Complex Numbers

Algebra Handout
12 Nov 2022

Summary of Extension to Complex Numbers:

$$\text{Peano Axioms} \to \mathbb{N} \xrightarrow{(-)} \mathbb{Z} \xrightarrow{(\div)} \underbrace{\mathbb{Q} \xrightarrow{\text{completion}} \mathbb{R} \xrightarrow{i^2=-1} \mathbb{C}}_{\text{fields}}$$

1 Basic Definitions

Just as -1 is not an element in $\mathbb{N}$ which satisfies $(-1)+1=0$, or $\sqrt{2}$ is not an element in $\mathbb{Q}$ which satisfies $\sqrt{2}^2-2=0$, we are going to let i be a new element that is not in $\mathbb{R}$ which satisfies $i^2+1=0$. Of course, after adding i into $\mathbb{R}$, we want to allow ourselves to do basic operations $(+,-,\times,\div)$ on it. Therefore, all numbers in the form $a+bi$ where a,b are real should be inside too. In fact, that is enough, as the sum, difference, product, or quotient of any two such numbers is also of the same form:

$$\begin{aligned}
(a+bi)\pm(c+di) &= (a\pm c)+(b\pm d)i \\
(a+bi)\cdot(c+di) &= (ac-bd)+(ad+bc)i \\
\frac{a+bi}{c+di} &= \frac{(a+bi)(c-di)}{(c+di)(c-di)} \\
&= \frac{ac+bd}{c^2+d^2}+\frac{bc-ad}{c^2+d^2}i
\end{aligned}$$

(Of course, the last one only holds except when $c^2+d^2=0$, but this means $(c,d)=(0,0)$ anyway). We hence define the set

$$\mathbb{C} = \{a+bi \mid a,b\in\mathbb{R}\}$$

of **complex numbers**. Note that real numbers are automatically complex numbers.

For a complex number $z=a+bi$ where $a,b\in\mathbb{R}$, we say $a=\mathrm{Re}(z)$ and $b=\mathrm{Im}(z)$ are the **real part** and **imaginary part** of z respectively. Two complex numbers are said to be equal if and only if their real **and** imaginary parts are equal.

Therefore a complex number is specified completely by two real numbers, its real and imaginary part. This looks familiar to the concept of coordinates, where each point in 2D space can be specified completely by two real numbers too, its $x-$ and $y-$ coordinate. In fact, we are going to introduce the **Argand Diagram**, which is a *number plane* instead of a number line. Every complex number is represented by a point on this plane:

Using this diagram, we can define a few more things:

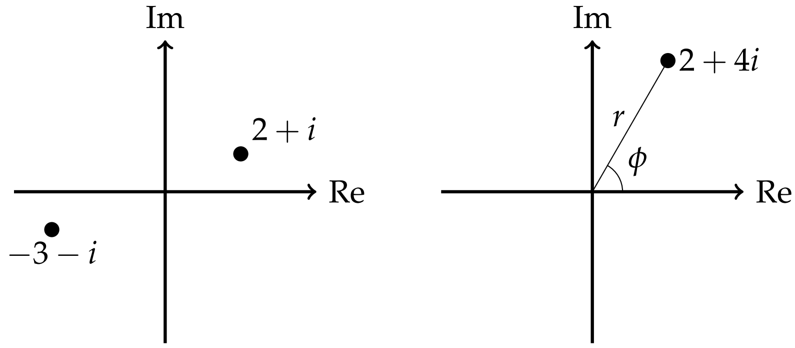

Figure 1: Complex Plane

Magnitude / Absolute Value
The magnitude $|z|$ of $z = a + bi$ $(a, b \in \mathbb{R})$ is defined to be its *distance from 0* on the Argand diagram. Therefore $\boxed{|a + bi| = \sqrt{a^2 + b^2}}$. It is represented by r in Figure 1(b).

Argument
The argument $\arg(z)$ of $z = a + bi$ $(a, b \in \mathbb{R})$ is defined to be the angle $\theta \in [-\pi, \pi)$ subtended by the complex number from the positive real axis (positive angles measured anticlockwise). It is represented by ϕ in Figure 1(b).

Complex Conjugate
The conjugate $\overline{z}$ of $z = a + bi$ $(a, b \in \mathbb{R})$ is simply the number $a - bi$. Note that $z\overline{z} = |z|^2$.

2 Geometry of Complex Numbers

Using the complex plane, complex numbers can be also seen as 'vectors', with the usual addition and subtraction.

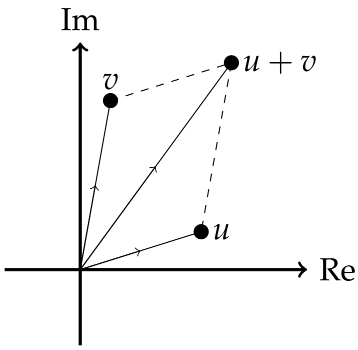

Figure 2: Complex Plane

However, it might not be so obvious how multiplication and division manifests on the plane. First of all, let's write the real and imaginary part in terms of $r = |z|$ and $\theta = \arg(z)$:

$$z = a + bi = (r \cos \theta) + (r \sin \theta)i = r(\cos \theta + i \sin \theta)$$

This is known as the **polar form** of complex numbers. If we take two numbers z_1, z_2:

$$\begin{aligned} z_1 &= r(\cos\theta + i\sin\theta) \\ z_2 &= R(\cos\phi + i\sin\phi) \\ \Rightarrow z_1 z_2 &= Rr(c_\theta + is_\theta)(c_\phi + is_\phi) \\ &= Rr\left[(c_\theta c_\phi - s_\theta s_\phi) + i\left(s_\theta c_\phi + c_\theta s_\phi\right)\right] \\ &= Rr(c_{\theta+\phi} + is_{\theta+\phi}) \end{aligned}$$

Therefore, when two complex numbers are multiplied,

Their magnitudes **multiply** but their arguments **add**.

$$|z_1 z_2| = |z_1||z_2|, \qquad \arg(z_1 z_2) \equiv \arg(z_1) + \arg(z_2) \pmod{2\pi}$$

The following diagram illustrates this:

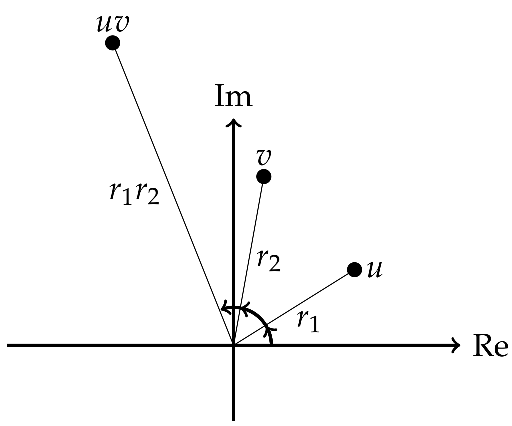

Figure 3: Multiplication

For geometry lovers, multiplying by a fixed complex number essentially applies a spiral similarity on the plane.

Question. How to rotate a complex number *clockwise* by 90 degrees about the origin?

Question. Solve the *three-square problem* ($\alpha + \beta + \gamma = ?$) by using complex numbers.

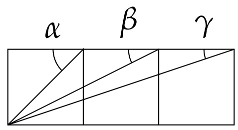

Figure 4: Three Squares

Question. Find a similar rule for division. How do you apply an inversion transformation with respect to the unit circle?

3 Conjugate

The conjugate has a very special property that it is closed under all four operations:

- $\overline{u+v} = \overline{u} + \overline{v}$
- $\overline{u-v} = \overline{u} - \overline{v}$
- $\overline{u \cdot v} = \overline{u} \cdot \overline{v}$
- $\overline{u \div v} = \overline{u} \div \overline{v}$

Exercise. Verify these.

4 Exponential Form

In $\mathbb{R}$, the function $e^x, \sin x, \cos x$ can be defined via Taylor Series by

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{k \geq 0} \frac{x^k}{k!} \\ \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots = \sum_{k \geq 0} \frac{(-1)^k x^{2k+1}}{(2k+1)!} \\ \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots = \sum_{k \geq 0} \frac{(-1)^k x^{2k}}{(2k)!} \end{aligned}$$

Results from *complex analysis* show that these power series all converge in $\mathbb{C}$ too (ratio test suffices). Therefore we can plug complex values[1] into the above series.

Exercise. Prove **Euler's Formula**:

$$e^{ix} = \cos x + i \sin x$$

Therefore, all complex numbers can be written as

$$r(\cos\theta + i\sin\theta) = re^{i\theta}.$$

This is the **exponential form** of complex numbers.

Exercise. Prove the geometric interpretation of multiplication again using exponential form.

[1]at this point, e^x where x is complex doesn't make much sense anymore. The best way is to not think of e^x as literally $e = 2.718\cdots$ raised to some power, but rather just a function that satisfies $e^{x+y} = e^x \cdot e^y$.

5 Fundamental Theorem of Algebra

Now we come to the algebraic beauty of $\mathbb{C}$:

Fundamental Theorem of Algebra. Every nonzero polynomial with complex coefficients can be factored into a product of linear factors.

As a result, **every complex polynomial has a root.** In fact, this statement is equivalent to the statement above, can you see why?

We will not prove this theorem in this handout (I might prove it in the lecture if we have time), but let's ponder the significance of this theorem. Even polynomials like $x^{40} + x^2 + 100$ which obviously doesn't cross the x-axis at all on a cartesian graph, do have solutions in $\mathbb{C}$! (In fact, forty roots because it can be factored into forty linear factors!)

However, there may be repeated roots. So we can at most say a complex polynomial of degree n has at most n roots. However, this lemma may help:

Lemma 1. A polynomial f has a repeated root if and only if f and f' have a common root.

Proof. ($\Rightarrow$) If $f(x) = (x-a)^2 g(x)$,

$$f'(x) = 2(x-a)g(x) + (x-a)^2 g'(x)$$

has a common root a with f. ($\Leftarrow$, contrapositive) If f has no repeated roots, then $f(x) = k\prod_i (x - a_i)$ where a_i are distinct. Then by the Generalised Product Rule

$$f'(x) = k \sum_i \prod_{j \neq i} (x - a_i)$$
$$\Rightarrow f'(a_t) = k \prod_{i \neq t} (a_t - a_i) \neq 0$$

therefore f' does not share any roots with f. $\square$

Example. The polynomial $P(x) = x^{40} + x^2 + 100$ has derivative $40x^{39} + 2x$, none of whose roots are roots of P (please check this), therefore P has forty distinct complex roots!

Question. How many roots does $x^n = c$ $(n \in \mathbb{N}, c \in \mathbb{C})$ have?

6 Roots of Unity and General Roots

We now look at the equation $x^n = 1$. According to the previous section this should have n distinct roots. Since $|x|^n = |x^n| = 1$ and $|x|$ is a positive real, $|x|$ can only be 1. Therefore $x = e^{i\theta}$. Then $e^{in\theta} = 1$. The obvious solutions are $\theta = 0, \frac{2\pi}{n}, \frac{4\pi}{n}, \cdots, \frac{(2n-2)\pi}{n}$. In fact, that's already n different solutions, and hence they are all of them! Therefore,

$$x^n = 1 \quad \Leftrightarrow \quad x = e^{2\pi i k/n} \ (k = 0, 1, \cdots, n-1).$$

These are called the n**-th roots of unity**. The one with the smallest positive argument, $e^{2\pi i/n}$, we will call it the *principal* n-th root of unity ζ_n. Therefore all the n-th roots of unity are just powers of ζ_n.

For the general case $x^n = c$ where $c \neq 0$, we can first write $c = Re^{i\theta}$ where $R \in \mathbb{R}$. Then

$$x = R^{1/n} e^{i\theta/n}$$

is one solution ($R^{1/n}$ is well-defined because $R > 0$ is real). In fact, if z is a solution then $z\zeta_n, z\zeta_n^2, \cdots, z\zeta_n^{n-1}$ are solutions. We've found n roots, and thus we've found all!

Note. Focus on $(i-1)^2 = (1-i)^2 = -2i$. Unlike $\mathbb{R}$ where we can define the non-negative square-root function $\sqrt{x}$, it makes no sense to say which one is $\sqrt{-2i}$ and which one is $-\sqrt{2i}$. Both $1-i$ and $i-1$ are roots to $x^2 = -2i$, so they are both square roots of $-2i$, and we'll leave it as that. However, you may define $\sqrt{x}$ with some rules, e.g. the one with the smallest argument, but we will not do that here.

7 Exercises

1. Let $n \in \mathbb{N}$. Simplify $(x-1)(x-e^{2\pi i/n})(x-e^{4\pi i/n})\cdots(x-e^{(2n-2)\pi i/n})$.

2. Let P be a real polynomial. Prove that if $x \in \mathbb{C}$ is a root of P, then $\overline{x}$ is a root of P too.

3. Prove that all real polynomials can be factored into a product of real linear factors and real irreducible quadratic factors.

4. Simplify $(2+e^{2\pi i/n})(2+e^{4\pi i/n})\cdots(2+e^{(2n-2)\pi i/n})$.

5. What's the conjugate of e^{ix} where x is real?

6. Write $\cos x$ and $\sin x$ in terms of e^{ix} and e^{-ix}. Prove $\cos 2x = 2\cos^2 x - 1$. In fact, prove any trigonometric identity you know.

7. Let x be real. Simplify the geometric series $\sum\limits_{k=0}^{n-1} e^{2\pi ik/n}$. Beware of pesky edge cases.

8. Write $\mathrm{Re}(x)$ and $\mathrm{Im}(x)$ in terms of x and $\overline{x}$.

9. Let θ and ϕ be real numbers. Find the real part of $\dfrac{1}{e^{i\theta}+e^{i\phi}}$. $\left[\text{Answer: } \dfrac{\cos\theta+\cos\phi}{1+\cos(\theta-\phi)}\right]$

10. Simplify $\binom{n}{0} + \binom{n}{3} + \binom{n}{6} + \cdots$.

