Lecture 2

Introduction to vector calculus

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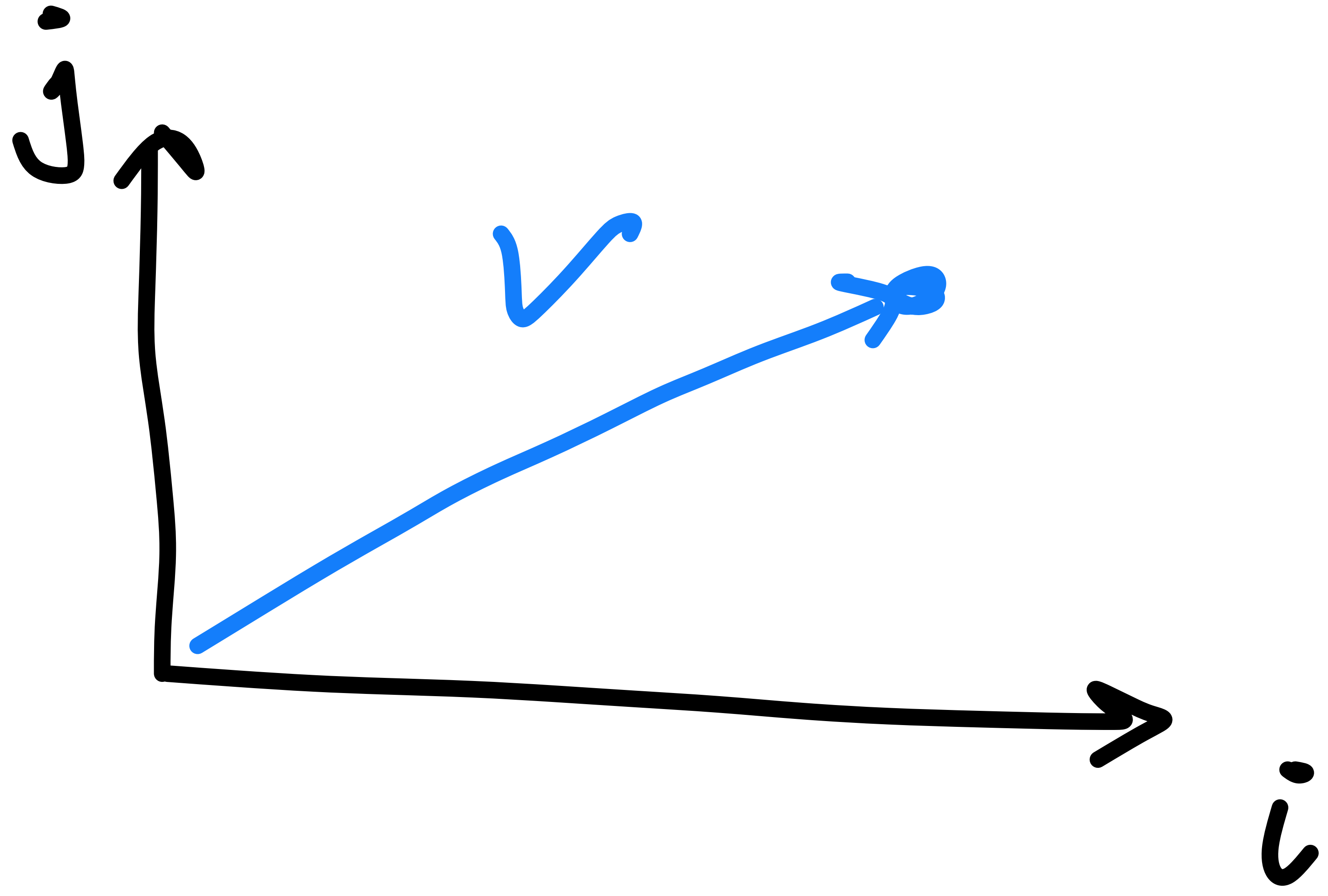
A) how to describe a vector?

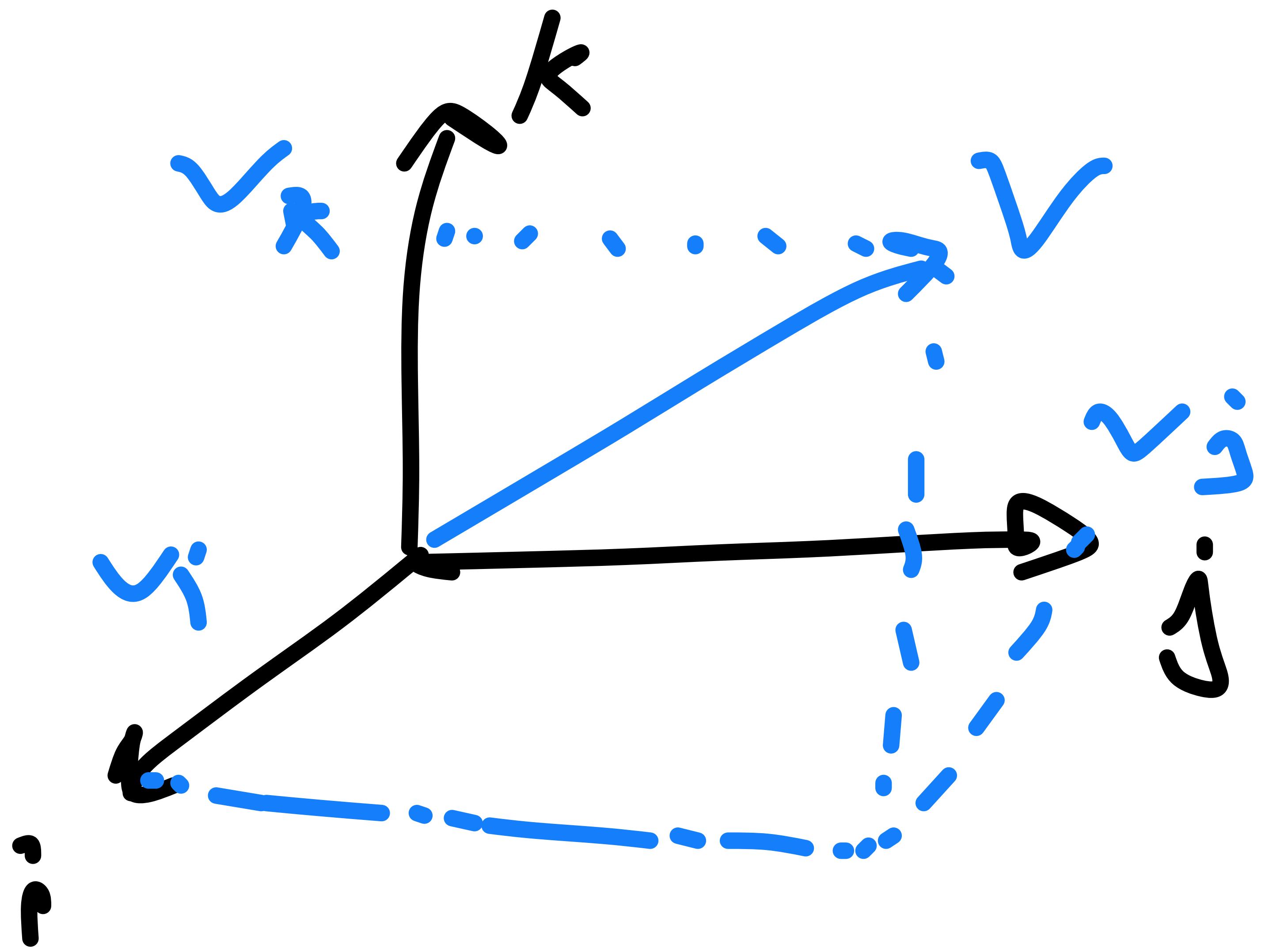
2D:



$$\vec{v} = (v_i)\,\hat{i} + (v_j)\,\hat{j}$$

$$= \begin{bmatrix} v_i \\ v_j \end{bmatrix}$$

3D



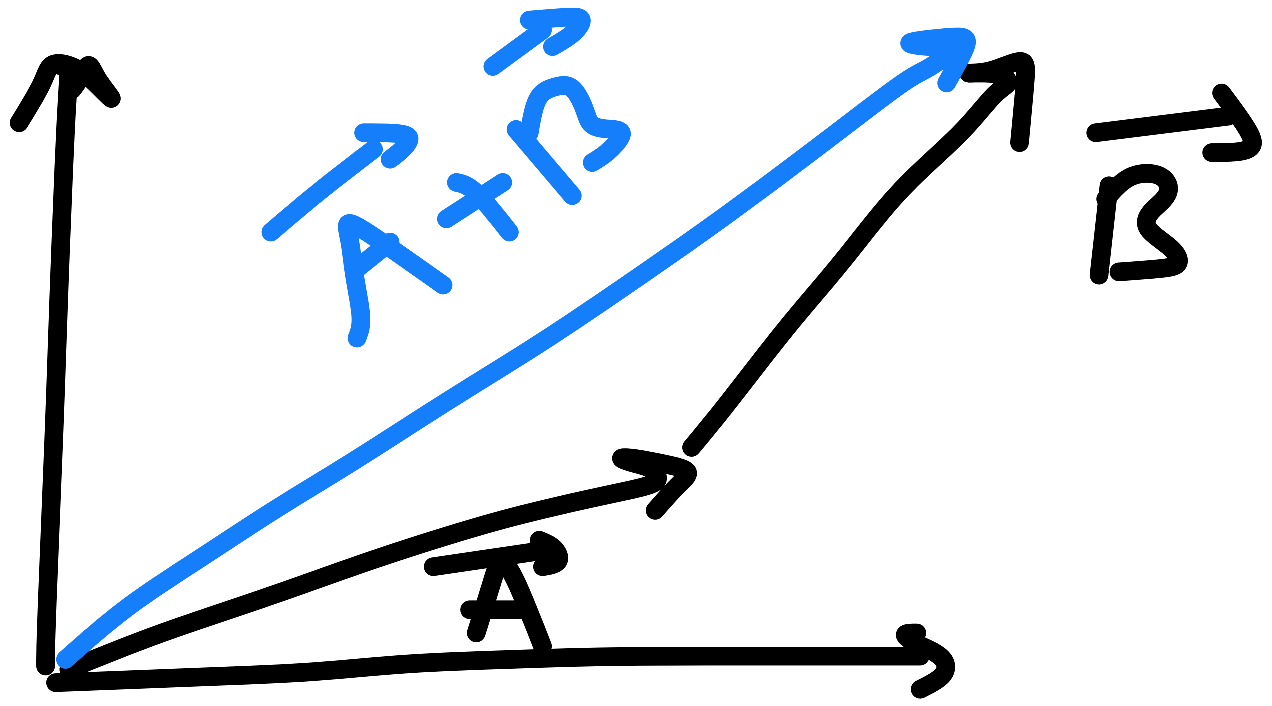
$$\vec{v} = v_i\,\hat{i} + v_j\,\hat{j} + v_k\,\hat{k}$$

$$= \begin{bmatrix} v_i \\ v_j \\ v_k \end{bmatrix}$$

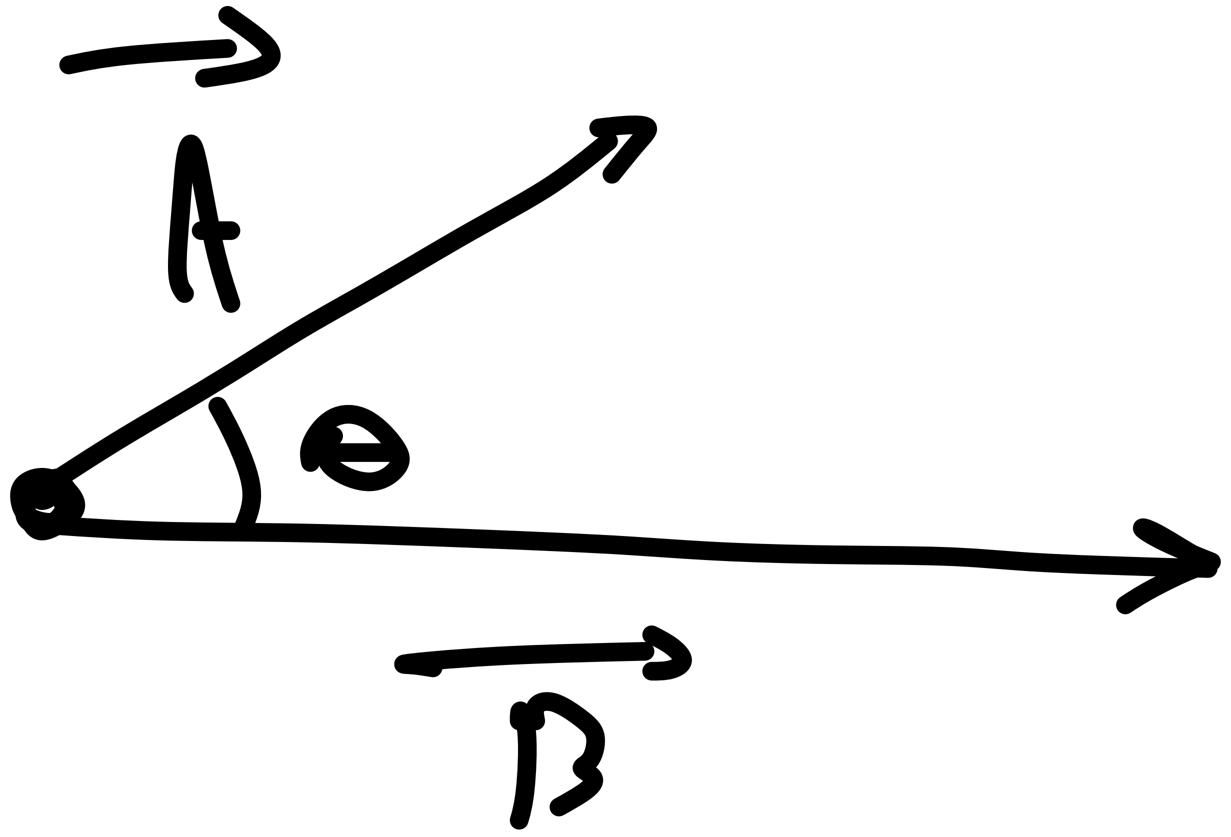
Some essential operations:

1. Addition: $\vec{A} + \vec{B}$

$= (a_i + b_i)\hat{i} + (a_j + b_j)\hat{j} + (a_k + b_k)\hat{k}$



2. Dot product: $\vec{A} \cdot \vec{B} = |A|\,|B| \cos\theta$



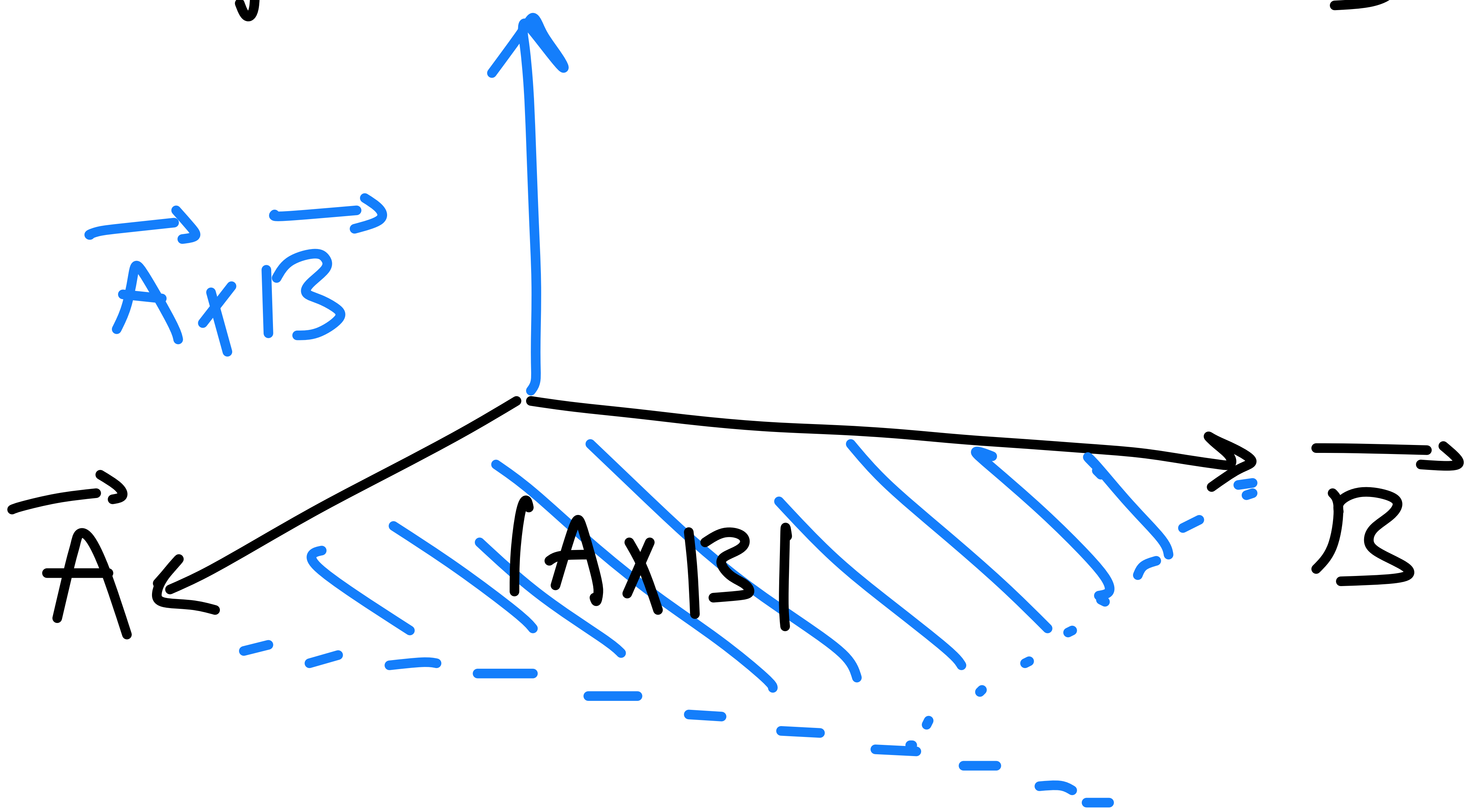
$A = \begin{bmatrix} a_i \\ a_j \\ a_k \end{bmatrix}$

$B = \begin{bmatrix} b_i \\ b_j \\ b_k \end{bmatrix}$

$A \cdot B = a_i b_i + a_j b_j + a_k b_k$

$= \text{Scalar}$

3. Cross product: $\vec{A} \times \vec{B}$: is a vector



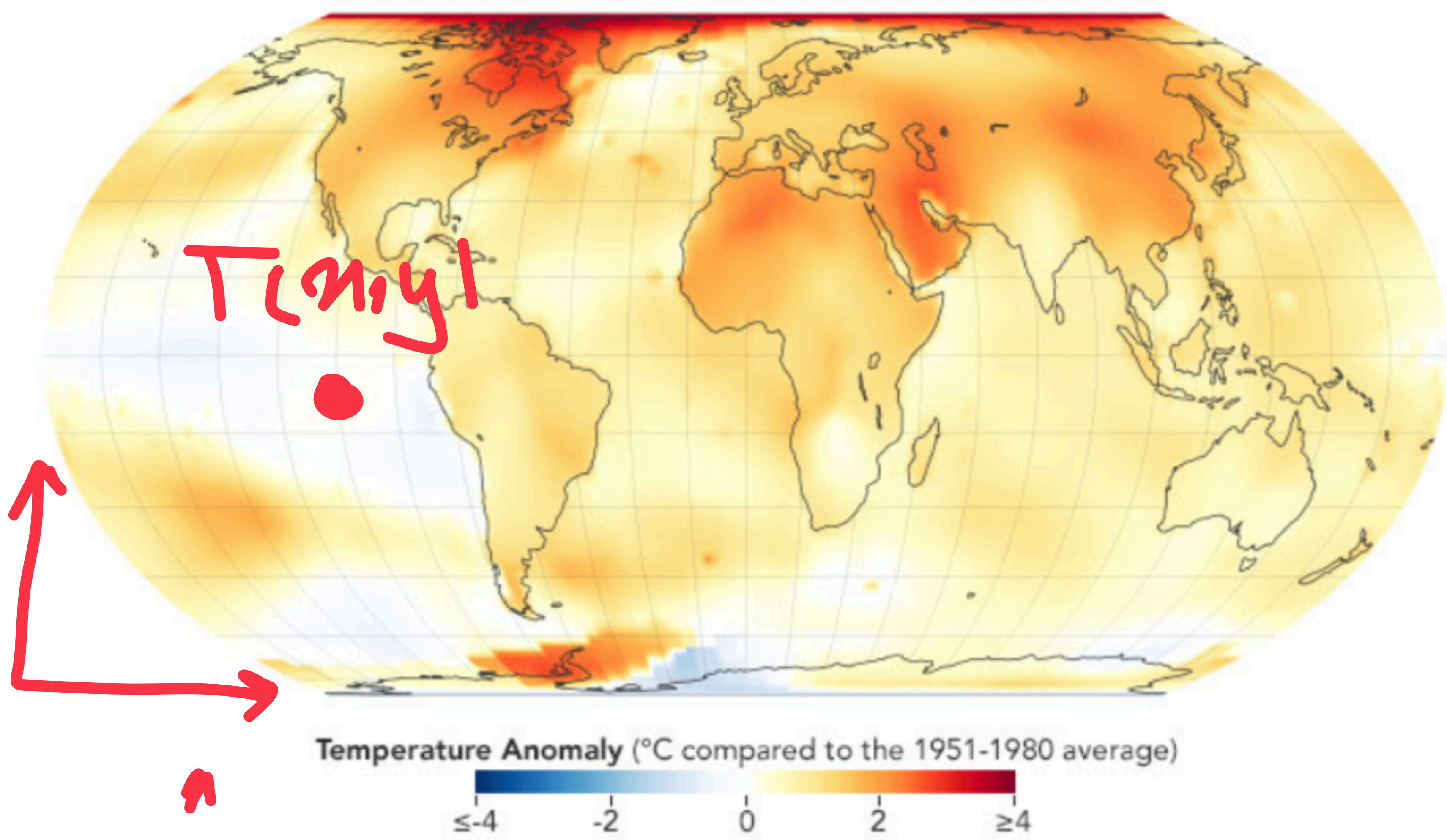
$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_i & a_j & a_k \\ b_i & b_j & b_k \end{vmatrix}_{det} = \begin{aligned} &+(a_j b_k - a_k b_j)\,\hat{i} \\ &-(a_i b_k - a_k b_i)\,\hat{j} \\ &+(a_i b_j - a_j b_i)\,\hat{k} \end{aligned}$$

B) Scalar fields:

Scalar fields provide a way to describe how scalar quantities vary in space, which is fundamental for understanding physical and biological processes.

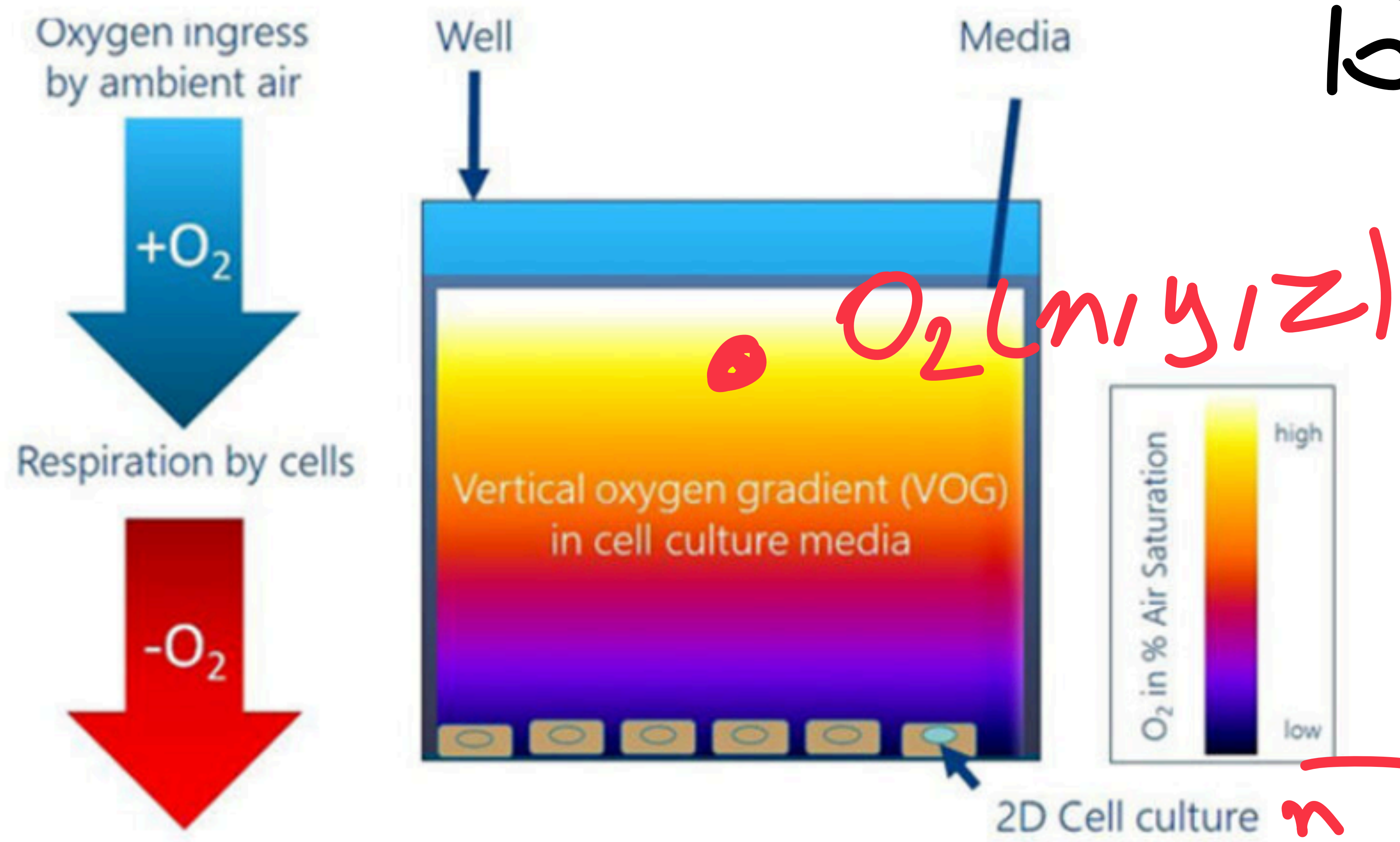
examples:

10^{+6} m



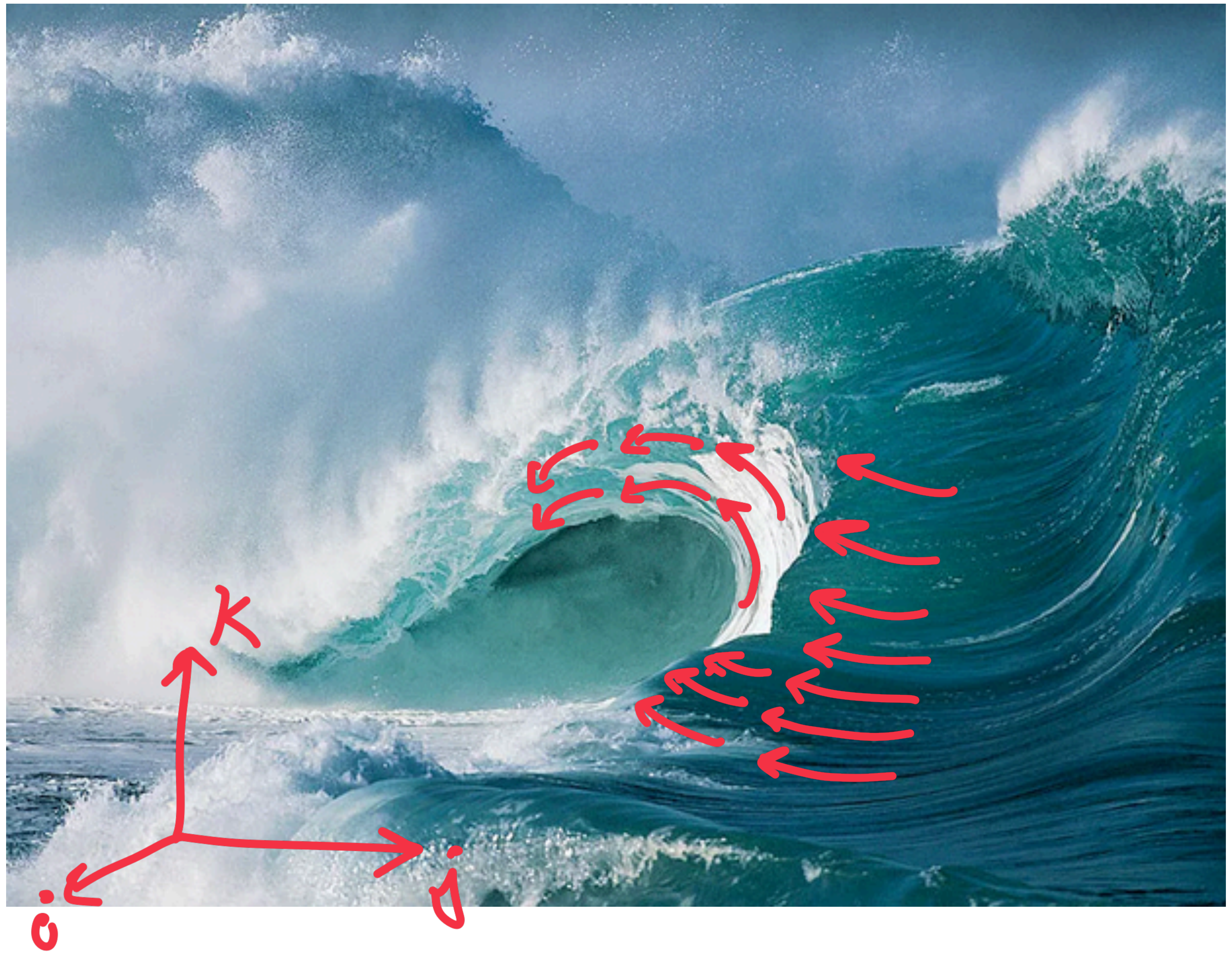
$T(x,y)$: scalar field
describes T on the
map

10^{-6} m



$O_2(x,y,z)$: scalar field
describes O_2 concentration
in a cell culture well

c)Vector fields



A vector field is a function that assigns a vector to every point in space. Commonly used in physics and engineering to represent quantities like velocity, force, or magnetic fields.

example: $\vec{V}(x,y,z)$

velocity vector field of water

$$\vec{V}(x,y,z) = V_x(x,y,z)\,\hat{i} + V_y(x,y,z)\,\hat{j} + V_z(x,y,z)\,\hat{k} \quad \text{or} \quad \begin{bmatrix} V_x(x,y,z) \\ V_y(x,y,z) \\ V_z(x,y,z) \end{bmatrix}$$

Before me move forward, think about this:

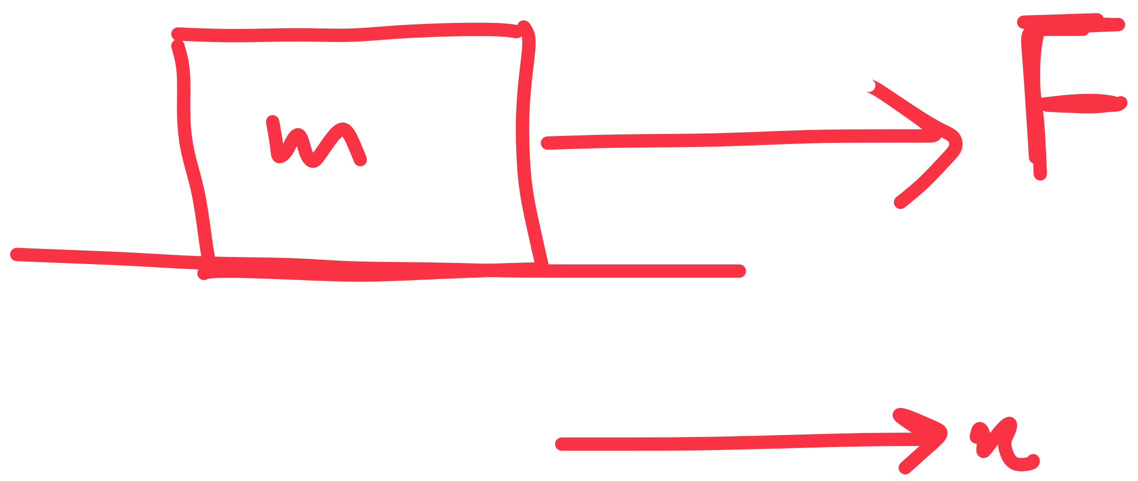
Scalar and **vector fields** are crucial in transport phenomena because they provide the mathematical framework to describe how quantities like temperature, pressure, velocity, and flux vary in space and time. These fields are essential for modeling physical systems, understanding conservation laws, analyzing fluxes and gradients, solving governing **partial differential equations**, handling **complex geometries**, and visualizing physical phenomena in biology. Mastery of these concepts is key to predicting and optimizing the behavior of systems in engineering and science.

Fields are functions, more accurately, multivariable functions. They can depend on spatial coordinates, for example, (x,y,z)(x,y,z), and also on time, t.

In physics, many laws are derived from the derivatives of functions.

For example:

Q: Calculate the velocity and position.



$$F = ma$$

$$F = m\frac{dV(t)}{t} \quad : \text{ODE} \quad \xrightarrow{\text{solve ODE by } \int}$$

$$\int \frac{F}{m}\,dt = \int \frac{dV}{dt}\,dt \Rightarrow$$

$$V = \frac{F}{m}t + V_0 \quad \xrightarrow{V_0 = 0 \text{ : bondary condition}} \quad V = \left(\frac{F}{m}\right)t$$

Similar to basic physics, where we work with derivatives, in this context, we will also deal with scalar and vector fields. We need to learn how to calculate different types of derivatives from these **fields** (function of x,y,z) and develop an intuition about what these derivatives represent.

D) Derivative of fields

The superstar of this show is the del operator (∇)

$$\vec{\nabla} = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k} = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix}$$

Del is a **vector** of <u>partial derivatives</u> (think about it, why partial?)

And we can apply this vector to fields (scalar or vector) and calculate different types of derivatives.

$\vec{\nabla} f$	$\vec{\nabla} \cdot \vec{f}$	$\vec{\nabla} \times \vec{f}$
Gradiant	divergence	Curl
f is scalar field	f is vector field	f is vector field

∇ Grad operator:

$$\underset{\text{scalar}}{f} \xrightarrow{\text{Grad}} \underset{\text{vector}}{\vec{\nabla} f}$$

$$\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k} :$$ vector field showing the direction and rate of the steepest increase in the scalar field

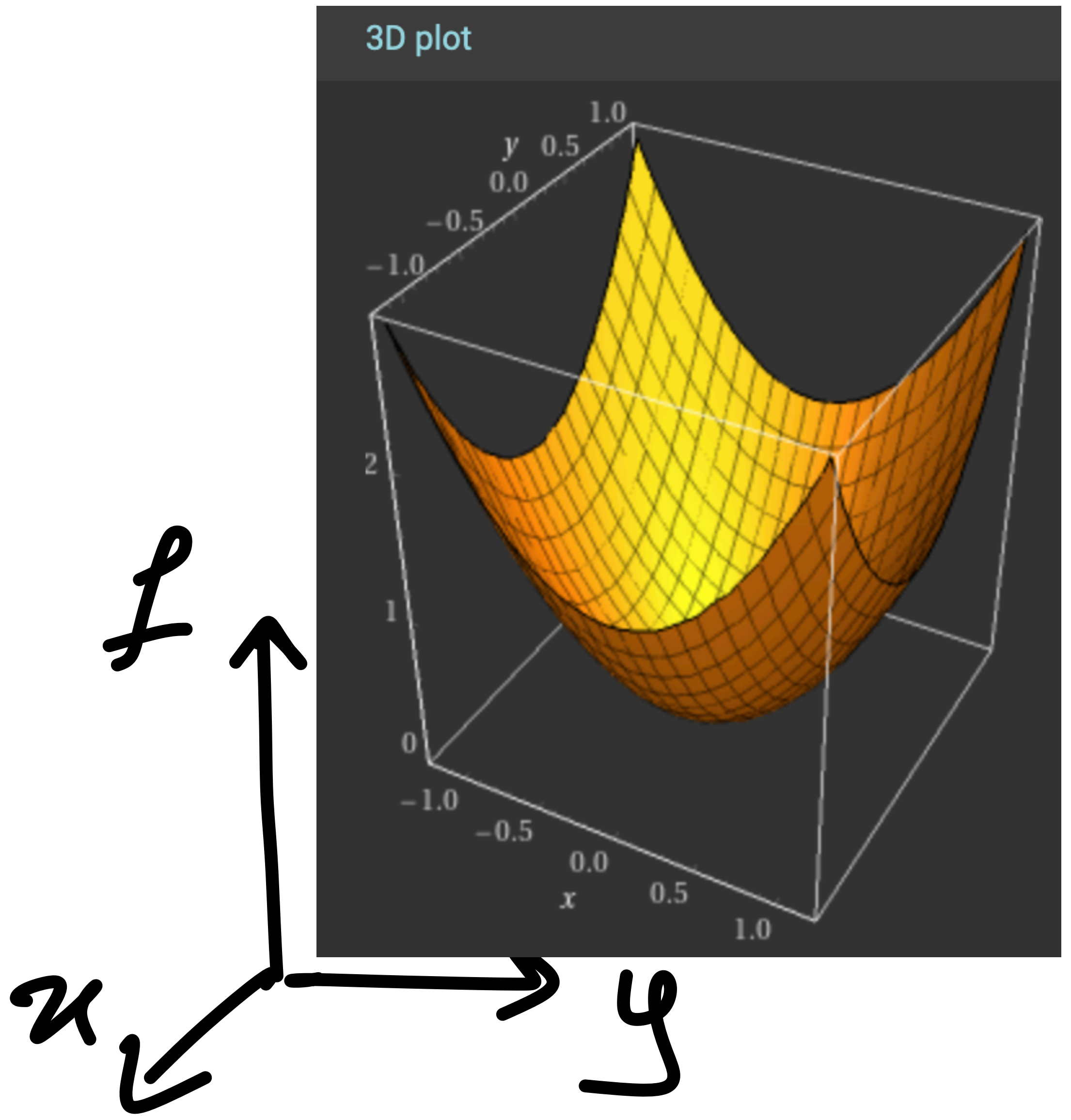
Example:

$$f = x^2 + y^2 \Rightarrow$$

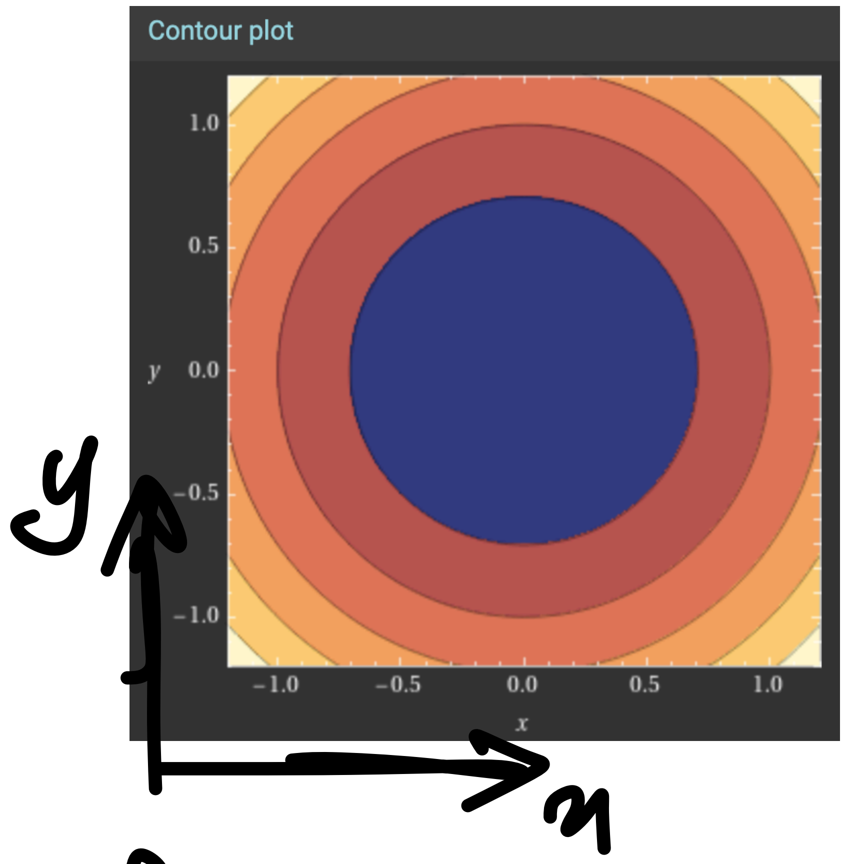
$$\vec{\nabla} f = 2x\,\hat{i} + 2y\,\hat{j}$$

$$= \begin{bmatrix} 2x \\ 2y \end{bmatrix}$$

f

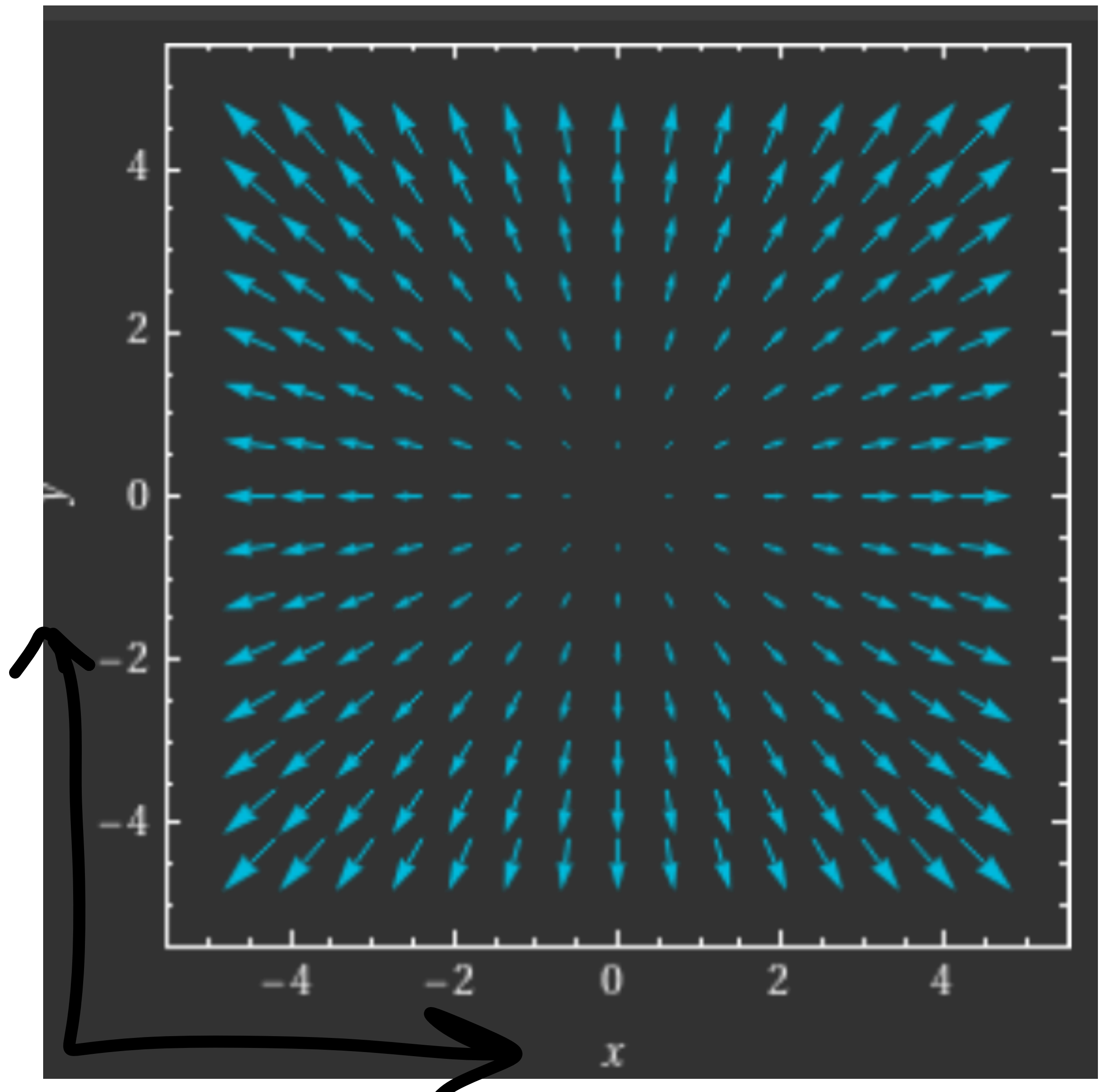


f



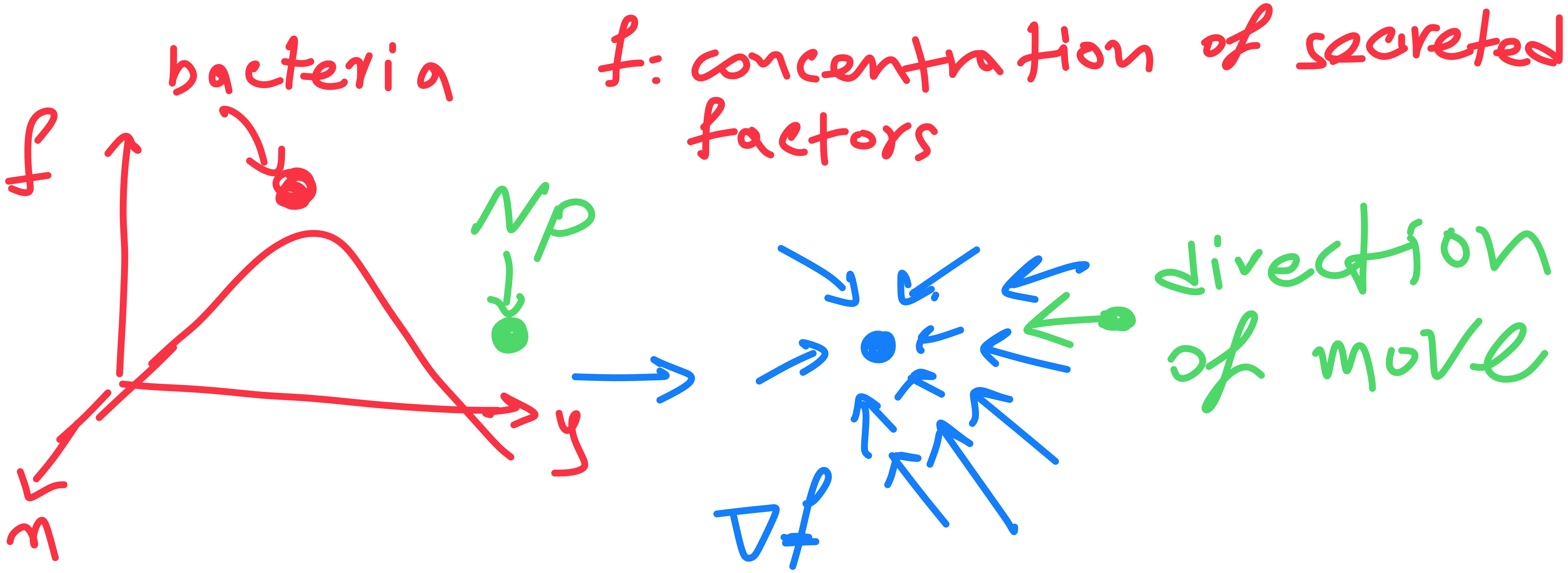
f: colour

$$\vec{\nabla} f = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$$



Example:

Chemotaxis is the process by which neutrophils, a type of white blood cell, move toward higher concentrations of chemical signals, such as chemokines, to reach sites of infection or injury. Neutrophils detect and calculate the gradient of the chemical concentration field using the gradient operator (∇), allowing them to determine the direction of the steepest increase in concentration. This directional movement is essential for their role in the immune response, helping them quickly locate and eliminate pathogens.



$\nabla \cdot$ div operator

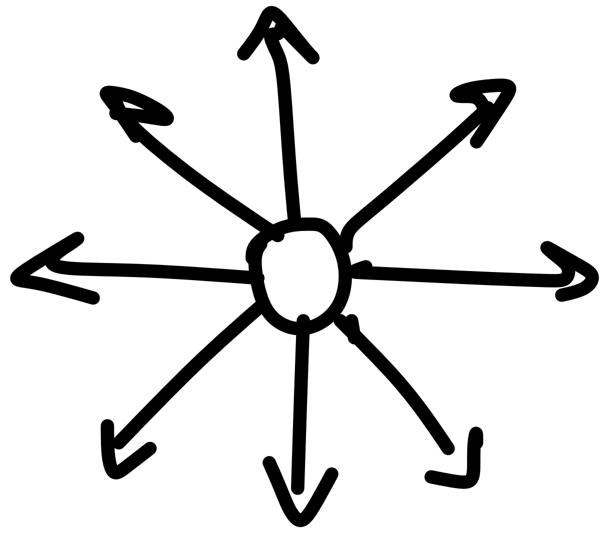
$$\vec{f} \xrightarrow{\nabla\cdot} \nabla\cdot f$$

Vector scalar

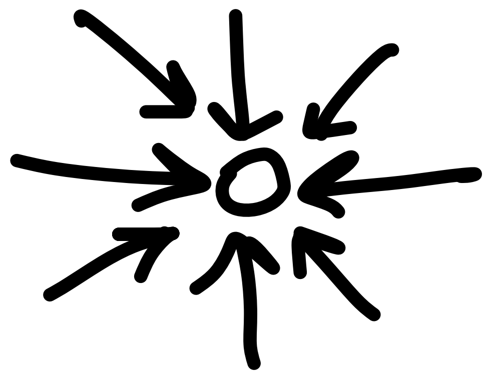
$$\nabla\cdot\vec{f} = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} \cdot \begin{bmatrix} f_x(x,y,z) \\ f_y(x,y,z) \\ f_z(x,y,z) \end{bmatrix} = \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + \frac{\partial f_z}{\partial z}$$

computes how much the vector field is locally contracting

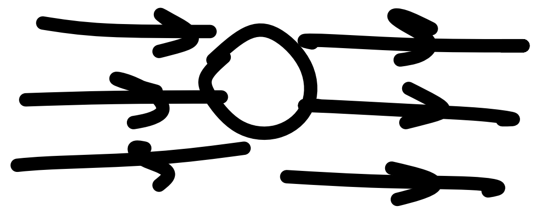
$\nabla\cdot\vec{f} > 0$



$\nabla\cdot\vec{f} < 0$

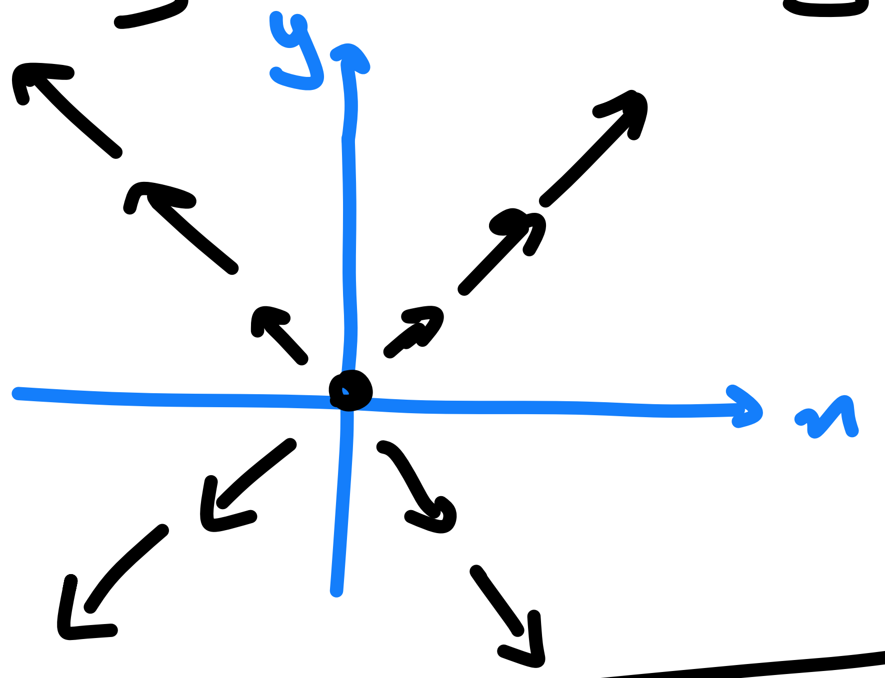


$\nabla\cdot\vec{f} = 0$



example:

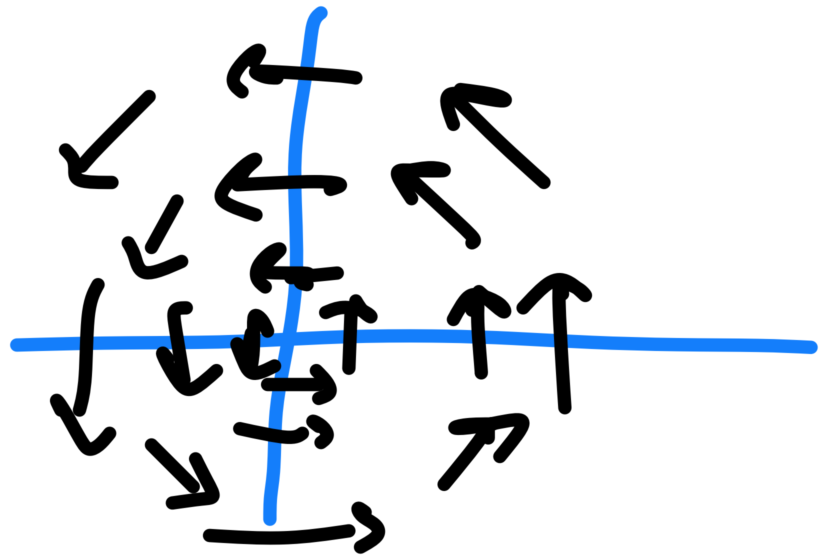
$$\vec{f}(x,y) = x\,\hat{i} + y\,\hat{j}$$



$$\nabla \cdot \vec{f} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} = 1 + 1 = 2$$

always divergence

$$\vec{f}(x,y) = -y\,\hat{i} + x\,\hat{j}$$



$$\nabla \cdot \vec{f} = \frac{-\partial y}{\partial x} + \frac{\partial x}{\partial y} = 0 + 0 = 0$$

divergence free

$\nabla \times$ Curl operator

$\vec{f} \quad \underline{\nabla \times} \quad \nabla \times f$

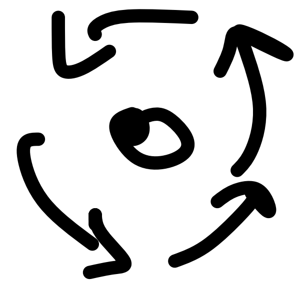
vector $\qquad\qquad$ vector

$$\nabla \times \vec{f} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_x & f_y & f_z \end{vmatrix}_{det}$$

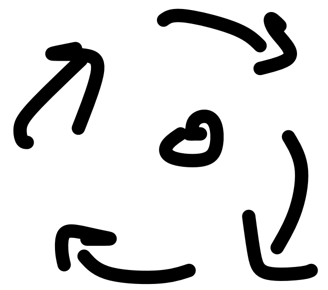
$$= \hat{i}\left(\frac{\partial f_z}{\partial y} - \frac{\partial f_y}{\partial z}\right) - \hat{j}\left(\frac{\partial f_z}{\partial x} - \frac{\partial f_x}{\partial z}\right) + \hat{k}\left(\frac{\partial f_z}{\partial x} - \frac{\partial f_x}{\partial x}\right)$$

Measures how much the vector field is locally rotating:

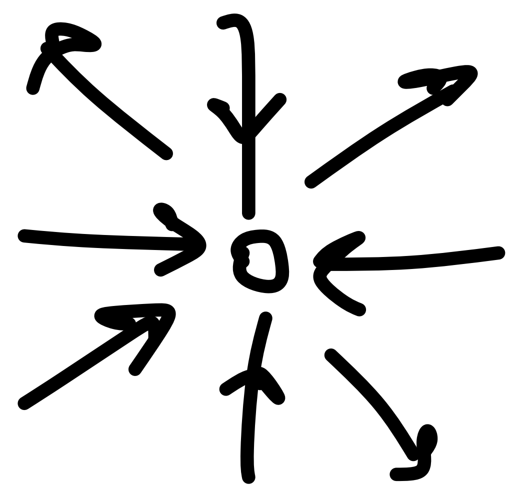
$\nabla \times \vec{f} > 0$



$\nabla \times \vec{f} < 0$



$\nabla \times \vec{f} = 0$



two important equations:

$$\begin{cases} \nabla \times \overrightarrow{\nabla f} = \vec{0} & f: \text{scalar field} \\ \nabla \cdot \nabla \times \vec{f} = 0 & \vec{f}: \text{vector field} \end{cases}$$

Think about why?

∇^2 Laplacian

$$f \xrightarrow{\nabla^2} \nabla^2 f$$

scalar → scalar

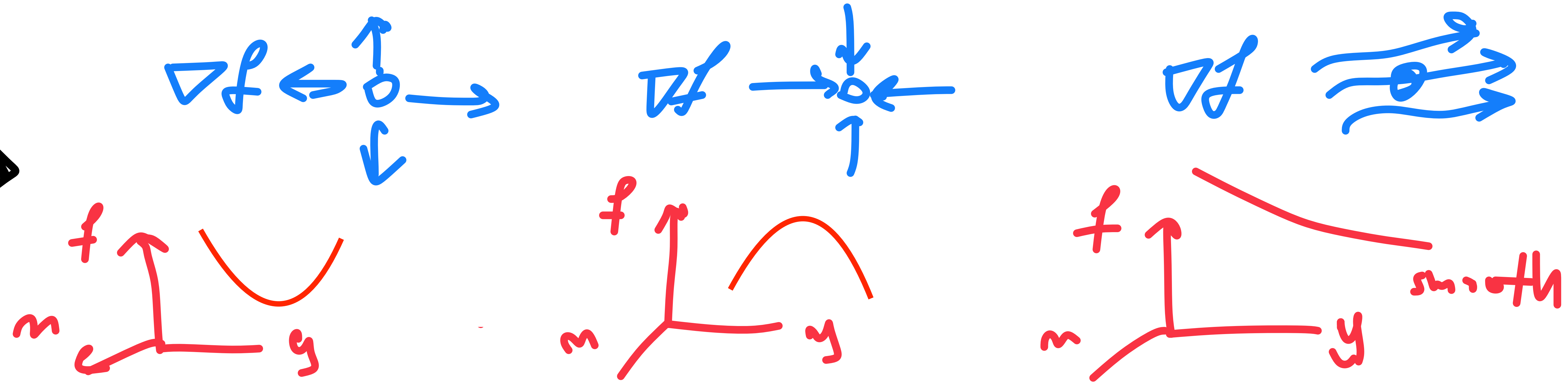
$\nabla^2 = \nabla \cdot \nabla$: div of grad

$$\nabla^2 f = \nabla \cdot \nabla f = \begin{bmatrix}\frac{\partial}{\partial x}\\ \frac{\partial}{\partial y}\\ \frac{\partial}{\partial z}\end{bmatrix} \cdot \begin{bmatrix}\frac{\partial}{\partial x}\\ \frac{\partial}{\partial y}\\ \frac{\partial}{\partial z}\end{bmatrix} f = \frac{\partial^2 f}{\partial^2 x} + \frac{\partial^2 f}{\partial^2 y} + \frac{\partial^2 f}{\partial^2 z}$$

The Laplacian measures the rate at which the average value of a function around a point differs from the value at that specific point, providing insight into the local curvature or "spread" of the function.

intuition: $f \xrightarrow{\nabla} \nabla f \xrightarrow{\nabla \cdot (\nabla f)}$

$\nabla^2 f > 0$ $\qquad$ $\nabla^2 f < 0$ $\qquad$ $\nabla^2 f = 0$



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