Lecture 12

A) Review of what we have learned about diffusion:

-Flux,

-First Fick's law

-Second Fick's law

B) Steady State approximation for solving Diffusion equation (part I)

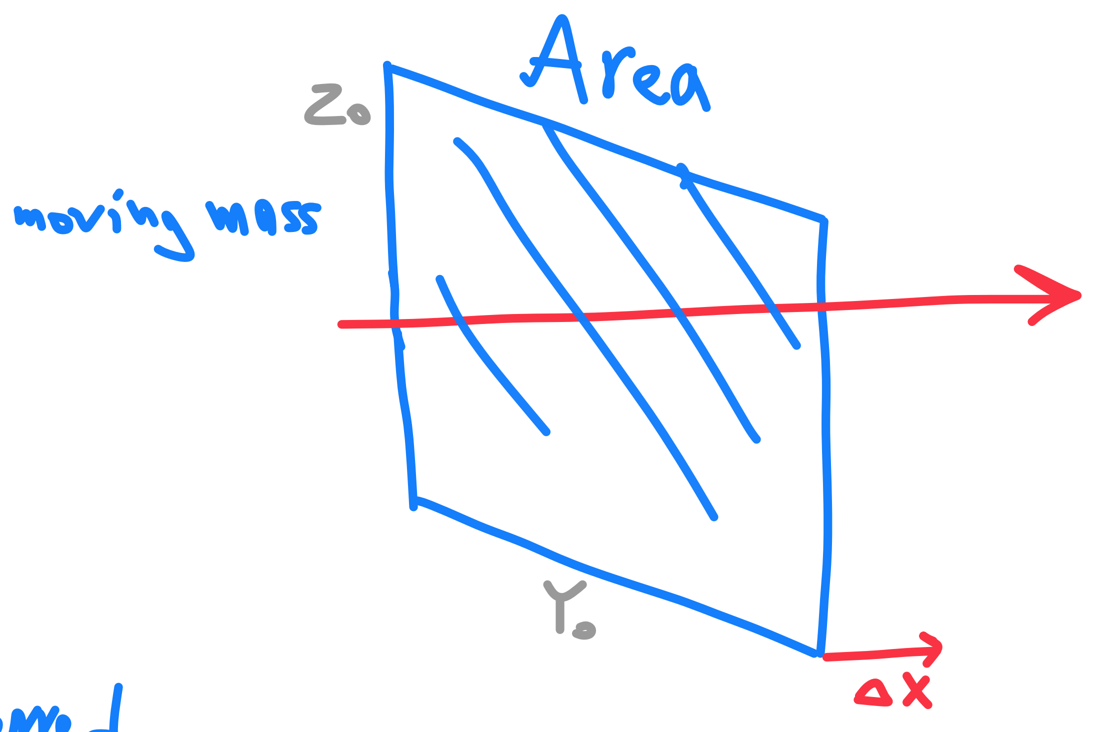
Bonus takeaway of this lecture:

Could sandworms simply be giant earthworms?

A) Review:

In Chapter 2, we begin examining the spatial aspects of mass transport. A fundamental concept that helps us understand mass transport across space is flux.

A1) Flux: Definition of Flux: Flux is a measure of the flow of a quantity (such as mass, energy, or momentum) through a given surface area per unit time. It describes how much of that quantity passes through the area in a certain time frame, providing insight into how substances move through space.



$$J = \frac{\text{mass transferred}}{\text{Area} \times \text{time}} \rightarrow$$

lets write it:

$$J = \frac{\overset{\text{density}}{\rho} \times \overset{(\Delta X \times Y_0 \times Z_0)}{\text{Volume}}}{\underset{(Z \times Y)}{\text{Area}} \times \Delta t} = \frac{\rho \times \Delta X \cdot Y_0 \cdot Z_0}{Y_0 Z_0 \times \Delta t} = \rho \frac{\Delta X}{\Delta t} \Rightarrow \vec{J} = \rho \vec{V}$$

Now, if our particles of interest are immersed in a fluid, their movement can generate flux, denoted as J. There are two main sources of flux:

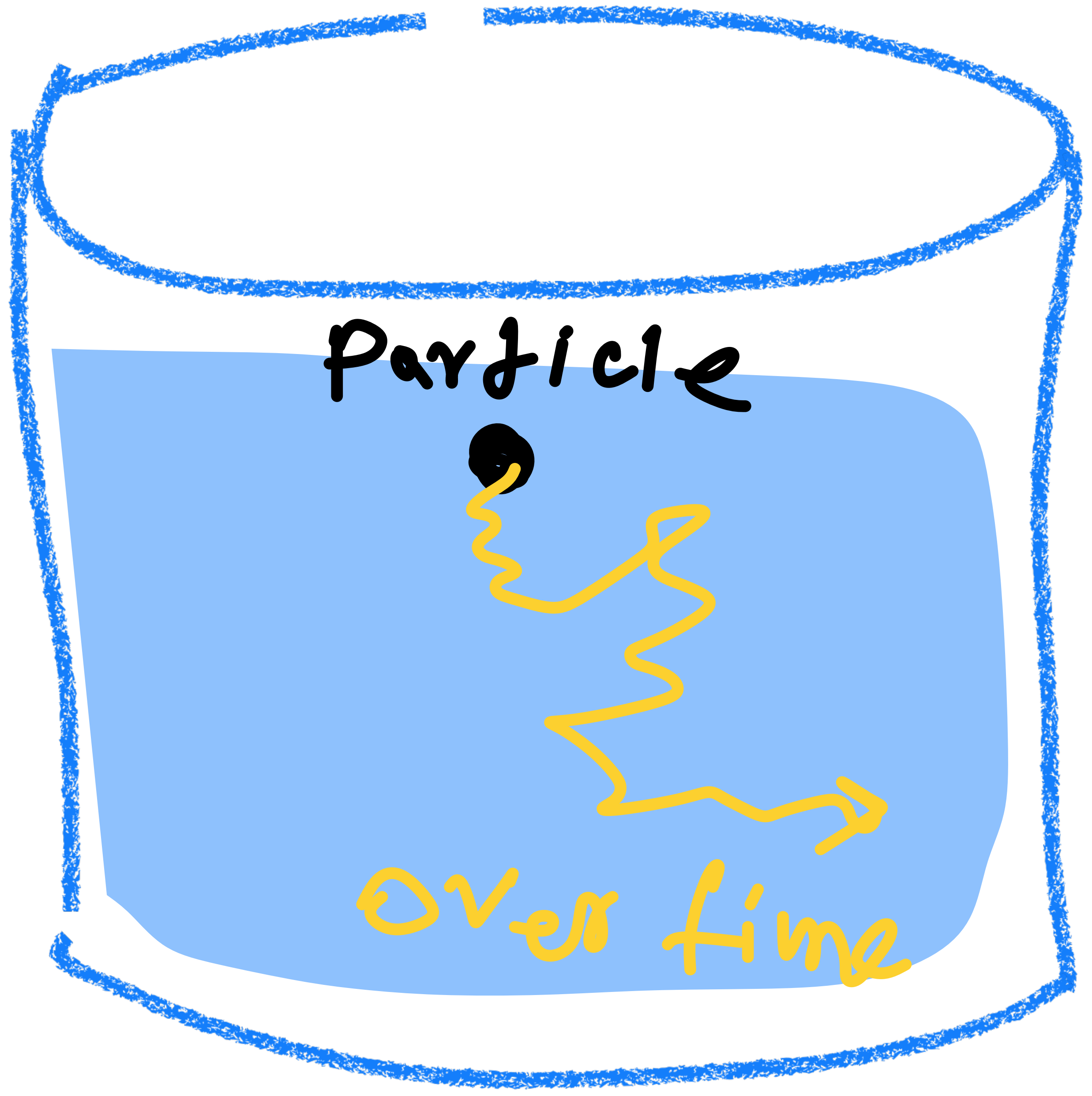
i) **Diffusion-driven Flux:** Particles move due to thermal energy and interactions with the surrounding medium (discussed in this chapter).

ii) **ii) Advection-driven Flux (Convection**): Flux occurs when the medium itself is flowing, carrying the particles along with it (covered in Chapter 3).

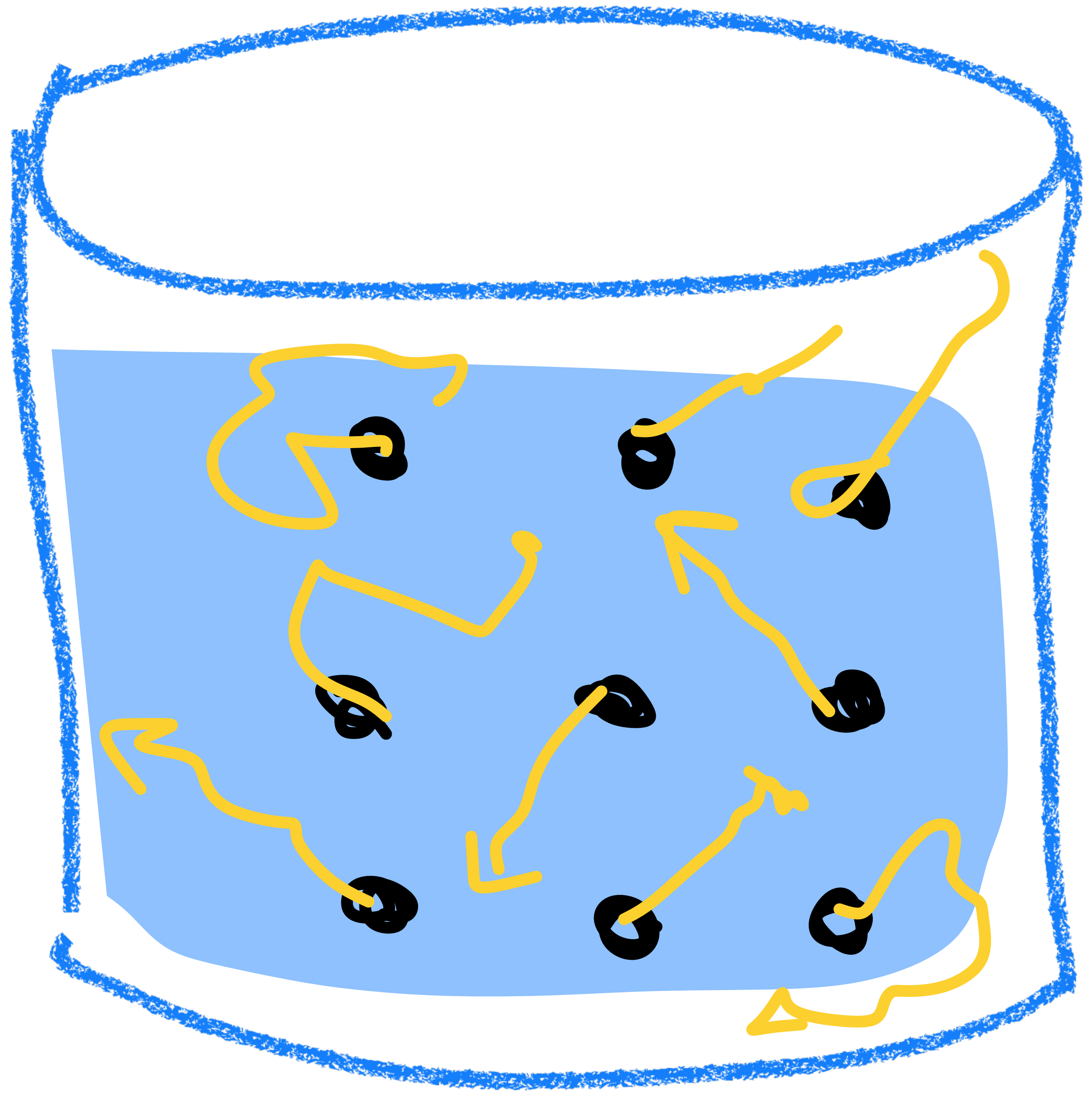
A2) Fick's First law

Let's examine diffusion-driven flux in an intuitive way. Consider the particles below:

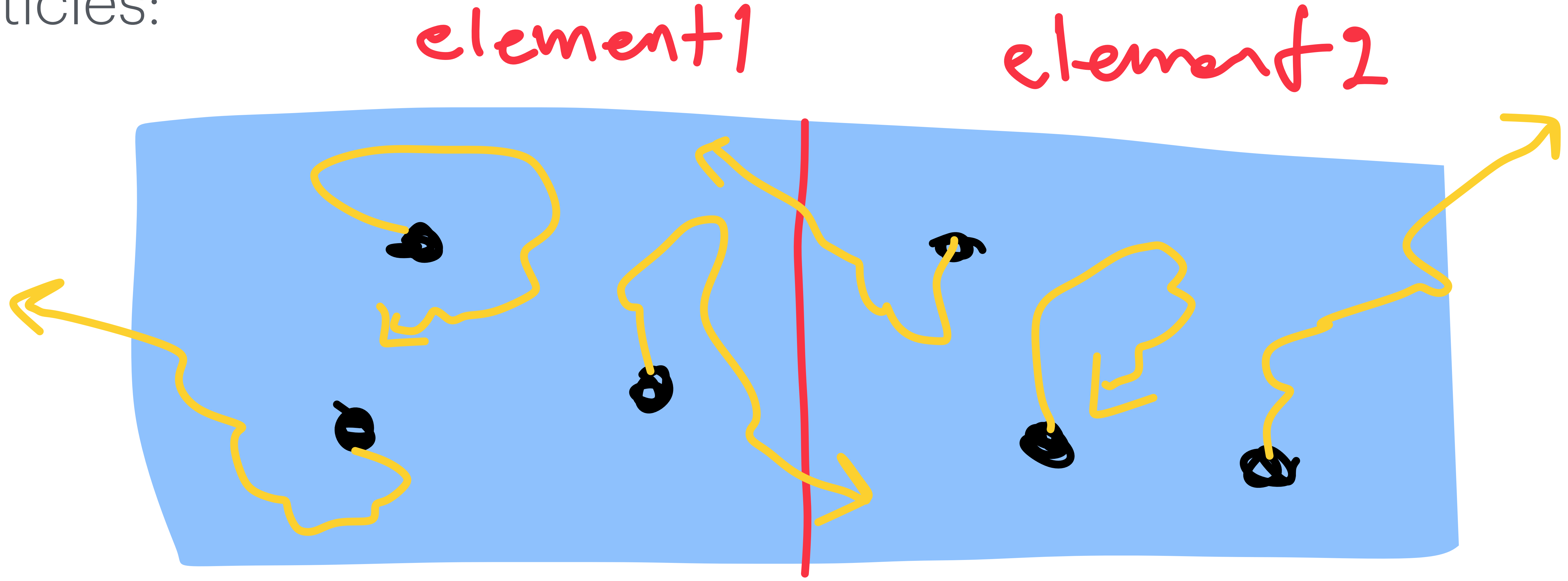
Following one particle randomly moving around



Following many particles randomly moving around



Now, let's focus on two neighboring small cubical elements (with similar volumes) in this medium containing our particles:

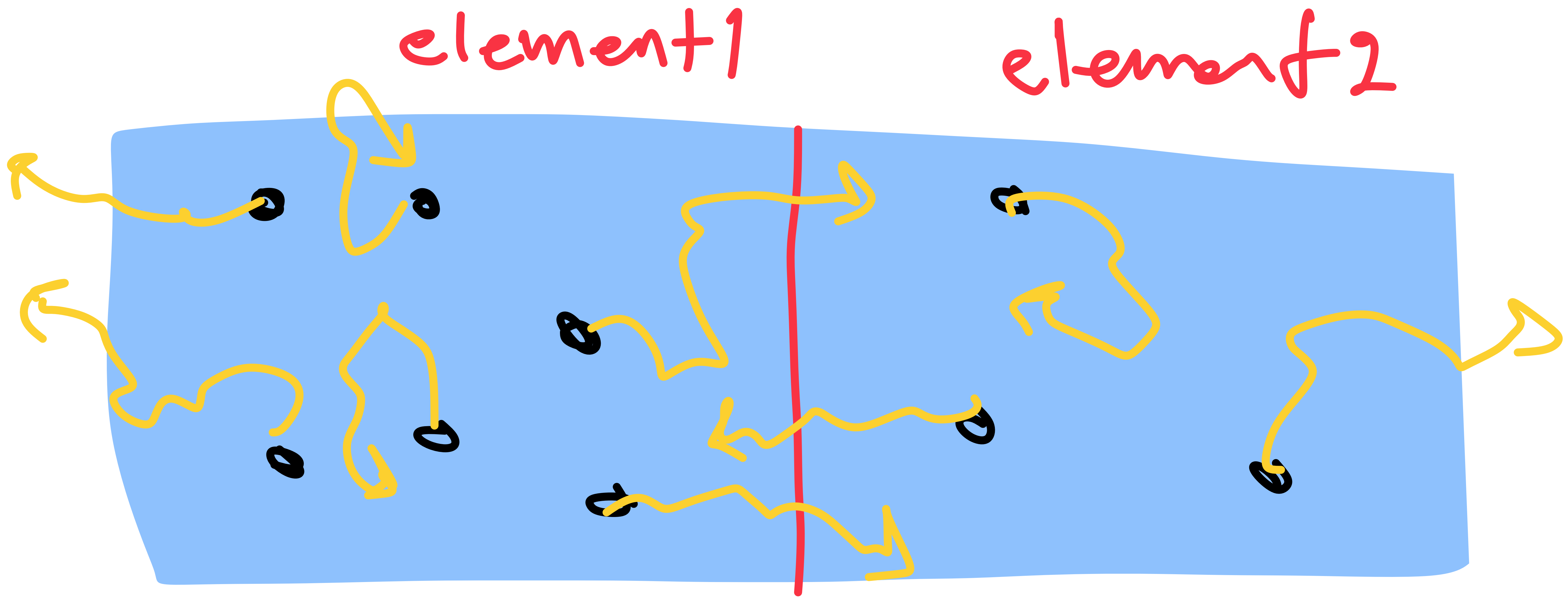


As we can see, these particles move around randomly. Some remain within their elements over time, while others leave. In particular, some particles will leave one element and enter another.

However, note that if the number of particles in both element 1 and element 2 is similar (or the concentrations are the same), the average number of particles moving from element 1 to element 2 will be equal to the number moving from element 2 to element 1. In other words:

$$\text{if } C_1 = C_2 \Longrightarrow |\vec{J}_{1,2}| = 0$$

But what happens if the concentration of particles is higher on one side than the other?



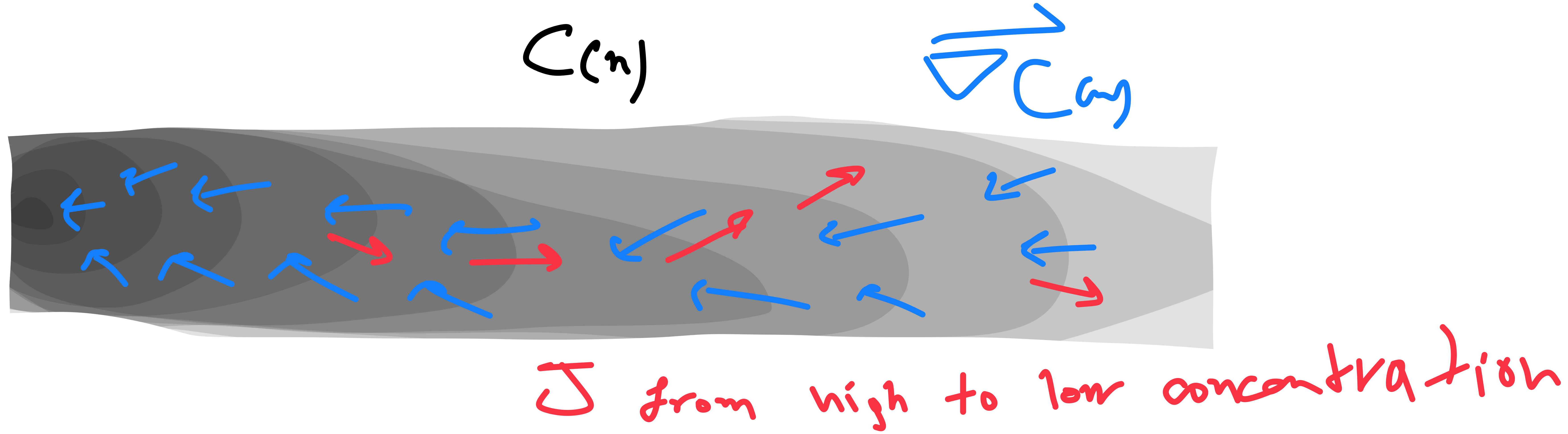
Effectively, more particles will move from element 1 to element 2 than in the opposite direction. In fact, we can express this as:

$$|\vec{J}| \propto C_1 - C_2$$

Since flux is proportional to the concentration difference, we define the diffusivity (diffusion coefficient) as D, which allows us to turn the above into an equation:

$$|\vec{J}| = |D(C_1 - C_2)|$$

Now, we can express this for the general case where we have a concentration field, **C(x)**, in space. This is essentially what we described for two neighboring elements but generalized to a continuous spatial domain. Note the negative sign, which indicates that flux, **J**, moves from regions of higher concentration to lower concentration, just like in our previous example.



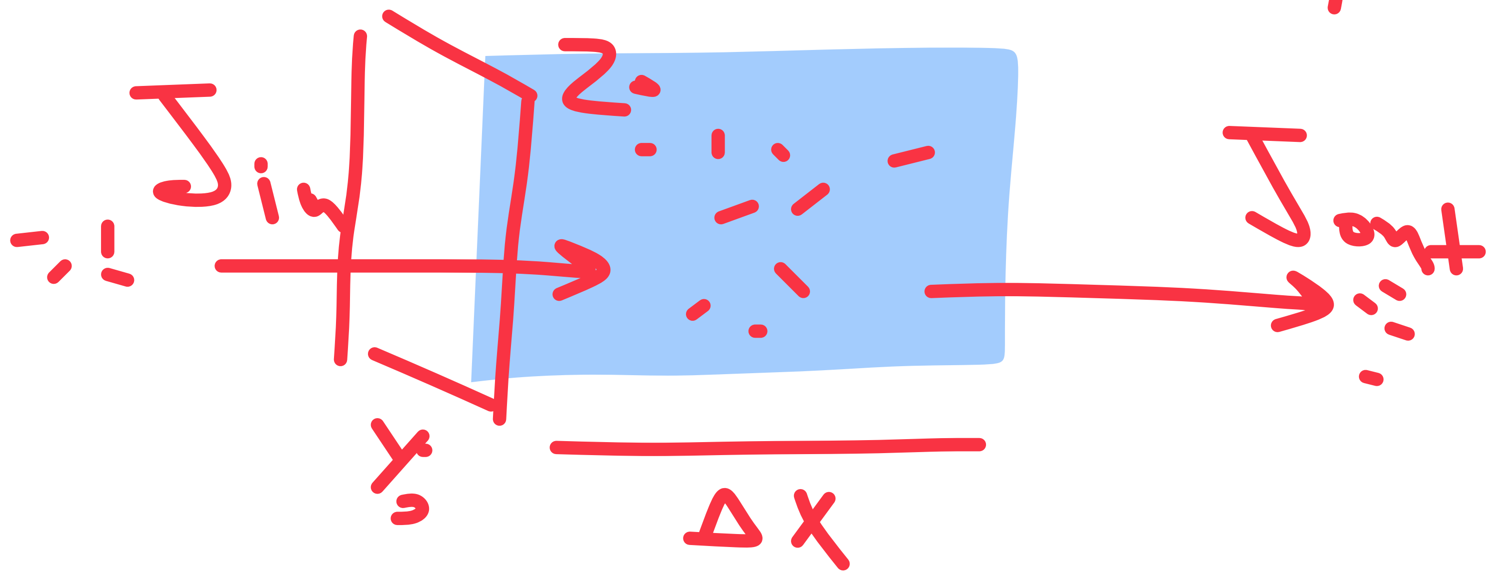
$$\vec{J}(x) = -D\,\overrightarrow{\nabla C(x)}$$

Fick's first law

A3) Fick's Second Law

Now, if we apply the mass balance principle in space, we can express it as follows: if there is a flux and no reactions generating or consuming particles, we can write the mass balance principle:

unite volume (1x1x1)



$$\frac{dm}{dt} = Area \times J_{in} - Area \times J_{out}$$

$$\Rightarrow \Delta x\, Y_0 Z_0 \frac{dc}{dt} = - Y_0 Z_0 (J_{out} - J_{in})$$

$$\Rightarrow \frac{dc}{dt} = - \frac{(J_{out} - J_{in})}{\Delta x} \equiv - \frac{\partial J}{\partial x}$$

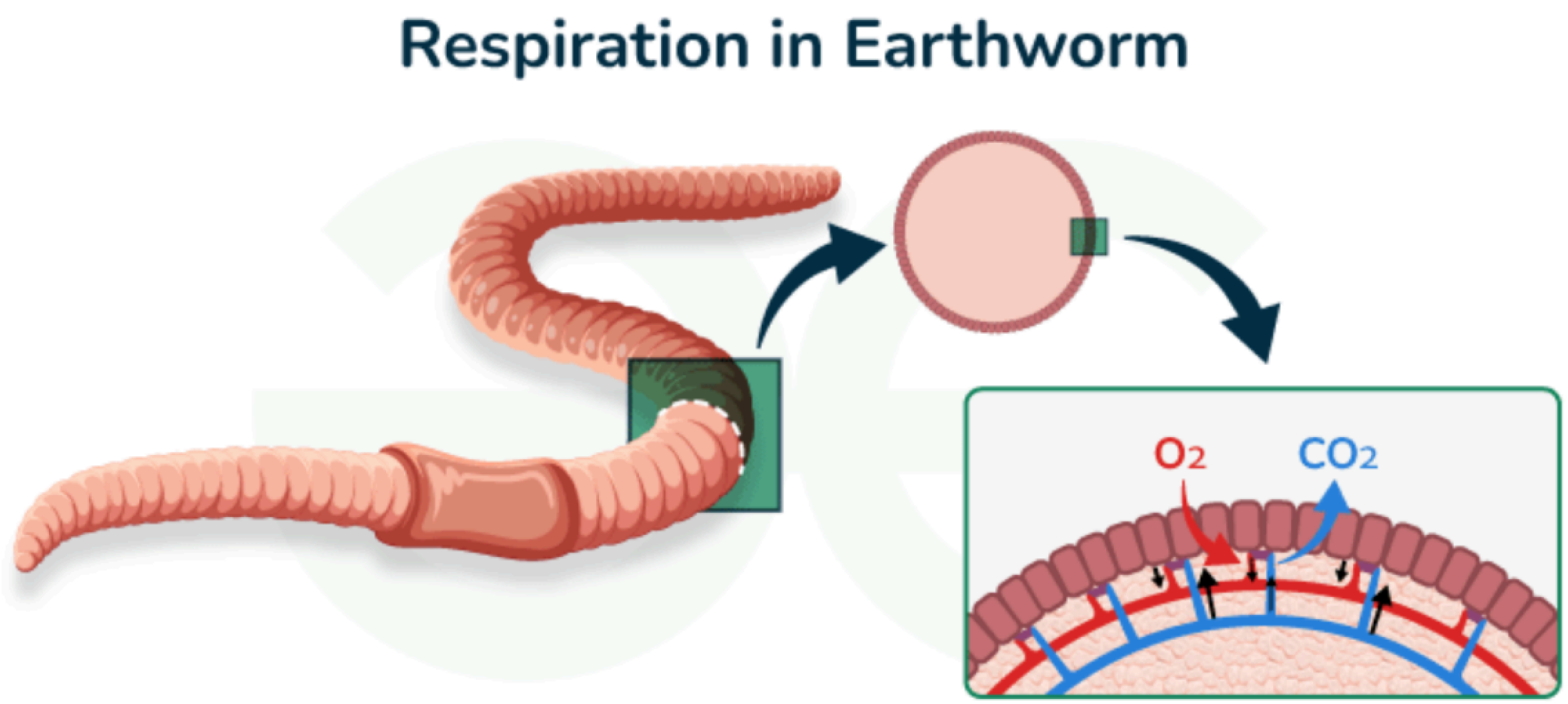
firs law

$$\longrightarrow \frac{\partial c(x,t)}{\partial t} = D \frac{\partial^2 c}{\partial^2 x} \xrightarrow{3D} \frac{\partial C(x,t)}{\partial t} = D \nabla^2 C(x,t)$$

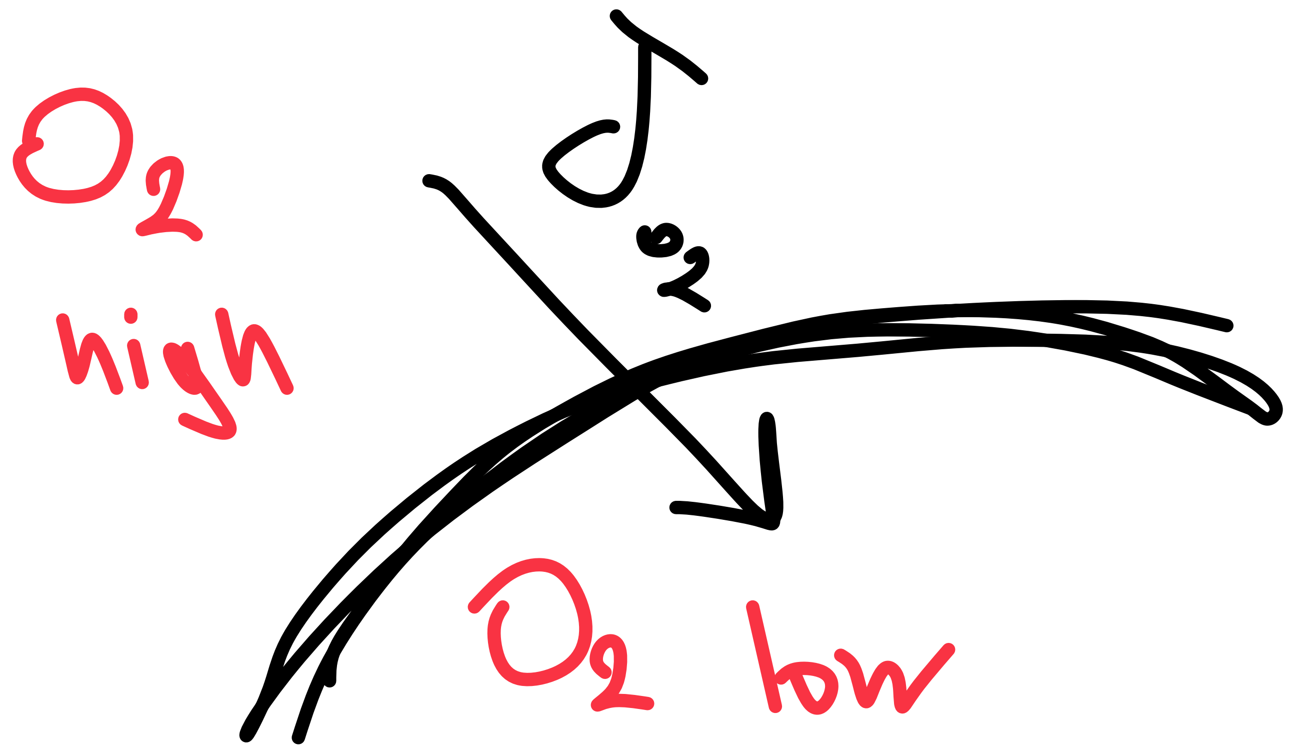
second law

Before moving on, let's pause and consider some implications of some this law for biology. For example, flux plays a crucial role in breathing and gas exchange, which are fundamental for animal survival.

Example 1: Gas exchange in worms:



/www.geeksforgeeks.org

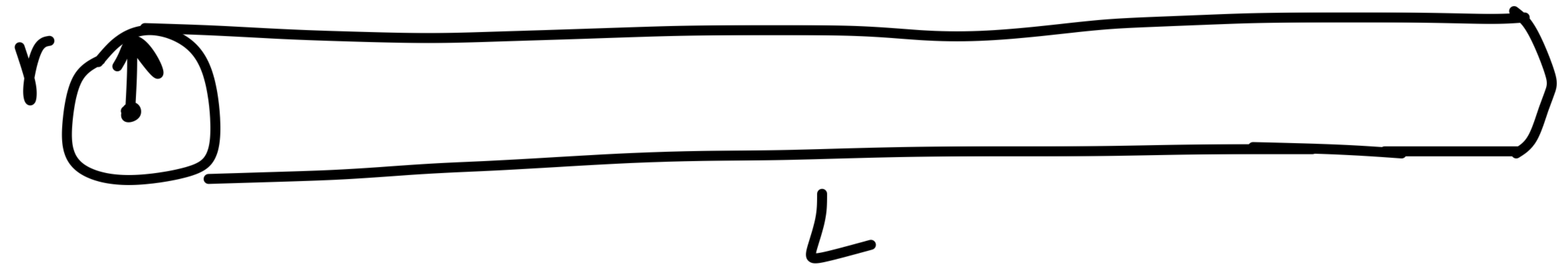


total rate of O_2 into the worm body:

$$\text{total rate} = J_{O_2} \times \text{Area}$$

So the Total rate of oxygen into a worm is proportional to Area:

modeling worm as a cylinder:



$$Area = 2\pi r \times L$$

$$Rate_{O_2,in} = 2\pi r L \times J_{O_2} \quad (1)$$

However, the rate of O_2 consumption by a worm is proportional to its body mass. Let's express this, where K is the O_2 consumption rate per volume for a worm:

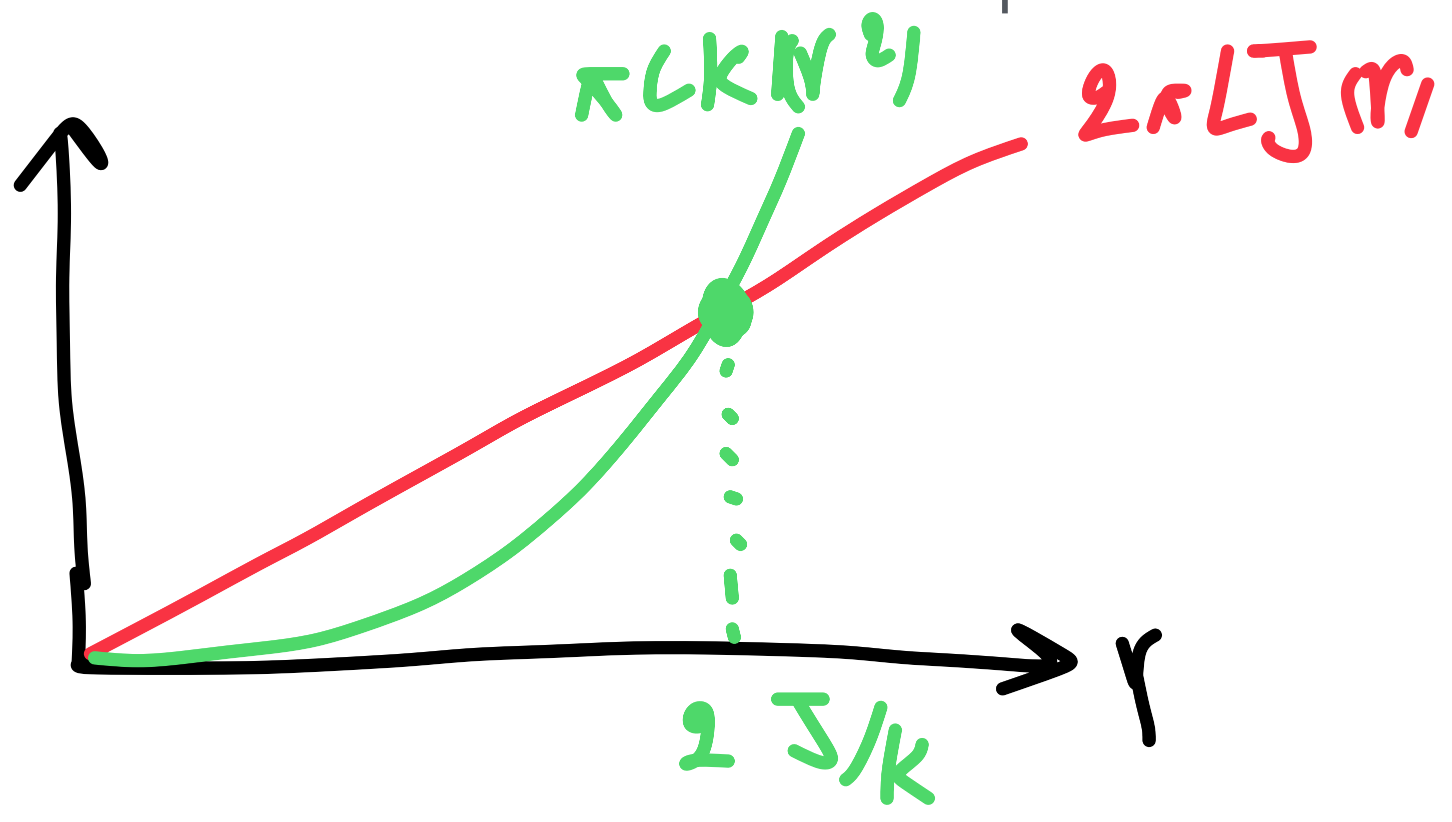
$$Rate_{O_2\,consume} = K \times \pi r^2 L$$

Now we can write the mass balance equation for O2 intake and consumption of a worm. if a worm wants to survive it's intake 02 should be more than or equal to its O2 consumption:

O_2 in $\downarrow$ $\quad$ O_2 consumed $\downarrow$

$$2\pi r L J_{O_2} \geq \pi r^2 L k$$

$$\Rightarrow r \leq \frac{2 J_{O_2}}{k}$$



Means that balance between oxygen supply and oxygen demand, together with the oxygen diffusion rate through tissues, determines the maximum dimensions of worms. Let's put some biologically relevant values for J and K for normal earthworms:

$$J \longrightarrow 0.5 \frac{\mu mol}{h * cm^2}$$

$$\rho_{worm} \approx 1 \frac{gr}{cm^3}$$

O2 consumption rate/ mass

$$\approx 1 \frac{\mu mol}{gr\ h} \Rightarrow K \approx 1 \frac{\mu mol}{cm^3\ h}$$

O2 consumption rate/ volume

$$\Rightarrow r \leq 2 \times \frac{5}{10} \times \frac{1}{1}$$

$$\boxed{r \leq 1\ cm}$$

In fact one of the biggest earthworms out there is:

Giant Gippsland earthworm

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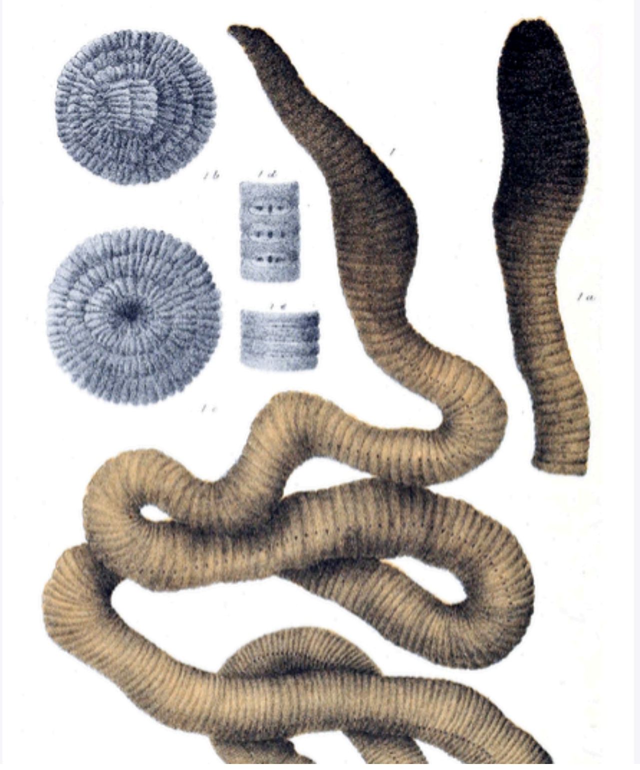
From Wikipedia, the free encyclopedia

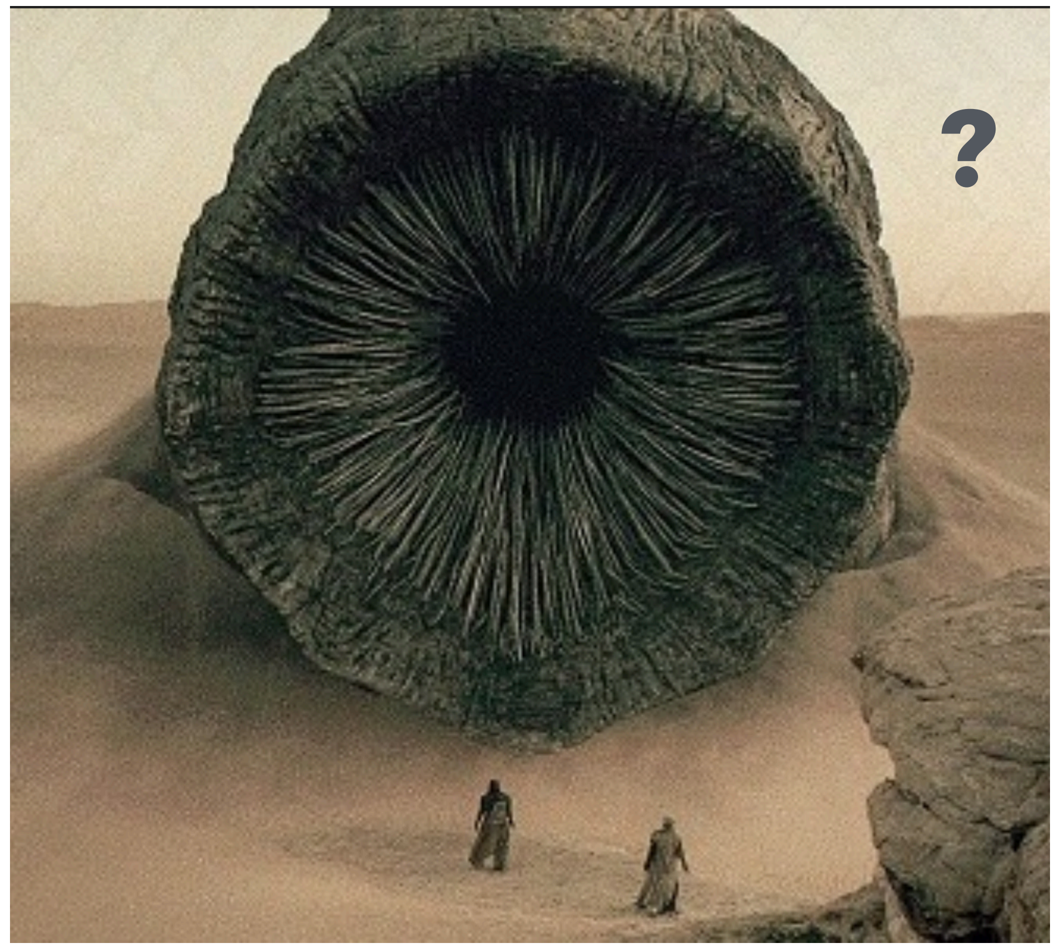
The **giant Gippsland earthworm** (***Megascolides australis***), is one of Australia's 1,000 native earthworm species.[2]

Giant Gippsland earthworm



Description [edit]

These giant earthworms average 1 metre (3.3 ft) long and 2 centimetres (0.79 in) in diameter and can reach 3 metres (9.8 ft) in length; however, their body is able to expand and contract making them appear much larger. On average they weigh about 200 grams (0.44 lb).[3][4] They have a dark purple head and a blue-grey body, and about 300 to 400 body segments.[2]



Now the question is: how were we humans able to grow so large? We don't breathe through our skin but rather through our lungs.

Let's use the values we discussed earlier to calculate the surface area needed to provide the O_2 required for our bodies as humans.

Example2: How much surface area is needed to provide us (humans) with our O2 needs?

$J \longrightarrow \frac{1}{2} \frac{\mu mol}{h \times cm^2}$

$\rho_{water} \approx 1 \frac{gr}{cm^3}$

O2 consumption rate/ mass $\approx 1 \frac{\mu mol}{gr\ h}$

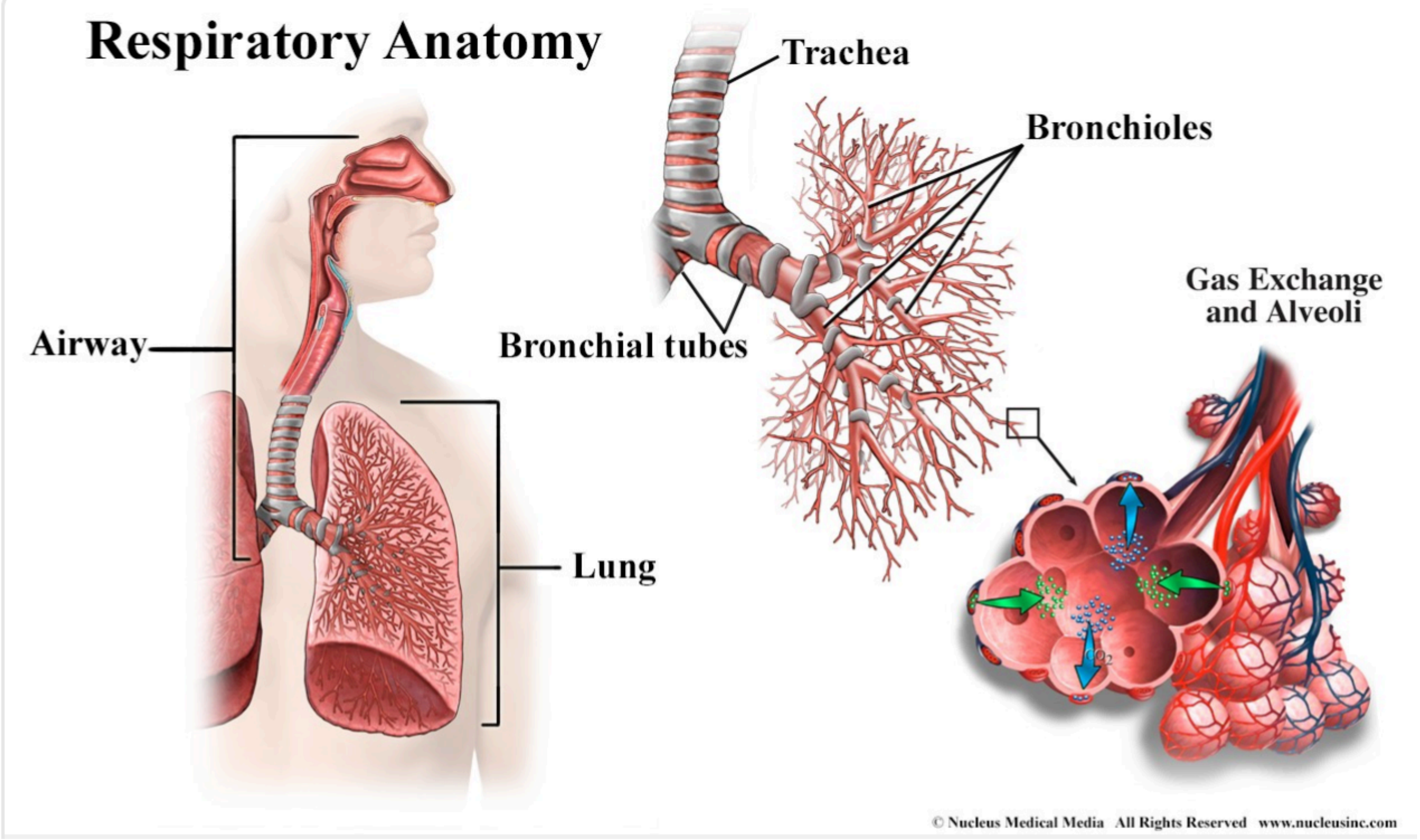
$\Longrightarrow$

mass = 100 kg = 100×10^3 gr

$\Rightarrow$ total O_2 consumption rate: $1 \frac{\mu mol}{h\ gr} \times 10^5 gr = 10^5 \frac{\mu mol}{h}$

$\Rightarrow$ Area needed: $A \times \frac{5}{5} \frac{\mu mol}{h\ cm^2} = 10^5 \frac{\mu mol}{h}$

$\Rightarrow A \sim \frac{1}{5} 10^6 cm^2$ or $\frac{1}{5} 10^2 m^2$



Historically, numerous experiments have tried to square in on the exact surface area available for gas exchange. From histological techniques to radiological approximations, all the methods that have been employed in the process have failed in reaching a consensus. In 1967, Thurlbeck reported that the area of emphysematous lungs ranged from 40 to 100 m^2, normally averaging around 63 m^2 [4]. Hasleton in 1972 estimated that the pulmonary surface area ranged from 23.56 to 68.76 m^2 [5]. Weibel provided three values at three different points in time: 150-180 m^2 in 1980, 80 m^2 in 1993, and 130 m^2 in 2009 [6,7].

https://www.ncbi.nlm.nih.gov/pmc/articles/PMC8863270/

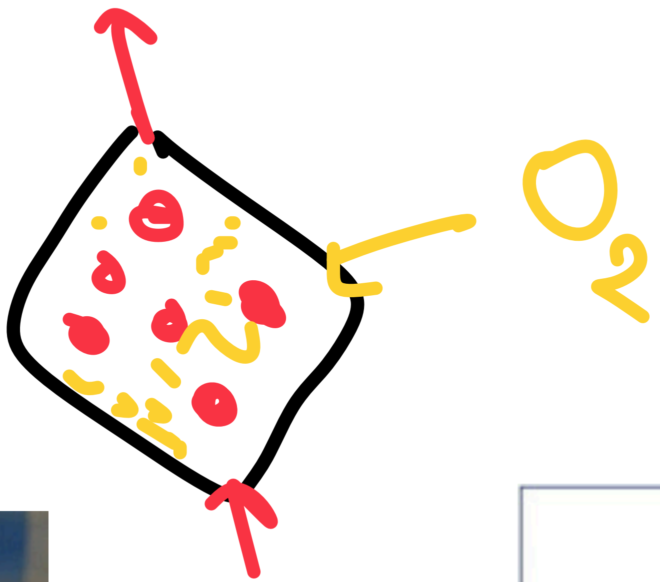
Bioengineering application: Creating Artificial Lung function: (AL)

Extracorporeal Membrane Oxygenation Circuitry (ECMO)

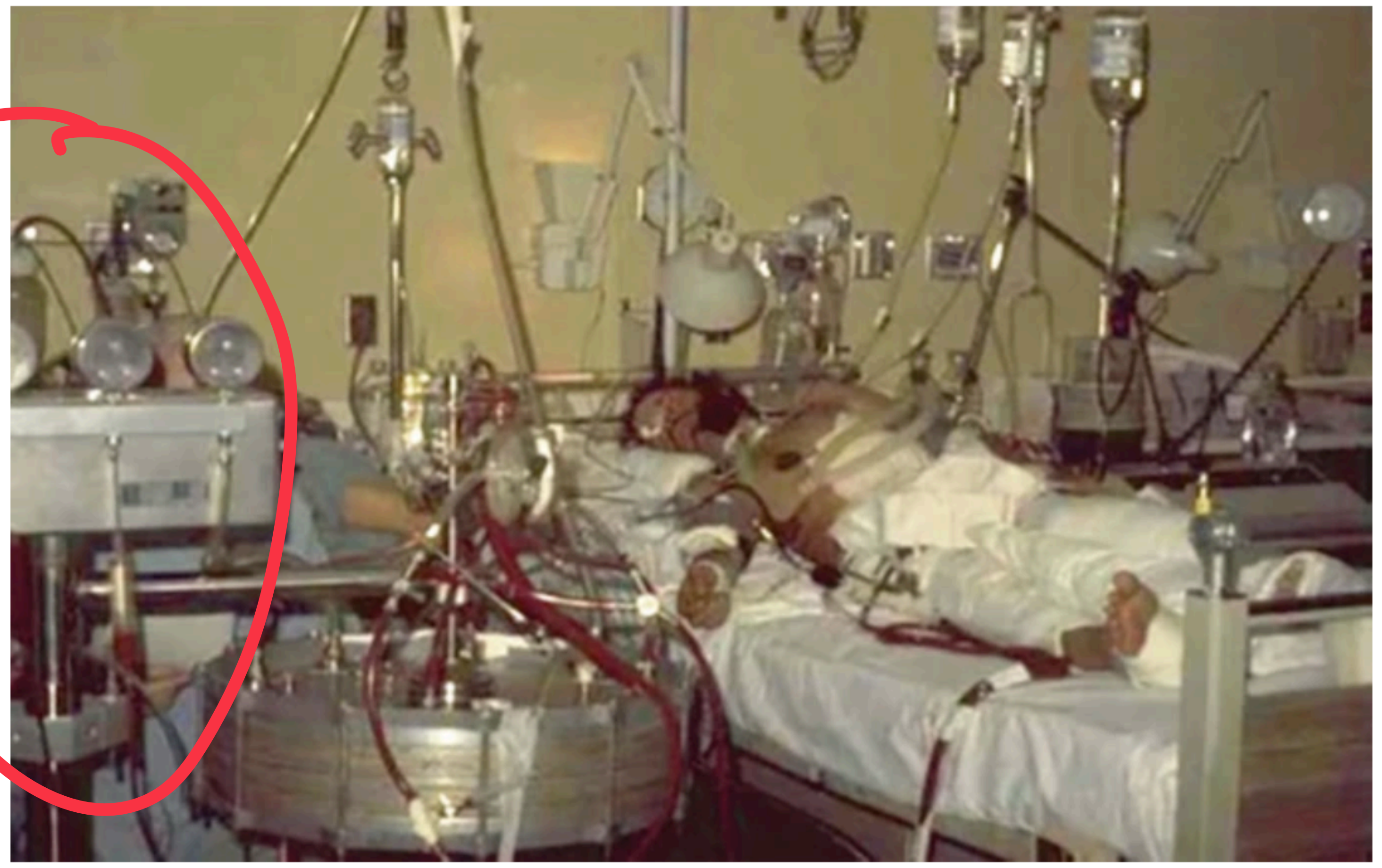
$$J = \frac{\text{total mass tras (Tmt)}}{\text{Area}} \Rightarrow |T.m.t| = |\text{Area} \times J| = \underbrace{\text{Area}} \times D \times |\nabla C|$$

Lung: high↑ × high↑ × medium↓

AL: low↓ × low↓ × Very high↑



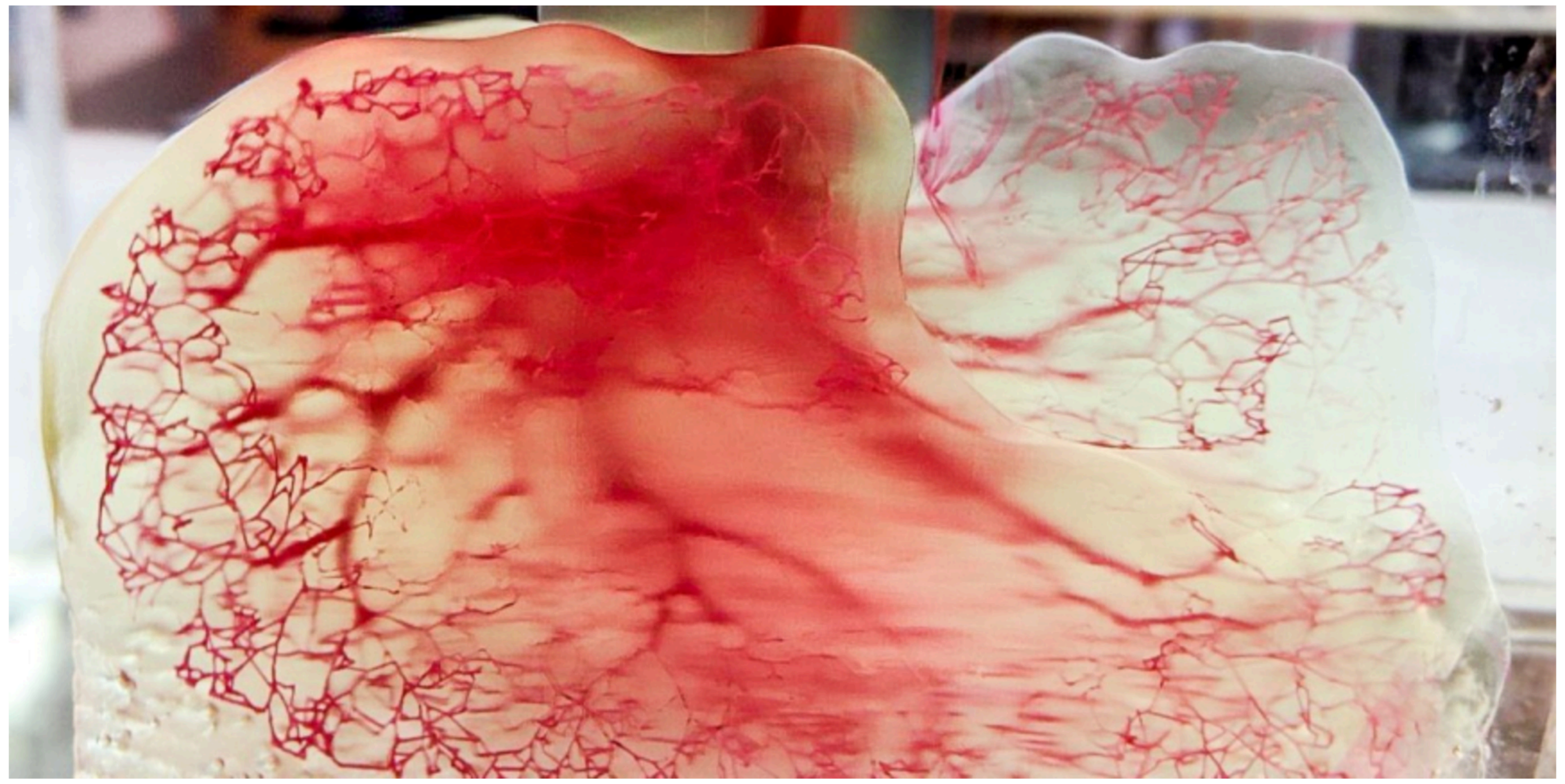
Ündar, Akif, et al. "Pediatric devices." *Mechanical Circulatory and Respiratory Support* (2018): 271-297.



The first reported survival after a motorcycle accident published by Donald Hill in 1972. (From Sangalli et al. History of ECMO. In ECMO, Extracorporeal Life Support in Adults by Sangalli F, Patroniti N, Pesenti A. pp. 3–10, 2014 with permission [24])

Camboni, Daniele, and Christof Schmid. "History of Extracorporeal Life Support: From Cardiopulmonary Bypass Towards ECMO." *ECMO Retrieval Program Foundation*. Cham: Springer International Publishing, 2023. 3-16.

Bioengineering application: Newer methods: creating a 3D bio-printed lung (with living cells)



Human vasculature model created using Print to Perfusion process (Image credit: United Therapeutics)

This involves using the custom materials and optimized process to rapidly create high-resolution hydrogel scaffolds which can be perfused with living cells to create tissues.

The lung scaffold has been heralded as the most complex part ever printed due to the 44 trillion voxels that lay out 4,000 kilometers of pulmonary capillaries and 200 million alveoli within it.

The 3D-printed scaffolds will be cellularized with a patient's own stem cells to create tolerable, transplantable human lungs that should not require immunosuppression to prevent rejection.

https://3dprinting.com/news/3d-printed-lung-demos-gas-exchange-in-animals/

Pause point

$$\begin{cases} J = \dfrac{T.m.t}{Area} \\[2ex] \vec{J} = -D \ \overrightarrow{\nabla c} \\[2ex] \dfrac{\partial c}{\partial t} = D \ \nabla^2 C \longrightarrow \text{this is a PDE} \end{cases}$$

B) Steady State approximation for solving Diffusion equation (part I)

What is steady state (s.s)?

Refers to the condition when the properties of interest at a point in space do not change with time, i.e. are not functions of time, and thus the time derivatives can be set to zero. This will make our calculations way much easier.

$$\frac{\partial C(x,t)}{\partial t} = 0 \quad \text{or} \quad C(\vec{x})$$

$$\cancel{\frac{\partial C(\vec{x},t)}{\partial t}} = D \nabla^2 C(x,t) \Rightarrow D \nabla^2 C(\vec{x}) = 0$$

this process will convert PDEs to ODEs....

In the next couple of lectures we will study this equation

End