In the previous lecture we reviewed the Navier-Stokes equation. Today we want to solve this equation for Steady-state, laminar, and Fully developed flow.

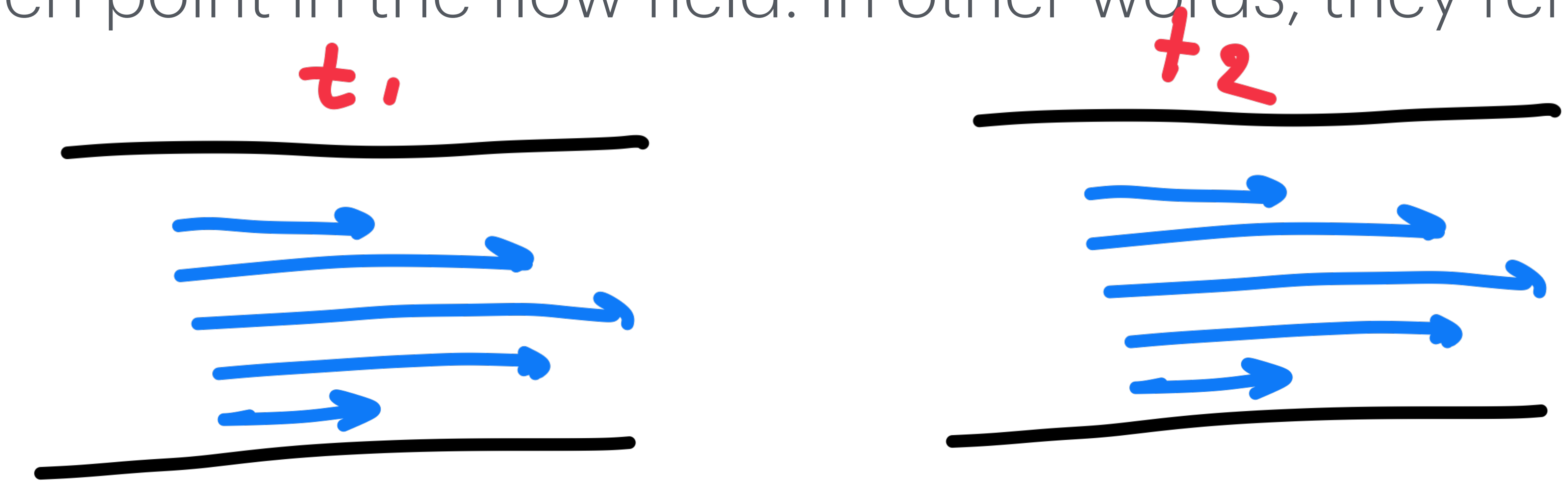
$$\boxed{\rho \frac{D\vec{V}}{Dt} = \rho\vec{g} - \vec{\nabla}P + \mu\vec{\nabla}^2\vec{V}}$$

Navier-Stokes eq.

what is $\vec{\nabla}^2\vec{V} \equiv (\nabla^2 V_x, \nabla^2 V_y, \nabla^2 V_z)$

if $\vec{V} = (V_x, V_y, V_z)$

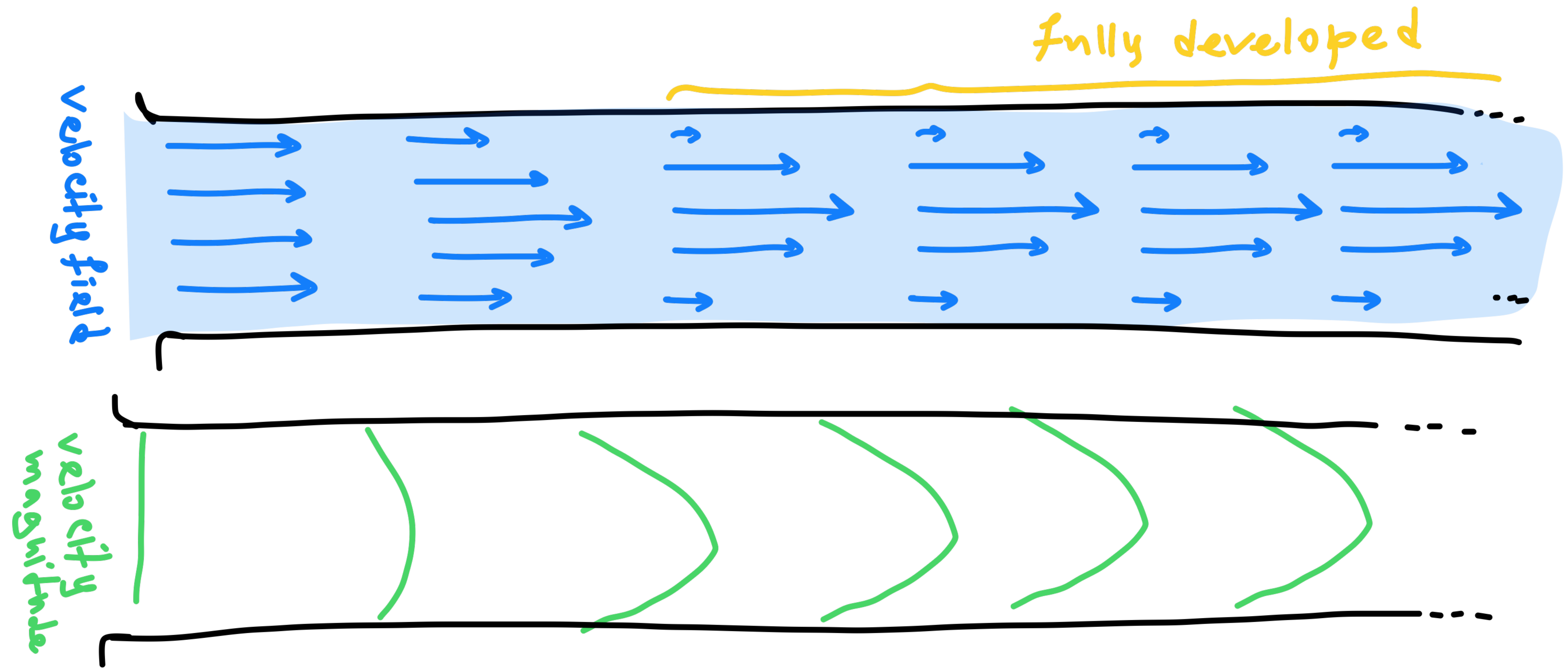
From previous lectures, we are already familiar with the concept of steady-state: for a flow at steady-state, properties such as velocity, pressure, and density do not change over time at any given point in the flow field. In other words, they remain constant with respect to time.



Velocity profile $= V(\vec{x}, t)$

$$\vec{V}(\vec{x}, t) \overset{s.s}{=} \vec{V}(\vec{x})$$

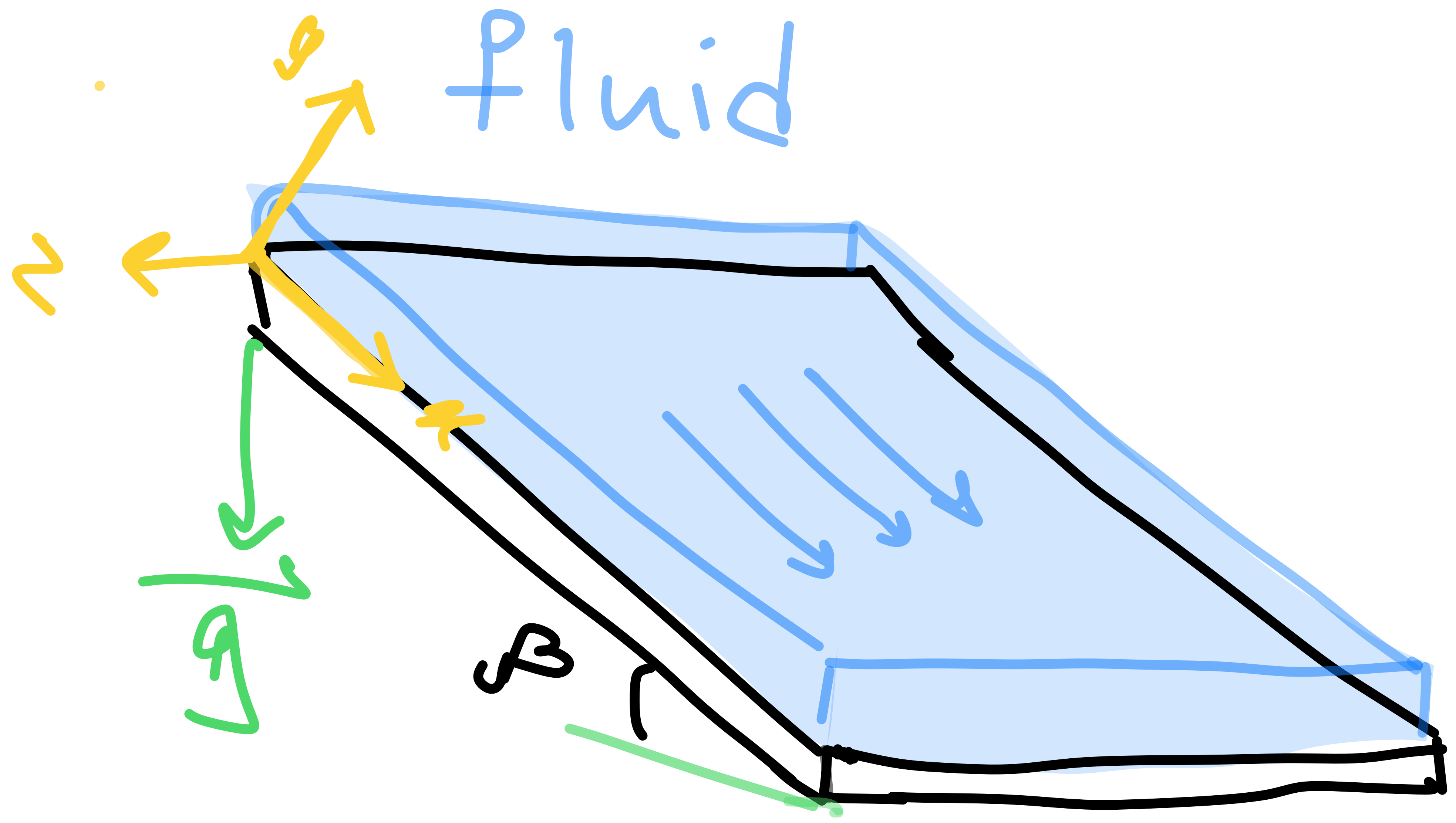
But what is a fully-developed flow? In fully-developed flow, the velocity profile does not change along the flow direction. This means that the flow has reached an equilibrium state, where the effects of viscosity have caused the velocity profile to stabilize. Therefor **no-slip boundary condition**



Also laminar flow means that velocity field is parallel and along the direction of the flow.

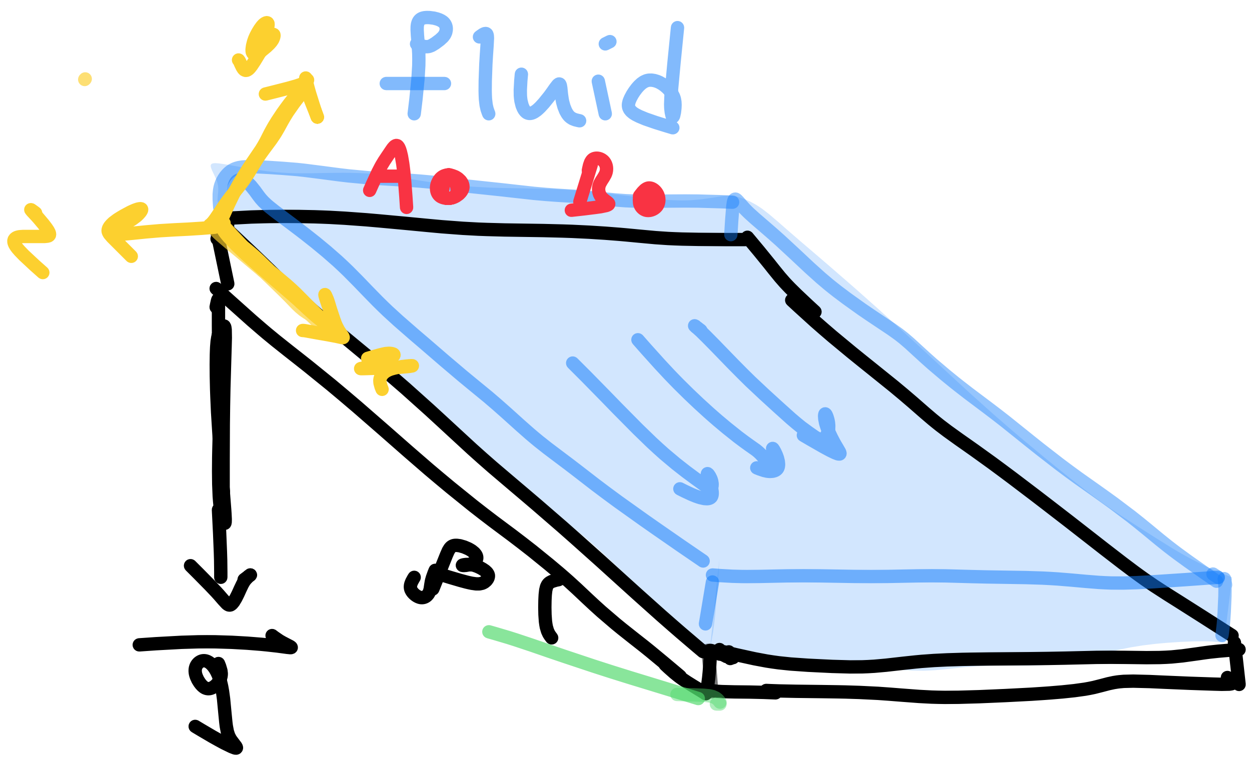
A) Laminar Flow in a falling film

A liquid flows down an inclined large plane at a steady, fully-developed, laminar, incompressible and Newtonian flow with a thickness d. The flow is driven solely by gravity, and there is no pressure gradient along the inclined surface.



Question A: What is the Velocity profile of the fluid?

Solution:



$\Rightarrow$

① Plate is large in z direction: Symmetry in z direction

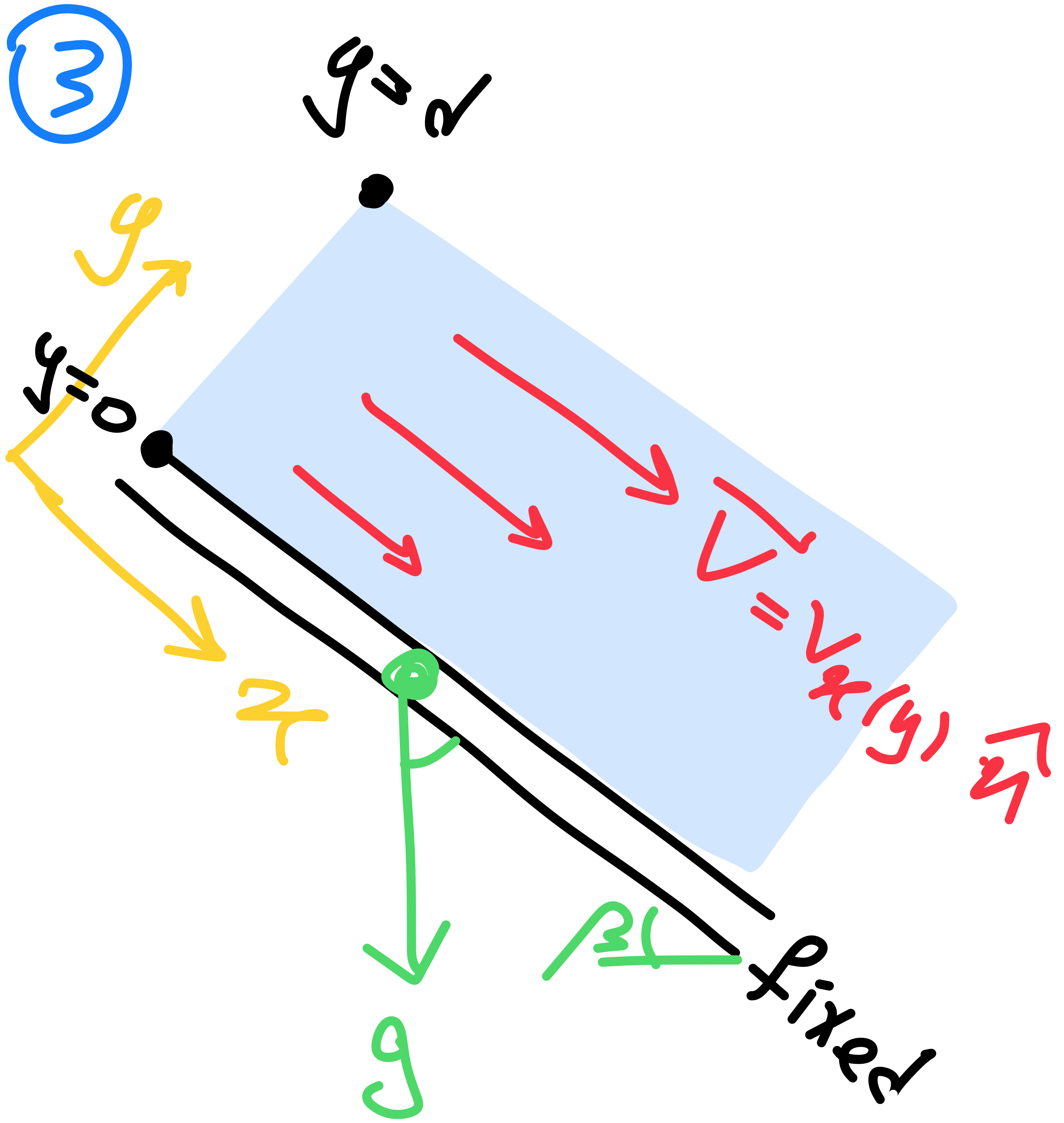
$V(x,y,z) \rightarrow V(x,y)$

② fully developed

$V(x,y) \rightarrow V_x(y)\ \hat{x}$

$\Rightarrow$

③



④ B.C: at $y=0$ we have no-slip: $V_x(y=0)=0$ ⓘ

at $y=d$ we have liquid-gas interface: $\dfrac{\partial V_x(y)}{\partial y}=0$ ⓘⓘ

5 So far we have significantly simplified the situation, our velocity is only the function of y, and also only in x direction! Now we are at a good place to use N-S equation, in this case, our coordinate system is cartesian.

N-S $\rho \frac{D\vec{V}}{Dt} = \rho \vec{g} - \vec{\nabla} P + \mu \vec{\nabla}^2 \vec{V}$ $\qquad \vec{V} = (V_x, \cancel{V_y}^{0}, \cancel{V_z}^{0}) \Rightarrow \vec{V} = V_x(y)$

so we just need N-S equation for $\hat{x}$ direction:

$$\underbrace{\rho \frac{D(V_x)}{Dt}}_{} = \underbrace{\rho g_x}_{\rho g \sin\beta} - \underbrace{\frac{\partial P}{\partial x}}_{0} + \mu \left(\underbrace{\frac{\partial^2 V_x}{\partial x^2}}_{0} + \frac{\partial^2 V_x}{\partial y^2} + \underbrace{\frac{\partial^2 V_x}{\partial z^2}}_{0} \right)$$

$V_x(y)$ $\qquad V_x(y)$

remember: $\frac{D\phi}{Dt} = \frac{\partial \phi}{\partial t} + V_x \frac{\partial \phi}{\partial x} + V_y \frac{\partial \phi}{\partial y} + V_z \frac{\partial \phi}{\partial z} \xrightarrow{\phi = V_x} \frac{DV_x}{Dt} = \underbrace{\frac{\partial V_x}{\partial t}}_{0 \ (\text{s.s})} + V_x \underbrace{\frac{\partial V_x}{\partial x}}_{0} + \underbrace{V_y}_{0} \frac{\partial V_x}{\partial y} + \underbrace{V_z}_{0} \underbrace{\frac{\partial V_x}{\partial z}}_{0} = 0$

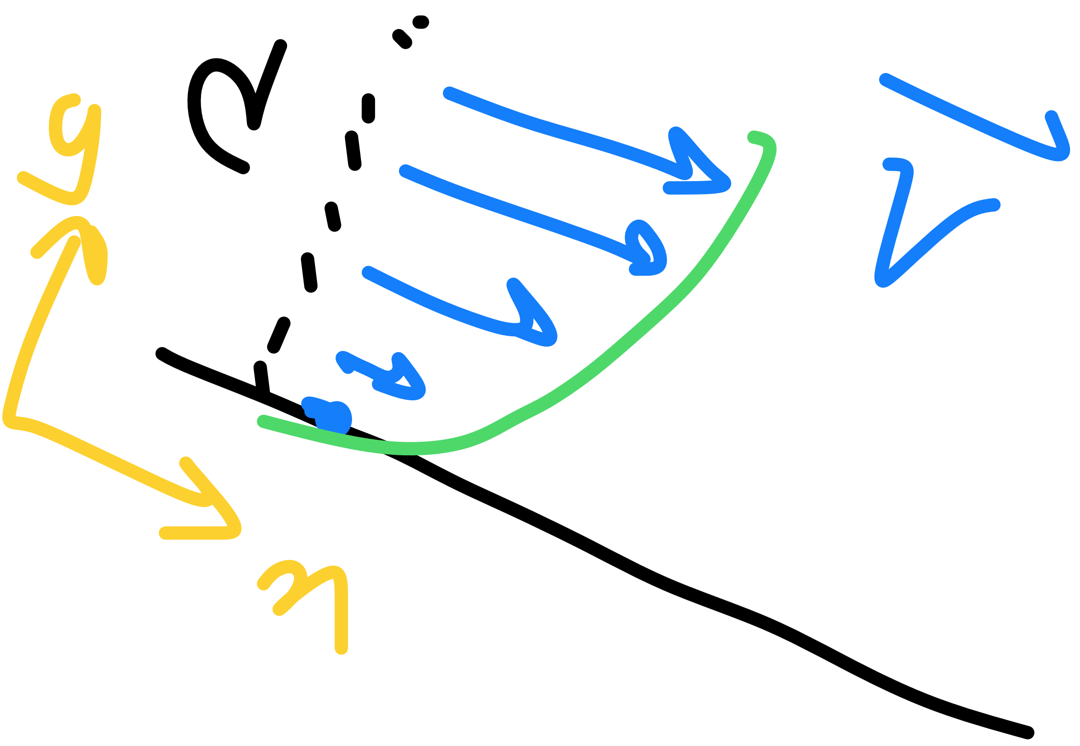
$$\mu \frac{\partial^2 V_n(y)}{\partial y^2} = -\rho g \sin\beta \longrightarrow \frac{\partial^2 V_n(y)}{\partial y^2} = \left(\frac{\rho g \sin\beta}{\mu}\right)$$

$$\xrightarrow{\int dy} \frac{\partial V_n(y)}{\partial y} = -\left(\frac{\rho g \sin\beta}{\mu}\right) y + C_1 \xrightarrow{\int dy} V_n(y) = -\left(\frac{\rho g \sin\beta}{2\mu}\right) y^2 + C_1 y + C_2$$

$$\xrightarrow{B.C(i)} V_n(y=0) = 0 \Rightarrow C_2 = 0$$

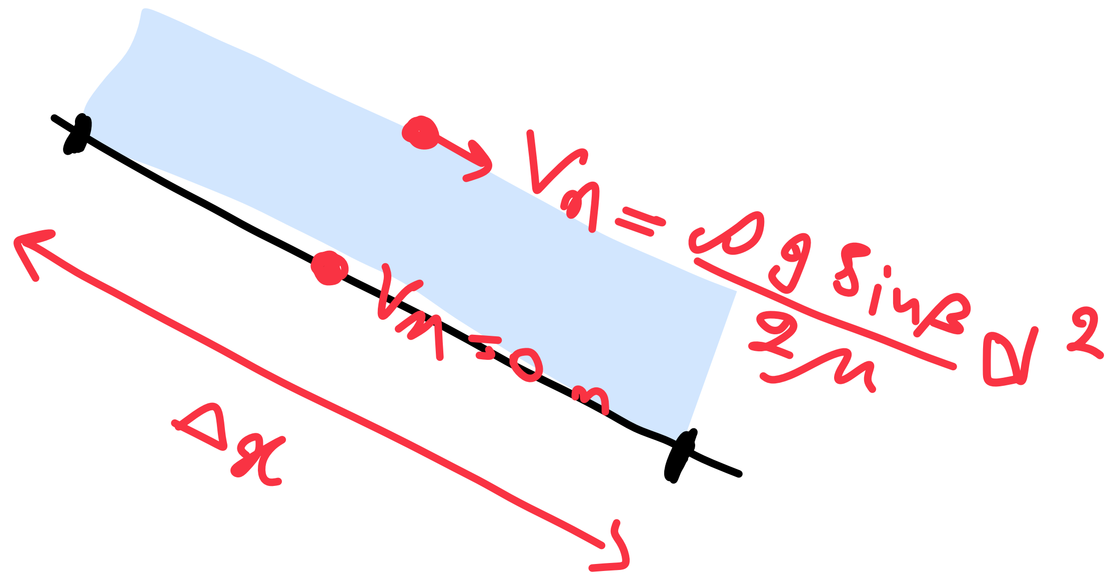
$$\xrightarrow{B.C(ii)} \frac{\partial V_n(y=d)}{\partial y} = 0 \Rightarrow -\frac{2\rho g \sin\beta}{2\mu} d + C_1 = 0 \Rightarrow C_1 = \frac{\rho g \sin\beta d}{\mu}$$

$$\Rightarrow \vec{V} = V_n(y)\hat{n} = \frac{\rho g \sin\beta}{\mu}\left(yd - \frac{y^2}{2}\right)\hat{n}$$



Activity!

Let's get a closer look: $V_x(y) = \frac{\rho g \sin\beta}{\mu}\left(yd - \frac{y^2}{2}\right)$

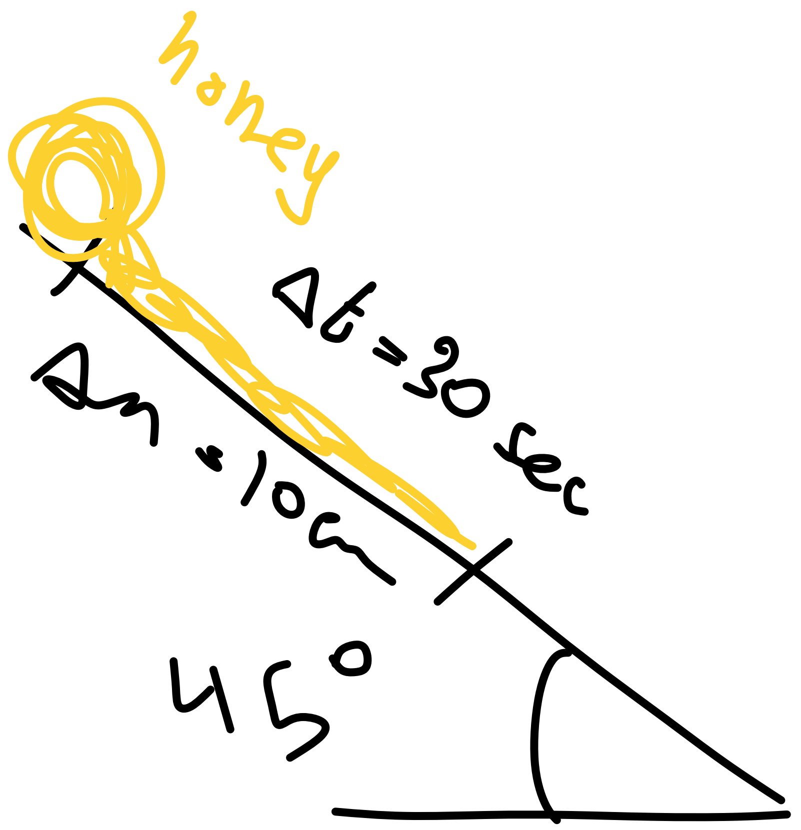


$$\bar V \simeq \frac{V_x(y=0) + V_x(y=d)}{2}$$

$$\bar V \simeq \frac{d^2 \rho g \sin\beta}{4\mu}$$

$$\Rightarrow \Delta x = \bar V \Delta t \Rightarrow \Delta x = \frac{d\rho g \sin\beta}{4\mu}\Delta t \Rightarrow \boxed{\mu \simeq \frac{d^2 \rho g \sin\beta\, \Delta t}{4\Delta x}}$$

We did this activity in class:



now let's put in some numbers:

$\Delta t = 3 \times 10$ s

$\Delta x = 10^{-1}$ m

$\rho \simeq 1.5\ 10^3 \frac{kg}{m^3}$

$g = 10 \frac{m}{s^2}$

$d \simeq 3\times 10^{-3}$

$\longrightarrow \mu_{\text{from in class experiment}} = \frac{\cancel{3}\times\cancel{10^{-3}} \times \cancel{3}\times\cancel{10^{-3}} \times 1.5\times\cancel{10^{3}} \times \cancel{10} \times 7\times\cancel{10^{-1}} \times \cancel{3}\times\cancel{10}}{\cancel{4}\times\cancel{10^{-1}}}$

$\Rightarrow \mu \simeq 7 \times 1.5$ Pa s $\simeq 11$ Pa.s $\simeq$ 11.000 mPa.s

!!

Here are some examples of viscosity values:

Water (20 °C)	1 mPas
Blood	4 bis 25 mPas
Engine oil	~100 mPas
Honey	~10.000 mPas
Glass	~10.000.000.000 mPas