Lecture 3: Coordinate systems

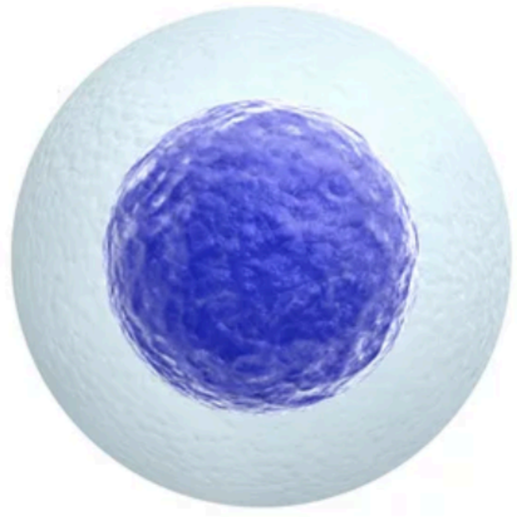
In the previous lecture we learned about vector derivatives:

	grad	div	Curl	Laplacian
Operator	$\vec{\nabla} f$	$\vec{\nabla} \cdot \vec{f}$	$\vec{\nabla} \times \vec{f}$	$\nabla^2 f$
formula in x,y,z	$\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$	$\frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + \frac{\partial f_z}{\partial z}$	$\left(\frac{\partial f_z}{\partial y} - \frac{\partial f_y}{\partial z}\right)\hat{i} - \left(\frac{\partial f_z}{\partial x} - \frac{\partial f_x}{\partial z}\right)\hat{j} + \left(\frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y}\right)\hat{k}$	$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$

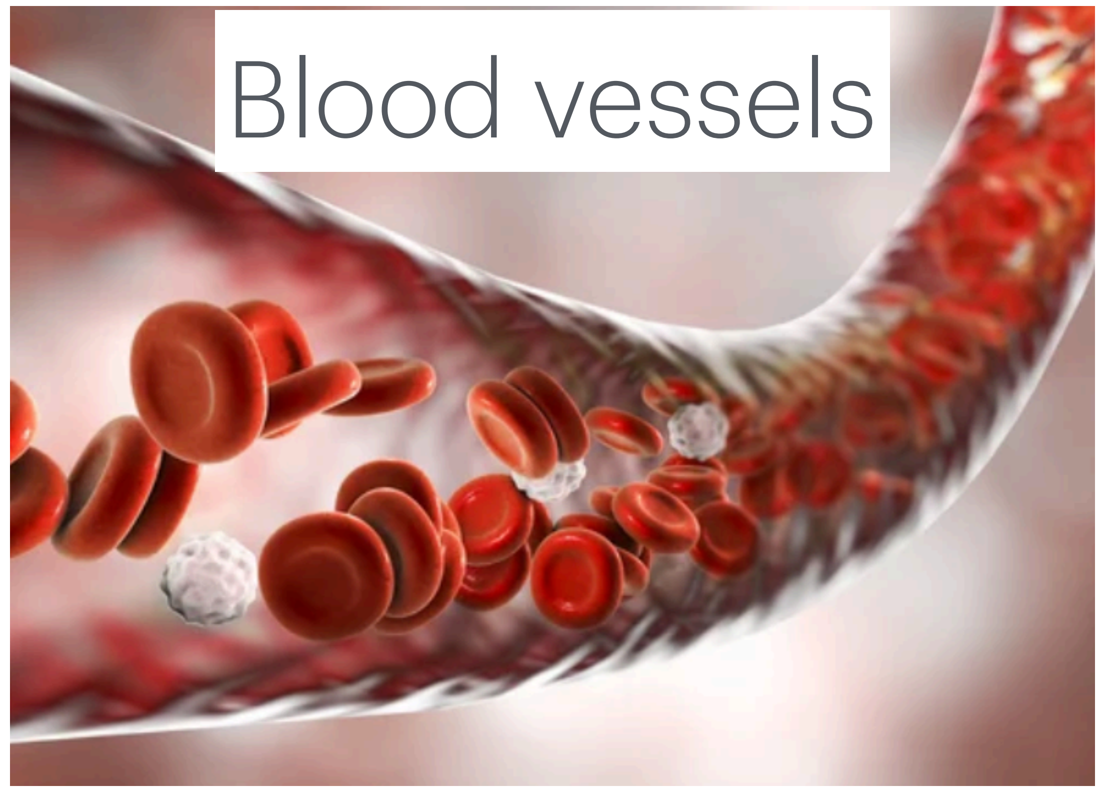
These exact formulas are only valid in Cartesian coordinate, but in many of our applications, spherical or cylindrical coordinates are preferred.

Many biological structures are naturally curved or exhibit symmetry that aligns more easily with **spherical** or **cylindrical** coordinates. This is also true for engineered devices, for example:

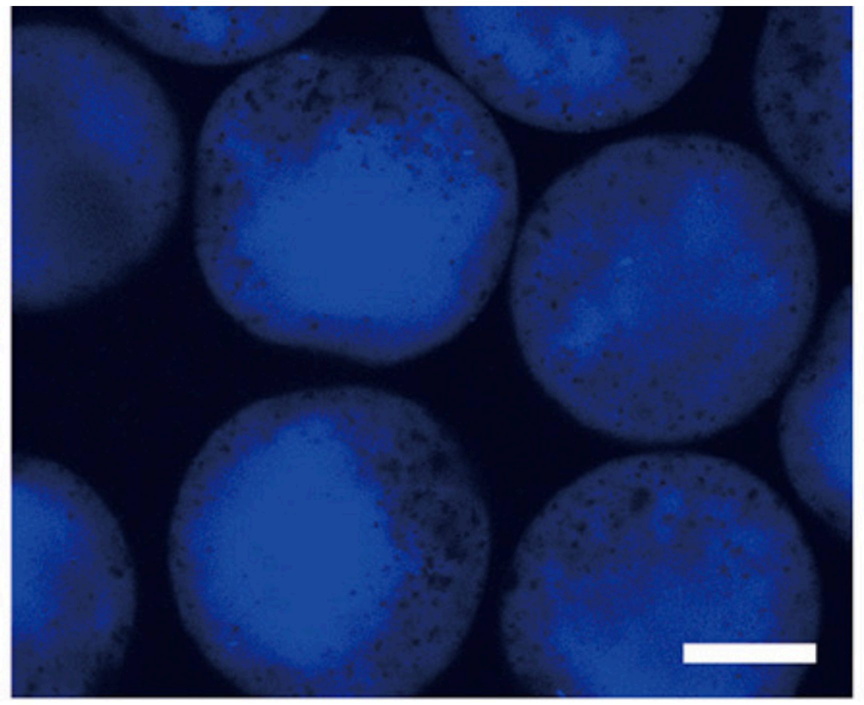
Cells



Blood vessels



Drug delivery mechanisms



Kubota, Takeshi, et al. "Ultrasound-triggered on-demand drug delivery using hydrogel microbeads with release enhancer." *Materials & Design* 203 (2021): 109580.

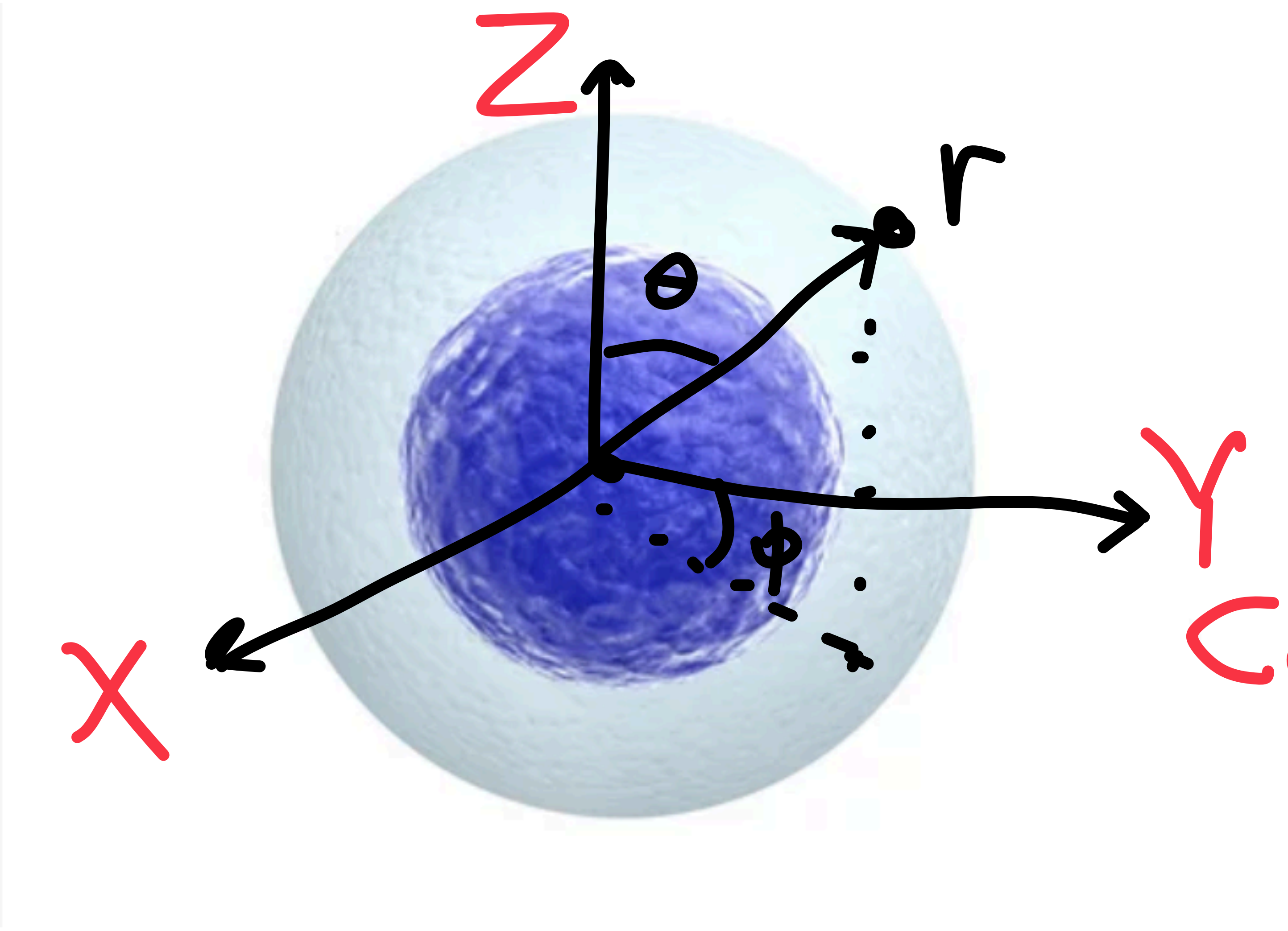
Bioreactors

Choosing the right coordinate system is like choosing the right language: it simplifies the problem and makes the underlying structure clearer. Just as using the appropriate language allows us to communicate complex ideas more efficiently, selecting the correct coordinate system aligns with the natural geometry of the system we are studying.

example :

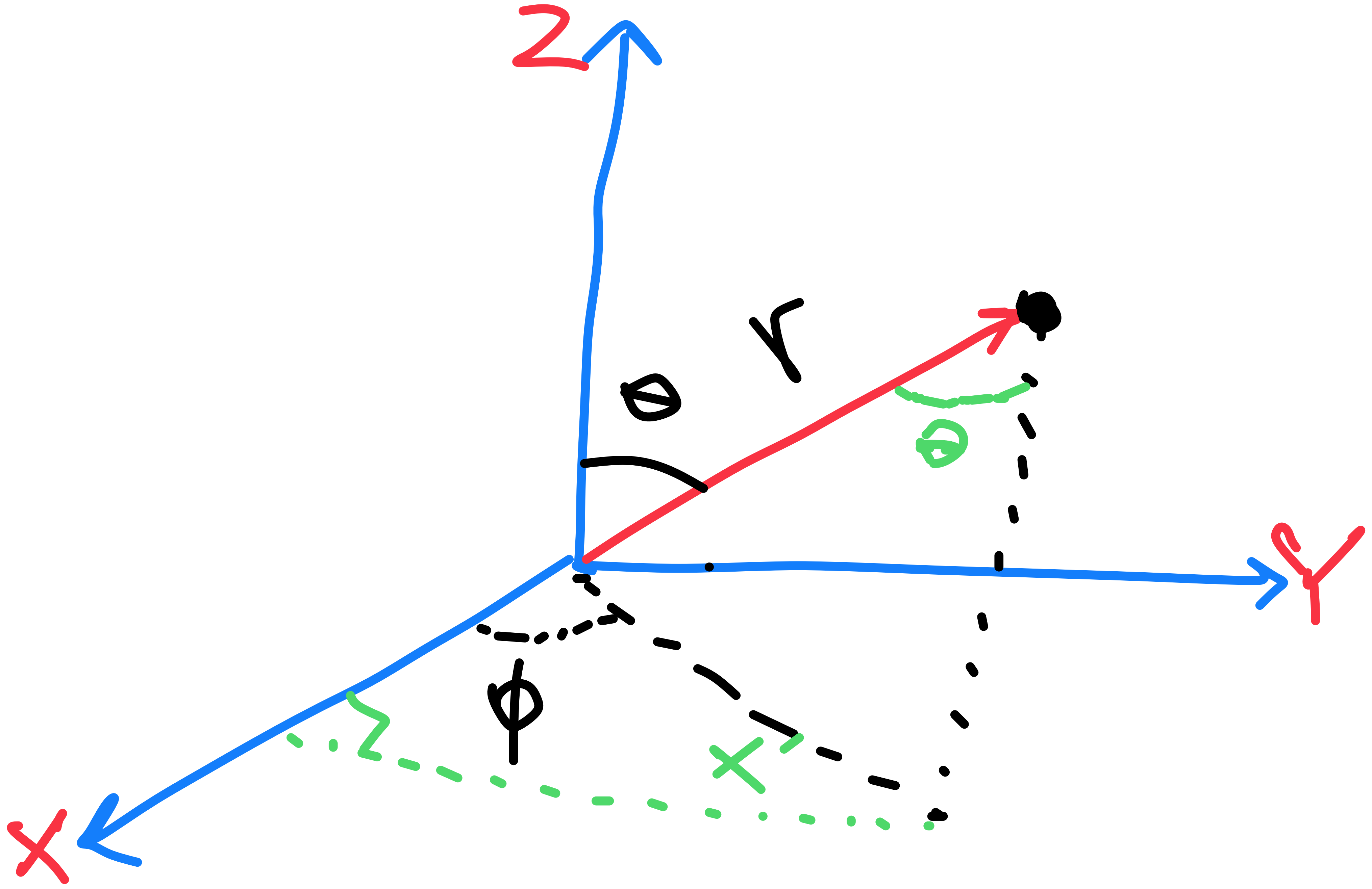
Cytokine Concentration Field (C) on the cell membrane

spherical: $\begin{cases} C(r,\theta,\phi,t) = C_0 e^{-\frac{Dt}{\alpha}} r \\ r = R \end{cases}$



Cartesian $\begin{cases} C(x,y,z,t) = C_0 e^{-\frac{Dt}{\alpha}} \sqrt{x^2+y^2+z^2} \\ \text{and} \\ x^2+y^2+z^2 = R \end{cases}$ → Solve!....

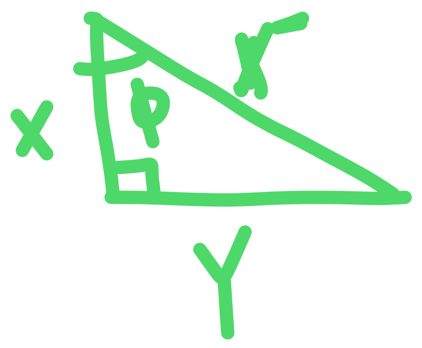
A) Spherical coordinate:



$\vec{V}(x, y, z)$

spherical → cartesian

$\sin\theta = \dfrac{x'}{r}$



$\cos\phi = \dfrac{X}{X'} = \dfrac{X}{r\sin\theta}$

$\Rightarrow X = r\sin\theta\cos\phi$ (1)

$\sin\phi = \dfrac{Y}{r\sin\theta}$

$Y = r\sin\theta\sin\phi$ (2)

$Z = r\cos\theta$ (3)

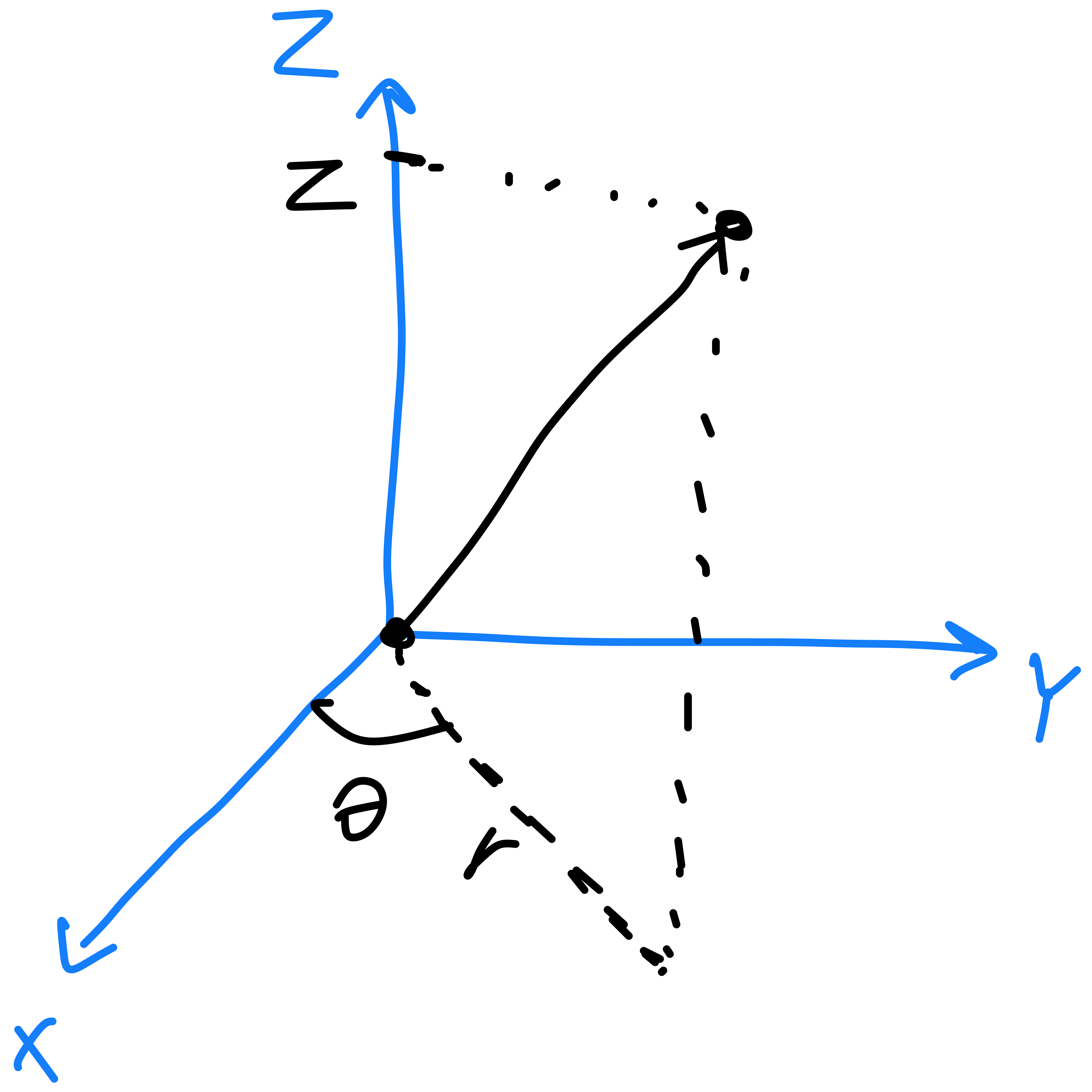
cartesian → spherical

$r = \sqrt{x^2 + y^2 + z^2}$

$\theta = \tan^{-1}\left(\dfrac{\sqrt{x^2+y^2}}{z}\right)$

$\phi = \tan^{-1}\dfrac{y}{x}$

B) Cylindrical Coordinate



Cyl → Cartesian

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$z = z$$

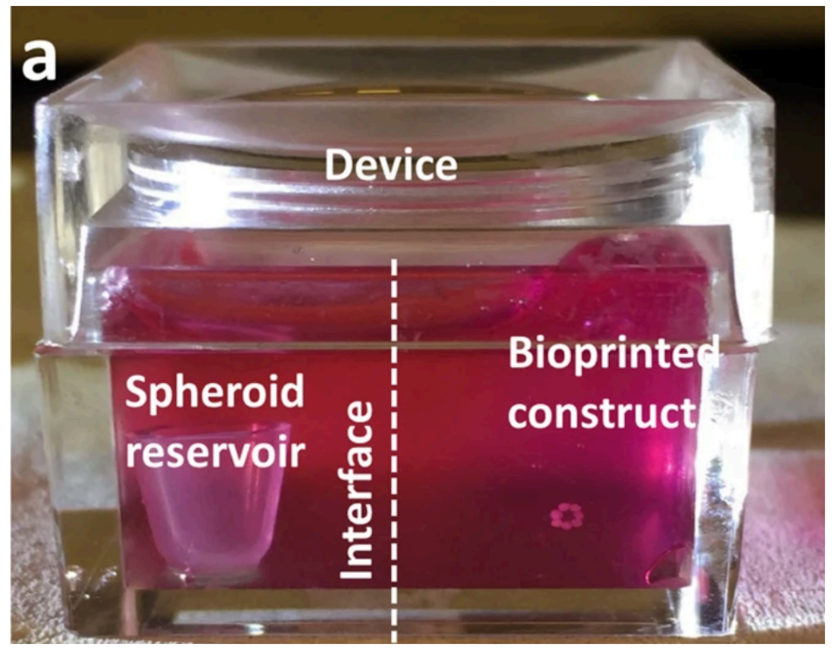
Cartesian → Cyl

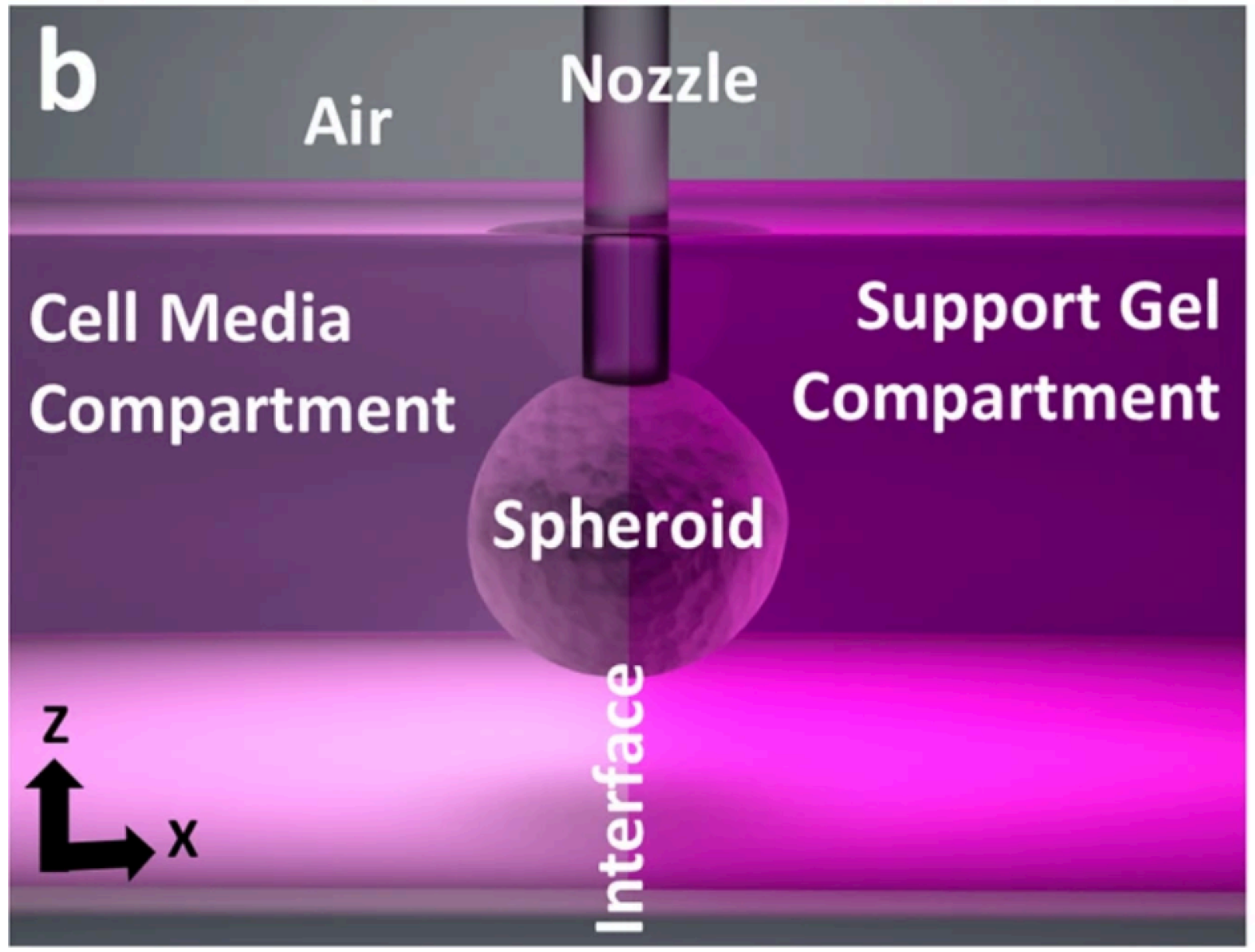
$$r = \sqrt{x^2 + y^2}$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$$z = z$$

Example 1: Scientists created a pre-fabricated tissue spheroids. They immersed these spheroids in a medium consisting of two compartments: Cell Media and Support Gel. This setup helps to keep the cells alive while also allowing for mechanical experiments on the spheroids. The shape of the spheroids is given below:

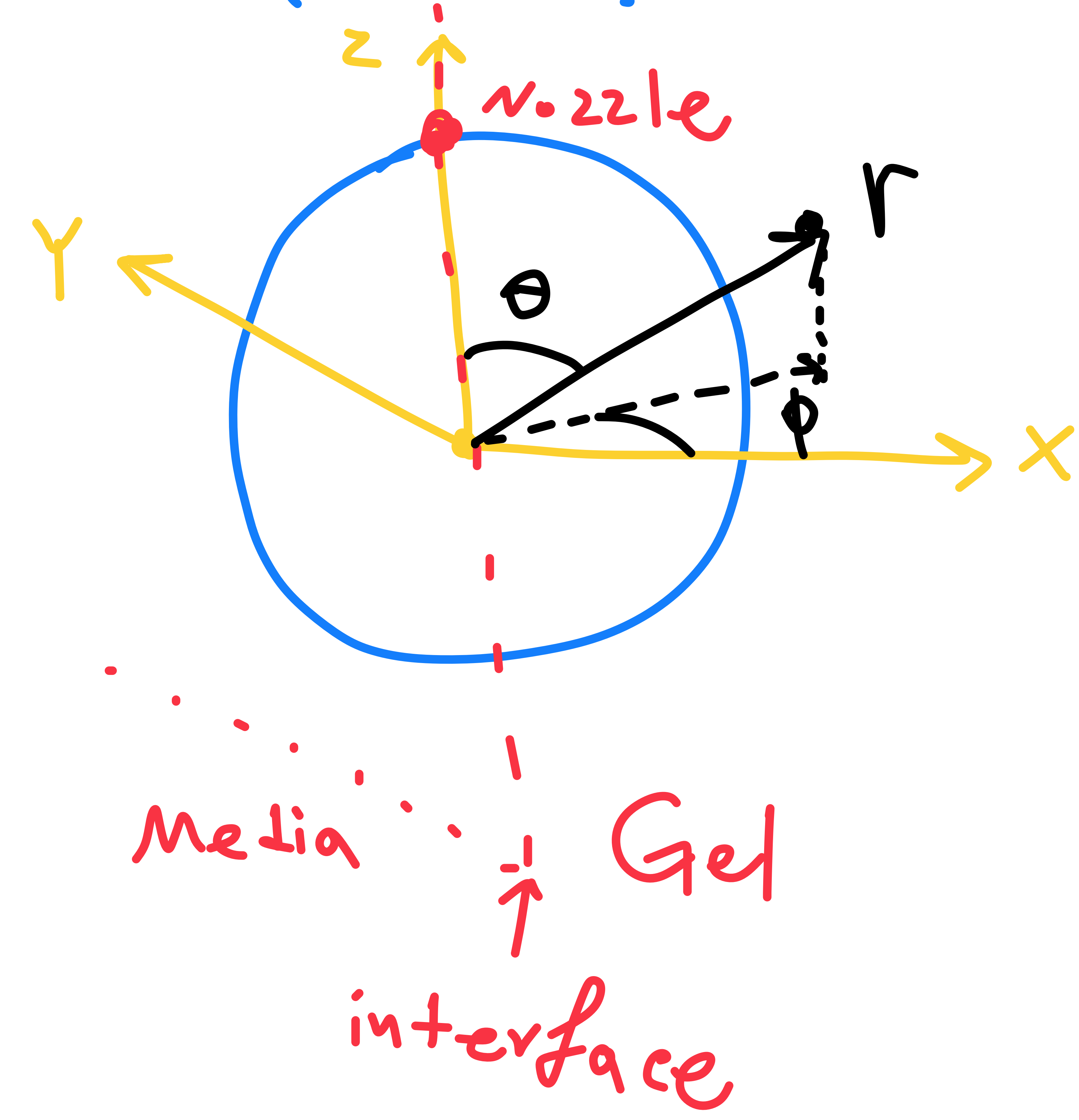




Adapted from: Ayan, B., Celik, N., Zhang, Z. et al. Aspiration-assisted freeform bioprinting of pre-fabricated tissue spheroids in a yield-stress gel. Commun Phys 3, 183 (2020). https://doi.org/10.1038/s42005-020-00449-4

Question 1: Write down the equation in spherical coordinates. Where is the spheroid? Where is the Cell Media? Where is the Support Gel? Where is the interface? (Assume that the center of the coordinate system is at the center of the spheroid.)

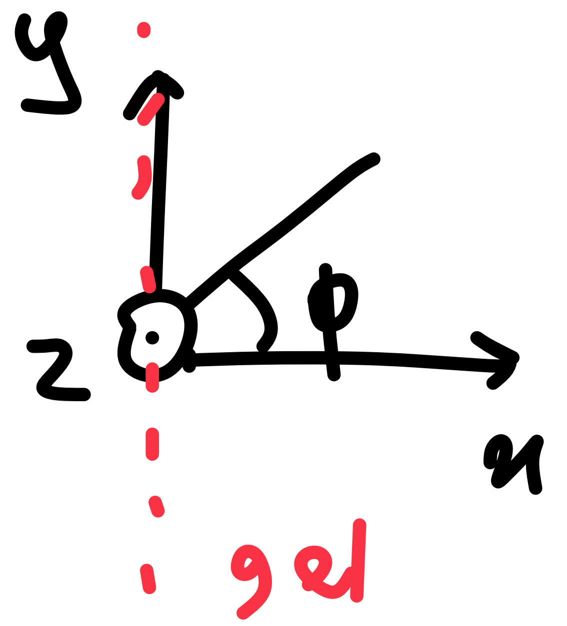
Solution:



① Spheroid: $r < R_{spheroid}$

② Nozzle: $\begin{cases} r = R_{spheroid} \\ \theta = 0 \end{cases}$

③ Gel:



$-\frac{\pi}{2} < \phi < \frac{\pi}{2}$

④ media: $\frac{\pi}{2} < \phi < \frac{3\pi}{2}$

⑤ interface: $\phi = \frac{\pi}{2}$ or $\phi = \frac{3\pi}{2}$

Scalar field

Note 1: The extremum (minimum or maximum) of $\boxed{f(u,v,w)}$ with respect to any variable is found by setting the partial derivative of the function with respect to that variable to zero. This gives the necessary condition for an extremum. Thus, to find the extremum of f(u,v,w) based on any variable, say u, we need to solve the following equation:

$$\frac{\partial f(u,v,w)}{\partial u} = 0$$

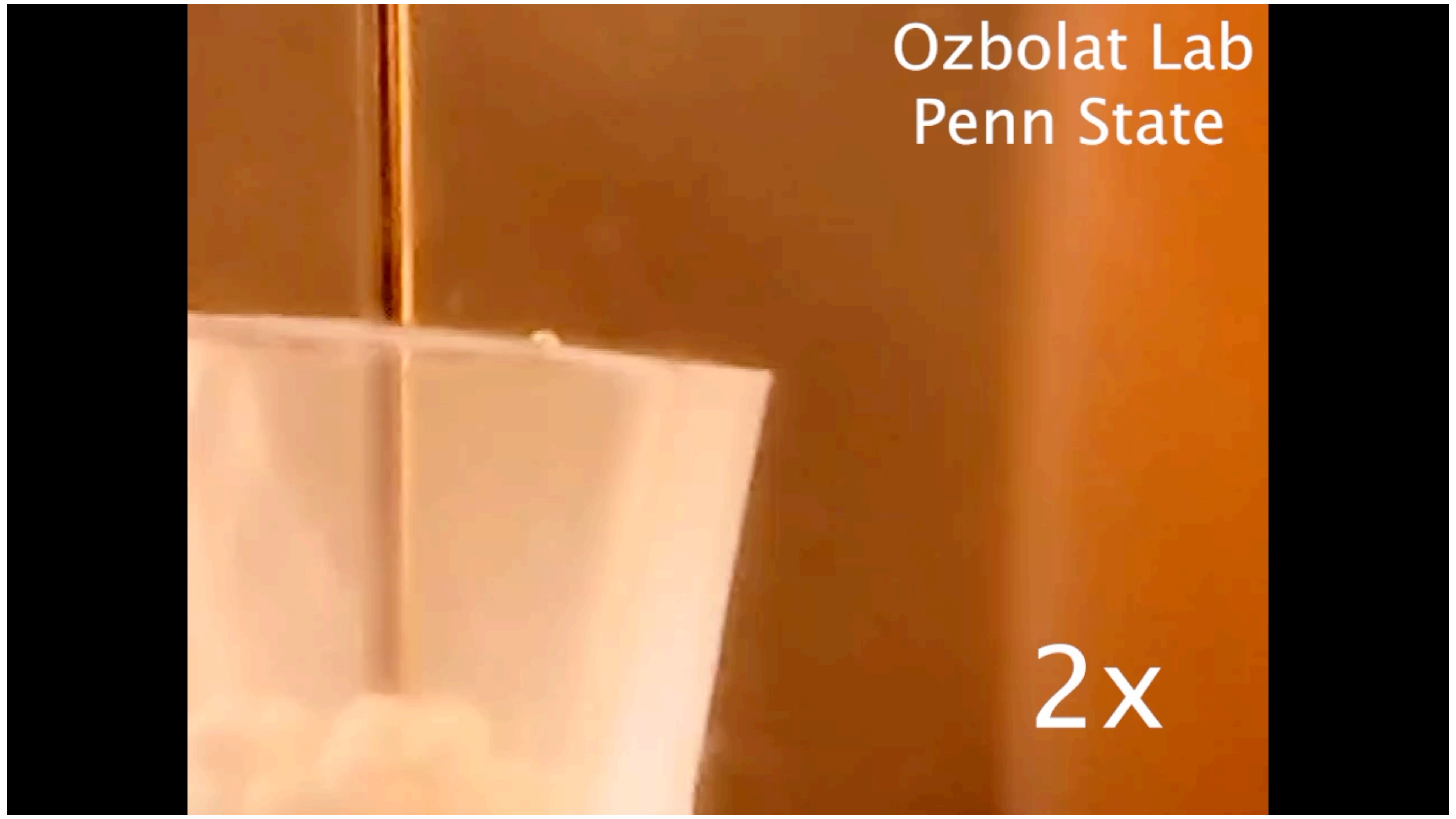
Similarly, if we want to find the extremum with respect to v or w, we solve:

$$\frac{\partial f}{\partial v} = 0 \quad , \quad \frac{\partial f}{\partial w} = 0$$

This approach applies regardless of the coordinate system used for u, v, and w, whether it is Cartesian, cylindrical, spherical, or any other system. The process of finding the extremum by setting the partial derivatives equal to zero remains valid because partial derivatives only depend on how f(u,v,w)f(u,v,w) changes with respect to each variable locally, not on the specific nature of the coordinate system.

Question 2 (from example 1): Next, they used a nozzle to grab the spheroid and transfer it into the medium. They estimated the force function (through some biophysical calculations). Below is a simplified version of the force. Which part of the 'spheroid's surface' is experiencing the maximum amount of force? How much is that force?

$$F(r,\phi,\theta) = C_0 R \cos^2\frac{\theta}{2} + C_1(\sin 2\phi)$$



Solution:

Step 1: compute partial derivatives: $\frac{\partial f}{\partial \theta}$ and $\frac{\partial f}{\partial \phi}$

Step 2: find the extremums: $\frac{\partial f}{\partial \theta}=0$, $\frac{\partial f}{\partial \phi}=0$

Step 3: making sure extremums are maximums (student's practice)

$$F = C_0 R \cos^2\frac{\theta}{2} + C_1 \sin 2\phi$$

1: $$\frac{\partial F}{\partial \theta} = 2 C_0 R \cos\frac{\theta}{2} \times \sin\frac{\theta}{2} \times \frac{1}{2} + 0 = -\frac{RC_0}{2}\left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)$$

$$= -\frac{RC_0}{2}\sin\theta$$

$$\frac{\partial f}{\partial \phi} = 0 + 2C_1 \cos 2\phi$$

$$2: \frac{\partial f}{\partial \theta} = 0 \Rightarrow -\frac{C_0 R}{2} \sin\theta = 0 \Rightarrow \theta = 0 \text{ or } \pi$$

$$\frac{\partial f}{\partial \phi} = 0 \Rightarrow 2C_1 \cos 2\phi = 0 \Rightarrow$$

$$2\phi = \frac{\pi}{2} + n\pi \Rightarrow \phi = \frac{\pi}{4} + \frac{n\pi}{2}$$

<u>end of solution</u>

What about derivative of fields?

we know how these are in Cartesian: $\vec{\nabla} f$, $\vec{\nabla} \cdot \vec{f}$, $\nabla^2 f$

now we want to find these in spherical coordinates; note that for example:

$$\nabla f = \frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y} + \frac{\partial f}{\partial z}\hat{z}$$

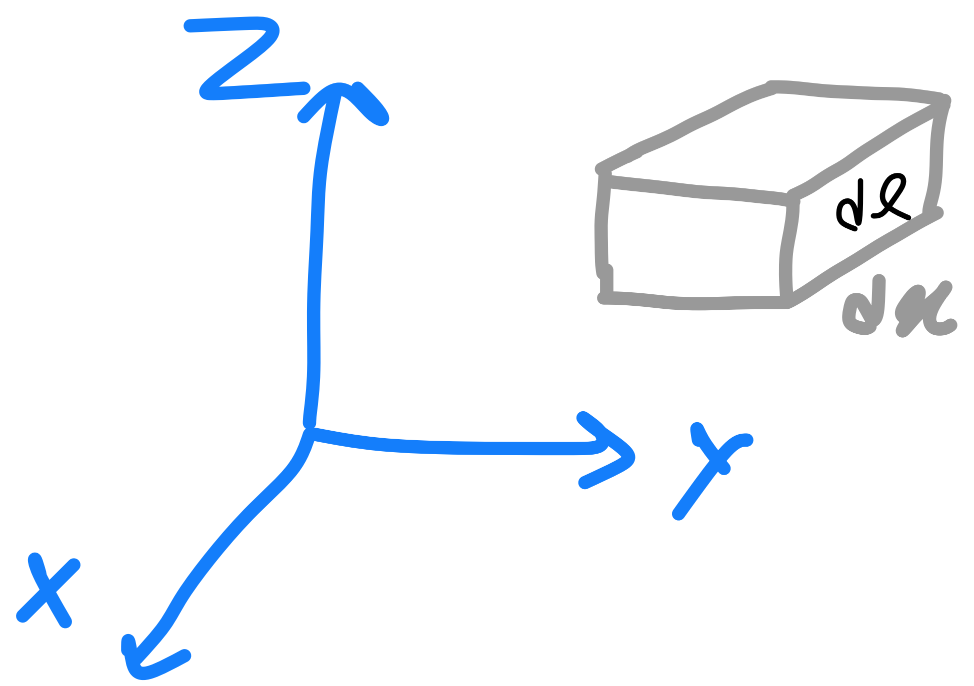
✓ this is our starting point

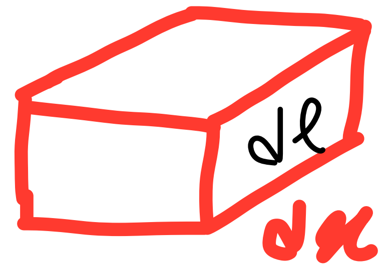
$$\nabla f \neq \frac{\partial f}{\partial r}\hat{r} + \frac{\partial f}{\partial \phi}\hat{\phi} + \frac{\partial f}{\partial \theta}\hat{\theta}$$

wrong!

When dealing with these vector calculus operations, the formulas in Cartesian coordinates are straightforward as shown above. However, in cylindrical and spherical coordinates (or any other coordinate system), the expressions for gradient, divergence, and curl need to be modified to reflect the geometry and metric of the specific coordinate system.

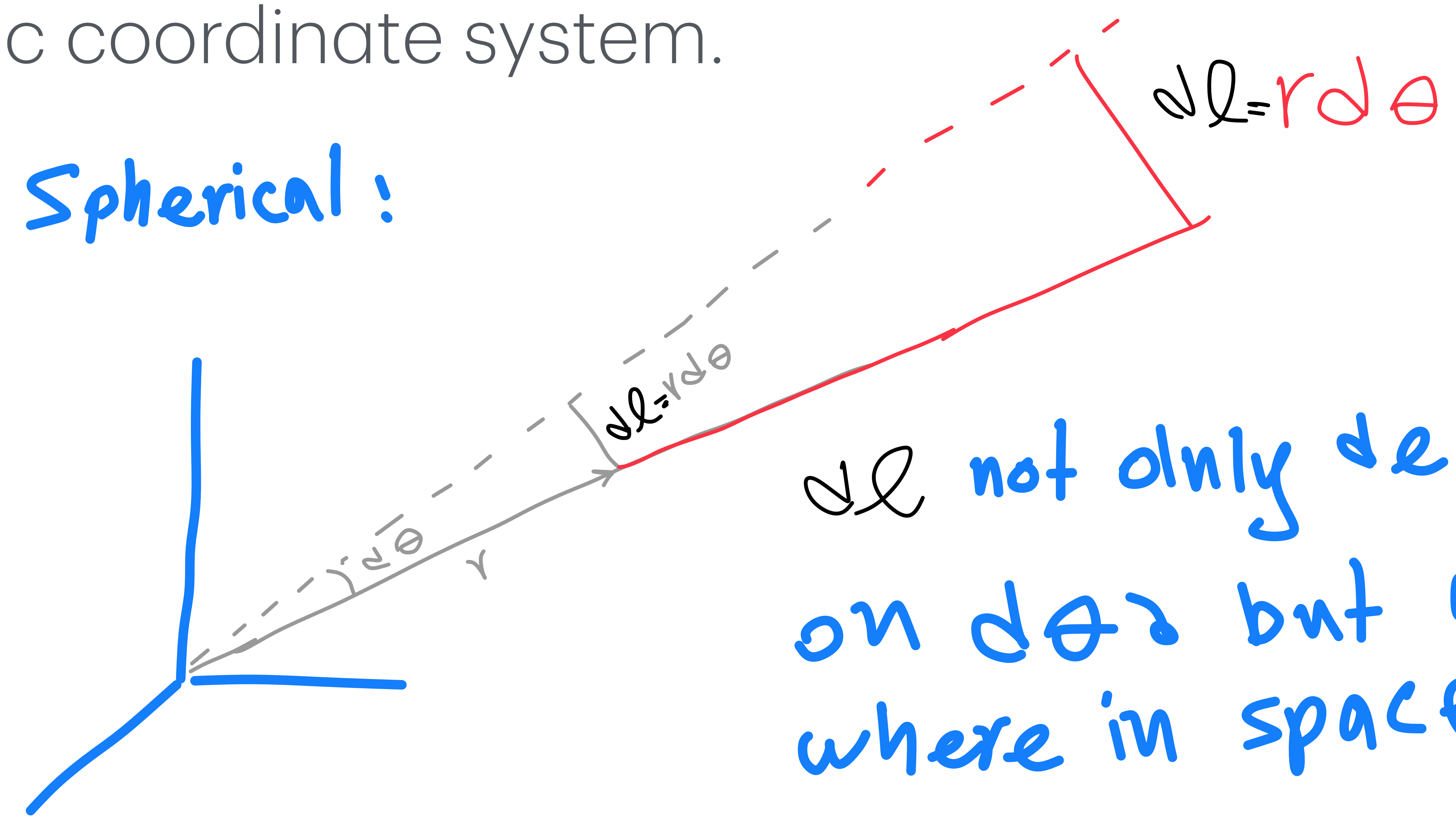
example: Cartesian :





dl is same regardless of where in space (x, y, z) and only depends on variation dx

Spherical :



dl not only depends on dθ but also where in space (r)

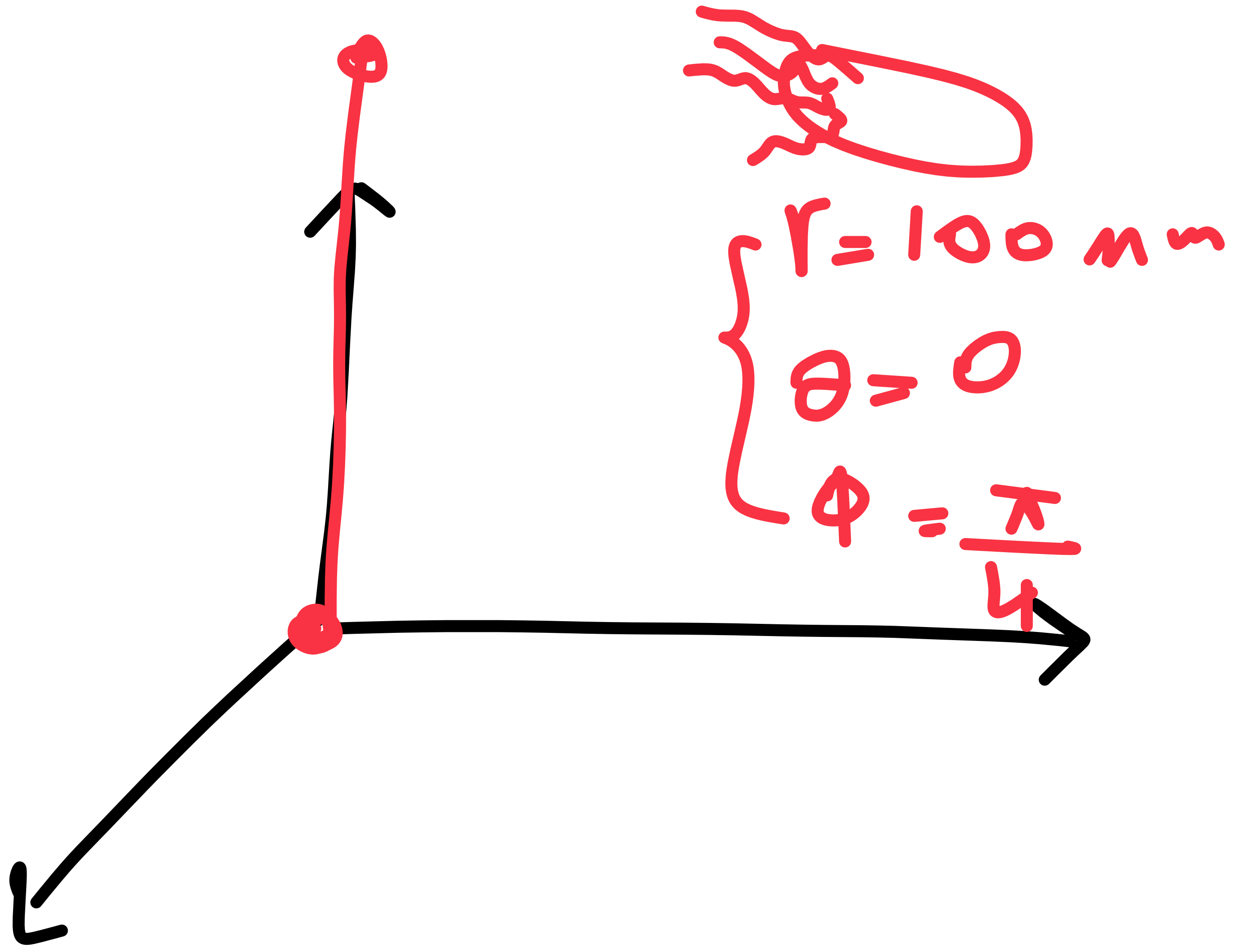
In the next slide, you will find the formulas for the gradient, divergence, curl, and Laplacian in three different coordinate systems: Cartesian, spherical, and cylindrical.

The detailed derivations of these formulas are not the focus here. What is important is understanding how to use these formulas and remembering that the gradient, divergence, curl, and Laplacian take different forms in cylindrical and spherical coordinates. Later in this lecture, we will go through some examples to demonstrate how to apply these formulas.

Grad, Div, Curl and the Laplacian

	Cartesian Coordinates	Cylindrical Coordinates	Spherical Coordinates
Conversion to Cartesian Coordinates		$x = \rho\cos\varphi \quad y = \rho\sin\varphi \quad z = z$	$x = r\cos\varphi\sin\theta \quad y = r\sin\varphi\sin\theta$ $z = r\cos\theta$
Vector A	$A_x i + A_y j + A_z k$	$A_\rho\widehat{\rho} + A_\varphi\widehat{\varphi} + A_z\widehat{z}$	$A_r\widehat{r} + A_\theta\widehat{\theta} + A_\varphi\widehat{\varphi}$
Gradient $\nabla\phi$	$\frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}k$	$\frac{\partial\phi}{\partial\rho}\widehat{\rho} + \frac{1}{\rho}\frac{\partial\phi}{\partial\varphi}\widehat{\varphi} + \frac{\partial\phi}{\partial z}\widehat{z}$	$\frac{\partial\phi}{\partial r}\widehat{r} + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\widehat{\theta} + \frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\varphi}\widehat{\varphi}$
Divergence $\nabla\cdot A$	$\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$	$\frac{1}{\rho}\frac{\partial(\rho A_\rho)}{\partial\rho} + \frac{1}{\rho}\frac{\partial A_\varphi}{\partial\varphi} + \frac{\partial A_z}{\partial z}$	$\frac{1}{r^2}\frac{\partial(r^2 A_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial A_\theta\sin\theta}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial A_\varphi}{\partial\varphi}$
Curl $\nabla\times A$	$\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$	$\begin{vmatrix} \frac{1}{\rho}\widehat{\rho} & \widehat{\varphi} & \frac{1}{\rho}\widehat{z} \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\varphi} & \frac{\partial}{\partial z} \\ A_\rho & \rho A_\varphi & A_z \end{vmatrix}$	$\begin{vmatrix} \frac{1}{r^2\sin\theta}\widehat{r} & \frac{1}{r\sin\theta}\widehat{\theta} & \frac{1}{r}\widehat{\varphi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\varphi} \\ A_r & rA_\theta & rA_\varphi\sin\theta \end{vmatrix}$
Laplacian $\nabla^2\phi$	$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$	$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\phi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\phi}{\partial\varphi^2} + \frac{\partial^2\phi}{\partial z^2}$	$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\phi}{\partial\varphi^2}$

Example 2: In an environment with a nutrient source, the concentration of the nutrient is given by the function below. Multiple bacteria are present in this environment. We know that these bacteria move towards higher concentrations of the nutrient by following the gradient. Determine the direction in which the given bacterium will move:



$$\begin{cases} C(r,\theta,\phi) = C_0\left(\dfrac{r}{R_0}\right)^2 e^{-\frac{r^2}{R^2}} \sin\theta \\ C_0 = 100 \ \text{ng/mL} \\ R_0 = 100 \ \text{nm} \end{cases}$$

Solution:

Grad, Div, Curl and the Laplacian

	Cartesian Coordinates	Cylindrical Coordinates	Spherical Coordinates
Conversion to Cartesian Coordinates		$x = \rho\cos\varphi \quad y = \rho\sin\varphi \quad z = z$	$x = r\cos\varphi\sin\theta \quad y = r\sin\varphi\sin\theta \quad z = r\cos\theta$
Vector A	$A_x i + A_y j + A_z k$	$A_\rho\hat{\rho} + A_\varphi\hat{\varphi} + A_z\hat{z}$	$A_r\hat{r} + A_\theta\hat{\theta} + A_\varphi\hat{\varphi}$
Gradient $\nabla\phi$	$\frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}k$	$\frac{\partial\phi}{\partial\rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial\phi}{\partial\varphi}\hat{\varphi} + \frac{\partial\phi}{\partial z}\hat{z}$	$\boxed{\frac{\partial\phi}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\varphi}\hat{\varphi}}$
Divergence $\nabla\cdot A$	$\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$	$\frac{1}{\rho}\frac{\partial(\rho A_\rho)}{\partial\rho} + \frac{1}{\rho}\frac{\partial A_\varphi}{\partial\varphi} + \frac{\partial A_z}{\partial z}$	$\frac{1}{r^2}\frac{\partial(r^2 A_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial A_\theta\sin\theta}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial A_\varphi}{\partial\varphi}$
Curl $\nabla\times A$	$\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$	$\begin{vmatrix} \frac{1}{\rho}\hat{\rho} & \hat{\varphi} & \frac{1}{\rho}\hat{z} \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\varphi} & \frac{\partial}{\partial z} \\ A_\rho & \rho A_\varphi & A_z \end{vmatrix}$	$\begin{vmatrix} \frac{1}{r^2\sin\theta}\hat{r} & \frac{1}{r\sin\theta}\hat{\theta} & \frac{1}{r}\hat{\varphi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\varphi} \\ A_r & rA_\theta & rA_\varphi\sin\theta \end{vmatrix}$
Laplacian $\nabla^2\phi$	$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$	$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\phi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\phi}{\partial\varphi^2} + \frac{\partial^2\phi}{\partial z^2}$	$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\phi}{\partial\varphi^2}$

$$\vec{\nabla C} = \frac{\partial C}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial C}{\partial\theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial C}{\partial\phi}\hat{\phi}$$

$$\begin{cases} \dfrac{\partial C}{\partial r} = C_0 \sin\theta \, \dfrac{2r}{R^2} e^{-\left(\frac{r}{R}\right)^2} \left(1-\left(\dfrac{r}{R}\right)^2\right) \\[2ex] \dfrac{\partial C}{\partial \theta} = C_0 \, \dfrac{r^2}{R_0^2} \, e^{-\frac{r^2}{R^2}} \cos\theta \\[2ex] \dfrac{\partial C}{\partial \phi} = 0 \end{cases}$$

$$\Rightarrow \vec{\nabla C} = C_0 \sin\theta \, \frac{2r}{R^2} e^{-\frac{r^2}{R^2}} \left(1-\left(\frac{r}{R}\right)^2\right) \hat{e}_r + C_0 \frac{r}{R^2} e^{-\left(\frac{r}{R}\right)^2} \cos\theta \, \hat{\phi}$$

$$\vec{\nabla C}\Big|_{\substack{r=100\,\mu m \\ \theta = 0}} = \frac{C_0\,(100\,\mu m)}{(100\,\mu m)^2} \, e^{-1} \, \hat{\phi} \Rightarrow \hat{\phi} \text{ is the direction}$$

Example 3: The fluid (water) in a Bio-fuild device has the velocity (V) given bellow (in cylindrical coordinate).

$$\vec{V}(r,\theta,z) = -k\, r^2\, \hat{r} + k\, r^2 \theta\, \hat{\theta} + 2k\, r z\, \hat{z}$$

calculate the divergence of this velocity field, does the obtained value makes sense?

Solution:

Grad, Div, Curl and the Laplacian

	Cartesian Coordinates	Cylindrical Coordinates	Spherical Coordinates
Conversion to Cartesian Coordinates		$x=\rho\cos\varphi \quad y=\rho\sin\varphi \quad z=z$	$x=r\cos\varphi\sin\theta \quad y=r\sin\varphi\sin\theta \quad z=r\cos\theta$
Vector A	$A_x i + A_y j + A_z k$	$A_\rho\hat\rho + A_\varphi\hat\varphi + A_z\hat z$	$A_r\hat r + A_\theta\hat\theta + A_\varphi\hat\varphi$
Gradient $\nabla\phi$	$\frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}k$	$\frac{\partial\phi}{\partial\rho}\hat\rho + \frac{1}{\rho}\frac{\partial\phi}{\partial\varphi}\hat\varphi + \frac{\partial\phi}{\partial z}\hat z$	$\frac{\partial\phi}{\partial r}\hat r + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\hat\theta + \frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\varphi}\hat\varphi$
Divergence $\nabla\cdot A$	$\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$	$\boxed{\frac{1}{\rho}\frac{\partial(\rho A_\rho)}{\partial\rho} + \frac{1}{\rho}\frac{\partial A_\varphi}{\partial\varphi} + \frac{\partial A_z}{\partial z}}$	$\frac{1}{r^2}\frac{\partial(r^2A_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial A_\theta\sin\theta}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial A_\varphi}{\partial\varphi}$
Curl $\nabla\times A$	$\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$	$\begin{vmatrix} \frac{1}{\rho}\hat\rho & \hat\varphi & \frac{1}{\rho}\hat z \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\varphi} & \frac{\partial}{\partial z} \\ A_\rho & \rho A_\varphi & A_z \end{vmatrix}$	$\begin{vmatrix} \frac{1}{r^2\sin\theta}\hat r & \frac{1}{r\sin\theta}\hat\theta & \frac{1}{r}\hat\varphi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\varphi} \\ A_r & rA_\theta & rA_\varphi\sin\theta \end{vmatrix}$
Laplacian $\nabla^2\phi$	$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$	$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\phi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\phi}{\partial\varphi^2} + \frac{\partial^2\phi}{\partial z^2}$	$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\phi}{\partial\varphi^2}$

$$\vec\nabla\cdot\vec V = \frac{1}{r}\frac{\partial}{\partial r}(rV_r) + \frac{1}{r}\frac{\partial V_\theta}{\partial\theta} + \frac{\partial V_z}{\partial z}$$

we have:

$$V = \underbrace{-kr^2}_{V_r}\,\hat r + \underbrace{kr^2\theta}_{V_\theta}\,\hat\theta + \underbrace{2krz}_{V_z}\,\hat z$$

$$\Rightarrow \vec\nabla\cdot\vec V = \underbrace{\frac{1}{r}\frac{\partial}{\partial r}\left(r(-kr^2)\right)}_{\frac{1}{r}\times(-3kr^2)} + \underbrace{\frac{1}{r}\frac{\partial}{\partial\theta}(kr^2\theta)}_{\frac{1}{r}(kr^2)} + \underbrace{\frac{\partial}{\partial z}(2krz)}_{2kr} = \underline{0}$$

Appendix: Derivation of Grad operator in spherical coordinate

(As an example of the lengthy procedure of deriving grad, div, curl in other coordinates)

Looking at this Derivative in cartesian coordinate:

if we want to change the coordinate to spherical we should know how to write these in spherical cordinates

$$\vec{\nabla} f = \frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y} + \frac{\partial f}{\partial z}\hat{z}$$

$\hat{x}, \hat{y}, \hat{z}$ in spherical coordinates ← unit vector

$\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ in spherical coordinates ← derivatives we kow that:

$$\vec{r} = x\,\hat{x} + y\,\hat{y} + z\,\hat{z}$$

$$x = r\cos\phi\,\sin\theta$$

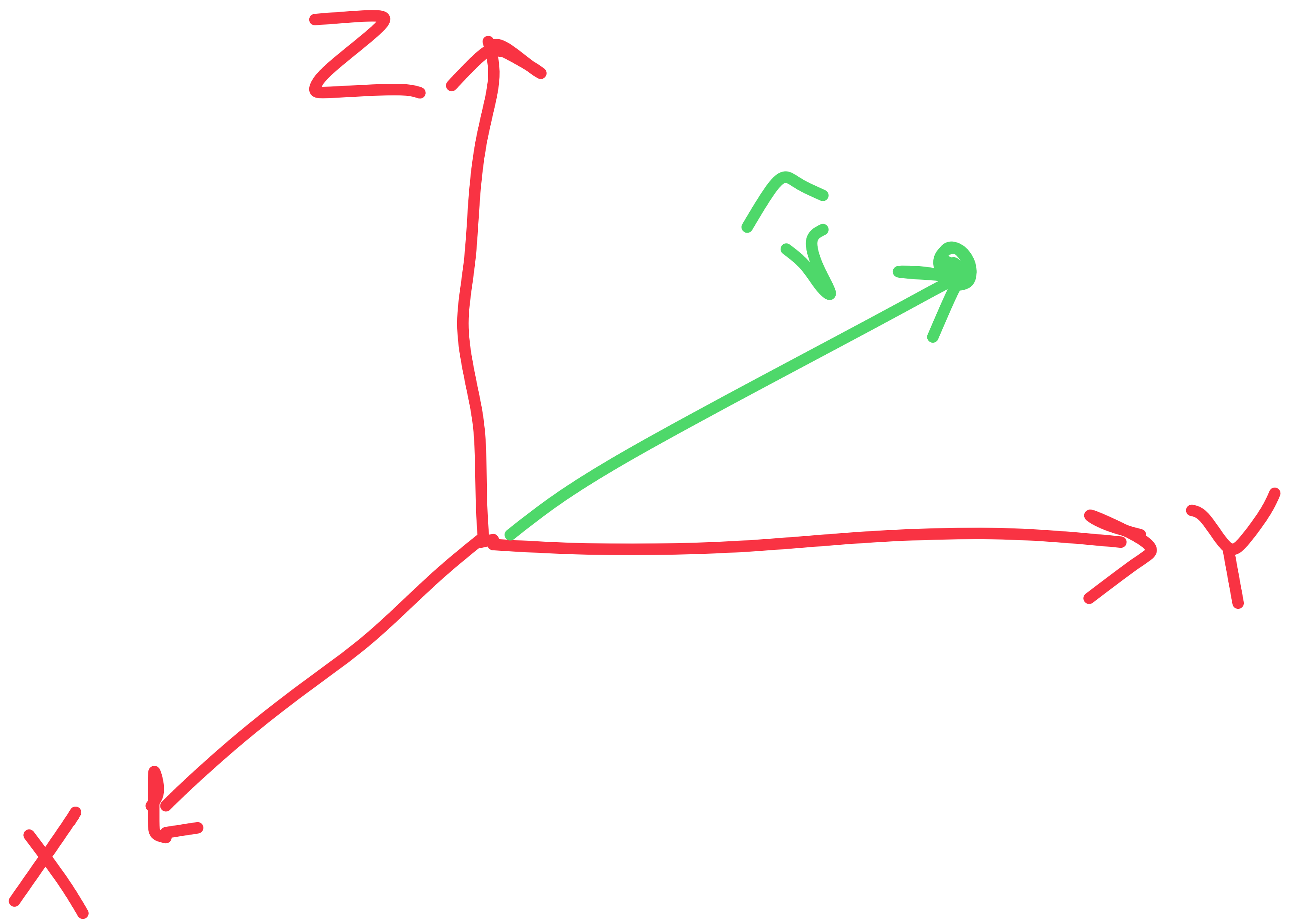
$$y = r\sin\phi\,\sin\theta$$

$$z = r\cos\theta$$

replace

$$\vec{r} = r\cos\phi\,\sin\theta\,\hat{x} + r\sin\phi\,\sin\theta\,\hat{y} + r\cos\theta\,\hat{z}$$

$$= r\,\hat{r}(\theta, \phi)$$



*Trick: if we have a field vector $\vec{f}(u, w, v)$ where u, w, v are arbitrary cordinate in an arbitrary coordinate systems we can find the unit vector $\hat{e}_u$ by doing this:

$$\vec{e}_u = \frac{\partial \vec{f}}{\partial u} \quad : \text{directional gradiant}$$

$$\hat{e}_u = \frac{\vec{e}_u}{|\vec{e}_u|}$$

now let's connect this to our case: $\vec{r} = \underbrace{r\ \hat{r}(\theta,\phi)}_{\vec{f}(r,\theta,\phi)}$

and we know $\hat{e}_u = \dfrac{\vec{e}_u}{|\vec{e}_u|}$, let's find $\hat{e}_\theta$ for example:

$$\vec{e}_\theta = \frac{\partial}{\partial\theta}\left(r\ \hat{r}(\theta,\phi)\right) = r\ \frac{\partial}{\partial\theta}\left(\hat{r}(\theta,\phi)\right)$$

$$= r\left[\cos\phi\cos\theta\ \hat{x} + \sin\phi\cos\theta\ \hat{y} - \sin\theta\ \hat{z}\right] \quad \text{i}$$

$$|\vec{e}_\theta| = \sqrt{r^2\left[\cos^2\phi\cos^2\theta + \sin^2\phi\cos^2\theta + \sin^2\theta\right]} = r\sqrt{\cos^2\theta + \sin^2\theta} = r \quad \text{ii}$$

$$\Rightarrow \boxed{\hat{e}_\theta = \cos\phi\cos\theta\ \hat{x} + \sin\phi\cos\theta\ \hat{y} - \sin\theta\ \hat{z}} \quad \checkmark$$

if we do the same for $\hat{e}_\phi$ we will get (try and see if you can obtain this):

$$\hat{e}_\phi = \sin\phi\,\hat{x} + \cos\phi\,\hat{y} \quad (2)$$

and $\hat{e}_r$:

$$\hat{e}_r = \sin\theta\cos\phi\,\hat{x} + \sin\theta\cos\theta\,\hat{y} + \cos\theta\,\hat{z} \quad (3)$$

(1), (2) and (3) are the unit vector conversions...

so we found unit vector conversions, next let's find derivatives in spherical coordinates: we have:

$$X = r\cos\phi\sin\theta$$
$$Y = r\sin\phi\sin\theta$$
$$Z = r\cos\theta$$

$$\frac{\partial X}{\partial r} = \cos\phi\sin\theta \qquad \frac{\partial Y}{\partial r} = \sin\phi\sin\theta \qquad \frac{\partial Z}{\partial r} = \cos\theta$$

$$\frac{\partial X}{\partial \theta} = r\cos\phi\cos\theta \qquad \frac{\partial Y}{\partial \theta} = r\sin\phi\cos\theta \qquad \frac{\partial Z}{\partial \theta} = -r\sin\theta$$

$$\frac{\partial X}{\partial \phi} = -r\sin\phi\cos\theta \qquad \frac{\partial Y}{\partial \phi} = r\cos\phi\sin\theta \qquad \frac{\partial Z}{\partial \phi} = 0$$

Now we can use these to calculate $\frac{\partial}{\partial r}$, $\frac{\partial}{\partial \theta}$ and $\frac{\partial}{\partial \phi}$ based on the chain rule.

$\boxed{\begin{array}{l}\text{reminder of chain rule in a function with multiple variables:}\\ f(u,w,v): \quad \frac{\partial f}{\partial \ell} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial \ell} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial \ell} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial \ell}\end{array}}$

$$\frac{\partial}{\partial r} = \frac{\partial x}{\partial r}\frac{\partial}{\partial x} + \frac{\partial y}{\partial r}\frac{\partial}{\partial y} + \frac{\partial z}{\partial r}\frac{\partial}{\partial z}$$

we know these terms

After some calculations:

$$\frac{\partial}{\partial x} = \cos\phi \sin\theta \frac{\partial}{\partial r} + \frac{\cos\phi \cos\theta}{r} \frac{\partial}{\partial \theta} + \frac{\sin\phi}{r\sin\theta} \frac{\partial}{\partial \phi} \qquad (4)$$

$$\frac{\partial}{\partial y} = \sin\phi \sin\theta \frac{\partial}{\partial r} + \frac{\sin\phi \cos\theta}{r} \frac{\partial}{\partial \theta} + \frac{\cos\phi}{r\sin\theta} \frac{\partial}{\partial \phi} \qquad (5)$$

$$\frac{\partial}{\partial z} = \cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r} \frac{\partial}{\partial \theta} \qquad (6)$$

now we can calcutate $\vec{\nabla} f$ in spherical by having

①, ②, ③, ④, ⑤, ⑥ :

$$\vec{\nabla} = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z} \xrightarrow{\text{Some long Calculations...}}$$

$$\boxed{\vec{\nabla} = \hat{r}\frac{\partial}{\partial r} + \frac{1}{r}\hat{\theta}\frac{\partial}{\partial \theta} + \frac{1}{r\sin\theta}\hat{\phi}\frac{\partial}{\partial \phi}}$$

Gradiant in spherical coordinate

Similar to spherical coordinates, we can perform the lengthy calculations to obtain the gradient, divergence, and Laplacian in cylindrical coordinates. The results for both coordinate systems will be provided at the end of the lecture.

The detailed derivations are not important. What is important is understanding how to use these formulas and remembering that the gradient, divergence, and Laplacian in cylindrical coordinates have different forms.

Grad, Div, Curl and the Laplacian

	Cartesian Coordinates	Cylindrical Coordinates	Spherical Coordinates
Conversion to Cartesian Coordinates		$x = \rho\cos\varphi \quad y = \rho\sin\varphi \quad z = z$	$x = r\cos\varphi\sin\theta \quad y = r\sin\varphi\sin\theta$ $z = r\cos\theta$
Vector A	$A_x i + A_y j + A_z k$	$A_\rho\widehat{\rho} + A_\varphi\widehat{\varphi} + A_z\widehat{z}$	$A_r\widehat{r} + A_\theta\widehat{\theta} + A_\varphi\widehat{\varphi}$
Gradient $\nabla\phi$	$\frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}k$	$\frac{\partial\phi}{\partial\rho}\widehat{\rho} + \frac{1}{\rho}\frac{\partial\phi}{\partial\varphi}\widehat{\varphi} + \frac{\partial\phi}{\partial z}\widehat{z}$	$\frac{\partial\phi}{\partial r}\widehat{r} + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\widehat{\theta} + \frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\varphi}\widehat{\varphi}$
Divergence $\nabla\cdot A$	$\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$	$\frac{1}{\rho}\frac{\partial(\rho A_\rho)}{\partial\rho} + \frac{1}{\rho}\frac{\partial A_\varphi}{\partial\varphi} + \frac{\partial A_z}{\partial z}$	$\frac{1}{r^2}\frac{\partial(r^2 A_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial A_\theta\sin\theta}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial A_\varphi}{\partial\varphi}$
Curl $\nabla\times A$	$\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$	$\begin{vmatrix} \frac{1}{\rho}\widehat{\rho} & \widehat{\varphi} & \frac{1}{\rho}\widehat{z} \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\varphi} & \frac{\partial}{\partial z} \\ A_\rho & \rho A_\varphi & A_z \end{vmatrix}$	$\begin{vmatrix} \frac{1}{r^2\sin\theta}\widehat{r} & \frac{1}{r\sin\theta}\widehat{\theta} & \frac{1}{r}\widehat{\varphi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\varphi} \\ A_r & rA_\theta & rA_\varphi\sin\theta \end{vmatrix}$
Laplacian $\nabla^2\phi$	$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$	$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\phi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\phi}{\partial\varphi^2} + \frac{\partial^2\phi}{\partial z^2}$	$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\phi}{\partial\varphi^2}$