1}{f(c)} + \tilde{f}(x)(x-c)$$

$$\tilde{f}(x)(x-c) = \frac{f(c)-f(x)}{f(x)f(c)}$$

$$\tilde{f}(x) = \frac{f(c)-f(x)}{f(x)f(c)\,(x-c)}$$

$$\tilde{f}(x) = \frac{-(f(x)-f(c))}{f(x)\,f(c)\,(x-c)}$$

Take a limit

$\lim_{x \to c} \tilde{f}(x) = \tilde{f}(c) \Rightarrow \tilde{f}(x)$ is cont.

$$\Rightarrow \lim_{x \to c} \tilde{f}(x) = -\frac{\lim_{x \to c} \frac{f(x)-f(c)}{(x-c)}}{\lim_{x \to c} f(x) \cdot f(c)}$$

$$= -\frac{f'(c)}{(f(c))^2}$$

$$\Rightarrow \boxed{\tilde{f}(c) = -\frac{f'(c)}{(f(c))^2}}$$

Chain Rule

Let $J \subseteq \mathbb{R}$ be an interval and c is a pt, $c \in J$.
Let $f : J \to \mathbb{R}$ be ~~a~~ diff at c. Assume that $f(J) \subseteq J_1$. Let $g : J_1 \to \mathbb{R}$ ~~at~~ be diff at $f(c)$.

Then

$$(g \circ f)'(c) = g'(f(c))\, f'(c)$$

Def^n

Let J be an interval. Let $f : J \to \mathbb{R}$ be a f^n.

A pt $c \in J$ is said to be a local maxima (minima).

(i) c is an interior pt
(i.e.) $\exists\, \delta > 0$ s.t
$V_\delta(c) \subseteq J$.

(ii) $f(x) \leq f(c) \;\forall\, x \in V_\delta(c)$ $\left(f(x) \geq f(c) \;\forall\, x \in V_\delta(c)\right)$

Def^n Let $f : J \to \mathbb{R}$ be a f^n. A pt $c \in J$ is said to be global maximum (minima) if $f(x) \leq f(c) \;\forall\, x \in J$.
$\left(f(x) \geq f(c)\right)$

Ex

$$f(x) = \begin{cases} \dfrac{\sin(x)}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}$$

$f(x) \leq 1$, $f(0) = 1$

local max $M = \{c : \sin c = 1\}$
$c \in M \Rightarrow c$ is a loc max.

Ex $f(x) = x \quad x \in [0, 1]$

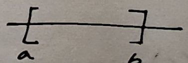
↳ No local maxima.
↳ has a global maxima at $x = 1$

Thm

$f : J \to \mathbb{R}$ and c is a local maxima. f is a differential func.

$\Rightarrow f'(c) = 0$

\# Note:

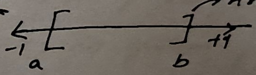


$a \to$ global max
$f'(a) \leq 0$

$b \to$ global max
$f'(b) \geq 0$

Normal

→ Assign a unit normal



$f'(a)\, \nu_a \geq 0$

Thm

$f: J \to \mathbb{R}$ & c is a local maxim.

$\Rightarrow f'(c) = 0$

Pf

Since $c \in J$ is a loc max, $\exists \delta > 0$

s.t.

(i) $(c-\delta, c+\delta) \subseteq J$

(ii) $f(x) \leq f(c) \quad \forall x \in (c-\delta, c+\delta)$.

Note

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

$$= \lim_{h \to 0^+} \frac{f(c+h) - f(c)}{h}$$

$$= \lim_{h \to 0^-} \frac{f(c+h) - f(c)}{h}$$

Now $f(c) \geq f(c+h)$

$$\lim_{h \to 0^+} \frac{f(c+h) - f(c)}{h} \leq 0$$

$$\lim_{h \to 0^-} \frac{f(c+h) - f(c)}{h} \geq 0$$

Now since f is differentiable at c.

$$\lim_{h \to 0^+} = \lim_{h \to 0^-}$$

$$\Rightarrow \quad 0 \leq \lim_{h \to 0^+} \frac{f(c+h) - f(c)}{h} \leq 0$$

$\Rightarrow \quad f'(c) = 0$

Rolle's Theorem

Let $f: [a,b] \longrightarrow \mathbb{R}$ be s.t.

(i) f cont on $[a,b]$

(ii) f is diff on (a,b).

(iii) $f(a) = f(b)$

then $\exists\, c \in (a,b)$ s.t. $f'(c) = 0$

Pf · It is enough to prove that f has either local max or local min.

Since $f: [a,b] \to \mathbb{R}$ is cont. $\exists\, x_1, x_2 \in [a,b]$ s.t.

$$f(x_1) \leq f(x) \leq f(x_2)$$

i.e. x_1 is global ~~max~~ minima and x_2 is global maxima.

Consider two case

1) $f(x_1) = f(x_2)$

$\Rightarrow f(x) = c$ where $c \in \mathbb{R}$

$\Rightarrow f'(x) = 0 \quad \forall x \in [a,b]$

2) $f(x_1) \neq f(x_2)$

then either x_1 or x_2 has to interior pt.
otherwise if x_1 and x_2 both are boundary pt $\Rightarrow f(x_1) = f(x_2) \Rightarrow$ Contradiction].

Then either x_1 or x_2 has to be interior pt.

either x_1 or x_2 has to be a local max or local minima.

$\Rightarrow$ either $f'(x_1) = 0$ or $f'(x_2) = 0$

Mean Value Thm (MVT)

Let $f:[a,b] \to \mathbb{R}$ be cont on $[a,b]$. and differentiable in (a,b), then $\exists\ c \in (a,b)$ s.t.

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$

$$g(x) = (b-a)\,f(x) + (f(b) - f(a))\,x$$

Thm (Cauchy's MVT)

Let $f, g : [a,b] \to \mathbb{R}$, be differentiable on (a,b) and continuous on $[a,b]$

Assume that $g'(x) \neq 0 \quad \forall x \in (a,b)$

Thus $\exists\ c \in (a,b)$ s.t.

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

Application of MVT

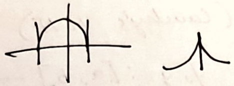
1. Let $f: J \to \mathbb{R}$ be diff and J be an interval.
Assume that
$$f'(x) = 0 \quad \forall x \in J$$
Then $f(x) = \text{const}. \quad \forall x \in J.$

2. Let $f' \geq 0$ Then
f is increasing.

3. f' is bdd $\Rightarrow f$ is Lipschitz. ✓

Ex: $f:[0,1]\to\mathbb{R}$ Find a fxn f which is differentiable

$f':[0,1]\to\mathbb{R}$ NOT Continuous



$\left(\frac{\sin x}{x}\right) = 1$

$\frac{x\cos x-\sin x}{x^2}$

$\frac{\cos x}{x}-\frac{\sin x}{x^2}$

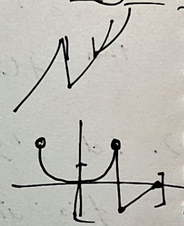
Darboux Thm

Let $f:[a,b]\to\mathbb{R}$ be diff. Let

$f'(a)<\lambda<f'(b)$

then $\exists\, c\in(a,b)$ s.t.

$$f'(c)=\lambda$$



Thm Let $f:[a,b]\to\mathbb{R}$ be diff s.t. $\exists\, M>0$

$$|f'(x)|\le M \quad \forall\, x\in[a,b]$$

then f satisfies

$$|f(x)-f(y)|\le M|x-y| \quad \forall\, x,y\in[a,b]$$

Pf

$$\left|\frac{f(x)-f(y)}{x-y}\right| = |f'(c)| \le M \text{ (by hypothesis)}$$

(MVT)

$$\Rightarrow |f(x)-f(y)|\le M|x-y| \quad \forall\, x,y\in[a,b]$$

Darboux's Thm

Pf Let us define

$g:[a,b]\to\mathbb{R}$

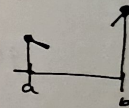
by $g(x)=f(x)-\lambda x$

Enough to find $c\in(a,b)$

s.t. $g'(c)=0$

Note $g'(a)=f'(a)-\lambda<0$

$g'(b)=f'(b)-\lambda>0$



Since g is continuous, it must have maxima or minima (EVT).

(*) let $x_0\in[a,b]$ s.t.

$$g(x_0)=\min\{g(x): x\in[a,b]\}$$

$x_0\in(a,b)$ since $g(a)$ or $g(b)$ can't be minima.

$\Downarrow$

Thus $\exists\, x_0\in(a,b)$

s.t.

$g(x_0)=\min\{g(x): x\in[a,b]\}$

$\Downarrow$

$g'(x_0)=0$ as $g(x_0)=\min$ in an open interval. (Local minima)

$\Rightarrow g'(x_0)=0$.

⊛ **CLAIM**

$x_0 \neq a$ or $x_0 \neq b$

Since $g'(a) < 0$

$\exists\ c \in V_\delta(a)$ for some $\delta > 0$

s.t. $g(a) > g(c)$

Since

$g(x_0) = \min\{ g(x) : x \in [a,b] \}$

$g(x_0) < g(c) < g(a)$

$\Rightarrow$ $x_0 \neq a$

similar proof for $x_0 \neq b$

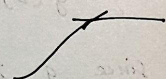
Since $g'(b) > 0$

$\exists\ c' \in V_\delta(b)$ for some $\delta > 0$

s.t. $g(b) > g(c')$

Thus $g(b) > g(x_0)$

$\Rightarrow b \neq x_0$



Inverse function Theorem

Thm. Let $f : (a,b) \to \mathbb{R}$, be diff and

$f'(x) \neq 0 \quad \forall x \in (a,b)$. $\quad I : (a,b)$

Then

(i) $f : (a,b) \to \mathbb{R}$ is strictly monotone (Darboux)

(iii) $f(I) = J$ is an interval

and $f^{-1} : J \to \mathbb{R}$ exists and continuous.

(ii) $f^{-1} : J \to \mathbb{R}$ is diff and

$$(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))}.$$

$f^{-1}(f(x)) = x$

$(f^{-1})'(f(x))f'(x) = 1$

$(f^{-1})'(f(x)) = \frac{1}{f'(x)}$

$(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))}$

Pf

either $f'(x) > 0 \ \forall x$

or $f'(x) < 0 \ \forall y$

By Darboux's Thm.

If not true $\exists\ x_0, y_0 \in I$

s.t. $f'(x_0) > 0$ and $f'(y_0) < 0$

By Darboux's $\exists\ m \in (\min\{x_0, y_0\}, \max\{x_0, y_0\})$

s.t. $f'(m) = 0$

$(f^{-1})'(d) =$ $f(c) = d$

$$\lim_{y \to d} \frac{f^{-1}(y) - f^{-1}(d)}{y - d}$$

$$= \lim_{f(x) \to f(c)} \frac{x - c}{f(x) - f(c)}$$

$f(x) \to f(c)$
$\Rightarrow x \to c$

$$\Rightarrow \lim_{x \to c} \frac{1}{\frac{f(x) - f(c)}{x - c}}$$

$$\Rightarrow \frac{1}{f'(c)} = \frac{1}{f'(f^{-1}(d))}$$

$$(f^{-1})'(d) = \frac{1}{f'(f^{-1}(d))}$$

L Hospital's Rule

Thm Let J be an interval. Let $a \in \partial J$ or let a be a boundary pt of J. Assume that

(i) $f, g : J \setminus \{a\} \to \mathbb{R}$ be differentiable.

(ii) $g'(x) \neq 0$ and $g(x) \neq 0$ $\forall x \in J \setminus \{a\}$

(iii) Assume that $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = A$ and A is either 0 or ∞

If $\lim_{x \to a} \frac{f'(x)}{g'(x)}$ exists, then

$$\lim_{x \to a} \frac{f'(x)}{g'(x)} = \lim_{x \to a} \frac{f(x)}{g(x)}$$

Pf Let $a \in J$.

Case 1 $0/0$

Assume that $\lim_{x \to a} f(x) = 0 = \lim_{x \to a} g(x)$.

Define $\tilde{f}(x) = \begin{cases} f(x) & x \in J \setminus \{a\} \\ 0 & x = a \end{cases}$

$\tilde{g}(x) = \begin{cases} g(x) & x \in J \setminus \{a\} \\ 0 & x = a \end{cases}$

then $\tilde{f}, \tilde{g} : J \to \mathbb{R}$ is cont.

Now $\dfrac{f(x)}{g(x)} = \dfrac{\tilde{f}(x)}{\tilde{g}(x)} = \dfrac{\tilde{f}(x) - \tilde{f}(a)}{\tilde{g}(x) - \tilde{g}(a)}$ $\forall x \in J \setminus \{a\}$

Obv domi.

Now by Cauchy's MVT

$\exists\, c_x$ lies between x and a $\forall x \in J \setminus \{a\}$

s.t.

$$\frac{f(x)}{g(x)} = \frac{\tilde{f}(x) - \tilde{f}(a)}{\tilde{g}(x) - \tilde{g}(a)} = \frac{f'(c_x)}{g'(c_x)}$$

$$\Rightarrow \lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{c_x \to a} \frac{f'(c_x)}{g'(c_x)}$$

$$= \lim_{x \to a} \frac{f'(x)}{f'(x)}.$$

Will show that

$$\lim_{x \to a+} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Let $x_n > a$ s.t. $x_n \to a$

Need to prove

$$\lim_{n \to \infty} \frac{f(x_n)}{g(x_n)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Now $\dfrac{f(x_n)}{g(x_n)} = \dfrac{\tilde{f}(x_n)}{\tilde{g}(x_n)} = \dfrac{\tilde{f}(x_n) - \tilde{f}(a)}{\tilde{g}(x_n) - \tilde{g}(a)}$

as $\tilde{g}(a) = 0 = \tilde{f}(a)$

Applying Cauchy's MVT $\exists$

$c_n \in (a, x_n)$ s.t. $a < c_n < x_n$

$$\frac{f(x_n)}{g(x_n)} = \frac{\tilde{f}'(c_n)}{\tilde{g}'(c_n)} = \frac{f'(c_n)}{g'(c_n)}$$

So $\displaystyle\lim_{n \to \infty} \frac{f(x_n)}{g(x_n)} = \lim_{n \to \infty} \frac{f'(c_n)}{g'(c_n)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$

Since x_n is arbitrary

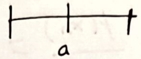
$$\lim_{x \to a+} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Similarly

$$\Rightarrow \lim_{x \to a-} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

$$\Rightarrow \lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Case-2 (∞/∞)



Motivation

$$\lim_{x\to a} f(x) = \infty = \lim_{x\to a} g(x)$$

$$\frac{f(x)-f(c)}{g(x)-g(c)} = \frac{f(x)}{g(x)} \frac{\left(1-f(c)/f(x)\right)}{\left(1-g(c)/g(x)\right)}.$$

$$\Rightarrow \frac{f(x)}{g(x)} = \left(\frac{f(x)-f(c)}{g(x)-g(c)}\right)\left(\frac{1-g(c)/g(x)}{1-f(c)/f(x)}\right)$$

$$\lim_{x\to a+} g(x) = \infty$$

$\Rightarrow \forall R \in \mathbb{R}^+, \exists \delta > 0$

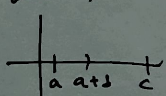
s.t. $x \in (a, a+\delta) \Rightarrow g(x) > R$

let $c \in J$ and $c > 0$

For $R = |g(c)|$, $\exists \delta$, s.t.

$x \in (a, a+\delta)$, $g(x) > |g(c)| \geq g(c)$

Just take $c \notin (a, a+\delta)$



Now using Cauchy MVT, m_n exists. s.t. $\frac{f'(m_n)}{g'(m_n)}$ Now prove like $0/0$.

Higher order derivative

ce^x

A differentiable f^n $f: J \to \mathbb{R}$ is twice differentiable at $c \in J$, if $f': J \to \mathbb{R}$ is differentiable.

Similarly, we define n^{th} order differentiability.

Defn

A f^n $f: J \to \mathbb{R}$ is infinitely differentiable if n^{th} derivative

$f^{(n)}$ exists $\forall n \in \mathbb{N}$.

Defn Smooth function $\uparrow$

Ex

$$f(x) = \begin{cases} e^{-1/x} & x > 0 \\ 0 & \leq 0 \end{cases}$$

Then f is smooth.

Note $f(0) = 0 = \lim_{x\to\infty} f(x)$

$\Rightarrow f$ is conti.

It is obv that f is diff for $x \in \mathbb{R}\setminus\{0\}$.

Now, $\lim_{x\to 0} \frac{f(x)-f(0)}{x} = \lim_{x\to 0} \frac{f(x)}{x}$

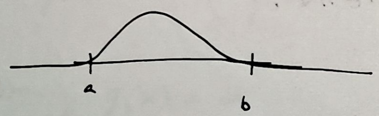
$$\lim_{x \to 0^-} \frac{f(x)}{x} = 0$$

$$\Rightarrow \lim_{x \to 0^+} \frac{f(x)}{x} = \lim_{x \to 0} \frac{e^{-1/x}}{x} \Rightarrow \lim_{y \to \infty} \frac{e^{-y}}{1/y}$$

$$= \lim_{y \to \infty} \frac{y}{e^y} = \lim_{y \to \infty} \frac{1}{e^y} = 0$$

(L'Hospital)

Similarly done for n derivative



Find a smooth $f: \mathbb{R} \to \mathbb{R}$

s.t. $f = 0$ on $[a,b]^c$

This does not have a Taylor series

IT IS NOT ANALYTIC!!

Leibnitz

Let $f, g: J \to \mathbb{R}$ be smooth, then

$$(fg)' = f'g + fg'$$

$$(fg)'' = f''g + 2f'g' + fg''$$

$$(fg)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k)}$$

Notation

Let $J \subseteq \mathbb{R}$ be an interval.

$C(J) = \{f: J \to \mathbb{R} : f \text{ is continuous}\}$

$C'(J) = \{f: J \to \mathbb{R} : f' \in C(J)\}$

$C^2(J) = \{f: J \to \mathbb{R} : f'' \in C(J)\}$

$\vdots$

$C^n(J) = \{f: J \to \mathbb{R} : f^{(n)} \in C(J)\}$

$C^k(J) \subset C^{k-1}(J)$

This should be trivial. If $f^{(k)}$ is cont.

Then $f^{(k)}$ exists $\Rightarrow f^{(k-1)}$ is cont.

However not all $C^{k-1}(J)$ functions are in $C^k(J)$ so. $C^k(J) \subset C^{k-1}(J)$

Q^n Construct a smooth func s.t. $f(x) = 0$ $\forall x \in [a,b]^c$, where a, b are given.

Ans

- construct $f_1: \mathbb{R} \to \mathbb{R}$ s.t.

  $f_1 \in C^\infty(\mathbb{R})$ and $f_1(x) = 0 \;\; \forall x \in (-\infty, a)$

- construct $f_2: \mathbb{R} \to \mathbb{R}$ s.t.

  $f_2 \in C^\infty(\mathbb{R})$ and $f_2(x) = 0 \;\forall x \in (b, \infty)$

$$f_1(x) = \begin{cases} e^{-1/x-a} & x > a \\ 0 & x \leq a \end{cases}$$

$$f_2(x) = \begin{cases} e^{1/x-b} & x < b \\ 0 & x \geq b \end{cases}$$

Then take

$$f(x) = f_1(x) f_2(x) \qquad (supp(f) = [a,b])$$

- Then $f \in C^\infty$

- $f(x) = 0 \quad \forall x \in [a,b]^c$

Support of a f_n

$$supp(f) = \overline{\{x \in \mathbb{R} : f(x) \neq 0\}}$$

= ~~largest closed set where f is non-zero~~

Eg.
$$f(x) = \begin{cases} c e^{\frac{1}{x^2-a^2}} & |x| < a \\ 0 & |x| \geq a \end{cases}$$

Then $f \in C^\infty(\mathbb{R})$

$f = 0 \quad \forall x, |x| \geq a$

- c is s.t. $\int_{\mathbb{R}} f(x) = 1$

$$f_n(x) = n f(nx)$$

~~support of~~ $supp(f_n) = |x| < \frac{1}{n}$

$$\int_{\mathbb{R}} f_n(x)\,dx = \int_{\mathbb{R}} n f(nx)\,dx = \int_{\mathbb{R}} f_{n}(y)\,dy$$

$= \underline{\underline{1}}$. (we have normalized like that only).

$supp(f_n) \longrightarrow \underset{\sim}{0}$

as $\underbrace{|x| < \frac{1}{n}}$

$$\lim |x_n| < \lim \left(\frac{1}{n}\right)$$

$\Rightarrow \underbrace{\lim |x_n| = 0}$

$\lim_{n\to\infty} (length(supp(f_n(x)))) = 0$

as $\exists \frac{1}{n} < x \ \forall x \in \mathbb{R}$

$$\lim_{n\to\infty} (f_n(0)) = \infty$$ [Haven't proved that the limit exists]

So essentially this approximates to a dirac delta function.

$$f_n(x) = \begin{cases} nCe^{\frac{1}{nx^2-a^2}} & |x| \leq \frac{1}{n} \\ 0 & |x| \geq \frac{1}{n} \end{cases}$$

$$\lim (f_n(x)) = \delta(x)$$

Taylor's Theorem (*) write it better !!

Let $f: J \to \mathbb{R}$ be smooth. Let $x, x_0 \in J$.
Then $\exists\, c$ lies between x and x_0

s.t.

$$f(x) = f(x_0) + (x-x_0) f'(x_0) + \dots + \frac{(x-x_0)^n}{n!} f^{(n)}(x_0)$$

$$+ \frac{(x-x_0)^{n+1}}{(n+1)!} f^{(n)}(c)$$

Ex: Let $f: \mathbb{R} \to \mathbb{R}$ be fn s.t.

$$f^{(k)} = 0 \quad \forall x \in \mathbb{R}$$

for some $k \in \mathbb{N}$, then

f is a polynomial of ~~deg~~ degree $(k-1)$

Pf Define

$F: J \to \mathbb{R}$ by

$$F(t) = f(t) + \sum_{k=1}^{n} \frac{(x-t)^k}{k!} f^{(k)}(t).$$

$$+ M (x-t)^{k+1} \quad \text{—— (i)}$$

We choose M s.t. (Ex)

$$F(x) = F(x_0)$$

$$F(x) = f(x)$$

~~Thus by~~ Thus by Rolle's Theorem,

$\exists\, c$ in between x and x_0

s.t. $f'(c) = 0$

Now calculate the derivative

$$F(t) = f'(t) + \sum_{k=1}^{n} \frac{(x-t)^k}{k!} f^{(k)}(t) \quad \text{—— (ii)}$$

$$F'(c) = 0$$

$$\Rightarrow f'(c) = 0$$

From (i) and (ii), we have the derived result.

Ex: Let $f: \mathbb{R} \to \mathbb{R}$ differential (or diff at c):
Let $c \in \mathbb{R}$ and $x_n < c < y_n$ s.t.
$y_n - x_n \to 0$ Then s.t.

$$\lim_{n\to\infty} \frac{f(y_n) - f(x_n)}{y_n - x_n} = f'(c)$$

For the differ. at c case make

$$\lim_{n\to\infty} \frac{f(y_n)-f(c)}{y_n - c} \underbrace{\left(\frac{y_n - c}{y_n - x_n}\right)}_{\alpha} + \frac{f(c) - f(x_n)}{c - x_n} \underbrace{\left(\frac{c - x_n}{y_n - x_n}\right)}_{1-\alpha}$$

Convex combination of both of them. So apply Sandwich theorem.

Holder continous [Random Def^n]

Let $\alpha \in (0,1)$. A f^x $f: J \longrightarrow \mathbb{R}$ is said to be Holder continuous of exponent "α". if

$$|f(x) - f(y)| \leq M|x-y|^{\alpha} \quad \forall\ x, y \in J$$

for some $M > 0$

Eg $f(x) = x^{1/2}$ $\quad$ $\alpha = 1/2$ ✓

$\underbrace{\alpha = 2/3 \ \times}$

Convex Functions

Following def^{ns} are equivalent

i) $f: [a,b] \longrightarrow \mathbb{R}$ be convex.

ii) If $x < z < y$,

$$\frac{f(z)-f(x)}{z-x} \leq \frac{f(y)-f(x)}{y-x} \leq \frac{f(y)-f(z)}{y-z}$$

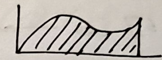
iii) f' is increasing.

iv) $f(x) \geq f(y) + f'(y)(x-y)$ $\forall\ x, y$

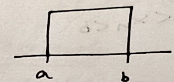
INTEGRATION

* area under a curve (given by f^n).
  known area of a square

$f: [a,b] \longrightarrow \mathbb{R}$.



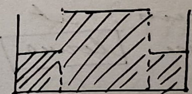
Not^n: area under the curve is denoted by $\int_a^b f(x)\,dx$ / $\int_a^b f$.



$$\int_a^b f = c(b-a)$$

$a = x_0 < x_1 < x_2 < x_3 = b$

$$f(x) = \begin{cases} e_1 & x \in [a, x_1) \\ e_2 & x \in [x_1, x_2) \\ e_3 & x \in [x_2, x_3] \end{cases}$$



$$\int_a^b f = e_1(x_1 - x_0) + e_2(x_2 - x_1) + e_3(x_3 - x_2)$$

Def^n [Partition]

Let $[a,b]$ given. A finite set $P = \{x_0, \ldots, x_n\}$ is called a partition of $[a,b]$ if

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b$$

-(Apparently non standard).

Defⁿ Let P and Q be two partitions of $[a,b]$. We say that Q is a refinement of P if

$$P \subsetneq Q$$

Assumption. $f : [a,b] \to \mathbb{R}$ is bounded.

Notⁿ $m = \inf\{f(x) : x \in [a,b]\}$

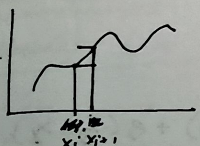
$M = \sup\{f(x) : x \in [a,b]\}$

$P = \{x_0, x_1, \dots x_n\}$ $\quad a = x_0 < x_1 < \dots < x_n < b$

$i = 0, 1, \dots, n-1$

$m_i(f) = \inf\{f(x) : x \in [x_i, x_{i+1})\}$

$M_i(f) = \sup\{f(x) : x \in [x_i, x_{i+1})\}$



If we use $m_i(f)$ for calculating sum, it is called the lower sum.

If we use $M_i(f)$ for calculating sum/integral it is called upper sum.

Defⁿ (Lower Sum)
Let p be a given partition of $[a,b]$, the lower sum of f w.r.t p is denoted by,

$$L(f,P) = \sum_{i=0}^{n-1} m_i(f)\,(x_{i+1} - x_i)$$

Defⁿ [Upper Sum]
We denote the upper sum given by $U(f,P)$ and defined by

$$U(f,P) = \sum_{i=0}^{n-1} M_i(f)\,(x_{i+1} - x_i)$$

Lemma

for any partition p of $[a,b]$

$$m(b-a) \le L(f,P) \le U(f,P) \le M(b-a)$$

Obviously

$m_i(f) \ge m, \quad M_i(f) \le M$

Defⁿ

$L(f) = \sup\{L(f,P) : p \text{ is a partition of } [a,b]\}$

Since $L(f,P)$ is bounded by $M(b-a)$,
$\exists \sup\{L(f,P)\}$

$L(f)$: lower integral of f.

$U(f) = \inf\{U(f,P) : p \text{ is a partition of } [a,b]\}$

Again inf exists as $U(f,P) \ge m(b-a)$

$U(f)$: upper integral of f.

Defn [Darboux integrability]
A bdd fxn $f:[a,b] \to \mathbb{R}$ is said to be Darboux integrable iff

$$L(f) = U(f)$$

If we take a refinement Q of a partition of P, we get $U(f)$ decreases and $L(f)$ increases.

Darboux Integrals

f is said to be Darboux integrable if

$$U(f) = L(f) = \int_a^b f.$$

Thm A $f:[a,b] \to \mathbb{R}$ is darboux integrable iff $\forall \varepsilon > 0$, $\exists$ a partition P_ε s.t.

$$U(f,P_\varepsilon) - L(f,P_\varepsilon) = |U(f,P_\varepsilon) - L(f,P_\varepsilon)| < \varepsilon.$$

Lemma Let P and Q be two partitions of $[a,b]$

(i) $L(f,P) \leq L(f,Q) \leq U(f,Q) \leq U(f,P)$ when $P \subseteq Q$

(ii) $L(f,P) \leq L(f,Q)$ $\forall P,Q$

(iii) $L(f,P) \leq L(f) \leq U(f) \leq U(f,Q)$ $\forall P,Q$

Pf of Lemma

(i) Let Q be a refinement of P.

Let $c \in [a,b] \setminus P$. Let $Q = P \cup \{c\}$.

Then Q is a refinement of P

Then, $\exists i$ s.t.

$$x_i < c < x_{i+1}$$

$$L(f,P) = m_0(f)(x_1 - x_0) + m_1(f)(x_2 - x_1) + \dots + m_{n-1}(f)(x_n - x_{n-1})$$

$$= A + m_i(f)(x_{i+1} - x_i)$$

$$L(f,Q) = A + \tilde{m}_1(f)(c - x_i) + \tilde{m}_2(f)(x_{i+1} - c)$$

$$\tilde{m}_1 = \inf_{x \in [x_i, c]} \{f\} \qquad \tilde{m}_2 = \inf_{x \in [c, x_{i+1}]} (f)$$

Note that $\tilde{m}_1, \tilde{m}_2 \geq m_i(f)$

Then

$$L(f,P) \leq L(f,Q)$$

We can then recursively continue, adding non-P elements each one by one with P,Q, $P \subseteq Q$

we have $L(P,P) \leq L(P,Q)$.

Similarly we can show that for

$$U(f,Q) \leq U(f,P).$$

(ii) Take the partition $\tilde{P} = P \cup Q$. Then $\tilde{P}$ is a partition of $[a,b]$.

using (i) as $P \subseteq \tilde{P}$ and $Q \subseteq \tilde{P}$

$$\underbrace{L(f,P) \leq L(f,P\cup Q)}\ .\ \underbrace{\leq U(f,\tilde{P}) \leq U(f,Q)}$$

Have been individually proved.

(iii) ~~Let~~ We can see that $L(f)$ is a lower bound of the set

$\tilde{U} = \{U(f,P) : P \text{ is a partition}\}$

$L(f) \leq U(f,P)$.

Since $U(f)$ is $\inf \tilde{U}$

$L(f) \leq U(f)$ [Proved]

Proof of Thm

Let $f : [a,b] \longrightarrow \mathbb{R}$ be integrable

Then $U(f) = L(f)$. Let $\varepsilon > 0$

Since $U(f) = \inf_{P \in I} \{U(f,P)\}$, $\exists\, P'_\varepsilon$ s.t.

$$U(f, P'_\varepsilon) < U(f) + \varepsilon.$$

Then $L(f) = \sup_{P \in J} L(f,P)$ $\exists\, \tilde{P}_\varepsilon$ s.t.

$$L(f,\tilde{P}_\varepsilon) > L(f) - \varepsilon$$

Let $P_\varepsilon = P'_\varepsilon \cup \tilde{P}_\varepsilon$

$$\underbrace{L(f,\tilde{P}_\varepsilon) \leq L(f,P_\varepsilon)}_{\tilde{P}_\varepsilon \leq P_\varepsilon} \leq L(f) = U(f)$$

$$\left.\begin{aligned} &\leq U(f,P_\varepsilon) \\ &\leq U(f,P'_\varepsilon). \end{aligned}\right\} P'_\varepsilon \subseteq P_\varepsilon$$

$$L(f) - \varepsilon \leq \underbrace{L(f,\tilde{P}_\varepsilon) \leq U(f,P_\varepsilon)}_{\text{Inner points of suppose an interval.}} < U(f) + \varepsilon$$

$\Rightarrow$ ~~$L(f) - \varepsilon \leq U(f) + \varepsilon$~~

$$|U(f,P_\varepsilon) - L(f,P_\varepsilon)| < 2\varepsilon \quad \text{[Proved]}$$

($\Leftarrow$) We know that $L(f) \leq U(f)$

Enough to prove that $L(f) \geq U(f)$

Let $\varepsilon > 0$, $\exists\, P_\varepsilon$ s.t.

$$U(f,P_\varepsilon) - L(f,P_\varepsilon) < \varepsilon$$

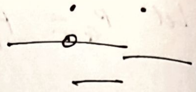
$$U(f) \leq U(f,P_\varepsilon) < L(f,P_\varepsilon) + \varepsilon \leq L(f) + \varepsilon$$

$\Rightarrow U(f) \leq L(f)$

$\Rightarrow U(f) = L(f)$.

Example

Dirichlet's func



$f : [0,1] \to \mathbb{R}$

Prove that Dirichlet func

$$f(x) = \begin{cases} 0 & x \in \mathbb{Q} \\ 1 & \text{otherwise} \end{cases}$$

Proof

Let p be any partition $p = \{x_0, \dots, x_n\}$ of $[0,1]$

then

$M_i(f) = 1 \qquad m_i(f) = 0$

$U(f,P) = 1 \qquad L(f,P) = 0$

$U(f) = 1 \qquad L(f) = 0$

~~Thus~~ $U(f) \neq L(f)$.

Thus f is not integrable.

The func is discontinuous at every pt in $[0,1]$.

Ex

$$f(x) = \begin{cases} 0 & x \in [0,1) \\ 100 & x = 1 \end{cases} \qquad f : [0,1] \to \mathbb{R}$$

Ans

$M_i(f) = 100 \sin$

$m_i(f) = 0$

$U(f,P) = 100 \underbrace{(x_n - x_{n-1})}_{h}$ ~~< ε~~ choose $(x_n - x_{n-1}) < \frac{\varepsilon}{100}$

$L(f,P) = 0$

$U(f, P_\varepsilon) - L(f, P_\varepsilon) < \varepsilon$

Let $\alpha_1, \dots, \alpha_m$ are real ~~f~~ no.s. Let ~~a~~

$0 < c_1 < c_2 < \dots < c_m = b$

$M_i < M$

$-m_i < -m$

Define

$$f(x) = \begin{cases} \alpha_1 & x \in [0, c_1) \\ \alpha_2 & x \in [c_1, c_2) \\ \vdots & \\ \alpha_m & x \in [c_{m-1}, c_m] \end{cases}$$

Then f is integrable.

Thm: Let $f : [a,b] \to \mathbb{R}$ be monotone. Then f is integrable.

Pf WLOG, assume f is monotonically increasing.

Let $\varepsilon > 0$. Let $P = \{x_0, \dots, x_n\}$ be a partn of func.

$$U(f,P) - L(f,P) = \sum_{i=0}^{n-1} (M_i(f) - m_i(f))(x_{i+1} - x_i)$$

~~since the function bounded~~,

$$= \sum_{i=0}^{n-1} (f(x_{i+1}) - f(x_i))(x_{i+1} - x_i)$$

$$\leq \max_i (x_{i+1} - x_i) \sum (f(x_{i+1}) - f(x_i))$$

$$= (f(b) - f(a)) \left(\max_i (x_{i+1} - x_i) \right)$$

Choose partition P s.t.

$$\max_i (x_{i+1} - x_i) < \frac{\varepsilon}{f(b) - f(a)}$$

choose P s.t.

- $x_{i+1} - x_i = h \ \forall i$
- $x_0 = a,\ x_1 = a+h,\ \dots,\ x_n = x_0 + nh = b$

$$h = \frac{b-a}{n} \quad n \in \mathbb{N}$$

choose

$$\frac{1}{n} < \frac{\varepsilon}{(b-a)(f(b)-f(a))}$$

Thus we have a partition P_ε

$$P_\varepsilon = \{x_0, \dots, x_n\} \quad \text{s.t. } U(f,P) - L(f,P) < \varepsilon$$

Thm

Let $f:[a,b] \to \mathbb{R}$ be continuous. Then f is continuous.

Pf Since f is defined on a compact and is continuous, f is uniformly continuous.

Let P be a partition. Let $\varepsilon > 0$

Now

$$U(f,P) - L(f,P) = \sum_{i=0}^{n-1} (M_i(f) - m_i(f))(x_{i+1} - x_i)$$

Uniform Continuity

A $f: [a,b] \to \mathbb{R}$ is UC if $\varepsilon > 0, \exists \delta > 0$

$x, y \in [a,b]$, $|x-y| < \delta$

$\Rightarrow |f(x) - f(y)| < \varepsilon$

Let us choose P s.t.

$$|x_{i+1} - x_i| < \delta \quad \forall i$$

Then $|f(\tilde{l}_i) - f(l_i)| < \varepsilon'$ Assume $\varepsilon' = \frac{\varepsilon}{(b-a)}$

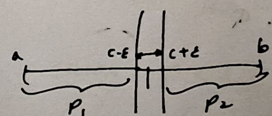
Then

$$U(f,P) - L(f,P) \leq \varepsilon' \sum (x_{i+1} - x_i)$$
$$= \varepsilon'(b-a)$$
$$= \varepsilon$$

Thm

Let $f:[a,b] \to \mathbb{R}$ be bdd. Let $c \in (a,b)$.
Assume that f is cont on $[a,b]$ except c.
Then f is integrable. [Works for finitely many pts]

Proof idea



Take $P = P_1 \cup P_2$ and arbitrarily take $\varepsilon \to 0$

Thm

Let $f:[a,b] \to \mathbb{R}$ cont & bdd except countable no of pts. then f is integrable.

↳ Like Thomae's Func.

Thm A bdd $f:[a,b] \to \mathbb{R}$ is integrable iff measure of a set of discontinuities is zero

---

Thomae's func

$f:[0,1] \longrightarrow \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 & x = 0 \\ 1/q & x = p/q, \ \gcd(p,q) = 1 \\ 0 & x \in [0,1] \setminus \mathbb{Q} \end{cases}$$

We will study the continuity of f.

We will show that

(i) f is continuous at $\mathbb{R} \setminus \mathbb{Q}$ and at 0

(ii) f is discontinuous at non-zero rational pts.

(ii) Let $c \in (0,1] \cap \mathbb{Q}$

Choose a seqn $c_n \in [0,1] \setminus \mathbb{Q}$

s.t. $c_n \longrightarrow c$

Then $f(c_n) = 0$ but $f(c) \neq 0$

$\Rightarrow f(c_n) \not\to f(c)$

$\Rightarrow f$ is disc at c.

(i) f is cont. at 0.

Observe that $\boxed{0 \le f(x) \le x}$ $\forall x \in [0,1]$

$\Rightarrow \lim_{x \to 0} f(x) = 0 = f(0)$

Continous at 0.

P-2

Let $c \in [0,1] \setminus \mathbb{Q}$

CLAIM: f is cont. at c.

Let $\varepsilon > 0$, Find $\delta > 0$ s.t.

$|f(x) - f(c)| < \varepsilon \quad \forall x \in (c-\delta, c+\delta) \subseteq (0,1)$

$\Rightarrow f(x) < \varepsilon \quad \forall x \in (c-\delta, c+\delta) \cap \mathbb{Q} \subseteq (0,1)$

if $(c-\delta, c+\delta) \setminus \mathbb{Q}$

Choose $\frac{1}{k} < \varepsilon$

$A_k = \bigcup_{i=2}^{k} \{x \mid f(x) = \frac{1}{i-1}\}$

A_k is a finite set ~~assume that~~

we have cardinality of $A_k = k$.

Note that $\forall x \in A_k^c$

$f(x) < \frac{1}{k}$

$r_i \in A_k$

- Choose $0 < \delta < \min\{|c - r_1|, \cdots |c - r_k|\}$
- $(c-\delta, c+\delta) \cap A_k = \phi$
- $(c-\delta, c+\delta) \subset A_k^c$

$\Rightarrow f(x) < \varepsilon \quad \forall x \in (c-\delta, c+\delta)$.

~~Thm~~ **Thm** [Thomae's func is integrable]

Pf Let $\varepsilon > 0$. Find P_ε s.t.

$$U(f, P_\varepsilon) - L(f, P_\varepsilon) < \varepsilon$$

Let p is any partition $\{x_0, \dots x_n\}$.

$$U(f, P) - L(f, P) = \sum_{i=0}^{n-1} (M_i(f) - m_i(f))(x_{i+1} - x_i)$$

$m_i(f) = 0$

$$= \sum_{i=0}^{n-1} (M_i(f))(x_{i+1} - x_i)$$

$\max(x_{i+1} - x_i) < \delta$

Since $\varepsilon > 0$, $\exists\, k \in \mathbb{N}$ s.t. $\frac{1}{k} < \varepsilon$.

consider the set A^k

Let $|A^k| = N$

$R = \{ i \mid [x_i, x_{i+1}] \cap A_k = \phi \}$

$R^c = \{ i \mid [x_i, x_{i+1}] \cap A_k \neq \phi \}$

$$U(f, P) - L(f, P) = \sum_{i \in R} M_i(f)(x_{i+1} - x_i) + \sum_{i \in R^c} M_i(f)(x_{i+1} - x_i)$$

$$\leq \frac{1}{k} + \delta N$$

Take $\delta = \frac{\epsilon}{2N}$

$$U(f, P) - L(f, P) < 3\epsilon/2$$

Let $k \in \mathbb{N}$, s.t. $2/k < \varepsilon$.

Choose $P_\varepsilon = \{x_0, \dots, x_n\}$ s.t.

$$\max_i \{(x_{i+1} - x_i)\} < \frac{1}{kN}$$

where $N = |A^k|$

and $A^k = \bigcup_{i=2}^{k} \{ f(x) = \frac{1}{k-1} \}$

$$U(f, P) - L(f, P) < \varepsilon.$$

Notation.

$\mathcal{R}([a, b]) = \{ f : [a, b] \to \mathbb{R} : f \text{ is integrable on } [a, b] \}$

$\mathcal{R} \to$ Riemann Integrable

Thm

(i) $f \in R[a,b] \Rightarrow cf \in R([a,b])$ $\forall\, x \in \mathbb{R}$

(ii) $f, g \in R[a,b] \Rightarrow f+g \in R([a,b])$ $\forall\, x \in \mathbb{R}$

(iii) $R([a,b])$ is a vector space

(iv) $C([a,b]) \subseteq R([a,b])$

(v) $T: R([a,b]) \to \mathbb{R}$ defined

$T(f) = \int_a^b f$ is a linear map

Pf [Scratch Work]

$$U(c,f) = \inf_P \{U(cf, P)\}.$$

$$= \begin{cases} \inf c\, U(f,P) & c > 0 \\ \inf c\, L(f,P) & c < 0 \end{cases}$$

$$= \begin{cases} c \inf_P U(f,P) & c > 0 \\ c \sup_P L(f,P) & c < 0 \end{cases}$$

$$= \begin{cases} c\, U(f) & c > 0 \\ c\, L(f) & c < 0 \end{cases}$$

since $f \in R([a,b])$

and $L(f) = U(f)$

$= cL(f) = cU(f)$.

$\Rightarrow L(cf) = U(cf)$.

Thm

Let $f, g : [a,b] \to \mathbb{R}$ be two integrable fn's.

Assume that $f(x) \le g(x)$ $\forall\, x \in [a,b]$.

Then $\int_a^b f \le \int_a^b g$

Pf [Exercise]

$g - f \geq 0$

$\int (g-f) \geq 0$

Thm) Let $f : [a,b] \to \mathbb{R}$ be integrable function.

Assume that $m \le f(t) \le M$ $\forall\, t \in [a,b]$.

Let $g : [m, M] \to \mathbb{R}$ be cont. Then

$g \circ f : [a,b] \to \mathbb{R}$ is integrable.

Rem: $g \in R([m,M])$ & $f \in R([a,b])$

$\not\Rightarrow g \circ f \in R([a,b])$

Suppose $g(x) := \begin{cases} 0 & x = 0 \\ 1 & x \neq 0 \end{cases}$

$f : [0,1] \to \mathbb{R}$ $f(x) := \{$ Thomaes function

$g \circ f(x) \approx$ Dirichlet

Pf Let $\varepsilon > 0$. Then we want to find a partition P_ε of $[a,b]$ s.t.

$$U(g\circ f, P_\varepsilon) - L(g\circ f, P_\varepsilon) < \varepsilon$$

Since g is uniformly continuous on $[m, M]$, $\exists$ $\delta_1 > 0$

s.t.

$$x, y \in [m, M], \; |x-y| < \delta_1$$

$$\Rightarrow |g(x) - g(y)| < \varepsilon$$

Define $\delta := \min\{\varepsilon, \delta_1\}$

Then for this $\delta > 0$,

$$|x-y| < \delta \Rightarrow |g(x) - g(y)| < \varepsilon$$

Since $f : [a,b] \to \mathbb{R}$ is integrable, $\exists$ $\delta_1 \in (0, \varepsilon)$, for $\eta > 0$ $\exists P_\eta$ s.t.

$$U(f, P_\eta) - L(f, P_\eta) < \eta$$

Let $P_\eta = \{t_0, \ldots, t_n\}$

Then

$$\sum_{i=0}^{n-1} (M_i(f) - m_i(f))(t_{i+1} - t_i) < \eta$$

Denote

$$A := \{ i \in \{0, 1, \ldots, n-1\} : M_i(f) - m_i(f) < \delta \}$$

For $i \in A$

$$\Rightarrow M_i(f) - m_i(f) < \delta$$

$$\Rightarrow |f(t) - f(s)| < \delta \qquad \forall\, t, s \in [t_i, t_{i+1}]$$

$$\Rightarrow |g\circ f(t) - g\circ f(s)| < \varepsilon \qquad \text{by uniform continuity}$$

$$\Rightarrow M_i(g\circ f) - m_i(g\circ f) \leq \varepsilon$$

$$\boxed{(i \in A) \quad M_i(g\circ f) - m_i(g\circ f) \leq \varepsilon}$$

Now let us take $i \in A^c$

$$\Rightarrow \eta > \sum_{i=0}^{n-1} (M_i(f) - m_i(f))(t_{i+1} - t_i)$$

$$\geq \sum_{i \in A^c} (M_i(f) - m_i(f))(t_{i+1} - t_i)$$

$$\geq \delta \sum_{i \in A^c} (t_{i+1} - t_i).$$

$$\Rightarrow \sum_{i \in A^c} (t_{i+1} - t_i) \leq \frac{\eta}{\delta}.$$

When we are given ε, and the function $g(x)$, we know that δ_1 and δ is well defined, so we can choose $\eta = \varepsilon\delta$

$\Rightarrow$ for $i \in A^c$

$$\sum_{i \in A^c} (t_{i+1} - t_i) \leq \varepsilon$$

$U(g\circ f, P) - L(g\circ f, P)$

$$= \sum_{i=0}^{n-1} \left(M_i(g\circ f(x)) - m_i(g\circ f(x))\right)(t_{i+1} - t_i)$$

$$= \sum_{i\in A} + \sum_{i\in A^c}$$

$$\leq \varepsilon \sum_{i\in A} (t_{i+1} - t_i) + \varepsilon \sum_{i\in A^c} \left(M_i(g\circ f(x)) - m_i(g\circ f(x))\right)$$

$$\leq \varepsilon(b-a) + \varepsilon(c) \text{ where}$$

$$c = \sup(g\circ f(x)) - \inf(g\circ f(x))$$

$$\leq \varepsilon(b-a + c)$$

(Proved)

Small note

P is the original partition in f's integrability.

$\underline{\underline{\text{Def}^{n}}}$ [tg] $P = \{x_0, \dots, x_n\}$ be a partition.

$t = \{t_0, \dots, t_{n-1}\}$ is a $\underline{\underline{\text{tag}}}$ of P if

$t_i \in [x_i, x_{i+1}]$ $x = 0, 1, \dots, n-1$

$\underline{\underline{\text{Def}^{n}}}$ [Riemann Sum]

$f : [a, b] \to \mathbb{R}$

$$S(f, P, t) = \sum_{i=0}^{n-1} f(t_i)(x_{i+1} - x_i)$$

Here S is called a Riemann Sum.

$\underline{\underline{\text{Def}^{n}}}$ [Riemann integrable]

A f^n $f : [a, b] \to \mathbb{R}$ is Riemann int if

$\epsilon > 0$, $\exists P_\epsilon$ s.t.

$$|S(f, Q, t) - A| < \epsilon$$

$\forall Q \supseteq P$ and $\forall$ tags t_i of Q. for some $A \in \mathbb{R}$

$\underline{\underline{\text{Thm 1}}}$

let $f : [a, b] \to \mathbb{R}$ is Riemann integrable. Then

f is bdd.

$\underline{\underline{\text{Thm 2}}}$

suppose for R we have A_1, A_2 s.t.

it satisfied all def^ns of Rieman int,

$$\underline{\underline{A_1 = A_2}}$$

Thm 3

Let $f:[a,b] \to \mathbb{R}$ be bdd. Then f is integrable iff f is riemann integrable. Moreover.

$$R(f) = {}^{b}\!\!\int f$$

Pf of Thm

Since f is Riemann int, for $\varepsilon = 1$, $\exists Q$ s.t.

$$|S(f, Q, t) - A| < 1. \quad \forall \text{ tag } t \text{ of } Q$$

Let $Q = \{x_0, \dots, x_n\}$.

Consider $[x_j, x_{j+1}]$ we claim that f is bdd $[x_j, x_{j+1}]$

$t = \{t_0, t_1, \dots, t_{n-1}\}$.

let $s = x_j$

$s = \{t_0, t_1, \dots, t_{j-1}, s, t_{j+1}, \dots, t_{n-1}\}$

$S(f, Q, t) - S(f, Q, s)$

$\Rightarrow (f(t) - f(s))(x_{j+1} - x_j)$

$= (f(t) - f(x_j))(x_{j+1} - x_j)$

we know

$$|S(f, Q, t) - A| < 1, \quad |S(f, Q, s) - A| < 1$$

$$\Rightarrow |S(f, Q, t) - S(f, Q, s)| < 2.$$

$$\Rightarrow |f(t) - f(x_j)| \leq \frac{2}{x_{j+1} - x_j}.$$

$$\Rightarrow |f(t)| < \underbrace{\frac{2}{x_{j+1} - x_j} + f(x_j)}_{\text{Take sup}\left(\frac{2}{x_{j+1} - x_j} + f(x_j)\right)}.$$

Thus f is bdd

Pf 3

Let f is int.

Let $\varepsilon > 0$, since

$$\inf_P (U(f, P)) = {}^{b}\!\!\int f.$$

$$\sup_P (L(f, P)) = {}^{b}\!\!\int f.$$

$\exists P_\varepsilon$

$$U(f, P_\varepsilon) < \int_a^b f + \varepsilon$$

$$L(f, P_\varepsilon) < \int_a^b f - \varepsilon$$

Let $Q \supseteq P_\varepsilon$ and t be a tag of Q.

$$L(f,Q) \leq S(f,Q,t) \leq U(f,Q)$$

$$\Rightarrow L(f,P_\varepsilon) \leq S(f,Q,t) \leq U(f,P_\varepsilon)$$

and

$$m_j(f) \leq f(t_j) \leq M_j(f)$$

$$\int_a^b f - \varepsilon \leq S(f,Q,t) \leq \int_a^b f + \varepsilon$$

$$\Rightarrow \left| S(f,Q,t) - \int_a^b f \right| < \varepsilon$$

($\Leftarrow$)

Let f be riemann integrable.

Let $\varepsilon > 0$, $\exists P$ s.t.

$$|S(f,P,t) - R(f)| < \varepsilon \quad \forall\ P \subseteq Q \text{ and } \forall \text{ tags } t \text{ of } Q.$$

Let $P = \{x_0, \dots, x_n\}$

$M_j(f) = \sup_{[x_j, x_{j+1}]} f$ $\exists\, t_i \in [x_i, x_{i+1}]$ and $s_i \in [x_i, x_{i+1}]$

$m_j(f) = \inf_{[x_j, x_{j+1}]} f$

$f(t_i) > M_j(f) - \varepsilon$

$f(s_i) < m_j(f) + \varepsilon$

$$M_i(f) - m_i(f) < f(t_i) - f(s_i) + 2\varepsilon$$

$$U(f,P) - L(f,P) < S(f,Q,t) - S(f,Q,s) + 2\varepsilon(b-a)$$

$$< |S(f,Q,t) - S(f,Q,s)| + 2\varepsilon(b-a)$$

$$< \varepsilon$$

Thm [Fundamental Thm of Calculus]

Let $f : [a,b] \to \mathbb{R}$ be diff. Assume that $f' : [a,b] \to \mathbb{R}$ is integrable. Then

$$\int_a^b f'\,dx = f(b) - f(a).$$

Pf Let $P = \{x_0, x_1, \dots, x_n\}$ be a partition.

$$f(b) - f(a) = \sum_{i=0}^{n-1} \left(f(x_{i+1}) - f(x_i) \right)$$

$$\overset{MVT}{=} \sum_{i=0} f'(t_i)\,(x_{i+1} - x_i)$$

for some $t_i \in [x_{i+1}, x_i]$

Note

$$\underbrace{L(f',P)} \le \underline{\sum f'(t_i)(x_{i+1}-x_i)} \le U(f',P).$$

$\Rightarrow \forall P$

$$L(f',P) \le f(b)-f(a) \le U(f',P)$$

since f' is integrable.

$$\int_a^b f' = \sup_P L(f',P) = \inf_P U(f',P)$$

$$\Rightarrow \sup_P L(f',P) \le f(b)-f(a) \le \inf_P U(f',P)$$

$$\Rightarrow f(b)-f(b) = \int_a^b f'\,dx$$

Thm (2nd FTC)

Let $f:[a,b]\to\mathbb{R}$ be integrable fn.

Define $F:[a,b]\to\mathbb{R}$ by

$$F(x) = \int_a^x f(t)\,dt$$

Then (i) $F(x)$ is ~~unif~~ lipschitz continuous

(ii) if $f(x)$ is cont. at c, then F is diff at c

$$F'(c) = f(c)$$

Some notes

Define $\frac{d}{dx}: C'([a,b]) \to C([a,b])$.

$$I(f)(x) = F(x) = \int_a^x f(t)\,dt$$

$$\underbrace{I: C([a,b]) \to C'([a,b])}$$

Now 2nd FTC $\Rightarrow$

$$\frac{d}{dx}(I)(f) = f \text{ on } C^1[a,b]\ \ C_0'([a,b])$$

$\subseteq C_0'([a,b]) = \{f\in C'([a,b]) : f(a)=0\}$

1st FTC $\frac{d}{dx}(I) = Id$ on $C([a,b])$

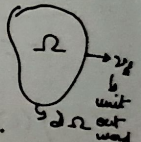
2nd FTC $I\left(\frac{d}{dx}\right) = Id \to$ on $C_0'([a,b])$.

Gauss

$$\int_\Omega \partial_{x_i} f = \int_{\partial\Omega} f\nu_i\,dS$$

$\Downarrow$

FTC.



$\nu_i \mapsto i$th component.

Pf (i) F is lipschitz continuous.

Since f is bdd, $\exists M>0$ s.t.

$$|f(x)| \le M \quad \forall x\in[a,b]$$

Now

$$F(x)-F(y) = \int_y^x f$$

$$|F(x)-F(y)| = \left|\int_y^x f\right| = \left|\int_{\min\{x,y\}}^{\max\{x,y\}} f(t)\,dt\right|$$

$$\le \int_{\min\{x,y\}}^{\max\{x,y\}} |f(t)|\,dt \le M|x-y|$$

Thus $F(x)$ is lipschitz

(ii) If f is cont at c, then

F is diff at c,

then $F'(c) = f(c)$

WLOG $x > c$

$$\frac{F(x) - F(c)}{x - c} = \frac{1}{x-c}\int_c^x f(t)\,dt$$

$$f(c) = \frac{1}{x-c}\int_c^x f(c)\,dt$$

$$\frac{F(x) - F(c)}{x - c} - f(c) = \frac{1}{x-c}\int_c^x (f(t) - f(c))\,dt$$

Now we know f is continuous.

$\forall \varepsilon > 0, \exists \delta > 0$ s.t.

$|f(t) - f(c)| < \varepsilon \quad \forall\, 0 \le |t-c| < \delta$,

Choose $\delta > 0$ s.t.

$|f(t) - f(c)| < \varepsilon \quad \forall t \in V_\delta(c)$

Try to prove this

$$\left| \frac{F(x) - F(c)}{x - c} - f(c) \right| = \frac{1}{x-c}\int_c^x (f(t) - f(c))\,dt$$

$$< \frac{1}{x-c}\int_c^x \varepsilon\,dt$$

$\Rightarrow\ < \varepsilon$ [Done]

Thm

Let u, v be two differentiable function s.t. u', v' are integrable. Then

$$\int_a^b uv' = uv\Big|_a^b - \int_a^b u'v$$

$$[uv]_a^b = u(b)v(b) - u(a)v(a)$$

Pf

Note uv is differentiable and using product rule.
$(uv)' = u'v + uv'$

We have proved. if f is integrable and g is integrable. , fg is integrable.

Thus $(uv)'$, $u'v$, uv' are all integrable

$$\int_a^b (uv)' = \int_a^b u'v + \int_a^b uv'$$

By using the first fundamental theorem of calculus,

$$[uv]_a^b = \int_a^b u'v + \int_a^b uv'$$

$$\Rightarrow \boxed{\int_a^b uv' = [uv]_a^b - \int_a^b uv'}$$

Thm (Substitution method)

Let $u : [a, b] \to \mathbb{R}$, $f : J \to \mathbb{R}$ and $u([a, b]) \subseteq J$, where J is a closed and bounded interval.

Assume that (i) $u : [a, b] \to \mathbb{R}$ be diff

(ii) u' is integrable

(iii) $u([a,b]) \subseteq J$ (iv) $f: J \to \mathbb{R}$ is continuous

Thm

$$\int_a^b f(u(x))\, u'(x)\, dx = \int_{u(a)}^{u(b)} f(t)\, dt$$

Pf

Let $F(x) = \int_c^x f(t)\,dt$ $\therefore c \in J$

Then by 2^{nd} FTC,

$$F'(x) = f(x)$$

$$\int_a^b f(u(x))\, u'(x)\, dx = \int_a^b F'(u(x))\, u'(x)\, dx$$

$$= \int_a^b \big(F(u(x))\, u(x)\big)'\, dx$$

by chain rule

$$= F(u(b))\cancel{u(b)} - F(u(a))\cancel{u(a)}$$

$$= \int_c^{u(b)} f(t)\, dt - \int_c^{u(a)}$$

$$= \int_{u(a)}^{u(b)} f(t)\, dt.$$

Thus

$$\int_a^b f(u(x))\, u'(x)\, dx = \int_{u(a)}^{u(b)} f(t)\, dt$$

$\therefore$

Mean Value Thm for int of $f^{x^n s}$

Thm

Let $f \in C([a,b])$. Then $\exists\, c \in [a,b]$ s.t.

$$f(c) = \frac{1}{b-a} \int_a^b f$$

Pf Let $M = \max_{[a,b]} f$, $m = \min_{[a,b]} f$. by extreme value theorem.

we can see

$$m \le f \le M$$

$$\Rightarrow m(b-a) \le \int_a^b f \le M(b-a)$$

$$\Rightarrow m \le \frac{1}{(b-a)} \int_a^b f \le M$$

Since $m, M \in f([a,b])$ by EVT, ~~then~~ we have by IVT, $\exists\, c \in [a,b]$ s.t.

$$f(c) = \frac{1}{b-a} \int_a^b f(x)\, dx$$

Thm (weighted MVT) Let $f \in C([a,b])$, $g \in \mathcal{R}([a,b])$. Assume that g does not change sign in $[a,b]$.

Then $\exists\, c \in [a,b]$ s.t.

$$f(c) \int_a^b g = \int_a^b fg\, dx$$

g is called the ~~fn for~~ weight ~~function~~ fx's.

Pf

Since f is continuous ~~s.t.~~ $\exists M, m$ ~~s.t.~~ $\in f([a,b])$

s.t. $m \leq f \leq M$

$$m g \leq fg \leq M g.$$

$$m\int_a^b g \leq \int_a^b fg \leq M\int_a^b g.$$

WLOG $g(x) \geq 0 \ \forall x \in [a,b]$

Thus Let

$\tilde{f}(x) = f(x)\int_a^b g(x)$, Apply IVT for $\tilde{f}$, ~~get~~

we get the result

$$\tilde{f}(c) = \int_a^b fg \, \cancel{dx}$$

$$\boxed{f(c)\int_a^b g(x)\,dx = \int_a^b f(x)\,g(x)\,dx}$$

<u>other way</u>

either $\int_a^b g(x) = 0$ or $\int_a^b g(x) \neq 0$

If $\int_a^b g(x) = 0$, the statement is trivially true.

otherwise take $\int_a^b g(x)\,dx = \lambda$ and then.

$m \leq \frac{1}{\lambda}\int_a^b f(x)g \leq M$, apply IVT on f and solve it [trivial]

Theorem (2nd MVT)

Let $f:[a,b] \to \mathbb{R}$ be continuously differentiable and $g:[a,b] \to \mathbb{R}$ be continuous. Assume that f' doesn't change sign on $[a,b]$. Then $\exists c \in [a,b]$ s.t.

$$\int_a^b f(x)g(x)\,dx = f(b)\int_c^b g(t)\,dt + f(a)\int_a^c g(t)\,dt$$

Pf Define $G = \int_a^x g(t)\,dt$. Since g is continuous we have

$$G'(x) = g(x) \ [\text{from } 2^{nd} \text{ FTC}]$$

Then,

$$\int_a^b f(x)g(x)\,dx = \int_a^b f(x)\,G'(x)\,dx$$

use integration by parts,

$$\int_a^b f(x)\,G'(x)\,dx = \int_a^b f(x)\,g(x)\,dx$$

$$= f(x)\,G(x)\Big|_a^b - \int_a^b f'(x)G(x)\,dx$$

$$= \big(f(b)G(b) - f(a)G(a)\big) - \int_a^b f'(x)\,G(x)\,dx$$

Now using the previous theorem as f' does not change sign in $[a,b]$, $\exists\, c$ s.t.

$$\int_a^b f'G = G(c)\int_a^b f' = G(c)\left[f(b)-f(a)\right]$$

$$\int_a^b f(x)g(x)\,dx = -G(c)\left[f(b)-f(a)\right] + f(b)G(b) - f(a)G(a)$$

$$= f(b)\left[G(b)-G(c)\right] + f(a)\left[G(c)-G(a)\right]$$

$$= f(b)\int_c^b G'(x)\,dx + f(a)\int_a^c G'(x)\,dx$$

$$= f(b)\int_c^b g(x)\,dx + f(a)\int_a^c g(x)\,dx$$

Thus we have seen,

$$\boxed{\int_a^b fg = f(b)\int_c^b g + f(a)\int_a^c g}$$

$\square$

Theorem

Let f be monotone in $[a,b]$. Then $\exists\, c\in[a,b]$ s.t.

$$\int_a^b f = f(a)(c-a) + f(b)(b-c)$$

Pf [Proof-1]

~~4~~ WLOG, let f be increasing,

Then,

$$f(a)\le f(x)\le f(b) \quad \forall x\in[a,b]$$

This implies

$$f(a)(b-a) \le \int_a^b f \le f(b)(b-a)$$

Let $H(x) := f(a)(x-a) + f(b)(b-x)$

$$H(b) = f(a)(b-a)$$

$$H(a) = f(b)(b-a)$$

$$\Rightarrow H(b) \le \int_a^b f \le H(a)$$

We know that $H(x)$ is continous.

By IVT, we ~~can~~ ~~get~~ have

$$\exists\, c\in[a,b] \Rightarrow H(c) = \int_a^b f$$

$$\Rightarrow \int_a^b f = f(a)(c-a) + f(b)(b-c)$$

$\square$

Proof - [2]

We have f is monotone, then f' does not change sign on $[a,b]$.

Let $\tilde{f}(x) := x f'(x)$, by prev thm we have $\exists c \in [a,b]$ s.t.

$$\Rightarrow c \int_a^b f'(x) = \int_a^b \tilde{f}(x)\,dx$$

$$\Rightarrow c\,[f(b)-f(a)] = \int_a^b x f'(x)$$

$$= x f(x)\Big|_a^b - \int_a^b f(x)\,dx$$

$$\Rightarrow c\,[f(b)-f(a)] = b f(b) - a f(a) - \int_a^b f(x)\,dx$$

$$\Rightarrow \int_a^b f = f(b)(b-c) + f(a)(c-a)$$

$\square$

We will now prove Taylor's theorem, in the form of an integral remainder.

Let $f : [a,b] \longrightarrow \mathbb{R}$ be smooth by FTC,

$$f(b) - f(a) = \int_a^b f'(t)\,dt$$

$$= -\int_a^b \frac{d}{dt}(b-t)\, f'(t)\,dt$$

We apply integration by parts,

$$= +\int_a^b (b-t) f''(t)\,dt - \left[(b-t) f'(t)\right]_a^b$$

$$\Rightarrow f(b) = f(a) + f'(a)(b-a) + \int_a^b (b-t) f''(t)\,dt$$

Now again we apply integration by parts,

$$\int_a^b (b-t) f''(t)\,dt = -\frac{1}{2}\int_a^b \frac{d}{dt}(b-t)^2 f''(t)\,dt$$

$$= +\frac{1}{2}(b-a)^2 f''(a) + \frac{1}{2}\int_a^b (b-t)^2 f'''(t)\,dt$$

$$\Rightarrow f(b) = f(a) + f^{(1)}(b-a) + \frac{1}{2}(b-a)^2 f^{(2)}(a) + \frac{1}{2}\int_a^b (b-t)^2 f^{(3)}(t)\,dt$$

By induction,

$$f(b) = f(a) + (b-a) f^{(1)}(a) + \frac{(b-a)^2}{2} f^{(2)}(a)$$

$$+ \dots + \frac{(b-a)^n}{n!} f^{(n)}(a)$$

$$\underbrace{+ \frac{1}{n!} \int_a^b (b-t)^n f^{(n+1)}(t)\, dt}$$

Integral form of ~~Tay~~ Taylor's theorem.

Improper Integral

⊛ **Prop^n**

Let $f:[a,b] \to \mathbb{R}$ be integrable,

Then $\lim_{d \to b^-} \lim_{c \to a^+} \int_c^d f\, dt = \int_a^b f$

$= \lim_{c \to a^+} \lim_{d \to b^-} \int_c^d f\, dt$

Pf:-

Fix $t_1 \in [a,b]$, then,

$$\int_c^d f = \int_c^{t_1} f + \int_{t_1}^d f$$

$$= G(c) + F(d)$$

Then

G & F are continuous on $[a,b]$. Hence,

$\lim_{c \to a^+} [G(c)]$ & $\lim_{d \to b^-} F(d)$ exists

Hence proved

Def^n [Improper Integrability]

Let (a,b) be an open interval (may be unbounded)

1. A $f x^n$ $f:(a,b) \to \mathbb{R}$ is locally integrable iff f is integrable on $[c,d]$ for every $[c,d] \subseteq (a,b)$

2. A local integrable $f x^n$ $f:(a,b) \to \mathbb{R}$ is said to be improper integrable if $\lim_{c \to a^+} \lim_{d \to b^-} \int_c^d f$ exists

If it exists,

$$\int_a^b f = \lim_{d \to b^-} \lim_{c \to a^+} \int_c^d f$$

Remark:-

If f is improper integrable,

$$\lim_{c \to a^+} \lim_{d \to b^-} \int_c^d f = \lim_{d \to b^-} \lim_{c \to a^+} \int_c^d f$$

Ex-1

$e^{-|x|} \quad x \in \mathbb{R}$

Let $c, d \in \mathbb{R}$, $c<0$ $d>0$

$$\int_c^d e^{-|x|}\,dx$$

$$= \int_c^0 e^{x}\,dx + \int_0^d e^{-x}\,dx$$

We want to compute

$$\int_{-\infty}^{\infty} e^{-|x|}\,dx$$

Part-1

$$\lim_{c\to-\infty} \int_c^0 e^{x}\,dx = \lim_{c\to\infty}\left[e^{x}\Big|_c^0\right]$$

$$= 1$$

Part.2

$$\lim_{d\to\infty}\int_0^d e^{-x}\,dx = -\lim_{d\to\infty}\left[e^{-x}\Big|_0^d\right]$$

$$= -\left[\lim_{d\to\infty} e^{-d} - 1\right]$$

$$= 1$$

Thus $\int_{-\infty}^{\infty} e^{-|x|}\,dx = 2.$

Ex

$f(x) = \frac{1}{x^p} \quad p>0 \quad x \in (0,1)$

f has a singularity near zero,

To compute $\epsilon \to 0$ $\lim_{\epsilon\to 0}\int_\epsilon^1 \frac{1}{x^p}\,dx$

$$\Rightarrow \int_\epsilon^1 \frac{1}{x^p}\,dx = \left[\frac{x^{-p+1}}{-p+1}\right]\Bigg|_\epsilon^1 \quad p \neq 1.$$

or $\left[\log x\right]_\epsilon^1 \quad p=1.$

Case -1

$$\Rightarrow \left[\frac{x^{-p+1}}{-p+1}\right]_\epsilon^1 = \frac{1}{-p+1} - \frac{1}{-p+1}\underline{\underline{\left(\frac{1}{\epsilon^{p-1}}\right)}}.$$

for the limit to exist

$p-1<0$

$\Rightarrow p<1$

Defn. Let $f:(a,b) \to \mathbb{R}$ be given s.t. f is locally integrable. We say that f has improper integrable if

$$\lim_{d \to b^-} \lim_{c \to a^+} \int_c^d f(x)\,dx \text{ exists.}$$

We then define

$$\int_a^b f := \lim_{d \to b^-} \lim_{c \to a^+} \int_c^d f(x)\,dx.$$

$$= \lim_{c \to a^+} \int_c^{t^*} f + \lim_{d \to b^-} \int_{t^*}^d f$$

Eg: For $p \in (1,\infty)$,

$$\int_a^\infty \frac{1}{t^p}\,dt < \infty \quad \text{where } a>0.$$

2. For $p \in (0,1)$

$$\int_0^a \frac{1}{t^p}\,dt < \infty$$

Both

$$\int_0^a \frac{1}{t}\,dt \quad , \quad \int_a^\infty \frac{1}{t}\,dt \quad \text{are not integrable.}$$

For $\int_a^\infty \frac{1}{t}\,dt$ use riemam sums then choose tags such that we get $\sum \frac{1}{n}$

Let I be an open interval of $\mathbb{R}$. Let $1 \leq p < \infty$

Define $L^p(I) = \{ f : I \to \mathbb{R} \mid \int |f|^p dx < \infty \}$

$\|f\|_{L^p(I)} = \left(\int_I |f|^p \right)^{1/p}$

we raise it to $\frac{1}{p}$ so $\|cf\| = |c| \, \|f\|$

(i) $\|cf\|_p = |c| \, \|f\|_p$ $\qquad \|f\|_{L^p} = \|f\|_p$

(ii) $\|f+g\|_{L^p} \leq \|f\|_{L^p} + \|g\|_{L^p}$ (Minkowski)

Semi Norm

(iii) $\|f\|_p \geq 0$

$\|f\|_p = 0 \iff f = 0$ almost everywhere.

$L^2(I) \Rightarrow$ Inner Product spaces.

$(f,g) = \int_I fg \, dx$

we can show completeness of the space.

L_p norm on Euclidean spaces

$\|x\|_p = \left(\sum |x_i|^p \right)^{1/p}$

Thm:

Let f, g be integrable on (a,b). Then $\alpha f + \beta g$ is also integrable on $\alpha f + \beta g$ is int on (a,b) and

$$\int_a^b \alpha f + bg = \alpha \int_a^b f + b \int_a^b g.$$

Thm: [Comparison Principle]

Let f, g be two locally integrable functions satisfying

$0 \leq f \leq g \quad \forall x \in (a,b)$

If g is int (improper) on (a,b), then f is also imp int on (a,b).

Pf Let $t^* \in (a,b)$ $[c,d] \subseteq (a,b)$.

Then $\int_c^d f = \int_c^{t^*} f + \int_{t^*}^d f$

Claim $\lim_{d \to b^-} \int_{t^*}^d$ exist

Now note $d \mapsto \int_{t^*}^d f$ is monotonically increasing as $f \geq 0$.

Also not $\int_{t^*}^d f \leq \int_{t^*}^d g \leq \int_{t^*}^b g \to$ exists

$\int_{c^*}^{d} f\,dx$ is bounded above and monotonic

Thus $\lim_{d \to b^-} \int_{c^*}^{d} f\,dx$ exists. Similarly c case also

Thus f is also imp integrable ■

Defⁿ :- A locally integrable f×n $f:(a,b) \to \mathbb{R}$ is absolutely integrable if $|f|$ is integrable on (a,b).

Theorem.

Let f and $|f|$ be locally integrable on (a,b). Assume that $|f|$ is integrable on (a,b). Then f is int on (a,b).

Moreover,

$$\left|\int_a^b f\right| \leq \int_a^b |f|$$

Pf Note

$$0 \leq f + |f| \leq 2|f|$$

We know $2|f|$ is integrable on (a,b)

By Comparison thm, $f+|f|$ is integrable on (a,b)

We have,

$|f|$ is integrable on (a,b).

⇒ we have,

$$f = (f+|f|) - |f|$$

⇒ f is integrable.

For all c, d s.t. $[c,d] \subseteq (a,b)$

$$\left|\int_c^d f\right| \leq \int_c^d |f|$$

taking limits $c \to a$, $d \to b$

$$\left|\int_a^b f\right| \leq \int_a^b |f|$$

Lemma

$\frac{\sin(x)}{x}$ is integrable on $[1,\infty)$ but not absolutely integrable on $[1,\infty)$

Pf Let $R \in \mathbb{R}$, $R > 1$.

$$\int_1^R \frac{\sin x}{x}\,dx = -\int_1^R \frac{\cos x}{x^2}\,dx - \left[\frac{\cos x}{x}\right]_1^R$$

we have

$$\lim_{R\to\infty} \frac{\cos R}{R} \text{ exists.}$$

we just need to show now,

$$\int_1^R \frac{\cos x}{x^2} \text{ is integrable.}$$

we have

$$0 \le \left|\frac{\cos x}{x^2}\right| \le \frac{1}{x^2}$$

we have $\frac{1}{x^2}$ is integrable on $[1, \infty)$

$\Rightarrow \left|\frac{\cos x}{x^2}\right|$ is integrable on $[1, \infty)$ by comparison test.

$\Rightarrow \frac{\cos x}{x^2}$ is integrable by prev thm.

Thus

$$\lim_{R\to\infty} \int_1^R \frac{\sin x}{x} \text{ exists}$$

$\Rightarrow \frac{\sin x}{x}$ is integrable on $[1, \infty)$

CLAIM $\left|\frac{\sin x}{x}\right|$ is not integrable over $[1, \infty)$

$$\int_1^{n\pi} \frac{|\sin x|}{x}\,dx \geqslant \int_\pi^{n\pi} \frac{|\sin x|}{x}\,dx$$

$$\int_\pi^{n\pi} \frac{|\sin x|}{x}\,dx = \int_\pi^{2\pi} + \int_{2\pi}^{3\pi} + \cdots + \int_{(n-1)\pi}^{n\pi}$$

$$= \sum_{k=1}^{n-1} \int_{k\pi}^{(k+1)\pi} \frac{|\sin x|}{x}\,dx$$

Now $\forall k \in \{1, \ldots, n-1\}$

$x < (k+1)\pi$ $\quad x \in [k\pi, (k+1)\pi)$

$$\Rightarrow \frac{1}{x} > \frac{1}{(k+1)\pi}$$

$$\int_\pi^{n\pi} \frac{|\sin x|}{x}\,dx > \sum_{k=1}^{n-1} \frac{1}{(k+1)\pi} \int_{k\pi}^{(k+1)\pi} |\sin x|\,dx$$

$$= \sum_{k=1}^{n-1} \frac{1}{(k+1)\pi} \int_0^\pi \sin x\,dx$$

$$= \underbrace{\frac{2}{\pi} \sum_{k=1}^{n-1} \frac{1}{k+1}}_{\text{Divergent.}}$$

Thus

$$\lim_{n\to\infty} \int_1^{n\pi} \left|\frac{\sin x}{x}\right| dx \geq \lim_{n\to\infty} \int_{\pi}^{n\pi} \left|\frac{\sin x}{x}\right| dx \geq \lim_{n\to\infty} \left(\frac{2}{\pi} \sum_{k=1}^{n-1} \frac{1}{k+1} \right) > \infty$$

$\Rightarrow$ ~~g~~ $\left|\frac{\sin x}{x}\right|$ is not integrable

---

Sequence of Functions

Defn

Let $X \subseteq \mathbb{R}$. A sequence of f^{x^n}s is $\{f_n\}_{n\in\mathbb{N}}$ where

$f_n : X \to \mathbb{R} \quad \forall n \in \mathbb{N}$

Defn [Pointwise Convergence]

Let $\{f_n\}$ be a sequence of f^{x^n}s on X. We say that f is the pointwise limit of f_n if

$$f_n(x) \longrightarrow f(x) \qquad \forall x \in X.$$

E-δ

$\forall \varepsilon > 0, \exists N_{\varepsilon,x}$ s.t.

$$|f_n(x) - f(x)| < \varepsilon \qquad \forall n \geq N_{\varepsilon,x}$$

$N_{\varepsilon,x}$ is dependent on x

If $N_{\varepsilon,x}$ is independent of x, then we have uniform convergence.

Defn [Uniform Convergence]

Let $\{f_n\}$ be a sequence of f^{x^n}s on X. We say that $f_n \to f$ or $f_n \rightrightarrows f$, uniformly converges iff $\forall \varepsilon > 0, \exists N_\varepsilon \in \mathbb{N}$

$$|f_n(x) - f(x)| < \varepsilon \qquad \forall n \geq N_\varepsilon \quad \forall x \in X.$$

Thm Uniform convergence $\Rightarrow$ Pointwise Convergence

Thm

Let f_n be a sequence of f^ns on X. Assume that $\sup_{x\in X} |f_n(x) - f(x)|$ exists.

Then $f_n \rightrightarrows f$ iff $\sup_{x\in X} |f_n(x) - f(x)| \longrightarrow 0$

Pf ($\Rightarrow$)

we have $f_n \rightrightarrows f$ ~~unif~~

$\forall \varepsilon > 0$,

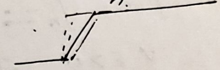
$$|f_n(x) - f(x)| < \varepsilon/2 \quad \forall n \geq N_\varepsilon \ \forall x \in X$$

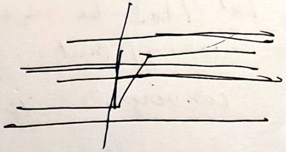
$$\Rightarrow \sup_{x\in X} |f_n(x) - f(x)| \leq \varepsilon/2 < \varepsilon \quad \forall n \geq N_\varepsilon$$

$$\Rightarrow \sup_{x\in X} |f_n(x) - f(x)| \longrightarrow 0$$

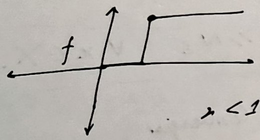
($\Leftarrow$) Easy

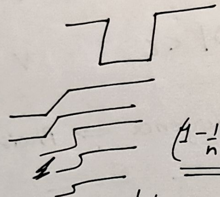
Ex : $f_n = \begin{cases} 0 & -\infty \le x \le 0 \\ nx & x \le x \le \frac{1}{n} \\ 1 & x \geq \frac{1}{n}. \end{cases}$

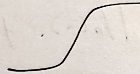


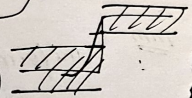


Ex $f_n(x) = x^n$









$x < 1.$

$\left(1-\frac{1}{n}, 1+\frac{1}{n}\right)$

$|f_N(x) \cancel{}| < \varepsilon$

$|x^N| < \varepsilon.$

Ex $f_n(x) = \begin{cases} nx & 0 \le x \le \frac{1}{n} \\ n\left(\frac{2}{n} - x\right) & \frac{1}{n} < x \le \frac{2}{n} \\ 0 & x \geq \frac{2}{n} \end{cases}$

$|x^N|$

$\left|\left(1-\frac{1}{n}\right)^N\right|$

~~Def~~ ~~Unif~~ Def^n [Uniform cauchy]

A seq of fx^ns $\{f_n\}$ on X is said to 'uniform cauchy' if given $\varepsilon > 0$ $\exists N \in \mathbb{N}$ s.t.

$|f_n(x) - f_m(x)| < \varepsilon \quad \forall\ n, m \geq N$

$\forall\ x \in X.$

Thm/ A sequence of fn's $\{f_n(x)\}$ on X is uniform cauchy iff it is uniform cgts.

Pf Unif convergent $\Rightarrow$ uniform Cauchy (Ex).

($\Leftarrow$)

Assume $\{f_n\}$ is uniform Cauchy. Then $\forall x \in X$, $\{f_n(x)\}$ is a Cauchy sequence. Then $\{f_n(x)\}$ is convergent. ~~The $\{f_n\}$~~.

Let $f : X \to \mathbb{R}$ be the ptwise limit.

Thus we have

$-\varepsilon < f_n(x) - f_m(x) < \varepsilon \quad \forall n, m \geq N, \forall x \in X.$

taking limit $m \to \infty$ $\quad f_m(x) \to f(x)$

$-\frac{\varepsilon}{2} \le f_n(x) - f(x) \le \frac{\varepsilon}{2}$

$\Rightarrow |f_n(x) - f(x)| \le \frac{\varepsilon}{2} < \varepsilon \quad \forall x \in X.$

$\Rightarrow |f_n(x) - f(x)| < \varepsilon \quad \forall x \in X, \forall n \geq N$

Thm

Let $f_n : J \to \mathbb{R}$ be seq of cont on J. Assume that, $f_n \rightrightarrows f$. Then $f : J \to \mathbb{R}$ is continuous.

Proof

Let $c \in J$. We will show that $f(x) \to f(c)$ as $x \to c$.

Let $\varepsilon > 0$. Since $f_n \rightrightarrows f$, $\exists N \in \mathbb{N}$ s.t.

$$|f_n(x) - f(x)| < \varepsilon \qquad \forall x \in X, \forall n \geq N$$

Let us fix $n = N$

$$|f_N(x) - f(x)| < \varepsilon$$

Since $f_N(x)$ is continuous at c, $\exists \delta > 0$ s.t. $|x-c| < \delta$,

$$\Rightarrow |f_N(x) - f_N(c)| < \varepsilon$$

Now for $x \in (c-\delta, c+\delta)$

$$|f(x) - f(c)| = |f(x) - f_N(x) + f_N(x) - f_N(c) + f_N(c) - f(c)|$$
$$< |f(x) - f_N(x)| + |f_N(x) - f_N(c)| + |f_N(c) - f(c)| < \varepsilon$$

Remark

$\left\{ \begin{array}{l} f \text{ is continuous,} \\ f_n \to f \text{ unif} \\ \Rightarrow f \text{ is continuous.} \end{array} \right.$

Let $\{x_m\}$ be a sequence s.t. $x_n \to c$, for some $c \in J$.

Then,

$$\lim_{n\to\infty} \lim_{m\to\infty} f_n(x_m) = \lim_{m\to\infty} \lim_{n\to\infty} f_n(x_m)$$

LHS

$$\lim_{n\to\infty} f_n(c) = f(c)$$

RHS

$$\lim_{m\to\infty} f(x_m) = f(c)$$ ← Requires continuity of f.

$\lim_{n\to\infty} \frac{d}{dx} f_n = \frac{d}{dx} \lim_{n\to\infty} f$

$$\lim_{n\to\infty} \int f_n = \int \lim_{m\to\infty} f_n$$

Recall

$$\frac{d}{dx}\left(\lim_{n\to\infty} f_n\right) = \lim_{n\to\infty} \frac{d}{dx}(f_n)$$

when $\{f_n\}$ and $\{f_n'\}$ are unif cgts.

$T_m = \begin{cases} 0 \\ 1+\frac{1}{(q)} \end{cases}$ $\quad + \frac{1}{q}$ $\quad 1+\left(\frac{1}{q}\right)^n$

Thm

Let $\{f_n\}$ be a seq of fns of $[a,b]$ s.t.

(i) f_n is integrable on $[a,b]$

(ii) $f_n \rightrightarrows f$.

Then f is integrable on $[a,b]$. Moreover,

$$\lim_{n\to\infty} \int_a^b f_n = \int_a^b f = \int_a^b \lim_{n\to\infty} f_n.$$

Remark Ptwise convergence is not enough

Ex $f(x) = \begin{cases} 1 & x \in [a,b]\cap\mathbb{Q} \\ 0 & \text{otherwise} \end{cases}$

Then f is NOT integrable.

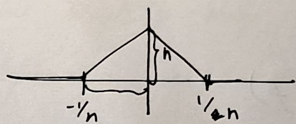
Let $[a,b]\cap\mathbb{Q} = \{r_n\}_{n\in\mathbb{N}}$

Define $f_n(x) = \begin{cases} 1 & x \in \{r_1, \dots, r_n\} \\ 0 & \text{otherwise} \end{cases}$

Eg $f_n(x) = \begin{cases} 0 & |x| \geq \frac{1}{n} \\ n - n^2x & 0 \leq x < \frac{1}{n} \\ n + n^2x & -\frac{1}{n} \leq x < 0 \end{cases}$

$\frac{y}{n} + nx = 1$

$y = 1 - nx$
$= n - n^2x$



$\int_{\mathbb{R}} f_n(x) = 1$

Pf Since $f_n \rightrightarrows f$, for given $\varepsilon > 0$, $\exists N \in \mathbb{N}$

$$|f_N(x) - f(x)| < \varepsilon \quad \forall \; x \in [a,b]$$

Since f_N is integ, $\exists P$ s.t.

$$U(f_N, P) - L(f_N, P) < \varepsilon$$

We know

$f_N - \varepsilon < f < f_N + \varepsilon$

$U(f_N) < U(f) + \varepsilon$

$$f_N(x) - f(x) > -\varepsilon$$

$$\Rightarrow f_N(x) > f(x) - \varepsilon$$

$$\Rightarrow M_i(f(x)) - \varepsilon \leq M_i(f_N(x))$$

$$U(f,P) - \varepsilon(b-a) \leq U(f_N, P)$$

$$L(f_N, P) \leq L(f,P) + \varepsilon(b-a)$$

$$\Rightarrow U(f,P) - L(f,P) \leq U(f_N,P) - L(f_N,P)$$

Claim

$$\lim_{n\to\infty} \int_a^b f_n = \int_a^b f = \int_a^b \lim_{n\to\infty} f_n .$$

Now

$$\int_a^b f_n - \int_a^b f = \int_a^b (f_n - f)$$

$$\Rightarrow 0 \le \left| \int_a^b (f_n - f) \right| \le \int_a^b |f_n - f| \le \left(\sup_{x\in[a,b]} |f_n(x) - f(x)| \right) (b-a)$$

take $n \to \infty$ on both side.

$$\Rightarrow \lim_{n\to\infty} \int_a^b f_n - f = 0$$

$$= \lim_{n\to\infty} \int_a^b f_n = \int_a^b \lim_{n\to\infty} f_n(x)$$

Series of fxns

Let $\{f_n\}$ be a seq of fns on X. Let

$$f_n : X \longrightarrow \mathbb{R}.$$

We want to study $\sum_{n=1}^{\infty} f_n(x)$

Define the partial sums,

$$S_n : X \longrightarrow \mathbb{R} \text{ by}$$

$$S_n(x) = \sum_{k=1}^{n} f_k(x)$$

Defn

1. We say that "$\sum_{n=1}^{\infty} f_n = f$ pointwise" if for each $x \in X$ $S_n(x) \to f(x)$.

2. We say that "$\sum_{n=1}^{\infty} f_n = f$ uniformly" if $\forall x \in X$ $S_n(x) \rightrightarrows f(x)$

3. "$\sum f_n$ converges absolutely" if $\forall x \in X$,

$$\lim_{n\to\infty} \sum_{k=1}^{n} |f_n(x)| \text{ exists}$$

Examples:- $X = [0,1]$ $\quad f_n(x) = \begin{cases} x^{n-1} & n \ge 2 \\ 1 & n = 1 \end{cases}$

$$S_n(x) = \sum_{k=1}^{n} x^{k-1} = \frac{1-x^{n}}{1-x}$$

$$S_n(x) = \begin{cases} \dfrac{1-x^n}{1-x} & x \in [0,1) \quad \text{not uniform} \Rightarrow \text{not conti} \\ n & x = 1 \quad \Rightarrow \text{continuous} \Rightarrow \text{uniform} \end{cases}$$

~~Conv~~

$S_n(x) \longrightarrow \dfrac{1}{1-x}$ ptwise on $[0,1)$.

This convergence is not uniform.

Prop^n Let f_n be a seq of f^ns on X. Then $\sum f_n$ converges uniformly iff $\forall \varepsilon > 0$ $\exists N \in \mathbb{N}$ s.t. $\forall m, n \geq N$ $(m \leq n)$

$$\left| \sum_{k=m+1}^{n} f_k(x) \right| < \varepsilon \quad \forall x \in X.$$

Thm [Weierstrass M-Test].

Let $\{f_n\}$ be a seq of f^ns on X. Assume that $\exists$ a positive sequence s.t. $\{M_n\}$ s.t.

$$|f_n(x)| \leq M_n, \ \forall x \in X, \ \forall n \in \mathbb{N}$$

Assume that $\sum M_n$ is cgt. ~~$\sum f_n$~~

Then $\sum f_n$ converges uniform and absolutely.

Dirichlet's theorem

Let $\{f_n\}$ and $\{g_n\}$ be two sequence of functions.

Define $F_n = \sum_{k=1}^{n} f_k$

Assume

(i) $\{F_n\}$ is uniformly bounded

$\exists M > 0, M \in \mathbb{R}$ s.t.

$|F_n(x)| \leq M \quad \forall n$.

(ii) g_n is monotonically decreasing

(iii) $g_n \to 0$ uniformly.

Then $\sum_{n=1}^{\infty} f_n g_n$ converges uniformly

Note (ii), (iii) imply $g_n \geq 0 \ \forall n$

Pf

Define

$$S_n = \sum_{k=1}^{n} f_k g_k$$

We have $f_k = F_k - F_{k-1} \quad \forall k = 2, 3 \ldots$

Then

$$S_n = \sum_{k=1}^{n} f_k g_k = \sum_{k=1}^{n} (F_k - F_{k-1}) g_k + F_1 g_1$$

$$= F_1 g_1 + F_n g_{n+1} - F_1 g_2 - \sum_{k=2}^{n} F_k (g_{k+1} - g_k)$$

$$S_n = F_n g_{n+1} - \sum_{k=1}^{n} F_k (g_{k+1} - g_k)$$

we want to show S_n is cauchy.

Let $m < n$ Then

$$S_n - S_m = -\sum_{k=m+1}^{n} F_k (g_{k+1} - g_k)$$

$$+ F_n g_{n+1} - F_m g_{m+1}$$

$$|S_n - S_m| \leq \left| -\sum_{k=m+1} F_k (g_{k+1} - g_k) \right| + |F_n g_{n+1} - F_m g_{m+1}|$$

$$\leq \sum_{k=m+1} |F_k| |(g_k - g_{k+1})| + |F_n g_{n+1} - F_m g_{m+1}|$$

$$\leq M \sum_{k=m+1}^{n} (g_k - g_{k+1}) + |F_n g_{n+1}| + |F_m g_{m+1}|$$

$$\leq M (g_{m+1} - g_{n+1})$$

$$+ M (g_{n+1} + g_{m+1})$$

$$\leq 2M (g_{m+1}) \leq 2M |g_{m+1}| < \varepsilon$$

Thus we know $\exists N \in \mathbb{N}$ s.t.

$$|S_n - S_M| < \varepsilon$$

~~$\sum \sin(nx)$~~ $\Rightarrow$ ~~$\sum_{n=1}^{k} e^{inx}$~~ Try to prove

if $f_n = \sin(nx)$

$F_n(x) = \sum_{k=1}^{n} f_k$

using complex numbers $\underline{e^{inx}}$.

$$\Rightarrow F_n(x) = \frac{\sin\left(\frac{nx}{2}\right)\sin\left(\frac{(n+1)x}{2}\right)}{\sin\left(\frac{x}{2}\right)}$$

Consider the interval $(0, 2\pi)$

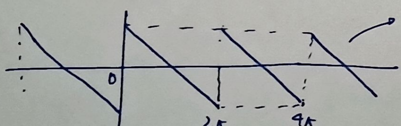
Let $\delta > 0$ s.t. $0 < \delta < 2\pi$

We now consider the interval $(\delta, 2\pi - \delta)$

Then for $0 < p \leq 1$

$\sum_{n=1}^{\infty} f_n$ converges uniformly on $(\delta, 2\pi - \delta)$

$$\pi - t = \frac{1}{2}\sum_{n=1}^{\infty} \frac{\sin(nt)}{n}$$ sine series only



fourier series

⇓

Here cosine terms cancel out so fourier series $\equiv$ sine series.

Power Series

Define $f_n(x) = a_n(x-a)^n$

$a_n \in \mathbb{R}, a \in \mathbb{R}, n \in \mathbb{N} \cup \{0\}$

we study $\sum_{n=0}^{\infty} a_n (x-a)^n$

Def^n The expression

$\sum_{n=0}^{\infty} a_n (x-a)^n$ is called a power series around a.

Let p be a polynomial

$p = \sum_{i=0}^{N} a_i x^i$ and $a_N \neq 0$

For simplicity we will assume $a = 0$.

So, we consider power series of the form $\sum_{n=1}^{\infty} a_n (x^n)$

Thm Consider the power series $\sum_{n=0}^{\infty} a_n x^n$. Then $\exists R \in [0, \infty]$ s.t. for all $0 < r < R$

(i) $\sum_{n=0}^{\infty} a_n x^n$ converges uniformly on $(-r, r)$

(ii) $\sum_{n=0}^{\infty} a_n x^n$ is continuous on $(-R, R)$

iii) $\sum_{n=0}^{\infty} a_n x^n$ diff on $(-R, R)$

and $\frac{d}{dx}\left(\sum_{n=0}^{\infty} a_n x^n\right) = \sum_{n=1}^{\infty} n a_n x^{n-1}$

(iv) For $a, b \in (-R, R)$
$\sum_{n=0}^{\infty} a_n x^n$ is int. and

$$\int_a^b \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n \int_a^b x^n$$

(v) If $|x| > R$ then
$\sum_{n=0} a_n x^n$ diverges.

Pf Consider the set (write the proof on your own).
$E = \{ |z| \mid \sum_{n=0}^{\infty} a_n z^n \text{ is cgt} \}$
Thus E is non-empty as $z = 0 \in E$.
Let $\sup E = R$.
If ~~is~~ E not bounded $R = \infty$.
Let $0 < r < R$
Let $|x| < r$, $\exists\ \alpha$ s.t. $r < \alpha \leq R$.
$|a_n x^n| \leq |a_n| r^n = |a_n \alpha^n| \left(\frac{r}{\alpha}\right)^n$
Now $\alpha \in E$ ~~&~~, thus
$\sum |a_n \alpha^n|$ is ~~dgt~~ cgt, $|a_n \alpha^n| \leq M$ for some $M \in \mathbb{R}$, $\forall n \in \mathbb{N}$

$\Rightarrow |a_n x^n| \leq M \left(\frac{r}{\alpha}\right)^n$

Now $\sum M \left(\frac{r}{\alpha}\right)^n$ converges as geometric series. Thus by M-Test we have $\sum a_n x^n$ cgt.

(iii) Let $0 < r < R$ will show that
$\sum n a_n x^{n-1}$ cgt uniformly.

Now
$|n a_n x^{n-1}| \leq n |a_n| r^{n-1} = n |a_n \alpha^n| \frac{r^{n-1}}{\alpha^n}$

Let $t_0 = \frac{r}{\alpha}$

$|n a_n x^{n-1}| \leq \frac{M}{\alpha} n t_0^{n-1}$

$\sum \frac{M}{\alpha} n t_0^{n-1}$ converges by ratio test.

$\frac{a_{n+1}}{a_n} = \left(1 + \frac{1}{n}\right) t$

$\Rightarrow \lim \frac{a_{n+1}}{a_n} = t < 1$. ; converges.

Note [To be proved later]

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \rho \qquad \lim_{n \to \infty} |a_n|^{1/n} = \rho$

$\Rightarrow R = \frac{1}{\rho} \qquad \Rightarrow R = \frac{1}{\rho}$

Thm

Consider the power series $\sum_{n=0}^{\infty} a_n x^n$. Then $\exists ! R \in [0,\infty]$ s.t.

(1) The series $\sum a_n x^n$ converges uniformly on compact subsets of $(-R, R)$.

Defn

The number R appearing in the prev thm is called the radius of convergence of $\sum a_n x^n$.

Assume one of the following

(i) $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \rho$ Proof [trivial]

(ii) $\lim_{n\to\infty} |a_n|^{1/n} = \rho$

Then $\frac{1}{\rho}$ is the radius of convergence

We will prove a diff thm now.

Abel's limit theorem.

Thm

Assume that $\sum a_n$ is cgt. Then $\sum a_n x^n$ cgt uniformly on $[0,1]$

Pf We will show that $\sum a_n x^n$ is uniformly cauchy on $[0,1]$

Consider $\cancel{\sum_{k=m+1} a_k x^k}$ Since $\sum_{n=0}^{\infty} a_n < \infty$, for $\varepsilon > 0$,

$\exists N$ s.t. $\left|\sum_{k=m+1}^{n} a_n\right| < \varepsilon \quad \forall N \le m < n$

Consider

$$\sum_{k=m+1}^{n} a_k x^k = \sum_{k=m+1}^{n} (A_k - A_{k-1}) x^k$$

$$A_k = \sum_{i=m+1}^{k} a_i$$

using summation by parts we have.

$$\sum_{k=m+1}^{n} a_k x^k = -\sum_{k=m+1}^{n} A_k (x^{k+1} - x^k) + \left(A_n x^{n+1} - A_m x^{m+1}\right)$$

$$\left|\sum_{k=m+1}^{n} a_k x^k\right| \le \sum_{k=m+1}^{n} |A_k|\, |x^{k+1} - x^k| + |A_n|\,|x^{n+1}| + |A_m|\,|x^{m+1}|$$

$$< \varepsilon \sum_{k=m+1}^{n} |x^{k+1} - x^k| + \varepsilon |x^{n+1}| + \varepsilon |x^{m+1}|$$

as $x > 0$ $= \varepsilon \sum_{k=m+1}^{n} x^k - x^{k+1} + \varepsilon x^{n+1} + \varepsilon x^{m+1}$

$$< 2\varepsilon x^{m+1} \le 2\varepsilon$$

$$\Rightarrow \left|\sum_{k=m+1}^{n} a_k x^k\right| < 2\varepsilon$$

Weierstrass approximation test

Thm

Let $f : [a,b] \to \mathbb{R}$ be a continuous fxn. Then $\exists$ a seq of polynomials $\{P_n\}$ s.t.

$P_n \to f$ unif on $[a,b]$.

In particular for $\varepsilon > 0$, $\exists$ a polynomial p s.t.

$$|P(x) - f(x)| < \varepsilon \quad \forall x \in [a,b]$$

Consider

$C([a,b]) : \{ f \,/\, f \text{ cont on } [a,b] \}$

$\mathbb{P} = \{$ set of all polynomials on $\mathbb{R} \}$

$$d(f,g) = \sup_{x \in [a,b]} |f(x) - g(x)|$$

$f, g \in C([a,b])$, $d(f,g) < \infty$

Thm says given $f \in C([a,b])$, $\exists P_n \in \mathbb{P}$ s.t.

$P_n \to f$ w.r.t d / uniformly.

Remark $\Rightarrow$ Thm is may not true for open sets

Example

Let us take $(0,1)$ open set

Take $f(x) = \frac{1}{x}$. Suppose $\exists$ a polynomial p s.t.

$$|P(x) - f(x)| < 1 \quad \forall x \in (0,1)$$

$$\Rightarrow |f(x)| \le |P(x)| + 1$$

Remark

May not work for unbounded sets.

Application

Let $f \in C([a,b])$ Assume

$$\int_a^b fg = 0 \quad \forall g \in C[a,b] \quad — ①$$

$\Rightarrow f = 0$

Pb. Let $f \in C([a,b])$

Assume that

$$\int_a^b f(x)\, x^n = 0 \quad \forall n = 0$$

$$\int f^2(x) + \varepsilon = 0$$

Then $f = 0$ on $[a,b]$

It is enough to prove

$\boxed{1 \Rightarrow 2}$

$$\int_a^b f(x)\, p(x)\, dx = 0 \quad \forall \; p \in \mathbb{P}$$

Let $g \in C([a,b])$

Then by Weierstrass approximation.

$$g(x) = \sum_{i=0}^{n} c_i x^{i}$$

$$\int_a^b \sum c_i f(x)\, x^i \, ~~dx~~ = 0$$

Thm

$a=0, b=1$

Let $f : [a,b] \to \mathbb{R}$ be a cts func. Then $\exists$ a sequence of ~~poly~~ polynomials $\{p_n\}$ s.t

$$p_n \rightrightarrows f \text{ unif on } [0,1]$$

In particular, for each $\epsilon > 0$ $\exists$ p s.t.

$$|p(x) - f(x)| < \epsilon.$$

Idea of the proof

Step-1 enough to prove $[a,b] = [0,1]$

Let $\psi : [a,b] \to [0,1]$ be the affine linear transformation

$$\tilde{f} : [0,1] \to \mathbb{R}.$$

~~$\tilde{f}$ ...~~

$$\tilde{f} = f \circ (\psi^{-1}) \qquad [0,1] \to \mathbb{R}$$

& by assumption $\exists$ P s.t

$$|\tilde{f}(x) - P(x)| < \epsilon \qquad \forall\, x \in [0,1]$$

$$\Rightarrow |f(\psi^{-1}(x)) - P(x)| < \epsilon \qquad \text{let } \psi^{-1}(x) = y$$

$$\Rightarrow |f(y) - P(\psi(y))| < \epsilon$$

$\psi(y) = ay + b$ $\quad P(\psi(y))$ ~~is~~ is also a polynomial.

S-2

enough to prove the thm for cts fn $f(a)=0=f(b)$

$$\tilde{f}(x) = \tilde{f}(x) - \tilde{f}(b) + (x-a)[\tilde{f}(a) - \tilde{f}(b)]$$

$\tilde{g} = g - H$

$\tilde{g}(a) = 0$

$\tilde{g}(b) = 0$

$H(x)$: st line joining $(a, g(a)), (b, g(b))$

$$H(x) - f(b) = \frac{f(a) - f(b)}{a - b}(x-a)$$

$H(x)$

Need to prove that for cts fn $f:[0,1] \to \mathbb{R}$ and $f(0)=f(1)=0$

Step 3

Approximation of Dirac delta using polynomials.

$$Q_n = c_n (1-x^2)^n$$

$$c_n = \frac{1}{\int_{-1}^{1} Q_n \, dx}.$$

$Q_n \to \delta$

fn $0<\delta<1$

$Q_n \rightrightarrows 0$

S-4 Convolution

Let $f: \mathbb{R} \to \mathbb{R}$ be cts s.t.

$$\tilde{f}(x) = \begin{cases} f(x) & x \in [0,1] \\ 0 & \text{other} \end{cases}$$

$$P_n(x) = \int_{\mathbb{R}} \tilde{f}(x+t) \, Q_n(t) \, dt$$

let $x \in [0,1]$

$$\int_{-1}^{1} [\tilde{f}(x+t) - \tilde{f}(x)] \, Q_n(t) \, dt$$

$$< \varepsilon + \int_{-1}^{-\delta} [\tilde{f}(x+t) - \tilde{f}(x)] \, Q_n(t) + \int_{\delta}^{1} [\tilde{f}(x+t) - \tilde{f}(x)] \, Q_n(t)$$

$$< \varepsilon + 2M \int_{\delta \le |t| < 1} Q_n(t)$$

$$< \varepsilon + 2M\varepsilon$$

$$P_n(x) = \int_{\mathbb{R}} f($$

$f(x)-$

$x+t = u$

$dt = du$

du

$t = u - x$

$\tilde{f}(u) \, Q_n(u-x) \, du$

$M = \sup(|f|)$

x^n

$(0, \delta)$

Dini's Test

Thm Let $f_n : [a,b] \to \mathbb{R}$ be monotone.

either

$$f_n(x) \le f_{n+1}(x) \quad \forall x \quad \forall n$$

or

$$f_n(x) \ge f_{n+1}(x) \quad \forall x \quad \forall n.$$

Assume

(i) f_n's are continuous

(ii) $f_n \to f$ ptwise

(iii) f is ~~unifo~~ continuous.

Then

$f_n \rightrightarrows f$ uniformly.

Pf WLOG,

Assume $f_n(x) \ge f_{n+1}(x) \quad \forall x, \forall n.$

$$f_1(x) \ge f_2(x) \ge \dots \ge f(x) \quad \forall x \in [a,b]$$

Define $g_n : [a,b] \to \mathbb{R}$

$$g_n(x) = f_n(x) - f(x)$$

Then (1)

(i) $g_n \ge 0$ (ii) $g_1 \ge g_2 \ge \dots \ge 0.$

To show $g_n \to 0$ uniformly

Let $x_n \in [a,b]$ s.t.

$$g_n(x_n) = \max_{x \in [a,b]} g_n(x) =: M_n$$

Need to show $M_n \to 0$

Note that M_n is decreasing.

Using Bolzano Weierstrass theorem. $\exists$ a

$\exists$ a subsequence x_{n_k}

s.t. $x_{n_k} \to x^*$

Then $x^* \in [a, b]$

Since $g_n(x^*) \longrightarrow 0$

For $\epsilon > 0$, $\exists N \in \mathbb{N}$

s.t. $g_N(x^*) < \epsilon$

Since g_N is cont at x^*

$\exists\ \delta > 0$, s.t.

$f^2(x) = x f(x)$

$f^*(x) = x$

Properties of Power series

Two facts

1. $f(x) = \begin{cases} e^{-1/x^2} & x > 0 \\ 0 & x \le 0 \end{cases}$

~~f is int~~ $f \in C^\infty(\mathbb{R})$ but f has no power series around 0.

* Take any open ball around 0, we will get ~~[illegible]~~ uncountably many 0's.

2. $f(x) = \sum_{n=0}^{\infty} a_n x^n$

$\{f_n : n x^n (1-x)\}$

$f_n(t) = \int_0^x \cos(nt)\,dt$

~~$f_n(x) =$~~

$f_n(t) = \left[\frac{\sin(nt)}{n}\right]_0^x$

$= \frac{\sin(nx)}{n}$

$f_n \to 0$

f_n is uniformly cts

$f_n \rightrightarrows f$

f is uniformly cts.