Lecture 24

Solving Navier-Stokes Equation for Steady-State and Fully-Developed Flow - Part 2: continue with Cartesian coordinate

A) Laminar Flow in a falling film (even more explanation)

B) Laminar Flow Between a Fixed and a Moving Plate: Example of rheology applications (e.g., rheometers).

In the previous lecture we reviewed the Navier-Stokes equation. Today we want to solve this equation for Steady-state, laminar, and Fully developed flow.

$$\boxed{\rho \frac{D\vec{V}}{Dt} = \rho \vec{g} - \vec{\nabla} P + \mu \vec{\nabla}^2 \vec{V}}$$

Navier-Stokes eq.

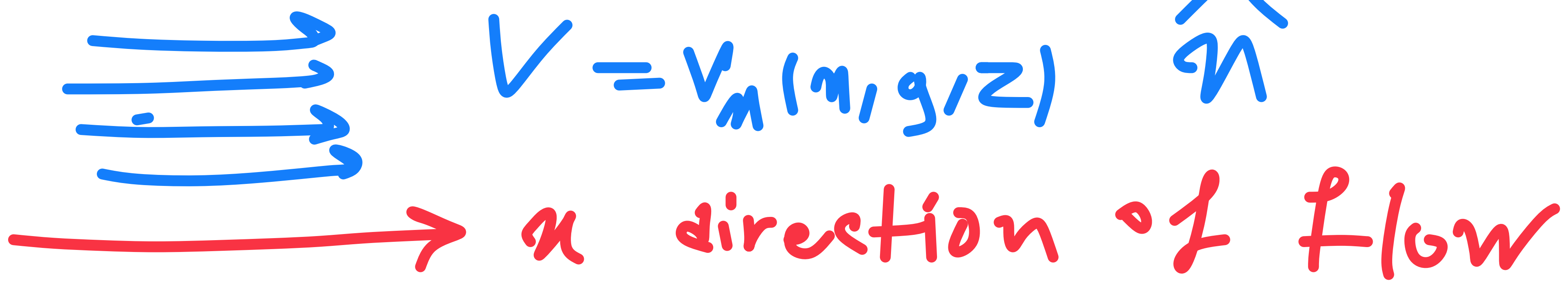
what is $\vec{\nabla}^2 \vec{V} \equiv (\nabla^2 V_x, \nabla^2 V_y, \nabla^2 V_z)$

if $\vec{V} = (V_x, V_y, V_z)$

Before continue with more examples of solving N-S, let's review some terms.

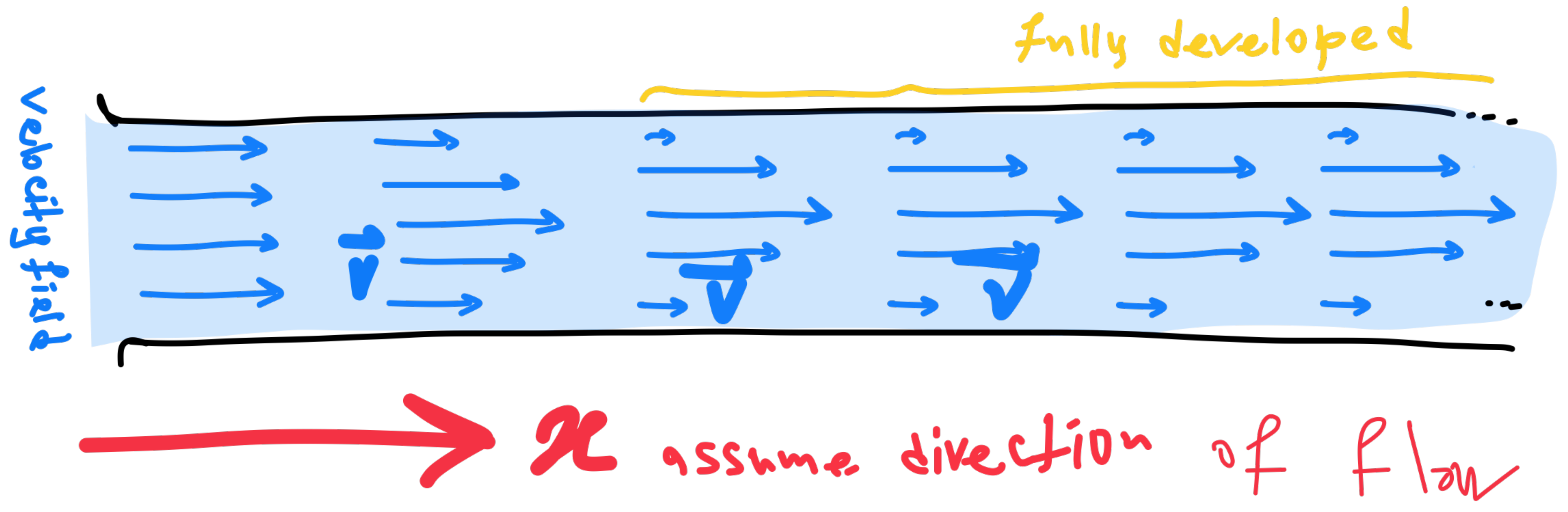
Review of Some terms, and their meanings:

i) laminar flow: means that velocity field is parallel and along the direction of the flow.



$$\vec{V} = V_x(x,y,z)\,\hat{x}$$

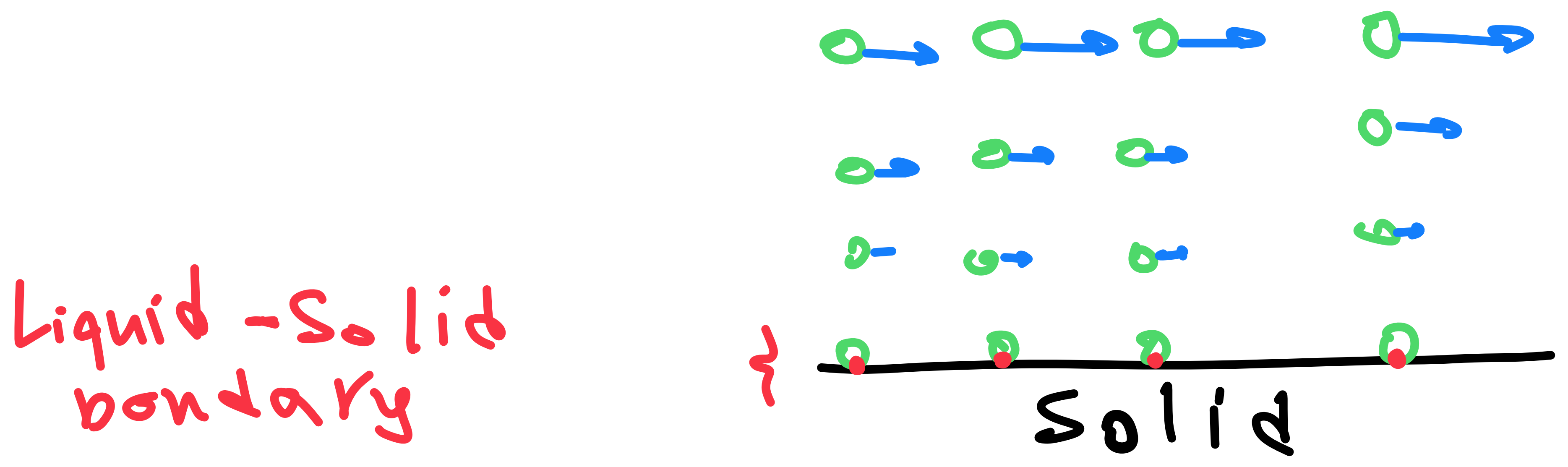
ii) Fully developed: Means there is **no spatial variation** for the velocity of the fluid in the **direction of the flow**



in fully developed section

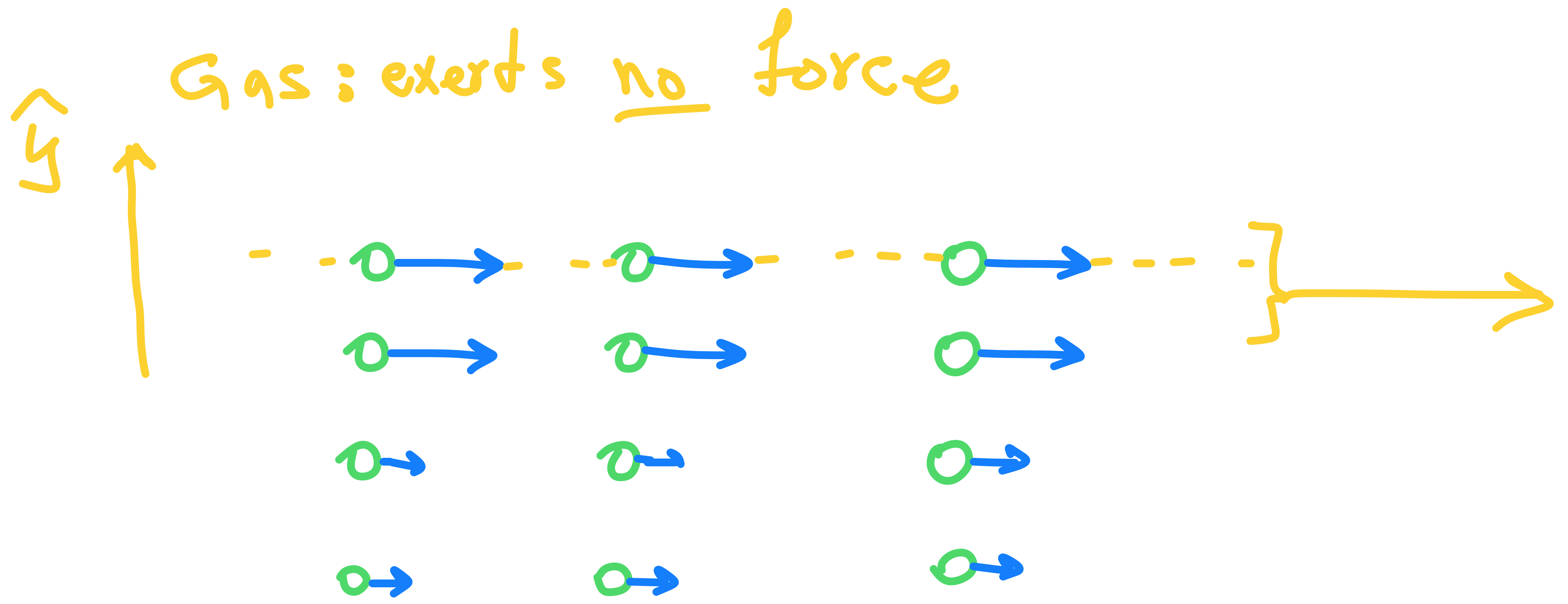
$$\frac{\partial \vec{V}}{\partial x} = 0$$

iii) Fluid-Solid boundary condition: When the fluid is fully developed, the velocity of fluid matches the **velocity of the solid** surface at the boundary:



$$V_{liquid} = V_{solid}\Big|_{boundary}$$

iv) Fluid-Gas boundary condition: Free-slip boundary condition. In this case, the derivative of the velocity a**long the direction of the interface** plane should be zero.

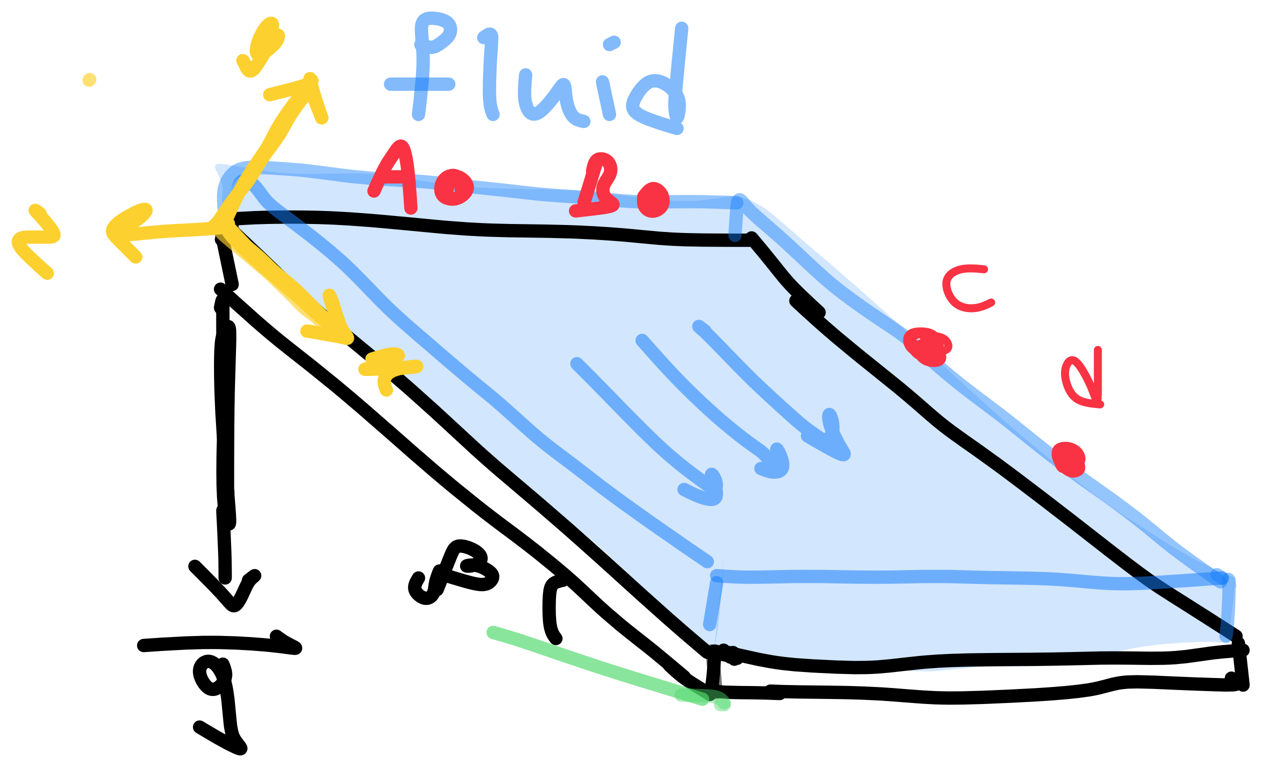


$$\frac{\partial V_{liquid}}{\partial y} = 0$$

In other words the velocity of the immediate layer of fluid in touch with air is exactly the same as the layer below it, because that's what deriving to forward and it doesn't receive any force from the gas

A) **Reminder: Laminar Flow in a falling film:** What we did...

A1) we applied the assumptions to simplify the situation (V and P)

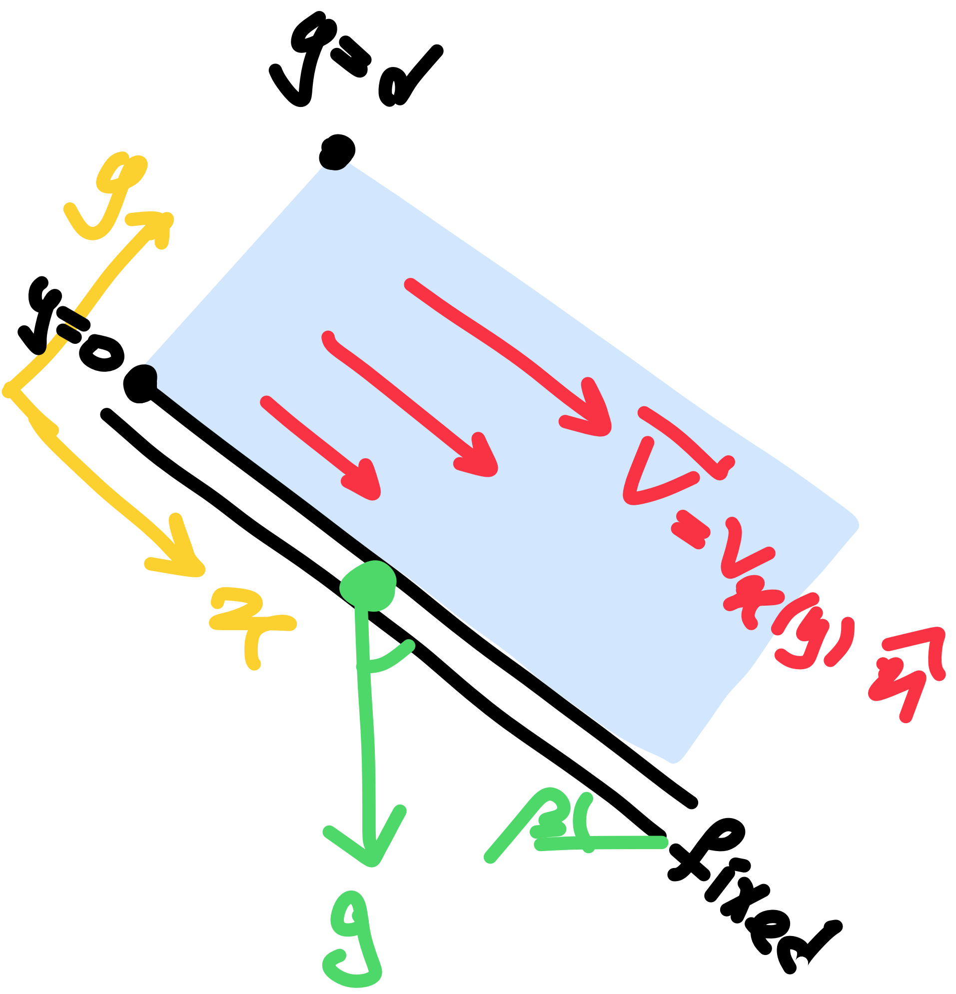


① Plane was large ⟶ symmetry in $\hat{z}$ $\Rightarrow \vec{V}(x,y,\cancel{z})$

$P(x,y,\cancel{z})$

② fully developed $\Rightarrow$ $\frac{\partial \vec{V}}{\partial x} = 0 \Rightarrow \vec{V}(\cancel{x},y)$

$\frac{\partial P}{\partial x} = 0 \Rightarrow P(\cancel{x},y)$



③ flow is Laminar: $\vec{V}(y) \overset{\text{only in } x \text{ direction}}{=\!=\!=} V_x(y)\,\hat{x}$

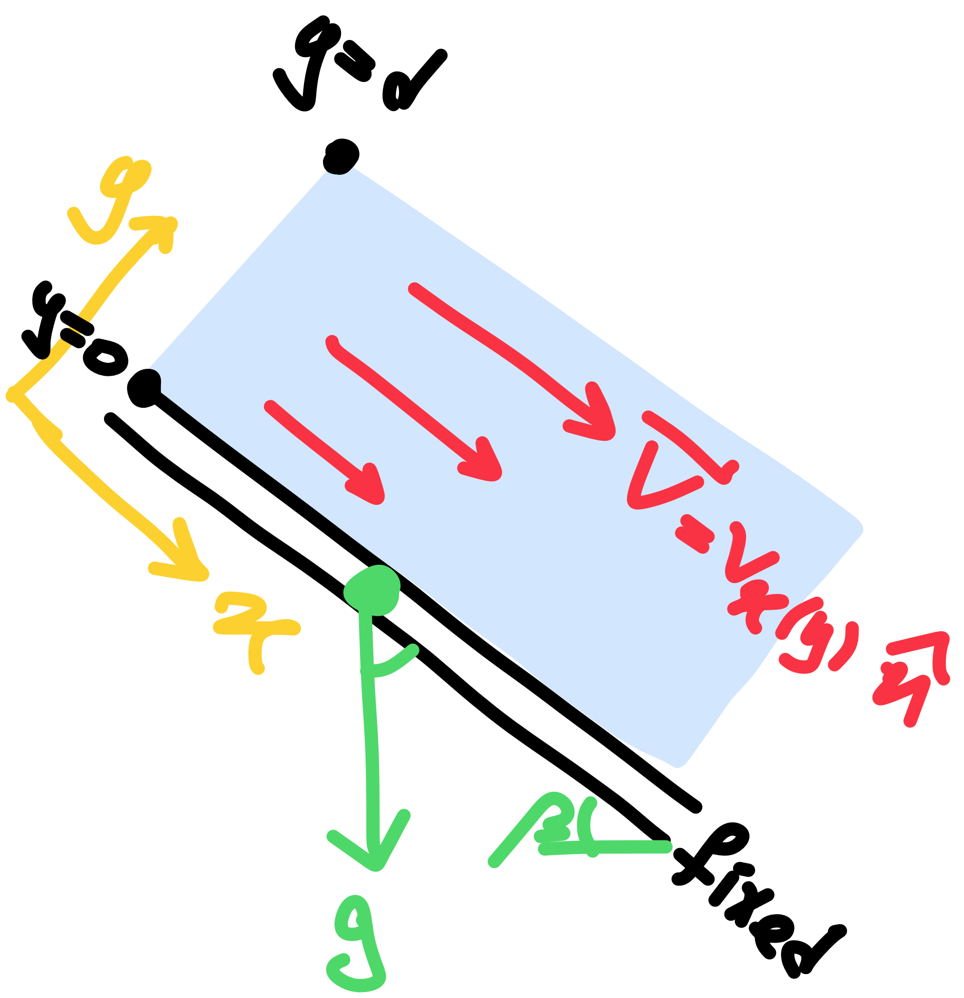
④ The only Pressure is from atmosphier;

$P|_{y=d} = P_{atm} \Longrightarrow P(\cancel{x},y,\cancel{z}) \Rightarrow P(y)$

only function of y

for any $\underline{x}$ and $\underline{z}$: $P(A) = P(B) = P(C) = P(D)$

A2: next we applied the boundary conditions, we have one Liquid-Solid and one Liquid-Gas boundaries



(5) B.C

Liquid-Solid $\Big|_{y=0}$ $\qquad v_x(y=0) = 0$ (i)

Liquid-Gas $\Big|_{y=d}$ $\qquad \dfrac{\partial v_x}{\partial y}\Big|_{y=d} = 0$ (ii)

A3: Now we are ready to try to solve N-S. But N-S is a vector equation, it means there are 3 equations each for one direction (Vx, Vy, Vz). Should we write down all?

Navier-Stokes in compact form: (true for all coordinate systems)

$$\rho \frac{D\vec{V}}{Dt} = \rho\vec{g} - \nabla P + \mu \vec{\nabla}^2 \vec{V}^2$$

Navier-Stokes expanded form in 'Cartesian-coordinate'

$$\begin{cases} \vec{g} = (g_x, g_y, g_z) \\ \vec{V} = (V_x, V_y, V_z) \end{cases}$$

$$\hat{x}\begin{cases} \partial x \\ V_x \\ g_x \end{cases} \quad \rho\left(\frac{\partial V_x}{\partial t} + V_x\frac{\partial V_x}{\partial x} + V_y\frac{\partial V_x}{\partial y} + V_z\frac{\partial V_x}{\partial z}\right) = \rho g_x - \frac{\partial P}{\partial x} + \mu\left(\frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_x}{\partial y^2} + \frac{\partial^2 V_x}{\partial z^2}\right)$$

$$\hat{y}\begin{cases} \partial y \\ V_y \\ g_y \end{cases} \quad \rho\left(\frac{\partial V_y}{\partial t} + V_x\frac{\partial V_y}{\partial x} + V_y\frac{\partial V_y}{\partial y} + V_z\frac{\partial V_y}{\partial z}\right) = \rho g_y - \frac{\partial P}{\partial y} + \mu\left(\frac{\partial^2 V_y}{\partial x^2} + \frac{\partial^2 V_y}{\partial y^2} + \frac{\partial^2 V_y}{\partial z^2}\right)$$

$$\hat{z}\begin{cases} \partial z \\ V_z \\ g_z \end{cases} \quad \rho\left(\frac{\partial V_z}{\partial t} + V_x\frac{\partial V_z}{\partial x} + V_y\frac{\partial V_z}{\partial y} + V_z\frac{\partial V_z}{\partial z}\right) = \rho g_z - \frac{\partial P}{\partial z} + \mu\left(\frac{\partial^2 V_z}{\partial x^2} + \frac{\partial^2 V_z}{\partial y^2} + \frac{\partial^2 V_z}{\partial z^2}\right)$$

*Look at Table **3.4-1** The equations of motion in rectangular Cartesian coordinates*

6. Let’s see how our efforts in steps **1-to-5** will help us in simplifying these equations:

we got: $\vec{V} = (v_x(y), \underline{0}, \underline{0})$, $\vec{g} = (g\sin\beta, -g\cos\beta, \underline{0})$, $P(y)$

(a) $$\rho\left(\overbrace{\frac{\partial v_x}{\partial t}}^{0} + \overbrace{v_x \frac{\partial v_x}{\partial x}}^{0} + \overbrace{v_y}^{0} \frac{\partial v_x}{\partial y} + \overbrace{v_z}^{0} \overbrace{\frac{\partial v_x}{\partial z}}^{0}\right) = \rho g_x - \overbrace{\frac{\partial P}{\partial x}}^{0} + \mu\left(\overbrace{\frac{\partial^2 v_x}{\partial x^2}}^{0} + \frac{\partial^2 v_x}{\partial y^2} + \overbrace{\frac{\partial^2 v_x}{\partial z^2}}^{0}\right)$$

(b) $$\underbrace{\rho\left(\frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} + v_z \frac{\partial v_y}{\partial z}\right)}_{0} = \rho g_y - \frac{\partial P}{\partial y} + \underbrace{\mu\left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2}\right)}_{0}$$

(c) $$\underbrace{\rho\left(\frac{\partial v_z}{\partial t} + v_x \frac{\partial v_z}{\partial x} + v_y \frac{\partial v_z}{\partial y} + v_z \frac{\partial v_z}{\partial z}\right)}_{0} = \underbrace{\rho g_z}_{0} - \underbrace{\frac{\partial P}{\partial z}}_{0} + \underbrace{\mu\left(\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2}\right)}_{0}$$

Well, we found that velocity is only in x direction. Therefor **IF** we are **only** interested in deriving the velocity field, we can only write down N-S in the x direction (**equation a**). But note that if you are interested in Pressure, P, it could vary in y direction (**equation b**)

(b) $\rho g_y = \frac{\partial P}{\partial y} \rightarrow \frac{\partial P}{\partial y} = -\rho g \cos\beta \overset{\int}{\Longrightarrow} P_{(y)} = P_0 - (\rho g \cos\beta) y \xrightarrow{\text{B.C: } P_{(y=d)}=0}$

$P_{(y)} = \rho g \cos\beta (d-y) \longrightarrow$ remember $P = \rho g h$. $h = (d-y)\cos\beta$ here

told us about pressure in fluid

(a) $\rho g \sin\beta + \mu \frac{\partial^2 V_x}{\partial y^2} = 0 \xrightarrow[\text{and B.C. Lecture 23}]{\text{solving by integration}} V_x = \frac{\rho g \sin\beta}{\mu}\left(yd - \frac{y^2}{2}\right)$

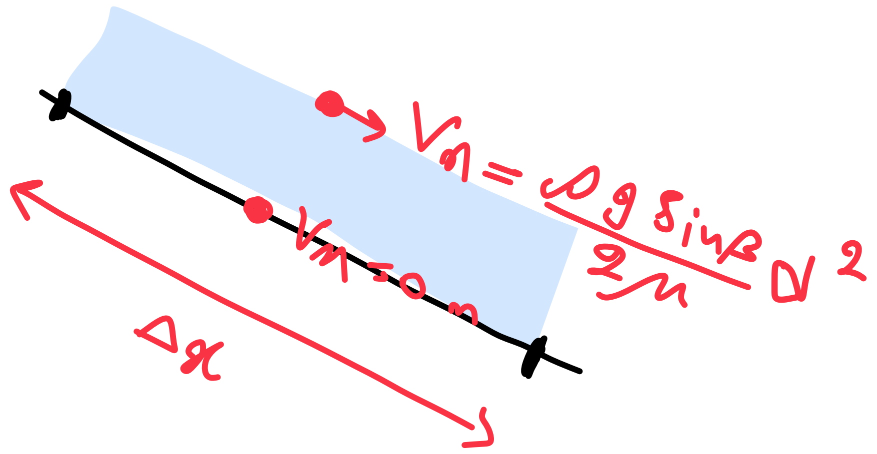
told us about velocity

Recap:

We made a ramp with 45 degrees, and did this experiment in class:

Activity!

Let's get a closer look: $V_x(y) = \frac{\rho g \sin\beta}{\mu}\left(yd - \frac{y^2}{2}\right)$

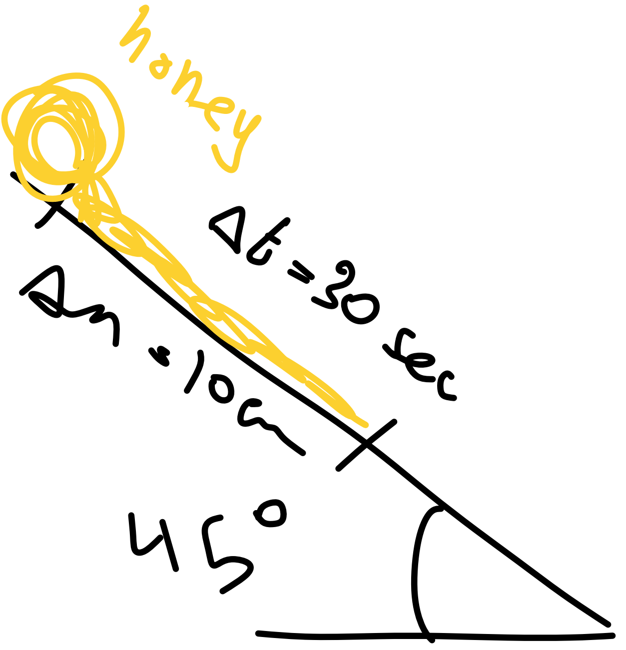


$\bar{V} \simeq \frac{V_m(y=0) + V_m(y=d)}{2}$

$\bar{V} \simeq \frac{d^2 \rho g \sin\beta}{4\mu}$

$\Rightarrow \Delta x = \bar{V}\Delta t \Rightarrow \Delta x = \frac{d\rho g \sin\beta}{4\mu}\Delta t \Rightarrow \mu \simeq \frac{d^2 \rho g \sin\beta\, \Delta t}{4\,\Delta x}$

we did this activity in class:



now let's put in some numbers:

$\Delta t = 3\times10$ s

$\Delta x = 10^{-1}$ m

$\rho \simeq 1.5\ 10^3 \frac{kg}{m^3}$

$g = 10 \frac{m}{s^2}$

$d \simeq 3\times10^{-3}$

$\longrightarrow \mu_{\text{from in class experiment}} = \frac{\cancel{3}\times\cancel{10^{-3}}\times\cancel{3}\times\cancel{10^{-3}}\times 1.5\times\cancel{10^{3}}\times\cancel{10}\times 7\times\cancel{10^{-1}}\times\cancel{3}\times\cancel{10}}{\cancel{4}\times\cancel{10^{-1}}}$

$\Rightarrow \mu \approx 7\times1.5$ Pa.s $\simeq 11$ Pa.s $\simeq 11.000$ mPa.s

Let’s see how our almost free and simple experimental setup compares with the measurement that is done with a 50000$+ equipment..

Viscosity of Flower Honey (Blended)

Commercially available sweetener

Description

Honey consists primarily of different kinds of sugar plus water. The main carbohydrates are fructose and glucose, f

The sampled honey is a commercially available, liquid bottled type produced in Western Europe. It is a blend from f parts carbohydrates and one part water.

Viscosity Table – Measurement data

Show 10 entries Search:

Temp. [°C]	Dyn. Viscosity [mPa.s]	Kin. Viscosity [mm²/s]	Density [g/cm³]
20	14095	9927.9	1.4198
22	10680	7528.8	1.4185
24	8156.4	5755.1	1.4173
26	6303.6	4451.8	1.416
28	4917.2	3475.7	1.4147
30	3872.6	2739.8	1.4134

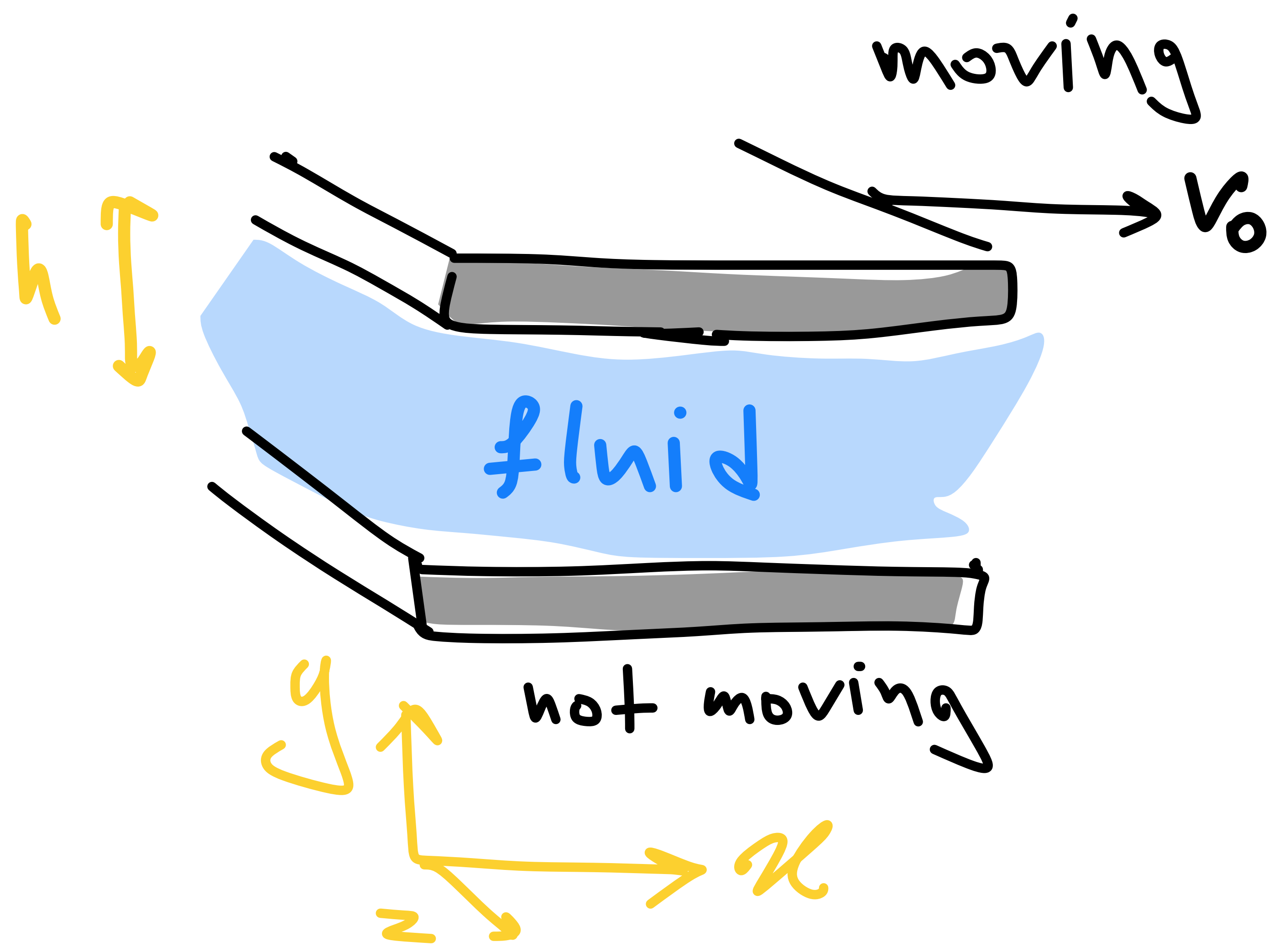
https://wiki.anton-paar.com/ca-en/flower-honey-blended/

Pause point!

B: Flow Between Fixed Plates with Upper Plate Moving (e.g., rheometers).

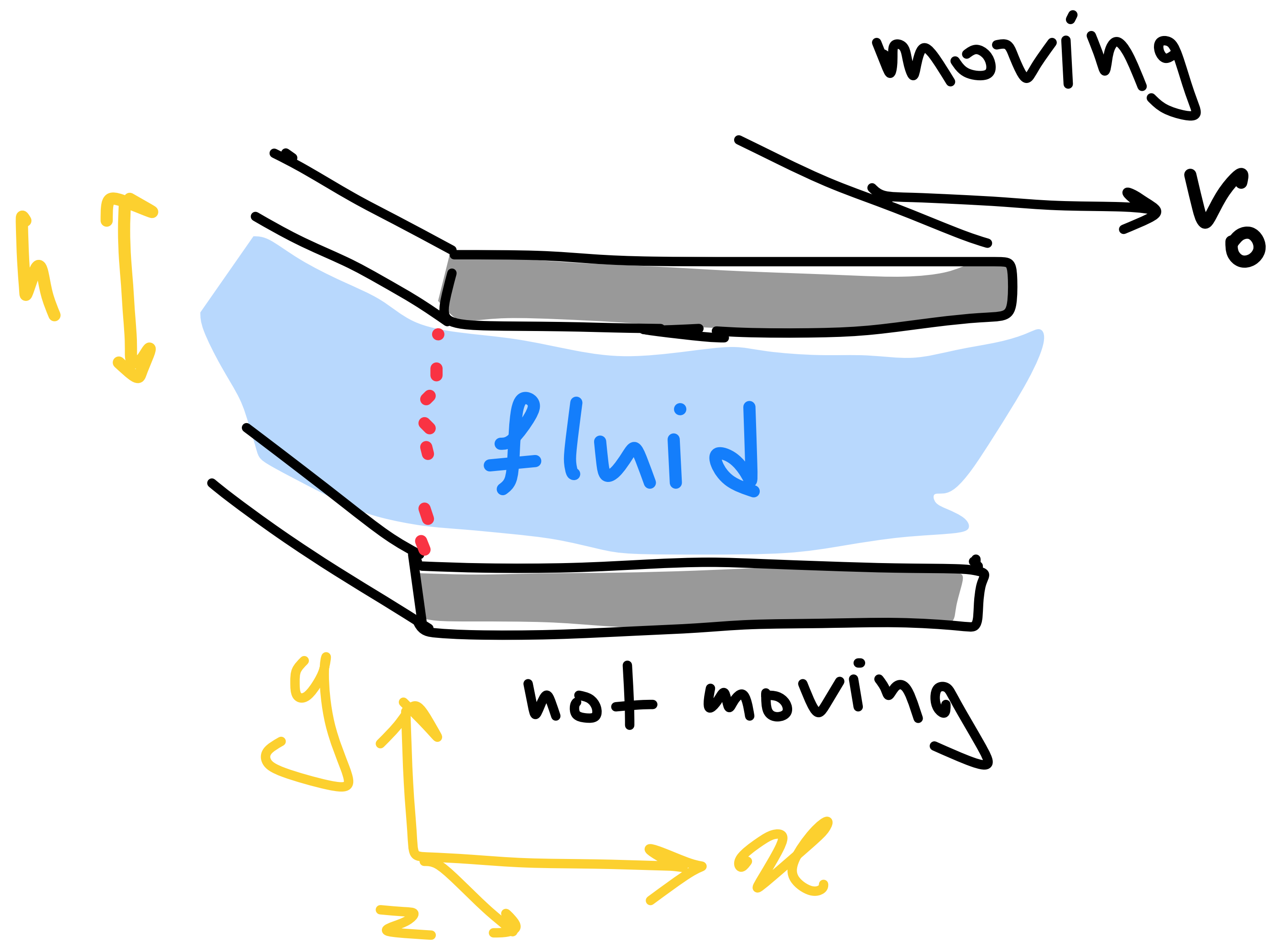
We have a flow between two very large parallel plates, separated by a small distance h, as found in a parallel-plate.

Assume that the flow is steady state, fully developed, laminar, incompressible and Newtonian. he upper plate moves with a constant velocity v0. There is no external pressure



Question B1: What is the Velocity profile of the fluid?

Solution:



$\Rightarrow$

1. Plate is large in z direction: Symmetry in z direction

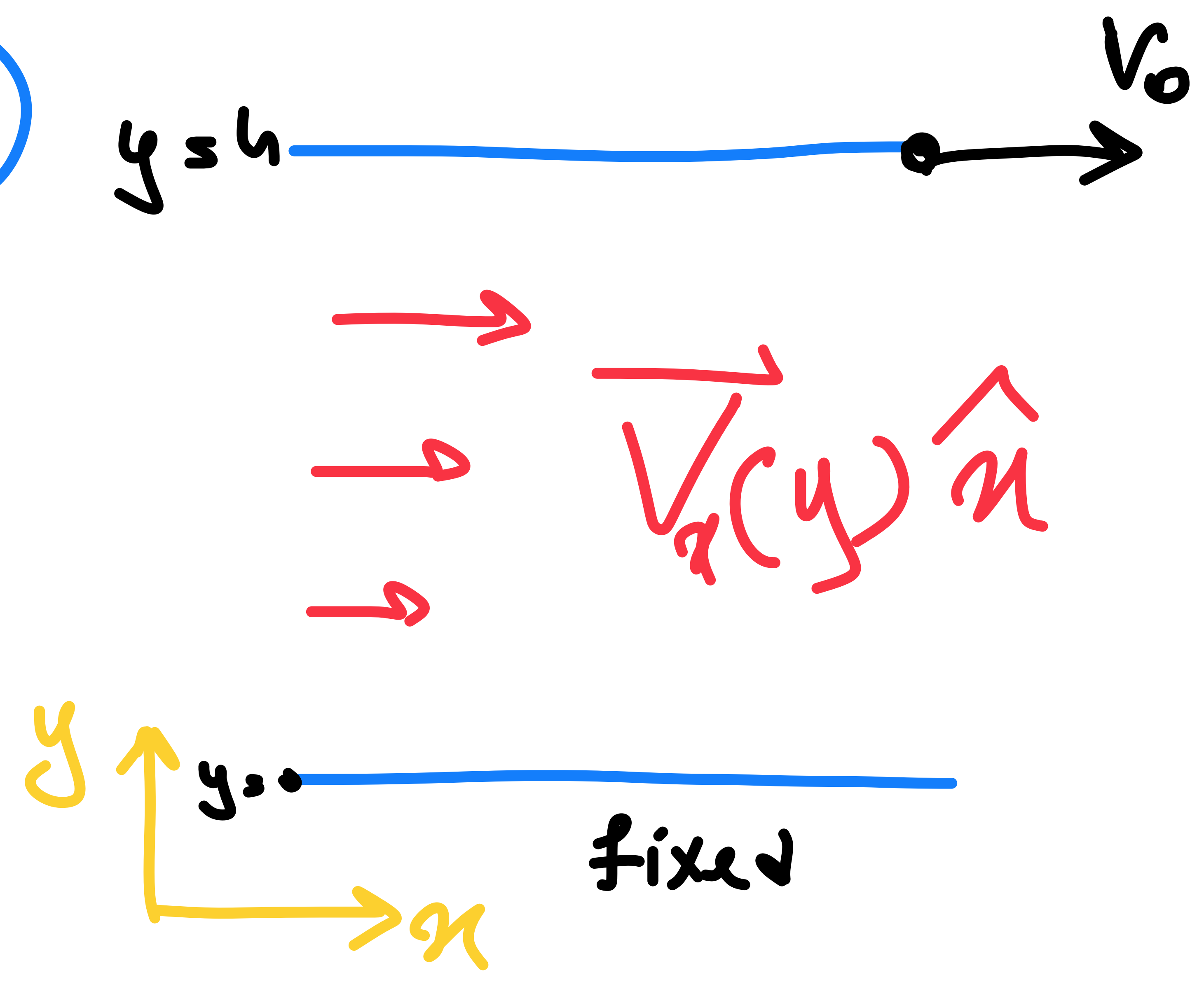
$$V(x,y,z) \to V(x,y)$$

2. fully developed

$$V(x,y) \to V_x(y)\,\hat{x}$$

$\Rightarrow$

3.



4. no external pressure $\Rightarrow$ $P(\cancel{x}, y, \cancel{z}) \to P(y)$

symmetry

5. Also boundary conditions (fully develope $\Rightarrow$ no-slip):

$$\vec{V}(y=0) = 0\,\hat{x}$$

$$\vec{V}(y=h) = V_0\,\hat{x}$$

6 So far we have significantly simplified the situation, our velocity is only the function of y, and also only in x direction! Now we are at a good place to use N-S equation, in this case, our coordinate system is cartesian.

we got: $\vec{V} = (v_x(y), 0, 0)$, $\vec{g} = (0, -g, 0)$, $P(y)$

a $$\rho\left(\overbrace{\frac{\partial v_x}{\partial t}}^{0} + \overbrace{v_x \frac{\partial v_x}{\partial x}}^{0} + \overbrace{v_y}^{0} \frac{\partial v_x}{\partial y} + \overbrace{v_z}^{0} \overbrace{\frac{\partial v_x}{\partial z}}^{0}\right) = \overbrace{\rho g_x}^{0} - \overbrace{\frac{\partial P}{\partial x}}^{0} + \mu\left(\overbrace{\frac{\partial^2 v_x}{\partial x^2}}^{0} + \frac{\partial^2 v_x}{\partial y^2} + \overbrace{\frac{\partial^2 v_x}{\partial z^2}}^{0}\right)$$

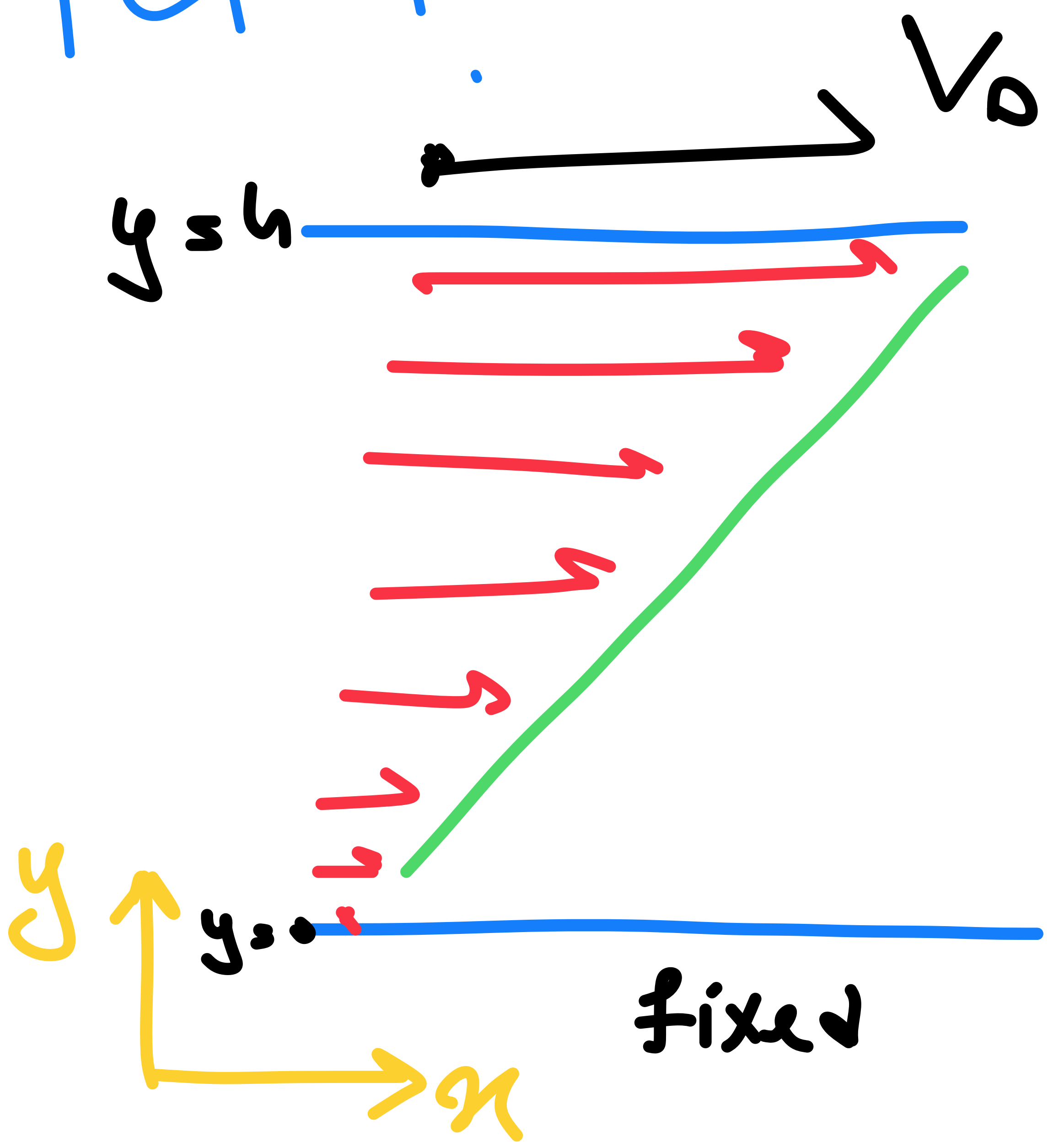
b $$\rho\underbrace{\left(\frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} + v_z \frac{\partial v_y}{\partial z}\right)}_{0} = \rho g_y - \frac{\partial P}{\partial y} + \underbrace{\mu\left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2}\right)}_{0}$$

c $$\underbrace{\rho\left(\frac{\partial v_z}{\partial t} + v_x \frac{\partial v_z}{\partial x} + v_y \frac{\partial v_z}{\partial y} + v_z \frac{\partial v_z}{\partial z}\right)}_{0} = \underbrace{\rho g_z}_{0} - \underbrace{\frac{\partial P}{\partial z}}_{0} + \underbrace{\mu\left(\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2}\right)}_{0}$$

(a) $\Rightarrow$

Therefor: $\mu \dfrac{\partial^2 V_x}{\partial^2 y} = 0$ and B.C $\begin{cases} V_x(0) = 0 \\ V_x(h) = V_0 \end{cases}$

$$\xRightarrow{\int dy} \frac{dV_x}{dy} = C_1 \xrightarrow{\int dy} V_x(y) = C_1 y + C_2 \xrightarrow{B.C} \begin{cases} C_2 = 0 \\ C_1 = \frac{V_0}{h} \end{cases}$$



$$\Rightarrow V(\vec{X}) = \frac{V_0}{h} y \, \hat{x}$$

Question B2: By having the velocity profile of the fluid, can you calculate the shear stress tensor inside the fluid?

Solution:

remember:

Shear stress tensor: $\tilde{\tau}$

$$\tilde{\tau} = \begin{pmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{pmatrix}$$

τ (face direction)(force direction)

also we had: $\vec{V} = (V_x, 0, 0)$

Table 3.4-4 Components of the stress tensor for Newtonian fluids in rectangular Cartesian coordinates

$$\tau_{xy} = \tau_{yx} = -\mu\left(\frac{\partial v_x}{\partial y} + \cancel{\frac{\partial v_y}{\partial x}}\right) = -\mu \frac{\partial V_x}{\partial y}, \quad V_x = \frac{V_0}{h} y \qquad \text{(A)}$$

$$\tau_{yz} = \tau_{zy} = -\mu\left(\cancel{\frac{\partial v_y}{\partial z}} + \cancel{\frac{\partial v_z}{\partial y}}\right) \qquad \text{(B)}$$

$$\tau_{zx} = \tau_{xz} = -\mu\left(\cancel{\frac{\partial v_z}{\partial x}} + \cancel{\frac{\partial v_x}{\partial z}}\right) = 0 \qquad \text{(C)}$$

$$\tau_{xx} = -2\mu\frac{\partial v_x}{\partial x} + \frac{2}{3}\mu\left(\cancel{\frac{\partial v_x}{\partial x}} + \cancel{\frac{\partial v_y}{\partial y}} + \cancel{\frac{\partial v_z}{\partial z}}\right) = 0 \qquad \text{(D)}$$

$$\tau_{yy} = -2\mu\cancel{\frac{\partial v_y}{\partial y}} + \frac{2}{3}\mu\left(\frac{\partial v_x}{\partial x} + \cancel{\frac{\partial v_y}{\partial y}} + \cancel{\frac{\partial v_z}{\partial z}}\right) = 0 \qquad \text{(E)}$$

$$\tau_{zz} = -2\mu\cancel{\frac{\partial v_z}{\partial z}} + \frac{2}{3}\mu\left(\cancel{\frac{\partial v_x}{\partial x}} + \cancel{\frac{\partial v_y}{\partial y}} + \cancel{\frac{\partial v_z}{\partial z}}\right) = 0 \qquad \text{(F)}$$

only non zero terms: $\tau_{xy} = \tau_{yx} = -\mu \frac{V_0}{h}$

$$\tau = \begin{pmatrix} 0 & -\mu\, v0/h & 0 \\ -\mu\, v0/h & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

End