Lecture 7

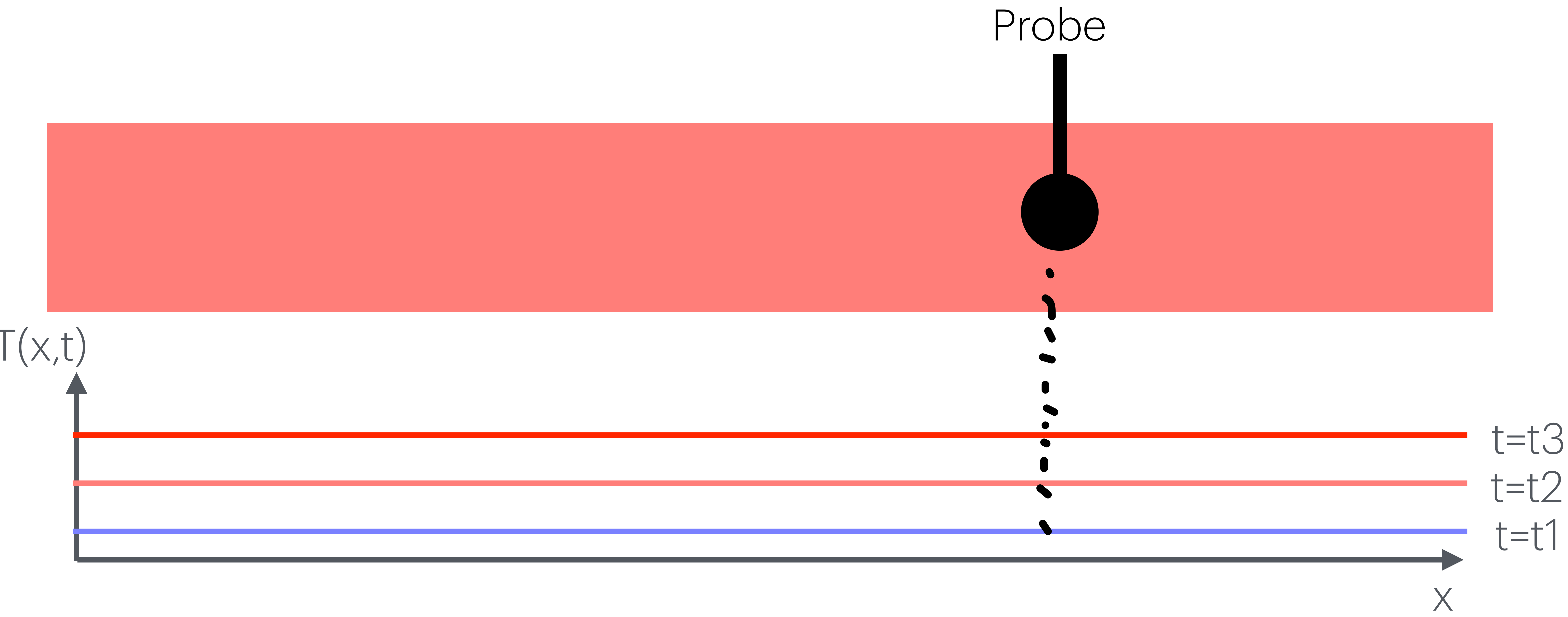
Total time derivative,

Continuity Equation

A) Total time derivative in fluids

In fluid dynamics, the total time derivative (also known as the material derivative or substantial derivative) represents the rate at which a physical quantity, such as velocity, temperature, or concentration, changes. . In the **Eulerian** perspective, it incorporates both the time-dependent change at a fixed spatial point and the variation resulting from the fluid's motion through space.

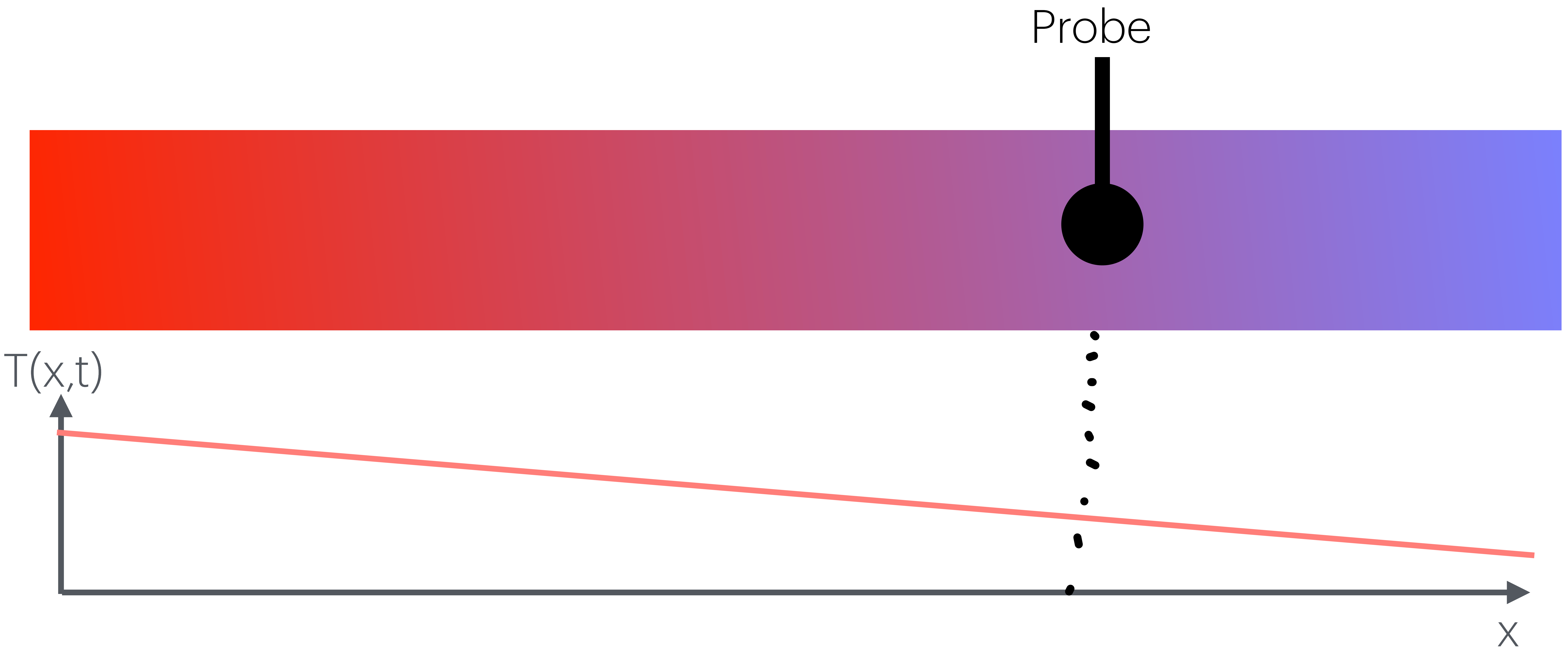
Scenario 1: The water temperature is uniform, and it is warming up. What will the probe measure as the rate of change of water's temperature?



$$\frac{\partial T_w}{\partial t}$$

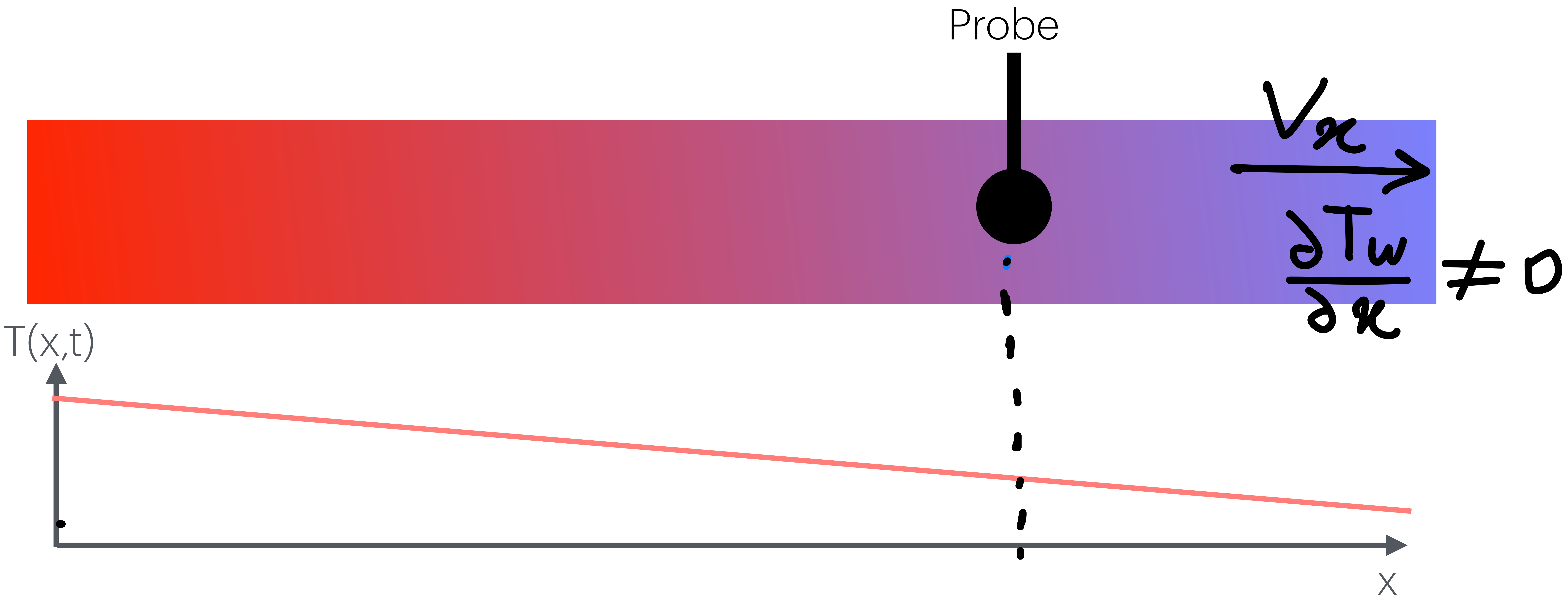
$$\frac{dT_{probe}}{dt} = \frac{\partial T_w}{\partial t}$$

Scenario 2: The water temperature is not uniform, but it is not changing by time. What will the probe measure as the rate of change of water's temperature if the water is not flowing (water is not moving).



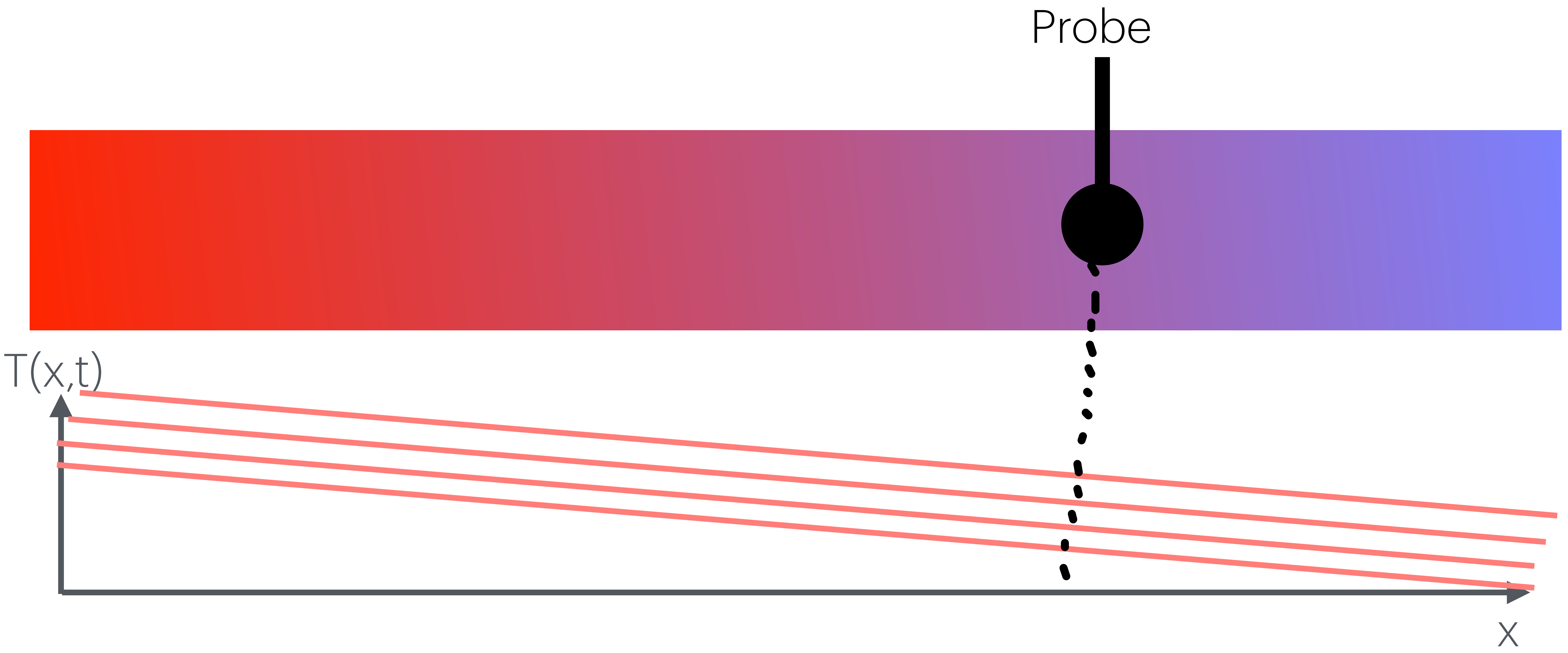
$$\frac{dT_{probe}}{dt} = 0$$

Scenario 3: The water temperature is not uniform, but it is not changing by time. What will the probe measure as the rate of change of water's temperature if the water is flowing (water is moving) with speed **vx**?



$$\frac{dT_{prob}}{dt} = \frac{\partial T_{water}}{dx} \times V_x$$

Put it all together: The water temperature is not uniform, and is changing by time. What will the probe measure as the rate of change of water's temperature if the water is flowing with speed **vx**?



$$\frac{dT_{probe}}{dt} = \frac{\partial T_w}{\partial t} + V_x \frac{\partial T_w}{\partial x}$$

in more dimenssions:

$$\frac{d\,T_{probe}}{dt} = \frac{\partial T_w}{\partial t} + V_x \frac{\partial T_w}{\partial x} + V_y \frac{\partial T_y}{\partial y} + V_z \frac{\partial T_z}{\partial z}$$

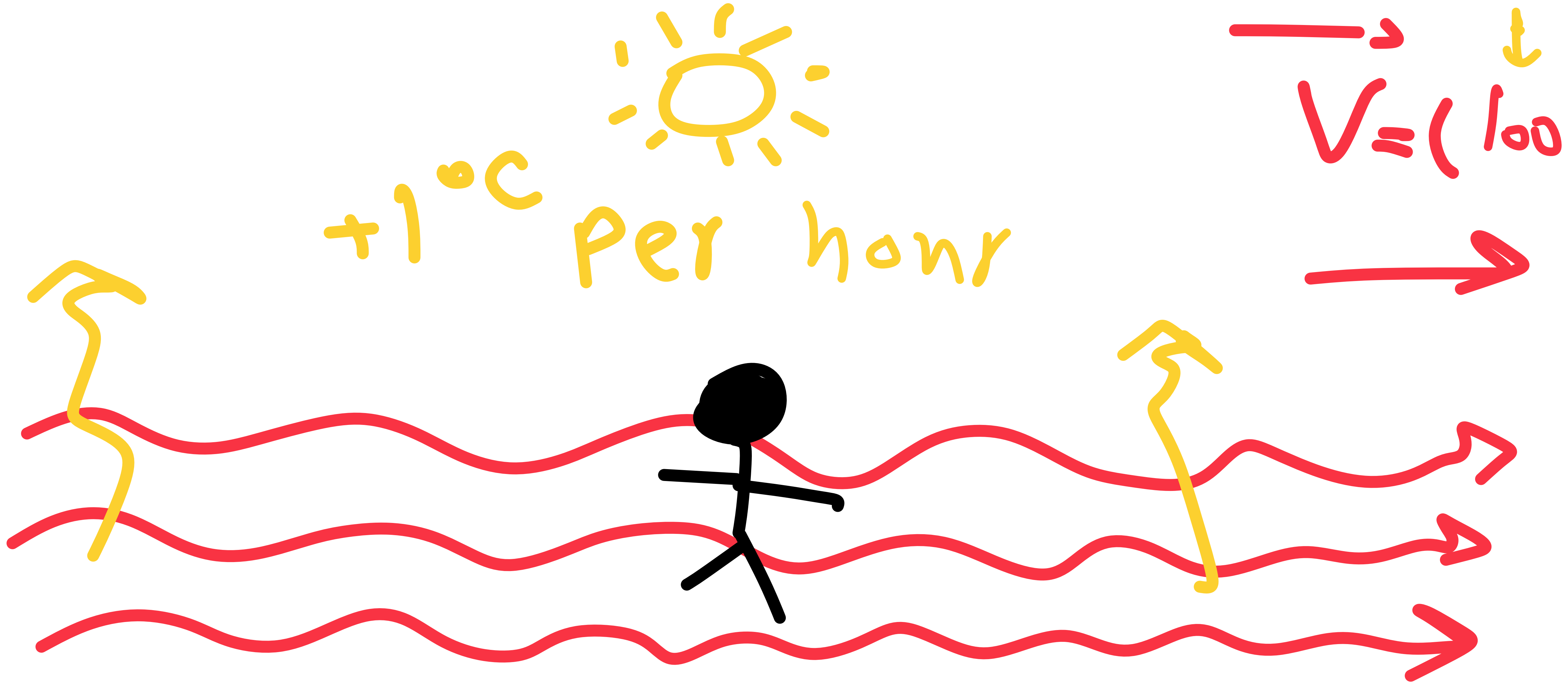
$\xrightarrow{\text{or}}$ $$\frac{d\,T_{probe}}{dt} = \frac{\partial T_w}{\partial t} + (\vec{V} \cdot \overrightarrow{\nabla T_w})$$

more general
Total derivative
$\longrightarrow$
of variable
c

$$\Rightarrow \boxed{\frac{Dc}{Dt} = \frac{\partial c}{\partial t} + (\vec{V} \cdot \overrightarrow{\nabla c})}$$

Pause point!

Example: The person below is standing in a river, what will they measure as the rate of change of temperature of water?



$$\vec{V} = (100, 0, 0) \quad \left[\frac{\text{meter}}{\text{hour}}\right]$$

$$T_w(t,x,y,z) = \underbrace{10}_{[°C]} + \underbrace{1}_{\left[\frac{°C}{t}\right]} \overset{\text{hour}}{t} + \underbrace{\frac{1}{100}}_{\left[\frac{°C}{m}\right]} \overset{\text{meter}}{x}$$

Person

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + \vec{V} \cdot \vec{\nabla} T$$

$$= +1 + (-100, 0, 0) \cdot \left(\frac{1}{100}, 0, 0\right)$$

$$= +1 - 1 = 0$$

mesures zero

B) Equation of Continuity

B-1) Background: Gauss's Divergence Theorem s a fundamental result in vector calculus that relates the flow (or flux) of a vector field through a closed surface to the behavior of the vector field inside the surface. It is widely used in fluid dynamics, electromagnetism, and other fields involving vector fields.

Theorem: The flux of a vector field **f** through a closed surface is equal to the integral of div(**f**) in the entire enclosed volume.

$\vec{f}$: vector field, continions

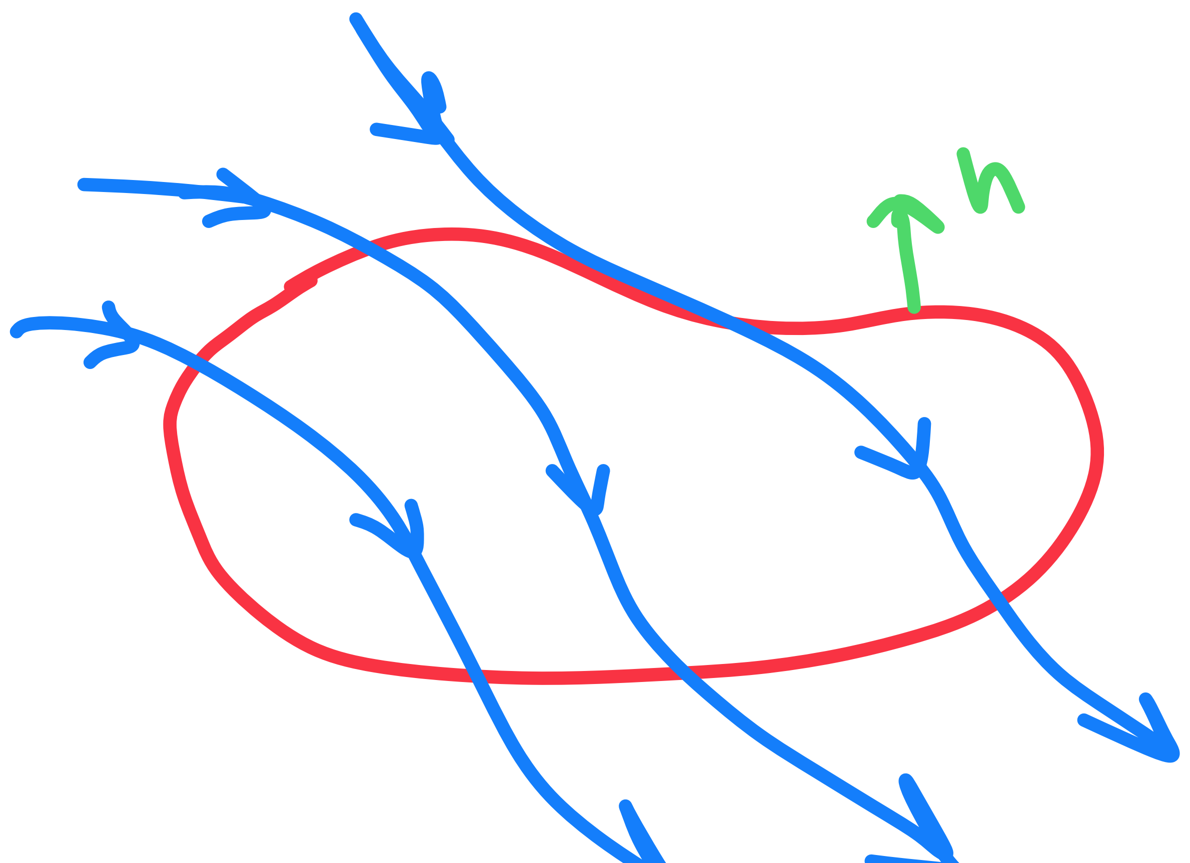
V: volume in space

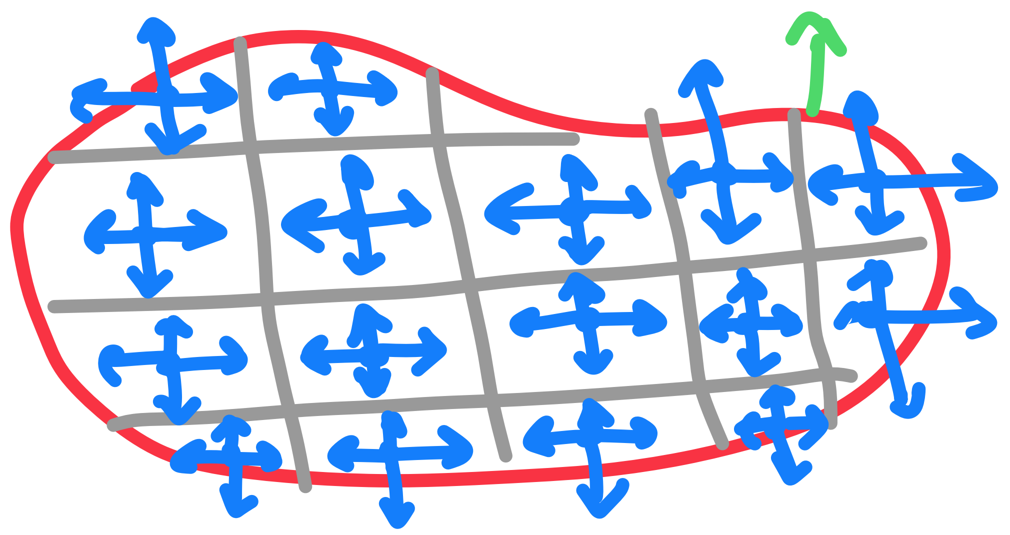
S: surface of the enclosed volume

$\vec{n}$: normal vector to the surface S

$$\oint_S \vec{f}\cdot\vec{n}\, ds = \int_V \nabla\cdot f \, dV$$

intuition:





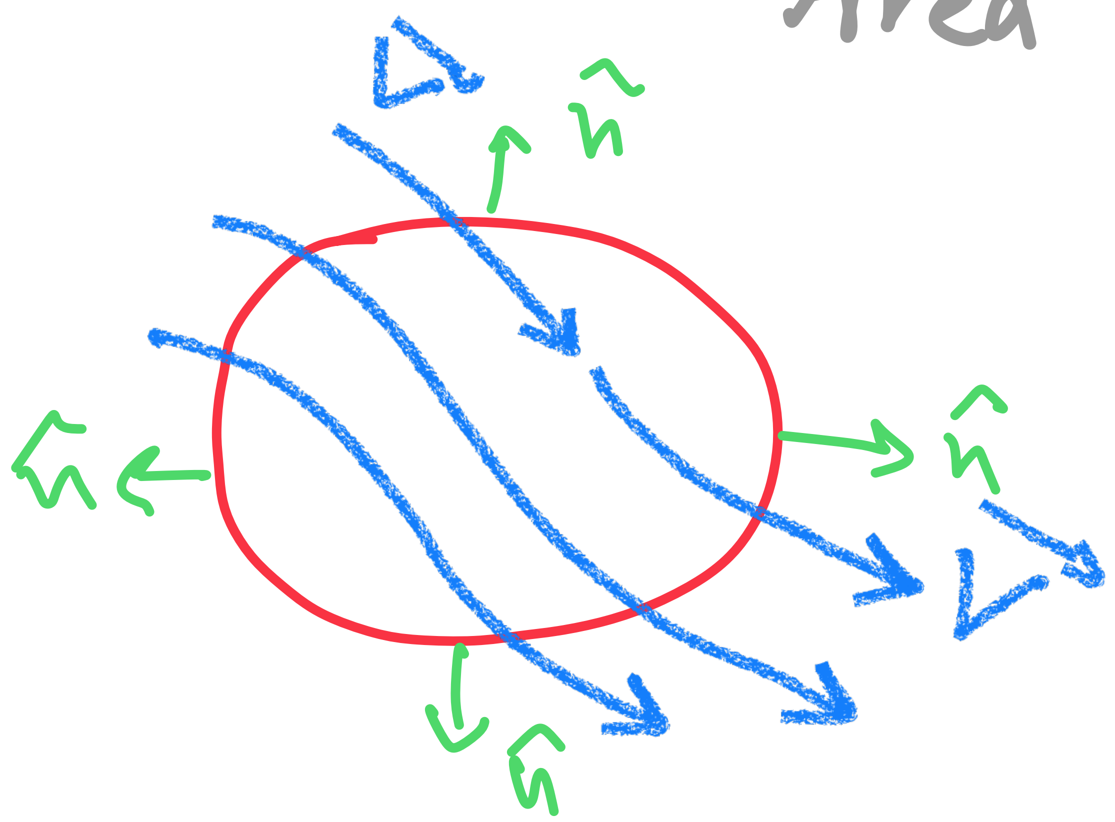
* all internal boxes cancell each other out!
* only the surface remains

B-2) Continuity Equation

Now let's write the old good mass balance equation for a fluid that is flowing in space (we assume no generation or consumption of mass):

$$\underbrace{\frac{dm}{dt}}_{\frac{d}{dt}\int_V \rho\, dV} = \overbrace{r_{in} - r_{out}}^{-\oint_S \rho \vec{V}\cdot\vec{n}} + \cancelto{0}{r_g - r_c}$$

$$\frac{\text{mass flux}}{\text{Area}} = -(\rho\vec{V})\cdot\vec{n}$$



$$\Rightarrow \frac{\partial}{\partial t}\int_V \rho\, dV = -\oint_S (\rho\vec{V})\cdot\vec{n} \xrightarrow{\text{div theorem}}$$

$$\frac{\partial}{\partial t}\int_V \rho\, dV = -\int_V \nabla(\rho\vec{V})\, dV \Rightarrow$$

$$\int_V \left(\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{V})\right) = 0 \xRightarrow{\text{true for all volumes}}$$

$$\boxed{\frac{\partial \rho}{\partial t} = -\vec{\nabla}\cdot(\rho\vec{V})}$$ continuity eq^n.

it is a fundament eq^n in fluid mechanics

Table 1.4.4-1 The equation of continuity

Rectangular coordinates
$$\frac{\partial \rho}{\partial t} + \left(\frac{\partial(\rho\, v_x)}{\partial x} + \frac{\partial(\rho\, v_y)}{\partial y} + \frac{\partial(\rho\, v_z)}{\partial z} \right) = 0 \qquad \text{(A)}$$
Cylindrical coordinates
$$\frac{\partial \rho}{\partial t} + \left(\frac{1}{r}\frac{\partial(\rho\, r\, v_r)}{\partial r} + \frac{1}{r}\frac{\partial(\rho\, v_\theta)}{\partial \theta} + \frac{\partial(\rho\, v_z)}{\partial z} \right) = 0 \qquad \text{(B)}$$
Spherical coordinates
$$\frac{\partial \rho}{\partial t} + \left(\frac{1}{r^2}\frac{\partial(\rho\, r^2 v_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\rho\, v_\theta \sin\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial(\rho\, v_\phi)}{\partial \phi} \right) = 0 \qquad \text{(C)}$$

Pause point!

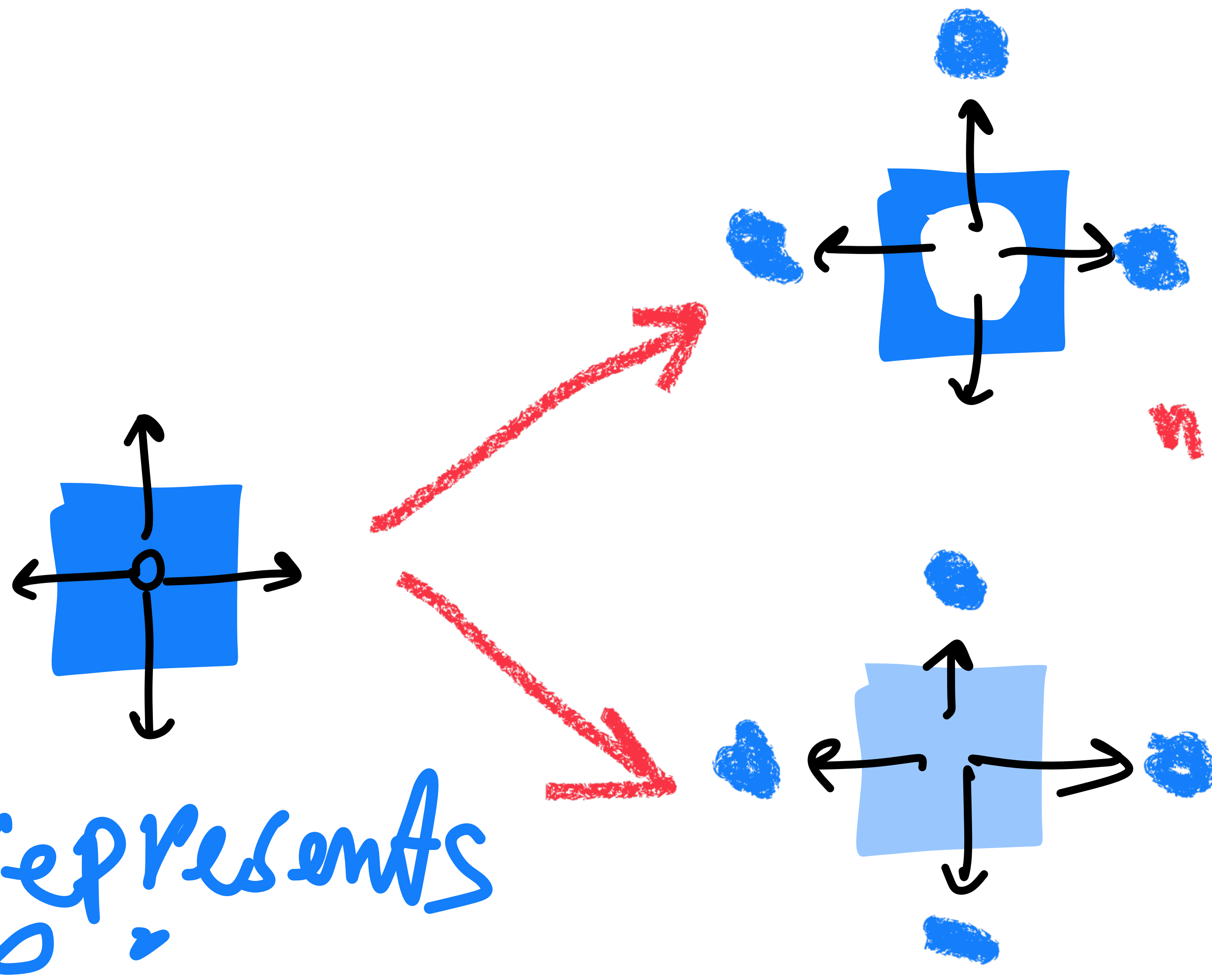
In a specific case that the fluid is not compressible:

$$\frac{\partial \rho}{\partial t} = - \vec{\nabla} \cdot (\rho \vec{V}) \xrightarrow{\rho = \text{cons}} \boxed{\vec{\nabla} \cdot \vec{V} = 0}$$

$$0 = -\rho \, \vec{\nabla} \cdot \vec{V} \Rightarrow$$

non compressible fluid.

intuitively: if $\vec{\nabla} \cdot \vec{V} \neq 0 \Rightarrow$



not allowed
not continious

not allowed
$\rho = \text{const}$

blue represents ρ

iClicker question:

Is this Velocity field possible for a water flow?

$$V(x,y,z) = 4x^2y\,\hat{x} - 2xy^2\,\hat{y} + 4xy\,\hat{z}$$

① yes becans $\vec{V}$ is continions

② yes becans $\vec{\nabla} \cdot \vec{V} = 0$

③ no, we can not say it based on $\vec{V}$

④ ① and ② together.

Review of the past 7 lectures... (if time allows…)

- **Lec 2** → Vector basics: ∇f, $\nabla \cdot F$, $\nabla \times F$, ∇f we learned what they mean and how to calculate them
- **Lec 3** → Coordinate systems; vector calculus in different coordinates. We learned that ∇f, $\nabla \cdot F$, $\nabla \times F$, ∇f are different in different coordinate systems and we should use a table for that.
- **Lec 4** → ODEs (first and second order), simple PDEs (separable ones, turn to ODE)
- **Lec 5** → Mass balance principle → application in steady-state (S.S.), for example bioreactors (book 1.1 and 1.3)
- **Lec 6** → Application of Lec 5 in pharmacokinetics. It was non-S.S., and we used Lec 4 to solve ODEs. (not in the book)
- **Lec 7** → Application of Lec 5 (mass balance) in fluids, we used Lec 2 to formulate the continuity equation. (book 1.4)