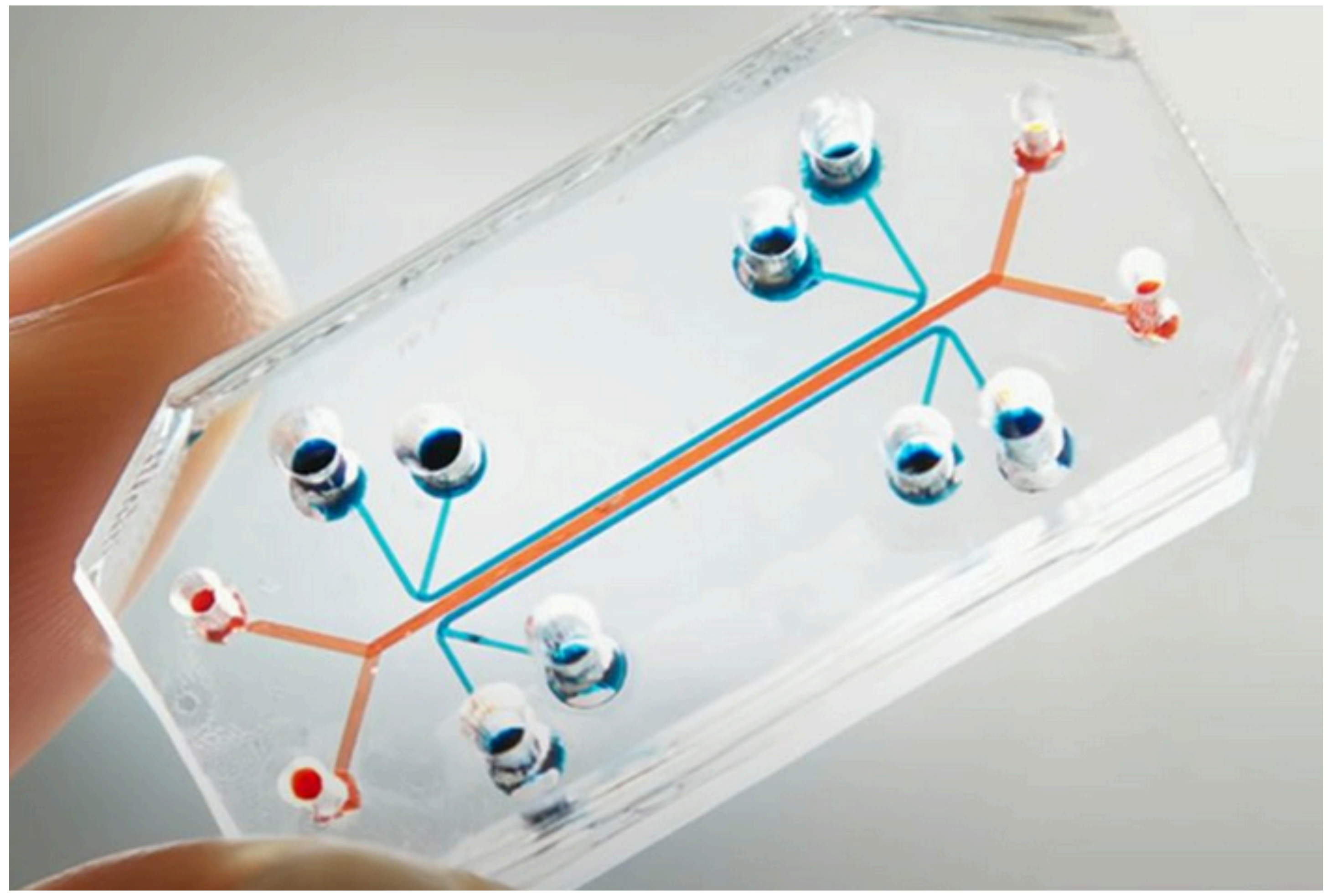
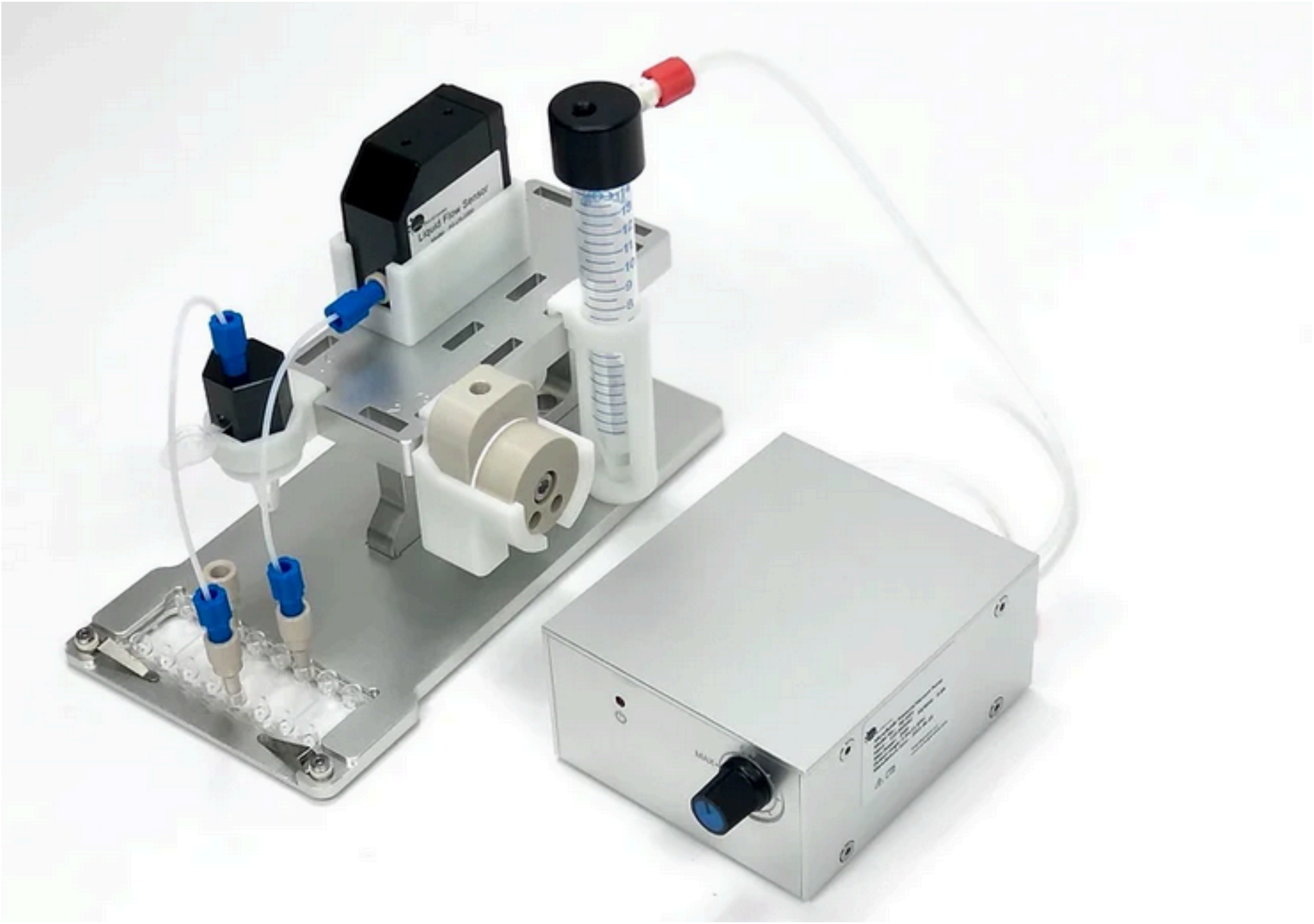
Lecture 25

Solving Navier-Stokes Equation for Steady-State and Fully-Developed Flow - Part 3: cylindrical coordinates

Analyzing flow in cylindrical coordinates is applicable in many cases, both in biology and in bioengineering.

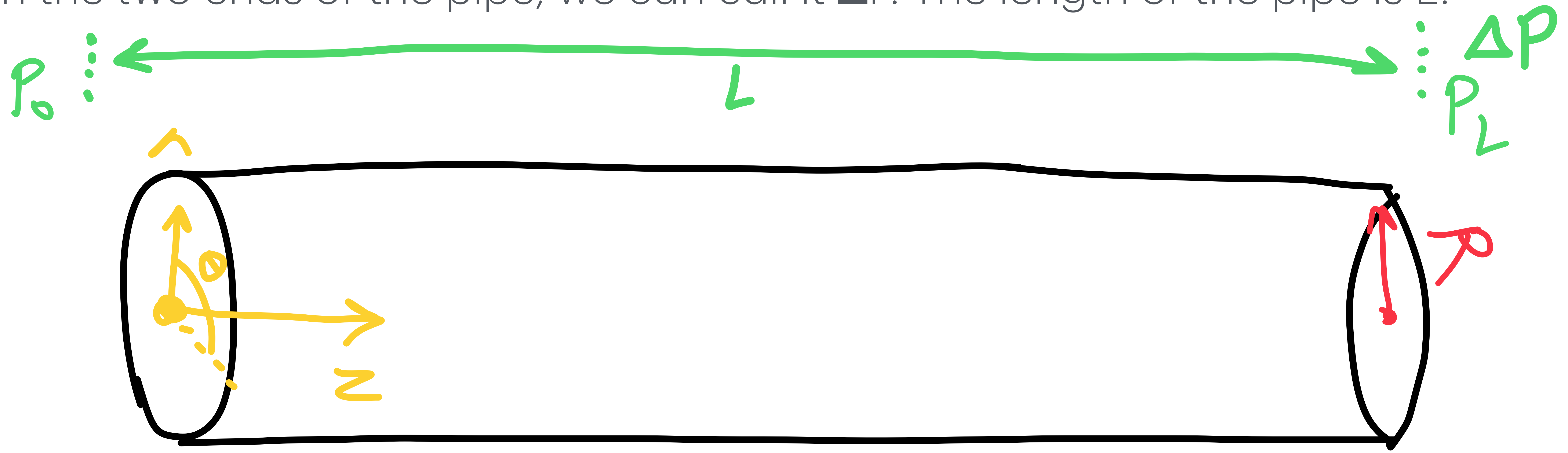




Flow in a Cylindrical Pipe

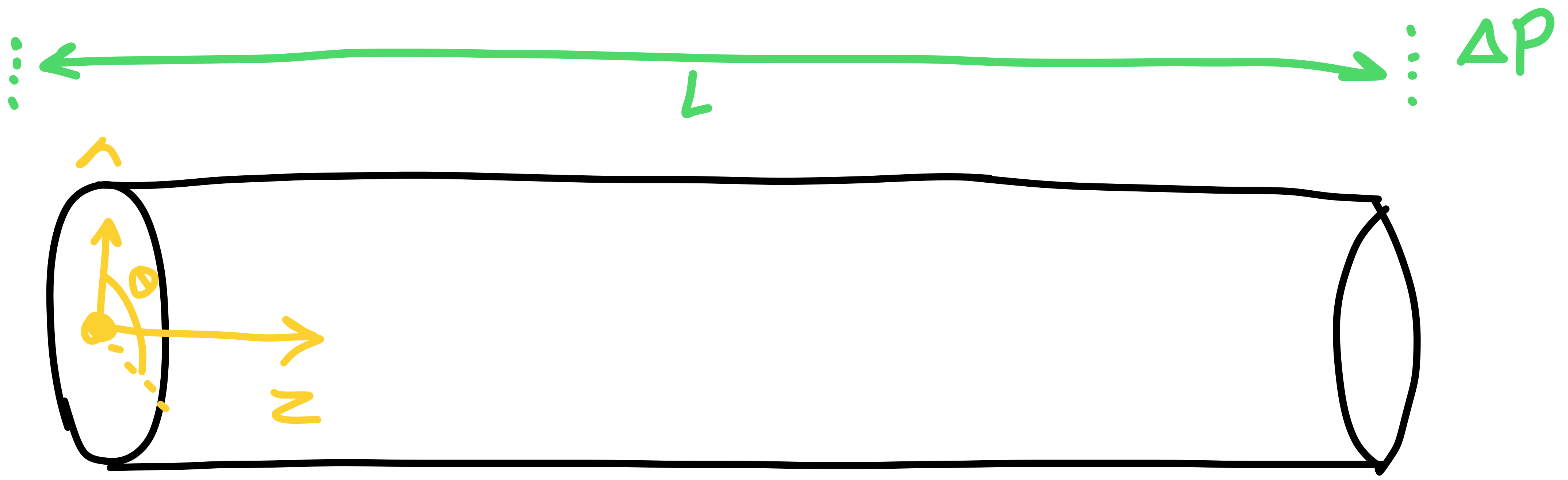
the laminar flow through a pipe of cylindrical cross-section. The results have significance in a variety of situations ranging from flow in micro-devices.

Here we have a S.S, laminar flow of a Newtonian fluid down a cylindrical pipe placed vertically. Let the flow be well-developed. The flow is running due to a pressure difference between the two ends of the pipe, we can call it ΔP. The length of the pipe is L.



Q1: Obtain the velocity profile of the viscous fluid inside the pipe.

Solution:



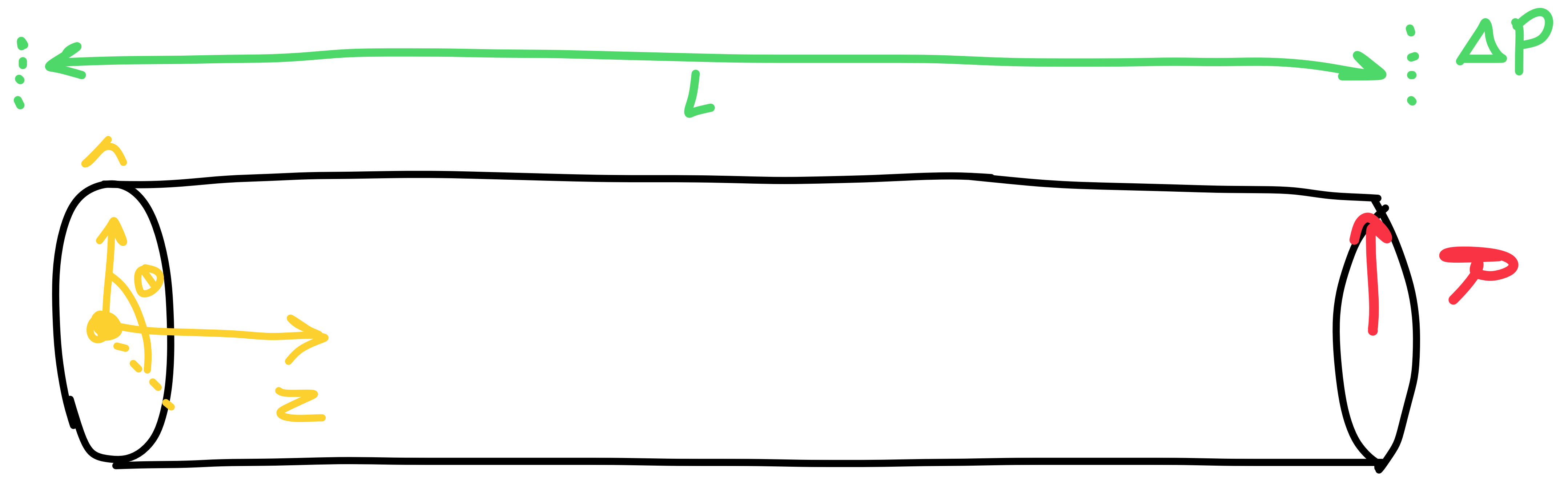
we are interested in

$$\vec{V}(r,\theta,z) \overset{\text{general form}}{=} \begin{cases} V_r(r,\theta,z)\ \hat{r} \\ V_\theta(r,\theta,z)\ \hat{\theta} \\ V_z(r,\theta,z)\ \hat{z} \end{cases}$$

① pipe symmetry in $\hat{\theta}$ (rotational symmetry): $\vec{V}(r,\cancel{\theta},z)$

② fully developed: $\vec{V}(r,z) \rightarrow \vec{V}(r)$

③ flow is laminar: $\vec{V}(r) = V_z(r)\ \hat{z}$



④ we have external pressure

so we know $\frac{\partial P}{\partial z} \neq 0$

Can't say more about $\frac{\partial P}{\partial r}$ and $\frac{\partial P}{\partial \theta}$ at the moment...

we come back to this later

⑤ B.C : Liquid-Solid on the surface: $V_z(r=R)=0$

6 now we can go to N-S equations in proper coordinate system:

we got $\vec{v} = \begin{cases} v_r = 0 \\ v_\theta = 0 \\ v_z(r) \end{cases}$, $v_z(r=R)=0$

Cylindrical coordinates

r direction

a
$$\rho\left(\underbrace{\frac{\partial v_r}{\partial t}}_{0} + \underbrace{v_r\frac{\partial v_r}{\partial r}}_{0} + \underbrace{\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}}_{0} - \underbrace{\frac{v_\theta^2}{r}}_{0} + \underbrace{v_z\frac{\partial v_r}{\partial z}}_{0}\right) = -\frac{\partial p}{\partial r} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\underbrace{\frac{\partial(r v_r)}{\partial r}}_{0}\right) + \underbrace{\frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2}}_{0} - \underbrace{\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}}_{0} + \underbrace{\frac{\partial^2 v_r}{\partial z^2}}_{0}\right] + \underbrace{\rho g_r}_{0}$$

θ direction

b
$$\rho\left(\underbrace{\frac{\partial v_\theta}{\partial t}}_{0} + \underbrace{v_r\frac{\partial v_\theta}{\partial r}}_{0} + \underbrace{\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}}_{0} + \underbrace{\frac{v_r v_\theta}{r}}_{0} + \underbrace{v_z\frac{\partial v_\theta}{\partial z}}_{0}\right) = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\underbrace{\frac{\partial(r v_\theta)}{\partial r}}_{0}\right) + \underbrace{\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2}}_{0} + \underbrace{\frac{2}{r^2}\frac{\partial v_r}{\partial \theta}}_{0} + \underbrace{\frac{\partial^2 v_\theta}{\partial z^2}}_{0}\right] + \underbrace{\rho g_\theta}_{0}$$

z direction

c
$$\rho\left(\underbrace{\frac{\partial v_z}{\partial t}}_{0} + \underbrace{v_r\frac{\partial v_z}{\partial r}}_{0} + \underbrace{\frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta}}_{0} + \underbrace{v_z\frac{\partial v_z}{\partial z}}_{0}\right) = -\frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \underbrace{\frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2}}_{0} + \underbrace{\frac{\partial^2 v_z}{\partial z^2}}_{0}\right] + \underbrace{\rho g_z}_{0}$$

now let's see what we get from each component of N-S equation:

ⓐ $\frac{\partial P}{\partial r} = 0 \Rightarrow P(r,\theta,z) \longrightarrow P(\theta,z)$ not function of r

ⓑ $\frac{1}{r}\frac{\partial P}{\partial \theta} = 0 \Rightarrow \frac{\partial P}{\partial \theta} = 0 \Rightarrow P(\theta,z) \to P(z)$ not function of θ

ⓒ $$\underbrace{\frac{\mu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V_z(r)}{\partial r}\right)}_{\substack{\text{only function of } r \\ \text{LHS}}} = \underbrace{\frac{\partial P}{\partial z}}_{\substack{\text{only function of } z \\ \text{RHS}}}$$

$\Rightarrow LHS = RHS = Const. \equiv C_1$

for RHS: $\frac{\partial P}{\partial z} = C_1 \Rightarrow P(z) = C_1 z + C_2$

from question $P(z=L) - P(z=0) = \Delta P$

$C_2 = P_0$, $C_1 = \frac{\Delta P}{L}$

$\Rightarrow$ $\boxed{P(z) = \frac{\Delta P}{L} z + P_0}$

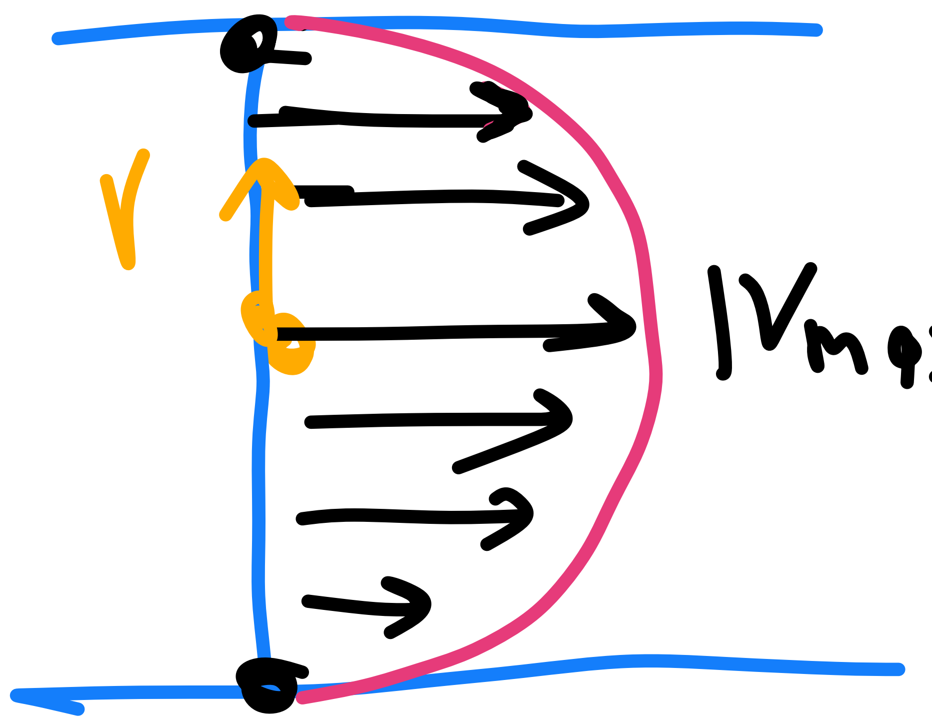
The LHS:

$$\frac{\mu}{r}\frac{\partial}{r}\left(r\frac{\partial V_z}{\partial r}\right) = C_1 = \frac{\Delta P}{L} \xRightarrow{\int} r\frac{\partial V_z}{\partial r} = \frac{\Delta P}{L}\frac{r^2}{2\mu} + C_3 \rightarrow$$

this should be true for $0 \leq r \leq R$, Looking at $r=0$ case: $0 = 0 + C_3 \Rightarrow C_3 = 0$

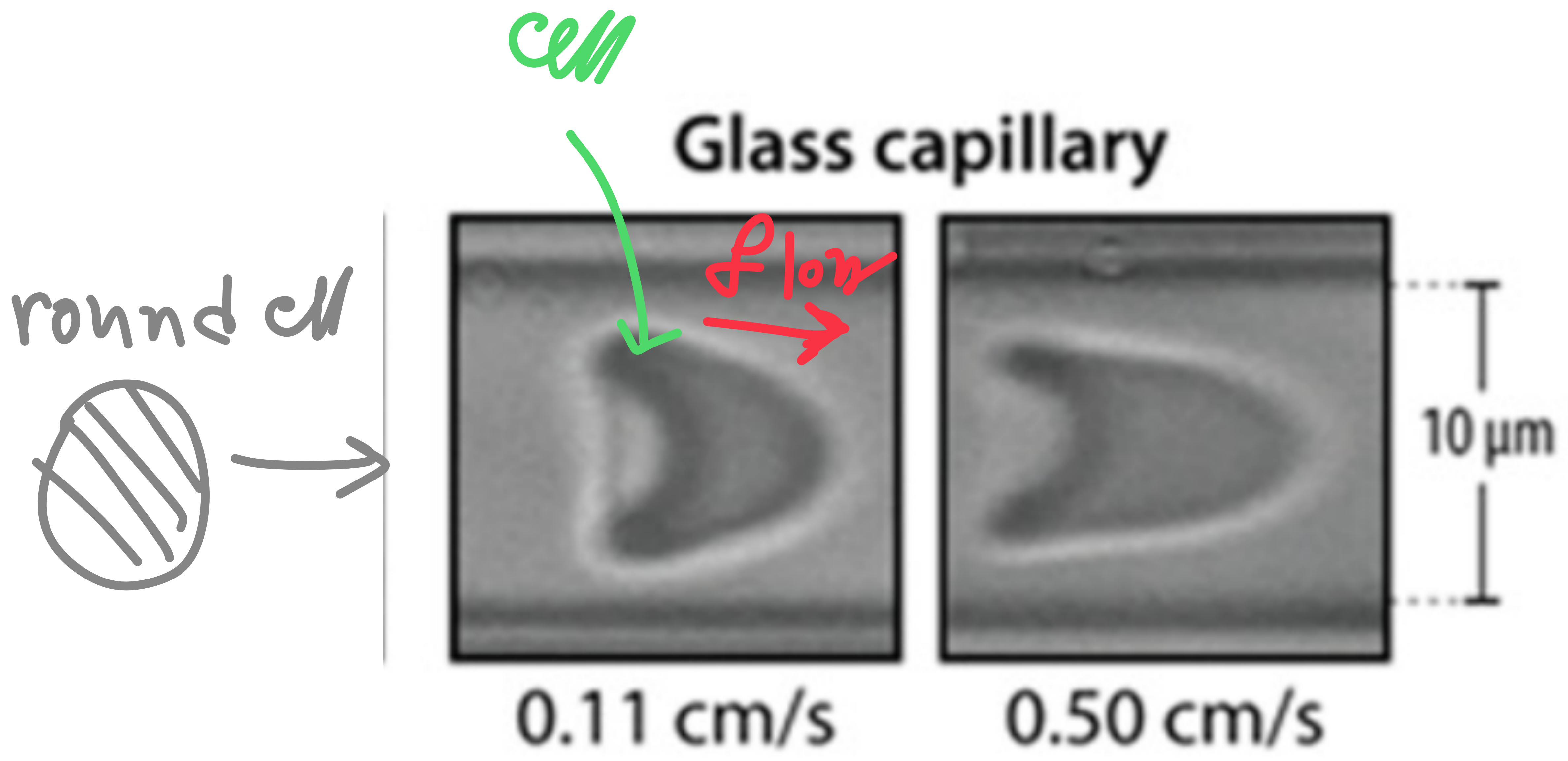
$$\xrightarrow{\int dr \text{ again}} V_z(r) = \frac{\Delta P}{4\mu L} r^2 + C_4 \,,\quad V_z(r=R) = 0 \Rightarrow C_4 = -\frac{\Delta P}{4\mu L} R^2$$

$$\rightarrow \vec{V} = V_z(r)\hat{z} = \frac{-\Delta P}{4\mu L}(r^2 - R^2)$$



$$|V_{max}| = \frac{\Delta P}{4\mu L} R^2$$

z



A Synergistic Overview between Microfluidics and Numerical Research for Vascular Flow and Pathological Investigations

Q2: What is the average velocity across the cross-section for the moving fluid?

$$\bar{V} \overset{\frac{\sum V_i}{n}}{=} \frac{\int_0^{2\pi}\int_0^R v_z \,(r\,dr)\,d\theta}{\int_0^{2\pi}\int_0^R r\,dr\,d\theta} \overset{\cdots}{=} \frac{\Delta P R^2}{8\mu L} = \frac{1}{2} V_{max}$$

Q3: What is the volumetric flow rate (the total rate of fluid flow) as the function of pressure, viscosity and radius of the tube?

$$Q \equiv \text{Area} \times \bar{V} = \pi R^2 \times \frac{\Delta P R^2}{8 \mu L} = \frac{\pi}{8 L \mu} \Delta P \; R^4$$

This has important implications for biological systems, particularly concerning blood flow, as the system is highly sensitive to radius changes. For example:

-During exercise, the body can increase blood flow by slightly expanding the radius of blood vessels.

-However, even minor blockages in the veins can cause significant problems due to this sensitivity.

I clicker questions: what are non zero elements in the shear stress tensor?

1) A
2) B
3) C
4) E
5) All are zero

Table 3.4-5 Components of the stress tensor for Newtonian fluids in cylindrical coordinates

$$\tau_{r\theta} = \tau_{\theta r} = -\mu\left(r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right) \quad \text{(A)}$$

$$\tau_{\theta z} = \tau_{z\theta} = -\mu\left(\frac{\partial v_\theta}{\partial z} + \frac{1}{r}\frac{\partial v_z}{\partial \theta}\right) \quad \text{(B)}$$

$$\tau_{zr} = \tau_{rz} = -\mu\left(\frac{\partial v_z}{\partial r} + \frac{\partial v_r}{\partial z}\right) \quad \text{(C)}$$

$$\tau_{rr} = -2\mu\frac{\partial v_r}{\partial r} + \frac{2}{3}\mu\left(\frac{1}{r}\frac{\partial (rv_r)}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z}\right) \quad \text{(D)}$$

$$\tau_{\theta\theta} = -2\mu\left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right) + \frac{2}{3}\mu\left(\frac{1}{r}\frac{\partial (rv_r)}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z}\right) \quad \text{(E)}$$

$$\tau_{zz} = -2\mu\frac{\partial v_z}{\partial z} + \frac{2}{3}\mu\left(\frac{1}{r}\frac{\partial (rv_r)}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z}\right) \quad \text{(F)}$$

Q4: What is the shear stress that is caused by this flow in the direction of the flow?

Solution:

τ(face direction)(force direction)

$$\tau_{rz} = -\mu \frac{\partial V_z}{\partial r} = -\mu \left(\frac{\Delta P}{4\mu L}\right) 2r = -\frac{\Delta P}{2L} r$$

Table 3.4-5 Components of the stress tensor for Newtonian fluids in cylindrical coordinates

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$$\tau_{zr} = \tau_{rz} = -\mu\left(\frac{\partial v_z}{\partial r} + \frac{\partial v_r}{\partial z}\right) \qquad \text{(C)}$$

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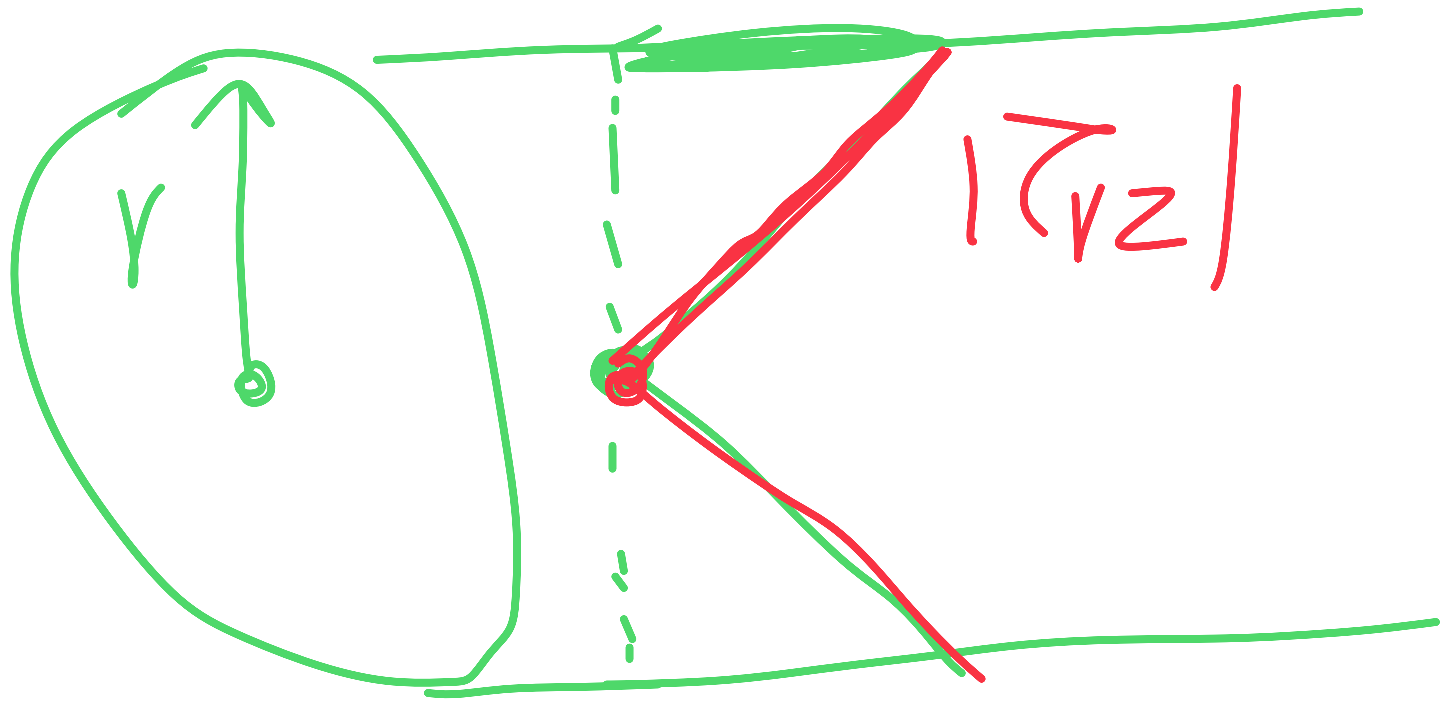
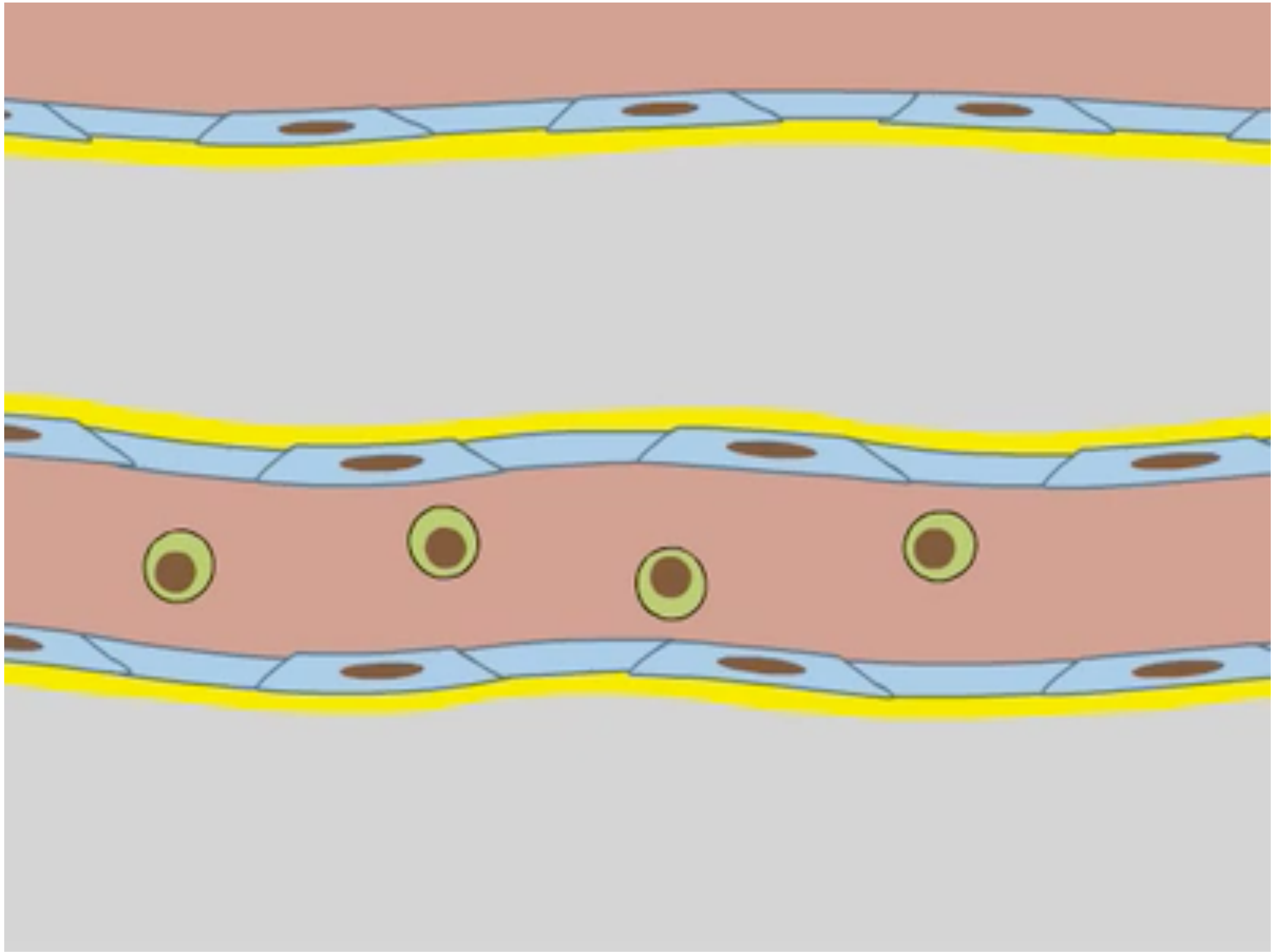
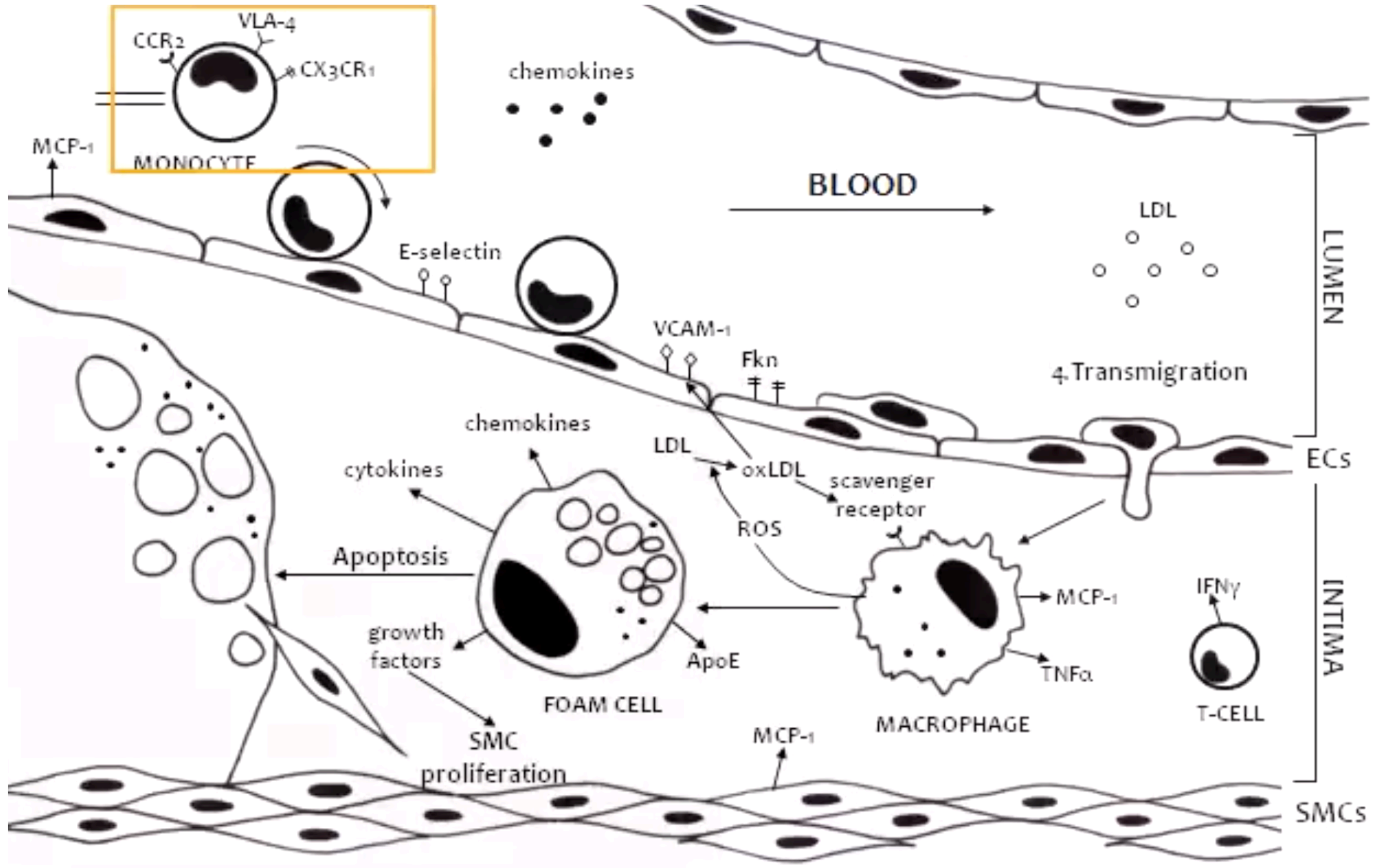
$$\tau_{zz} = -2\mu\frac{\partial v_z}{\partial z} + \frac{2}{3}\mu\left(\frac{1}{r}\frac{\partial (r v_r)}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z}\right) \qquad \text{(F)}$$

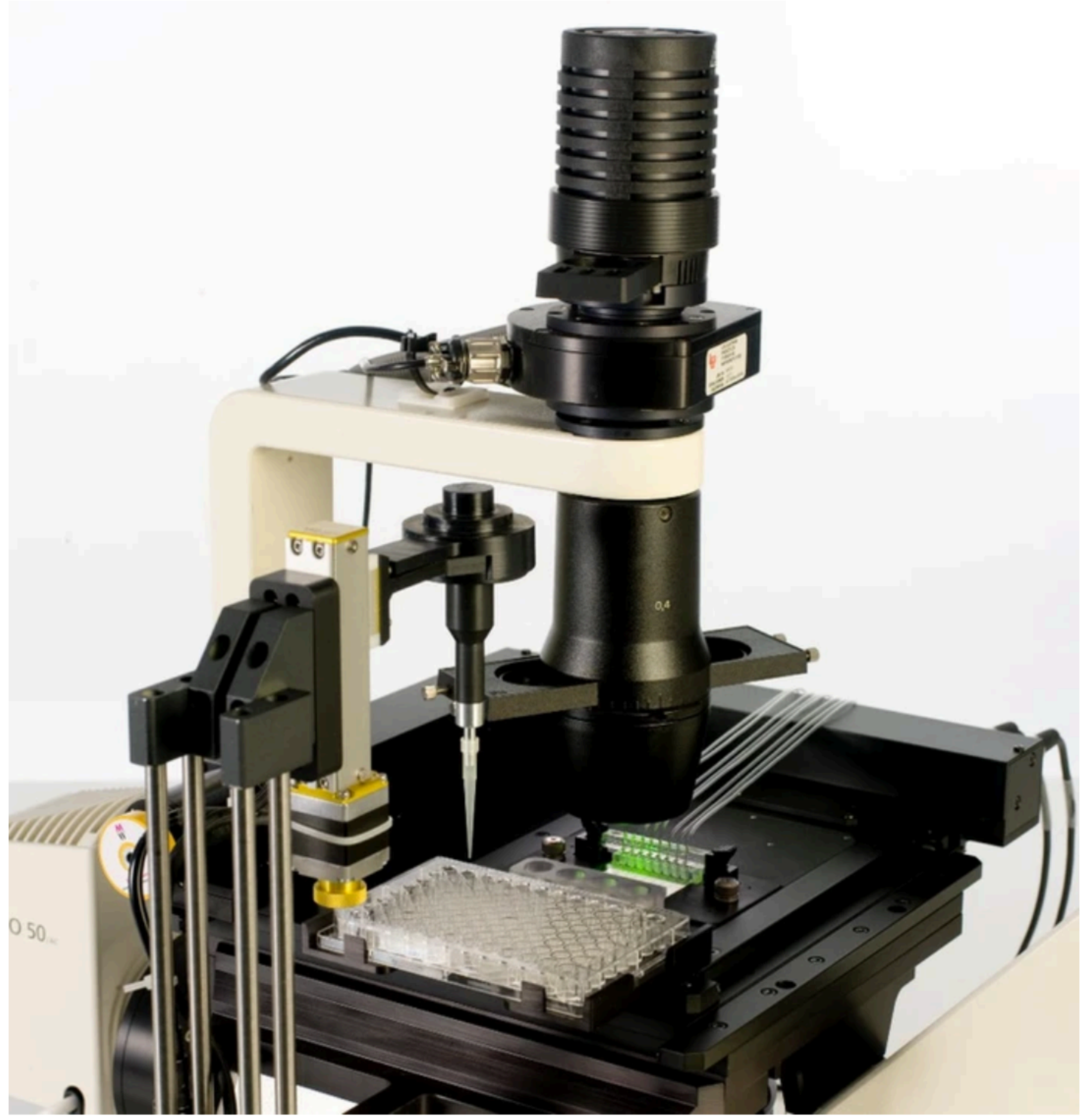


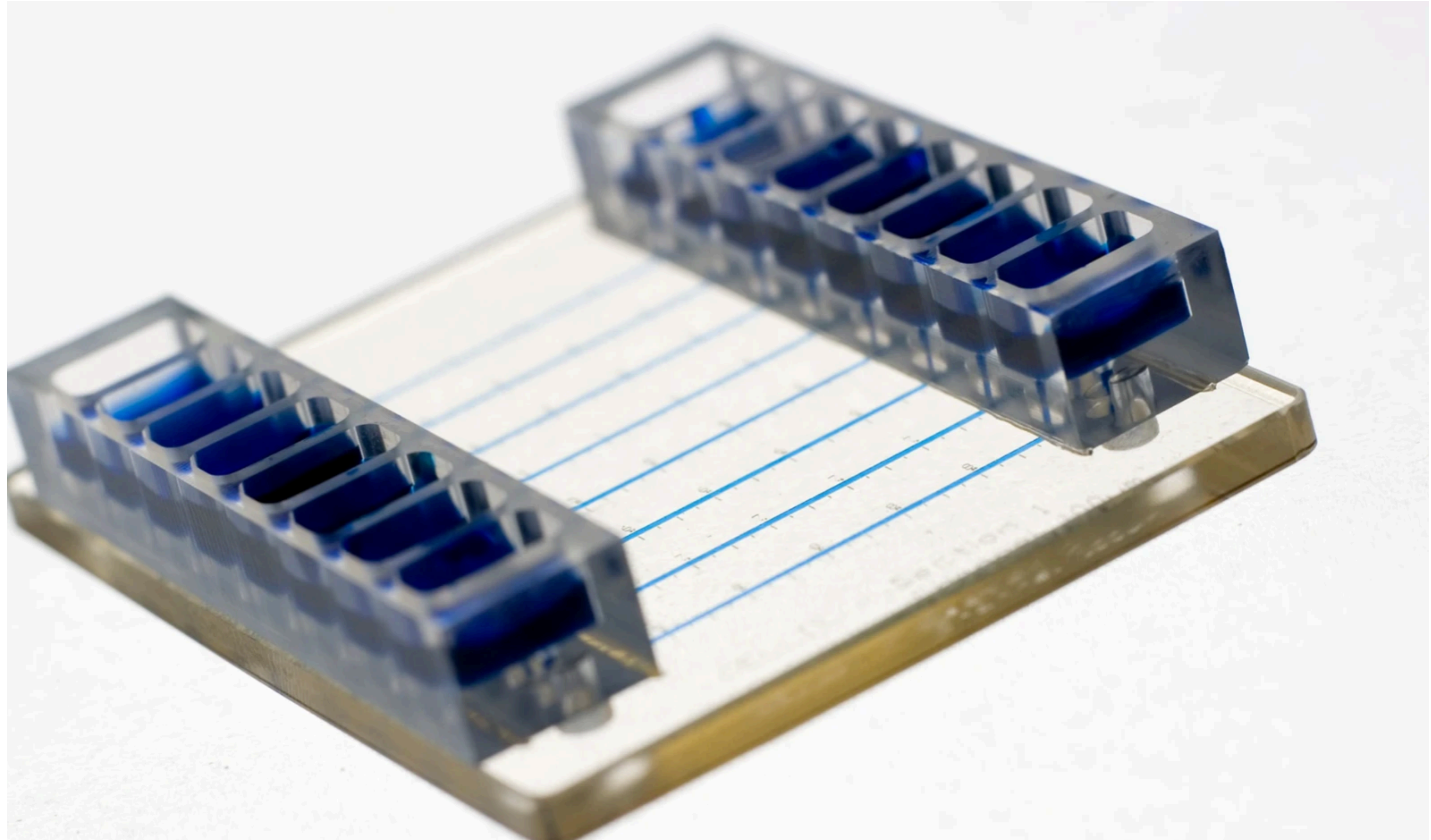
How biology uses this? Example



How engineers use this? Example:



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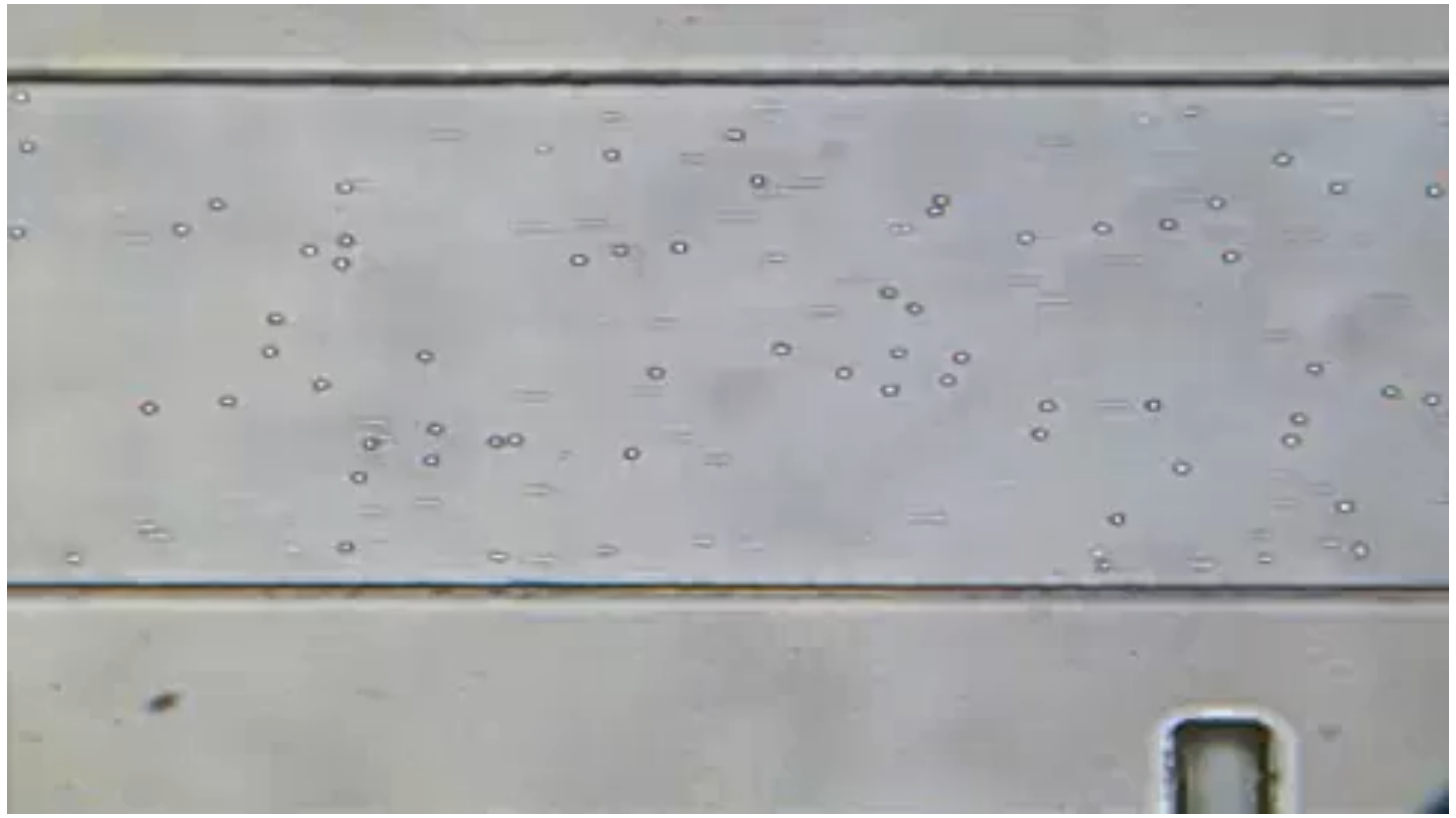

End

FYI: https://www.wearecellix.com/_files/ugd/735c22_07a73a473d2b4124b912b6d2773ab7a2.pdf

Technical specifications	
Material	Topas
Number of channels per biochip	8
Volume of each channel	1.12 µL
Dimensions of each channel	400 µm (W) x 100 µm (D) x 28 mm (L)
Dead volume at input port	0.1 µL
Thickness of bottom substrate	0.17 mm