Lecture 22

Fluid mechanics - Review

-Motivation

-Some important concepts

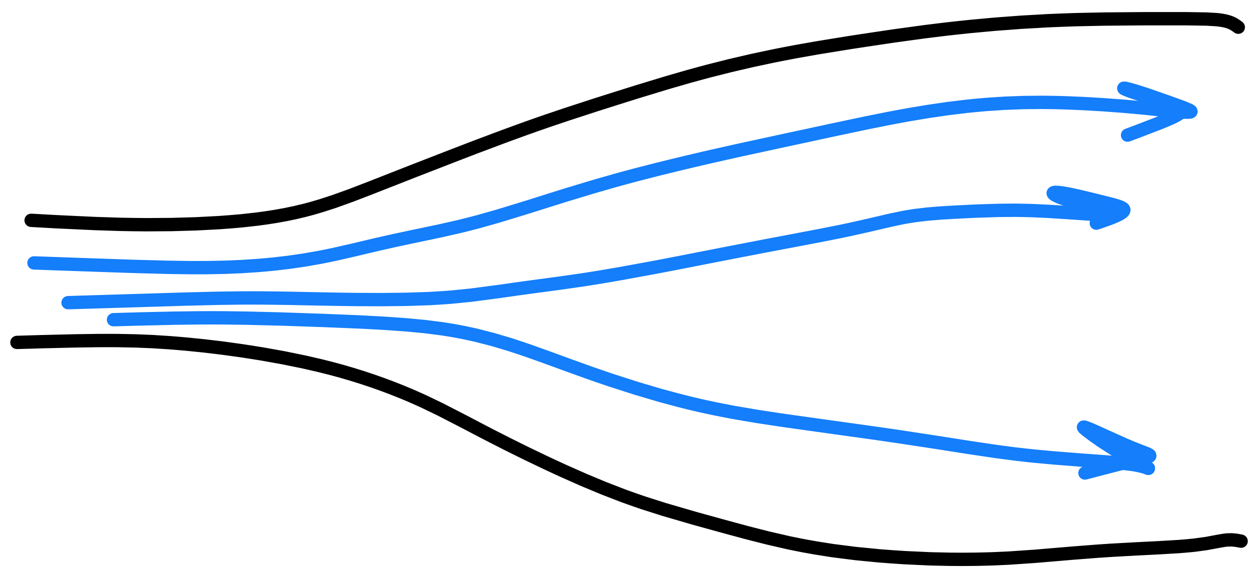
A)equation of motion

B) Rheology

C)Navier Stokes equation

This chapter has focused on momentum transfer—in simpler terms, how fluids flow. In biological systems, fluid flow plays a crucial role in transporting materials between cells and organs, a process known as convection.

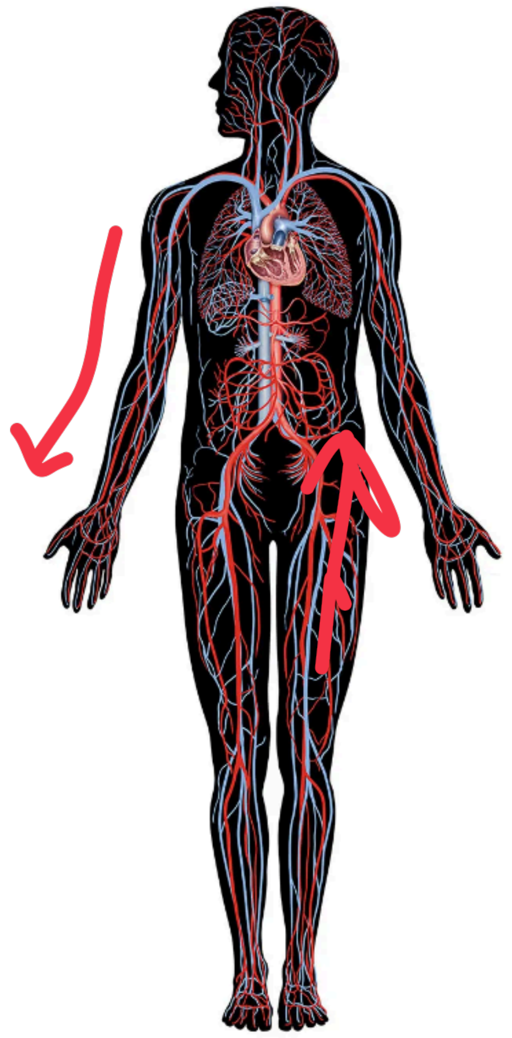
Before examining convection and its impact on particle transport in detail, we need to understand the carrier medium itself: the fluid. In Chapter 3, we will explore fluid **mechanics**, focusing on how fluids flow and their velocity profiles under various forces, all influenced by fluid properties.



Examples of Fluid Mechanics Applications Across Engineering Fields

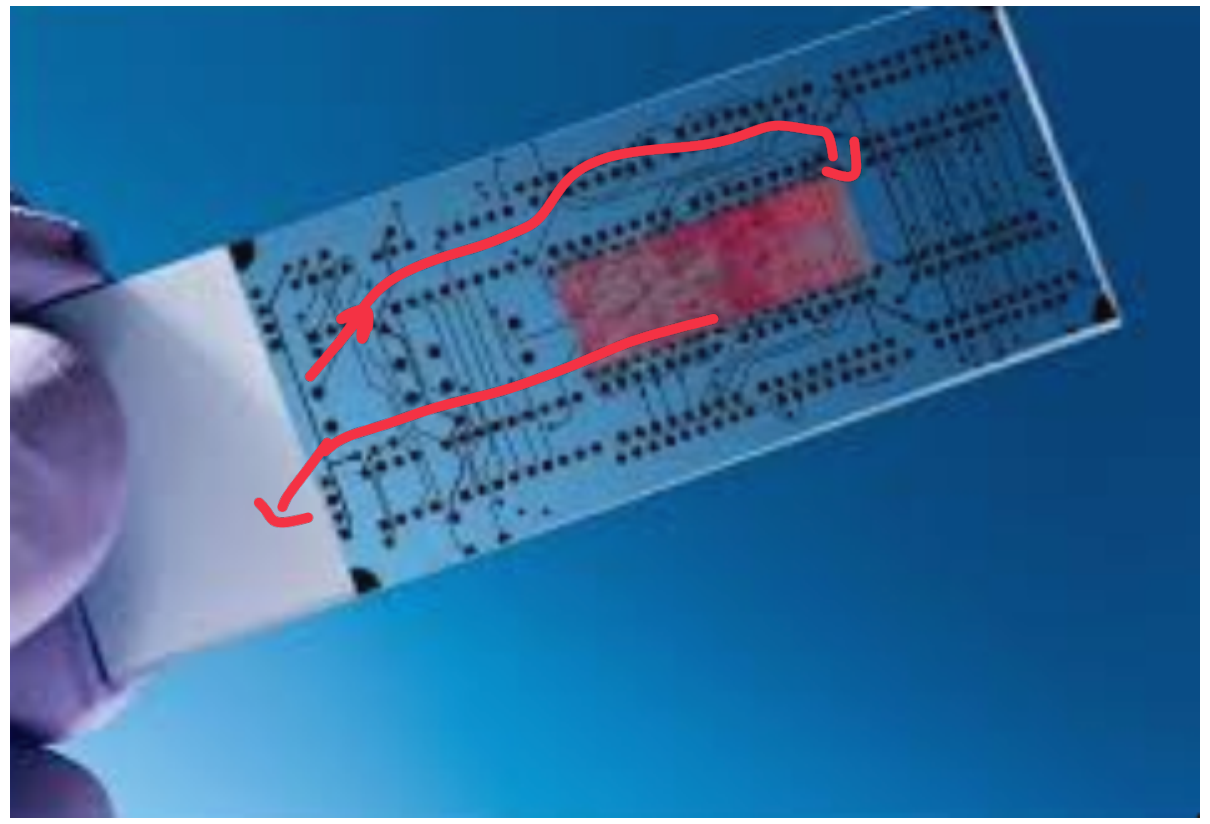
10^{2} m

10^{0} m

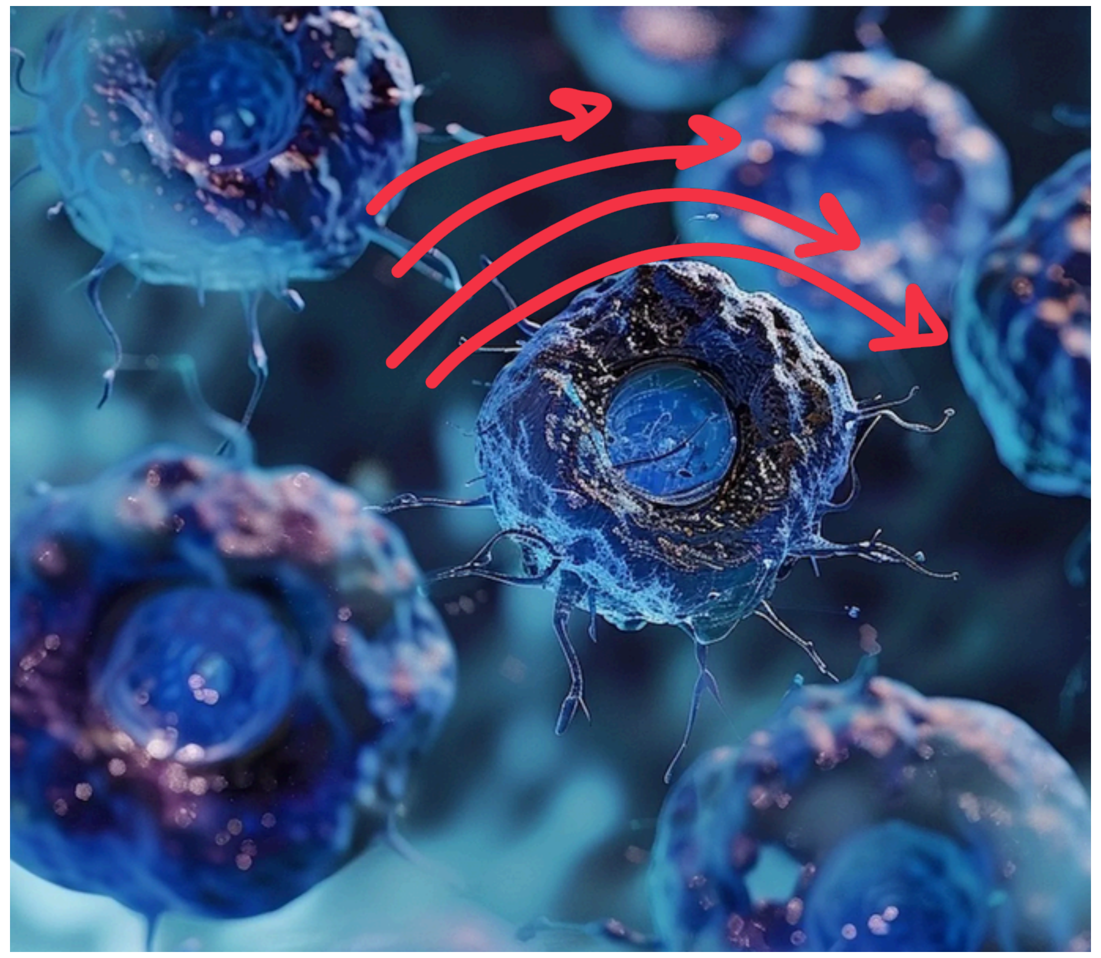


10^{-1} m

10^{-2} m



10^{-6} m

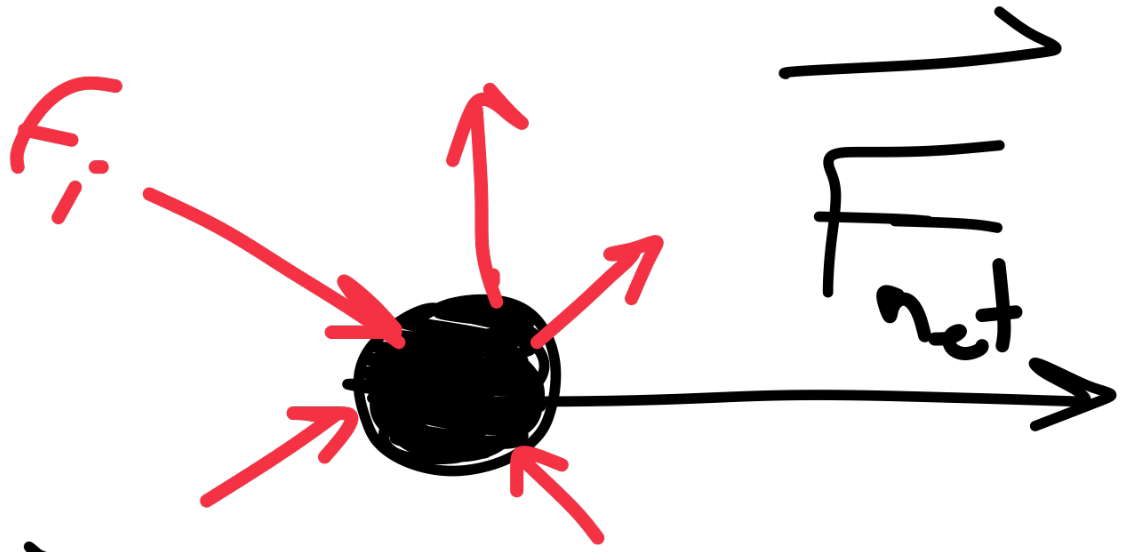


A) Equation of motion

An Intuitive (and not necessarily complete) introduction to Fluid **Mechanics:**

Let’s start with what we know from classic Newtonian mechanics—Physics 101:

Newton’s second law for a point like particle:

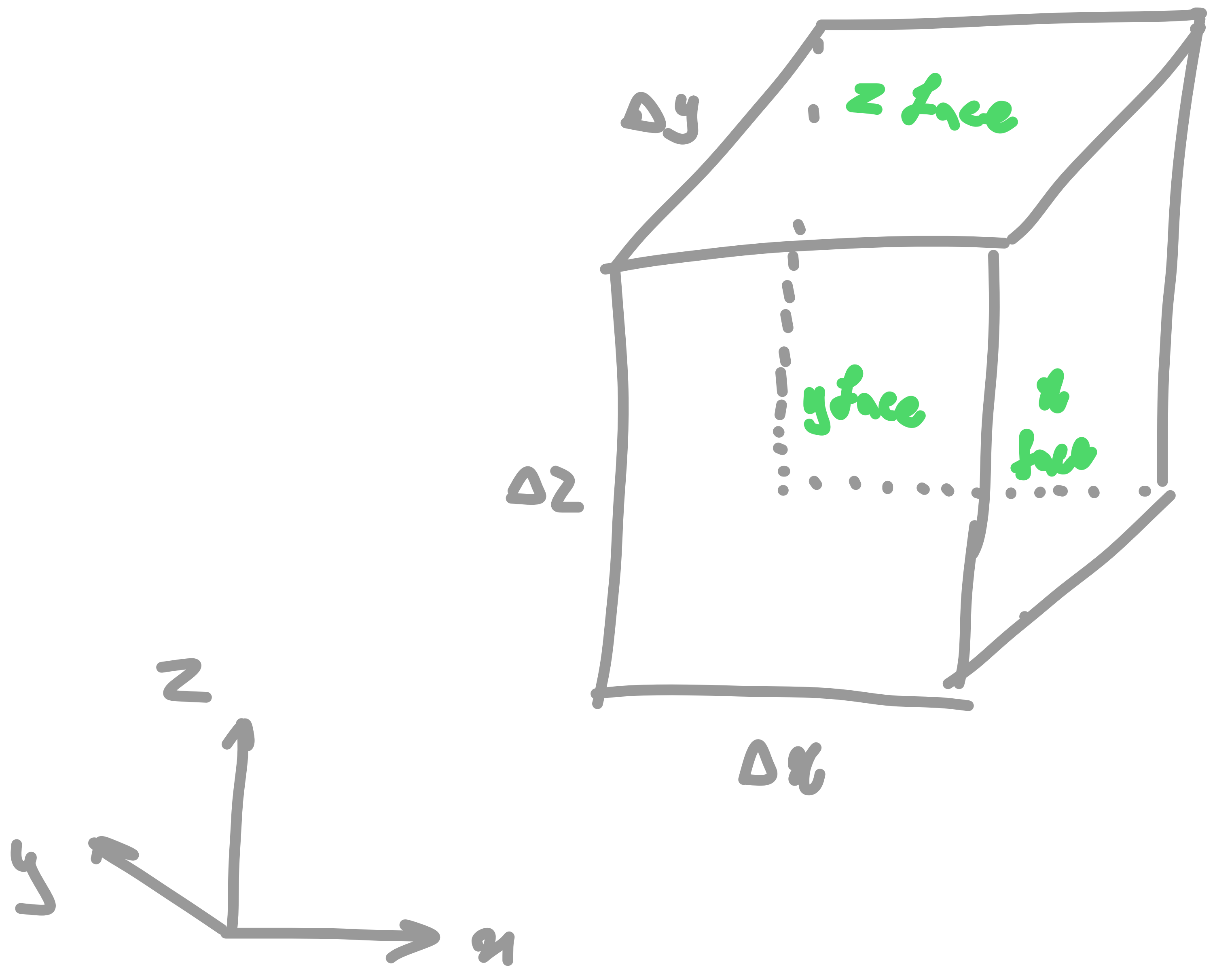


$$\vec{F}_{net} = \sum_i \vec{F}_i$$

$$\vec{F}_{net} = m\,\vec{a} = m\,\frac{d\vec{V}}{dt}$$

we are dealing with continuous systems, it means they consists of a large number of point like particles, let’s try to rewrite this for a volume of fluid:

Newton's second law but for a cubic element of **fluid**:



for a unit volume: $m \longrightarrow \rho \, \Delta x \, \Delta y \, \Delta z$

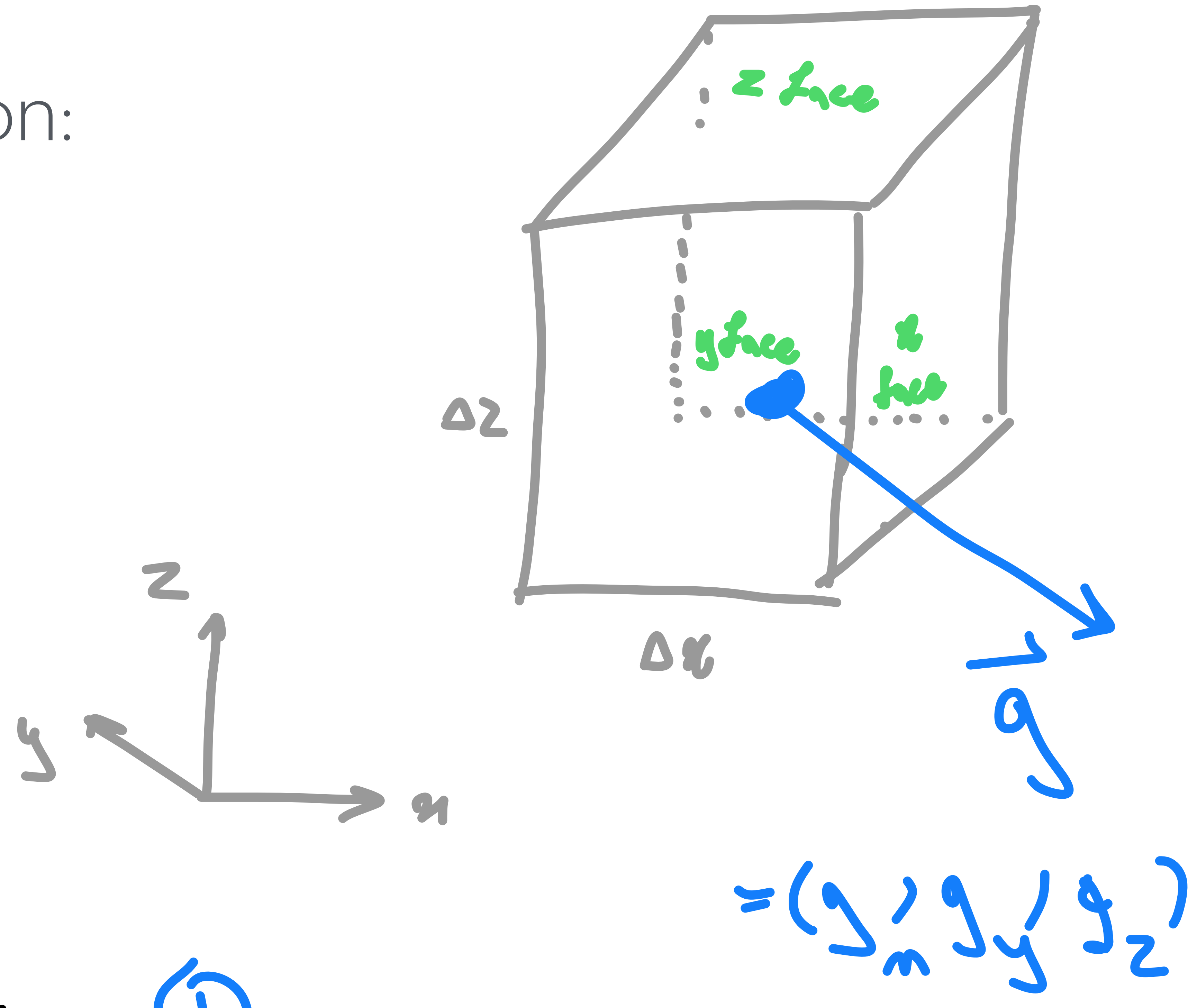
fluid is moving: $\dfrac{dv}{dt} \longrightarrow \dfrac{Dv}{Dt}$

$$\Delta x \, \Delta y \, \Delta z \, \rho \frac{DV}{Dt} = F_{net}$$

This **net** force can come from different sources, we will discuss the main sources in a typical fluid.

To make math more simple, let's consider the cartesian coordinate, and only let's consider the forces in the **x direction.**

Force #1: <u>Gravitational</u> force on the cubic element in the x direction:

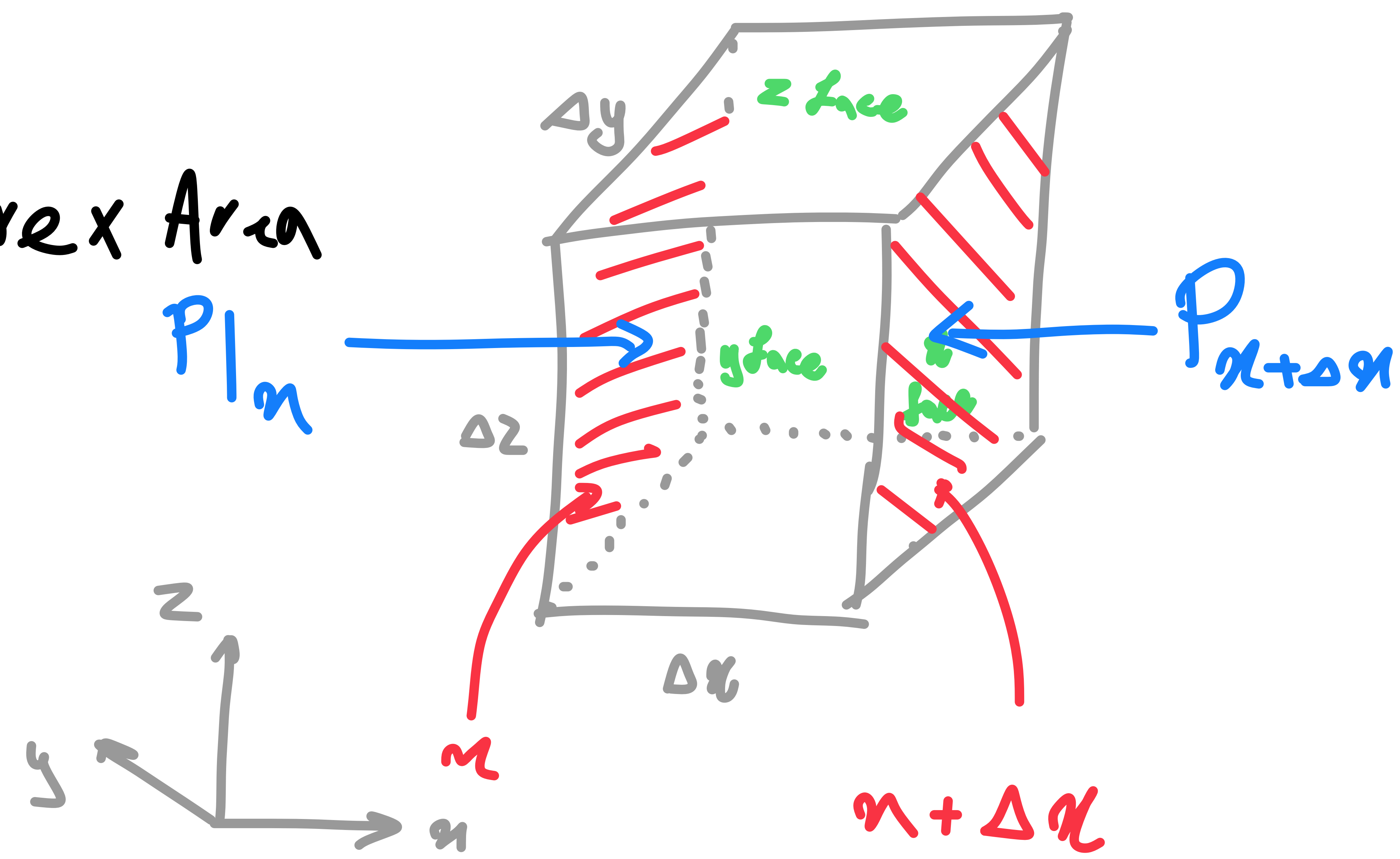


for the element: $\vec{f_g} = \rho \Delta x \Delta y \Delta z \; g_x$

for the $\hat{x}$ component: $\vec{f_{g_x}} = \rho \Delta x \Delta y \Delta z \; g_x$ (1)

Force #2: The force that is due to pressure difference in the x direction:

for the element: recall that force = Pressure x Area

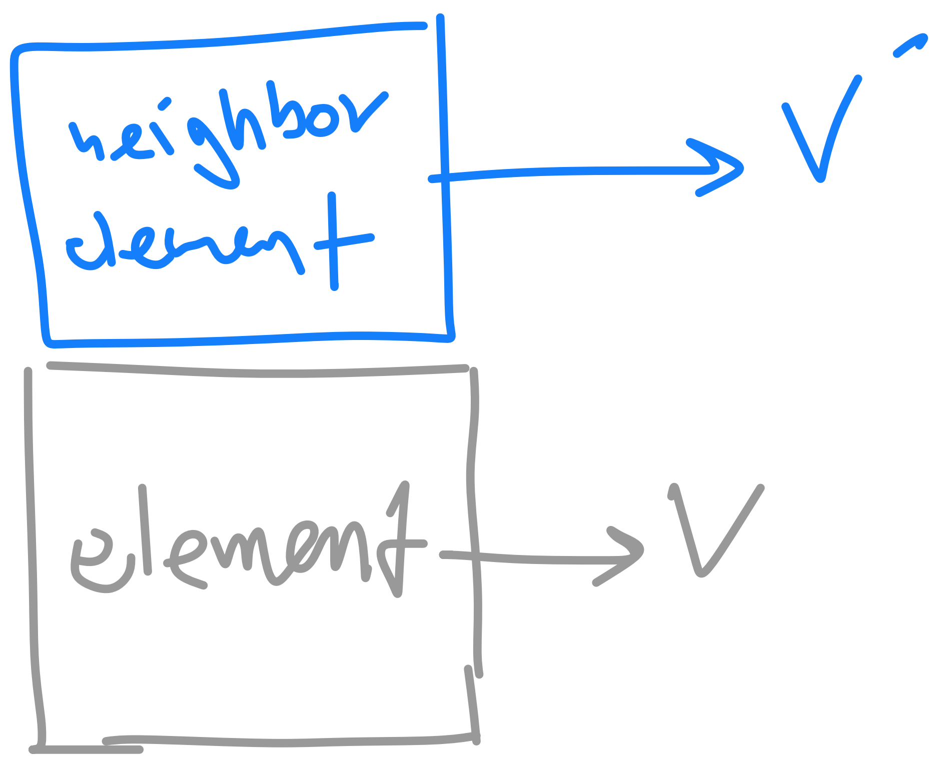


$$f_{P_x} = \Delta y \Delta z \, P|_x - \Delta y \Delta z \, P|_{x+\Delta x}$$

$$f_{P_x} = \Delta y \Delta z \left(P|_x - P|_{x+\Delta x}\right) \quad (2)$$

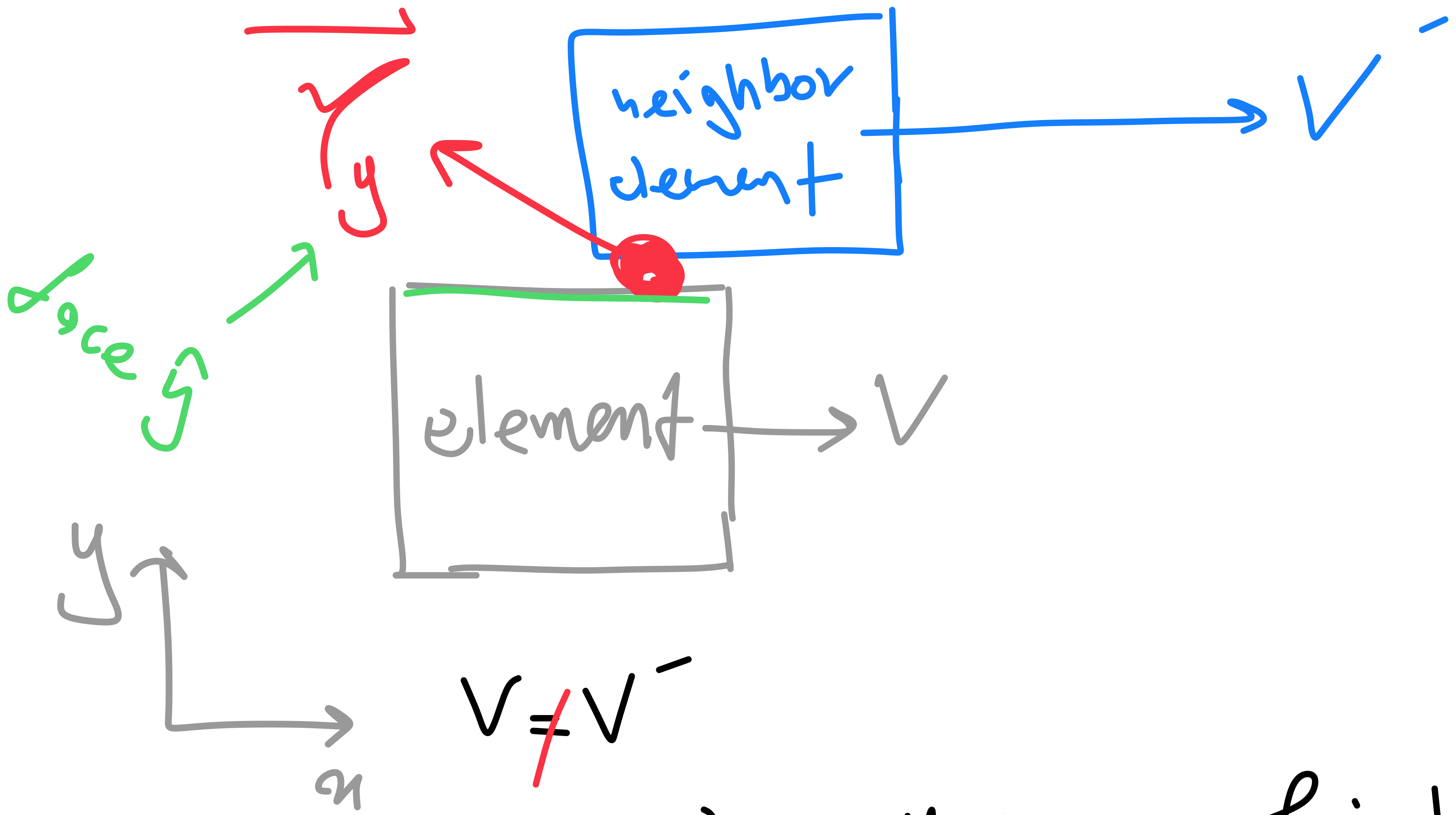
Force #3: The force that is due to shear stress.

What is a shear stress? Shear stress arises in a fluid when there are changes in velocity across space, creating a velocity gradient. When the fluid velocity varies from one layer to another (for example, faster flow at the center and slower near the edges), and the fluid has viscosity (internal friction), these differences in speed create internal frictional forces.



$V = V'$

• force



$V \neq V'$

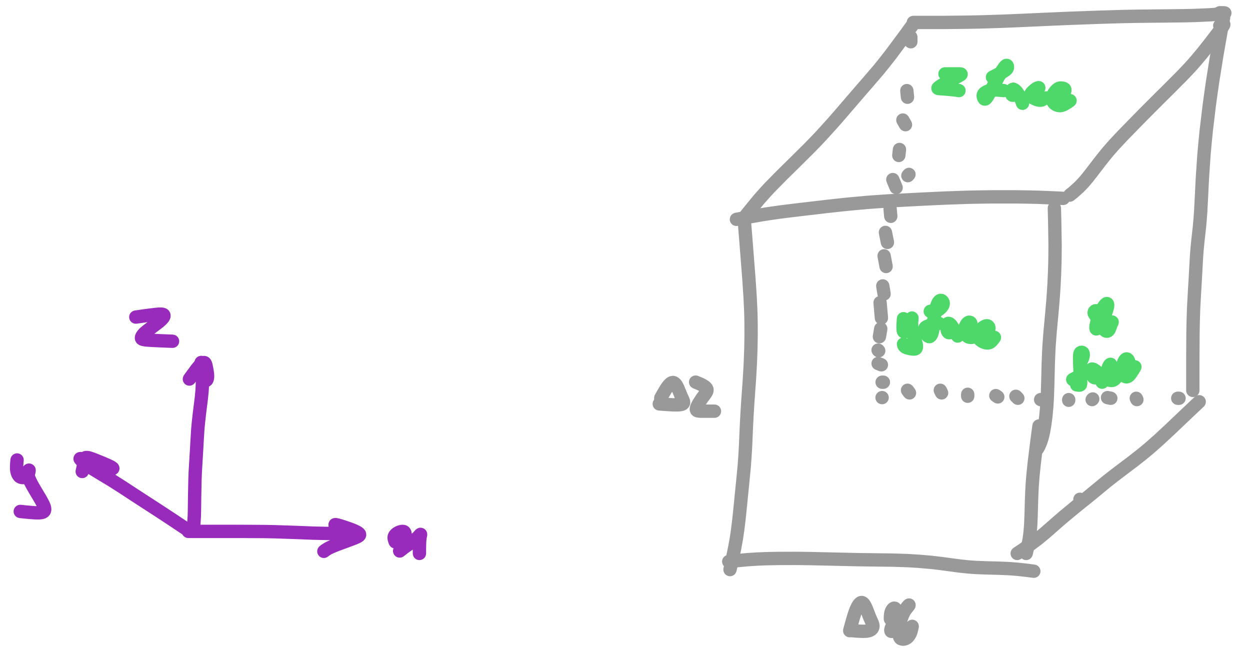
two element will have friction

Force #3: The force that is due to shear stress.

The shear force Tau (**T**) is can act on any of the face directions of the cube (x,y,z), and it can be at any direction (x,y,z).

Shear stress: τ

tensor



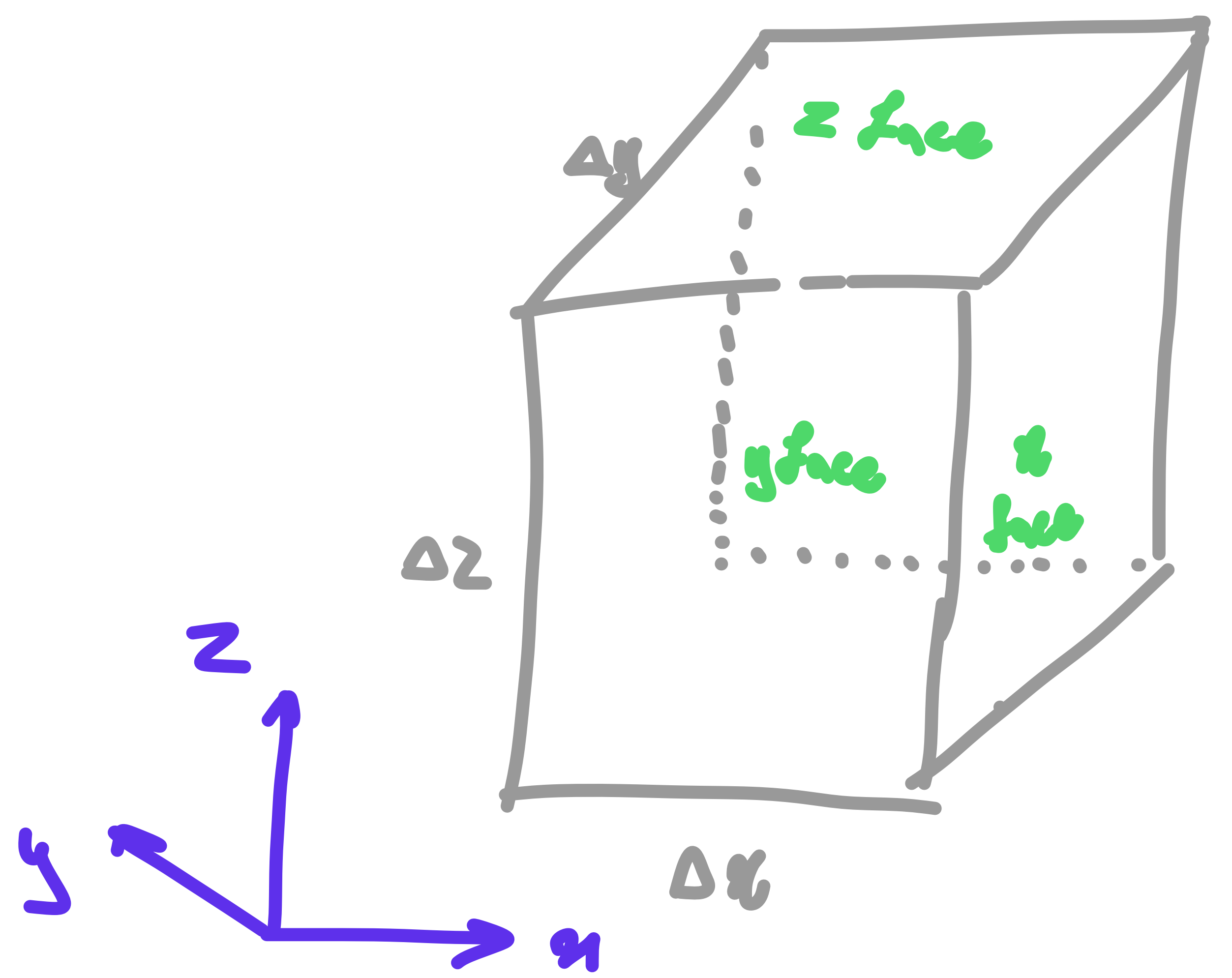
$$\tau = \begin{pmatrix} \tau_{xx} & \tau_{xy} & \tau_{xy} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{pmatrix}$$

τ(face direction)(force direction)

questions?

Force #3: The force that is due to shear stress.

Now let's continue with finding the shear stress forces on the element in the x directions, they can be exerted on any face directions.



$$f_{sx} = \Delta y \Delta z \, \tau_{xx}\big|_x - \Delta y \Delta z \, \tau_{xx}\big|_{x+\Delta x} + \Delta x \Delta z \, \tau_{yx}\big|_y - \Delta x \Delta z \, \tau_{yx}\big|_{y+\Delta y} + \Delta x \Delta y \, \tau_{zx}\big|_z - \Delta x \Delta y \, \tau_{zx}\big|_{z+\Delta z}$$

$$\Rightarrow f_{sx} = \Delta y \Delta z \left(\tau_{xx}\big|_x - \tau_{xx}\big|_{x+\Delta x} \right) + \Delta x \Delta z \left(\tau_{yx}\big|_y - \tau_{yx}\big|_{y+\Delta y} \right) + \Delta x \Delta y \left(\tau_{zx}\big|_z - \tau_{zx}\big|_{z+\Delta z} \right) \quad (3)$$

Now we can put all the forces in x direction (#1, #2, and #3) together.

$$\Delta x \Delta y \Delta z \rho \frac{DV_x}{Dt} = f_{g_x} + f_{p_x} + f_{s_x} = \Delta x \Delta y \Delta z \rho g_x$$

$$+ \Delta y \Delta z \left(P|_x - P|_{x+\Delta x} \right)$$

$$+ \Delta y \Delta z \left(\tau_{xx}|_x - \tau_{xx}|_{x+\Delta x} \right) + \Delta x \Delta z \left(\tau_{yx}|_y - \tau_{yx}|_{y+\Delta y} \right) + \Delta x \Delta y \left(\tau_{zx}|_z - \tau_{zx}|_{z+\Delta z} \right)$$

$\div \Delta x \Delta y \Delta z$

$$\longrightarrow \rho \frac{DV_x}{Dt} = \rho g_x + \frac{\left(P|_x - P|_{x+\Delta x} \right)}{\Delta x}$$

$$+ \frac{\tau_{xx}|_x - \tau_{xx}|_{x+\Delta x}}{\Delta x} + \frac{\left(\tau_{yx}|_y - \tau_{yx}|_{y+\Delta y} \right)}{\Delta y} + \frac{\left(\tau_{zx}|_z - \tau_{zx}|_{z+\Delta z} \right)}{\Delta z}$$

$\Delta x \rightarrow 0$

$\Delta y \rightarrow 0$

$\Delta z \rightarrow 0$

$\longrightarrow$

$$\rho \frac{DV_x}{Dt} = \rho g_x - \frac{\partial P}{\partial x} - \left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} \right) \quad (A)$$

Similar for y and z direction:

$$\rho \frac{DV_y}{Dt} = \rho g_y - \frac{\partial P}{\partial y} - \left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} \right) \quad (B)$$

$$\rho \frac{DV_z}{Dt} = \rho g_z - \frac{\partial P}{\partial z} - \left(\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} \right) \quad (C)$$

or (A), (B), (C) $\Rightarrow$

$$\boxed{\rho \frac{D\vec{V}}{Dt} = \underbrace{\rho \vec{g}}_{\text{gravity}} - \underbrace{\vec{\nabla} P}_{\text{pressure}} - \underbrace{\nabla \cdot \tilde{\tau}}_{\text{viscosity}}}$$

This is called Equation of motion

B) Rheology

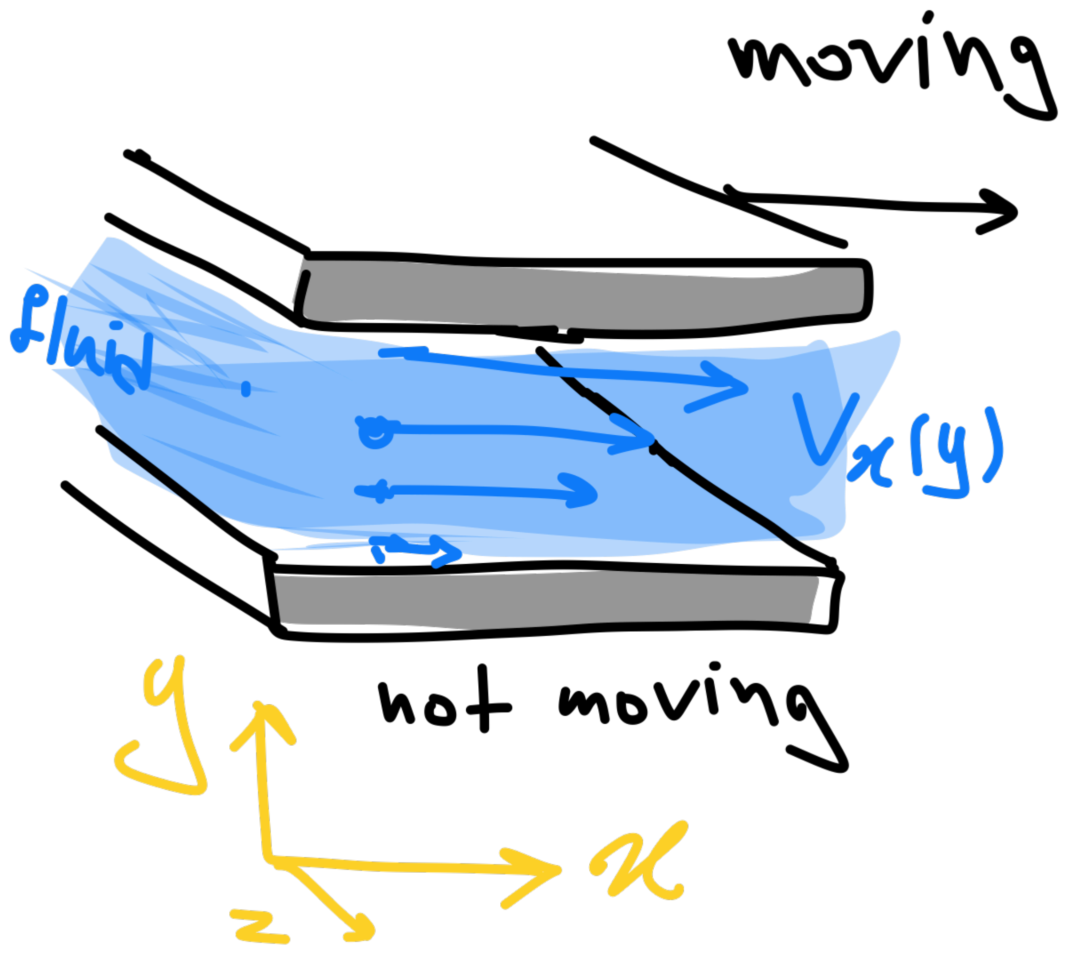
But Equation of motion is not complete and solvable on its own. What does it mean? It means we cannot solve it to find the velocity over time, because we have no idea what shear stress tensor (**T**) is.

But we mentioned that the shear stress tensor (**T**) itself should be related to the velocity variation is space, therefore it is a function of velocity, **T**(v). But how is this dependency?

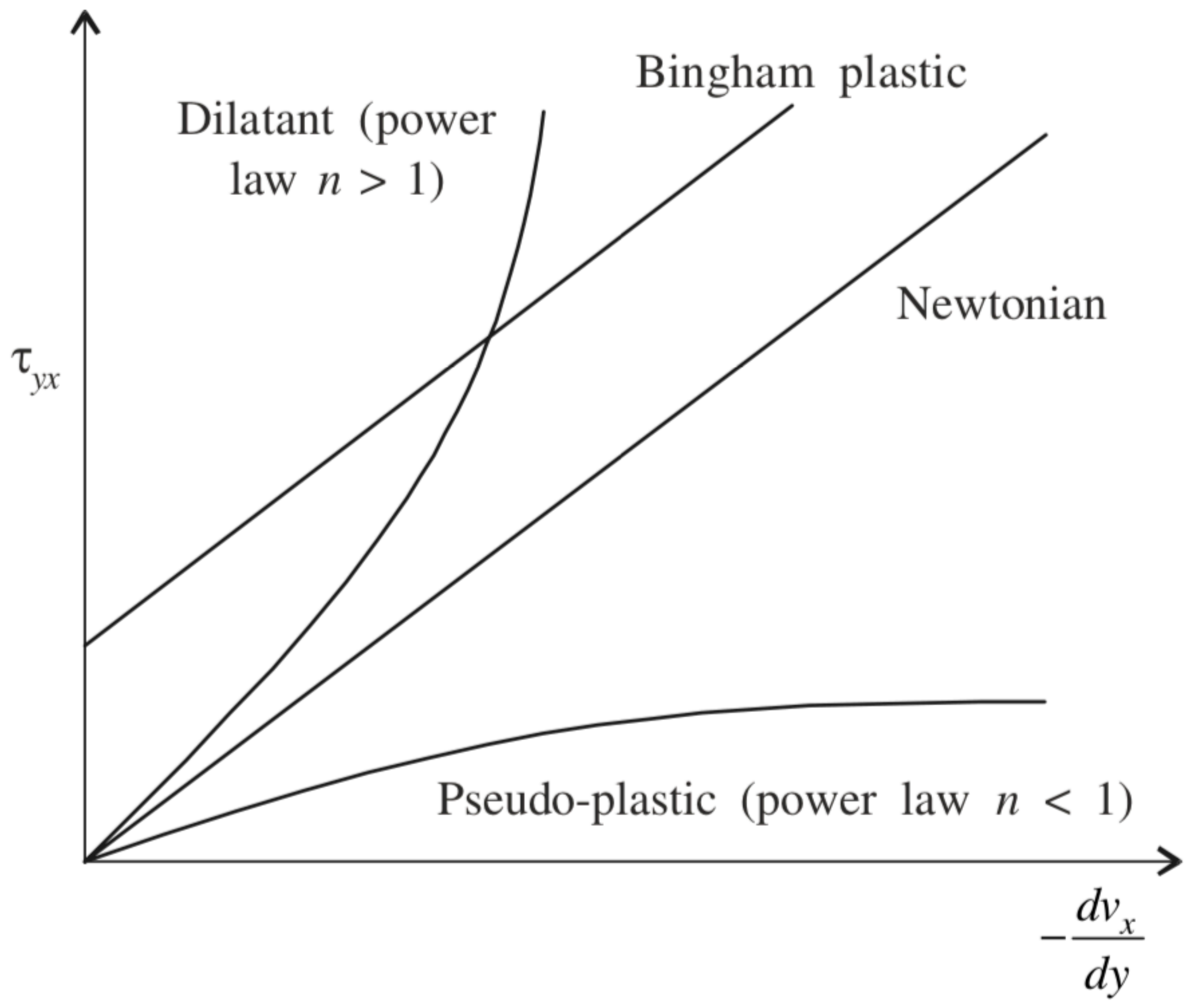
Rheology Experiments: these experiments connect shear stress to strain rates (velocity changes)

Rheology Equipment:

mechanism:



results:



Specific case:

for Newtonian fluids:

$$\tau_{yx} = -\mu \frac{\partial V_x}{dy}$$

μ → viscosity

More precisely, for a newtonian fluid, the shear stress tensor and velocity are connected like bellow:

Table 3.4-4 Components of the stress tensor for Newtonian fluids in rectangular Cartesian coordinates

$$\tau_{xy} = \tau_{yx} = -\mu\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right) \qquad \text{(A)}$$

$$\tau_{yz} = \tau_{zy} = -\mu\left(\frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y}\right) \qquad \text{(B)}$$

$$\tau_{zx} = \tau_{xz} = -\mu\left(\frac{\partial v_z}{\partial x} + \frac{\partial v_x}{\partial z}\right) \qquad \text{(C)}$$

$$\tau_{xx} = -2\mu\frac{\partial v_x}{\partial x} + \frac{2}{3}\mu\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right) \qquad \text{(D)}$$

$$\tau_{yy} = -2\mu\frac{\partial v_y}{\partial y} + \frac{2}{3}\mu\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right) \qquad \text{(E)}$$

$$\tau_{zz} = -2\mu\frac{\partial v_z}{\partial z} + \frac{2}{3}\mu\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right) \qquad \text{(F)}$$

Now if we substitute these into our equation of motion, we will get rid of shear stress and will find an equation that is only for velocities, called Navier-Stokes equation

C) Navier-Stokes equation

$$\rho \frac{D\vec{V}}{Dt} = \rho\vec{g} - \vec{\nabla}P - \nabla\cdot\tilde{T}$$

for Newtonian and incompressible fluid: $-\nabla\cdot T = \mu\vec{\nabla}^2\vec{V}$

$$\boxed{\rho \frac{D\vec{V}}{Dt} = \rho\vec{g} - \vec{\nabla}P + \mu\vec{\nabla}^2\vec{V}}$$

Navier-Stokes eq.

what is $\vec{\nabla}^2\vec{V} \equiv (\nabla^2 V_x, \nabla^2 V_y, \nabla^2 V_z)$

if $\vec{V} = (V_x, V_y, V_z)$