Lecture 26

A) Applying the Law of Flow in Cylindrical Pipes (lecture 25) to Study Fluid Networks: Analogy with Ohm's law

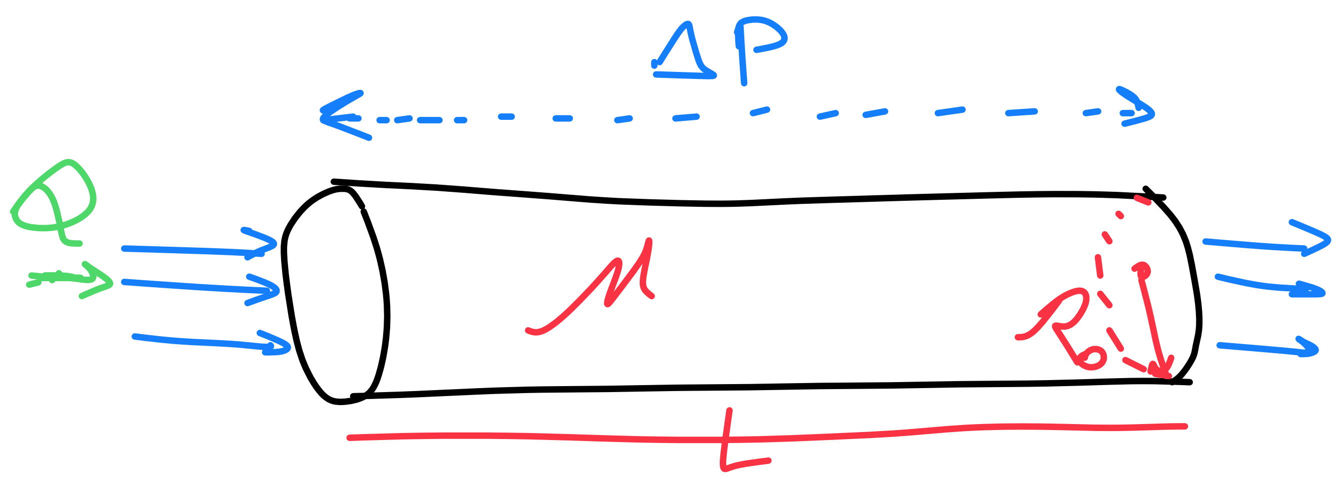
B) Briefly exploring another important example using cylindrical coordinates

C) Reynolds number: A deeper look

A) Applying the Law of Flow in Cylindrical Pipes to Study Fluid Networks

In the previous lecture we drove 'Hagen–Poiseuille' Law for Cylindrical Pipe Flow:

it describes laminar flow rate of an incompressible, Newtonian fluid through a long cylindrical pipe



$$Q = \frac{\pi R^4}{8\mu L}\Delta P \Rightarrow Q = \frac{\Delta P}{\left(\frac{8\mu L}{\pi R_0^4}\right)}$$

Analogy:

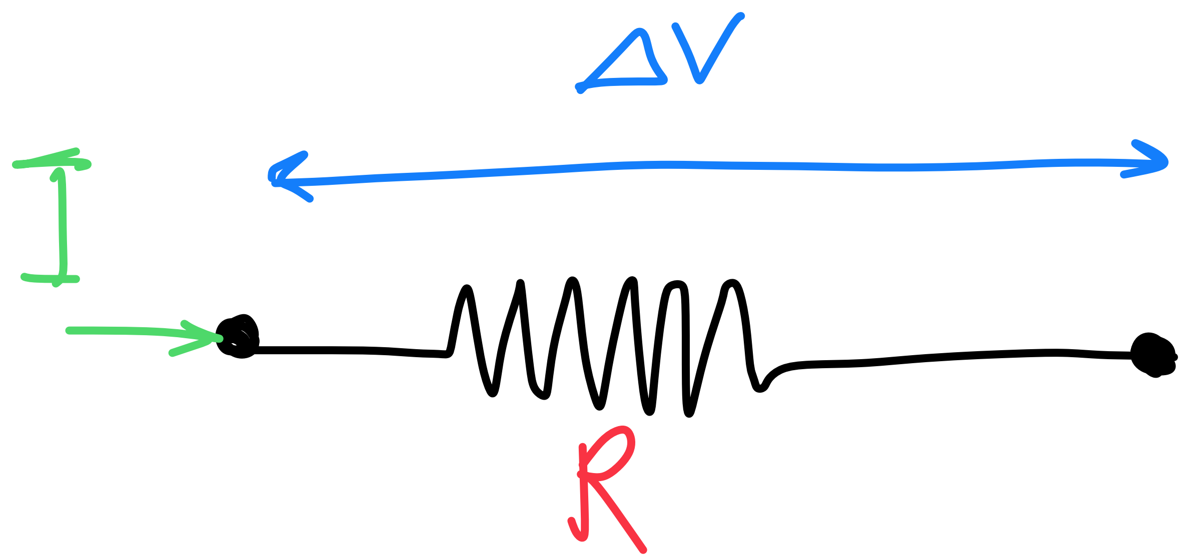
$$Q \longleftrightarrow I$$

$$\Delta P \longleftrightarrow \Delta V$$

$$\underbrace{\left(\frac{8\mu L}{\pi R^4}\right)}_{\equiv R_f} \longleftrightarrow R$$

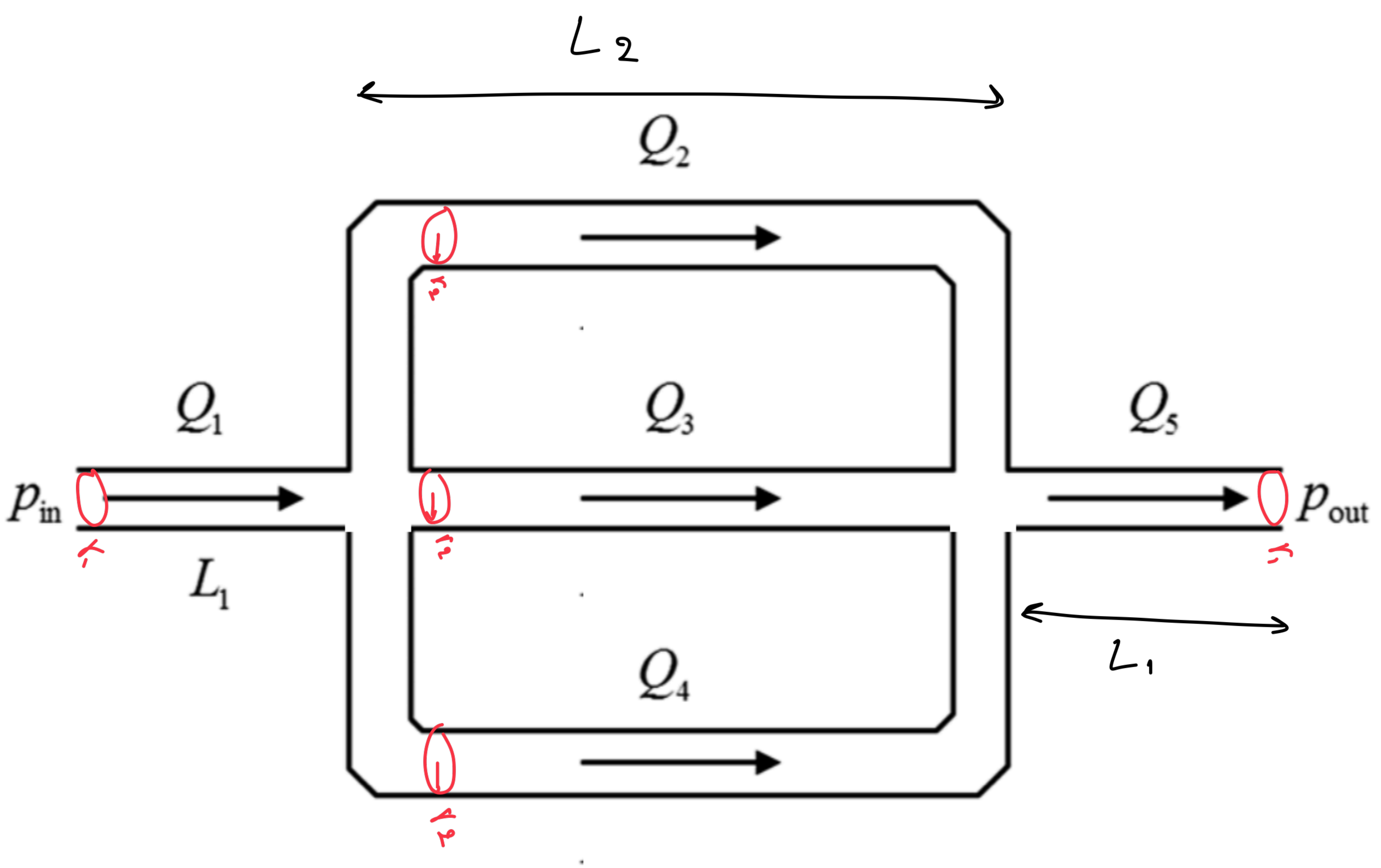
Now remember from you previous courses, the simple Ohm's law:

It describes the electrical current (electrical flow) through a resistor:



$$I = \frac{\Delta V}{R}$$

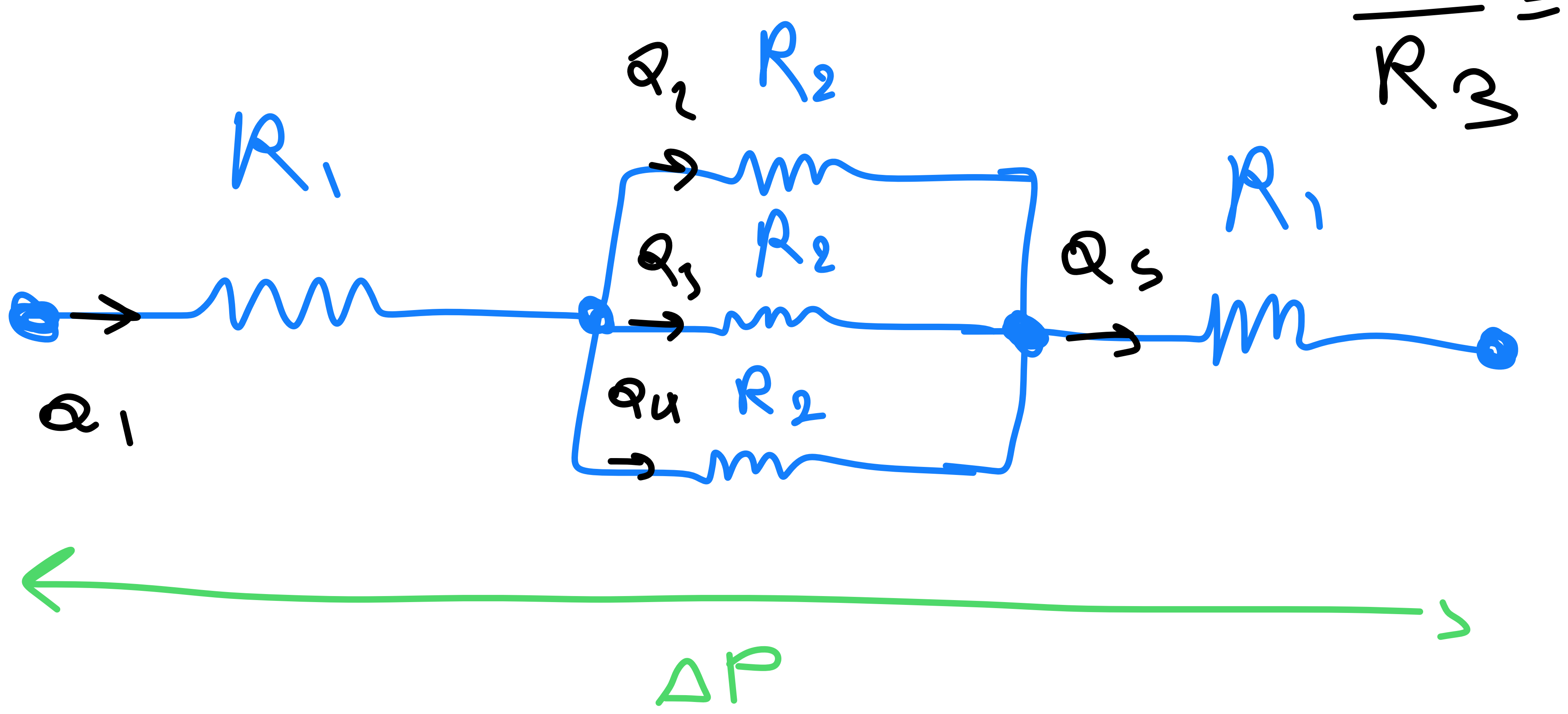
Example: We are designing a simple microfluidic device. What is the fluid flows Q2, Q3, Q4, Q5 if we know apply the pressure difference shown bellow?



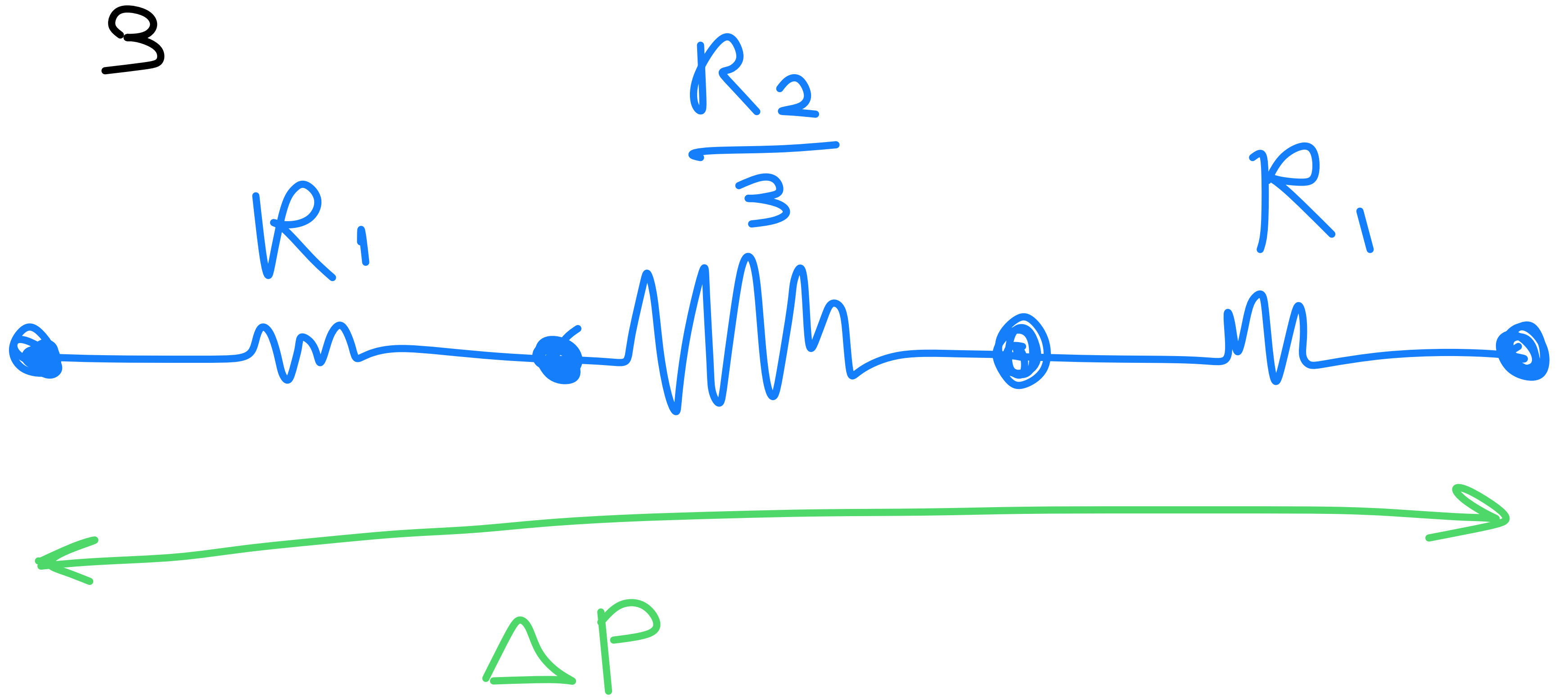
Solution: Let's define:

$$R_1 = \frac{8\mu L_1}{\pi r_1^4} \qquad R_2 = \frac{8\mu L_2}{\pi r_2^4}$$

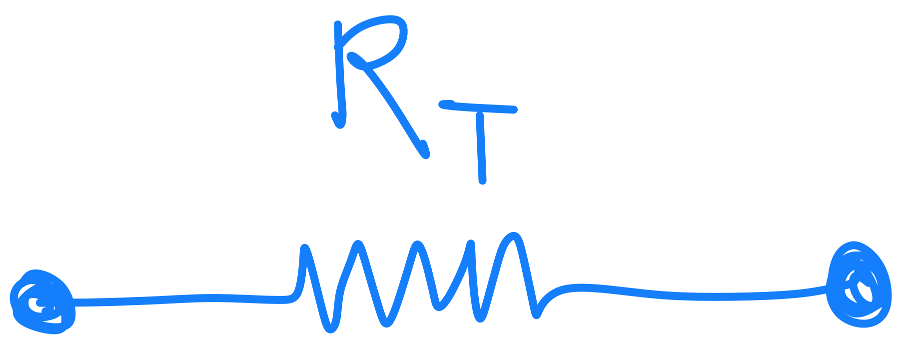
we have then:



$$\frac{1}{R_3} = \frac{1}{R_2} + \frac{1}{R_2} + \frac{1}{R_2} = \frac{1}{\frac{R_2}{3}}$$



$$R_T = 2R_1 + \frac{R_2}{3}$$

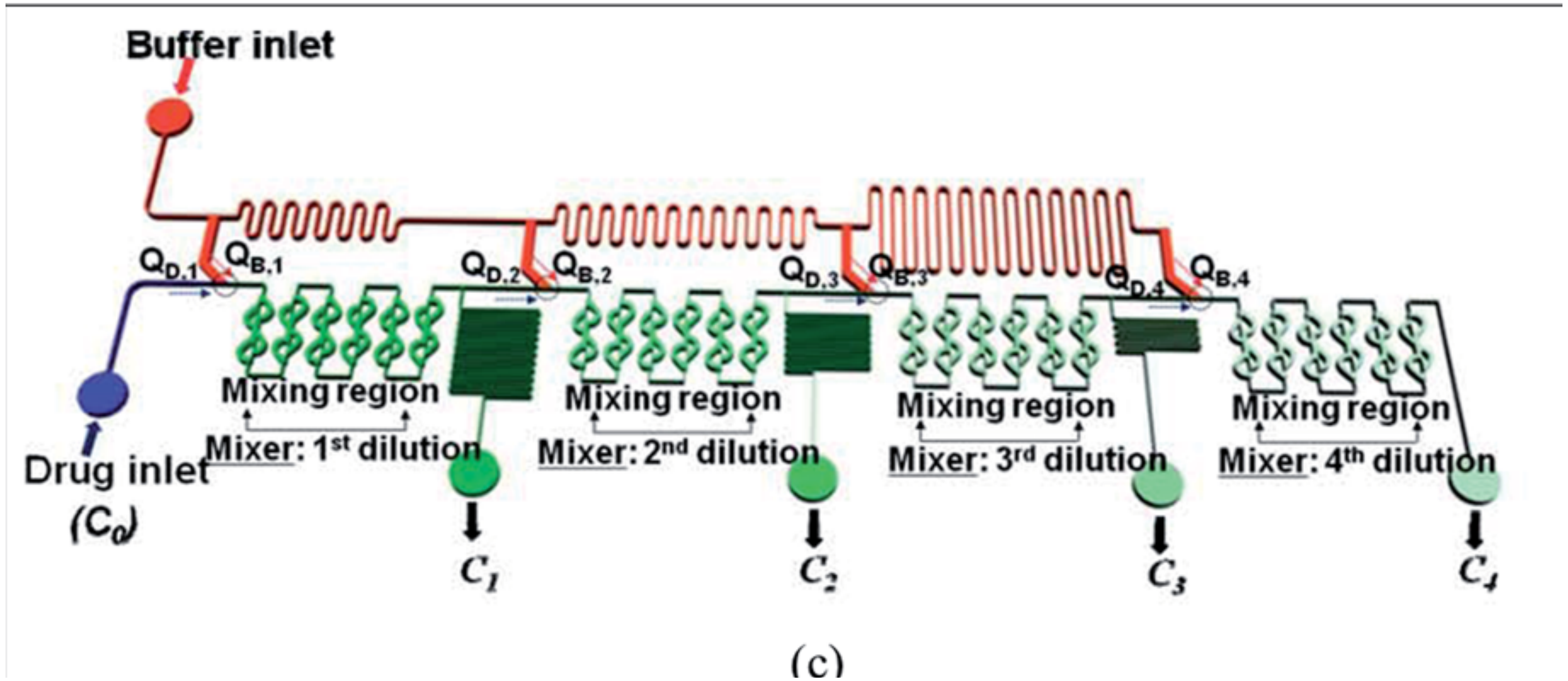


$$\Rightarrow Q_1 = Q_5 = \frac{\Delta P}{R_T}$$

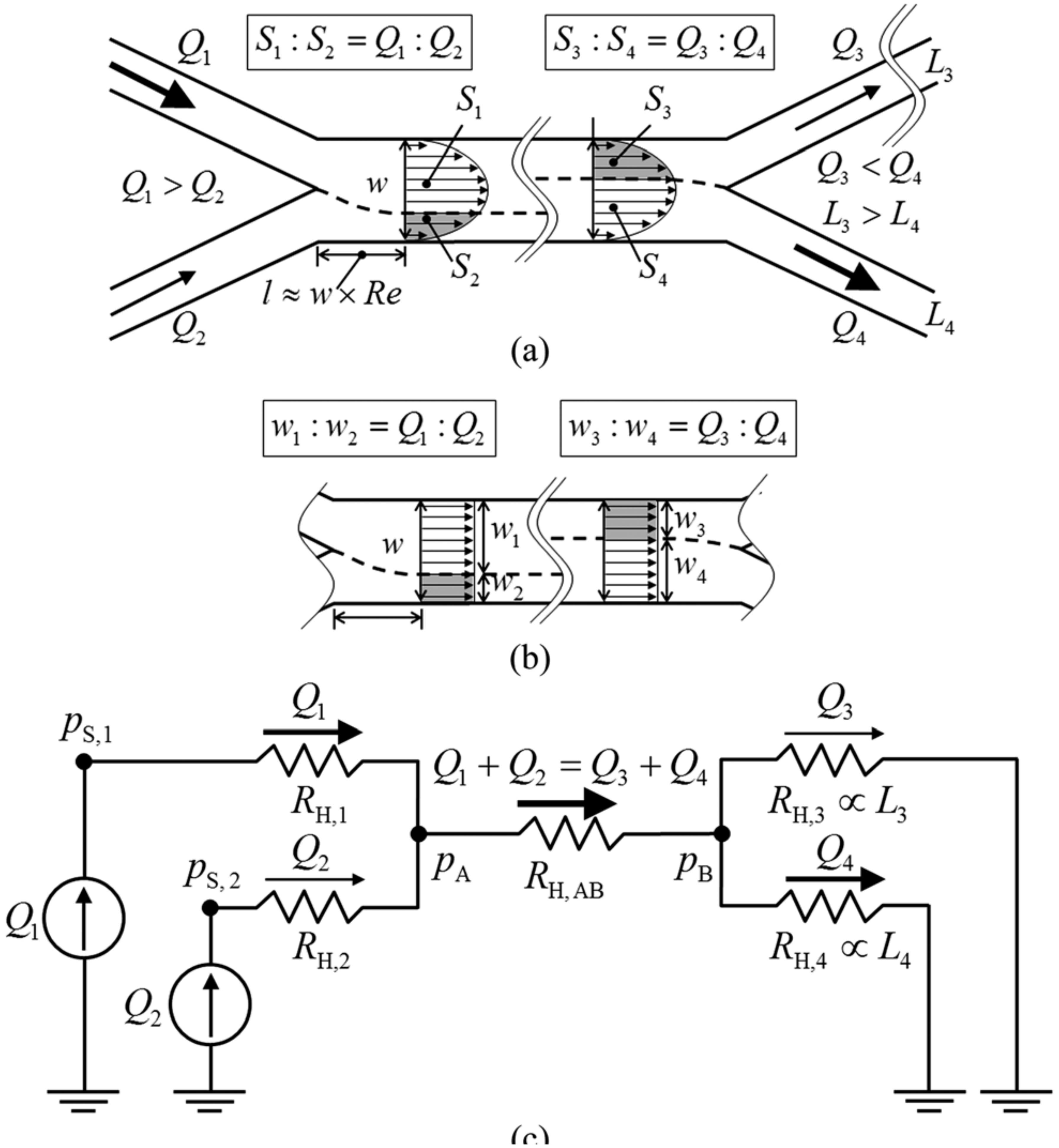
$$Q_2 = Q_3 = Q_4 = \frac{Q_1}{3}$$

solved!

Please read this for more information, now you know everything to understand this paper: Design of pressure-driven microfluidic networks using electric circuit analogy (2011)

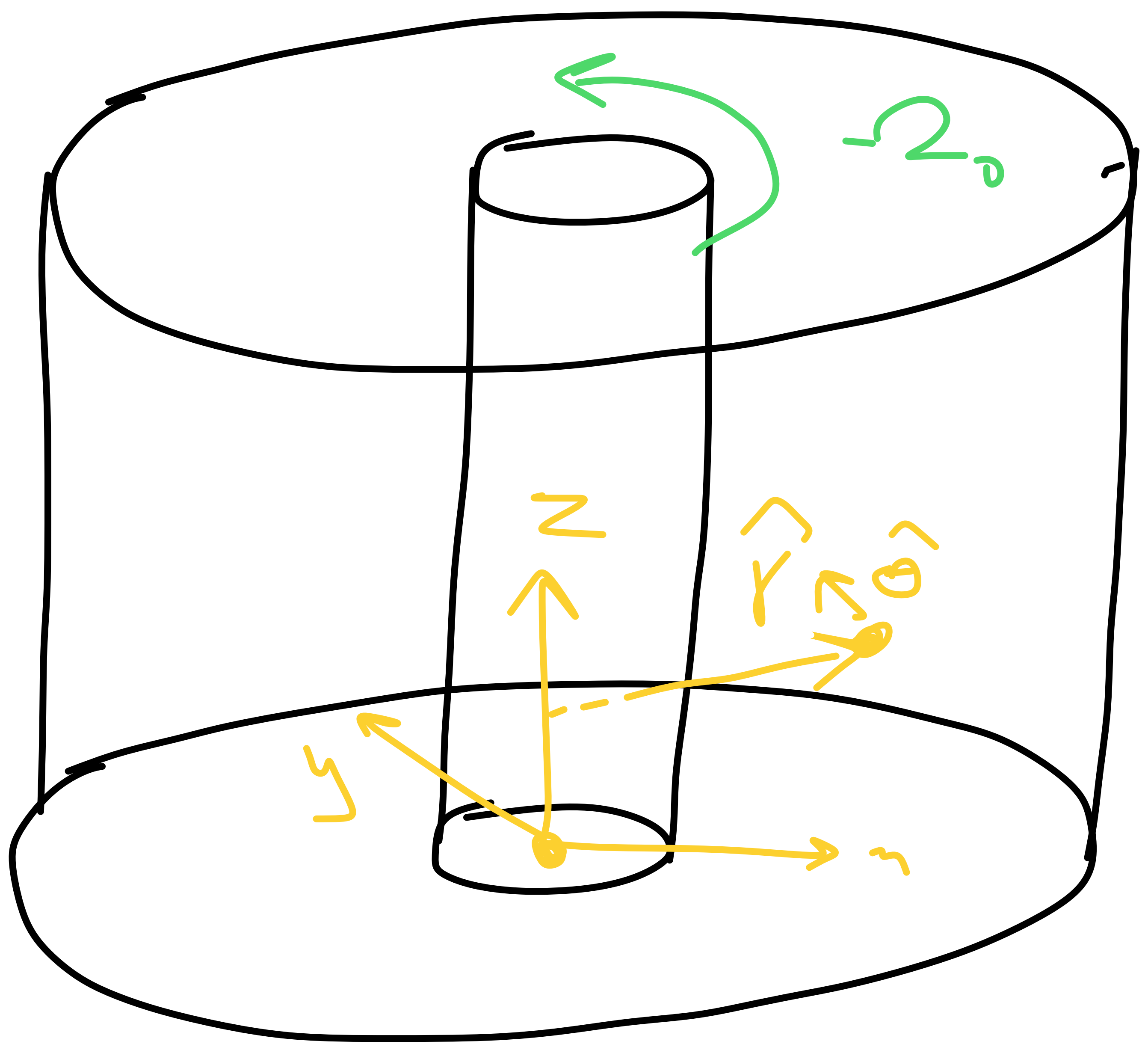


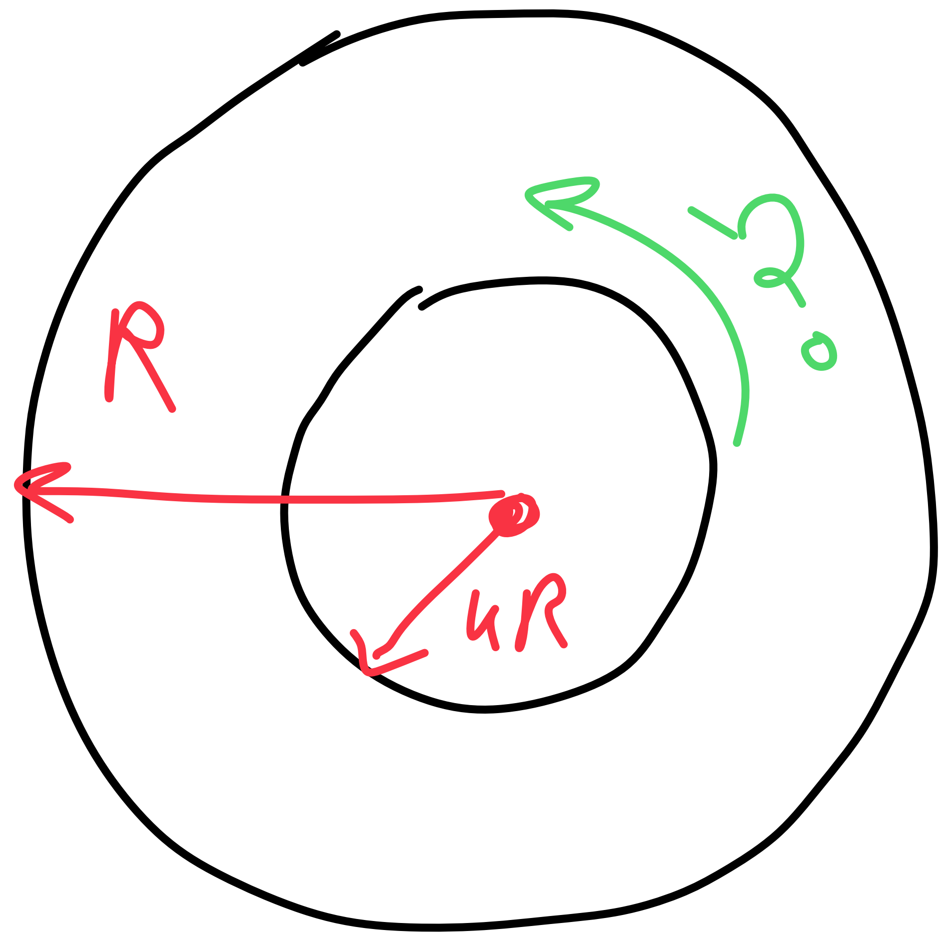
(c)



B) Briefly exploring Tangential Annular Flow (full solution in textbook 3.4.3)

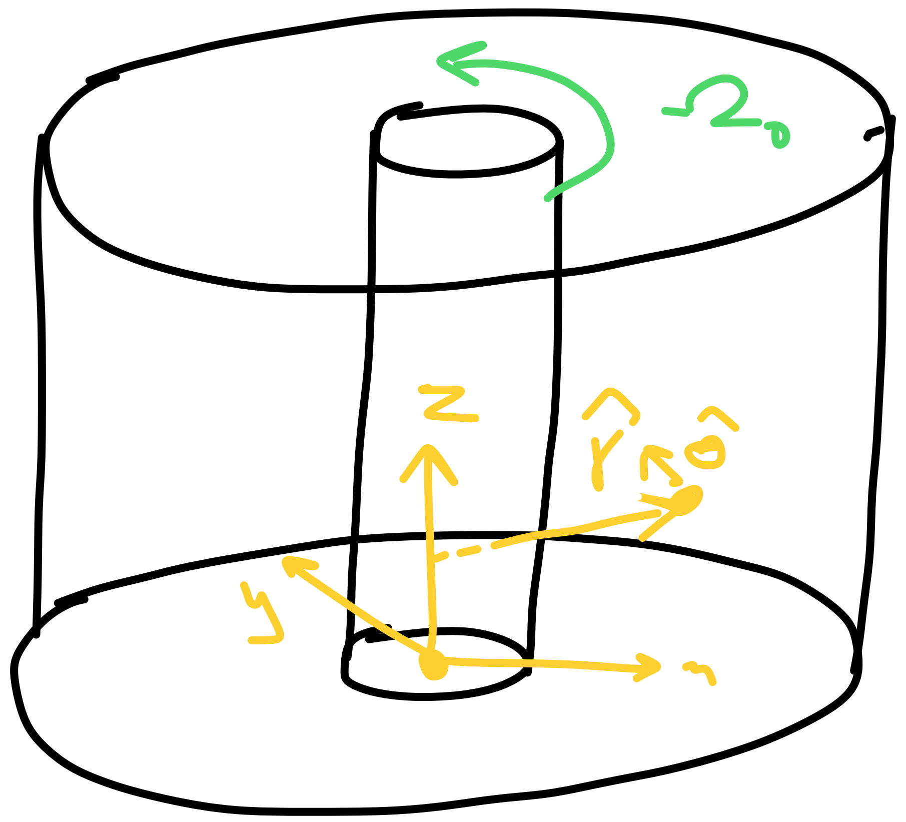
Tangential annular flow between two concentric cylinders: Here we have a S.S, laminar flow of a Newtonian fluid in a cylindrical system placed vertically. Let the flow be well-developed. The flow is running due to the rotation of the inner cylinder with the angular velocity give bellow:

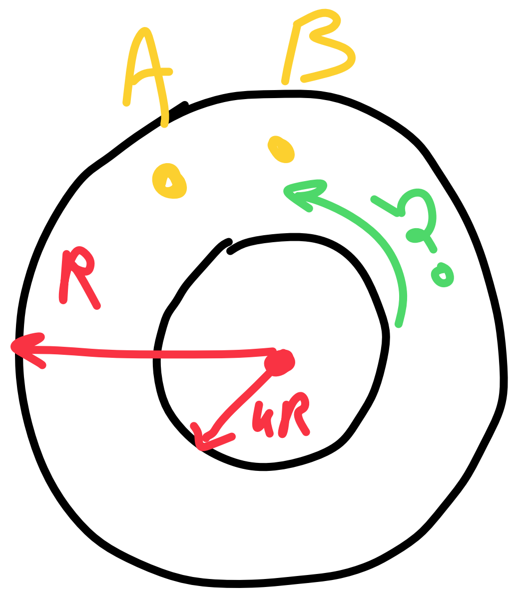




Q: Find the velocity profile of the fluid.

Steps :





We are interested in

$$\vec{V}(r,\theta,z) \overset{\text{general form}}{=} \begin{cases} v_r(r,\theta,z)\,\hat{r} \\ v_\theta(r,\theta,z)\,\hat{\theta} \\ v_z(r,\theta,z)\,\hat{z} \end{cases}$$

⓪ S.S $\Rightarrow \frac{\partial}{\partial t} = 0$

① Laminar : $\vec{V} = V_\theta\,\hat{\theta}$

② symmetry in $\hat{z}$: $\vec{V}(r,\theta,\cancel{z}) \rightarrow \vec{V}(r,\theta) = V_\theta(r,\theta)\,\hat{\theta}$

③ Symmetry in $\hat{\theta}$ $\Rightarrow V_A = V_B \Rightarrow V_\theta(r,\cancel{\theta})$

④ Symmetry in $\hat{\theta}$ $\longrightarrow P_A = P_B \Rightarrow P(r,\cancel{\theta},z)$

⑤ no-slip B.C $\begin{cases} V_\theta(r=R) = 0 \\ V_\theta(r=kR) = -\Omega_0 \end{cases}$

⑥ Solving N-S:

we had $\vec{v} = \begin{vmatrix} 0 & \hat{r} \\ v_\theta(r) & \hat{\theta} \\ 0 & \hat{z} \end{vmatrix}$, $P(r,z)$

Cylindrical coordinates

r direction

(a) $$\rho\left(\underset{0}{\frac{\partial v_r}{\partial t}} + \underset{0}{v_r \frac{\partial v_r}{\partial r}} + \underset{0}{\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}} - \frac{v_\theta^2}{r} + \underset{0}{v_z \frac{\partial v_r}{\partial z}}\right) = -\frac{\partial p}{\partial r} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\underset{0}{\frac{\partial (r v_r)}{\partial r}}\right) + \underset{0}{\frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2}} - \underset{0}{\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}} + \underset{0}{\frac{\partial^2 v_r}{\partial z^2}}\right] + \underset{0}{\rho g_r} \rightarrow -\rho\frac{v_\theta^2}{r} = -\frac{\partial p}{\partial r}$$

θ direction

(b) $$\rho\left(\underset{0}{\frac{\partial v_\theta}{\partial t}} + v_r \frac{\partial v_\theta}{\partial r} + \underset{0}{\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}} + \underset{0}{\frac{v_r v_\theta}{r}} + \underset{0}{v_z} \frac{\partial v_\theta}{\partial z}\right) = -\underset{0}{\frac{1}{r}\frac{\partial p}{\partial \theta}} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial (r v_\theta)}{\partial r}\right) + \underset{0}{\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2}} + \underset{0}{\frac{2}{r^2}\frac{\partial v_r}{\partial \theta}} + \underset{0}{\frac{\partial^2 v_\theta}{\partial z^2}}\right] + \underset{0}{\rho g_\theta} \rightarrow \mu\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_\theta)\right)\right) = 0$$

z direction

(c) $$\rho\left(\underset{0}{\frac{\partial v_z}{\partial t}} + \underset{0}{v_r \frac{\partial v_z}{\partial r}} + \underset{0}{\frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta}} + \underset{0}{v_z \frac{\partial v_z}{\partial z}}\right) = -\frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\underset{0}{\frac{\partial v_z}{\partial r}}\right) + \underset{0}{\frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2}} + \underset{0}{\frac{\partial^2 v_z}{\partial z^2}}\right] + \rho g_z \rightarrow \frac{\partial p}{\partial z} = \rho g_z$$

full solution in textbook 3.4.3

C) Reynolds number: A deeper look

remember N-S equation:

$$\rho \frac{DV}{Dt} = \rho \vec{g} - \vec{\nabla} P + \mu \nabla^2 V$$

$$\Rightarrow \rho\left(\frac{\partial \vec{V}}{\partial t} + \underbrace{(\vec{V}\cdot\nabla)\vec{V}}_{\substack{\text{non linear } (v^2) \\ \text{and unstable} \\ A}}\right) = (\rho\vec{g} - \vec{\nabla}P) + \underbrace{\mu \nabla^2 V}_{\substack{\text{Viscose and linear} \\ B}}$$

let's compare A and B magnitudes: $\nabla \rightarrow \frac{d}{\partial x} \rightarrow \frac{1}{l}$

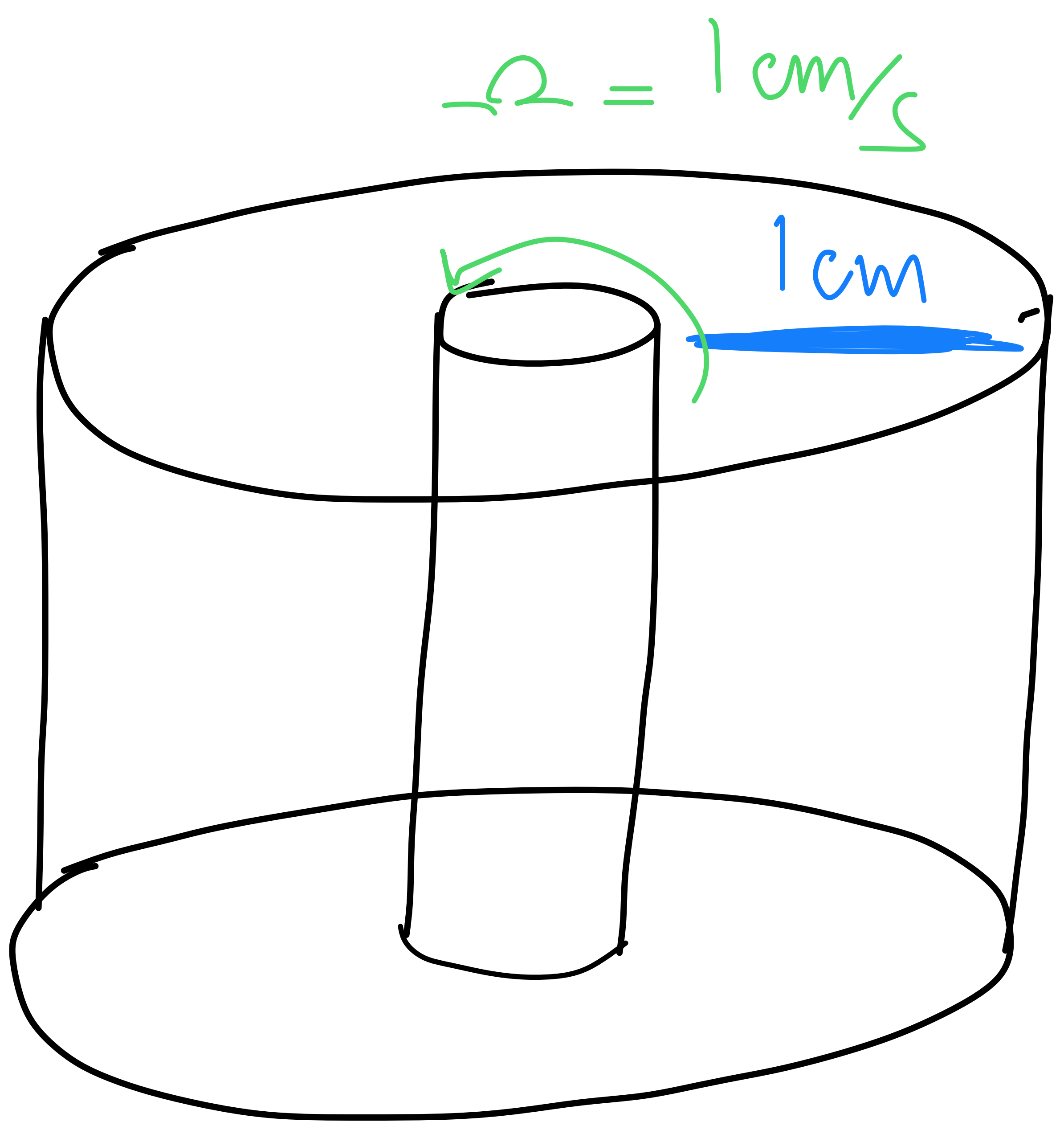
$$\frac{|A|}{|B|} \propto \frac{\rho V \frac{V}{l}}{\mu \frac{V}{l^2}} = \frac{\rho V l}{\mu} \equiv \text{Reynold's number}$$

big: $\gg 1$: A is dominant system turbolant

small: $\ll 1$: B is dominant System laminar

Activity: What is the Re for the setup bellow:

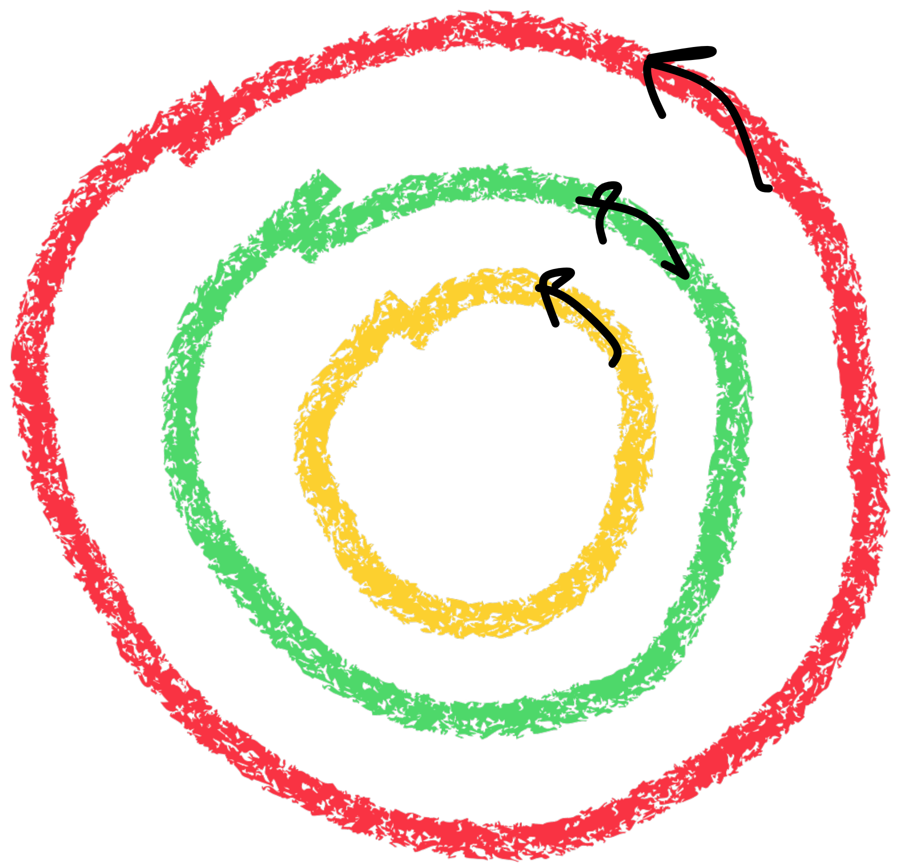
We have a chamber, inner radius 3 cm, outer radius 4 cm, the inner rod's surface is rotating moving with approximately 1 cm/s. The system is filled with corn syrup!



$\mu = 1500$
corn syrup

$$Re \cong \frac{1000 \frac{kg}{m^3} \times \frac{1}{100} \frac{m}{s} \times \frac{1}{100} m}{1500 \times 10^{-3} \frac{kg}{m\,s}} = 0.01$$

Low Re ⇒ viscose, Laminar.



layers

End