Lecture 4: Introduction to Differential Equations

Differential equations (DEs) are crucial in both physics and biology because they provide a powerful mathematical framework to describe and predict the behavior of natural systems. DEs can be derived from fundamental laws of nature; for example, Newton's second law of motion, which leads to equations that describe how objects move under the influence of forces.

Second law: $$m \frac{d^2 x(t)}{dt^2} = F(x, t)$$

Alternatively, DEs can be formulated based on the intuition of a modeler about how a process works. For instance, the logistic growth equation in biology models how populations grow and stabilize over time, reflecting the limited resources in an environment.

growth model: $$\frac{dP(t)}{dt} = r P(t) \left(1 - \frac{P(t)}{K}\right)$$

Regardless of how DEs are obtained, solving them is an essential skill for engineers and scientists. Understanding how to work with DEs enables you to analyze and predict the behavior of complex systems, whether in engineering, physics, biology, or other fields. In this lecture, we will provide a brief overview of differential equations, exploring their significance and applications.

Two main types of DE:

Ordinary Differential Equations (ODEs) and Partial Differential Equations (PDEs)

A) ODEs contain derivatives with respect to only one variable

example:

$$\frac{df}{dx} + \frac{d^2 f}{dx^2} = f + e^{x}$$

order = 2

B) PDEs involve derivatives with respect to more than one variable

order = 3

example:

$$\frac{\partial f}{\partial x} + \frac{\partial^3 f}{\partial^3 y} = f^2(x,y) + xy$$

A) Some useful solution methods for ODEs.

are about functions with one variable

Note: dont get confused by naming. find the function an variables first:

$$\frac{d^2 f}{d^2 x} = 0$$ ✓

$f(x)$

$$\frac{d^2 y}{d^2 t} + 2\frac{dy}{dt} = y$$ ✓

$y(t)$

$$\frac{\partial^2 t}{\partial^2 x} + \frac{\partial^2 t}{\partial^2 y} = t$$ X

$t(x,y)$

A1) Solving **a first order** ODE: you can solve them by integration

General form: $$\frac{dy}{dx} + a(x)\,y = h(x)$$

Solve: define: $$P \equiv e^{\int a(x)\,dx} \longrightarrow \frac{dP}{dx} = \frac{d}{dx}\left(\int a(x)\,dx\right) \times e^{\int a(x)\,dx}$$

or: $$\frac{1}{P}\frac{dP}{dx} = a(x)$$

re-write the ODE: $\longrightarrow$

$$P\frac{dy}{dx} + \underbrace{P\,a(x)}_{\frac{1}{P}\frac{dP}{dx}}\,y = P\,h(x)$$

$$\underbrace{\qquad\qquad\qquad\qquad}$$

$$\frac{d(Py)}{dx} = P\,h(x) \Longrightarrow \boxed{y = \frac{1}{P}\int P\,h\,dx + \frac{C}{P}}$$

C: constant TBD @ $x = x_0$, $y = y_0$

Example 1 : Solving this ODE, with the given boundary conditions (B.Cs):

$$x\frac{dy}{dx} + (1-x)y = xe^{x}, \quad B.C\begin{cases} x=1 \\ y=10 \end{cases}$$

Solution:

① First, re-arranging the equation to our familiar form:

$$\frac{dy}{dx} + \underbrace{\left(\frac{1}{x}-1\right)}_{a} y = \underbrace{e^{x}}_{h} \Rightarrow \begin{cases} p = e^{\int\left(\frac{1}{x}-1\right)dx} \\ \quad = e^{\ln(x)-x} = xe^{-x} \end{cases}$$

② Apply solution:

$$\Rightarrow y(x) = \frac{e^{x}}{x}\int x e^{-x} e^{x}\, dx + \frac{Ce^{x}}{x} = \frac{xe^{x}}{2} + C\frac{e^{x}}{x}$$

$$B.C \Rightarrow \frac{e}{2} + C\frac{e}{2} = 10 \Rightarrow C = \frac{20-e}{e}$$

A2) Solving a linear, second order, homogenous ODE: There is a formula

$$\frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_2 y = 0$$

General Solution:

$y = C e^{rx}$ or more general $y = \sum_i c_i e^{r_i x}$

now we substitute ce^{rx} with y in the ODE:

$$C e^{rx} \underbrace{(r^2 + a_1 r + a_2)}_{\text{must be zero}} = 0$$

$$r_{1,2} = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2}$$

because has two roots

therefor:

$$\boxed{\begin{cases} y = C_1 e^{r_1 x} + C_2 e^{r_2 x} \\ r_{1,2} = \dfrac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2} \\ C_1, C_2 : \text{TBD by B.C} \end{cases}}$$

$\leftarrow r_{1,2} \in \mathbb{R}$ or $\mathbb{C}$

this is called characteristic eqnation

Example 2 : Solving this ODE, with the given boundary conditions (B.Cs):

$$\frac{d^2y}{dx^2} = m^2 y \quad , \quad B.C \begin{cases} x=0 \ , \ y=-1 & (1) \\ x=\infty \quad y=0 & (2) \end{cases}$$

based on the solution : $a_1 = 0$, $a_2 = -m^2$

$\Rightarrow \quad r_{1,2} = \pm m$

$\Rightarrow \quad y = C_1 e^{mx} + C_2 e^{-mx}$,

B.C (1) $\Rightarrow \quad C_1 + C_2 = -1$

B.C (2) $\quad C_1 \times \infty + C_2 \times 0 = 0 \Rightarrow C_1 = 0$

$\Rightarrow C_2 = -1$

$\Rightarrow \boxed{y = -e^{-mx}}$

A3) Solving a linear, second order, homogenous ODE a special case: we can use integration to solve it

$$\frac{\partial^2 y}{\partial x^2} = b$$

or in $\frac{\partial^2 y}{\partial x^2} + a_1 \frac{\partial y}{\partial x} + a_2 y = b$

$a_1 = a_2 = 0$

Solve by integration: $\xrightarrow{\int dx}$

$$\frac{\partial y}{\partial x} = bx + C \xrightarrow{\int dx} y = \frac{b}{2}x^2 + Cx + D$$

C and D TBD by B.C

B) PDEs.

PDEs are abont multivariate functions

$$f(u, v, w, \dots)$$

We study the solution of PDEs in case by case basis, we mainly review a method that is useful for this course called separation of variables

Let's try to solve the PDE given bellow:

function

Partial derivative

$$\frac{\partial T}{\partial x} + \frac{\partial T}{\partial y} = 0 \quad , \; T(x, y)$$

variables

$$\text{B.C:} \quad T(0,0) = 1, \quad T(1,0) = e^{2}$$

Separation of variables: IF the PDE is **linear**, that is the best first method. With this method we will try to convert PDEs to 'multiple independent ODEs' that we know how to solve.

This is the '101 solving PDEs' method, very useful and important. Note that it doesn't always work.

Step ①

To use the separation of variable method first we should try and see if our variables separate.

what does it mean?

$$T(x,y) = X(x)\, Y(y) \rightarrow$$

if we can write T as this and it satisfies the PDE, separation of variable works.

insert to PDE $\longrightarrow$ $$\frac{\partial (X(x) Y(y))}{\partial x} + \frac{\partial (X(x) Y(y))}{\partial y} = 0$$

$$\Rightarrow \quad Y(x) \frac{\partial X(x)}{\partial x} + X(x) \frac{\partial Y(y)}{\partial y} = 0$$

but this is still looking like PDEs. note that:

ⓐ $\frac{\partial^2 X(x)}{\partial x^2} = \frac{d^2 X(x)}{d x^2}$, since X only depends on x, same for Y

ⓑ we can rearrange the equation as this:

$$\frac{1}{X(x)} \frac{d X(x)}{d x} = -\frac{1}{Y(y)} \frac{d Y(y)}{d y} = \lambda \implies \begin{cases} \frac{d X(x)}{d x} = \lambda X(x) \\ \frac{d Y(y)}{d y} = -\lambda Y(y) \end{cases}$$

arbitrary constant

these are two independant ODEs

Step ② if step ① worked then finding ODE solutions

the general solutions are:

$$\begin{cases} X(x) = a\,e^{\lambda x} \\ Y(y) = b\,e^{-\lambda y} \end{cases} \quad , \quad a, b \text{ are constants}$$

$$\Rightarrow \quad T(x,y) = \overset{a \times b}{\overset{\downarrow}{C}}\, e^{\lambda(x-y)}$$

step ③ finding constants (C, λ) by applying B.C

B.C:

$$T(0,0) = c\,e^{0} = 1 \Rightarrow c = 1$$

$$T(1,0) = \overset{1}{\cancel{c}}\,e^{\lambda(1)} = e^{2} \Rightarrow \lambda = 2$$

$$\Rightarrow \quad T(x,y) = e^{2x-2y}$$

Note that this was a specific solution to demonstrate the method of separation of variables in a simple case, for more general solution of this PDE we can use the method of 'Characteristic Curves'....

Conclusion: A Simple workflow for the simple DEs that we will see regularly in this course, they are usually **homogenous** and **linear**.