Polynomials

Algebra Handout
12 Nov 2022

---

1 Definitions

Definition 1. Let $n \geq 0$ be an integer. A polynomial of degree n is a function of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = \sum_{k=0}^{n} a_k x^k$$

where $a_n, a_{n-1}, \cdots, a_1, a_0$ are constants and $a_n \neq 0$. We also include the zero function ($f(x) = 0$ for all x) as a polynomial and usually say it has degree $-\infty$ (or just agree it has lower degree than any other polynomial).

Definition 2. The numbers $a_n, \cdots, a_0$ defined above are called the **coefficients** of f.

Definition 3. The solution(s) to the equation $f(x) = 0$ are called the **root/zero(s)** of $f(x)$.

Definition 4. A polynomial is said to be **monic** if the **leading coefficient** a_n is 1.

Exercise. Which of the following are polynomials? Which are monic?

- $2x^2 + 1$
- 2
- $1 + x + x^2 + \cdots$
- $x^{100} - \pi x + 1$
- 2^x
- x^{-1}

Exercise. A polynomial of degree 101 has no real root. Is this possible?

Exercise. A polynomial of degree 102 has no real root. Is this possible?

Exercise. $\deg(fg) = \deg f + \deg g$. Can you see why we say $\deg 0 = -\infty$ now?

Exercise. $\deg(f + g) \leq \max(\deg f, \deg g)$.

2 Division Algorithm

Given two positive integers m and n, we can always write $m = nq + r$ uniquely where $r < n$. This is called the **division algorithm**. There is also an analogous statement for polynomials:

Theorem 1. Given nonzero polynomials f, g, there exist unique polynomials q, r such that

$$f = gq + r, \qquad \deg r < \deg g$$

Example. Say $f(x) = x^5 + 3x^2 - 2x + 1$ and $g(x) = x^2 - 1$, then

$$x^5 + 3x^2 - 2x + 1 = (x^2 - 1)(x^3 + x + 3) + (-x + 4)$$

You have probably learnt how to find q and r via long division, but being able to find some q and r doesn't guarantee *uniqueness*. So let's prove it.

Proof. **[Existence]** Stating the exact procudure of long division is enough to prove existence, but let's prove it another way. We induct on the degree of f. The case $\deg f = 0$ is easy (left as an exercise). Assume for all f, g with $\deg f \le k$ there exists such q, r.

Let's say $\deg f = k + 1$. If $\deg g > \deg f$, then $q = 0, r = f$ works. Or else when $\deg g \le \deg f$, say $g(x) = g_n x^n + \cdots$ and $f(x) = f_m x^m + \cdots$, notice that

$$f(x) - (f_m / g_n) x^{m-n} g(x)$$

has degree $\le k$ because the leading term of f is annihilated. Therefore, by inductive hypothesis there exists q_1, r_1 such that

$$f - (f_m/g_n)x^{m-n} g = gq_1 + r_1, \qquad \deg r_1 < \deg g.$$

Rewriting the equation gives

$$f = \left((f_m/g_n)x^{m-n} + q_1\right) g + r_1, \qquad \deg r_1 < \deg g.$$

Therefore $q = (f_m/g_n)x^{m-n} + q_1$ and $r = r_1$ works.

[Uniqueness] The most common way of proving uniqueness is performing subtraction in some way. Say there exist $(q_1, r_1), (q_2, r_2)$ such that

$$\begin{aligned} f &= gq_1 + r_1, \qquad \deg r_1 < \deg g \\ f &= gq_2 + r_2, \qquad \deg r_2 < \deg g \end{aligned}$$

then by subtracting we get

$$0 = g(q_1 - q_2) + (r_1 - r_2).$$

We see that $q_1 - q_2 = 0$, otherwise $\deg g(q_1 - q_2) \ge \deg g > \deg(r_1 - r_2)$ and their sum cannot be equal to 0 on the LHS. But when $q_1 - q_2 = 0$ the equation forces $r_1 - r_2 = 0$. Therefore $q_1 = q_2, r_1 = r_2$. □

Note. The above theorem needs an extra condition to hold if, let's say, we are only allowed to have integer coefficients. What is the condition? (Hint, it is about g).

Corollary 1 (Remainder/Factor Theorem). If f is a polynomial with root a, then $f(x) = (x - a)g(x)$ for some polynomial g. If, on the other hand, $f(a) = r$, then $f(x) = (x - a)g(x) + r$.

A result of the factor theorem is,

Theorem 2. A nonzero polynomial of degree n cannot have more than n roots. □

Corollary 2. If two nonzero polynomials f, g with degree $\leq n$ agree at more than n points (meaning: there exist $n+1$ distinct x_i such that $f(x_i) = g(x_i)$ for all i), then $f = g$.

3 Lagrange Interpolation

Question: Can you find a polynomial f with real coefficients such that

$$f(1) = 2, f(2) = 3, f(3) = 4?$$

How about a polynomial g with real coefficients such that

$$g(1) = 2, g(2) = 3, g(3) = \pi?$$

Before we answer these questions, let's notice from Corollary 2 that if we only allow f or g to have degree ≤ 2, then they must be unique (if they exist, of course). Let's prove that they must exist.

- For the first example, $f(x) = x + 1$ obviously works. Therefore by Corollary 2 there are no other polynomials of degree ≤ 2 that work.
- For the second example, existence is not so easy. Let's break down the problem into several pieces, and please try them yourself:
  - First we find three polynomials g_1, g_2, g_3 with deg ≤ 2 satisfying:

$$\begin{gathered} g_1(1) = 2, g_1(2) = 0, g_1(3) = 0 \\ g_2(1) = 0, g_2(2) = 3, g_2(3) = 0 \\ g_3(1) = 0, g_3(2) = 0, g_3(3) = \pi \end{gathered}$$

  - Then we stitch them together to form our final g:

$$g(x) = g_1(x) + g_2(x) + g_3(x)$$

  You should get something like

$$g(x) = 2\frac{(x-2)(x-3)}{(1-2)(1-3)} + 3\frac{(x-1)(x-3)}{(2-1)(2-3)} + \pi\frac{(x-1)(x-2)}{(3-1)(3-2)}$$

More generally,

Theorem 3 (Lagrange Interpolation). If $x_1, \cdots, x_n$ are distinct numbers and $y_1, \cdots, y_n$ are numbers (not necessarily distinct) then the (unique) polynomial f of deg $< n$ such that

$f(x_i) = y_i$ for all i is

$$f(x) = \sum_{i=1}^{n} y_i \prod_{j \neq i} \frac{x - x_j}{x_i - x_j}$$

Exercise. $f(x)$ leaves a remainder of 2 when divided by $(x-3)$ whereas $f(x)$ leaves a remainder of 1 when divided by $(x-4)$. Find the remainder polynomial when $f(x)$ is divided by $(x-3)(x-4)$.

Exercise. Let $f_0(x)$ be the polynomial obtained by interpolation. Find a way to express all polynomials $f(x)$ such that $f(x_i) = y_i$ for all i.

4 An IMO Example

IMOSL2019A5. Let $x_1, \cdots, x_n$ be distinct reals. Prove that

$$\sum_{i=1}^{n} \prod_{j \neq n} \frac{1 - x_i x_j}{x_i - x_j} = \begin{cases} 0 & \text{if } n \text{ even} \\ 1 & \text{if } n \text{ odd} \end{cases}$$

We first want a polynomial which we can easily incorporate into the Interpolation Formula. Therefore we want something like $y_i = f(x_i) = \prod_{j \neq i}(1 - x_i x_j)$, but the closest thing we can have is probably by choosing

$$f(x) = (1 - x_1 x)(1 - x_2 x) \cdots (1 - x_n x)$$

except this generates a new factor $(1 - x_i^2)$. Let's just see what happens when we apply interpolation on $f(x)$ anyway:

$$f(x) = \sum_{i=1}^{n} (1 - x_i^2) \prod_{j \neq i} \frac{(1 - x_i x_j)(x - x_j)}{x_i - x_j}.$$

How do we 'erase' the $(1 - x_i^2)$? The idea turns out to be adding more 'nodes' to the interpolation. If we interpolate not just on $x_1, \cdots, x_n$ but also on $1, -1$, then the denominator in the above expression has new factors of $x_i - 1$ and $x_i + 1$, cancelling with the $(1 - x_i^2)$. Let's do exactly that then. Redoing interpolation on $1, -1, x_1, \cdots, x_n$,

$$f(x) = f(1)\frac{x+1}{1+1}\prod_{j} \frac{x - x_j}{1 - x_j} + f(-1)\frac{x-1}{-1-1}\prod_{j} \frac{x - x_j}{-1 - x_j} + \sum_{i=1}^{n} f(x_i) \frac{(x-1)(x+1)}{(x_i - 1)(x_i + 1)} \prod_{j \neq i} \frac{x - x_j}{x_i - x_j}$$

Each term above is a polynomial with degree $n+1$, but we know by definition that f has degree n. That means the coefficient of x^{n+1} must miraculously vanish after summing everything. By looking at the leading coefficients:

$$0 = f(1)\frac{1}{1+1}\prod_{j} \frac{1}{1 - x_j} + f(-1)\frac{1}{-1-1}\prod_{j} \frac{1}{-1 - x_j} + \sum_{i=1}^{n} f(x_i) \frac{1}{(x_i - 1)(x_i + 1)} \prod_{j \neq i} \frac{1}{x_i - x_j}$$

Substituing $f(1) = \prod_j (1 - x_j)$, $f(-1) = \prod_j (1 + x_j)$ and $f(x_i)$ gives

$$0 = \frac{1}{2} - \frac{(-1)^n}{2} - \sum_{i=1}^{n} \prod_{j \neq i} \frac{1 - x_i x_j}{x_i - x_j}. \qquad \square$$

5 Exercises

1. Find the remainder when x^{100} is divided by $x^2 - x - 6$.

2. If $a \in \mathbb{R}$ is a root of a polynomial with integer coefficients, then there is a *unique* monic polynomial with integer coefficients, with a as a root, with minimal degree. This is called the **minimal polynomial** of a.

3. Find a polynomial in $\mathbb{Z}[x]$ with root $\sqrt{2} + \sqrt{3}$.

4. Prove

   $$\frac{a^2}{(a-b)(a-c)} + \frac{b^2}{(b-a)(b-c)} + \frac{c^2}{(c-a)(c-b)} = 1.$$

   (As Peter Lorre would say, '*Do it ze kveek vay, Johnny!*') How about

   $$\frac{a^3}{(a-b)(a-c)} + \frac{b^3}{(b-a)(b-c)} + \frac{c^3}{(c-a)(c-b)}?$$

5. Consider the system of equations

   $$\begin{aligned} a + 8b + 27c + 64d &= 1 \\ 8a + 27b + 64c + 125d &= 27 \\ 27a + 64b + 125c + 216d &= 125 \\ 64a + 125b + 216c + 343d &= 343 \end{aligned}$$

   Find the value of $64a + 27b + 8c + d$.

6. Show that for all $n \in \mathbb{N}$, there is a polynomial P_n such that $\cos nx = P_n(\cos x)$.

7. Let P be a polynomial with positive coefficients. Prove that if

   $$P\left(\frac{1}{x}\right) \geq \frac{1}{P(x)}$$

   holds for $x = 1$, then it holds for all $x > 0$.